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Stage comparison, fixed points, and least fixed points in Kripke-Platek environments

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Abstract Let T be Kripke-Platek set theory with infinity extended by the axiom (Beta) plus the schema that claims that every set-bounded Σ definable monotone operator from the collection of all sets to $Pow(a)$ for some set a has a fixed point. Then T proves that every such operator has a least fixed point. This result is obtained by following the proof of an analogous result for von Neumann-Bernays-Gödel set theory in Sato [6], with some minor modifications.

1 Introduction

The main result of this article is the following: Take Kripke-Platek set theory with infinity and add the axiom (Beta) plus the axiom schema that every Σ definable monotone operator F from the collection of all sets to $Pow(a)$ for some set a has a fixed point. Then it can be shown that any such operator has a least fixed point.

In Sato [6] an analogous result has been proved, among other things, for von Neumann-Bernays-Gödel set theory. Although working in a much weaker environment, we can follow [6] in large parts and, therefore, the present paper can be understood as a supplement to [6]. However, in a few cases some technical definitions and some proofs have to be modified.

The main technical ingredients of the following approach are on the one hand the formulation and proof of the stage comparison theorem in Moschovakis [5] and on the other hand Sato's abstract approach to well-founded stage comparison relations. Also some unpublished notes [8] of Steila have been useful.

In Jäger and Steila [2] the systematic investigation of fixed point axioms and related principles in Kripke-Platek environments has been initiated, and further results along these lines are presented in Jäger and Steila [3]. It is planned that [4] provides a complete analysis of the situation in the presence of the axiom (Beta). In this context the results of this article will play a central role.

2 The general syntactic environment

All theories considered in this article are extensions of Kripke-Platek set theory with infinity. However, we do not work in standard first order KP but turn to its class extension KP^c, introduced in Jäger and Steila [2]. The reason for this step is notational convenience: As we shall see, this theory is very well suited for speaking about operators.

Let $\mathcal{L}$ be the standard language of first order set theory with $\in$ and $=$ as the only non-logical symbols and countably many set variables $a, b, c, \dots$ (possibly with subscripts). The formulas

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of $\mathcal{L}$ and the syntactic categories of Δ_0, Σ, Π, Σ_n, and Π_n formulas are defined as usual. $\mathcal{L}^c$ is the extension of $\mathcal{L}$ by countably many class variables $F, G, H, U, V, W, X, Y, Z$ (possibly with subscripts).

The atomic formulas of $\mathcal{L}^c$ are those of $\mathcal{L}$ plus all expressions of the form $(a \in U)$, and the formulas of $\mathcal{L}^c$ are built up from these atomic formulas by use of the propositional connectives and quantification over sets and classes. Equality of classes is defined by

$$(U = V) \;:=\; \forall x(x \in U \;\leftrightarrow\; x \in V)$$

and not treated as an atomic formula.

We say that an $\mathcal{L}^c$ formula is elementary iff it contains no class quantifiers. The Δ^c_0, Σ^c, Π^c, Σ^c_n, and Π^c_n formulas of $\mathcal{L}^c$ are defined in analogy to $\mathcal{L}$ but now permitting subformulas of the form $(a \in U)$.

The theory KP^c is formulated in $\mathcal{L}^c$ and based on classical logic for sets and classes with equality. The non-logical axioms of KP^c comprise the following formulas:

(A1) *Extensionality.* $\forall x(x \in u \;\leftrightarrow\; x \in v) \;\rightarrow\; u = v$.

(A2) *Pair.* $\exists x(u \in x \;\wedge\; v \in x)$.

(A3) *Union.* $\exists z(\forall x \in u)(\forall y \in x)(y \in z)$.

(A4) Δ^c_0 *Separation.* For all Δ^c_0 formulas φ in which y does not occur free:

$$\exists y \forall x(x \in y \;\leftrightarrow\; x \in u \;\wedge\; \varphi[x]). \qquad (\Delta^c_0\text{-}\mathsf{Sep})$$

(A5) Δ^c_0 *Collection.* For all Δ^c_0 formulas φ in which z does not occur free:

$$(\forall x \in u)\exists y \varphi[x, y] \;\rightarrow\; \exists z(\forall x \in u)(\exists y \in z)\varphi[x, y]. \qquad (\Delta^c_0\text{-}\mathsf{Col})$$

(A6) Δ^c_1 *Comprehension.* For all Σ^c_1 formulas $\varphi[x]$ and all Π^c_1 formulas $\psi[x]$ in which X does not occur free:

$$\forall x(\phi[x] \;\leftrightarrow\; \psi[x]) \;\rightarrow\; \exists X \forall x(x \in X \;\leftrightarrow\; \varphi[x]). \qquad (\Delta^c_1\text{-}\mathsf{CA})$$

(A7) *Elementary $\in$-induction.* For all elementary $\mathcal{L}^c$ formulas $\varphi[x]$:

$$\forall x((\forall y \in x)\varphi[y] \;\rightarrow\; \varphi[x]) \;\rightarrow\; \forall x \varphi[x]. \qquad (\mathsf{El}\text{-}\mathsf{I}_\in)$$

(A8) *Infinity.* $\exists y(\varnothing \in y \;\wedge\; (\forall x \in y)(x \cup \{x\} \in y))$.

In the following we will frequently make use of standard set-theoretic abbreviations and terminology. Actually, we have done so already in the formulation of (A8). In particular, we write $\langle x, y\rangle$ for the Kuratowski pair of the sets x, y and $a \times b$ for the product of a and b, i.e.

$$a \times b = \{\langle x, y\rangle : x \in a \;\wedge\; y \in b\}.$$

Also, given sets a and x, let $(a)_x$ be the set $\{y : \langle y, x\rangle \in a\}$. This notion makes most sense, of course, in case that a is a binary relation.

In Jäger and Steila [2] it is shown that KP^c is conservative over Kripke-Platek set theory KP. So from the proof-theoretic perspective KP^c is as good (or bad) as KP. But from a notational perspective, it is more convenient to work in KP^c. From what is said in [2] it is also clear that $(\Delta^c_0\text{-}\mathsf{Sep})$ and $(\Delta^c_1\text{-}\mathsf{CA})$ can be extended to $(\Delta^c\text{-}\mathsf{Sep})$ and $(\Delta^c\text{-}\mathsf{CA})$, respectively.

KP^c is a natural framework for speaking about operators. We call a class U an *operator* iff all its elements are right-unique ordered pairs:

$$Op[U] \;:=\; \begin{cases} (\forall x \in U)(\exists y, z(x = \langle y, z\rangle \;\wedge \\ \forall y, z_0, z_1(\langle y, z_0\rangle \in U \;\wedge\; \langle y, z_0\rangle \in U \;\rightarrow\; z_0 = z_1). \end{cases}$$

In the following we let the letters F, G, and H range over operators. We say that x belongs to the domain of F iff there exists a y such that $\langle x, y\rangle \in F$. If x belongs to the domain of F, then $F(x)$ denotes the unique y such that $\langle x, y\rangle \in F$. Then we can work with $F(x)$ as if it were an "ordinary" term, for example,

$$\begin{aligned}
F(a) = b &\ :=\ \langle a, b\rangle \in F,\\
F(a) \in b &\ :=\ (\exists x \in b)(\langle a, x\rangle \in F),\\
b \in F(a) &\ :=\ \exists x(\langle a, x\rangle \in F \wedge b \in x),\\
F(a) \subseteq b &\ :=\ \exists x(\langle a, x\rangle \in F \wedge x \subseteq b),\\
b \subseteq F(a) &\ :=\ \exists x(\langle a, x\rangle \in F \wedge b \subseteq x),\\
F(a) = G(b) &\ :=\ \exists x(\langle a, x\rangle \in F \wedge \langle b, x\rangle \in G),\\
F(a) \subseteq G(b) &\ :=\ \exists x, y(\langle a, x\rangle \in F \wedge \langle b, y\rangle \in G \wedge x \subseteq y),\\
F(a) \in U &\ :=\ (\exists x \in U)(\langle a, x\rangle \in F).
\end{aligned}$$

If F is an operator whose domain is of the form $\mathbb{V}^n$ for some natural number n, we often write $F(x_1, \ldots, x_n)$ instead of $F(\langle x_1, \ldots, x_n\rangle)$; here $\mathbb{V}$ stands for the collection of all sets.

