Two unfoldings of finitist arithmetic

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2 Defining unfolding

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4 Interlude: Ramified analysis and the ordinal $\Gamma_0$

5 Unfolding finitist arithmetic

6 Unfolding finitist arithmetic with bar rule

7 Appendix: Wellfoundedness of exponentiation
Unfolding schematic formal systems (Feferman ’96)

Given a schematic formal system $S$, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted $S$?
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Example (Non-finitist arithmetic NFA)

Logical operations: $\neg$, $\land$, $\forall$.

1. $x' \neq 0$
2. $\text{Pd}(x') = x$
3. $P(0) \land (\forall x)(P(x) \rightarrow P(x')) \rightarrow (\forall x)P(x)$.
Schematic formal systems

- The informal philosophy behind the use of schemata is their open-endedness.
- Implicit in the acceptance of a schemata is the acceptance of any meaningful substitution instance.
- Schematas are applicable to any language which one comes to recognize as embodying meaningful notions.
Background and previous approaches

General background: Implicitness program (Kreisel '70)

Various means of extending a formal system by principles which are implicit in its axioms.

- Reflection principles, transfinite recursive progressions (Turing '39, Feferman '62)
- Autonomous progressions and predicativity (Feferman, Schütte '64)
- Reflective closure based on self-applicative truth (Feferman '91)
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- Operations are not bound to any specific mathematical domain
The full unfolding $\mathcal{U}(S)$

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- Each relation symbol $R$ of $S$ together with $U_S$ determines a predicate $R^*$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^*$. 

Operations on predicates, such as e.g. conjunction, are just special kinds of operations. Each logical operation $l$ of $S$ determines a corresponding operation $l^*$ on predicates. Families or sequences of predicates given by an operation $f$ form a new predicate $\text{Join}(f)$, the disjoint union of the predicates from $f$. 

T. Strahm (IAM, Univ. Bern)
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The substitution rule

Substitution rule (Subst)

\[
\frac{A[\bar{P}]}{A[\bar{B}/\bar{P}]} \tag{Subst}
\]

\(\bar{P} = P_1, \ldots, P_m\): sequence of free predicate symbols

\(\bar{B} = B_1, \ldots, B_m\): sequence of formulas

\(A[\bar{B}/\bar{P}]\) denotes the formula \(A[\bar{P}]\) with \(P_i\) replace by \(B_i\) \((1 \leq i \leq n)\)
The three unfolding systems

Definition \((U(S), U_0(S), U_1(S))\)

- \(U(S)\): full (predicate) unfolding of \(S\)
- \(U_0(S)\): operational unfolding of \(S\) (no predicates)
- \(U_1(S)\): \(U(S)\) without (Join)
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Remark: The original formulation of unfolding made use of a background theory of typed operations with general Least Fixed Point operator. The present formulation is a simplification of this approach.
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The proof theory of the three unfolding systems for NFA

Theorem (Feferman, Str.)

We have the following proof-theoretic characterizations.

1. $\mathcal{U}_0(\text{NFA})$ is proof-theoretically equivalent to $\text{PA}$.
2. $\mathcal{U}_1(\text{NFA})$ is proof-theoretically equivalent to $\text{RA}_{<\omega}$.
3. $\mathcal{U}(\text{NFA})$ is proof-theoretically equivalent to $\text{RA}_{<\Gamma_0}$.

In each case we have conservation with respect to arithmetic statements of the system on the left over the system on the right.
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Ramified analysis

\( \mathcal{L}_2 \): Language of second-order arithmetic.

Given a collection \( \mathcal{M} \) of sets of natural numbers, define \( \mathcal{M}^* \) to consist of all sets \( S \subseteq \mathbb{N} \) such that for some condition \( A(x) \in \mathcal{L}_2 \) we have

\[
\forall x(x \in S \leftrightarrow A^\mathcal{M}(x))
\]
Ramified analysis

$L_2$: Language of second-order arithmetic.

Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^*$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in L_2$ we have

$$\forall x(x \in S \leftrightarrow A^M(x))$$

Definition (Ramified analytic hierarchy)

$$\mathcal{M}_0 := \text{arithmetically definable sets}$$

$$\mathcal{M}_{\alpha+1} := \mathcal{M}^\alpha$$

$$\mathcal{M}_\lambda := \bigcup_{\beta<\lambda} \mathcal{M}_\beta$$
The systems $\text{RA}_\alpha$

We let $\text{RA}_\alpha$ denote a (semi) formal system for $\mathcal{M}_\alpha$.

