Two unfoldings of finitist arithmetic

Thomas Strahm (joint work with Solomon Feferman, Stanford)

Institut für Informatik und angewandte Mathematik, Universität Bern

Swansea, April 13, 2010

Unfolding schematic formal systems (Feferman '96)

Given a schematic formal system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S ?

Unfolding schematic formal systems (Feferman '96)

Given a schematic formal system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S ?

Example (Non-finitist arithmetic NFA)

Logical operations: $\neg$, $\wedge$, $\forall$.

(1) $x' \neq 0$

(2) $\mathsf{Pd}(x') = x$

(3) $P(0) \wedge (\forall x)(P(x) \to P(x')) \to (\forall x)P(x)$.

Schematic formal systems

- The informal philosophy behind the use of schemata is their open-endedness
- Implicit in the acceptance of a schemata is the acceptance of any meaningful substitution instance
- Schematas are applicable to any language which one comes to recognize as embodying meaningful notions

Background and previous approaches

General background: Implicitness program (Kreisel '70)

Various means of extending a formal system by principles which are implicit in its axioms.

- Reflection principles, transfinite recursive progressions (Turing '39, Feferman '62)
- Autonomous progressions and predicativity (Feferman, Schütte '64)
- Reflective closure based on self-applicative truth (Feferman '91)

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:

  (1) $\mathsf{k}ab = a$,

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:
  (1) $\mathsf{k}ab = a$,
  (2) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$,

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:

  (1) $\mathsf{k}ab = a$,
  (2) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$,
  (3) $\mathsf{p}_0(a, b) = a \wedge \mathsf{p}_1(a, b) = b$,

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:
  1. $\mathsf{k}ab = a$,
  2. $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$,
  3. $\mathsf{p}_0(a, b) = a \wedge \mathsf{p}_1(a, b) = b$,
  4. $\mathsf{d}ab\top = a \wedge \mathsf{d}ab\bot = b$.

How is the unfolding of a schematic system S defined ?

- We have a general notion of (partial) operation and predicate
- Predicates are just special kinds of operations, equipped with an $\in$ relation
- Underlying partial combinatory algebra with pairing and definition by cases:
  (1) $\mathsf{k}ab = a$,
  (2) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$,
  (3) $\mathsf{p}_0(a, b) = a \wedge \mathsf{p}_1(a, b) = b$,
  (4) $\mathsf{d}ab\top = a \wedge \mathsf{d}ab\bot = b$.
- Operations are not bound to any specific mathematical domain

The full unfolding $\mathcal{U}(\mathsf{S})$

- The universe of S has associated with it an additional unary relation symbol, $\mathsf{U_S}$, and the axioms of S are to be relativized to $\mathsf{U_S}$.

The full unfolding $\mathcal{U}(\mathsf{S})$

- The universe of S has associated with it an additional unary relation symbol, $\mathsf{U_S}$, and the axioms of S are to be relativized to $\mathsf{U_S}$.
- Each function symbol f of S determines an element $f^\star$ of our partial combinatory algebra.

The full unfolding $\mathcal{U}(\mathsf{S})$

- The universe of S has associated with it an additional unary relation symbol, $\mathsf{U_S}$, and the axioms of S are to be relativized to $\mathsf{U_S}$.
- Each function symbol f of S determines an element $f^\star$ of our partial combinatory algebra.
- Each relation symbol R of S together with $\mathsf{U_S}$ determines a predicate $R^\star$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^\star$.

The full unfolding $\mathcal{U}(\mathsf{S})$

- The universe of S has associated with it an additional unary relation symbol, U_S, and the axioms of S are to be relativized to U_S.
- Each function symbol f of S determines an element $f^\star$ of our partial combinatory algebra.
- Each relation symbol R of S together with U_S determines a predicate $R^\star$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^\star$.
- Operations on predicates, such as e.g. conjunction, are just special kinds of operations. Each logical operation l of S determines a corresponding operation $l^\star$ on predicates.