It is easy to see that any Σ function symbol in the sense of Barwise [1] defines an operator whose domain is of the form $\mathbb{V}^n$. We can, therefore, identify Σ function symbols with the operators they define.

Let us now turn to fixed point and least fixed point assertions. First we consider their first order formalizations and then their analogues in $\mathcal{L}^c$. The notational advantage of the richer framework with classes will be evident.

Let $\varphi[x, y]$ be a Σ_1 formula of $\mathcal{L}$ with distinguished free variables x, y and set

$$\mathcal{B}_\varphi[a] \ :=\ \forall x \exists! y \varphi[x, y] \ \wedge\ \forall x, y(\varphi[x, y] \ \rightarrow\ y \subseteq a).$$

$\mathcal{B}_\varphi[a]$ states that $\varphi[x, y]$ describes a Σ_1 definable operator that maps all sets to subsets of a; in this sense it is bounded by a. Further, we write $\mathcal{M}_\varphi[a]$ for the conjunction of $\mathcal{B}_\varphi[a]$ and the monotonicity assertion

$$\forall x_0, x_1, y_0, y_1(\varphi[x_0, y_0] \wedge \varphi[x_1, y_1] \wedge x_0 \subseteq x_1 \ \rightarrow\ y_0 \subseteq y_1).$$

The axioms for (least) fixed points of monotone and set-bounded Σ_1 operators are the following two schemas:

$$\mathcal{M}_\varphi[a] \ \rightarrow\ \exists x \varphi[x, x], \tag{Σ_1-FP}$$

$$\mathcal{M}_\varphi[a] \ \rightarrow\ \exists x(\varphi[x, x] \wedge \forall y(\varphi[y, y] \ \rightarrow\ x \subseteq y)), \tag{Σ_1-LFP}$$

where $\varphi[x, y]$ ranges over all Σ_1 formulas, possibly containing additional parameters. In $\mathcal{L}^c$ life is (notationally) easier.

Definition 1 For any set a and any operator F we introduce the following shorthand notations:

$$\begin{aligned}
Mon[a, F] &\ :=\ \forall x(F(x) \subseteq a) \ \wedge\ \forall x, y(x \subseteq y \ \rightarrow\ F(x) \subseteq F(y)),\\
FP[F, b] &\ :=\ F(b) = b,\\
LFP[F, b] &\ :=\ F(b) = b \ \wedge\ \forall x(F(x) = x \ \rightarrow\ b \subseteq x).
\end{aligned}$$

Then we consider the following two axiom schemas:

$$Mon[a, F] \ \rightarrow\ \exists x FP[F, x], \tag{FP^c}$$

$$Mon[a, F] \ \rightarrow\ \exists x LFP[F, x]. \tag{LFP^c}$$

If $\varphi[x, y]$ is a Σ_1 formula such that $\mathcal{M}_\varphi[a]$, then, by (Δ^c_1-CA) there exists a operator F such that, for all x and y,

$$F(x) = y \ \leftrightarrow\ \varphi[x, y].$$

Obviously, we have $Mon[a, F]$. Therefore, (Σ_1-FP) and (Σ_1-LFP) follow from (FP^c) and (LFP^c), respectively. In view of the model construction in Jäger and Steila [2] it is also clear that KP^c+(FP^c) is a conservative extension of KP+(Σ_1-FP) and that KP^c+(LFP^c) is conservative over KP+(Σ_1-LFP).

We end this section by recalling the notion of well-foundedness and stating the famous axiom (Beta).

A set r is a relation iff every element of r is an ordered pair. A relation r is called *well-founded on a* iff every non-empty subset b of a has an r-minimal element:

$$Wf[a,r] \ :=\ (\forall b \subseteq a)(b \neq \varnothing \ \rightarrow\ (\exists m \in b)(\forall n \in b)(\langle n,m\rangle \notin r)).$$

Definition 2 If we write $Dom[f,a]$ to express that f is a function with domain a, then *the axiom* (Beta) is the universal closure of the formula

$$Wf[a,r] \ \rightarrow\ \exists f(Dom[f,a] \wedge (\forall m \in a)(f(m) = \{f(n) : n \in a \wedge \langle n,m\rangle \in r\})).$$

In Barwise [1] the function f is said to be *collapsing* for r.

Since the reversal of this implication is obvious, the axiom (Beta) has the effect of making the Π_1 predicate $Wf[a,r]$ a Δ_1 predicate.

Let r be well-founded on a. By means of the axiom (Beta) we can then go from the structure $\langle a,r\rangle$ to $\langle\{f(m) : m \in a\}, \in\rangle$ where f is collapsing for r. Since $\in$-induction is available in KP^c for all elementary formulas we obtain the following form of transfinite induction.

Theorem 3 (Transfinite induction) *Let $\varphi[m]$ be an elementary formula. Then $\mathsf{KP}^c + (\mathsf{Beta})$ proves that $Wf[a,r]$ implies*

$$(\forall m \in a)((\forall n \in a)(\langle n,m\rangle \in r \ \rightarrow\ \varphi[n]) \ \rightarrow\ \varphi[m]) \ \rightarrow\ (\forall m \in a)\varphi[m].$$

Proof We assume $Wf[a,r]$ and let f be collapsing for r according to the axiom (Beta). Given the elementary $\varphi[m]$, we assume, in addition, that

$$(\forall m \in a)((\forall n \in a)(\langle n,m\rangle \in r \ \rightarrow\ \varphi[n]) \ \rightarrow\ \varphi[m])$$

and set

$$\psi[x] \ :=\ (\forall m \in a)(x = f(m) \ \rightarrow\ \varphi[m]).$$

Then $\forall x\psi[x]$ is proved by $\in$-induction. $(\forall m \in a)\varphi[m]$ is immediate from that. □

In the following we will frequently use this theorem without mentioning it again.

3 Well-founded parts and the order types of their elements

From now on we work in the theory $\mathsf{KP}^c + (\mathsf{Beta})$. Our aim is to show that the well-founded part of a binary relation r on base set a is a set. This is a well-know result, proved also, for example, in Sato and Zumbrunnen [7]. Below we present it in a form tailored for what we need later. We also define the corresponding order type of each element of this well-founded part.

Let a be any set and r a relation (not necessarily on a). We say that the element m of a belongs to the *well-founded part of r on a* iff r is well-founded on the r^+-predecessors of m in a,

$$Wf[a,r,m] \ :=\ Wf[\{n \in a : \langle n,m\rangle \in r^+\}, r],$$

where r^+ is the transitive closure of r.

