Weak theories of operations and types

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General aims of this talk

In this talk we will discuss

- weak systems of operations and types in the spirit of Feferman’s explicit mathematics
- uniform proof-theoretic characterizations of various classes of computational complexity in this setting
- relationship to traditional bounded arithmetic
- issues of feasibility in higher types
- some aspects of self-referential truth
Explicit mathematics

Systems of explicit mathematics have been introduced by Feferman in 1975. They have been employed in foundational works in various ways:

- foundations of constructive mathematics
- proof theory of subsystems of second order arithmetic and set theory; foundational reductions
- logical foundations of functional programming languages
- universes and higher reflection principles
- formal proof-theoretic framework for abstract computations from ordinary and generalized recursion theory
1. Introduction
2. The axiomatic framework
3. Characterising complexity classes
4. Higher type issues
5. Adding types and names
6. Partial truth
7. Conclusions
Informal applicative setting
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- Untyped universe of operations or rules, which can be freely applied to each other.
- Self-application is meaningful, though not necessarily total.
- The computational engine of these rules is given by a partial combinatory algebra, featuring partial versions of Curry’s combinators k and s.
- In addition, there is a ground “urelement” structure of the binary words or strings with certain natural operations on them.
Informal applicative setting (ctd.)

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- \( \ast \): word concatenation
- \( \times \): word multiplication
The logic of partial terms

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  \[ A(t) \land t \downarrow \rightarrow (\exists x)A(x) \]
- Strictness axioms claim that terms occurring in positive atoms are defined.
The basic applicative language $\mathcal{L}$

$\mathcal{L}$ is a first order language for the logic of partial terms:
- constants $k, s, p, p_0, p_1, d_W, \epsilon, s_0, s_1, p_W, s_\ell, p_\ell, c_\subseteq, l_W$ ...
- relation symbols $=, \downarrow, W$
- arbitrary term application $\circ$
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- $t_1 t_2 \ldots t_n := (\ldots (t_1 \circ t_2) \circ \ldots \circ t_n)$
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- $t : W^k \rightarrow W := (\forall x_1 \ldots x_k \in W) t x_1 \ldots x_k \in W$
The basic applicative language $\mathcal{L}$

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- constants $k, s, p, p_0, p_1, d_W$, $\epsilon, s_0, s_1, p_W, s_\ell, p_\ell, c_\subseteq, l_W$ ...
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- $t \in W := W(t)$
- $t : W^k \rightarrow W := (\forall x_1 \ldots x_k \in W)tx_1 \ldots x_k \in W$
- $t : W^W \times W \rightarrow W := (\forall f \in W \rightarrow W)(\forall x \in W)tfx \in W$
The basic theory of operations and words B

The logic of B is the logic of partial terms. The non-logical axioms of B include:

- partial combinatory algebra:
  \[ k_{xy} = x, \quad s_{xy} \lor s_{xyz} \simeq xz(yz) \]

- pairing \( p \) with projections \( p_0 \) and \( p_1 \)
- defining axioms for the binary words \( W \) with \( \epsilon \), the successors \( s_0, s_1 \), \( s_\ell \) an the predecessor \( p_W \) and \( p_\ell \)
- definition by cases \( d_W \) on \( W \)
- initial subword relation \( c \subseteq \), length of words \( l_W \)
Consequences of the partial combinatory algebra axioms

As usual in untyped applicative settings we have:

**Lemma (Explicit definitions and fixed points)**

1. *For each* $\mathcal{L}$ *term* $t$ *there exists an* $\mathcal{L}$ *term* $(\lambda x. t)$ *so that*

   $$
   B \vdash (\lambda x. t) \downarrow \land (\lambda x. t)x \simeq t
   $$

2. *There is a closed* $\mathcal{L}$ *term* $\text{fix}$ *so that*

   $$
   B \vdash \text{fix}g \downarrow \land \text{fix}gx \simeq g(\text{fix}g)x
   $$
Standard models

Example (Recursion-theoretic model \textit{PRO})

