# Intuitionistic fixed point theories for strictly positive operators

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**Abstract**

In this paper it is shown that the intuitionistic fixed point theory $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ for $\alpha$ times iterated fixed points of strictly positive operator forms is conservative for negative arithmetic and $\Pi^0_2$ sentences over the theory $\mathsf{ACA}^{-i}_\alpha$ for $\alpha$ times iterated arithmetic comprehension without set parameters. This generalizes results previously due to Buchholz [5] and Arai [2].

**Keywords:** Intuitionistic fixed point theories, strictly positive operators, accessibility operators, Heyting arithmetic.

**Mathematics Subject Classification:** 03F, 03F50, 03F55, 03F30, 03D70.

# 1 Introduction

The study of fixed points of positive arithmetic operators has long been central to proof theory. Whereas the original interest was in *least* definable fixed points and corresponding theories of iterated inductive definitions (cf. [6] for a survey), it has turned out later that already the axiomatization of the fixed point property alone, without claiming leastness of fixed points, is of significant proof-theoretic interest.

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Probably the first result in this direction is Aczel's proof in [1] that the *classical* (non-iterated) fixed point theory $\widehat{\mathsf{ID}}_1$ is equivalent to the classical subsystem $\Sigma^1_1$-$\mathsf{AC}$ of analysis and to Martin-Löf's type theory with one universe. This result was subsequently generalized by Feferman [7] in his proof of Hancock's conjecture; in particular, finitely iterated fixed point theories $\widehat{\mathsf{ID}}_n$ for $n < \omega$ are introduced whose limit $\widehat{\mathsf{ID}}_{<\omega}$ is shown to be of the same strength as predicative analysis. The transfinite iterations $\widehat{\mathsf{ID}}_\alpha$ of Feferman's theories have been analyzed only recently in Jäger, Kahle, Setzer, and Strahm [9]. For further recent work on *classical* fixed point theories the reader is advised to consult [3, 10, 12, 14].

Not long ago it has been observed by Buchholz [5] that the choice of logic is crucial in the context of fixed point theories, i.e., the corresponding systems based on *intuitionistic* logic are much weaker than their classical counterparts. In [5] a certain intuitionistic theory $\widehat{\mathsf{ID}}^i_1(\mathit{strong})$ for strongly positive operator forms is introduced; see the next section for an exact definition of this fixed point theory. It is shown loc. cit. that $\widehat{\mathsf{ID}}^i_1(\mathit{strong})$ is conservative over Heyting arithmetic $\mathsf{HA}$ for almost negative sentences by an embedding of $\widehat{\mathsf{ID}}^i_1(\mathit{strong})$ into $\mathsf{HA}$ augmented by Church's thesis $\mathsf{CT}_0$. Buchholz' result was subsequently extended in Arai [2], where it is proved that even the finite iterations $\widehat{\mathsf{ID}}^i_n(\mathit{strong})$ for $n < \omega$ are conservative over $\mathsf{HA}$ *for all arithmetic sentences*. Arai's proof makes crucial use of Goodman's theorem.

In this paper we will generalize the results of Buchholz and Arai in two directions. First of all, we show that one can even allow so-called *strictly positive operator forms* so that the corresponding theories $\widehat{\mathsf{ID}}^i_n(\mathit{strict})$ do not exceed the strength of Heyting arithmetic $\mathsf{HA}$. Strictly positive operators extend the strongly positive ones; they have been widely studied in the context of intuitionistic inductive definability and capture many natural examples, cf. [6]. Secondly, in this paper we will also establish the strength of the *transfinitely* iterated fixed point theories $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ for $\alpha \geq \omega$. In particular, it is proved that $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ is conservative over $\mathsf{ACA}^{-i}_\alpha$ for all negative arithmetic and $\Pi^0_2$ sentences, where $\mathsf{ACA}^{-i}_\alpha$ denotes the natural theory of $\alpha$ times iterated arithmetic comprehension without set parameters.