Theorem 4 (Well-founded part) *There exists an operator WP such that for all sets a and relations r,*

$$\mathsf{WP}(a,r) \ =\ \{m \in a : Wf[a,r,m]\}.$$

Proof In view of the axiom (Beta), the formula $Wf[a,r,m]$ is equivalent to

$$\begin{aligned}\exists x, f(x = \{n \in a : \langle n,m\rangle \in r^+\} \wedge Dom[f,x] \wedge \\ (\forall k \in x)(f(k) = \{f(n) : n \in x \wedge \langle n,k\rangle \in r\})).\end{aligned}$$

Thus our claim follows by a simple application of $(\Delta^c\text{-}\mathsf{CA})$. □

We call the set $\mathsf{WP}(a,r)$ the well-founded part of r on a and observe that it is well-founded with respect to r.

Lemma 5 *We have $Wf[\mathsf{WP}(a,r), r]$ for all sets a and relations r.*

Proof Suppoe that b is a non-empty subset of $\mathsf{WP}(a,r)$. We have to show that b has an r-minimal element. Since $b \neq \varnothing$, there exists an $m \in b$. Keep in mind that

$$Wf[a,r,m]. \tag{1}$$

If $(\forall n \in b)(\langle n,m\rangle \notin r)$, then m is such an r-minimal element. Otherwise, there exists an $n_0 \in b$ with $\langle n_0,m\rangle \in r$. Then consider the set

$$c \;:=\; b \cap \{n \in a : \langle n,m\rangle \in r^+\}$$

which is a non-empty subset of $\{n \in a : \langle n,m\rangle \in r^+\}$. Because of (1) there exists a $k \in b$ with

$$\langle k,m\rangle \in r^+ \;\wedge\; (\forall n \in c)(\langle n,k\rangle \notin r). \tag{2}$$

Now it is easy to see that (2) yields

$$(\forall n \in b)((\langle n,k\rangle \notin r).$$

This means that k is an r-minimal element of b. □

Following Barwise [1] we can introduce operators rk and B – both are Σ operation symbols in the terminology of [1] – such that rk is the usual rank function and

$$\mathsf{B}(a,r) = 0 \quad \text{iff } r \text{ is not well-founded on } a,$$

but if r is well-founded on a, then $\mathsf{B}(a,r)$ is the uniquely determined function f with domain a such that $f(m) = \{f(n) : n \in a \wedge \langle n,m\rangle \in r)$ for all $m \in a$.

Definition 6 For all sets a and all relations r we define the *order type function otype* by setting, for all $m \in \mathsf{WP}(a,r)$,

$$otype(a,r,m) \;:=\; rk(f(m))$$

if f is the collapsing function $\mathsf{B}(\mathsf{WP}(a,r),r)$. Otherwise, if m does not belong to $\mathsf{WP}(a,r)$, we do not care for the value of $otype(a,r,m)$; set, for example, $otype(a,r,m) := \langle a,r\rangle$ for all $m \notin \mathsf{WP}(a,r)$.

We end this section by providing an alternative characterization of well-founded parts. Given a and r, it is obtained by iterating the operator $Acc_{(a,r)}$, defined by

$$Acc_{(a,r)}(x) \;:=\; \{m \in a : (\forall n \in a)(\langle n,m\rangle \in r \;\rightarrow\; n \in x)\}$$

for all $x \subseteq a$, along the ordinals.

Theorem 7 *Given a set a and a relation r use Σ recursion to define the operator F on the ordinals such that*

$$F(\alpha) \;=\; Acc_{(a,r)}(\bigcup_{\xi<\alpha} F(\xi))$$

for all ordinals α. Then we have:

1. $\forall\alpha(F(\alpha) \subseteq \mathsf{WP}(a,r))$.
2. $\forall\alpha(\forall m \in \mathsf{WP}(a,r))(otype(a,r,m) \leq \alpha \;\rightarrow\; m \in F(\alpha))$.
3. *For $\beta := \sup(otype(a,r,m)+1 : m \in \mathsf{WP}(a,r))$ we have*

$$\mathsf{WP}(a,r) \;=\; \bigcup_{\xi<\beta} F(\xi) \;=\; \bigcup_{\xi} F(\xi).$$

Proof The first and second assertion are proved by induction on α. The third assertion is an immediate consequence of the second, where we observe that the existence of β is a direct consequence of Σ reflection. □

4 The stage comparison relation

In the theory of monotone inductive definitions the stage comparison relations play an important role. Let us begin with following Moschovakis [5] to recall some elementary notions and standard results. Pick an arbitrary set a. An operator

$$F : Pow(a) \to Pow(a).$$

is called *monotone* iff $F(x) \subseteq F(y)$ for all $x \subseteq y \subseteq a$. For each such operator F and each ordinal α we define the set I_F^α by the transfinite recursion

$$I_F^\alpha \;=\; F(I_F^{<\alpha}) \quad \text{with} \quad I_F^{<\alpha} \;:=\; \bigcup_{\xi<\alpha} I_F^\xi$$

and set $I_F := \bigcup_{\xi \in On} I_F^\xi$. Then I_F is the least fixed point of F, and there exists an ordinal α such that

$$I_F \;=\; I_F^{<\alpha} \;=\; \bigcap\{x \subseteq a : F(x) = x\}.$$

The stages I_F^α of I_F can be used to assign ordinals to the elements of I_F by setting

$$|m|_F \;:=\; \text{least } \xi \text{ such that } m \in I_F^\xi.$$

Based on the stages of I_F we can now define two binary relations on a:

$$m \le_F^* n \;:=\; m \in I_F \;\wedge\; (n \notin I_F \;\vee\; |m|_F \le |n|_F),$$

$$m <_F^* n \;:=\; m \in I_F \;\wedge\; (n \notin I_F \;\vee\; |m|_F < |n|_F).$$

Then the stage comparison theorem states that both relations, $\le_F^*$ and $<_F^*$, are least fixed points of suitable monotone operators.

Until the end of this section we work in the theory KP^c. We fix an arbitrary set a and an operator F that maps all sets to subsets of a and is monotone with respect to the subset relation. We are primarily interested in fixed points and least fixed points of such F.

Definition 8 For any set a and any operator F we define:

$$Mon[a,F] \;:=\; \forall x(F(x) \subseteq a) \;\wedge\; \forall x,y(x \subseteq y \;\to\; F(x) \subseteq F(y)),$$

$$FP[F,b] \;:=\; F(b) = b,$$

$$LFP[F,b] \;:=\; F(b) = b \;\wedge\; \forall x(F(x) = x \;\to\; b \subseteq x).$$

In his definitions of the relations $\le_F^*$ and $<_F^*$ Moschovakis explicitly refers to ordinals to index the stages of the inductive definition generated by F. Sato [6], on the other hand, comes up with an abstract approach to stage comparison, avoiding the use of ordinals. It is only based on properties of the relation $<_F^*$ above.

This definition will be central for what follows. In principle, the stage comparison relation would make sense for arbitrary operators from $Pow(a)$ to $Pow(a)$. However, in our context it will only be used in connection with monotone such operators.

Definition 9 (Stage comparison relation) Assume $Mon[a,F]$. A relation $r \subseteq a \times a$ is called a *stage comparison relation for* F – in symbols $SC_{(a,F)}[r]$ – iff the following two properties are satisfied for all $m, n \in a$:

(sc.1) $(\exists k \in a)(m \in F((r)_k) \;\wedge\; n \notin F((r)_k)) \;\to\; m \in (r)_n$.

(sc.2) $m \in (r)_n \;\leftrightarrow\; (\exists k \in a)(k \in (r)_n \;\wedge\; m \in F((r)_k))$.

