# Classical Categories and Deep Inference

Richard M^c^Kinley*

University of Bath

**Abstract.** Deep inference is a proof-theoretic notion in which proof rules apply arbitrarily deeply inside a formula. We show that the essence of deep inference is the bifunctoriality of the connectives. We demonstrate that, when given an inequational theory that models cut-reduction, a deep inference calculus for classical logic (SKSg) is a categorical model of the classical sequent calculus LK in the sense of Führmann and Pym. We observe that this gives a notion of cut-reduction for derivations in SKSg, for which the usual notion of cut in SKSg is a special case. Viewing SKSg as a model of the sequent calculus uncovers new insights into the Craig interpolation lemma and intuitionistic provability.

## 1 Introduction

In recent years, the received wisdom that classical logic is uninteresting from a proof-theoretic point of view has been seriously challenged by a number of successful attempts to give a denotational semantics to classical proofs. The difficulties these approaches overcome are illustrated by the example below.

Given two proofs $\Phi_1$ and $\Phi_2$ of the same sequent,the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi_1}{\vdots}}{\Gamma \vdash \Delta}}{\Gamma \vdash \phi, \Delta}\,WR \qquad \dfrac{\dfrac{\dfrac{\Phi_2}{\vdots}}{\Gamma \vdash \Delta}}{\Gamma, \phi \vdash \Delta}\,WL}{\dfrac{\Gamma, \Gamma \vdash \Delta, \Delta}{\Gamma \vdash \Delta}\,CL, CR,}\,Cut \tag{1}$$

(usually attributed to Lafont [1]), reduces (essentially) to either $\Phi_1$ or $\Phi_2$; the choice is non-deterministic. In a model which admits cut-reduction as equality, $\Phi_1$ and $\Phi_2$ acquire the same denotation. Note that this example does not rely on negation.

Certain previous attempts to avoid this collapse have relied on moving to a sub-logic (i.e., intuitionistic or linear logic). Others rely on a restricted cut-reduction system which makes a systematic choice of the left- or right-hand reduction in (1): for example, classical natural deduction systems [2, 3]. None of these supply what we want of a semantics: a faithful representation of the structure of cut-reduction, with concrete models that illuminate the proof theory.

Of the numerous recent candidates, Führmann and Pym [4–6] seem to give the best synthesis of those two requirements. Their *classical categories* model classical proofs

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as morphisms in a special kind of poset-enriched linearly distributive category. The poset enrichment models cut-reduction, so that whenever a proof $\Phi$ cut-reduces to a proof $\Psi$, we have $\lfloor \Phi \rfloor \leq \lfloor \Psi \rfloor$, where $\lfloor \Phi \rfloor$ is the denotation of $\Phi$ in the category. Classical categories admit all commuting conversions of the sequent calculus as equalities; the ordering represents certain cuts against structural rules. The semantics sheds new light on the status of the MIX law in classical logic: the two obvious ways of defining it (as a cut against either $\bot$ or $\top$) acquire the same denotation, and the rule can be eliminated.

Meanwhile, work by Brünnler [7], building on the calculus of structures of Guglielmi [8, 9] has led to a new proof system (SKSg) for classical logic that promises a finer-grained analysis of proofs. The calculus of structures uses "deep inference" (inference rules operating arbitrarily deeply inside formulæ) to dispense with the tree- like structure of sequent proofs. The result is a precise duality: inference rules are unary, and come in dual pairs (or are self dual). A derivation can be dualized by "inverting" each inference rule to obtain a proof of the contrapositive derivation

Much of the work in deep inference at present lies in finding new proof-theoretic systems ("Formalism A" [10] and "Formalism B" [11]) that generalize the calculus of structures by eliminating "bureaucracy": proofs which are essentially the same but differ syntactically. For example,

$$\cfrac{\cfrac{(A,[B,C])}{[(A,B),C]}\,\mathsf{s}}{[(A',B),C]}\,\Phi \qquad \text{and} \qquad \cfrac{\cfrac{(A,[B,C])}{(A',[B,C])}\,\Phi}{[(A',B),C]}\,\mathsf{s} \tag{2}$$

would, in Formalism B, be represented by the same syntactic structure.

**Table 1.** System LK

$$\frac{}{\varphi \vdash \varphi}\,Ax_{LK} \qquad \frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma', \varphi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,CUT_{LK} \qquad \frac{\Gamma, \varphi, \varphi \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,CL \qquad \frac{\Gamma \vdash \Delta, \varphi, \varphi}{\Gamma \vdash \Delta, \varphi}\,CR$$

$$\frac{}{\vdash \top}\,\top L \qquad \frac{}{\bot \vdash}\,\bot R \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \varphi \vdash \Delta}\,WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \varphi}\,WR$$

$$\frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma' \vdash \Delta', \psi}{\Gamma, \Gamma' \vdash \Delta, \Delta', \varphi \wedge \psi}\,\wedge R \qquad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\,\wedge L \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\,\neg L \qquad \frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\,\neg R$$

$$\frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\,\vee R \qquad \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma', \psi \vdash \Delta'}{\Gamma, \Gamma', \varphi \vee \psi \vdash \Delta, \Delta'}\,\vee L \qquad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \psi, \varphi \vdash \Delta}\,EL \qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \psi, \varphi, \Delta}\,ER$$

This paper sets out to show, using classical categories as a case study, that the well established notions of proof equality arising from categorical logic may be applied with great ease to deep inference formalisms. (Indeed, one may view the calculus of structures as a categorical logic without coherence, functoriality and naturality [12], the challenge being to reintegrate functoriality and naturality (and perhaps coherence)).