Our proof of the above results exhibits in a very perspicuous manner why iterated intuitionistic fixed point theories for strictly positive operators simply correspond to iterated arithmetical comprehension without parameters. A crucial first step in the argument below goes back to an observation of Buchholz [4] in the context of iterated inductive definitions, where it is shown that the "realizability content" of strictly positive operators corresponds to certain general forms of accessibility inductive definitions. In a second step one observes that such accessibility operators allow for very simple arithmetic solutions obtained by diagonalization, as long as one is only interested in their fixed point property. This second observation is closely related to Feferman's sketch at the end of [7], where he shows how to produce a $\Pi^0_2$ fixed point solution for the operator corresponding to Kleene's $\mathcal{O}$, an argument apparently going back to a mistake of Kleene himself, in his first attempt in 1944 to establish the complexity of his $\mathcal{O}$.

## 2 The theories $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ and $\mathsf{ACA}^{-i}_\alpha$

In this section we want to give a precise definition of the fixed point theories $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ for strictly positive operator forms and their subsystems $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strong})$ as well as $\widehat{\mathsf{ID}}^i_\alpha(\mathit{acc})$ for strongly positive and accessibility operator forms, respectively. Moreover, we introduce the theories $\mathsf{ACA}^{-i}_\alpha$ for iterated arithmetic comprehension without parameters.

All systems are based on suitable extensions of the language $\mathcal{L}_1$ of intuitionistic first order arithmetic. We assume that $\mathcal{L}_1$ includes *number variables* $(a, b, c, u, v, w, x, y, z, \dots)$ and symbols for all primitive recursive functions and relations. The *number terms* $(r, s, t, \dots)$ and *formulas* $(A, B, C, \dots)$ of $\mathcal{L}_1$ are defined as usual. Let us agree that $\mathcal{L}_1$ is based on $\rightarrow$, $\wedge$, $\vee$, $\forall$, $\exists$, and that $\neg$ is defined in terms of $\rightarrow$ and 0=1. Further, recall that an $\mathcal{L}_1$ formula is called *negative*, if it does not contain $\vee$ and $\exists$. Finally, the *almost negative* $\mathcal{L}_1$ formulas are exactly those formulas which do not contain $\vee$, and $\exists$ only in front of atomic formulas.

In order to define the class of strictly positive and accessibility operator forms we let $P$ and $Q$ denote fresh unary relation symbols and denote by $\mathcal{L}_1(P, Q)$ the extension of $\mathcal{L}_1$ by $P$ and $Q$; the language $\mathcal{L}_1(Q)$ is defined similarly. The *strictly positive (with respect to $P$)* formulas of $\mathcal{L}_1(P, Q)$ (cf. e.g. [6]) are now inductively generated as follows:

1. The formulas of $\mathcal{L}_1(Q)$ are strictly positive.
2. The formulas of the form $P(t)$ are strictly positive.
3. The strictly positive formulas are closed under $\wedge$, $\vee$, $\forall$, and $\exists$.
4. If $A$ is an $\mathcal{L}_1(Q)$ formula and $B$ is strictly positive, then $(A \rightarrow B)$ is strictly positive.

The *strongly positive (with respect to $P$)* formulas of $\mathcal{L}_1(P, Q)$ (cf. Arai [2] and Buchholz [5]) are defined to be the subclass of the strictly positive formulas which is generated by clauses (1)–(3) in the above inductive definition. A strictly positive $\mathcal{L}_1(P, Q)$ formula which contains at most $x$ and $y$ free is called a *strictly positive operator form*, and we let $\mathcal{A}(P, Q, x, y)$ range over such forms; the *strongly positive operator forms* are defined analogously.

A very important further subclass of the strictly positive operator forms are operators for accessibility inductive definitions. Accordingly, we call $\mathcal{A}(P, Q, x, y)$ an *accessibility operator form*, if it is of the shape

$$A \wedge (\forall z)(B(z) \rightarrow P(z)),$$

for formulas $A$ and $B$ belonging to the language $\mathcal{L}_1(Q)$. Observe that $A$ and $B$ in general depend on the parameters $x$ and $y$ in such operator forms.