The following theorem and its proof are more or less directly taken from Sato [6].

Theorem 10 (Least fixed point) *Assume that*

$$Mon[a,F] \;\wedge\; SC_{(a,F)}[r] \;\wedge\; Wf[a,r].$$

Then the set $b := \{m \in a : (\exists n \in a)(m \in F((r)_n))\}$ *is the least fixed point of* F, *i.e.* $LFP[F,b]$.

Proof In view of (Δ_0^c-Sep) it is obvious that

$$b \;:=\; \{m \in a : (\exists n \in a)(m \in F((r)_n))\}$$

is a set. We first prove

$$b \subseteq x \text{ for all } x \text{ such that } F(x) \subseteq x. \tag{1}$$

So assume that $F(x) \subseteq x$ for some $x \subseteq a$. We show that

$$(r)_m \subseteq x \;\text{ for all } m \in a. \tag{2}$$

Assume this is not the case. Because of $Wf[a,r]$, we can then pick an r-minimal $n \in a$ such that $(r)_n \not\subseteq x$. Thus (sc.2) yields

$$(r)_n \;=\; \bigcup_{k \in (r)_n} F((r)_k) \;\subseteq\; F(x) \;\subseteq\; x.$$

This is a contradiction, and (2) is established. From (2) we obtain

$$F((r)_n) \subseteq F(x) \subseteq x$$

for all $n \in a$, and that yields (1). So we know that b is contained in all fixed points of F. It remains to show that b is a fixed point of F. To do so, we distinguish the following to cases:

(i) $(\forall m \in a)(\exists n \in a)(m \in F((r)_n))$. Then $b = a$ and

$$b = a \subseteq \bigcup_{n \in a} F((r)_n) \subseteq F(a) = F(a),$$

where the second inclusion holds since $(r)_n \subseteq a$ for all $n \in a$ and F is monotone. Clearly, $F(a) \subseteq a$ and, therefore, a – and that means b – is a fixed point of F.

(ii) $(\exists m \in a)(\forall n \in a)(m \notin F((r)_n))$. Let $m \in a$ be such that $m \notin F((r)_n)$ for all $n \in a$. Then we obtain

$$F((r_m)) \;\subseteq\; b \;=\; \{k \in a : (\exists n \in a)((k \in F((r)_n))\} \;\subseteq\; (r)_m \tag{3}$$

where the first inclusion is obvious and the second a consequence of (sc.1). According to (2) we thus have

$$(\forall n \in a)((r)_n \subseteq (r)_m), \tag{4}$$

implying that

$$b \;\subseteq\; F((r)_m). \tag{5}$$

Finally, from (sc.2) we conclude that

$$(r)_m \;\subseteq\; \bigcup_{n \in (r)_m} F((r)_n) \;\subseteq\; b. \tag{6}$$

Now (3), (5), and (6) give us that

$$(r)_m = b \quad \text{and} \quad F(b) = b.$$

Thus b is a fixed point of F, as we had to show. □

Still following Sato [6], we now discuss several useful structural properties of stage comparison relations. We begin with a simple observation and refer to [6] for its proof.

Lemma 11 *$Mon[a,F] \wedge SC_{(a,F)}[r] \wedge Wf[a,r]$ implies:*

1. $(\forall k,m,n \in a)(k \in (r)_m \wedge m \in (r)_n \rightarrow k \in (r)_n)$.
2. $(\forall m,n \in a)(m \in (r)_n \rightarrow (\exists k \in a)(m \in F((r)_k) \wedge n \notin F((r)_k))$.

Given a set h, a relation r, and an element m of a, we write $(h)_{r \restriction m}$ for the union of the sets $(h)_n$ with n ranging over the predecessors of m with respect to r in a,

$$(h)_{r \restriction m} \;:=\; \{k \in a : (\exists n \in a)(n \in (r)_m \wedge k \in (h)_n)\}.$$

With this notation at hand we can now turn to some definitions that will play a central role in connection with what Sato calls a “sandwich property”.

Definition 12

$$\begin{aligned} It_{(a,F)}[h,r] &:= (\forall m \in a)((h)_m = F((h)_{r\upharpoonright m})); \\ \Phi_{(a,F)}[m,n,h,r] &:= It_{(a,F)}[h,r] \land (\exists k \in a)(m \in (h)_k \land n \notin (h)_k); \\ \Psi_{(a,F)}[m,n,h,r] &:= It_{(a,F)}[h,r] \to (\forall k \in a)(n \in (h)_k \to m \in (h)_{r\upharpoonright k}). \end{aligned}$$

This definition is so that the following two lemmas, their corollary, and the final theorem of this section are (more or less) straightforward. For detailed proofs we refer to [6] once more.

Lemma 13 *$SC_{(a,F)}[r]$ implies, for all sets h,*

$$h = \{\langle m,n\rangle \in a^2 : m \in F((r)_n)\} \to It_{(a,F)}[h,r].$$

Lemma 14 *$Mon[a,F] \land Wf[a,r] \land Wf[a,s]$ implies:*

1. *$It_{(a,F)}[g,r] \land It_{(a,F)}[h,s] \to (\forall m,n \in a)((g)_m \subseteq (h)_{s\upharpoonright n} \lor (h)_n \subseteq (g)_m)$.*
2. *$(\forall m,n \in a)(\Phi_{(a,F)}[m,n,g,r] \to \Psi_{(a,F)}[m,n,h,s])$.*

Corollary 15 *We have for all $m,n \in a$ that*

$$Mon[a,F] \land \exists g,r(Wf[a,r] \land \Phi_{(a,F)}[m,n,g,r]) \to \forall h,s(Wf[a,s] \to \Psi_{(a,F)}[m,n,h,s]).$$

Theorem 16 (Uniqueness of well-founded stage comparison relation) *We have for all a, F, r, s:*

$$Mon[a,F] \land Wf[a,r] \land Wf[a,s] \land SC_{(a,F)}[r] \land SC_{(a,F)}[s] \to r = s.$$

5 A short interlude: least closed points

The content of this section is not needed in the following. It is about a delicate difference between *least fixed points* of monotone operators and their *least closed points*.

As before, we assume $Mon[a,F]$. Then a set b is called *F-closed* iff $F(b) \subseteq b$. It is obvious that any fixed point of F is F-closed, but that the converse is not true in general. Now let us look at least such F-closed sets; we call them least closed points,

$$LCP[F,b] := F(b) \subseteq b \land \forall x(F(x) \subseteq x \to b \subseteq x).$$

The following observation is easy to prove.

Lemma 17 *If F has a least closed point, then this point is the least fixed point of F; i.e.* KP^c *proves that*

$$LCP[F,b] \to LFP[F,b].$$

Proof Let b be the least closed point of F. We first convince ourselves that b is a fixed point of F. Since $F(b) \subseteq b$ we obtain by the monotonicity of F that $F(F(b)) \subset F(b)$. Hence $F(b)$ is F-closed, thus $b \subseteq F(b)$ because of the leastness of b. Therefore, $F(b) = b$.

Furthermore, every fixed point of F is F-closed. Hence b is contained in all fixed points of F. □

Now we turn to the converse direction and introduce an axiom schema[1] that claims that every monotone operator has a fixed point:

$$Mon[a,F] \to \exists x FP[F,x], \qquad (\mathsf{FP}^c)$$

where a ranges over all sets and F over all operators.