The author hopes that these insights will act as a guide to those designing future formalisms.

- In §4 we give an inequational theory on proofs in SKSg, making it a classical category. This category:
  - Captures the essence of Formalism A as bifunctoriality of the connectives;
  - Captures the essence of Formalism B by requiring that certain inference rules are (lax) natural;
  - Does not collapse to a boolean algebra;
  - Is a model of the two-sided classical sequent calculus, LK.
- We identify two distinct forms of cut in SKSg (§5.1):
  1. A cut equivalent to that of a one-sided sequent system; and
  2. A cut equivalent to that of a two-sided sequent system.

  We show that the first is well-definable in terms of the second.
- We give a characterization of intuitionistically valid calculus of structures derivations (Lemma 3). Using this, we give a refinement of the Craig interpolation lemma for propositional classical logic (Corollary 1).

## 2 Classical categories

First, we introduce *classical categories* [4–6]. We do not give details of the diagrams required for coherence (although many can be inferred later from the equational parts of the theory given below for SKSg). Classical categories are a sound and complete semantics of the classical sequent calculus, developed by Führmann and Pym, that unlike previous attempts can distinguish between proofs and model all cut-reductions. A classical category is a poset-enriched category with extra structure, including two symmetric monoidal products $\otimes$ and $\oplus$, with units $1$ and $0$. Mediating between these functors is a natural transformation $\delta : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$, making it a symmetric linearly distributive category ([13]). Various coherence conditions hold for these categories, which in addition to giving coherence model the structure of cut-reduction in a two-sided sequent calculus. The category also has for each object $A$ a complement $A^{\perp}$, and morphisms $A^{\perp} \otimes A \to 0$ (contradiction) and $1 \to A \oplus A^{\perp}$ (excluded middle) which, with certain coherence conditions make it a symmetric linearly distributive category with negation. It is, by virtue of this, equivalent to a $*$-autonomous category, and a model of multiplicative linear logic, in the sense that objects model propositions and morphisms model proofs. We model $\phi \wedge \psi$ inductively as $\lfloor \phi \rfloor \otimes \lfloor \psi \rfloor$, and similarly disjunction is modelled by $\oplus$.

How the cut rule is modelled is of particular interest. Given a proof $\Phi$ of $\Gamma \vdash \Delta, \phi$ and a proof $\Psi$ of $\phi, \Gamma' \vdash \Delta'$, with denotations $\mathcal{C}\lfloor \Phi \rfloor, \mathcal{C}\lfloor \Psi \rfloor$ in a classical category $\mathcal{C}$, we denote the cutting together of these two proofs by:

$$\lfloor \Gamma \rfloor \otimes \lfloor \Gamma' \rfloor \xrightarrow{\mathcal{C}\lfloor \Phi \rfloor \otimes \mathbf{id}} (\lfloor \Delta \rfloor \oplus \lfloor \phi \rfloor) \otimes \lfloor \Gamma' \rfloor \xrightarrow{\delta'} \lfloor \Delta \rfloor \oplus (\lfloor \phi \rfloor \otimes \lfloor \Gamma' \rfloor) \xrightarrow{\mathbf{id} \oplus \mathcal{C}\lfloor \Psi \rfloor} \lfloor \Delta \rfloor \oplus \lfloor \Delta' \rfloor, \quad (3)$$

where $\delta'$ is the evident morphism obtained from $\delta$ and symmetric monoidal isomorphisms, and $\mathbf{id}$ is identity. In this setting, cut is a generalized composition.

In addition to this structure, a classical category carries the structure necessary to model weakening and contraction on the right. Every object has a *symmetric monoid* — a multiplication $\nabla_A : A \oplus A \to A$ and unit $[\,]_A : 0 \to A$, satisfying equations that state the associativity and symmetry of $\nabla_A$ and the neutrality of $[\,]_A$. In addition, we require that $\nabla_{A \oplus B}$ is definable pointwise; that $[\,]_{A \oplus B}$ is definable pointwise; and that $[\,]_0 = \mathbf{id}$. We say a symmetric monoidal category *has symmetric monoids* provided it satisfies these three laws. Dually, a classical category has symmetric comonoids given by $\Delta_A : A \to A \otimes A$ and $\langle\rangle_A : A \to 1$ for weakening and contraction on the right.