Take the universe of binary words and interpret application $\circ$ as partial recursive function application in the sense of o.r.t.
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Example (The open term model $\mathcal{M}(\lambda \eta)$)
- Take the universe of open terms
- Consider the usual reduction of the extensional untyped lambda calculus $\lambda \eta$
- Application is juxtaposition
- Two terms are equal if they have a common reduct
- $W$ denotes those terms that reduce to a “standard” word $\overline{w}$
Natural induction principles
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$\Sigma^b_W$-formulas

Formulas $A(x)$ of the form

$$(\exists y \in W)(y \leq fx \land B(f, x, y))$$

for $B$ positive and $W$-free
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$\Sigma_W^b$ notation induction on $W$, $(\Sigma_W^b \text{-I}_W)$

$f : W \rightarrow W \land A(\epsilon) \land (\forall x \in W)(A(x) \rightarrow A(s_0 x) \land A(s_1 x)) \rightarrow (\forall x \in W)A(x)$
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**Σ_{W}^{b} lexicographic induction on $W$, (Σ_{W}^{b}-I_{l})**

$$f : W \rightarrow W \land A(\epsilon) \land (\forall x \in W)(A(x) \rightarrow A(s_{\ell}x)) \rightarrow (\forall x \in W)A(x)$$
Deriving bounded recursions

Using the fixed point theorem one proves the following lemma:

**Bounded recursion on notation**

There exists a closed \( \mathcal{L} \) term \( r_W \) so that \( B + (\Sigma^b_W \text{I}_W) \) proves

\[
\begin{align*}
f : W &\rightarrow W \land g : W^3 \rightarrow W \land b : W^2 \rightarrow W \\
&\rightarrow \\
\{ & \\
& r_W \ fgb : W^2 \rightarrow W \land \\
& x \in W \land y \in W \land y \neq \epsilon \land h = r_W \ fgb \rightarrow \\
& \ h x \epsilon = f x \land h x y = g x y (h x (p_W y)) \mid b x y
\end{align*}
\]

Here \( t \mid s \) is \( t \) if \( t \leq s \) and \( s \) otherwise.

Similarly, bounded unary recursion is derivable in \( B + (\Sigma^b_W \text{I}_\ell) \).
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Definition

A function $F : \mathbb{W}^n \to \mathbb{W}$ is called provably total in an $\mathcal{L}$ theory $T$, if there exists a closed $\mathcal{L}$ term $t_F$ such that

(i) $T \vdash t_F : \mathbb{W}^n \to \mathbb{W}$ and, in addition,

(ii) $T \vdash t_F \overline{w_1} \cdots \overline{w_n} = F(w_1, \ldots, w_n)$ for all $w_1, \ldots, w_n$ in $\mathbb{W}$.

Let $\tau(T) = \{ F : F$ provably total in $T \}$. 
Four natural applicative systems

The four systems PT, PTLS, PS, LS

\[
\begin{align*}
PT & := B(\ast, \times) + (\Sigma^b_W - I_W) \\
PTLS & := B(\ast) + (\Sigma^b_W - I_W) \\
PS & := B(\ast, \times) + (\Sigma^b_W - I_\ell) \\
LS & := B(\ast) + (\Sigma^b_W - I_\ell)
\end{align*}
\]

Theorem (S '03)

We have the following lower bounds:

1. \(\text{FPtime} \) is contained in \( \tau(PT) \),
2. \(\text{FPtimeLinspace} \) is contained in \( \tau(PTLS) \),
3. \(\text{FPspace} \) is contained in \( \tau(PS) \),
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3. \(FP_{\text{space}}\) is contained in \(\tau(PS)\),
4. \(FL_{\text{LinSpace}}\) is contained in \(\tau(LS)\).
Classical systems of bounded arithmetic and PT

- Ferreira’s system $\text{PTCA}^+$ is directly contained in $\text{PT}$
- $\text{PTCA}^+$ corresponds to Buss’ $S^1_2$
- The Melhorn-Cook-Urquhart basic feasible functionals resp. the system $\text{PV}^\omega$ are directly contained in $\text{PT}$ (see later)
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- Since the main formulas in the non-logical axioms and rules are positive, we can reduce all non-positive cuts; $\vdash_*$ denotes provability restricted to positive cuts.
- We establish upper bounds directly for an extension of our systems by the axioms of totality of application and extensionality of operations.
Upper bounds: realizability

Definition (Realizability for positive formulas)

Let $A$ be a positive formula and $\rho \in \mathbb{W}$.