In order to formulate transfinitely iterated fixed point theories below we have to fix an ordinal notation system. In the following we confine ourselves to the standard notation system of order type $\Gamma_0$ which is based on the binary $\varphi$ or Veblen function (cf. e.g. [11, 13]). We let $\prec$ stand for the corresponding primitive recursive wellordering and assume without loss of generality that the field of $\prec$ is the set of all natural numbers and that 0 is the least element with respect to $\prec$. Finally, we set for all formulas $A(x)$ and number terms $s$:

$$\mathsf{TI}(s, A) \;:=\; (\forall x)[(\forall y)(y \prec x \rightarrow A(y)) \rightarrow A(x)] \;\rightarrow\; (\forall x \prec s)A(x).$$

The stage is now set in order to introduce the theories $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ for each $\alpha$ less than $\Gamma_0$.[1] $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ is formulated in the language $\mathcal{L}_{\mathsf{fix}}$, which extends $\mathcal{L}_1$ by a new unary relation symbol $P^{\mathcal{A}}$ for each strictly positive operator form $\mathcal{A}(P, Q, x, y)$. In the following let us write $P^{\mathcal{A}}_s(t)$ for $P^{\mathcal{A}}(\langle t, s\rangle)$ and $P^{\mathcal{A}}_{\prec s}(t)$ for $t = \langle (t)_0, (t)_1\rangle \wedge (t)_1 \prec s \wedge P^{\mathcal{A}}(t)$. Here $\langle \cdot, \cdot \rangle$ denotes a primitive recursive coding function with associated projections $(\cdot)_0$ and $(\cdot)_1$; in the sequel we will write, for example, $\langle t_1, t_2, t_3\rangle$ for $\langle\langle t_1, t_2\rangle, t_3\rangle$.

The theory $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strict})$ is based on intuitionistic logic with equality and comprises the following axioms: (i) the axioms of Heyting arithmetic $\mathsf{HA}$ with the scheme of complete induction for all formulas of $\mathcal{L}_{\mathsf{fix}}$, (ii) the fixed point axioms

$$(\forall a \prec \alpha)(\forall x)[P^{\mathcal{A}}_a(x) \leftrightarrow \mathcal{A}(P^{\mathcal{A}}_a, P^{\mathcal{A}}_{\prec a}, x, a)]$$

for all strictly positive operator forms $\mathcal{A}(P, Q, x, y)$, as well as (iii) the axioms $\mathsf{TI}(\alpha, A)$ for all $\mathcal{L}_{\mathsf{fix}}$ formulas $A$. We write $\widehat{\mathsf{ID}}^i_{<\alpha}(\mathit{strict})$ for the union of the theories $\widehat{\mathsf{ID}}^i_\beta(\mathit{strict})$ for $\beta$ less than $\alpha$. The theories $\widehat{\mathsf{ID}}^i_\alpha(\mathit{strong})$ as well as

[1] Of course, the restriction to ordinals less than $\Gamma_0$ is not essential; it just stems from the choice of our notation system for the purpose of this article.

$\widehat{\mathsf{ID}}^i_\alpha(acc)$ are defined in the same manner as $\widehat{\mathsf{ID}}^i_\alpha(strict)$, but only for strongly positive and accessibility operator forms, respectively.

Finally, we want to introduce the theories $\mathsf{ACA}^{-i}_\alpha$ for $\alpha$ times iterated arithmetic comprehension *without parameters*. It is most convenient to formulate $\mathsf{ACA}^{-i}_\alpha$ as a first order system. For that purpose we let $\mathcal{L}_{\mathsf{it}}$ denote the extension of $\mathcal{L}_1$ by new unary relation symbols $H^A$ for each $\mathcal{L}_1(Q)$ formula $A(Q,x,y)$ with at most $x,y$ free. The formulas $H^A_s(t)$ and $H^A_{\prec s}(t)$ are defined similarly as above.