**Table 2.** Inequalities of a classical category

$$
\begin{array}{ccccccccc}
 & A \oplus C & \xrightarrow{\Delta} & (A \oplus C) \otimes (A \oplus C) & & A \oplus C & \xrightarrow{\langle\rangle} & 1 & \\
\Delta\nabla & \mathbf{id} \oplus \Delta \big\downarrow & \leq & \big\downarrow \hat{\delta} & & \mathbf{id} \oplus \langle\rangle \big\downarrow & \leq & \big\downarrow \cong & \langle\rangle[\,] \\
 & A \oplus (C \otimes C) & \xleftarrow{\nabla \oplus \mathbf{id}} & (A \oplus A) \oplus (C \otimes C) & & A \oplus 1 & \xleftarrow{[\,] \oplus \mathbf{id}} & 0 \oplus 1 & \\
 & A \otimes C & \xleftarrow{\nabla} & (A \otimes C) \oplus (A \otimes C) & & A \otimes C & \xleftarrow{[\,]} & 0 & \\
\nabla\Delta & \mathbf{id} \otimes \nabla \big\uparrow & \leq & \big\uparrow \check{\delta} & & \mathbf{id} \otimes [\,] \big\uparrow & \leq & \big\uparrow \cong & [\,]\langle\rangle \\
 & A \otimes (C \oplus C) & \xrightarrow{\Delta \oplus \mathbf{id}} & (A \otimes A) \otimes (C \oplus C) & & A \otimes 0 & \xrightarrow{\langle\rangle \oplus \mathbf{id}} & 1 \otimes 0 &
\end{array}
$$

**Definition 1.** *A* classical category *is an order-enriched symmetric linearly distributive category with negation such that:*

1. *The symmetric monoidal category* $(\mathcal{C}, \oplus, 0)$ *has symmetric monoids;*
2. *The symmetric monoidal category* $(\mathcal{C}, \otimes, 1)$ *has symmetric comonoids;*
3. *The object indexed families of maps* $\Delta_A$, $\nabla_A$, $\langle\rangle_A$ *and* $[\,]_A$ *are lax natural transformations, in the sense that for every morphism* $f$ *we have*

$$
\begin{array}{rl}
\Delta \circ f \leq (f \otimes f) \circ \Delta & \qquad f \circ \nabla \leq \nabla \circ (f \oplus f) \\
\langle\rangle \circ f \leq \langle\rangle & \qquad f \circ [\,] \leq [\,]
\end{array}
$$

4. *The inequalities in Table 2 hold, where* $\hat{\delta}$ *and* $\check{\delta}$ *are the evident morphisms obtained from* $\delta$ *and symmetric monoidal isomorphisms*
5. *Composition of morphisms, and the functors* $\oplus$, $\otimes$ *are monotonic in all arguments.*

*Example 1.* $\mathbf{Rel}_\otimes$ is a classical category with objects sets and morphisms binary relations, in which both $\otimes$ and $\oplus$ are given by the set theoretic product. Both 0 and 1 are given by the singleton set $\{*\}$. Negation is identity on objects, and the excluded middle on a set A is the relation $\{(*, (x, x)) : x \in A\}$ from $\{*\}$ to $A \times A$. The map $\nabla_A$ is $\{((x, x), x) : x \in A\}$ and $[\,]_A$ is $\{(*, x) : x \in A\}$. The order on hom-sets is set-theoretic inclusion of relations.

*Example 2.* A boolean lattice $\mathbf{B}$ is a classical category, with meet as $\otimes$ and join as $\oplus$.

*Example 3.* If $\mathcal{C}$ and $\mathcal{C}'$ are classical categories, then so are $\mathcal{C}^{op}$ and $\mathcal{C} \times \mathcal{C}'$. In particular, the product of a classical category with non-trivial hom-sets (e.g. $\mathbf{Rel}_{\otimes}$) and a boolean algebra $\mathcal{B}$ is a non-compact, non-trivial classical category.

Before stating soundness and completeness, we need a notion of theory. For the sake of simplicity we consider only pure logic.

**Definition 2.** *A* sequent theory *over a collection of atoms $\mathcal{A}$ is a set of inequalities $\Phi \preccurlyeq \Psi$, where both $\Phi$ and $\Psi$ are proofs of a sequent $\Gamma \vdash \Delta$ over $\mathcal{A}$, such that:*

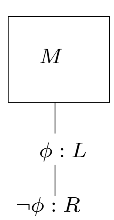
1. *The relation $\preccurlyeq$ is reflexive, transitive, and compatible (i.e. all inference rules are "monotonic w.r.t. $\preccurlyeq$);*
2. *The relation holds for both directions of the usual cut-reduction rules for eliminating logical cuts. It also holds in both directions for a number of coherence rules, including axiom expansions (for details, see [4]);*
3. *The usual rules for eliminating cut against weakening and contraction hold in only one direction: from redex to reduct.*

Führmann and Pym prove the following [4]:

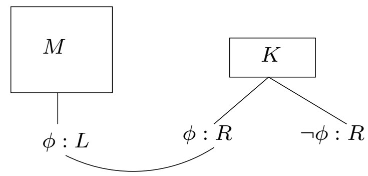
**Theorem 1 (Soundness [4]).** *Let $\mathcal{P}$ be a set of proofs over a set of atoms $\mathcal{A}$. Then for every interpretation $\mathcal{C}\lfloor - \rfloor$ in a classical category $\mathcal{C}$, the judgements $\Phi \preccurlyeq \Psi$ such that $\lfloor \Phi \rfloor \leq \lfloor \Psi \rfloor$ form a sequent theory.*

**Theorem 2 (Completeness [4]).** *Let $\mathcal{T}$ be a sequent theory, and suppose that $\lfloor \Phi \rfloor \leq \lfloor \Psi \rfloor$ holds for every interpretation in a classical category $\mathcal{C}$. Then $\Phi \preccurlyeq \Psi$ is in $\mathcal{T}$.*