- $\rho \, r \, W(t)$ if $\mathcal{M}(\lambda \eta) \models t = \bar{\rho}$,
- $\rho \, r \, (t_1 = t_2)$ if $\rho = \epsilon$ and $\mathcal{M}(\lambda \eta) \models t_1 = t_2$,
- $\rho \, r \, (A \land B)$ if $\rho = \langle \rho_0, \rho_1 \rangle$ and $\rho_0 \, r \, A$ and $\rho_1 \, r \, B$,
- $\rho \, r \, (A \lor B)$ if $\rho = \langle i, \rho_0 \rangle$ and either $i = 0$ and $\rho_0 \, r \, A$ or $i = 1$ and $\rho_0 \, r \, B$,
- $\rho \, r \, (\forall x)A(x)$ if $\rho \, r \, A(u)$ for a fresh variable $u$,
- $\rho \, r \, (\exists x)A(x)$ if $\rho \, r \, A(t)$ for some term $t$.

If $\Delta$ denotes a sequence $A_1, \ldots, A_n$, then $\rho \, r \, \Delta$ iff $\rho = \langle i, \rho_0 \rangle$ for some $1 \leq i \leq n$ and $\rho_0 \, r \, A_i$. 
Upper bounds: Main Lemma

Lemma (Realizability for PT)

Let $\Gamma \Rightarrow \Delta$ be a sequent of positive formulas with $\Gamma = A_1, \ldots, A_n$ and assume that $\text{PT}^+ \vdash_\ast \Gamma[\vec{u}] \Rightarrow \Delta[\vec{u}]$. Then there exists a function $F : W^n \rightarrow W$ in $\text{FPtime}$ so that we have for all terms $\vec{s}$ and all $\rho_1, \ldots, \rho_n \in W$:

$$\text{For all } 1 \leq i \leq n : \rho_i \vdash A_i[\vec{s}] \quad \Rightarrow \quad F(\rho_1, \ldots, \rho_n) \vdash \Delta[\vec{s}].$$

Similar realizability theorems hold for the systems PTLS, PS, and LS.
The main theorem (concluded)

Theorem (S ’03)

We have the following characterizations:

1. $\tau(PT)$ equals $FPtime$,
2. $\tau(PTLS)$ equals $FPtimeLinspace$,
3. $\tau(PS)$ equals $FPspace$,
4. $\tau(LS)$ equals $FLinspace$. 
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Basic feasible functionals (Melhorn-Cook-Urquhart)

General area of higher type complexity theory.
In particular: feasible functionals of higher type.
Most robust class: basic feasible functionals BFF.

Various kinds of interesting characterizations:
- function algebra, typed lambda calculus (Melhorn, Cook-Urquhart)
- programming languages (Cook-Kapron, Irwin-Kapron-Royer)
- oracle Turing machines (Cook-Kapron, Seth)
- bounded arithmetic (Seth, Ignjatovic)
The system $\text{PV}^\omega$

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- extensionality (optional)
- NP induction

The 1-section of $PV^\omega$ coincides with the polytime functions.
Results

Theorem (S ’04)

1. The system PV^ω is contained in PT; i.e., the basic feasible functionals in all finite types are provably total in PT.

2. The provably total type 2 functionals of PT coincide exactly with the basic feasible functionals of type 2.
Results

Theorem (S ’04)

1. The system $\text{PV}^\omega$ is contained in $\text{PT}$; i.e., the basic feasible functionals in all finite types are provably total in $\text{PT}$

2. The provably total type 2 functionals of $\text{PT}$ coincide exactly with the basic feasible functionals of type 2

Conjecture

PT characterizes the BFF’s in all finite types.