**Problem**

How do we justify the ordinals $\alpha$ in the generation of $\mathcal{M}_\alpha$ respectively $\text{RA}_\alpha$?
The systems $\text{RA}_\alpha$

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**Problem**

How do we justify the ordinals $\alpha$ in the generation of $\mathcal{M}_\alpha$ respectively $\text{RA}_\alpha$?

**Autonomy condition**

$\text{RA}_\alpha$ is only justified if $\alpha$ is a recursive ordinal so that $\text{RA}_{<\alpha}$ proves the wellfoundedness of $\alpha$. 
The ordinal $\Gamma_0$

**Question**

Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process?
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**Definition (The ordinal $\Gamma_0$)**

- $\varphi_0(\beta) := \omega^\beta$
- $\varphi_\alpha(\beta) := \beta$th common fixed point of $(\varphi_\xi)_{\xi < \alpha}$
- $\Gamma_0 :=$ least ordinal $> 0$ that is closed under $\varphi$
The ordinal $\Gamma_0$

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\varphi_0(\beta) := \omega^\beta \\
\varphi\alpha(\beta) := \beta\text{th common fixed point of } (\varphi\xi)_{\xi<\alpha} \\
\Gamma_0 := \text{least ordinal } > 0 \text{ that is closed under } \varphi
\]

**Theorem (Feferman, Schütte)**

\[\text{Aut(RA)} = \Gamma_0\]
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Finitist arithmetic

**Question:** What principles are implicit in the actual finitist conception of the set of natural numbers?
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**Example (Finitist arithmetic FA)**

Logical operations: $\land$, $\lor$, $\exists$.

1. $x' = 0 \rightarrow \bot$
2. $Pd(x') = x$
3. $\Gamma \rightarrow P(0)$, $\Gamma, P(x) \rightarrow P(x')$

(3) $\Gamma \rightarrow P(x)$.

Note that the statements proved are sequents $\Sigma$ of the form $\Gamma \rightarrow A$, where $\Gamma$ is a finite sequence (possibly empty) of formulas. The logic is formulated in Gentzen-style.
Generalization of the substitution rule (Subst)

We have to generalize the substitution rule (Subst) to rules of inference:

Substitution rule (Subst’)

Given that the rule of inference

\[
\frac{\Sigma_1, \ldots, \Sigma_n}{\Sigma}
\]

is derivable, we can adjoin each of its substitution instances

\[
\frac{\Sigma_1[\bar{B}/\bar{P}], \ldots, \Sigma_n[\bar{B}/\bar{P}]}{\Sigma[\bar{B}/\bar{P}]}
\]

as a new rule of inference.
The proof theory of the three unfolding systems for FA

The full unfolding of FA includes the basic logical operations as operations on predicates as well as \textit{Join}.

\textbf{Theorem (Feferman, Str.)}

\textit{All three unfolding systems for finitist arithmetic, }$\mathcal{U}_0(\text{FA})$, $\mathcal{U}_1(\text{FA})$ \textit{and }$\mathcal{U}(\text{FA})$ \textit{are proof-theoretically equivalent to Skolem’s Primitive Recursive Arithmetic PRA.}

Support of Tait’s informal analysis of finitism.
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Aim of this section

In the following

- We will study a natural bar rule BR leading to extensions $U_0^+(FA)$, $U_1^+(FA)$ and $U^+(FA)$ of our unfolding systems for finitism.
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- We will study a natural bar rule $BR$ leading to extensions $\mathcal{U}_0^+(FA)$, $\mathcal{U}_1^+(FA)$ and $\mathcal{U}^+(FA)$ of our unfolding systems for finitism.
- The so-obtained extensions will all have the strength of Peano arithmetic $PA$. 
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In the following

- We will study a natural bar rule $BR$ leading to extensions $\mathcal{U}^+_0(FA)$, $\mathcal{U}^+_1(FA)$ and $\mathcal{U}^+(FA)$ of our unfolding systems for finitism.
- The so-obtained extensions will all have the strength of Peano arithmetic $PA$.
- This shows one way how Kreisel’s analysis of extended finitism fits in our framework.
Defining $\mathcal{U}_0^+(FA)$: Preliminaries