The proof in [4] of the latter relies heavily on the use of proof nets for classical logic, introduced by Robinson [14], to construct a term model. These proof-nets are two-sided, and correspond very closely to the sequent calculus. (In [6], equivalent proof nets in style of Blute et. al. [15] are used, since they are closer to the categorical structure). The calculations involved are significantly simplified, since proof nets admit commuting conversions as syntactic equalities. One other very useful refinement of the presentation of LK (which is best presented in proof nets) is the use of cut against constants instead of rules to present $\neg L$, $\neg R$, and a variety of other proof rules: the proof net



which corresponds to an application of the LK rule $\neg R$ to a derivation of a sequent $\Gamma, \varphi \vdash \Delta$ $\neg R$, is taken to be shorthand for




which corresponds to that same LK derivation followed by cut against the (canonical) proof of the sequent $\vdash \varphi, \neg\varphi$. The $\wedge R$ is rule is modelled by cut against a proof net for $A, B \vdash A \wedge B$, and similarly for $\vee L$.

## 3 The calculus of structures for classical logic

We summarize Brünnler's SKSg [7], a deep symmetric system in the style of the calculus of structures for classical propositional logic. We give translations into a single-sided sequent system. Finally, we discuss Guglielmi's notions of "Formalism A" and "Formalism B" [10, 11], which generalize the calculus of structures.

### 3.1 System SKSg

The *calculus of structures* [8, 9] is a formalism that employs deep inference. The sequent calculus LK does not exhibit deep inference. Consider the sequent $\vdash C \wedge (A \vee A),\ B,\ B$. We can apply contraction across the comma, but not across the disjunction. Semantically, comma on the right is the same as disjunction: the syntactic distinction is required for completeness of the sequent calculus. The calculus of structures removes this distinction, and any calculus of structures inference rule operates arbitrarily deeply in a formula: this is deep inference. Proofs in the calculus of structures are not trees as in the sequent calculus, but are linear.

We now present the syntax for classical logic in a deep inference setting:

**Definition 3.** *Given a set V of propositional variables, we consider formulæ given by the grammar*

$$F ::= \mathsf{f} \,|\, \mathsf{t} \,|\, v \,|\, [F, \ldots, F] \,|\, (F, \ldots, F) \,|\, \bar{v},$$

*where v is a variable,* $\mathsf{t}$ *and* $\mathsf{f}$ *are* true *and* false*,* $[\ldots]$ *and* $(\ldots)$ *are disjunction and conjunction, and* $\bar{v}$ *is the negation of* $v$ *(negation on general formulæbeing inductively defined). We use* $\{\}$ *to denote a hole in a formula: a context* $S\{\}$ *is a formula with one occurrence of the hole, and* $S\{R\}$ *is that context with the hole filled with a formula* $R$*. We will omit the notation for the empty context where it can be inferred, for example writing* $S(A, B)$ *for* $S\{(A, B)\}$

*We take associativity and commutativity of connectives and the usual behaviour of units, to hold at the level of syntactic equivalence. The set* $S$ *of* structures *is the quotient of the set* $F$ *by the smallest relation containing these syntactic equivalences and closed under formation of formulæ from contexts.*

The deep symmetric system SKSg for classical propositional logic is given in Table 3. Each rule is unary, and is either self dual or comes as one of a dual pair. A *derivation*

**Table 3.** System SKSg

$$\frac{S[R,R]}{S\{R\}}\,\mathsf{c}\!\downarrow \qquad \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]}\,\mathsf{i}\!\downarrow \qquad \frac{S\{\mathsf{f}\}}{S[R]}\,\mathsf{w}\!\downarrow \qquad \frac{S([R,U],T)}{S[(R,T),U]}\,\mathsf{s} \qquad \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}\,\mathsf{i}\!\uparrow \qquad \frac{S(R)}{S\{\mathsf{t}\}}\,\mathsf{w}\!\uparrow \qquad \frac{S\{R\}}{S(R,R)}\,\mathsf{c}\!\uparrow$$

is a sequence of applications of rules. A *proof* of $A$ is a derivation from $\mathsf{t}$ to $A$.

The rules correspond closely to those of the one-sided sequent system GS1p. (GS1p differs from LK in its axiom and cut rules, as given in Table 4. The other rules of GS1p are the rules $\wedge R$, $\vee L$, $CL$ and $WR$ of LK, restricted to right handed sequents.) For example:

$$\frac{\vdash \Gamma,\varphi \qquad \vdash \Gamma',\psi}{\vdash \Gamma,\Gamma',\varphi\wedge\psi}\,\wedge R \rightarrow \dfrac{\dfrac{([\Gamma,\varphi],[\Gamma',\psi])}{[\Gamma,(\varphi,[\Gamma',\psi])]}\,\mathsf{s}}{[\Gamma,\Gamma',(\varphi,\psi)]}\,\mathsf{s}$$

and

$$\frac{\vdash \Gamma,\varphi \qquad \vdash \Gamma',\neg\varphi}{\vdash \Gamma,\Gamma'}\,CUT \rightarrow \dfrac{\dfrac{\dfrac{\dfrac{([\Gamma,\varphi],[\Gamma',\bar{\varphi}])}{[\Gamma,(\varphi,[\Gamma',\bar{\varphi}])]}\,\mathsf{s}}{[\Gamma,\Gamma',(\varphi,\bar{\varphi})]}\,\mathsf{s}}{[\Gamma,\Gamma',\mathsf{f}]}\,\mathsf{i}\!\uparrow}{[\Gamma,\Gamma']}=$$

