Unfolding schematic formal systems

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Outline

1. Introduction
2. Defining unfolding
3. Unfolding non-finitist arithmetic
4. Interlude: Ramified analysis and the ordinal $\Gamma_0$
5. Unfolding finitist arithmetic
6. Future work
Unfolding schematic formal systems (Feferman ’96)

Given a schematic formal system $S$, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted $S$?
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**Example (Non-finitist arithmetic NFA)**

Logical operations: $\neg$, $\land$, $\forall$.

(1) $x' \neq 0$

(2) $Pd(x') = x$

(3) $P(0) \land (\forall x)(P(x) \rightarrow P(x')) \rightarrow (\forall x)P(x)$. 

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Schematic formal systems

- The informal philosophy behind the use of schemata is their open-endedness.
- Implicit in the acceptance of a schemata is the acceptance of any meaningful substitution instance.
- Schematas are applicable to any language which one comes to recognize as embodying meaningful notions.
Background and previous approaches

General background: Implicitness program (Kreisel ’70)

Various means of extending a formal system by principles which are implicit in its axioms.

- Reflection principles, transfinite recursive progressions (Turing ’39, Feferman ’62)
- Autonomous progressions and predicativity (Feferman, Schütte ’64)
- Reflective closure based on self-applicative truth (Feferman ’91)
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- Operations are not bound to any specific mathematical domain.
The full unfolding $\mathcal{U}(S)$

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- Each relation symbol $R$ of $S$ together with $U_S$ determines a predicate $R^*$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^*$. 

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- Families or sequences of predicates given by an operation $f$ form a new predicate $\text{Join}(f)$, the disjoint union of the predicates from $f$. 

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The substitution rule

\[
\frac{A[\overline{P}]}{A[\overline{B}/\overline{P}]} \quad \text{(Subst)}
\]

\(\overline{P} = P_1, \ldots, P_m\): sequence of free predicate symbols

\(\overline{B} = B_1, \ldots, B_m\): sequence of formulas

\(A[\overline{B}/\overline{P}]\) denotes the formula \(A[\overline{P}]\) with \(P_i\) replace by \(B_i\) \((1 \leq i \leq n)\)
The three unfolding systems

Definition \((\mathcal{U}(S), \mathcal{U}_0(S), \mathcal{U}_1(S))\)

- \(\mathcal{U}(S)\): full (predicate) unfolding of \(S\)
- \(\mathcal{U}_0(S)\): operational unfolding of \(S\) (no predicates)
- \(\mathcal{U}_1(S)\): \(\mathcal{U}(S)\) without \((Join)\)
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Remark: The original formulation of unfolding made use of a background theory of typed operations with general Least Fixed Point operator. The present formulation is a simplification of this approach.
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3 Unfolding non-finitist arithmetic

4 Interlude: Ramified analysis and the ordinal $\Gamma_0$

5 Unfolding finitist arithmetic

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Theorem (Feferman, Strahm)

We have the following proof-theoretic characterizations.

1. $\mathcal{U}_0(\text{NFA})$ is proof-theoretically equivalent to $\text{PA}$.
2. $\mathcal{U}_1(\text{NFA})$ is proof-theoretically equivalent to $\text{RA}_{<\omega}$.
3. $\mathcal{U}(\text{NFA})$ is proof-theoretically equivalent to $\text{RA}_{<\Gamma_0}$.

In each case we have conservation with respect to arithmetic statements of the system on the left over the system on the right.
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Ramified analysis

$\mathcal{L}_2$: Language of second-order arithmetic.

Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^*$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in \mathcal{L}_2$ we have

$$\forall x (x \in S \leftrightarrow A^\mathcal{M}(x))$$
Ramified analysis

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Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^*$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in L_2$ we have

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Definition (Ramified analytic hierarchy)

$\mathcal{M}_0 :=$ arithmetically definable sets

$\mathcal{M}_{\alpha+1} := \mathcal{M}_\alpha^*$

$\mathcal{M}_\lambda := \bigcup_{\beta < \lambda} \mathcal{M}_\beta$
The systems $\text{RA}_\alpha$

We let $\text{RA}_\alpha$ denote a (semi) formal system for $\mathcal{M}_\alpha$.

