First steps into metapredicativity in explicit mathematics

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Abstract

The system EMU of explicit mathematics incorporates the *uniform* construction of universes. In this paper we give a proof-theoretic treatment of EMU and show that it corresponds to transfinite hierarchies of fixed points of positive arithmetic operators, where the length of these fixed point hierarchies is bounded by ε_0.

1 Introduction

Metapredicativity is a new general term in proof theory which describes the analysis and study of formal systems whose proof-theoretic strength is beyond the Feferman-Schütte ordinal Γ_0 but which are nevertheless amenable to purely predicative methods. Typical examples of formal systems which are apt for scaling the initial part of metapredicativity are the transfinitely iterated fixed point theories $\widehat{\mathsf{ID}}_\alpha$ whose detailed proof-theoretic analysis is given by Jäger, Kahle, Setzer and Strahm in [18]. In this paper we assume familiarity with [18]. For natural extensions of Friedman's ATR that can be measured against transfinitely iterated fixed point theories the reader is referred to Jäger and Strahm [20].

In the mid seventies, Feferman [3, 4] introduced systems of *explicit mathematics* in order to provide an alternative foundation of constructive mathematics. More precisely, it was the origin of Feferman's program to give a logical account of Bishop-style constructive mathematics. Right from the beginning, systems of explicit mathematics turned out to be of general interest for proof theory, mainly in connection with the proof-theoretic analysis of subsystems of first and second order arithmetic and set theory, cf. e.g. Jäger [15] and Jäger and Pohlers [19]. More recently, systems of explicit mathematics have been used to develop a general logical framework for functional programming and type theory, where it is possible to derive correctness and termination properties of functional programs. Important references in this connection are Feferman [6, 7, 9] and Jäger [17].

Universes are a frequently studied concept in constructive mathematics at least since the work of Martin-Löf, cf. e.g. Martin-Löf [23] or Palmgren [27] for a survey. They can be considered as types of types (or names) which are closed under previously recognized type formation operations, i.e. a universe *reflects* these operations. Hence, universes are closely related to reflection principles in classical and admissible set theory. Universes were first discussed in the framework of explicit mathematics in Feferman [5] in connection with his proof of Hancock's conjecture. In Marzetta [25, 24] they are introduced via a so-called (non-uniform) limit axiom, thus providing a natural framework of explicit mathematics which has exactly the strength of predicative analysis, cf. also Marzetta and Strahm [26] and Kahle [22].

In this paper we discuss the system EMU of explicit mathematics which contains a *uniform* universe construction principle and includes full formula induction on the natural numbers. Our universes are closed under elementary comprehension and join (disjoint union), and there is an operation which uniformly takes a given type (name) and yields a universe containing that name. We show that EMU is proof-theoretically equivalent to the transfinitely iterated fixed point theory $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ with proof-theoretic ordinal $\varphi 1 \varepsilon_0 0$ for φ a ternary Veblen function. Independently and very recently, similar results have been obtained in the context of Frege structures by Kahle [21] and in the framework of Martin-Löf type theory by Rathjen [29].

The plan of the paper is as follows. In Section 2 we give the formal definition of the system EMU. Section 3 is devoted to a wellordering proof for EMU, i.e. we establish $\varphi 1 \varepsilon_0 0 \leq |\mathsf{EMU}|$. In Section 4 we describe a proof-theoretic reduction of EMU to $\widehat{\mathsf{ID}}_{<\varepsilon_0}$. We conclude with some remarks concerning subsystems of EMU containing restricted induction principles on the natural numbers.

2 The theory EMU

In this section we introduce the theory EMU of explicit mathematics with a natural principle for the uniform construction of universes. We present EMU in the framework of types and names of Jäger [16] together with the finite axiomatization of elementary comprehension of Feferman and Jäger [10].

Let us first introduce the language $\mathcal{L}$ of EMU. It is a two-sorted language with countable lists of *individual variables* $a, b, c, f, g, h, x, y, z, \dots$ and *type variables* $A, B, C, X, Y, Z, \dots$ (both possibly with subscripts). $\mathcal{L}$ includes the following *individual constants*: k, s (combinators), $\mathsf{p}, \mathsf{p_0}, \mathsf{p_1}$ (pairing and projection), $\mathsf{0}$ (zero), $\mathsf{s_N}$ (successor), $\mathsf{p_N}$ (predecessor), $\mathsf{d_N}$ (definition by numerical cases), nat (natural numbers), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image), j (join) and u (universe construction). There is only one binary function symbol $\cdot$ for (partial) application of individuals to individuals.

The relation symbols of $\mathcal{L}$ include equality $=$ for both individuals and types, the unary predicate symbols $\downarrow$ (defined) and N (natural numbers) on individual terms, U (universes) on types, and the binary relation symbols $\in$ (membership) and $\Re$ (naming, representation relation) between individuals and types.

The *individual terms* $(r, s, t, \dots)$ of $\mathcal{L}$ are built up from individual variables and individual constants by means of $\cdot$, with the usual conventions for application in combinatory logic or λ calculus. We write (s,t) for $\mathsf{p}st$, s' for $\mathsf{s_N}s$, $\mathsf{1}$ instead of $\mathsf{0}'$ and so on. The *type terms* are just the type variables.

The *atoms* of $\mathcal{L}$ have one of the following forms: $s = t$, $A = B$, $s\downarrow$, $\mathsf{N}(s)$, $\mathsf{U}(A)$, $s \in A$, or $\Re(s, A)$. The *formulas* of $\mathcal{L}$ $(E, F, G, \dots)$ are generated from the atoms by closing against the usual connectives as well as quantification in both sorts. The following table contains a useful list of abbreviations:

$$
\begin{array}{rcl}
s \simeq t & := & s\downarrow \vee\, t\downarrow \rightarrow s = t, \\
x \in \mathsf{N} & := & \mathsf{N}(x), \\
(\exists x \in A)F & := & (\exists x)(x \in A \wedge F), \\
(\forall x \in A)F & := & (\forall x)(x \in A \rightarrow F), \\
A \subset B & := & (\forall x \in A)(x \in B), \\
A \in B & := & (\exists x)(\Re(x, A) \wedge x \in B), \\
s \in t & := & (\exists X)(\Re(t, X) \wedge s \in X), \\
\Re(s) & := & (\exists X)\Re(s, X), \\
\mathsf{U}(s) & := & (\exists X)(\Re(s, X) \wedge \mathsf{U}(X)).
\end{array}
$$

The logic of EMU is the classical logic of partial terms of Beeson [1] for the individuals, and classical logic with equality for the types[1]. The non-logical axioms of EMU are divided into the following groups.

I. Applicative axioms

Partial combinatory algebra

(1) $\mathsf{k}xy = x$,

(2) $\mathsf{s}xy\downarrow \,\wedge\, \mathsf{s}xyz \simeq xz(yz)$.

Pairing and projection

(3) $\mathsf{p_0}(x,y) = x \,\wedge\, \mathsf{p_1}(x,y) = y$.