The cases for axiom, weakening and contraction are similar. We now have the essential tools for the proof of the following theorem from [7]:

**Theorem 3 (GS1p to SKSg).** *For every proof $\Phi$ of $\vdash \Delta$ in GS1p there exists an SKSg proof of $\Delta$ that does not involve* $\mathsf{c}\!\uparrow$ *or* $\mathsf{w}\!\uparrow$ *, and has the same number of occurrences of* $\mathsf{i}\!\uparrow$ *as there are occurrences of cut in $\Phi$.*

*Remark 1.* Only the translation of the cut rule requires an application of $\mathsf{i}\!\uparrow$ : hence the preservation result. Since the $\mathsf{i}\!\uparrow$ rule is so closely related to the cut rule in GS1p, it is often referred to as the cut rule for the calculus of structures.

The following theorem (from [7]) gives us a partial converse — partial, since the number of cuts is not preserved:

**Theorem 4 (SKSg to GS1p).** *For every proof of a structure $S$ in SKSg there exists a proof of $\vdash S$ in GS1p.*

Given a proof in SKSg which begins with $\mathsf{t}$, we can translate it into a GS1p proof. Since cut is admissible in GS1p, we can obtain a cut-free proof. Translating that proof back into SKSg, the proof we obtain contains no instance of $\mathsf{i}\!\uparrow$ . This proves the following, (for which there is also a direct proof [16, 7]):

**Theorem 5 (Cut-elimination).** *Any structure provable from $\mathsf{t}$ in SKSg is provable without the use of $\mathsf{i}\!\uparrow$ (and without $\mathsf{w}\!\uparrow$ or $\mathsf{c}\!\uparrow$ ).*

### 3.2 "Formalism A" and "Formalism B"

The calculus of structures allows more freedom in the application of inference rules. As a result, it lays bare more bureaucracy than many other systems. Formalisms A [10] and B [11] are suggestions for ways to design new systems which lack this bureaucracy. Here we will describe the ideas behind these systems and the types of bureaucracy they avoid. Later, we will ask what light a categorical semantics sheds on the issues involved.

When dealing with the sequent calculus, one has to deal with an enormous number of commuting conversions; this is why proof nets, which validate these conversions as identities, are so useful. In the calculus of structures, the situation is made worse. For example, in the sequent calculus derivation,

$$\frac{\begin{array}{c}\vdots\,\Phi \\ \vdash \Gamma, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\Psi \\ \vdash \Gamma', \psi\end{array}}{\vdash \Gamma, \Gamma', \varphi \wedge \psi}\wedge R \tag{4}$$

the two sub-proofs act in parallel. In the calculus of structures, the same derivation can be written by applying the sub-proofs sequentially, as either

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{t}}{(\mathsf{t},\mathsf{t})}=}{([\Gamma,\varphi],\mathsf{t})}\Phi}{([\Gamma,\varphi],[\Gamma,\psi])}\Psi}{[\Gamma,(\varphi,[\Gamma',\psi])]}\mathsf{s}}{[\Gamma,\Gamma',(\varphi,\psi)]}\mathsf{s} \quad \text{or} \quad \frac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{t}}{(\mathsf{t},\mathsf{t})}=}{(\mathsf{t},[\Gamma,\psi])}\Psi}{([\Gamma,\varphi],[\Gamma,\psi])}\Phi}{[\Gamma,(\varphi,[\Gamma',\psi])]}\mathsf{s}}{[\Gamma,\Gamma',(\varphi,\psi)]}\mathsf{s}, \tag{5}$$

or by interleaving the inference rules from each proof, in a number of possible ways. All of these should, morally, represent the same proof. Formalism A would be a system in which applications of inference rules can be made in parallel, as they are in the sequent calculus or proof nets.

Formalism B caters to another, more subtle, form of bureaucracy. Suppose we have a derivation $\Phi$ of $A'$ from $A$ in the calculus of structures. The syntactic objects

$$\frac{\dfrac{(A,[B,C])}{[(A,B),C]}\mathsf{s}}{[(A',B),C]}\Phi \quad \text{and} \quad \frac{\dfrac{(A,[B,C])}{(A',[B,C])}\Phi}{[(A',B),C]}\mathsf{s} \tag{6}$$

should denote the same proof. The solution posed by Guglielmi [11] is to have inference rules acting not on formulæ, or on structures, but on derivations. Thus

$$\frac{(\Delta, [\Delta', \Delta''])}{[(\Delta, \Delta'), \Delta'']}\,\mathsf{s} \tag{7}$$

would be the canonical expression for the both expressions in (6). Recent progress has also been made on a version of the formalism called *wired deduction*.

**Table 4.** System GS1p: Axiom and Cut

$$\frac{}{\vdash \varphi, \bar{\varphi}}\,Ax_{GS1} \qquad \frac{\vdash \Gamma, \varphi \qquad \vdash \Gamma', \bar{\varphi}}{\vdash \Gamma, \Gamma'}\,CUT_{GS1p}$$

## 4 SKSg forms a classical category

In this section, we show that SKSg admits an inequational theory such that it forms a classical category.

We should note that deep inference proof theory regards each formula/structure as its own identity derivation.