The conjecture holds for the intuitionistic version of $\text{PT}$ as follows from work by Cantini.
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Types and names in explicit mathematics

Types are collections of individuals and can have quite complicated defining properties. Types are represented by operations or names. Each type may have several different names or representations. The interplay of names and types on the level of operations witnesses the explicit character of explicit mathematics. In the following we use a formalization of the types-and-names-paradigm due to Jäger.
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The language of types and names

The language $\mathcal{L}_T$ is a two-sorted language extending $\mathcal{L}$ by type variables $U$, $V$, $W$, $X$, $Y$, $Z$, ... binary relation symbols $\Re$ (naming) and $\in$ (elementhood) new (individual) constants $w$ (initial segment of $W$), $\text{id}$ (identity), $\text{dom}$ (domain), $\text{un}$ (union), $\text{int}$ (intersection), and $\text{inv}$ (inverse image) The formulas $A$, $B$, $C$, ... of $\mathcal{L}_T$ are built from the atomic formulas of $\mathcal{L}$ as well as formulas of the form $(s \in X)$, $\Re(s, X)$, $(X = Y)$ by closing under the boolean connectives and quantification in both sorts.
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The formulas $A, B, C, \ldots$ of $\mathcal{L}_T$ are built from the atomic formulas of $\mathcal{L}$ as well as formulas of the form

$$(s \in X), \quad \Re(s, X), \quad (X = Y)$$

by closing under the boolean connectives and quantification in both sorts.
Ontological axioms

We use the following abbreviations:

\[
\begin{align*}
\mathcal{R}(s) & := \exists X \mathcal{R}(s, X), \\
s \in t & := \exists X (\mathcal{R}(t, X) \land s \in X). 
\end{align*}
\]
Ontological axioms

We use the following abbreviations:

\[ \Re(s) := \exists X \Re(s, X), \]
\[ s \in t := \exists X (\Re(t, X) \land s \in X). \]

Ontological axioms (explicit representation and extensionality)

(O1) \[ \exists x \Re(x, X) \]
(O2) \[ \Re(a, X) \land \Re(a, Y) \rightarrow X = Y \]
(O3) \[ \forall z (z \in X \leftrightarrow z \in Y) \rightarrow X = Y \]
The system PET

Define $W_a(x) := W(x) \land x \leq a$.

Type existence axioms

($w_a$) $a \in W \rightarrow R(w(a)) \land \forall x(x \in w(a) \leftrightarrow W_a(x))$

(id) $R(id) \land \forall x(x \in id \leftrightarrow \exists y(x = (y, y)))$

(inv) $R(a) \rightarrow R(inv(f, a)) \land \forall x(x \in inv(f, a) \leftrightarrow fx \in a)$

(un) $R(a) \land R(b) \rightarrow R(un(a, b)) \land \forall x(x \in un(a, b) \leftrightarrow (x \in a \lor x \in b))$

(int) $R(a) \land R(b) \rightarrow R(int(a, b)) \land \forall x(x \in int(a, b) \leftrightarrow (x \in a \land x \in b))$

(dm) $R(a) \rightarrow R(dom(a)) \land \forall x(x \in dom(a) \leftrightarrow \exists y((x, y) \in a))$
Type induction on $W$

$$\epsilon \in X \land (\forall x \in W)(x \in X \rightarrow s_0 x \in X \land s_1 x \in X) \rightarrow (\forall x \in W)(x \in X)$$

**Definition (The theory PET)**

PET is the extension of the first-order applicative theory $B(\ast, \times)$ by

- the ontological axioms
- the above type existence axioms
- type induction on $W$
Proof-theoretic strength of PET

Let $\mathsf{PT}^-$ be $\mathsf{PT}$ without universal quantifiers in induction formulas.

**Theorem (Spescha, S. ’08)**

1. $\mathsf{PET}$ is a conservative extension of $\mathsf{PT}^-$.  
2. $\tau(\mathsf{PT}^-) = \mathsf{FPTime}$.

The lower bounds use an involved embedding of $\mathsf{PT}^-$ into $\mathsf{PET}$.

The upper bounds proceed via a model-theoretic argument.
Additional principles I

Totality, extensionality, choice

Totality of application:

\((\text{Tot})\) \quad \forall x \forall y (x y \downarrow)

Extensionality of operations:

\((\text{Ext})\) \quad \forall f \forall g (\forall x (fx \simeq gx) \rightarrow f = g)

Axiom of choice:

\((\text{AC})\) \quad (\forall x \in W)(\exists y \in W) A[x, y] \rightarrow (\exists f : W \rightarrow W)(\forall x \in W) A[x, fx]

where \(A[x,y]\) is a positive elementary formula.
Additional principles II

Uniformity, universal quantification

Uniformity principle (Cantini)

\[(UP) \quad \forall x (\exists y \in W) A[x, y] \rightarrow (\exists y \in W)(\forall x) A[x, y]\]

where \(A[x, y]\) is positive elementary.