The theory $\mathsf{ACA}^{-i}_\alpha$ is based on intuitionistic logic with equality and comprises the following axioms: (i) the axioms of Heyting arithmetic $\mathsf{HA}$ with the scheme of complete induction for all formulas of $\mathcal{L}_{\mathsf{it}}$, (ii) the iterated arithmetical comprehension without parameters axioms

$$(\forall a \prec \alpha)(\forall x)[H^A_a(x) \leftrightarrow A(H^A_{\prec a}, x, a)]$$

for all $\mathcal{L}_1(Q)$ formulas $A(Q,x,y)$, as well as (iii) the axioms $\mathsf{TI}(\alpha, A)$ for all $\mathcal{L}_{\mathsf{it}}$ formulas $A$. We write $\mathsf{ACA}^{-i}_{<\alpha}$ for the union of the theories $\mathsf{ACA}^{-i}_\beta$ for $\beta$ less than $\alpha$.

In the next section we will also refer to the classical version $\mathsf{ACA}^{-}_\alpha$ of $\mathsf{ACA}^{-i}_\alpha$. It should not surprise the reader that $\mathsf{ACA}^{-}_\alpha$ and $\mathsf{ACA}^{-i}_\alpha$ prove the same negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$. This can be established by employing standard double negation and Friedman translation (cf. e.g. [6]).

**Theorem 1** *We have that $\mathsf{ACA}^{-}_\alpha$ is conservative over $\mathsf{ACA}^{-i}_\alpha$ for all negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$.*

## 3 Reduction of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\mathsf{ACA}^{-i}_\alpha$

It is the aim of this section to give a reduction of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\mathsf{ACA}^{-i}_\alpha$ which preserves negative arithmetic and $\Pi^0_2$ sentences. Our proof-theoretic analysis of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ proceeds in two steps. First, we refer to Buchholz [4] in order to reduce $\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\widehat{\mathsf{ID}}^i_\alpha(acc)$ using recursive realizability. Then we show how to model $\widehat{\mathsf{ID}}^i_\alpha(acc)$ directly in the classical theory $\mathsf{ACA}^{-}_\alpha$ by means of certain diagonalization techniques inherent in Feferman [7].

Let us now first turn to the reduction of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\widehat{\mathsf{ID}}^i_\alpha(acc)$. Here we can literally follow Buchholz [4], paragraph 6, where a reduction of $\mathsf{ID}^i_\alpha(strict)$ to $\mathsf{ID}^i_\alpha(acc)$ is carried through by means of a standard recursive realizability interpretation. The reader can readily observe that this procedure carries over to our theories "with hat" in order to yield the desired reduction of

$\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\widehat{\mathsf{ID}}^i_\alpha(acc)$. Since the realizability argument is given in [4] in full detail, we will confine ourselves in the sequel to describing the main steps and ideas of the reduction only.

The notion $e\,\mathbf{r}\,C$ ("$e$ realizes $C$") for $C$ a formula in $\mathcal{L}_1(P,Q)$ is defined by induction on the complexity of $C$ in the following manner, where $\{e\}$ denotes a standard enumeration of the partial recursive functions.

$$\begin{array}{lcl}
e\,\mathbf{r}\,C & := & C \quad \text{if } C \text{ is an atomic formula of } \mathcal{L}_1 \\
e\,\mathbf{r}\,P(t) & := & P(\langle t, e\rangle) \\
e\,\mathbf{r}\,Q(t) & := & Q(\langle (t)_0, (e)_1, (t)_1\rangle) \\
e\,\mathbf{r}\,A \wedge B & := & (e)_0\,\mathbf{r}\,A \wedge (e)_1\,\mathbf{r}\,B \\
e\,\mathbf{r}\,A \vee B & := & ((e)_0 = 0 \rightarrow (e)_1\,\mathbf{r}\,A) \wedge ((e)_0 \neq 0 \rightarrow (e)_1\,\mathbf{r}\,B) \\
e\,\mathbf{r}\,A \rightarrow B & := & (\forall u)(u\,\mathbf{r}\,A \rightarrow \{e\}(u)\,\mathbf{r}\,B) \\
e\,\mathbf{r}\,(\forall x)A(x) & := & (\forall x)(\{e\}(x)\,\mathbf{r}\,A(x)) \\
e\,\mathbf{r}\,(\exists x)A(x) & := & (e)_1\,\mathbf{r}\,A((e)_0).
\end{array}$$

