Unfolding schematic formal systems

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Unfolding schematic formal systems (Feferman '96)

Given a schematic formal system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S ?

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Example (Non-finitist arithmetic NFA)

Logical operations: $\neg$, $\wedge$, $\forall$.

(1) $x' \neq 0$

(2) $\mathsf{Pd}(x') = x$

(3) $P(0) \wedge (\forall x)(P(x) \rightarrow P(x')) \rightarrow (\forall x)P(x)$.

Schematic formal systems

- The informal philosophy behind the use of schemata is their open-endedness
- Implicit in the acceptance of a schemata is the acceptance of any meaningful substitution instance
- Schematas are applicable to any language which one comes to recognize as embodying meaningful notions

Background and previous approaches

General background: Implicitness program (Kreisel '70)

Various means of extending a formal system by principles which are implicit in its axioms.

- Reflection principles, transfinite recursive progressions (Turing '39, Feferman '62)
- Autonomous progressions and predicativity (Feferman, Schütte '64)
- Reflective closure based on self-applicative truth (Feferman '91)

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- Operations are not bound to any specific mathematical domain

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- Each function symbol f of S determines an element $f^\star$ of our partial combinatory algebra.
- Each relation symbol R of S together with $\mathsf{U_S}$ determines a predicate $R^\star$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^\star$.

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- Operations on predicates, such as e.g. conjunction, are just special kinds of operations. Each logical operation l of S determines a corresponding operation $l^\star$ on predicates.
- Families or sequences of predicates given by an operation f form a new predicate $Join(f)$, the disjoint union of the predicates from f.

The substitution rule

Substitution rule (Subst)

$$\frac{A[\bar{P}]}{A[\bar{B}/\bar{P}]} \qquad \text{(Subst)}$$

$\bar{P} = P_1, \ldots, P_m$: sequence of free predicate symbols

$\bar{B} = B_1, \ldots, B_m$: sequence of formulas

$A[\bar{B}/\bar{P}]$ denotes the formula $A[\bar{P}]$ with P_i replace by B_i $(1 \leq i \leq n)$

The three unfolding systems

Definition ($\mathcal{U}(\mathsf{S})$, $\mathcal{U}_0(\mathsf{S})$, $\mathcal{U}_1(\mathsf{S})$)

- $\mathcal{U}(\mathsf{S})$: full (predicate) unfolding of S
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Remark: The original formulation of unfolding made use of a background theory of typed operations with general Least Fixed Point operator. The present formulation is a simplification of this approach.

The proof theory of the three unfolding systems for NFA

Theorem (Feferman, Strahm)

We have the following proof-theoretic characterizations.

1. $\mathcal{U}_0(\mathsf{NFA})$ *is proof-theoretically equivalent to* PA.
2. $\mathcal{U}_1(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\omega}$.
3. $\mathcal{U}(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\Gamma_0}$.

In each case we have conservation with respect to arithmetic statements of the system on the left over the system on the right.

Ramified analysis

$\mathcal{L}_2$: Language of second-order arithmetic.

Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^\star$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in \mathcal{L}_2$ we have

$$\forall x(x \in S \leftrightarrow A^{\mathcal{M}}(x))$$

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Definition (Ramified analytic hierarchy)

$$\begin{aligned}
\mathcal{M}_0 &:= \text{arithmetically definable sets} \\
\mathcal{M}_{\alpha+1} &:= \mathcal{M}_\alpha^\star \\
\mathcal{M}_\lambda &:= \bigcup_{\beta<\lambda} \mathcal{M}_\beta
\end{aligned}$$

The systems RA_α

We let RA_α denote a (semi) formal system for $\mathcal{M}_\alpha$.

Problem

How do we justify the ordinals α in the generation of $\mathcal{M}_\alpha$ respectively RA_α ?

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Autonomity condition

RA_α is only justified if α is a recursive ordinal so that $\mathrm{RA}_{<\alpha}$ *proves* the wellfoundedness of α.

The ordinal Γ_0

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Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process ?

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Definition (The ordinal Γ_0)

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\varphi_0(\beta) &:= \omega^\beta \\
\varphi_\alpha(\beta) &:= \beta\text{th common fixed point of } (\varphi_\xi)_{\xi<\alpha} \\
\Gamma_0 &:= \text{least ordinal} > 0 \text{ that is closed under } \varphi
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Theorem (Feferman, Schütte)

$$\mathrm{Aut}(\mathrm{RA}) = \Gamma_0$$

Finitist arithmetic

Question: What principles are implicit in the actual finitist conception of the set of natural numbers ?

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Example (Finitist arithmetic FA)

Logical operations: $\wedge$, $\vee$, $\exists$.

(1) $u' = 0 \rightarrow Q$,

(2) $\mathsf{Pd}(u') = u$,

(3) $$\frac{Q \rightarrow P(0) \qquad Q \rightarrow (P(u) \rightarrow P(u'))}{Q \rightarrow P(v)} \quad (u \text{ fresh}).$$

Implications at the top-level are used to form relative assertions.

Primary and secondary formulas

- Primary formulas $(A, B, C, \dots)$ are built from the atomic formulas by means of $\wedge$, $\vee$ and $\exists$
- Secondary formulas $(F, G, H, \dots)$ are of the form

$$A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B) \dots)$$

where $n \geq 0$ and $A_1, A_2, \dots, A_n, B$ are primary formulas.

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where $n \geq 0$ and $A_1, A_2, \dots, A_n, B$ are primary formulas.

Remark: The original formulation of unfolding finitist arithmetic made use of sequent-style formalization of logic. The present formulation is a simplification of this approach and uses a Hilbert-style system.

Generalization of the substitution rule (Subst)

We have to generalize the substitution rule (Subst) to rules of inference:

Substitution rule (Subst')

Given that the rule of inference

$$\frac{F_1, F_2, \ldots, F_n}{F}$$

is *derivable*, we can adjoin each of its substitution instances

$$\frac{F_1[\bar{B}/\bar{P}],\ F_2[\bar{B}/\bar{P}], \ldots, F_n[\bar{B}/\bar{P}]}{F[\bar{B}/\bar{P}]}$$

as a new rule of inference.

The proof theory of the three unfolding systems for FA

The full unfolding of FA includes the basic logical operations as operations on predicates as well as *Join*.

Theorem (Feferman, Strahm)

All three unfolding systems for finitist arithmetic, $\mathcal{U}_0(\mathrm{FA})$, $\mathcal{U}_1(\mathrm{FA})$ and $\mathcal{U}(\mathrm{FA})$ are proof-theoretically equivalent to Skolem's Primitive Recursive Arithmetic PRA.

Support of Tait's informal analysis of finitism (Tait '81).

Future work

Unfolding of

- Finitist arithmetic with ordinals
- Feasible arithmetic
- Arithmetic with choice functionals
- Second order arithmetic
- Set-theoretical systems