Theorem 18 *In* $\mathsf{KP}^c + (\mathsf{FP}^c)$ *we can prove that, for all a and F,*

$$Mon[a,F] \to \forall x(LFP[F,x] \to LCP[F,x]).$$

Proof Assume $Mon[a, F]$ and let b be the least fixed point of F. Since b is F-closed, all we have to do is to show that

$$F(c) \subseteq c \;\rightarrow\; b \subseteq c \qquad (*)$$

for any set c. Given a c with $F(c) \subseteq c$, we consider the operator G_c defined by, for any x,

$$G_c(x) \;:=\; F(x) \cap c.$$

$Mon[a, G_c]$ is obvious. Therefore, the schema (FP^c) gives us a fixed point d of G_c, i.e.

$$d \;=\; G_c(d) \;=\; F(d) \cap c,$$

in particular, $d \subseteq c$. Therefore, $F(d) \subseteq F(c) \subseteq c$. Hence,

$$F(d) \;=\; F(d) \cap c \;=\; G_c(d) \;=\; d.$$

Since d is a fixed point of F we have

$$b \;\subseteq\; d \;\subseteq\; c.$$

This finishes the proof of (*) and thus also that of our theorem. □

6 Double-negated operators

Let us come back to the beginning of the previous section and the stages I_F^α of the least fixed point I_F of the monotone operator F. The following line of argument is implicit in Moschovakis [5].

For all $m, n \in I_F$ we have

$$\begin{aligned} m <_F^* n \;&\leftrightarrow\; |m|_F < |n|_F \\ &\leftrightarrow\; |n|_F \not\leq |m|_F \\ &\leftrightarrow\; n \notin I_F^\alpha \;\text{ for } \alpha = |m|_F \\ &\leftrightarrow\; n \notin F(I_F^{<\alpha}) \;\text{ for } \alpha = |m|_F \\ &\leftrightarrow\; n \notin F(\{k \in a : k <_F^* m\}) \end{aligned}$$

This is a kind of fixed point characterization of $<_F^*$. However, in this characterization there is a negative reference to F and monotonicity could be lost. But now an idea implicit in [5] delivers a solution. The point is is to replace the assertion "$k <_F^* m$" within F on the right-hand side by the same sort of equivalence.

Definition 19 (Double-negated F) Depending on F, we introduce a new operator $(\neg F)$ by setting for all sets x,

$$(\neg F)(x) \;:=\; \{k \in a : (\exists m, n \in a)(k = \langle m, n\rangle \;\wedge\; n \notin F((x)_m))\}.$$

The double-negated F then is defined to be the operator $F^* := (\neg F) \circ (\neg F)$.

It is a first and immediate consequence of this definition, that

$$Mon[a, F] \;\rightarrow\; Mon[a, F^*].$$

The following somewhat technical lemma – its proof is again in [6] – is a central building block in the proof of Theorem 21 below.

Lemma 20 *Under the assumptions*

(i) $Mon[a, F]$,

(ii) $(\neg F)(f) = g \;\wedge\; (\neg F)(g) = f$,

(iii) $It_{(a,F)}[h, s] \;\wedge\; Wf[a, s]$,

(iv) $b \;=\; \{\langle\langle m, n\rangle, k\rangle \in a^2 \times a : m \in (h)_k \;\wedge\; n \notin (h)_k\}$,

(v) $c \;=\; \{\langle\langle m, n\rangle, k\rangle \in a^2 \times a : n \in (h)_k \;\rightarrow\; m \in (h)_{s\restriction k}\}$,

we have that

$$(\forall k \in a)((b)_k \subseteq f \;\wedge\; g \subseteq (c)_k).$$

Proof We begin this proof with the following two observations that are obvious from the definitions of b and c:

$$(\forall m,\ell \in a)(m \notin (h)_\ell \ \rightarrow\ (h)_\ell \subseteq ((b)_\ell)_m), \tag{1}$$

$$(\forall m,k \in a)(m \in (h)_k \ \rightarrow\ ((c)_k)_m \subseteq (h)_{s\restriction k}). \tag{2}$$

We also have

$$(\forall \ell \in a)(g \subseteq (c)_\ell \ \rightarrow\ (b)_\ell \subseteq f). \tag{3}$$

Proof of (3). Assume $g \subseteq (c)_\ell$ and $\langle m,n\rangle \in (b)_\ell$. Then $m \in (h)_\ell$ and $n \notin (h)_\ell$. Therefore, (2) yields

$$(g)_m \subseteq ((c)_\ell)_m \subseteq (h)_{s\restriction\ell}$$

and that implies

$$F((g)_m) \subseteq F((h)_{s\restriction\ell}) = (h)_\ell.$$

Because of $n \notin (h)_\ell$ we thus have $n \notin F((g)_m)$. By the definition of $(\neg F)$, we, threfore, have $\langle m,n\rangle \in (\neg F)(g) = f$. ⊣

After these preparatory steps it is now fairly easy to prove

$$(\forall k \in a)(g \subseteq (c)_k) \tag{4}$$

by induction on k along s. Let $\langle m,n\rangle \in g = (\neg F)(f)$, i.e. $n \notin F((f)_m)$. Given $k \in a$, $\langle m,n\rangle \in (c)_k$ is equivalent to

$$m \notin (h)_{s\restriction k} \ \rightarrow\ n \notin (h)_k. \tag{5}$$

Asssume $m \notin (h)_{s\restriction k}$. By the induction hypothesis we have

$$(\forall \ell \in (s)_k)(g \subseteq (c)_l),$$

thus, in view of (3), also

$$(\forall \ell \in (s)_k)((b)_\ell \subseteq f).$$

This yields

$$(h)_{s\restriction k} \ = \ \bigcup\nolimits_{\ell \in (s)_k} (h)_\ell \ \subseteq \ \bigcup\nolimits_{\ell \in (s)_k} ((b)_\ell)_m \ \subseteq \ (f)_m,$$

where the first "$\subseteq$" holds because of (1). Hence

$$(h)_k \ = \ F((h)_{s\restriction k}) \ \subseteq \ F((f)_m).$$

Since $n \notin F((f)_m)$ we obtain $n \notin (h)_k$, as we had to show in order to establish (5). Hence (4) is established. Clearly, (3) and (4) imply what we have to prove. □

Observe that this lemma is symmetric in f and g so that we can swap their roles. Later we will need the following consequence of this lemma. It describes what Sato calls the "sandwich property": a fixed point of the double-negation of F sits between $\Phi_{(a,F)}$ and $\Psi_{(a,F)}$.

Theorem 21 *$F^*(r) = r \wedge Wf[a,s]$ imply for arbitrary h:*

1. $(\forall m,n \in a)(\Phi_{(a,F)}[m,n,h,s] \ \rightarrow\ \langle m,n\rangle \in r)$.
2. $(\forall m,n \in a)(\langle m,n\rangle \in r \ \rightarrow\ \Psi_{(a,F)}[m,n,h,s])$.

Proof Assume $F^*(r) = r$, $Wf[a,s]$, and $It_{(a,F)}[h,s]$. In order to establish the first assertion, we have to show that, for all $m,n,k \in a$,

$$m \in (h)_k \wedge n \notin (h)_k \ \rightarrow\ \langle m,n\rangle \in r.$$

But this is a direct consequence of the previous lemma with $f := r$ and $g := (\neg F)(r)$.