The full unfolding $\mathcal{U}(\mathsf{S})$

- The universe of S has associated with it an additional unary relation symbol, $\mathsf{U_S}$, and the axioms of S are to be relativized to $\mathsf{U_S}$.
- Each function symbol f of S determines an element $f^\star$ of our partial combinatory algebra.
- Each relation symbol R of S together with $\mathsf{U_S}$ determines a predicate $R^\star$ of our partial combinatory algebra with $R(x_1, \ldots, x_n)$ if and only if $(x_1, \ldots, x_n) \in R^\star$.
- Operations on predicates, such as e.g. conjunction, are just special kinds of operations. Each logical operation l of S determines a corresponding operation $l^\star$ on predicates.
- Families or sequences of predicates given by an operation f form a new predicate $Join(f)$, the disjoint union of the predicates from f.

The substitution rule

Substitution rule (Subst)

$$\frac{A[\bar{P}]}{A[\bar{B}/\bar{P}]} \qquad \text{(Subst)}$$

$\bar{P} = P_1, \ldots, P_m$: sequence of free predicate symbols

$\bar{B} = B_1, \ldots, B_m$: sequence of formulas

$A[\bar{B}/\bar{P}]$ denotes the formula $A[\bar{P}]$ with P_i replace by B_i $(1 \leq i \leq n)$

The three unfolding systems

Definition ($\mathcal{U}(\mathsf{S})$, $\mathcal{U}_0(\mathsf{S})$, $\mathcal{U}_1(\mathsf{S})$)

- $\mathcal{U}(\mathsf{S})$: full (predicate) unfolding of S
- $\mathcal{U}_0(\mathsf{S})$: operational unfolding of S (no predicates)
- $\mathcal{U}_1(\mathsf{S})$: $\mathcal{U}(\mathsf{S})$ without (*Join*)

The three unfolding systems

Definition ($\mathcal{U}(\mathsf{S})$, $\mathcal{U}_0(\mathsf{S})$, $\mathcal{U}_1(\mathsf{S})$)

- $\mathcal{U}(\mathsf{S})$: full (predicate) unfolding of S
- $\mathcal{U}_0(\mathsf{S})$: operational unfolding of S (no predicates)
- $\mathcal{U}_1(\mathsf{S})$: $\mathcal{U}(\mathsf{S})$ without (*Join*)

Remark: The original formulation of unfolding made use of a background theory of typed operations with general Least Fixed Point operator. The present formulation is a simplification of this approach.

The proof theory of the three unfolding systems for NFA

Theorem (Feferman, Str.)

We have the following proof-theoretic characterizations.

1. $\mathcal{U}_0(\mathsf{NFA})$ *is proof-theoretically equivalent to* PA.
2. $\mathcal{U}_1(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\omega}$.
3. $\mathcal{U}(\mathsf{NFA})$ *is proof-theoretically equivalent to* $\mathsf{RA}_{<\Gamma_0}$.

In each case we have conservation with respect to arithmetic statements of the system on the left over the system on the right.

Ramified analysis

$\mathcal{L}_2$: Language of second-order arithmetic.

Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^\star$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in \mathcal{L}_2$ we have

$$\forall x(x \in S \leftrightarrow A^{\mathcal{M}}(x))$$

Ramified analysis

$\mathcal{L}_2$: Language of second-order arithmetic.

Given a collection $\mathcal{M}$ of sets of natural numbers, define $\mathcal{M}^\star$ to consist of all sets $S \subseteq \mathbb{N}$ such that for some condition $A(x) \in \mathcal{L}_2$ we have

$$\forall x(x \in S \leftrightarrow A^{\mathcal{M}}(x))$$

Definition (Ramified analytic hierarchy)

$$\begin{aligned}
\mathcal{M}_0 &:= \text{arithmetically definable sets} \\
\mathcal{M}_{\alpha+1} &:= \mathcal{M}_\alpha^\star \\
\mathcal{M}_\lambda &:= \bigcup_{\beta<\lambda} \mathcal{M}_\beta
\end{aligned}$$

The systems RA_α

We let RA_α denote a (semi) formal system for $\mathcal{M}_\alpha$.

Problem

How do we justify the ordinals α in the generation of $\mathcal{M}_\alpha$ respectively RA_α ?

The systems RA_α

We let RA_α denote a (semi) formal system for $\mathcal{M}_\alpha$.