Natural numbers

(4) $\mathsf{0} \in \mathsf{N} \,\wedge\, (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

[1]All the results of this paper also hold in the presence of intuitionistic logic.

(5) $(\forall x \in \mathsf{N})(x' \neq \mathsf{0} \wedge \mathsf{p_N}(x') = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq \mathsf{0} \;\rightarrow\; \mathsf{p_N}x \in \mathsf{N} \wedge (\mathsf{p_N}x)' = x)$.

Definition by numerical cases

(7) $a \in \mathsf{N} \wedge b \in \mathsf{N} \;\wedge\; a = b \rightarrow \mathsf{d_N}xyab = x$,

(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \;\wedge\; a \neq b \rightarrow \mathsf{d_N}xyab = y$.

As usual one derives from the axioms of a partial combinatory algebra a theorem about λ abstraction as well as a form of the recursion theorem, cf. e.g. Beeson [1] or Feferman [3] for a proof of these standard facts. The axioms for types in general are given in the next block.

II. General axioms for types

Extensionality

(9) $(\forall x)(x \in A \leftrightarrow x \in B) \rightarrow A = B$.

Ontological axioms

(10) $\Re(a, B) \wedge \Re(a, C) \rightarrow B = C$,

(11) $(\exists x)\Re(x, A)$.

Axiom (10) tells us that there are no homonyms, i.e., different types have different names (representations), whereas axiom (11) states that every type has a name.

Natural numbers

(12) $(\exists X)[\Re(\mathsf{nat}, X) \;\wedge\; (\forall x)(x \in X \leftrightarrow \mathsf{N}(x))]$.

Identity

(13) $(\exists X)[\Re(\mathsf{id}, X) \;\wedge\; (\forall x)(x \in X \leftrightarrow (\exists y)(x = (y, y)))]$.

Complements

(14) $\Re(a, A) \;\rightarrow\; (\exists X)[\Re(\mathsf{co}\, a, X) \;\wedge\; (\forall x)(x \in X \leftrightarrow x \notin A)]$.

Intersections

(15) $\Re(a, A) \wedge \Re(b, B) \;\rightarrow$
$$(\exists X)[\Re(\mathsf{int}(a, b), X) \;\wedge\; (\forall x)(x \in X \leftrightarrow x \in A \wedge x \in B)].$$

Domains

(16) $\Re(a, A) \rightarrow (\exists X)[\Re(\mathsf{dom}\, a, X) \wedge (\forall x)(x \in X \leftrightarrow (\exists y)((x, y) \in A))]$.

Inverse images

(17) $\Re(a, A) \rightarrow (\exists X)[\Re(\mathsf{inv}(f, a), X) \wedge (\forall x)(x \in X \leftrightarrow fx \in A)]$.

An $\mathcal{L}$ formula is called *elementary*, if it contains no bound type variables nor the naming relation $\Re$. Axioms (12)-(17) provide a finite axiomatization of the scheme of uniform elementary comprehension, i.e. the usual scheme of elementary comprehension is derivable from (12)-(17), cf. Feferman and Jäger [10]. The final general type axiom is the principle of join. For its formulation, let us write $A = \Sigma(B, f)$ for the statement

$$(\forall x)(x \in A \leftrightarrow x = (\mathsf{p_0}x, \mathsf{p_1}x) \wedge \mathsf{p_0}x \in B \wedge \mathsf{p_1}x \in f(\mathsf{p_0}x)).$$

Join (disjoint sum)

(18) $\Re(a, A) \wedge (\forall x \in A)(\exists Y)\Re(fx, Y) \rightarrow (\exists Z)(\Re(\mathsf{j}(a, f), Z) \wedge Z = \Sigma(A, f))$.

Let us now turn to the axioms about universes, which are divided into three subsections.

III. Axioms for universes

Ontological axioms

(19) $\mathsf{U}(A) \wedge x \in A \rightarrow \Re(x)$,

(20) $\mathsf{U}(A) \wedge \mathsf{U}(B) \wedge A \in B \rightarrow A \subset B$.

The crucial axiom (19) claims that universes contain only names, and axiom (20) states a kind of transitivity condition.[2] Universes obey the following natural closure conditions.

Closure conditions

(21) $\mathsf{U}(A) \rightarrow \mathsf{nat} \in A$,

(22) $\mathsf{U}(A) \rightarrow \mathsf{id} \in A$,

(23) $\mathsf{U}(A) \wedge b \in A \rightarrow \mathsf{co}\, b \in A$,

(24) $\mathsf{U}(A) \wedge b \in A \wedge c \in A \rightarrow \mathsf{int}(b, c) \in A$,

[2] In [25, 24, 26, 22] a further ontological axiom for universes is present; it claims that $\in$ is total on universes. Totality is not an official axiom of EMU, but it can be added without raising its strength.

(25) $\mathsf{U}(A) \wedge b \in A \;\rightarrow\; \mathsf{dom}\, b \in A$,

(26) $\mathsf{U}(A) \wedge b \in A \;\rightarrow\; \mathsf{inv}(f, b) \in A$,

(27) $\mathsf{U}(A) \wedge b \in A \wedge (\forall x \in b)(fx \in A) \;\rightarrow\; \mathsf{j}(b, f) \in A$.

So far we have no axioms which guarantee the existence of universes at all. Therefore, we add the following principle of uniform universe construction, which uniformly for a given name yields a universe which contains that name.

Universe construction

(28) $\Re(a) \;\rightarrow\; \mathsf{U}(\mathsf{u}a) \wedge a \in \mathsf{u}a$.

In EMU we assume the induction schema, i.e. complete induction on the natural numbers is available for arbitrary statements of $\mathcal{L}$.

IV. Formula induction on N

For each $\mathcal{L}$ formula $F(x)$:

(29) $F(\mathsf{0}) \wedge (\forall x \in \mathsf{N})(F(x) \rightarrow F(x')) \;\rightarrow\; (\forall x \in \mathsf{N})F(x)$.

This finishes the description of the systems EMU. In the next section we turn to the wellordering proof for EMU.

3 A wellordering proof for EMU

In this section we sketch the main lines of a wellordering proof for EMU. More precisely, we show that EMU proves transfinite induction for each initial segment of the ordinal $\varphi 1 \varepsilon_0 0$. This is also the proof-theoretic ordinal of the theory $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ analyzed in Jäger, Kahle, Setzer and Strahm [18]; in the following we assume that the reader is familiar with the wellordering proofs for the theories $\widehat{\mathsf{ID}}_\alpha$ as they are presented in detail in Section 5 of [18].