Let $\prec$ be a binary relation whose characteristic function is given by a closed term $t_\prec$ so that $\mathcal{U}_0(FA)$ proves $t_\prec : \mathbb{N}^2 \rightarrow \{0, 1\}$. We write $x \prec y$ instead of $t_\prec xy = 0$ and further assume that $\prec$ is a linear ordering with least element 0, provably in $\mathcal{U}_0(FA)$. 
Defining $\mathcal{U}_0^+(\text{FA}):$ Preliminaries

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- Let $f$ denote a new constant of our applicative language. There are no non-logical axioms for $f$; it serves as an anonymous function from $\mathbb{N}$ to $\mathbb{N}$, representing a possibly infinite descending sequence along a given ordering.
Expressing wellfoundedness

The rule $\text{NDS}(f, \prec)$ says that for each possibly infinite descending chain $f$ w.r.t. $\prec$ there is an $x$ such that $f^x = 0$. Formally, it is given as follows:
Expressing wellfoundedness

The rule $\text{NDS}(f, \prec)$ says that for each possibly infinite descending chain $f$ w.r.t. $\prec$ there is an $x$ such that $fx = 0$. Formally, it is given as follows:

**The rule NDS($f$, $\prec$)**

\[
\begin{align*}
    u \in \mathbb{N} & \rightarrow fu \in \mathbb{N}, \\
    u \in \mathbb{N}, fu \neq 0 & \rightarrow f(u') \prec fu, \\
    u \in \mathbb{N}, fu = 0 & \rightarrow f(u') = 0 \\
    (\exists x \in \mathbb{N})(fx = 0)
\end{align*}
\]
Formulating the bar rule

Let \( \overline{s}^r = s_1^r, \ldots, s_n^r \) and \( \overline{s}^p = s_1^p, \ldots, s_n^p \) be sequences of terms of length \( n \). Accordingly, let \( \overline{t}^r = t_1^r, \ldots, t_m^r \) and \( \overline{t}^p = t_1^p, \ldots, t_m^p \) be sequences of terms of length \( m \). The superscripts ‘\( r \)’ and ‘\( p \)’ stand for recursion and parameter, respectively.
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### The bar rule BR

Whenever we have derived the four premises

1. \( \text{NDS}(f, \prec) \)
2. \( x, y \in \mathbb{N} \rightarrow \bar{s}^r \in \mathbb{N} \land \bar{s}^p \in \mathbb{N} \)
3. \( x, y \in \mathbb{N}, \bigwedge_i [s_i^r \prec x \supset P(s_i^r, s_i^p)] \rightarrow \bar{t}^r \in \mathbb{N} \land \bar{t}^p \in \mathbb{N} \)
4. \( x, y \in \mathbb{N}, \bigwedge_i [s_i^r \prec x \supset P(s_i^r, s_i^p)], \bigwedge_j [t_j^r \prec x \supset P(t_j^r, t_j^p)] \rightarrow P(x, y) \)

we can infer \( x \in \mathbb{N} \land y \in \mathbb{N} \rightarrow P(x, y) \).
How to use the rule: nested recursion

In $\mathcal{U}_0^+(\text{FA})$, whenever we have derived $\text{NDS}(f, \prec)$, then we can use the bar rule $\text{BR}$ in order to justify \textit{nested recursion} along $\prec$. 

Example (Justifying nested recursion using BR)

As usual, $(r)x$ is $r$ if $r \prec x$ and 0 otherwise. Define $F$ by

$F(0, y) \equiv H(y)$

$F(x, y) \equiv G(x, y, F(k(x, y), F(l(x, y), x, p(x, y))))$

We set $n = 2$ and $m = 1$ and choose the following terms:

- $s_{r_1} = l(x, y) \times x$
- $s_{p_1} = y$
- $s_{r_2} = m(x, y) \times x$
- $s_{p_2} = y$
- $t_{r_1} = k(x, y, F(l(x, y), x, y)) \times x$
- $t_{p_1} = p(x, y, F(m(x, y), x, y))$
How to use the rule: nested recursion

In $\mathcal{U}_0^+(\text{FA})$, whenever we have derived $\text{NDS}(f, \prec)$, then we can use the bar rule $\text{BR}$ in order to justify *nested* recursion along $\prec$.