Problem

How do we justify the ordinals α in the generation of $\mathcal{M}_\alpha$ respectively RA_α ?

Autonomity condition

RA_α is only justified if α is a recursive ordinal so that $\mathsf{RA}_{<\alpha}$ *proves* the wellfoundedness of α.

The ordinal Γ_0

Question

Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process ?

The ordinal Γ_0

Question

Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process ?

Definition (The ordinal Γ_0)

$$\begin{aligned} \varphi_0(\beta) &:= \omega^\beta \\ \varphi_\alpha(\beta) &:= \beta\text{th common fixed point of } (\varphi_\xi)_{\xi<\alpha} \\ \Gamma_0 &:= \text{least ordinal} > 0 \text{ that is closed under } \varphi \end{aligned}$$

The ordinal Γ_0

Question

Where does this procedure stop, i.e. which ordinals can be reached by such an autonomous process ?

Definition (The ordinal Γ_0)

$$
\begin{aligned}
\varphi_0(\beta) &:= \omega^\beta \\
\varphi_\alpha(\beta) &:= \beta\text{th common fixed point of } (\varphi_\xi)_{\xi<\alpha} \\
\Gamma_0 &:= \text{least ordinal} > 0 \text{ that is closed under } \varphi
\end{aligned}
$$

Theorem (Feferman, Schütte)

$$\mathrm{Aut}(\mathrm{RA}) = \Gamma_0$$

Finitist arithmetic

Question: What principles are implicit in the actual finitist conception of the set of natural numbers ?

Finitist arithmetic

Question: What principles are implicit in the actual finitist conception of the set of natural numbers ?

Example (Finitist arithmetic FA)

Logical operations: $\wedge$, $\vee$, $\exists$.

(1) $x' = 0 \rightarrow \bot$

(2) $\mathsf{Pd}(x') = x$

(3) $\dfrac{\Gamma \rightarrow P(0) \qquad \Gamma, P(x) \rightarrow P(x')}{\Gamma \rightarrow P(x)}$.

Note that the statements proved are sequents Σ of the form $\Gamma \rightarrow A$, where Γ is a finite sequence (possibly empty) of formulas. The logic is formulated in Gentzen-style.

Generalization of the substitution rule (Subst)

We have to generalize the substitution rule (Subst) to rules of inference:

Substitution rule (Subst')

Given that the rule of inference

$$\frac{\Sigma_1, \Sigma_2, \ldots, \Sigma_n}{\Sigma}$$

is *derivable*, we can adjoin each of its substitution instances

$$\frac{\Sigma_1[\bar{B}/\bar{P}],\ \Sigma_2[\bar{B}/\bar{P}], \ldots, \Sigma_n[\bar{B}/\bar{P}]}{\Sigma[\bar{B}/\bar{P}]}$$

as a new rule of inference.

The proof theory of the three unfolding systems for FA

The full unfolding of FA includes the basic logical operations as operations on predicates as well as *Join*.

Theorem (Feferman, Str.)

All three unfolding systems for finitist arithmetic, $\mathcal{U}_0(\mathsf{FA})$, $\mathcal{U}_1(\mathsf{FA})$ and $\mathcal{U}(\mathsf{FA})$ are proof-theoretically equivalent to Skolem's Primitive Recursive Arithmetic PRA.

Support of Tait's informal analysis of finitism.

Aim of this section

In the following

- We will study a natural bar rule BR leading to extensions $\mathcal{U}_0^+(\mathsf{FA})$, $\mathcal{U}_1^+(\mathsf{FA})$ and $\mathcal{U}^+(\mathsf{FA})$ of our unfolding systems for finitism

Aim of this section

In the following

- We will study a natural bar rule BR leading to extensions $\mathcal{U}_0^+(\mathsf{FA})$, $\mathcal{U}_1^+(\mathsf{FA})$ and $\mathcal{U}^+(\mathsf{FA})$ of our unfolding systems for finitism
- The so-obtained extensions will all have the strength of Peano arithmetic PA