In the sequel we presuppose the same ordinal-theoretic facts as given in Section 2 of [18]. Namely, we let Φ_0 denote the least ordinal greater than 0 which is closed under all n-ary φ functions, and we assume that a standard notation system of order type Φ_0 is given in a straightforward manner. We write $\prec$ for the corresponding primitive recursive wellordering with least element 0. When working in EMU in this section, we let $a, b, c, \dots$ range over the field of $\prec$ and ℓ denote limit notations. There exist primitive recursive functions acting on the codes of this notation system which correspond to the usual operations on ordinals. In the sequel it is often convenient in order to simplify notation to use ordinals and ordinal operations instead of their codes and primitive recursive analogues. Then (for example) ω and $\omega + \omega$ stand for the natural numbers whose order type with respect to $\prec$ are ω and $\omega + \omega$. Finally, let us put as usual:

$$\begin{array}{rcl} \mathit{Prog}(F) & := & (\forall a)[(\forall b \prec a)F(b) \rightarrow F(a)], \\ \mathit{TI}(F,a) & := & \mathit{Prog}(F) \rightarrow (\forall b \prec a)F(b). \end{array}$$

If we want to stress the relevant induction variable of a formula F, we sometimes write $\mathit{Prog}(\lambda a.F(a))$ instead of $\mathit{Prog}(F)$. If X is a type and x a name of a type, then $\mathit{Prog}(X)$ and $\mathit{Prog}(x)$ have their obvious meaning; $\mathit{TI}(X,a)$ and $\mathit{TI}(x,a)$ read analogously.

In the sequel it is our aim to derive $(\forall X)\,\mathit{TI}(X,\alpha)$ in EMU for each ordinal α less than $\varphi 1 \varepsilon_0 0$. A crucial step towards that aim is the following: given a type X with a name x, we can build a transfinite hierarchy of universes above a universe containing x along $\prec$, and indeed such a hierarchy can be shown to be well-defined up to each fixed α less than ε_0. The hierarchy h (depending on x) is given by the recursion theorem in order to satisfy the following recursion equations:

$$\begin{array}{rcl} \mathsf{h}x\mathsf{0} & \simeq & \mathsf{u}x, \\ \mathsf{h}x(a+1) & \simeq & \mathsf{u}(\mathsf{h}xa), \\ \mathsf{h}x\ell & \simeq & \mathsf{u}(\mathsf{j}(\{a : a \prec \ell\}, \mathsf{h}x)). \end{array}$$

In other words, the hierarchy starts with a universe containing x, at successor stages one puts a universe on top of the hierarchy defined so far, and at limit stages a universe above the disjoint union of the previously defined hierarchy is taken.

Lemma 1 *For each ordinal α less than ε_0, the following are theorems of* EMU*:*

1. $(\forall x)[\Re(x) \;\rightarrow\; (\forall a \prec \alpha)\mathsf{U}(\mathsf{h}xa)]$,
2. $(\forall x)[\Re(x) \;\rightarrow\; (\forall a \prec \alpha)(\forall b \prec a)(\mathsf{h}xb \in \mathsf{h}xa)]$.

Proof. For the proof of this lemma it is crucial to observe that we have transfinite induction up to each α less than ε_0 available in EMU with respect to *arbitrary* statements of $\mathcal{L}$. This is due to the fact that EMU includes the scheme of formula induction on the natural numbers. Hence, both claims can be proved by transfinite induction up to an $\alpha \prec \varepsilon_0$. For the first assertion this is immediate. For the second one makes use of the transitivity axiom (20). For example, assume that ℓ is a limit notation, and we want to establish that $\mathsf{h}xb \in \mathsf{h}x\ell$ for a specific $b \prec \ell$. Since ℓ is limit one also has $b+1 \prec \ell$, and of course it is $\mathsf{h}xb \in \mathsf{h}x(b+1)$. On the other hand, one easily sees that there is a name of the universe denoted by $\mathsf{h}x(b+1)$ which belongs to $\mathsf{h}x\ell$, since we have by definition $\mathsf{j}(\{c : c \prec \ell\}, \mathsf{h}x) \in \mathsf{h}x\ell$. But then $\mathsf{h}xb \in \mathsf{h}x\ell$ is immediate by transitivity. $\square$

Crucial for carrying out the wellordering proof in EMU is the very natural notion $I^c_x(a)$ of *transfinite induction up to a for all types (respectively names) belonging to a universe* $\mathsf{h}xb$ *for* $b \prec c$, which is given as follows:

$$I^c_x(a) \;:=\; (\forall b \prec c)(\forall u \in \mathsf{h}xb)\, TI(u, a).$$

The next lemma tells us that $I^\ell_x(a)$ can be represented by a type in $\mathsf{h}x\ell$.

Lemma 2 *For each ordinal α less than ε_0, the following is a theorem of* EMU*:*

$$(\forall x, \ell)[\Re(x) \wedge \ell \preceq \alpha \;\rightarrow\; (\exists y \in \mathsf{h}x\ell)(\forall a)(a \in y \leftrightarrow I^\ell_x(a))].$$

Proof. We sketch the proof of this claim by working informally in EMU. Assuming $\Re(x)$ and $\ell \preceq \alpha \prec \varepsilon_0$, we know by the definition of $\mathsf{h}x\ell$ that $\mathsf{j}(\{b : b \prec \ell\}, \mathsf{h}x) \in \mathsf{h}x\ell$. By closure of $\mathsf{h}x\ell$ under join this readily entails that also (a name of) the type

$$\{(b, u, v) : b \prec \ell \wedge u \in \mathsf{h}xb \wedge v \in u\}$$

belongs to $\mathsf{h}x\ell$. Therefore, by closure of $\mathsf{h}x\ell$ under elementary comprehension, there exists a y in $\mathsf{h}x\ell$ which satisfies the condition claimed by the lemma. $\square$

The next lemma is used for the base case in Main Lemma I below. We do not give its proof here, since the relevant arguments can easily be extracted and adapted to the present context from Feferman [5, 8] or Schütte [30].

Lemma 3 *For each ordinal α less than ε_0 the following is a theorem of* EMU*:*

$$(\forall x, \ell, a)[\Re(x) \wedge \ell \preceq \alpha \wedge I^\ell_x(a) \;\rightarrow\; I^\ell_x(\varphi a \mathsf{0})].$$

The following corollary is an immediate consequence:

Corollary 4 *For each ordinal α less than ε_0 the following is a theorem of* EMU*:*

$$(\forall x, \ell)[\Re(x) \wedge \ell \preceq \alpha \;\rightarrow\; Prog(\lambda a. I^\ell_x(\Gamma_a))].$$

Main Lemma I below makes crucial use of the binary relation $\uparrow$, which reads as follows:

$$a \uparrow b \;:=\; (\exists c, \ell)(b = c + a \cdot \ell).$$

We are now in a position to state Main Lemma I. It corresponds exactly to Main Lemma I in Jäger, Kahle, Setzer and Strahm [18], formulated in the framework of explicit mathematics with universes. Given the preparations outlined in this section, chiefly the last corollary and Lemma 2, its proof is very much the same as the proof given in [18] and, therefore, we omit it here.