**Example (Justifying nested recursion using BR)**

As usual, $(r)_x$ is $r$ if $r \prec x$ and 0 otherwise. Define $F$ by $(x \neq 0)$

$$F(0, y) \simeq H(y)$$

$$F(x, y) \simeq G(x, y, F(k(x, y, F(l(x, y)_x, y))_x, p(x, y, F(m(x, y)_x, y))))$$

We set $n = 2$ and $m = 1$ and choose the following terms:

$$s_1^r = l(x, y)_x, \quad s_1^p = y$$

$$s_2^r = m(x, y)_x, \quad s_2^p = y$$

$$t_1^r = k(x, y, F(l(x, y)_x, y))_x, \quad t_1^p = p(x, y, F(m(x, y)_x, y))$$
We summarize our previous findings in the following theorem.

**Theorem**

Assume that $\text{NDS}(f, \prec)$ is derivable in $\mathcal{U}_0^+(\text{FA})$. Then $\mathcal{U}_0^+(\text{FA})$ justifies nested recursion along $\prec$. 

Summarizing ...
Tait’s seminal 1961 paper

Tait’s seminal 1961 paper


- For each ordinal $\alpha < \varepsilon_0$ let $\prec_{\alpha}$ be a primitive recursive standard wellordering $\prec_{\alpha}$ of ordertype $\alpha$. 
Tait’s seminal 1961 paper


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- Let us write $\text{NDS}(f, \alpha)$ instead of $\text{NDS}(f, \prec_\alpha)$
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- Aim at showing that $\mathcal{U}_0^+(\text{FA})$ derives $\text{NDS}(f, \alpha)$ for each $\alpha < \varepsilon_0$
- Use one direction of Tait’s famous result, i.e. that nested recursion on $\omega^\alpha$ entails ordinary recursion on $\omega^\alpha$ or, more useful in our setting, nested recursion on $\omega^\alpha$ entails $\text{NDS}(f, \omega^\alpha)$
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- Tait’s argument can be directly formalized in $\mathcal{U}_0^+(\text{FA})$
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- Tait’s argument can be directly formalized in $\mathcal{U}_0^+(\text{FA})$
- For more details, see the Appendix
\( \mathcal{U}_0^+ (FA) \): Lower bounds

**Theorem**

*Provably in \( \mathcal{U}_0^+ (FA) \), nested recursion along \( \omega \alpha \) entails NDS(f, \( \omega^\alpha \)).*

**Corollary**

*We have for each \( \alpha < \varepsilon_0 \) that \( \mathcal{U}_0^+ (FA) \) derives NDS(f, \( \alpha \)).*
Upper bounds

\( \mathcal{U}_0^+ (FA) \) is readily interpretable in the subsystem of second order arithmetic \( \text{ACA}_0 \) as follows:

- Fix a *function variable* \( f \) in \( \mathcal{L}_2 \) and translate \( (a \cdot b) \) as \( \{a\}^f (b) \), where \( \{n\}^f \) for \( n = 0, 1, 2, \ldots \) is a enumeration of the functions that are partial recursive in \( f \).

- The *constant* \( f \) is interpreted as a natural number \( i \) so that \( \{i\}^f (x) \simeq f(x) \).

- The translation of BR is validated by observing that \( \text{ACA}_0 \) proves \( \text{WF}(\prec) \rightarrow \text{TI}(\prec, A) \) for each arithmetic formula \( A \).

On top of this interpretation, one models predicates (including join) to show that even the strength of \( \mathcal{U}_0^+ (FA) \) does not go beyond \( \text{PA} \).
The proof theory of the three unfolding systems for FA with bar rule

Theorem (Feferman, Str.)

All three unfolding systems for finitist arithmetic with bar rule, $\mathcal{U}_0^+(FA)$, $\mathcal{U}_1^+(FA)$ and $\mathcal{U}^+(FA)$ are proof-theoretically equivalent to Peano arithmetic PA.

Support of Kreisel’s analysis of extended finitism.
Appendix: Wellfoundedness of exponentiation

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Tait’s argument in a nutshell

(based on a compact presentation of W. Tait in a personal communication with S. Feferman)
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We want to show that nested recursion on $\omega^\delta$ entails NDS($f, \omega^\delta$).

In the following we will work with (codes of) ordinals below $\varepsilon_0$ and assume that $<$ denotes the corresponding ordering relation on them.
Tait’s argument in a nutshell

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In the following we will work with (codes of) ordinals below $\varepsilon_0$ and assume that $<$ denotes the corresponding ordering relation on them.