Aim of this section

In the following

- We will study a natural bar rule BR leading to extensions $\mathcal{U}_0^+(\mathsf{FA})$, $\mathcal{U}_1^+(\mathsf{FA})$ and $\mathcal{U}^+(\mathsf{FA})$ of our unfolding systems for finitism
- The so-obtained extensions will all have the strength of Peano arithmetic PA
- This shows one way how Kreisel's analysis of extended finitism fits in our framework

Defining $\mathcal{U}_0^+(\mathsf{FA})$: Preliminaries

- Let $\prec$ be a binary relation whose characteristic function is given by a closed term $t_\prec$ so that $\mathcal{U}_0(\mathsf{FA})$ proves $t_\prec : \mathsf{N}^2 \to \{0, 1\}$. We write $x \prec y$ instead of $t_\prec xy = 0$ and further assume that $\prec$ is a linear ordering with least element 0, provably in $\mathcal{U}_0(\mathsf{FA})$.

Defining $\mathcal{U}_0^+(\mathsf{FA})$: Preliminaries

- Let $\prec$ be a binary relation whose characteristic function is given by a closed term $t_\prec$ so that $\mathcal{U}_0(\mathsf{FA})$ proves $t_\prec : \mathsf{N}^2 \to \{0, 1\}$. We write $x \prec y$ instead of $t_\prec xy = 0$ and further assume that $\prec$ is a linear ordering with least element 0, provably in $\mathcal{U}_0(\mathsf{FA})$.
- Let f denote a new constant of our applicative language. There are no non-logical axioms for f; it serves as an anonymous function from N to N, representing a possibly infinite descending sequence along a given ordering.

Expressing wellfoundedness

The rule NDS$(\mathsf{f}, \prec)$ says that for each possibly infinite descending chain f w.r.t. $\prec$ there is an x such that $\mathsf{f}x = 0$. Formally, it is given as follows:

Expressing wellfoundedness

The rule $\mathsf{NDS}(\mathsf{f}, \prec)$ says that for each possibly infinite descending chain f w.r.t. $\prec$ there is an x such that $\mathsf{f}x = 0$. Formally, it is given as follows:

The rule $\mathsf{NDS}(\mathsf{f}, \prec)$

$$\frac{\begin{array}{c} u \in \mathsf{N} \;\rightarrow\; \mathsf{f}u \in \mathsf{N}, \\ u \in \mathsf{N},\, \mathsf{f}u \neq 0 \;\rightarrow\; \mathsf{f}(u') \prec \mathsf{f}u, \\ u \in \mathsf{N},\, \mathsf{f}u = 0 \;\rightarrow\; \mathsf{f}(u') = 0 \end{array}}{(\exists x \in \mathsf{N})(\mathsf{f}x = 0)}$$

Formulating the bar rule

Let $\bar{s^r} = s^r_1, \ldots, s^r_n$ and $\bar{s^p} = s^p_1, \ldots, s^p_n$ be sequences of terms of length n. Accordingly, let $\bar{t^r} = t^r_1, \ldots, t^r_m$ and $\bar{t^p} = t^p_1, \ldots, t^p_m$ be sequences of terms of length m. The superscripts 'r' and 'p' stand for recursion and parameter, respectively.

Formulating the bar rule

Let $\bar{s^r} = s_1^r, \ldots, s_n^r$ and $\bar{s^p} = s_1^p, \ldots, s_n^p$ be sequences of terms of length n. Accordingly, let $\bar{t^r} = t_1^r, \ldots, t_m^r$ and $\bar{t^p} = t_1^p, \ldots, t_m^p$ be sequences of terms of length m. The superscripts 'r' and 'p' stand for recursion and parameter, respectively.

The bar rule BR

Whenever we have derived the four premises

(1) $\mathsf{NDS}(\mathsf{f}, \prec)$

(2) $x, y \in \mathsf{N} \;\rightarrow\; \bar{s^r} \in \mathsf{N} \land \bar{s^p} \in \mathsf{N}$

(3) $x, y \in \mathsf{N}, \; \bigwedge_i [s_i^r \prec x \supset P(s_i^r, s_i^p)] \;\rightarrow\; \bar{t^r} \in \mathsf{N} \land \bar{t^p} \in \mathsf{N}$

(4) $x, y \in \mathsf{N}, \; \bigwedge_i [s_i^r \prec x \supset P(s_i^r, s_i^p)], \; \bigwedge_j [t_j^r \prec x \supset P(t_j^r, t_j^p)] \;\rightarrow\; P(x, y)$

we can infer $x \in \mathsf{N} \land y \in \mathsf{N} \;\rightarrow\; P(x, y)$.