Lemma 5 (Main Lemma I) *Let $Main_\alpha(a)$ be defined as follows:*

$$Main_\alpha(a) \;:=\; (\forall x, b, c)[\Re(x) \land c \preceq \alpha \;\land\; \omega^{1+a} \uparrow c \;\land\; I_x^c(b) \;\rightarrow\; I_x^c(\varphi 1ab)].$$

Then EMU *proves $Prog(\lambda a.Main_\alpha(a))$ for each ordinal α less than ε_0.*

Using Main Lemma I, we are now in a position to derive the main theorem of this section.

Theorem 6 EMU *proves $(\forall X)\, TI(X, \alpha)$ for each ordinal α less than $\varphi 1 \varepsilon_0 0$.*

Proof. It is enough to show that EMU proves $(\forall X)\, TI(X, \varphi 1 \alpha \mathsf{0})$ for each $\alpha < \varepsilon_0$. For that purpose, fix an arbitrary $\alpha < \varepsilon_0$. Then we also have $\omega^{1+\alpha} \cdot \omega < \varepsilon_0$ and, hence, we have $Prog(\lambda a.Main_{\omega^{1+\alpha}\cdot\omega}(a))$ as a theorem of EMU by Main Lemma I. Since transfinite induction below ε_0 is available in EMU with respect to arbitrary statements of $\mathcal{L}$, we obtain that EMU proves $Main_{\omega^{1+\alpha}\cdot\omega}(\alpha)$, i.e. the statement

$$(\forall x, b, c)[\Re(x) \land c \preceq \omega^{1+\alpha} \cdot \omega \;\land\; \omega^{1+\alpha} \uparrow c \;\land\; I_x^c(b) \;\rightarrow\; I_x^c(\varphi 1 \alpha b)].$$

By choosing c as $\omega^{1+\alpha} \cdot \omega$ and b as $\mathsf{0}$ in this assertion, one derives the following as a theorem of EMU:

$$(\forall x)[\Re(x) \;\rightarrow\; I_x^{\omega^{1+\alpha}\cdot\omega}(\varphi 1 \alpha \mathsf{0})].$$

But now we can immediately derive $\mathsf{EMU} \vdash (\forall X)\, TI(X, \varphi 1 \alpha \mathsf{0})$ as claimed. □

We finish this section by mentioning that it would be possible to obtain $\varphi 1 \varepsilon_0 0$ as a lower bound for EMU even without assuming the transitivity axiom (20). However, the wellordering proof would require more "coding". Since transitivity of universes is a natural condition which holds in the standard structures of EMU discussed in the next section, we included (20) in the axioms of EMU.

4 Reduction of EMU to $\widehat{\mathsf{ID}}_{<\varepsilon_0}$

In this section we sketch a proof-theoretic reduction of EMU to the transfinitely iterated fixed point theory $\widehat{\mathsf{ID}}_{<\varepsilon_0}$; the latter theory is shown to possess proof-theoretic ordinal $\varphi 1 \varepsilon_0 0$ in [18] and, hence, together with the results of the previous section, we obtain that $\varphi 1 \varepsilon_0 0$ is also the proof-theoretic ordinal of EMU. Our reduction proceeds in two steps: first, we sketch a Tait-style reformulation of EMU which includes a form of the ω rule and, therefore, allows us to establish a partial cut elimination theorem for EMU, yielding quasi-normal derivations of length bounded by ε_0. In a second step we provide partial models for EMU which will subsequently be used in order to prove an asymmetric interpretation theorem for quasinormal derivations. It is argued that the whole procedure can be formalized in $\widehat{\mathsf{ID}}_{<\varepsilon_0}$; in particular, the partial models needed for an interpretation of EMU are available in $\widehat{\mathsf{ID}}_{<\varepsilon_0}$.

Let us start with an infinitary Tait-style reformulation of EMU. Since Tait formulations of systems of explicit mathematics are rather familiar from the literature, we confine ourselves to a sketchy description of the Tait calculus T_∞ of EMU. For more detailed expositions the reader is referred to Glaß and Strahm [13], or Marzetta and Strahm [26].

As usual, the language appropriate for setting up a Tait-style calculus for EMU presupposes complementary relation symbols for each relation of $\mathcal{L}$. Formulas are then generated from the positive and negative literals by closing against conjunction and disjunction as well as existential and universal quantification in both sorts. Negation is defined as usual by applying the law of double negation and De Morgan's laws. In the sequel we identify formulas of $\mathcal{L}$ and their translations in the Tait-style language corresponding to $\mathcal{L}$. Important classes of formulas are the so-called Σ^+ and Π^- formulas, cf. [13, 26]. A formula in the Tait-style language of $\mathcal{L}$ is called Σ^+, if it contains no negations of $\Re$ as well as no universal type quantifiers. Negations of Σ^+ formulas are called Π^- formulas. The rank $rn(F)$ of a formula F is defined in such a way that it is 0 if F is a Σ^+ or Π^- formula and it is computed as usual for more complex formulas, cf. [13, 26]. Axioms and rules of inference of T_∞ are formulated for finite sets of formulas, which have to be interpreted disjunctively. The capital greek letters $\Gamma, \Lambda, \dots$ denote finite sets of formulas, and we write, e.g., Γ, Λ, F, G for the union of Γ, Λ and $\{F, G\}$.

The logical axioms and rules of inference of T_∞ are now as usual, cf. [13] for a detailed exposition. In particular, T_∞ includes the cut rule. As far as the non-logical axioms and rules are concerned, we notice that all axioms of EMU except axioms (18) and (29) can easily been written in a Tait style manner so that the relevant main formulas are always either in Σ^+ or in Π^-. For example, the universe construction axiom just reads as

$$\Gamma,\ \neg\Re(s),\ \mathsf{U}(\mathsf{u}s) \wedge s \in \mathsf{u}s.$$

Axiom (18) is replaced by the following two rules of inference, cf. [26].

$$\frac{\Gamma,\ t{\downarrow} \wedge \Re(s,A) \wedge (\forall x \in A)(\exists X)\Re(tx,X)}{\Gamma,\ (\exists Z)(\Re(\mathsf{j}(s,t),Z) \wedge Z{\subset}\Sigma(A,t))}\ (J_1)$$

$$\frac{\Gamma, t{\downarrow} \wedge \Re(s,A) \wedge (\forall x \in A)(\exists X)\Re(tx,X)}{\Gamma,\ (\exists Z)(\Re(\mathsf{j}(s,t),Z) \wedge Z{\supset}\Sigma(A,t))}\ (J_2)$$

where $Z{\subset}\Sigma(A,t)$ abbreviates

$$(\forall z)(z \in Z \ \rightarrow\ z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in A \wedge (\exists X)(\Re(t(\mathsf{p}_0 z), X) \wedge \mathsf{p}_1 z \in X)),$$

and $Z \supset \Sigma(A, t)$ is spelled out as

$$(\forall z)(z = (\mathsf{p_0} z, \mathsf{p_1} z) \wedge \mathsf{p_0} z \in A \wedge (\forall X)(\Re(t(\mathsf{p_0} z), X) \rightarrow \mathsf{p_1} z \in X) \;\rightarrow\; z \in Z).$$

Finally, we replace the schema of formula induction (29) by the following version of the ω rule, cf. [13]. Here $\overline{n}$ denotes the nth numeral of $\mathcal{L}$.

$$\frac{\Gamma,\, \overline{n} \neq t \quad \text{for all } n < \omega}{\Gamma,\, \neg\mathsf{N}(t)}$$

$\mathsf{T}_\infty \,|\!\frac{\alpha}{k}\, \Gamma$ expresses that there is a derivation of the finite set Γ of $\mathcal{L}$ formulas such that α is an upper bound for the proof length and k is a strict upper bound for the ranks (in the sense of rn) of cut formulas occurring in the derivation.