The key observation made by Buchholz is that the assertion $e\mathbf{r}C$ for a *strictly positive* $\mathcal{L}_1(P,Q)$ formula $C$ is equivalent in intuitionistic logic to an "accessibility statement" of the form $A \wedge (\forall z)(B(z) \rightarrow P(z))$, with $A$ and $B$ formulas of $\mathcal{L}_1(Q)$. The reader should not have difficulties in verifying this assertion and in case of doubt can consult Buchholz [4], Lemma 6.1. Using this crucial fact, the realizability interpretation of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ to $\widehat{\mathsf{ID}}^i_\alpha(acc)$ is rather straightforward. More precisely, one associates to a given *strictly positive operator form* $\mathcal{A}(P,Q,x,y)$ the *accessibility operator form* $\mathcal{A}^+(P,Q,x,y)$,

$$\mathcal{A}^+(P,Q,x,y) \;:=\; (x)_1\,\mathbf{r}\,\mathcal{A}(P,Q,(x)_0,y).$$

Then one defines the realizability relation $e\,\mathbf{r}\,C$ for $C$ being a formula in the language $\mathcal{L}_{\mathsf{fix}}$ by choosing $e\,\mathbf{r}\,P^{\mathcal{A}}(t)$ as $P^{\mathcal{A}^+}(\langle (t)_0, e, (t)_1\rangle)$ and using the same clauses as above in the remaining cases. It is now a matter of routine to check that if the $\mathcal{L}_{\mathsf{fix}}$ sentence $A$ is derivable in $\widehat{\mathsf{ID}}^i_\alpha(strict)$, then there exists a natural number $n$ so that $\widehat{\mathsf{ID}}^i_\alpha(acc)$ proves the assertion $n\,\mathbf{r}\,A$; for details cf. [4]. As usual, realizability yields conservativity for almost negative formulas (cf. [15]) and, hence, we can summarize our discussion in the following theorem.

**Theorem 2** *We have that $\widehat{\mathsf{ID}}^i_\alpha(strict)$ is conservative over $\widehat{\mathsf{ID}}^i_\alpha(acc)$ for all almost negative sentences of $\mathcal{L}_1$.*

The second step of the reduction of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ consists in the proof-theoretic analysis of $\widehat{\mathsf{ID}}^i_\alpha(acc)$. More precisely, we now show how to embed the classical version $\widehat{\mathsf{ID}}_\alpha(acc)$ of $\widehat{\mathsf{ID}}^i_\alpha(acc)$ into $\mathsf{ACA}^-_\alpha$. The crucial step in this embedding is a simple *diagonalization* argument, similar in spirit to Aczel's analysis of $\widehat{\mathsf{ID}}_1$ by means of the $\Sigma^1_1$ axiom of choice, cf. [7]. But in contrast to full $\widehat{\mathsf{ID}}_1$, which forces a $\Sigma^1_1$ solution for the required fixed point, we can already do with an *arithmetic* fixed point due to the presence of *accessibility inductive definitions*. This observation is closely related to the argument given at the end of Feferman [7], where it is shown how to produce a $\Pi^0_2$ fixed point for the definition of Kleene's $\mathcal{O}$; indeed, $\mathcal{O}$ is just a special kind of accessibility inductive definition.

**Theorem 3** *We have that* $\widehat{\mathsf{ID}}_\alpha(acc)$ *is conservative over* $\mathsf{ACA}^-_\alpha$ *for all sentences of* $\mathcal{L}_1$.