How to use the rule: nested recursion

In $\mathcal{U}_0^+(\mathsf{FA})$, whenever we have derived $\mathsf{NDS}(\mathsf{f}, \prec)$, then we can use the bar rule BR in order to justify *nested* recursion along $\prec$.

How to use the rule: nested recursion

In $\mathcal{U}_0^+(\mathrm{FA})$, whenever we have derived $\mathrm{NDS}(\mathrm{f}, \prec)$, then we can use the bar rule BR in order to justify *nested* recursion along $\prec$.

Example (Justifying nested recursion using BR)

As usual, $(r)_x$ is r if $r \prec x$ and 0 otherwise. Define F by $(x \neq 0)$

$$F(0, y) \simeq H(y)$$

$$F(x, y) \simeq G(x, y, \quad F(\ k(x, y, F(l(x, y)_x, y))_x, \quad p(x, y, F(m(x, y)_x, y))\ ))$$

We set $n = 2$ and $m = 1$ and choose the following terms:

$$\begin{aligned} s_1^r &= l(x, y)_x, & s_1^p &= y \\ s_2^r &= m(x, y)_x, & s_2^p &= y \\ t_1^r &= k(x, y, F(l(x, y)_x, y))_x, & t_1^p &= p(x, y, F(m(x, y)_x, y)) \end{aligned}$$

Summarizing ...

We summarize our previous findings in the following theorem.

Theorem

Assume that $\mathsf{NDS}(\mathsf{f}, \prec)$ *is derivable in* $\mathcal{U}_0^+(\mathsf{FA})$. *Then* $\mathcal{U}_0^+(\mathsf{FA})$ *justifies nested recursion along* $\prec$.

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α
- Let us write $\mathsf{NDS}(\mathsf{f}, \alpha)$ instead of $\mathsf{NDS}(\mathsf{f}, \prec_\alpha)$

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α
- Let us write $\mathsf{NDS}(\mathsf{f}, \alpha)$ instead of $\mathsf{NDS}(\mathsf{f}, \prec_\alpha)$
- Aim at showing that $\mathcal{U}_0^+(\mathsf{FA})$ derives $\mathsf{NDS}(\mathsf{f}, \alpha)$ for each $\alpha < \varepsilon_0$

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α
- Let us write $\mathsf{NDS}(\mathsf{f}, \alpha)$ instead of $\mathsf{NDS}(\mathsf{f}, \prec_\alpha)$
- Aim at showing that $\mathcal{U}_0^+(\mathsf{FA})$ derives $\mathsf{NDS}(\mathsf{f}, \alpha)$ for each $\alpha < \varepsilon_0$
- Use one direction of Tait's famous result, i.e. that nested recursion on $\omega\alpha$ entails ordinary recursion on ω^α or, more useful in our setting, nested recursion on $\omega\alpha$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\alpha)$

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α
- Let us write $\mathsf{NDS}(\mathsf{f}, \alpha)$ instead of $\mathsf{NDS}(\mathsf{f}, \prec_\alpha)$
- Aim at showing that $\mathcal{U}_0^+(\mathsf{FA})$ derives $\mathsf{NDS}(\mathsf{f}, \alpha)$ for each $\alpha < \varepsilon_0$
- Use one direction of Tait's famous result, i.e. that nested recursion on $\omega\alpha$ entails ordinary recursion on ω^α or, more useful in our setting, nested recursion on $\omega\alpha$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\alpha)$
- Tait's argument can be directly formalized in $\mathcal{U}_0^+(\mathsf{FA})$

Tait's seminal 1961 paper

William Tait: Nested recursion, Mathematische Annalen, 143 (1961).