For the second assertion we need that, for any $m,n,k \in a$,

$$\langle m,n\rangle \in r \ \rightarrow\ (n \in (h)_k \ \rightarrow\ m \in (h)_{s\restriction k}).$$

Also this is immediate from the previous lemma, but now with $g := r$ and $f := (\neg F)(r)$. □

Please observe that for the previous theorem we need a well-founded relation s on a along which F is iterated. Working in KP^c, a fixed point of F^* would not necessarily provide us with such a well-founded relation in general. But since the axiom (Beta) is available, we are on the safe side; see next lemma.

7 Main Lemma and the existence of the stage comparison relation

Recall that we work in $\mathsf{KP}^c + (\mathsf{Beta})$. As before, a is an arbitrary but fixed non-empty set and F is an operator that maps all sets to subsets of a. We also assume that F is monotone with respect to the subset relation.

Our aim in this section is to show that from a fixed point of the double-negation F^* of F and a suitable form of the sandwich property a well-founded stage comparison relation for F can be defined.

Lemma 22 *Let r be an arbitrary relation on a, i.e. $r \subseteq a \times a$. Then we can define sets r^*, p, q, and h satisfying:*

(i) r^* *is the reflexive and transitive closure of* r;

(ii) $p \;=\; \{m \in \mathsf{WP}(a,r) : (\forall n \in (r^*)_m)(n \in F((r)_n))\}$;

(iii) $q \;=\; (r \cap (p \times p)) \cup (p \times a \setminus p)$;

(iv) $h \;=\; \begin{cases} \{\langle m,n\rangle \in a^2 : n \in p \wedge m \in F((q)_n)\} \;\cup \\ \{\langle m,n\rangle \in a^2 : n \notin p \wedge m \in F(\bigcup_{k\in p} F((q)_k))\}. \end{cases}$

Furthermore, we have $Wf[a,q]$.

Proof r^* is easily defined in KP^c from r. The axiom (Beta) guarantees that the well-founded part $\mathsf{WP}(a,r)$ of r on a is a set. From that the existence of the sets p, q, and h is clear by $(\Delta^c\text{-}\mathsf{Sep})$. p is a subset of $\mathsf{WP}[a,r]$, hence well-founded. Consequently, $r \cap (p \times p)$ is well-founded, and in q only the elements of $a \setminus p$ are added as maximal elements. Therefore, $Wf[a,q]$ is established. □

The following Main Lemma is tailored according to the corresponding lemma in Sato [6]. However, some auxiliary definitions and some technical intermediate steps had to be modified.[2]

Lemma 23 (Main Lemma) *Suppose that $r \subseteq a \times a$ and that r^*, p, q, and h are as in the previous lemma. Suppose further, that depending on these sets we write $[m]$ instead of $(r^*)_m$ for all $m \in a$ and define*

- *for $m \in p$:*

$$q|m := \{\langle n,k\rangle \in q : k \in [m]\},$$

$$h|m := \{\langle n,k\rangle \in a^2 : (n \in (h)_k \wedge k \in [m]) \vee (n \in F(\varnothing) \wedge k \notin [m])\};$$

- *for $m \in a \setminus p$: $\quad q|m := q \quad$ and $\quad h|m := h$.*

Finally, assume that we have:

$(+)$ $(\forall k,m,n \in a)(\Phi_{(a,F)}[m,n,h|k,q|k] \;\rightarrow\; \langle m,n\rangle \in r)$,

$(-)$ $(\forall k,m,n \in a)(\langle m,n\rangle \in r \;\rightarrow\; \Psi_{(a,F)}[m,n,h|k,q|k])$.

Then we also have:

$$SC_{(a,F)}[q] \;\wedge\; p = \{m \in a : (\exists n \in a)(m \in F((q)_n))\}.$$

Proof These two assertions are proved by a sequence of auxiliary claims. We begin with a list of rather direct consequences of our definitions.

Claim 1. We have for all $m,n \in a$:

(i) $m \in [m]$.

(ii) $m \in (r)_n \;\rightarrow\; [m] \subseteq [n]$.

(iii) $m \in p \;\rightarrow\; (r)_m \subseteq p \wedge [m] \subseteq p$.

(iv) $n \in p \;\rightarrow\; (r)_n = (q)_n$.

(v) $n \in p \wedge n \in (r)_m \;\rightarrow\; n \in (q)_m$.

(vi) $n \in (q)_m \;\rightarrow\; n \in p$.

(vii) $m \in (q|n)_n \;\leftrightarrow\; m \in (q)_n$.

(viii) For all $k \in [m]$: $(h)_k = (h|m)_k = (h|k)_k$.

(ix) For all $m \in p$ and $k \in [m]$: $(h)_{q \upharpoonright k} = (h|m)_{(q|m) \upharpoonright k} = (h|k)_{(q|k) \upharpoonright k}$.

(x) $(h|m)_{(q|m) \upharpoonright m} = (h)_{q \upharpoonright m}$.

Proof of Claim 1. Immediate consequences of the definitions above. ⊣

Claim 2. For all $m \in p$:

(i) $(h)_{q \upharpoonright m} = (q)_m$.

(ii) $(\forall n \in [m]) It_{(a,F)}[h|n, q|n]$.

Proof of Claim 2. Simultaneously by induction on $m \in p$ along q.
(i) Assume $n \in (q)_m$. Then $n \in p$ according to Claim 1-(vi) and, therefore, $n \in F((r)_n)$. In view of Claim 1-(iv) we have $(r)_n = (q)_n$, hence

$$n \in F((r)_n) = F((q)_n) = (h)_n \subseteq (h)_{q \upharpoonright m}.$$

For the converse direction, assume $n \in (h)_{q \upharpoonright m}$. Then there exists a $k \in (q)_m$ such that $n \in (h)_k$. Now we distinguish two cases:

- $m \notin (h)_k$. Then, by Claim 1-(viii), $n \in (h|k)_k$ and $m \notin (h|k)_k$. Applying the induction hypothesis (ii) for k we thus obtain

  $$\Phi_{(a,F)}[n, m, h|k, q|k].$$

  Hence (+) yields

  $$n \in (r)_m = (q)_m,$$

  where this equality follows from $m \in p$ and Claim 1-(iv).

- $m \in (h)_k$. Then, as before, $m \in (h|k)_k$. Since $k \in (q)_m$, we also have $k \in (r)_m$; see above. Now we apply (−) and obtain

  $$\Psi_{(a,F)}[k, m, h|k, q|k].$$

  Since $It_{(a,F)}[h|k, q|k]$ according to induction hypothesis (ii) for k, we can proceed with

  $$m \in (h|k)_k \;\rightarrow\; k \in (h|k)_{(q|k) \upharpoonright k}$$

  and thus obtain $k \in (h|k)_{(q|k) \upharpoonright k}$. By Claim 1-(x) this yields $k \in (h)_{q \upharpoonright k}$. Now we apply induction hypothesis (i) for k and obtain $k \in (q)_k$. However, this is in contradiction to $Wf[a, q]$ and – as a consequence – the case "$m \in (h)_k$" is ruled out. This finishes the proof of (i).

(ii) Assume $n \in [m]$. If $n \neq m$, then there exists an $m_0 \in (r)_m$ with $n \in [m_0]$, and $It_{(a,F)}[h|n, q|n]$ follows from the induction hypothesis. So it remains to prove that $It_{(a,F)}[h|m, q|m]$, i.e.

$$(h|m)_k = F((h|m)_{(q|m) \upharpoonright k}) \tag{*}$$

for all $k \in a$. To do so, we distinguish three cases.