Proof. First of all it is trivial to figure out an embedding of $\mathsf{ACA}^-_\alpha$ into $\widehat{\mathsf{ID}}_\alpha(acc)$, since accessibility operator forms of $\mathcal{L}_1(P,Q)$ cover arbitrary $\mathcal{L}_1(Q)$ formulas $A(Q,x,y)$. Therefore, let us see how we can obtain an embedding of $\widehat{\mathsf{ID}}_\alpha(acc)$ into $\mathsf{ACA}^-_\alpha$ preserving sentences in $\mathcal{L}_1$.

For that purpose we fix an arbitrary accessibility operator form $\mathcal{A}(P,Q,x,y)$. Hence, there are $\mathcal{L}_1(Q)$ formulas $A$ and $B$, so that $\mathcal{A}(P,Q,x,y)$ has the form

$$A \wedge (\forall z)(B(z) \rightarrow P(z)). \tag{1}$$

Note that $A$ and $B$ in general depend on the variables $x$ and $y$. First, we choose a natural number $n$ so that $A$ and $B$ are equivalent to a $\Pi^0_n$, respectively a $\Sigma^0_n$ formula of $\mathcal{L}_1(Q)$. Our aim now is to find a fixed point solution by means of diagonalization of a universal $\Pi^0_n$ predicate. Towards that aim, let $E(Q,u,x,y,z)$ be a *universal* $\Pi^0_n$ formula of $\mathcal{L}_1(Q)$ which enumerates for $u = 0,1,2,\ldots$ all $\Pi^0_n$ formulas $C(x,y,z)$ of $\mathcal{L}_1(Q)$. Let us now have a look at the $\mathcal{L}_1(Q)$ formula

$$\mathcal{A}(E(Q,u,u,\cdot,y),Q,x,y), \tag{2}$$

which results from $\mathcal{A}(P,Q,x,y)$ by replacing each subformula of the form $P(t)$ by $E(Q,u,u,t,y)$. Due to the special form (1) of our accessibility operator form $\mathcal{A}(P,Q,x,y)$, one readily sees that (2) is trivially equivalent to a $\Pi^0_n$ formula of $\mathcal{L}_1(Q)$. Hence, there is a natural number $k$ effectively depending on $\mathcal{A}$, so that (2) is equivalent to $E(Q,k,u,x,y)$. But now indeed, by substituting $k$ for $u$, we have established the equivalence of the two formulas

$$E(Q,k,k,x,y) \quad \text{and} \quad \mathcal{A}(E(Q,k,k,\cdot,y),Q,x,y). \tag{3}$$

Towards formalizing our argument in $\mathsf{ACA}^-_\alpha$, choose now the $\mathcal{L}_1(Q)$ formula $D(Q,x,y)$ as $E(Q,k,k,x,y)$. Using this definition, the hierarchy axioms of $\mathsf{ACA}^-_\alpha$ for $D$ readily yield

$$(\forall a \prec \alpha)(\forall x)[H^D_a(x) \leftrightarrow D(H^D_{\prec a},x,a) \leftrightarrow E(H^D_{\prec a},k,k,x,a)]. \tag{4}$$

Moreover, straightforward formalization of our discussion above shows that $\mathsf{ACA}^-_\alpha$ proves

$$(5) \qquad (\forall a \prec \alpha)(\forall x)[E(H^D_{\prec a}, k, k, x, a) \leftrightarrow \mathcal{A}(E(H^D_{\prec a}, k, k, \cdot, a), H^D_{\prec a}, x, a)].$$

By combining (4) and (5) we get that $\mathsf{ACA}^-_\alpha$ derives

$$(6) \qquad (\forall a \prec \alpha)(\forall x)[H^D_a(x) \leftrightarrow \mathcal{A}(H^D_a, H^D_{\prec a}, x, a)].$$

Clearly, (6) reveals that we now get an embedding of $\widehat{\mathsf{ID}}_\alpha(acc)$ into $\mathsf{ACA}^-_\alpha$ by interpreting $P^{\mathcal{A}}$ as $H^D$ and leaving the arithmetic part of $\mathcal{L}_{\mathsf{fix}}$ untouched. This is as desired and ends our proof of the theorem. $\square$

We are now in a position to combine Theorem 1, Theorem 2, and Theorem 3 in order to obtain the following main result of this article.