**Problem**

How do we justify the ordinals $\alpha$ in the generation of $\mathcal{M}_\alpha$ respectively $\text{RA}_\alpha$?
The systems $\text{RA}_\alpha$

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**Problem**

How do we justify the ordinals $\alpha$ in the generation of $M_\alpha$ respectively $\text{RA}_\alpha$?

**Autonomy condition**

$\text{RA}_\alpha$ is only justified if $\alpha$ is a recursive ordinal so that $\text{RA}_{<\alpha}$ proves the wellfoundedness of $\alpha$. 
The ordinal $\Gamma_0$

**Question**

Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process?
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**Definition (The ordinal $\Gamma_0$)**

\[
\varphi_0(\beta) := \omega^\beta \\
\varphi_\alpha(\beta) := \text{$\beta$th common fixed point of $(\varphi_\xi)_{\xi<\alpha}$} \\
\Gamma_0 := \text{least ordinal $> 0$ that is closed under $\varphi$}
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\Gamma_0 := \text{least ordinal } > 0 \text{ that is closed under } \varphi
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**Theorem (Feferman, Schütte)**

\[\text{Aut(RA)} = \Gamma_0\]
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Question: What principles are implicit in the actual finitist conception of the set of natural numbers?
Finitist arithmetic

**Question:** What principles are implicit in the actual finitist conception of the set of natural numbers?

**Example (Finitist arithmetic FA)**

Logical operations: $\land, \lor, \exists$.

1. $u' = 0 \rightarrow Q$,
2. $Pd(u') = u$,
3. $Q \rightarrow P(0)$  $Q \rightarrow (P(u) \rightarrow P(u'))$  $Q \rightarrow P(v)$ (u fresh).

Implications at the top-level are used to form relative assertions.
Primary and secondary formulas

- **Primary formulas** \((A, B, C, \ldots)\) are built from the atomic formulas by means of \(\land, \lor\) and \(\exists\)

- **Secondary formulas** \((F, G, H, \ldots)\) are of the form

\[
A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B)\ldots)
\]

where \(n \geq 0\) and \(A_1, A_2, \ldots, A_n, B\) are primary formulas.
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where \(n \geq 0\) and \(A_1, A_2, \ldots, A_n, B\) are primary formulas.

**Remark:** The original formulation of unfolding finitist arithmetic made use of sequent-style formalization of logic. The present formulation is a simplification of this approach and uses a Hilbert-style system.
Generalization of the substitution rule (Subst)

We have to generalize the substitution rule (Subst) to rules of inference:

**Substitution rule (Subst’)**

Given that the rule of inference

\[
\frac{F_1, F_2, \ldots, F_n}{F}
\]

is *derivable*, we can adjoin each of its substitution instances

\[
\frac{F_1[\bar{B}/\bar{P}], F_2[\bar{B}/\bar{P}], \ldots, F_n[\bar{B}/\bar{P}]}{F[\bar{B}/\bar{P}]}
\]

as a new rule of inference.
The full unfolding of FA includes the basic logical operations as operations on predicates as well as Join.

**Theorem (Feferman, Strahm)**

All three unfolding systems for finitist arithmetic, $U_0(FA)$, $U_1(FA)$ and $U(FA)$ are proof-theoretically equivalent to Skolem’s Primitive Recursive Arithmetic PRA.

Support of Tait’s informal analysis of finitism (**Tait ’81**).
Future work

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Future work

Unfolding of

- Finitist arithmetic with ordinals
- Feasible arithmetic
- Arithmetic with choice functionals
- Second order arithmetic
- Set-theoretical systems