Universal quantification:

\[(all) \quad \mathcal{R}(a) \rightarrow \mathcal{R}(all(a)) \land \forall x (x \in all(a) \leftrightarrow \forall y ((x, y) \in a))\]
Results

**Theorem**

*The provably total functions of PET augmented by any combination of the principles *(all)*, *(UP)*, *(AC)*, *(Tot)*, and *(Ext)* coincide with the polynomial time computable functions.*
The Join axiom

The Join axioms are given by the following assertions (J.1) and (J.2):

(J.1)  \( \mathcal{R}(a) \wedge (\forall x \in a) \mathcal{R}(fx) \rightarrow \mathcal{R}(j(a, f)) \)

(J.2)  \( \mathcal{R}(a) \wedge (\forall x \in a) \mathcal{R}(fx) \rightarrow \forall x (x \in j(a, f) \leftrightarrow \Sigma[f, a, x]) \)

where \( \Sigma[f, a, x] \) is the formula

\[ \exists y \exists z (x = (y, z) \wedge y \in a \wedge z \in fy) \]

Conjecture

Join does not increase the proof-theoretic strength of PET.
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3 Characterising complexity classes

4 Higher type issues

5 Adding types and names

6 Partial truth

7 Conclusions
Extensios of PT by a partial truth predicate

Andrea Cantini has studied various extensions of PT by

- a (form of) self-referential truth (à la Aczel, Feferman, Kripke, etc.), providing a fixed point theorem for predicates
- an axiom of choice for operations and a uniformity principle, restricted to positive conditions

These extensions do not alter the proof-theoretic strength of PT.
Truth axioms

New (atomic) formula: \( T(t) \)

\[ x \in a := T(ax) \]

\[ \{ x : A \} := \lambda x.[A] \quad (\text{[A] term with the same free variables as A}) \]
Choice and uniformity

Positive choice and uniformity in the truth theoretic setting:

(AC) \( (\forall x \in W)(\exists y \in W)T(axy) \rightarrow (\exists f : W \rightarrow W)(\forall x \in W)T(ax(fx)) \)

(UP) \( \forall x(\exists y \in W)T(axy) \rightarrow (\exists y \in W)(\forall x)T(axy) \)

**Theorem (Cantini)**

\( \tau(PT + \text{truth axioms} + \textbf{AC} + \textbf{UP}) = \text{FP}_{\text{TIME}} \)

Proof methods used by Cantini: internal forcing semantics, non-standard variants of realizability, cut elimination.
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7 Conclusions
Let \((\text{Pos-I}_W)\) denote the schema of induction on \(W\) for positive formulas.

**Theorem (Cantini)**

\[ \tau(B + (\text{Pos-I}_W)) \text{ coincides with the primitive recursive functions.} \]

Cantini’s original proof uses a formalized asymmetric interpretation in \(I\Sigma_1\). Alternatively, one can use the realizability techniques outlined in this talk.
Addendum: Positive safe induction

Andrea Cantini has also devised natural applicative systems for $\text{FPTIME}$ that are inspired by the work of Leivant and Cook-Bellantoni in implicit computational complexity.

According to this approach, one uses two tiers (or sorts) $W_0$ and $W_1$ of binary words and allows induction over $W_1$ with respect to formulas which are positive and do only mention $W_0$.

In this way, applicative theories based on combinatory logic provide a natural basis also in the context of implicit computational complexity.
Future work

Future topics for research include:

- clarify the role of further type-theoretic principles such as join
- study theories of types and names for complexity classes other than \( \text{FPTIME} \)
- weak universes and reflection principles
- etc.
Selected References

2. G. Jäger, Induction in the elementary theory of types and names, *Computer Science Logic ’87*, LNCS 329, 1988
3. A. Cantini, Choice and uniformity in weak applicative theories, *Logic Colloquium ’01*, LNL 20, ASL, 2005