We observe that EMU can be embedded into T_∞ in a straightforward manner; as usual, complete induction on the natural numbers is derivable by making use of the ω rule and at the price of infinite derivation lengths, cf. e.g. [13] for details.

Lemma 7 (Embedding of EMU into T_∞) *Assume that F is an $\mathcal{L}$ formula which is provable in EMU. Then there exist $\alpha < \omega + \omega$ and $k < \omega$ so that $\mathsf{T}_\infty \,|\!\frac{\alpha}{k}\, F$.*

Further, we observe that the axioms and rules of inference of T_∞ are tailored so that all main formulas are either Σ^+ or Π^-. Hence, usual cut elimination techniques from predicative proof theory (cf. e.g. [28, 30]) apply in order to show that all cuts of rank greater than 0 can be eliminated. The derivation lengths of the so-obtained quasinormal derivations can be measured as usual by the terms $\omega_k(\alpha)$, where we set $\omega_0(\alpha) = \alpha$ and $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$. We summarize our observations in the following partial cut elimination lemma.

Lemma 8 (Partial cut elimination for T_∞) *Assume that Γ is a finite set of $\mathcal{L}$ formulas so that $\mathsf{T}_\infty \,|\!\frac{\alpha}{1+k}\, \Gamma$. Then we have that $\mathsf{T}_\infty \,|\!\frac{\omega_k(\alpha)}{1}\, \Gamma$.*

A combination of the previous two lemmas yields the following corollary.

Corollary 9 *Assume that F is an $\mathcal{L}$ formula which is provable in EMU. Then there exists an $\alpha < \varepsilon_0$ so that $\mathsf{T}_\infty \,|\!\frac{\alpha}{1}\, F$.*

The second main step of our reduction of EMU to $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ consists in setting up partial models $\mathfrak{M}(\alpha)$ for EMU, which will be used in order to prove an asymmetric interpretation theorem for quasinormal T_∞ derivations.

First, let us consider a fixed interpretation of the applicative (type-free) fragment of $\mathcal{L}$. We choose as universe for our operations the set of natural numbers $\mathbb{N}$ and interpret N by $\mathbb{N}$; term application $\cdot$ is interpreted as partial recursive function application, i.e. $a \cdot b$ just means $\{a\}(b)$. By ordinary recursion theory, it is now straightforward to find interpretations for $\mathsf{k}, \mathsf{s}, \mathsf{p}, \mathsf{p_0}, \mathsf{p_1}, \mathsf{0}, \mathsf{s_N}, \mathsf{p_N}, \mathsf{d_N}$ so that the applicative axioms of EMU are satisfied. In order to get an interpretation of the remaining individual constants of $\mathcal{L}$ we proceed as follows. Choose pairwise different natural numbers $\widehat{\mathsf{nat}}, \widehat{\mathsf{id}}, \widehat{\mathsf{co}}, \widehat{\mathsf{int}}, \widehat{\mathsf{dom}}, \widehat{\mathsf{inv}}, \widehat{\mathsf{j}}, \widehat{\mathsf{u}}$; interpret nat and id by $\langle \widehat{\mathsf{nat}} \rangle$ and $\langle \widehat{\mathsf{id}} \rangle$, respectively; interpret co by a natural number co so that $\{co\}(a) = \langle \widehat{\mathsf{co}}, a \rangle$; for int choose a natural number int so that $\{int\}(\langle a, b \rangle) = \langle \widehat{\mathsf{int}}, a, b \rangle$; the constants $\mathsf{dom}, \mathsf{inv}, \mathsf{j}, \mathsf{u}$ are interpreted analogously. Here we have used $\langle \ldots \rangle$ to denote standard sequence coding.

In a next step we now want to describe partial models $\mathfrak{M}(\alpha)$, $\mathfrak{N}(\alpha), \ldots$ of EMU. These are defined in such a way that they easily fit into the framework of iterated positive inductive definitions. Basically, one defines *codes* for types together with an *extension* and a *co-extension* for each such code. Essential use is made of fixed points of a positive arithmetic operator $\Phi_{X,\alpha}$ from the power set of $\mathbb{N}$ to the power set of $\mathbb{N}$, depending on a parameter set $X \subset \mathbb{N}$ and an ordinal α. We give the formal specification of $\Phi_{X,\alpha}$ first and afterwards comment on its informal meaning. For that purpose, fix naturals $r, \varepsilon, \bar{\varepsilon}$ which are different from all interpretations so far. Further, fix a parameter set $X \subset \mathbb{N}$ and an ordinal α. For $Y \subset \mathbb{N}$ we put $a \in \Phi_{X,\alpha}(Y)$, if there exist naturals b, c, d, f so that one of the following clauses (1)-(28) applies:

1. $a = \langle r, b \rangle \wedge a \in X$,
2. $a = \langle \varepsilon, b, c \rangle \wedge a \in X \wedge \langle r, c \rangle \in X$,
3. $a = \langle \bar{\varepsilon}, b, c \rangle \wedge a \in X \wedge \langle r, c \rangle \in X$,
4. $\alpha \in Suc$[3] $\wedge\, a = \langle r, \langle \widehat{\mathsf{u}}, b \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in X$,
5. $\alpha \in Suc \wedge a = \langle \varepsilon, b, \langle \widehat{\mathsf{u}}, c \rangle \rangle \wedge \langle r, \langle \widehat{\mathsf{u}}, c \rangle \rangle \notin X \wedge \langle r, c \rangle \in X \wedge \langle r, b \rangle \in X \wedge$ $(\forall y)[\langle \varepsilon, y, b \rangle \in X \leftrightarrow \langle \bar{\varepsilon}, y, b \rangle \notin X]$,
6. $\alpha \in Suc \wedge a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{u}}, c \rangle \rangle \wedge \langle r, \langle \widehat{\mathsf{u}}, c \rangle \rangle \notin X \wedge \langle r, c \rangle \in X \wedge \langle r, b \rangle \notin X$,
7. $\alpha \in Suc \wedge a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{u}}, c \rangle \rangle \wedge \langle r, \langle \widehat{\mathsf{u}}, c \rangle \rangle \notin X \wedge \langle r, c \rangle \in X \wedge \langle r, b \rangle \in X \wedge$ $\neg(\forall y)[\langle \varepsilon, y, b \rangle \in X \leftrightarrow \langle \bar{\varepsilon}, y, b \rangle \notin X]$,
8. $\alpha = 0 \wedge a = \langle r, \langle \widehat{\mathsf{nat}} \rangle \rangle$,
9. $\alpha = 0 \wedge a = \langle \varepsilon, b, \langle \widehat{\mathsf{nat}} \rangle \rangle$,
10. $\alpha = 0 \wedge a = \langle r, \langle \widehat{\mathsf{id}} \rangle \rangle$,

[3] *Suc* denotes the class of successor ordinals.