**Theorem 4** *We have that $\widehat{\mathsf{ID}}^i_\alpha(strict)$, $\widehat{\mathsf{ID}}^i_\alpha(strong)$ and $\widehat{\mathsf{ID}}^i_\alpha(acc)$ are conservative over $\mathsf{ACA}^{-i}_\alpha$ for all negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$.*

It is not difficult to see that the theories $\mathsf{ACA}^{-i}_n$ for $n < \omega$ are directly interpretable in Heyting arithmetic $\mathsf{HA}$; hence, we have conservativity of $\widehat{\mathsf{ID}}^i_{<\omega}(strict)$ over $\mathsf{HA}$ for negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$.

Moreover, observe that the theory $\mathsf{ACA}^{-i}_\omega$ is in fact just a first order version of the second order system $\mathsf{ACA}^i$ based on arithmetic comprehension *with set parameters* and the *full schema of induction* on the natural numbers. The reader will not find it difficult to provide mutual embeddings between $\mathsf{ACA}^{-i}_\omega$ and $\mathsf{ACA}^i$. Summing up, we are in a position to state the following corollary to our main theorem.

**Corollary 5** *We have the following conservation results:*

1. *$\widehat{\mathsf{ID}}^i_{<\omega}(strict)$, $\widehat{\mathsf{ID}}^i_{<\omega}(strong)$ and $\widehat{\mathsf{ID}}^i_{<\omega}(acc)$ are conservative over $\mathsf{HA}$ for all negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$.*

2. *$\widehat{\mathsf{ID}}^i_\omega(strict)$, $\widehat{\mathsf{ID}}^i_\omega(strong)$ and $\widehat{\mathsf{ID}}^i_\omega(acc)$ are conservative over $\mathsf{ACA}^i$ for all negative and $\Pi^0_2$ sentences of $\mathcal{L}_1$.*

Recall that Arai [2] has proved conservativity of $\widehat{\mathsf{ID}}^i_{<\omega}(strong)$ over $\mathsf{HA}$ for *all* sentences of $\mathcal{L}_1$. Hence, it is a natural question to ask whether conservativity of $\widehat{\mathsf{ID}}^i_\alpha(strict)$ over $\mathsf{ACA}^{-i}_\alpha$ indeed also holds for all $\mathcal{L}_1$ sentences.

# 4 Proof-theoretic ordinals of $\mathsf{ACA}_\alpha^-$

Since systems with parameter-free arithmetic comprehension are probably less well-known than the corresponding systems based on arithmetic comprehension *with* parameters, let us briefly state the proof-theoretic ordinal $|\mathsf{ACA}_\alpha^-|$ of $\mathsf{ACA}_\alpha^-$, cf. e.g. [9] for a precise definition of the notion of proof-theoretic ordinal.

**Theorem 6** *Assume that $\alpha < \Gamma_0$ is of the form $\omega^{\alpha_n} + \omega^{\alpha_{n-1}} + \cdots + \omega^{\alpha_1} + m$ for ordinals $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 > 0$ and $m < \omega$, and let $\varepsilon(\alpha)$ denote the least $\varepsilon$ number greater than $\alpha$. Then we have:*

$$|\mathsf{ACA}_\alpha^-| = \varphi\alpha_n(\varphi\alpha_{n-1}(\cdots\varphi\alpha_1\varepsilon(\alpha))\cdots).$$

We omit a rigorous proof of this theorem, since the relevant arguments and techniques are well-established in the literature. The lower bounds are obtained by well-known predicative well-ordering techniques, cf. e.g. [8, 13]. Upper proof-theoretic bounds can be derived using suitable predicative cut-elimination in a semi-formal system of ramified analysis, e.g the system $\mathsf{RA}^\star$ of Schütte [13], and the fact that iterated arithmetic comprehension can be modeled in $\mathsf{RA}^\star$.

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