- $k \in [m] \wedge k \neq m$. Then, as above, $k \in [m_0]$ for some $m_0 \in (r)_m$. By the induction hypothesis we thus have $It_{(a,F)}[h|k, q|k]$, in particular

  $$(h|k)_k = F((h|k)_{(q|k) \upharpoonright k}).$$

  Now we make use of Claim 1-(viii),(ix) and immediately obtain (*).

- $k = m$. From (i) we know that $(h)_{q \upharpoonright m} = (q)_m$, hence the definition of h yields

  $$(h)_m = F((q)_m) = F((h)_{q \upharpoonright m}).$$

  It remains to apply Claim 1-(viii),(ix) again which gives us

  $$(h|m)_m = (h)_m = F((q)_m) = F((h)_{q \upharpoonright m}) = F((h|m)_{(q|m) \upharpoonright m}).$$

- $k \notin [m]$. Then $(h|m)_k = F(\varnothing)$ and $(h|m)_{(q|m) \upharpoonright k} = \varnothing$ such that we have (*) also in this case.

So also (ii) of Claim 2 has been proved. ⊣

Claim 3.

(i) $It_{(a,F)}[h, q]$.

(ii) $(\forall m \in a) It_{(a,F)}[h|m, q|m]$.

Proof of Claim 3. (ii) is an immediate consequence of Claim 2-(ii), assertion (i) of this claim, and the definition of $h|m$. In order to establish (i), we have to show that, for all $m \in a$,

$$(h)_m = F((h)_{q\upharpoonright m}). \tag{**}$$

If $m \in p$, then Claim 2-(i) gives us $(q)_m = (h)_{q\upharpoonright m}$ and, therefore, (**) follows from the definition of h. If $m \notin p$, then $(q)_m = p$ and, consequently,

$$(h)_m = F(\bigcup_{k\in p} F((q)_k)) = F(\bigcup_{k\in(q)_m} (h)_k) = F((h)_{q\upharpoonright m}).$$

This means that we have (**) also for all $m \notin p$, and Claim 3 is proved. ⊣

Claim 4. $(\forall m \in a)((h)_m \subseteq p)$.

Proof of Claim 4. We proceed by induction on m along q. Assume

$$n \in (h)_m \tag{1}$$

and distinguish the following two cases:

- $n \in (h)_{q\upharpoonright m}$. Then there exists an $m_0 \in (q)_m$ such that $n \in (h)_{m_0}$, and the induction hypothesis implies $n \in p$.
- $n \notin (h)_{q\upharpoonright m}$. We first convince ourselves that n belongs to $\mathsf{WP}(a, r)$, and for that it is sufficient to check that $(r)_n \subseteq p$. So let $k \in (r)_n$. Then assumption $(-)$ gives us

  $$\Psi_{(a,F)}[k, n, h|m, q|m].$$

  Together with Claim 3-(ii) we thus obtain

  $$n \in (h|m)_m \;\rightarrow\; k \in (h|m)_{(q|m)\upharpoonright m}.$$

  Since $n \in (h)_m = (h|m)_m$ in view of Claim 1-(viii), we can continue with

  $$k \in (h|m)_{(q|m)\upharpoonright m} = (h)_{q\upharpoonright m}$$

  according to Claim 1-(x). Thus $k \in (h)_{m_0}$ for some $m_0 \in (q)_m$, and the induction hypothesis implies $k \in p$. This proves

  $$(r)_n \subseteq p, \tag{2}$$

  and thus we have

  $$n \in \mathsf{WP}(a, r). \tag{3}$$

  It remains to show that $(\forall k \in [n])(k \in F((r)_k))$. For that we first observe that

  $$(h)_{q\upharpoonright m} \subseteq (r)_n. \tag{4}$$

  Proof of (4). Pick a $k \in (h)_{q\upharpoonright m}$ and recall that $(h)_{q\upharpoonright m} = (h|m)_{(q|m)\upharpoonright m}$ according to Claim 1-(x). Since $n \notin (h)_{q\upharpoonright m}$ and because of Claim 3-(ii) we have $\Phi_{(a,F)}[k, n, h|m, q|m]$. Hence assumption $(+)$ yields $k \in (r)_n$. So we have (4).

  From Claim 3-(i), (1), and (4) we now conclude that

  $$n \in (h)_m = F((h)_{q\upharpoonright m}) \subseteq F((r)_n). \tag{5}$$

  From (2) and Claim 1-(iii) we also obtain

  $$(\forall k \in [n])(k \neq n \;\rightarrow\; k \in p) \tag{6}$$

  and, therefore, by the definition of p,

  $$(\forall k \in [n])(k \neq n \;\rightarrow\; k \in F((r)_k)). \tag{7}$$

  Finally, from (5) and (6) we deduce $(\forall k \in [n])(k \in F((r)_k))$, and that finishes the proof of Claim 4. ⊣

Claim 5. $p = \bigcup_{m\in p} F((q)_m) = F(p)$.

Proof of Claim 5. We prove this claim in three steps.

(s1) $p \subseteq \bigcup_{m\in p} F((q)_m)$: For $m \in p$ we have

$$m \in F((r)_m) = F((q)_m)$$

by the definition of p and Claim 1-(iv).

(s2) $\bigcup_{m\in p} F((q)_m) \subseteq F(p)$: For $m \in p$ we have

$$F((q)_m) \subseteq F((h)_{q\restriction m}) \subseteq F(p)$$

by Claim 2-(i) and Claim 4.

(s3) $F(p) \subseteq p$: If $a \subseteq p$, then the assertion is obvious. Otherwise there exists an $n \in a$ with $n \notin p$. By the definition of p we then have

$$(h)_n = F(\bigcup_{m\in p} F((q)_m)).$$

As a consequence, we have

$$F(p) \subseteq F(\bigcup_{m\in p} F((q)_m)) \subseteq (h)_n \subseteq p$$

by (s1) and Claim 4.

Clearly, (s1) – (s3) imply Claim 5. ⊣

Claim 6. For all $m, n \in a$,

$$(\exists k \in a)(m \in F((q)_k) \wedge n \notin F((q)_k)) \;\rightarrow\; m \in (q)_n.$$

Proof of Claim 6. Assume $m \in F((q)_k)$ and $n \notin F((q)_k)$ for some $k \in a$. If $k \notin p$, then $(q)_k = p$ and, therefore,

$$m \in F((q)_k) = F(p) = p \quad \text{and} \quad n \notin F((q)_k) = F(p) = p$$

according to Claim 5. Hence $m \in (q)_n$ by the definition of q. Otherwise, if $k \in p$, the definition of h tells us that $m \in (h)_k$ and $n \notin (h)_k$. Therefore, in view of Claim 1-(viii), $m \in (h|k)_k$ and $n \notin (h|k)_k$. Making use of Claim 2-(ii) we thus obtain $\Phi_{(a,F)}[m, n, h|k, q|k]$ from which

$$m \in (r)_n \tag{8}$$

follows by means of assumption (+). Furthermore, $m \in (h)_k$ implies $m \in p$ by Claim 4. Hence $m \in (q)_n$ by Claim 1-(v). ⊣

Claim 7. For all $m, n \in a$,

$$m \in (q)_n \;\leftrightarrow\; (\exists k \in a)(k \in (q)_n \wedge m \in F((q)_k)).$$

Proof of Claim 7. If $n \in p$, then Claim 2-(i) and the definition of h imply

$$(q)_n = (h)_{q\restriction n} = \bigcup\nolimits_{k\in(q)_n} (h)_k = \bigcup\nolimits_{k\in(q)_n} F((q)_k)$$

since $(q)_n \subseteq p$ by Claim 1-(vi). If $n \notin p$, then $(q)_n = p$ and Claim 5 yields

$$(q)_n = \bigcup\nolimits_{k\in p} F((q)_k) = \bigcup\nolimits_{k\in(q)_n} F((q)_k).$$

Thus in both cases we have what we want. ⊣

Claim 8. $p = \{m \in a : (\exists n \in a)(m \in F((q)_n))\}$.