11. $\alpha = 0 \wedge a = \langle \varepsilon, b, \langle \widehat{\mathsf{id}} \rangle \rangle \wedge (\exists x)(b = \langle x, x \rangle)$,

12. $\alpha = 0 \wedge a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{id}} \rangle \rangle \wedge (\forall x)(b \neq \langle x, x \rangle)$,

13. $a = \langle r, \langle \widehat{\mathsf{co}}, b \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in Y$,

14. $a = \langle \varepsilon, b, \langle \widehat{\mathsf{co}}, c \rangle \rangle \wedge \langle \widehat{\mathsf{co}}, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle \bar{\varepsilon}, b, c \rangle \in Y$,

15. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{co}}, c \rangle \rangle \wedge \langle \widehat{\mathsf{co}}, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle \varepsilon, b, c \rangle \in Y$,

16. $a = \langle r, \langle \widehat{\mathsf{int}}, b, c \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in Y \wedge \langle r, c \rangle \in Y$,

17. $a = \langle \varepsilon, b, \langle \widehat{\mathsf{int}}, c, d \rangle \rangle \wedge \langle \widehat{\mathsf{int}}, c, d \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle r, d \rangle \in Y \wedge$
$\langle \varepsilon, b, c \rangle \in Y \wedge \langle \varepsilon, b, d \rangle \in Y$,

18. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{int}}, c, d \rangle \rangle \wedge \langle \widehat{\mathsf{int}}, c, d \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle r, d \rangle \in Y \wedge$
$[\langle \bar{\varepsilon}, b, c \rangle \in Y \vee \langle \bar{\varepsilon}, b, d \rangle \in Y]$,

19. $a = \langle r, \langle \widehat{\mathsf{dom}}, b \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in Y$,

20. $a = \langle \varepsilon, b, \langle \widehat{\mathsf{dom}}, c \rangle \rangle \wedge \langle \widehat{\mathsf{dom}}, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge (\exists x)(\langle \varepsilon, \langle b, x \rangle, c \rangle \in Y)$,

21. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{dom}}, c \rangle \rangle \wedge \langle \widehat{\mathsf{dom}}, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge (\forall x)(\langle \bar{\varepsilon}, \langle b, x \rangle, c \rangle \in Y)$,

22. $a = \langle r, \langle \widehat{\mathsf{inv}}, f, b \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in Y$,

23. $a = \langle \varepsilon, b, \langle \widehat{\mathsf{inv}}, f, c \rangle \rangle \wedge \langle \widehat{\mathsf{inv}}, f, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle \varepsilon, \{f\}(b), c \rangle \in Y$,

24. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{inv}}, f, c \rangle \rangle \wedge \langle \widehat{\mathsf{inv}}, f, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \{f\}(b)\uparrow$,

25. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{inv}}, f, c \rangle \rangle \wedge \langle \widehat{\mathsf{inv}}, f, c \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge \langle \bar{\varepsilon}, \{f\}(b), c \rangle \in Y$,

26. $a = \langle r, \langle \widehat{\mathsf{j}}, b, f \rangle \rangle \wedge a \notin X \wedge \langle r, b \rangle \in Y \wedge$
$(\forall x)[\langle \bar{\varepsilon}, x, b \rangle \notin Y \rightarrow \langle r, \{f\}(x) \rangle \in Y]$,

27. $a = \langle \varepsilon, b, \langle \widehat{\mathsf{j}}, c, f \rangle \rangle \wedge \langle \widehat{\mathsf{j}}, c, f \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge$
$(\forall x)[\langle \bar{\varepsilon}, x, c \rangle \notin Y \rightarrow \langle r, \{f\}(x) \rangle \in Y] \wedge$
$b = \langle (b)_0, (b)_1 \rangle \wedge \langle \varepsilon, (b)_0, c \rangle \in Y \wedge \langle \varepsilon, (b)_1, \{f\}((b)_0) \rangle \in Y$,

28. $a = \langle \bar{\varepsilon}, b, \langle \widehat{\mathsf{j}}, c, f \rangle \rangle \wedge \langle \widehat{\mathsf{j}}, c, f \rangle \notin X \wedge \langle r, c \rangle \in Y \wedge$
$(\forall x)[\langle \bar{\varepsilon}, x, c \rangle \notin Y \rightarrow \langle r, \{f\}(x) \rangle \in Y] \wedge$
$[b \neq \langle (b)_0, (b)_1 \rangle \vee \langle \bar{\varepsilon}, (b)_0, c \rangle \in Y \vee \langle \bar{\varepsilon}, (b)_1, \{f\}((b)_0) \rangle \in Y]$.

Natural numbers belonging to $\Phi_{X,\alpha}(Y)$ have one of the three forms $\langle r, a\rangle$, $\langle \varepsilon, b, a\rangle$ or $\langle \bar{\varepsilon}, b, a\rangle$ with the associated informal meaning, "a is a representation or name for a type", "b belongs to the type coded by a", and "b does not belong to the type coded by a", respectively. Clauses (1)-(3) inherit all type codes, ε relations and $\bar{\varepsilon}$ relations in X to $\Phi_{X,\alpha}(Y)$. In the case of α being a successor ordinal, clauses (4)-(7) associate to each type code in X a new type (universe), which contains exactly those type codes in X on which ε and $\bar{\varepsilon}$ are complementary. Clauses (8)-(28) state closure conditions for types in the sense of axioms (12)-(18) of EMU; in each case ε and $\bar{\varepsilon}$ are defined separately.

A sequence of sets of natural numbers $(X_\beta)_{\beta\leq\alpha}$ is called a Φ *sequence*, if it satisfies the following conditions for each $\beta \leq \alpha$:

(1) if $\beta = 0$, then X_β is a fixed point of $\Phi_{\emptyset,0}$;

(2) if β is a successor ordinal $\gamma + 1$, then X_β is a fixed point of $\Phi_{X_\gamma,\beta}$;

(3) if β is a limit ordinal, then X_β is a fixed point of $\Phi_{\bigcup_{\gamma<\beta} X_\gamma, \beta}$.

A Φ sequence $(X_\beta)_{\beta\leq\alpha}$ determines an interpretation $\mathfrak{M}(\alpha)$ of $\mathcal{L}$ as follows:

(i) the applicative fragment of $\mathcal{L}$ is interpreted as described above.