**Definition 4 (The theory $\mathcal{T}$).**

*The theory $\mathcal{T}$ is a set of expressions $\Phi \leq \Psi$, where $\Phi$ and $\Psi$ are derivations in the calculus of structures. We write $\equiv$ for the symmetric closure of $\leq$. We give the inequations in a shallow form, with the understanding that the theory is closed under formation of contexts.*

*We deal first with permutation of non-interfering rules. Given two inference rules $p$ and $q$, the following holds:*

$$\cfrac{\cfrac{(A, A')}{(A, B')}\,q}{(B, B')}\,p \equiv \cfrac{\cfrac{(A, A')}{(B, A')}\,p}{(B, B')}\,q (:= (p, q)), \tag{8}$$

*and similarly for disjunction.*

*The nesting of derivations and switch is also part of our theory:*

$$\cfrac{\cfrac{(A, [B, C])}{(E, [F, G])}\,(p, [q, r])}{[(E, F), G)]}\,s \equiv \cfrac{\cfrac{(A, [B, C])}{[(A, B), C]}\,s}{[(E, F), G]}\,[(p, q), r], \tag{9}$$

*along with several equalities relating different ways of nesting switches, for example:*

$$\dfrac{(A,B,[C,D])}{[(A,B,C),D]}\,s \equiv \dfrac{\dfrac{(A,B,[C,D])}{(A,[(B,C),D])}\,s}{[(A,B,C),D]}\,s\,. \tag{10}$$

*The following rule and its dual govern interactions between negations:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A}{(A,\mathsf{t})}=}{(A,[\bar{A},A])}\,\mathsf{i}{\downarrow}}{[(A,\bar{A}),A]}\,\mathsf{s}}{[\mathsf{f},A]}\,\mathsf{i}{\uparrow}}{A}= \;\equiv\; \dfrac{A}{A}\,\mathbf{id} \tag{11}$$

*The remaining equations are given in Table 5 and the inequations of the theory are given in Table 6.*

**Table 5.** Equalities: weakening and contraction

$$\dfrac{\dfrac{\dfrac{S[[P,Q],[P,Q]]}{S[[P,P],[Q,Q]]}=}{S[[P,P],Q]}\downarrow c}{S[P,Q]}\downarrow c \equiv \dfrac{S[[P,Q],[P,Q]]}{S[P,Q]}\downarrow c$$

$$\dfrac{\dfrac{[[A,A],A]}{[A,A]}\,\mathsf{c}{\downarrow}}{A}\,\mathsf{c}{\downarrow} \equiv \dfrac{\dfrac{[A,[A,A]]}{[A,A]}\,\mathsf{c}{\downarrow}}{A}\,\mathsf{c}{\downarrow}$$

$$\dfrac{\dfrac{\dfrac{A}{[A,\mathsf{f}]}=}{[A,A]}\,\mathsf{w}{\downarrow}}{A}\,\mathsf{c}{\downarrow} \equiv \dfrac{A}{A}\,\mathbf{id} \qquad \dfrac{\dfrac{\mathsf{f}}{[\mathsf{f},\mathsf{f}]}=}{[A,B]}\,[\mathsf{w}{\downarrow}\,,\mathsf{w}{\downarrow}\,] \equiv \dfrac{\mathsf{f}}{[A,B]}\,\mathsf{w}{\downarrow} \qquad \dfrac{\mathsf{f}}{\mathsf{f}}\,\mathsf{w}{\downarrow} \equiv \dfrac{\mathsf{f}}{\mathsf{f}}\,\mathbf{id}$$

We now prove the main technical theorem of the paper.

Since we have an identity derivation on structures, we can form a category from SKSg, with structures as objects and derivations between two structures as morphisms; in particular, each inference rule is a morphism. Composition is given by concatenation. We have extended conjunction and disjunction to the inference rules (Equation 8) and we can extend that definition inductively to all derivations. We have required that the connectives preserve identity derivations. By induction on the length of derivation, we can show that both are bifunctorial. That they are monoidal follows from the syntactic equalities of associativity, symmetry and units (i.e, they are *strong* monoidal products).

SKSg is a symmetric linearly distributive category, $\delta$ being given by the switch rule, which is natural by (9). The required coherences are those typified by (10) (for further details of the coherences see [13]). Along with (11), $\mathsf{i}\uparrow$ and $\mathsf{i}\downarrow$ are precisely what is required to model negation in a linearly distributive category. The equations in Table 5 show that $\mathsf{c}\downarrow$ and $\mathsf{w}\downarrow$ form a symmetric monoid ($\nabla$, $[\,]$) (and dually). Finally, the inequations in Table 6 plus their duals give SKSg the structure of a classical category.

**Theorem 6.** *SKSg with $\mathcal{T}$ forms a classical category.*

**Table 6.** Inequalities: weakening and contraction

$$\dfrac{\dfrac{A}{(A,A)}\,\mathsf{c}\uparrow}{(B,B)}\,(p,p) \geq \dfrac{\dfrac{A}{B}\,p}{(B,B)}\,\mathsf{c}\uparrow \qquad \dfrac{\dfrac{\mathsf{f}}{A}\,\mathsf{w}\downarrow}{B}\,p \leq \dfrac{\mathsf{f}}{B}\,\mathsf{w}\downarrow$$

$$\dfrac{[A,C]}{[A,(C,C)]}\,\mathsf{c}\uparrow \leq \dfrac{\dfrac{\dfrac{[A,C]}{([A,C],[A,C])}\,\mathsf{c}\uparrow}{[A,A,(C,C)]}\,\mathsf{s}'}{[A,(C,C)]}\,\mathsf{c}\downarrow \qquad \dfrac{(A,\mathsf{f})}{(A,B)}\,\mathsf{w}\downarrow \leq \dfrac{\dfrac{\dfrac{(A,\mathsf{f})}{(\mathsf{t},\mathsf{f})}\,\mathsf{w}\uparrow}{\mathsf{f}}\,=}{(A,B)}\,\mathsf{w}\downarrow$$