A possibly infinite descending sequence $f$ in $\omega^\delta$

Let $f$ be a fixed function from $\omega$ to $\omega^\delta$ satisfying for all natural numbers $n$ the condition

\[
f(n) > 0 \rightarrow f(n+1) < f(n) \quad \text{and} \quad f(n) = 0 \rightarrow f(n+1) = 0. \quad (\star)
\]
Ordinal-theoretic preliminaries

Given an ordinal $\alpha < \omega^\delta$ in its normal form

$$\alpha = \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n$$

where $\delta > \alpha_1 > \cdots > \alpha_n$ and $a_i < \omega$ ($1 \leq i \leq n$), we set

$$\alpha\{i\} = \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_k} a_k \quad (k = \min(n, i))$$

$$\alpha[i] = \begin{cases} \omega \alpha_i + a_i & \text{if } 0 < i \leq n \\ 0 & \text{if } n < i \end{cases}$$

Clearly, $\alpha[i] < \omega^\delta$ and $0\{i\} = 0[i] = 0$. Further, we have the following important property.
Lemma

We have that $\alpha\{i + 1\} < \beta\{i + 1\}$ if and only if

$\alpha\{i\} < \beta\{i\} \lor (\alpha\{i\} = \beta\{i\} \land \alpha[i + 1] < \beta[i + 1])$. 
The crucial property

The crucial step in Tait’s argument is to define a function $\mu : \omega^2 \to \omega$ such that (writing $\mu_i(j)$ for $\mu(i, j)$)

The property (⋆⋆)

$$f(j + \mu_i(j)) = 0 \lor f(j + \mu_i(j))\{i\} < f(j)\{i\}$$  (⋆⋆)

It will then suffice to choose $\mu_0(0)$ as a root of $f$, since according to (⋆⋆), $f(\mu_0(0)) = 0$. 
Defining $\mu_i(j)$

The definition of $\mu_i(j)$ will be by nested recursion on $f(j)[i + 1] < \omega \delta$. 
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- Let $n$ be the number of summands in the normal form of $f(j)$. If $i \geq n$, we may simply set $\mu_i(j) = 1$; then (**) holds due to property (*) of our given function $f$.

- So assume $0 \leq i < n$. Because $f(j)[i + 2] < f(j)[i + 1]$, we can use $\mu_{i+1}(j) = \bar{\mu}$ in the definition of $\mu_i(j)$. Hence, according to (**) we have for $\bar{\mu}$ that either (1) or (2) holds:

$$f(j + \bar{\mu}) = 0 \quad (1)$$

$$f(j + \bar{\mu})\{i + 1\} < f(j)\{i + 1\} \quad (2)$$

If (1) holds, we set $\mu_i(j) = \bar{\mu}$. 
Defining $\mu_i(j)$ (ctd.)

- In case of (2), we use the lemma above to obtain one of the following properties (3) or (4):

  \[ f(j + \mu_i)\{i\} < f(j)\{i\} \quad (3) \]
  \[ f(j + \mu_i)\{i\} = f(j)\{i\} \land f(j + \mu_i)[i + 1] < f(j)[i + 1] \quad (4) \]

In case of (3), we again set $\mu_i(j) = \bar{\mu}$.

- If (4) holds, then clearly $\mu_i(j + \bar{\mu}) = \bar{\bar{\mu}}$ is defined. In this case we set $\mu_i(j) = \mu_i + \bar{\bar{\mu}}$. Then we can verify, using property (***) for $\bar{\bar{\mu}}$, that one of the following conditions (5) or (6) holds:

  \[ f(j + \mu_i(j)) = f((j + \mu_i) + \bar{\bar{\mu}}) = 0 \quad (5) \]
  \[ f(j + \mu_i(j))\{i\} < f(j + \mu_i)\{i\} = f(j)\{i\} \quad (6) \]

This is as desired and concludes the definition of $\mu_i(j)$. 
Summarizing...

Summarizing, $\mu_i(j)$ has been defined to satisfy the following equation:

The recursive definition of $\mu_i(j)$

$$
\mu_i(j) = \begin{cases} 
1 & \text{if } i \geq n \\
\mu_{i+1}(j) & \text{if } f(j + \mu_{i+1}(j)) = 0 \text{ or } f(j + \mu_{i+1}(j))\{i\} < f(j)\{i\} \\
\mu_{i+1}(j) + \mu_i(j + \mu_{i+1}(j)) & \text{else}
\end{cases}
$$

It is now easy to explicitely express the definition of $\mu_i(j)$ as a nested recursion on $\omega\delta$. 
The end

Thank you for your attention!