(ii) the types in $\mathfrak{M}(\alpha)$ range over the set T_α of natural numbers m so that $\langle r, m\rangle$ belongs to X_α and $\varepsilon, \bar{\varepsilon}$ are complementary with respect to m, i.e.

$$(\forall x)(\langle \varepsilon, x, m\rangle \in X_\alpha \leftrightarrow \langle \bar{\varepsilon}, x, m\rangle \notin X_\alpha).$$

(iii) the elementhood relation for T_α is $\in_\alpha$, and we have that $m \in_\alpha n$ if n belongs to T_α and $\langle \varepsilon, m, n\rangle$ is an element of X_α. Equality between types is just extensional equality.

(iv) the naming relation R_α of $\mathfrak{M}(\alpha)$ is given by pairs (m, n) so that m, n belong to T_α and are extensionally equal with respect to $\in_\alpha$.

(v) the collection of universes $U_\alpha \subset T_\alpha$ is obtained by taking those m for which there exists an $\langle \widehat{\mathsf{u}}, n\rangle$ in T_α that is extensionally equal to m.

This finishes the specification of $\mathfrak{M}(\alpha) = (T_\alpha, \in_\alpha, R_\alpha, U_\alpha)$. For each $\beta < \alpha$ we obtain an obvious restriction $\mathfrak{M}(\beta) = (T_\beta, \in_\beta, R_\beta, U_\beta)$ of $\mathfrak{M}(\alpha)$ by defining $T_\beta, \in_\beta, R_\beta, U_\beta$ from X_β analogously to (ii)-(v).

It is important to notice here that two structures $\mathfrak{M}(\alpha)$ and $\mathfrak{N}(\alpha)$ are in general different since they can be generated from two different Φ sequences. As we will see, however, our asymmetric interpretation theorem below is independent of a particular choice of a Φ sequence.

We are now ready to provide an asymmetrical interpretation of T_∞ into the structures $\mathfrak{M}(\alpha)$ for suitable α. In particular, we show that if a Σ^+ sentence A is provable in EMU, then there exists an ordinal α less than ε_0 so that A holds in each structure $\mathfrak{M}(\alpha)$. Asymmetrical interpretations are a well-known technique in proof theory, cf. e.g. [2, 14, 30]. They have previously been applied in the context of explicit mathematics e.g. in [11, 12, 13, 25, 24, 26].

Before we turn to the interpretation itself, let us state essential persistency properties of Σ^+ and Π^- formulas w.r.t. the structures $\mathfrak{M}(\alpha)$. The proof of the following lemma is immediate from the definition of the structures $\mathfrak{M}(\alpha)$.

Lemma 10 *Let $\mathfrak{M}(\alpha) = (T_\alpha, \ldots)$ be a structure for $\mathcal{L}$, and let $\gamma \leq \beta \leq \alpha$, $\vec{u} \in T_\gamma$ and $\vec{m} \in \mathbb{N}$. Then we have for all Σ^+ formulas $F[\vec{A}, \vec{a}]$[4] and all Π^- formulas $G[\vec{A}, \vec{a}]$:*

1. $\mathfrak{M}(\gamma) \models F[\vec{u}, \vec{m}] \quad \Longrightarrow \quad \mathfrak{M}(\beta) \models F[\vec{u}, \vec{m}]$.
2. $\mathfrak{M}(\beta) \models G[\vec{u}, \vec{m}] \quad \Longrightarrow \quad \mathfrak{M}(\gamma) \models G[\vec{u}, \vec{m}]$.

In the sequel let us assume that $\Gamma[\vec{A}, \vec{a}]$ is a set of Σ^+ and Π^- formulas. Further, let $\mathfrak{M}(\alpha)$ be a structure for $\mathcal{L}$ and let $\gamma \leq \beta \leq \alpha$. Then we write

$$\mathfrak{M}(\gamma, \beta) \models \Gamma[\vec{u}, \vec{m}] \qquad (\vec{u} \in T_\gamma, \vec{m} \in \mathbb{N}),$$

provided that one of the following conditions is satisfied:

(1) there is a Π^- formula $F[\vec{A}, \vec{a}]$ in Γ so that $\mathfrak{M}(\gamma) \models F[\vec{u}, \vec{m}]$;

(2) there is a Σ^+ formula $G[\vec{A}, \vec{a}]$ in Γ so that $\mathfrak{M}(\beta) \models G[\vec{u}, \vec{m}]$.

The asymmetric interpretation result mentioned above now reads as follows.

Lemma 11 (Main Lemma II) *Let γ be a fixed ordinal and $\mathfrak{M}(\omega^\gamma)$ an arbitrary $\mathcal{L}$ structure. Further assume that $\Gamma[\vec{A}, \vec{a}]$ is a finite set of Σ^+ and Π^- formulas so that $\mathsf{T}_\infty \,\frac{\alpha}{1}\, \Gamma$ for an ordinal $\alpha < \gamma$. Then we have for all ordinals $\beta < \omega^\gamma$:*

$$\vec{u} \in T_\beta \text{ and } \vec{m} \in \mathbb{N} \quad \Longrightarrow \quad \mathfrak{M}(\beta, \beta + 2^\alpha) \models \Gamma[\vec{u}, \vec{m}].$$

Proof. The assertion is proved by induction on $\alpha < \gamma$. As an example we discuss the axiom about universe construction as well as the cut rule. In all other cases the claim follows from the construction of $\mathfrak{M}(\omega^\gamma)$, the induction hypothesis and the persistency lemma. In particular, observe that the complement property of the element relation is preserved by all type constructors.

[4] We write $F[\vec{A}, \vec{a}]$ in order to indicate that *all* parameters of F come from the list $\vec{A}, \vec{a}$.

Let us first assume that $\Gamma[\vec{A},\vec{a}]$ is an axiom about universe construction (28). Then $\Gamma[\vec{A},\vec{a}]$ has the form

$$\Lambda[\vec{A},\vec{a}],\ \neg(\exists X)\Re(s[\vec{a}],X),\ (\exists X)[\Re(\mathsf{u}s[\vec{a}],X) \wedge \mathsf{U}(X) \wedge s[\vec{a}] \in X]. \tag{1}$$

Now fix an ordinal $\beta < \omega^\gamma$, $\vec{u} \in T_\beta$ and $\vec{m} \in \mathbb{N}$. Further suppose that $\mathfrak{M}(\beta)$ models $(\exists X)\Re(s[\vec{m}],X)$, i.e. we have that $s[\vec{m}]$ belongs to T_β. If $\mathsf{u}s[\vec{m}]$ already belongs to T_β, then it is easily seen that $s[\vec{m}] \in_\beta \mathsf{u}s[\vec{m}]$. Hence, assume that $\mathsf{u}s[\vec{m}]$ is not in T_β. But then we have by construction of $\mathfrak{M}(\beta+1)$ that $\mathsf{u}s[\vec{m}]$ belongs to $T_{\beta+1}$, and also $s[\vec{m}] \in_{\beta+1} \mathsf{u}s[\vec{m}]$. All together we obtain by persistency for all ordinals $\alpha < \gamma$:

$$\mathfrak{M}(\beta,\beta+2^\alpha) \models \Gamma[\vec{u},\vec{m}]. \tag{2}$$

As a second illustrative example let us consider the case where $\Gamma[\vec{A},\vec{a}]$ is the conclusion of a cut rule. Then the cut formula has rank 0, i.e. there is a Σ^+ formula $F[\vec{A},\vec{a}]$ and $\alpha_0,\alpha_1 < \alpha < \gamma$ so that

$$\mathsf{T}_\infty \,\overset{\alpha_0}{\underset{1}{\vdash}}\, \Gamma[\vec{A},\vec{a}],\ F[\vec{A},\vec{a}] \quad \text{and} \quad \mathsf{T}_\infty \,\overset{\alpha_1}{\underset{1}{\vdash}}\, \Gamma[\vec{A},\vec{a}],\ \neg F[\vec{A},\vec{a}]. \tag{3}$$