Proof of Claim 8. The inclusion "$\subseteq$" is obvious from Claim 5. For "$\supseteq$" we pick an arbitrary $n \in a$ and distinguish between $n \in p$ and $n \notin p$. If $n \in p$, then, clearly, Claim 5 yields $F((q)_n) \subseteq p$. If $n \notin p$, then $(q)_n = p$ and we have $F((q)_n) = F(p) = p$, also by Claim 5. ⊣

Since Claim 6 and Claim 7 imply that $SC_{(a,F)}[q]$, Claim 8 has been all we need to complete the proof of our Main Lemma, and we are done. ☐

The previous result is the central technical tool for showing that, working in $\mathsf{KP}^c + (\mathsf{Beta})$, any monotone operators F that maps all sets to subsets of a has a well-founded stage comparison relation – provided that the double negation F^* of F has a fixed point.

Theorem 24 ($\mathsf{KP}^c + (\mathsf{Beta})$) *We have the following implication:*

$$\exists x FP[F^*, x] \;\rightarrow\; \exists q(Wf[a, q] \wedge SC_{(a,F)}[q]),$$

where, as always in this section, F is supposed to be an operator that satisfies $Mon[F, a]$.

Proof Suppose $FP[F^*, r]$ for some set r. Now we turn to Lemma 22, define r^*, p, q, and h as there and conclude that

$$Wf[a,q]. \qquad (*)$$

Threfore, clearly, also $Wf[a, q|k]$ for all $k \in a$. By Theorem 21 we see that

(+) $(\forall m, n \in a)(\Phi_{(a,F)}[m, n, h|k, q|k] \;\rightarrow\; \langle m, n\rangle \in r)$,

(−) $(\forall m, n \in a)(\langle m, n\rangle \in r \;\rightarrow\; \Psi_{(a,F)}[m, n, h|k, q|k])$

for all $k \in a$. Hence the Main Lemma implies $SC_{(a,F)}[q]$. Together with (*) this concludes our proof. □

8 Least fixed points in KP^c + (Beta) + (FP^c) and their stages

Recall the two fixed point principles (FP^c) and (LFP^c) introduced in Definition 1. Clearly, every instance of (FP^c) follows from (LFP^c). Now we will see that over KP^c + (Beta) also the converse is true.

Theorem 25 *In* KP^c + (Beta) + (FP^c) *we can prove, for any* a *and* F,

$$Mon[a, F] \;\rightarrow\; \exists q(\,Wf[a, q] \,\wedge\, SC_{(a,F)}[q]).$$

Proof Given a and F with $Mon[a, F]$, we assign to F its double-negated F^*. Then, as remarked earlier, we have $Mon[a, F^*]$, and thus (FP^c) tells us that there is a fixed point of F^*. Hence Theorem 24 yields the existence of a well-founded relation q on a such that $SC_{(a,F)}[q]$. □

Corollary 26 *In* KP^c + (Beta) + (FP^c) *every instance of* (LFP^c) *is provable.*

Proof Pick any a and F such that $Mon[a, F]$. Using the previous result and Theorem 10 we immediately see that there exists a least fixed point b of F. □

We end this article by characterizing the resolution of the least fixed point of a monotone operator into its ordinal stages by means of its stage comparison relation.

Working in KP^c + (Beta) + (FP^c) we let a be a set and F an operator with $Mon[a, F]$. From the previous considerations we know that there exists a $q \subseteq a^2$ such that $SC_{(a,F)}[q]$ and $Wf[a, q]$. We also know that

$$b \;:=\; \{m \in a : (\exists n \in a)(m \in F((q)_n))\}.$$

is the least fixed point of F.

Turning to ordinals, recall that any $m \in a$ has the order type

$$otype(a, q, m) \;=\; rk(f(m))$$

where f with

$$f(m) \;=\; \{f(n) : n \in a \,\wedge\, n \in (q)_m\}$$

for all $m \in a$ is the collapsing function for q. The ordinal

$$\beta \;:=\; \sup(otype(a, q, m) + 1 : m \in a),$$

which exists by Σ reflection, is the least upper bound of all order types in connection with $Wf[a, q]$.

As mentioned above, for any ordinal α the α-th iterate of F is defined by Σ recursion such that

$$I^\alpha_F \;=\; F(I^{<\alpha}_F) \quad \text{where} \quad I^{<\alpha}_F \;:=\; \bigcup\nolimits_{\xi<\alpha} I^\xi_F.$$

The following statement characterizes the relationship relationship between these stages and the stage comparison relation:

$$(\forall \alpha < \beta)(\forall m \in a)(otype(a, q, m) = \alpha \;\rightarrow\; (q)_m = I^{<\alpha}_F). \qquad (*)$$

The proof of (*) is by induction on α; we omit details. It follows that

$$b \;=\; \{m \in a : (\exists \xi < \beta)(m \in I^\xi_F)\}$$

is the description of the least fixed point of F "from below".

To sum up: It has been shown that in the theory KP^c + (Beta) + (FP^c) every operator that

(i) maps all elements of the universe to subsets of a given set a,

(ii) is monotone with respect to the subset relation possesses a least fixed point. In Jäger and Steila [4] we will use this fact in order to prove a slightly stronger result: Whenever we have a – not necessarily monotone – operator F from the universe to the power set of some a, then there exists an ordinal α such that after α-many iterations of F nothing new will be generated; called *maximal iteration principle* in Jäger and Steila [2].

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Notes

1. We will come back to this schema in Section 8.

2. Consider, in particular, the definitions of $q|m$ and $h|m$.

References

[1] J. Barwise. *Admissible Sets and Structures.* Number 7 in Perspectives in Mathematical Logic. Springer, 1975. 3, 4, 5

[2] G. Jäger and S. Steila. About some fixed point axioms and related principles in Kripke-Platek environments. *The Journal of Symbolic Logic*, 83(2):642–668, 2018. 1, 2, 3, 16

[3] G. Jäger and S. Steila. From Mathesis Universalis to fixed points and related set-theoretic concepts. In S. Centrone, S. Negri, S. Sarikaya, and P.M. Schuster, editors, *Mathesis Universalis, Computability and Proof*, volume 412 of *Synthese Library*, pages 113–129. Springer, 2019. 1

[4] G. Jäger and S. Steila. About some fixed point axioms and related principles in Kripke-Platek environments: adding the axiom (Beta). in preparation. 1, 16

[5] Y. N. Moschovakis. *Elementary Induction on Abstract Structures.* Number 77 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1974. 1, 6, 9

[6] K. Sato. Full and hat inductive definitions are equivalent in NBG. *Archive for Mathematical Logic*, 54(1-2):75–112, 2015. 1, 6, 7, 8, 9, 11

[7] K. Sato and R. Zumbrunnen. A new model construction by making a detour via intuitionistic theories I: operational set theory without choice is Π_1-equivalent to KP. *Annals of Pure and Applied Logic*, 166(2):121–186, 2015. 4

[8] S. Steila. FP in KP + Beta. unpublished manuscript, 2019. 1

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