*Remark 2.* The structure we have given to SKSg models precisely the equalities of Formalisms A and B, and we would expect these formalisms (when they appear) to form a classical category. Conversely, by looking at classical categories we can infer what additional structure these formalisms will need to include. Categorically, Formalism A amounts to bifunctoriality of disjunction and conjunction. (Deep inference in the calculus of structures is the ordinary functoriality of the connectives with respect to each argument.) The behaviour of Formalism B is modelled by naturality of switch (and lax naturality of the structural rules).

**Theorem 7.** *SKSg + $\mathcal{T}$ is not equivalent to a boolean algebra; it is a non-trivial.*

*Proof.* The quotient of the category $\mathbf{Rel}_\otimes$ by the equivalence relation $\mathcal{R}$, which identifies objects along the symmetric monoidal isomorphisms, is a nontrivial classical category. Define a functor $\mathcal{G}$ from $\mathbf{Rel}_\otimes/\mathcal{R}$ inductively on objects by mapping each propositional variable to a unique set, and inductively on morphisms by mapping each inference rule to the corresponding classical category morphism in $\mathbf{Rel}_\otimes$. This is well defined, as $\mathbf{Rel}_\otimes$ is monoidal. Then, for any proof $\Phi$ of SKSg, $\mathcal{G}(f \circ \mathsf{i}\downarrow) \neq \mathsf{i}\downarrow$ .

*Remark 3.* The need to quotient $\mathbf{Rel}_\otimes$ arises from the equalities in SKSg. If, instead, we add new (invertible) rules corresponding to the symmetric monoidal isomorphisms, we obtain a category equivalent to the category of proof nets for classical logic.

## 5 Insights into SKSg and the sequent calculus

### 5.1 The meaning of cut in SKSg

We have seen that when we refer to cut in SKSg, we mean the rule i↑ . This corresponds well to the definition of cut in a one-sided sequent system such as GS1p, and can be eliminated, either directly or via a translation into GS1p. However, in the previous section we saw SKSg as a classical category, and thereby a model of the two sided system LK. The order given on proofs is a model of cut-reduction, and yet it is independent of the rule i↑ . In this section we explore the relationship between these two notions of cut.

First, notice that any sequent in GS1p is a sequent in LK; we show that we can embed any GS1p proof into LK.

**Lemma 1.** *Any proof $\Phi$ in GS1p can be transformed into a proof in LK.*

*Proof.* Suppose we have a GS1p proof $\Phi$. Since GS1p and LK differ only on cut and axiom, we transform only instance of these rules.

We replace each occurrence of an axiom

$$\frac{}{\vdash \neg\varphi, \varphi} Ax_{GS1p} \quad \text{with} \quad \frac{\frac{}{\varphi \vdash \varphi} Ax_{LK}}{\vdash \neg\varphi, \varphi} \neg R \tag{12}$$

The case of cut is rather more complicated. There are three obvious ways of defining the GS1p cut rule in LK:

$$\frac{\vdash \varphi, \Delta \qquad \frac{\vdash \neg\varphi, \Delta' \qquad \frac{\frac{}{\varphi \vdash \varphi} Ax_{LK}}{\neg\varphi, \varphi \vdash} \neg L}{\varphi \vdash \Delta'} Cut_{LK}}{\vdash \Delta, \Delta'} Cut_{LK}, \tag{13}$$

$$\frac{\vdash \neg\varphi, \Delta', \quad \frac{\vdash \varphi, \Delta}{\neg\varphi \vdash \Delta} \neg L}{\vdash \Delta, \Delta'} Cut_{LK}, \tag{14}$$

$$\frac{\frac{\vdash \neg\,\varphi, \Delta' \qquad \vdash \varphi, \Delta}{\vdash (\neg\varphi \wedge \varphi), \Delta, \Delta'} \wedge R \qquad \frac{\frac{\frac{}{\varphi \vdash \varphi} Ax_{LK}}{\neg\varphi, \varphi \vdash} \neg L}{\neg\varphi \wedge \varphi \vdash} \wedge L}{\vdash \Delta, \Delta'} Cut_{LK}. \tag{15}$$

These definitions are coherent, in the sense that they are identified in our semantics. Eliminating the logical cut in (15), and shifting to proof nets, with negation as cut against a constant as in (**??**, §2), all three are represented by the same proof net.

As SKSg is a classical category, the usual notion of interpretation gives us a translation from LK to SKSg. The following is a corollary of soundness:

**Theorem 8 (From LK to SKSg).** *For every proof $\Phi$ of a sequent $\phi \vdash \psi$ in LK , there is a derivation $\lfloor \Phi \rfloor$ from $\phi$ to $\psi$ in SKSg. For any cut-reduction $\Phi \preccurlyeq \Psi$ in LK, there is a corresponding inequality in SKSg +$\mathcal{T}$*

*Remark 4.* This translation maps a proof $\psi$, containing no occurrences of $\neg L$, to a "cut-free" proof in the sense of SKSg, regardless of the number of instances of cut in $\psi$. In particular, consider the following lemma:

**Lemma 2.** *The sequent $\vdash \Gamma$, if provable in LK, is provable without recourse to rules operating on the left hand side of the sequent.*

The SKSg cut-elimination theorem (Theorem 5) is a corollary of this result.