Choose $\beta < \omega^\gamma, \vec{u} \in T_\beta$ and $\vec{m} \in \mathbb{N}$. We have to show $\mathfrak{M}(\beta,\beta+2^\alpha) \models \Gamma[\vec{u},\vec{m}]$. If we apply the induction hypothesis to (3) with β and $\beta + 2^{\alpha_0}$, respectively, then we get

$$\begin{aligned} \mathfrak{M}(\beta,\beta+2^{\alpha_0}) &\models \Gamma[\vec{u},\vec{m}],\ F[\vec{U},\vec{m}], && (4)\\ \mathfrak{M}(\beta+2^{\alpha_0},\beta+2^{\alpha_0}+2^{\alpha_1}) &\models \Gamma[\vec{u},\vec{m}],\ \neg F[\vec{U},\vec{m}]. && (5) \end{aligned}$$

Observe that $\beta + 2^{\alpha_0} + 2^{\alpha_1} \leq \beta + 2^\alpha$. Hence, if it is

$$\text{(i) } \mathfrak{M}(\beta,\beta+2^{\alpha_0}) \models \Gamma[\vec{u},\vec{m}] \quad \text{or} \quad \text{(ii) } \mathfrak{M}(\beta+2^{\alpha_0},\beta+2^{\alpha_0}+2^{\alpha_1}) \models \Gamma[\vec{u},\vec{m}],$$

then our assertion immediately follows by persistency. But one of (i) and (ii) applies, since otherwise (4) and (5) imply

$$\mathfrak{M}(\beta+2^{\alpha_0}) \models F[\vec{u},\vec{m}] \quad \text{and} \quad \mathfrak{M}(\beta+2^{\alpha_0}) \models \neg F[\vec{u},\vec{m}]. \tag{6}$$

This, however, is not possible, and hence our claim is proved. □

Together with Corollary 9 we have thus established the following result.

Corollary 12 *Assume that the Σ^+ sentence F is provable in* EMU*. Then there exists an ordinal $\alpha < \varepsilon_0$ so that $\mathfrak{M}(\alpha) \models F$ for arbitrary $\mathcal{L}$ structures $\mathfrak{M}(\alpha)$.*

This finishes the treatment of quasinormal T_∞ derivations by means of asymmetric interpretation into partial models of EMU. We finish this section by briefly addressing how the reduction procedure for EMU described so far can be formalized in the transfinitely iterated fixed point theory $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ of [18] in order to yield conservativity of EMU over $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ with respect to arithmetic statements. Together with the results of the previous section and the fact that $|\widehat{\mathsf{ID}}_{<\varepsilon_0}| = \varphi 1 \varepsilon_0 0$ (cf. [18]) this shows the proof-theoretic equivalence of EMU and $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ as desired.

The *first step* in reducing EMU to $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ is provided by Corollary 9. Here we observe that a straightforward formalization of infinitary derivations and cut elimination procedures is required within $\widehat{\mathsf{ID}}_{<\varepsilon_0}$, cf. e.g. Schwichtenberg [31] for similar arguments. The *second step* of our reduction consists in formalizing Main Lemma II in $\widehat{\mathsf{ID}}_{<\varepsilon_0}$. Recall that this lemma holds for structures $\mathfrak{M}(\omega^\gamma)$ which are given by an arbitrary fixed point hierarchy of a (parameterized) positive arithmetic operator, and exactly such arbitrary fixed point hierarchies of length bounded below ε_0 are available in $\widehat{\mathsf{ID}}_{<\varepsilon_0}$; observe that we can do with structures of a fixed level less than ε_0 in Main Lemma II, since we are always working with a fixed EMU derivation. Of course, some straightforward formal truth definitions have to be described in $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ for a proper formalization of Main Lemma II. Summing up, we have established the following result.

Theorem 13 EMU *can be embedded into* $\widehat{\mathsf{ID}}_{<\varepsilon_0}$*; moreover, arithmetic sentences are preserved under this embedding.*

Together with Theorem 6 we can thus state the following main corollary.

Corollary 14 EMU *is proof-theoretically equivalent to* $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ *and has proof-theoretic ordinal* $\varphi 1 \varepsilon_0 0$*.*

5 Final remarks

In this paper we have given a proof-theoretic analysis of EMU, a system of explicit mathematics with a principle for *uniform* universe construction and including the schema of formula induction. Let us now briefly look at subsystems of EMU with restricted forms of complete induction on the natural numbers. Let $\mathsf{EMU}\!\restriction$ denote EMU with complete induction restricted to types, and $\mathsf{EMU}\!\restriction + (\Sigma^+\text{-}\mathsf{I_N})$ be EMU with complete induction restricted to formulas in the class Σ^+, cf. the previous section. Then the methods of the last section can be applied in order to get a reduction of $\mathsf{EMU}\!\restriction$ and $\mathsf{EMU}\!\restriction + (\Sigma^+\text{-}\mathsf{I_N})$ to $\widehat{\mathsf{ID}}_{<\omega}$ and $\widehat{\mathsf{ID}}_{<\omega^\omega}$, respectively, and indeed it can be shown that these bounds are sharp. The equivalence $\mathsf{EMU}\!\restriction \equiv \widehat{\mathsf{ID}}_{<\omega}$ has previously been obtained in

Kahle [22], who relied heavily on the treatment of a non-uniform formulation of the limit axiom in Marzetta [25, 24] and Marzetta and Strahm [26]. Let us summarize all these results in the following theorem.

Theorem 15 *We have the following proof-theoretic equivalences:*

1. $\mathsf{EMU}{\upharpoonright} \equiv \widehat{\mathsf{ID}}_{<\omega}$,
2. $\mathsf{EMU}{\upharpoonright} + (\Sigma^+\text{-}\mathsf{I_N}) \equiv \widehat{\mathsf{ID}}_{<\omega^\omega}$,
3. $\mathsf{EMU} \equiv \widehat{\mathsf{ID}}_{<\varepsilon_0}$.

The corresponding proof-theoretic ordinals are $\Gamma_0, \varphi 1\omega 0$, *and* $\varphi 1\varepsilon_0 0$, *respectively.*

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