Each inference rule in SKSg, when read from top to bottom, is a valid entailment $\phi \rightarrow \psi$ in classical logic, and there is therefore a cut-free proof of $\phi \vdash \psi$. Composing these proofs using the cut rule, we obtain:

**Theorem 9 (From SKSg to LK).** *For every derivation $\Phi$ from $A$ to $B$ in SKSg there exists an LK proof of the sequent $A \vdash B$, with a number of instances of cut equal to the number of rule applications in $\Phi$ minus one.*

*Remark 5.* This theorem embodies the notion of cut as composition. A proof is cut free in this sense only if it is an instance of an inference rule: in particular $\mathsf{i}\uparrow$ translates to a cut-free proof in LK.

Remarks 4 and 5 show that the translations between LK and SKSg respect the notion of cut given by composition and the ordering on proofs, but not that given by instances of the $\mathsf{i}\downarrow$ rule. The presence of this rule is clearly not sufficient to identify non-normal derivations of SKSg. We can, however, generalise the notion of a normal proof to a normal derivation in the following way:

**Definition 5.** *A derivation is* normal *if it contains no $\uparrow$ rule below a $\downarrow$ rule.*

(This observation was also made recently (and independently) by Brünnler.) This definition agrees with the equivalent notion for LK, in the sense that any normal LK derivation (or proof net) translates to a normal SKSg derivation (taking care to note that since we have (implicit) cuts against constants in our proof nets, a normal proof net is one without *essential cuts*). The notion of a normal SKSg proof is easily seen to be a special case of this definition, since any $\uparrow$ rule with premise $\mathtt{t}$ has conclusion equivalent to $\mathtt{t}$. Notice also that each equational law in $\mathcal{T}$ involves only $\uparrow$ or $\downarrow$ rules: That is, application of these rules to a normal derivation yields another normal derivation. Meanwhile, the lax naturalities clearly show that moving an $\mathsf{w}\uparrow$ or $\mathsf{c}\uparrow$ rule above some other derivation generates some change in denotation: The change corresponds to a move closer to a normal form. By switching to a local presentation of the inequalities on proofs (i.e. an inequational theory on SKSg) the author hopes in the future to remove these inequalities from the theory and understand cut-reduction purely as moving $\uparrow$ rules above $\downarrow$ rules.

### 5.2 Intuitionistic validity and decomposition

Which proofs in SKSg are intuitionistically valid? Considering each inference rule as an implication, we find that only $\mathsf{i}\downarrow$ is invalid. (Recall that negation is derived, and therefore all the formulae in a derivation are assumed to be in negation normal form. This excludes de Morgan duality from the list of syntactic equivalences we include. All the other equivalences are intuitionistically valid). We have, therefore:

**Lemma 3.** *A calculus of structures derivation is intuitionistically valid if it contains no application of the* $\mathsf{i}\downarrow$ *rule.*

We call a derivation which has this property *intuitionistic*, and a derivation which has no instance of $\mathsf{i}\uparrow$ *co-intuitionistic*. (This terminology derives from the fact that if there is a co-intuitionistic derivation $S$ from $A$ to $B$, the negation normal form (*nnf*) of $\bar{B}$ intuitionistically entails the nnf of $\bar{A}$, and the de-Morgan dual of $S$ is such an intuitionistic derivation.) In particular, the $\downarrow$ fragment of SKSg is co-intuitionistic. Consider the following decomposition lemma from [7]:

**Theorem 10.** *For every derivation of B from A in SKSg there is a derivation that is of the form*

$$\begin{array}{l} A \\ \vdots\, SKSg \setminus \{\mathsf{i}\downarrow, \mathsf{w}\downarrow\} \\ C \\ \vdots\, SKSg \setminus \{\mathsf{i}\uparrow, \mathsf{w}\uparrow\} \\ B \end{array}$$

Using that the first phase of the above proof introduces no new propositional variables, and the second phase introduces no new propositional variables when viewed bottom up, Brünnler obtains the Craig interpolation lemma [7]; using Lemma 3, we may now strengthen that result:

**Corollary 1 (Refined Craig interpolation).** *For all propositional formulæ $\phi$ and $\psi$ in nnf, if $\phi$ implies $\psi$ then there is an interpolant $\gamma$ such that $\phi$ intuitionistically implies $\gamma$, $\gamma$ co-intuitionistically implies $\psi$, and $\gamma$ contains only propositional variables common to A and B. Given a* normal *derivation $S$ from $\phi$ to $\psi$, we can choose the intuitionistic and co-intuitionistic derivations such that their composition equals $S$*

*Proof.* A normal derivation is already $\uparrow - \downarrow$ factored, and clearly the composition of these factors is the original derivation.

McKinley and Brünnler have observed this behaviour in the sequent calculus (where it can be demonstrated by an additional observation on top of the usual structural induction), but the property is (in the opinion of the author) perspicuous in the calculus of structures.

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