- For each ordinal $\alpha < \varepsilon_0$ let $\prec_\alpha$ be a primitive recursive standard wellordering $\prec_\alpha$ of ordertype α
- Let us write $\mathsf{NDS}(\mathsf{f}, \alpha)$ instead of $\mathsf{NDS}(\mathsf{f}, \prec_\alpha)$
- Aim at showing that $\mathcal{U}_0^+(\mathsf{FA})$ derives $\mathsf{NDS}(\mathsf{f}, \alpha)$ for each $\alpha < \varepsilon_0$
- Use one direction of Tait's famous result, i.e. that nested recursion on $\omega\alpha$ entails ordinary recursion on ω^α or, more useful in our setting, nested recursion on $\omega\alpha$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\alpha)$
- Tait's argument can be directly formalized in $\mathcal{U}_0^+(\mathsf{FA})$
- For more details, see the *Appendix*

$\mathcal{U}_0^+(\mathsf{FA})$: Lower bounds

Theorem

Provably in $\mathcal{U}_0^+(\mathsf{FA})$, nested recursion along $\omega\alpha$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\alpha)$.

Corollary

We have for each $\alpha < \varepsilon_0$ that $\mathcal{U}_0^+(\mathsf{FA})$ derives $\mathsf{NDS}(\mathsf{f}, \alpha)$.

Upper bounds

$\mathcal{U}_0^+(\mathsf{FA})$ is readily interpretable in the subsystem of second order arithmetic ACA_0 as follows:

- Fix a *function variable f* in $\mathcal{L}_2$ and translate $(a \cdot b)$ as $\{a\}^f(b)$, where $\{n\}^f$ for $n = 0, 1, 2, \ldots$ is a enumeration of the functions that are partial recursive in f
- The constant f is interpreted as a natural number i so that $\{i\}^f(x) \simeq f(x)$
- The translation of BR is validated by observing that ACA_0 proves $\mathsf{WF}(\prec) \rightarrow \mathsf{TI}(\prec, A)$ for each arithmetic formula A

On top of this interpretation, one models predicates (including join) to show that even the strength of $\mathcal{U}^+(\mathsf{FA})$ does not go beyond PA.

The proof theory of the three unfolding systems for FA with bar rule

Theorem (Feferman, Str.)

All three unfolding systems for finitist arithmetic with bar rule, $\mathcal{U}_0^+(\mathsf{FA})$, $\mathcal{U}_1^+(\mathsf{FA})$ and $\mathcal{U}^+(\mathsf{FA})$ are proof-theoretically equivalent to Peano arithmetic PA.

Support of Kreisel's analysis of extended finitism.

Tait's argument in a nutshell

(based on a compact presentation of W. Tait in a personal communication with S. Feferman)

Tait's argument in a nutshell

(based on a compact presentation of W. Tait in a personal communication with S. Feferman)

We want to show that nested recursion on $\omega\delta$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\delta)$.

In the following we will work with (codes of) ordinals below ε_0 and assume that $<$ denotes the corresponding ordering relation on them.

Tait's argument in a nutshell

(based on a compact presentation of W. Tait in a personal communication with S. Feferman)

We want to show that nested recursion on $\omega\delta$ entails $\mathsf{NDS}(\mathsf{f}, \omega^\delta)$.

In the following we will work with (codes of) ordinals below ε_0 and assume that $<$ denotes the corresponding ordering relation on them.

A possibly infinite descending sequence f in ω^δ

Let f be a fixed function from ω to ω^δ satisfying for all natural numbers n the condition

$$f(n) > 0 \rightarrow f(n+1) < f(n) \quad \text{and} \quad f(n) = 0 \rightarrow f(n+1) = 0. \qquad (\star)$$

Ordinal-theoretic preliminaries

Given an ordinal $\alpha < \omega^\delta$ in its normal form

$$\alpha = \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n$$

where $\delta > \alpha_1 > \cdots > \alpha_n$ and $a_i < \omega$ $(1 \le i \le n)$, we set

$$\begin{array}{rcl} \alpha\{i\} & = & \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_k \quad (k = \min(n, i)) \\ \alpha[i] & = & \begin{cases} \omega\alpha_i + a_i & \text{if } 0 < i \le n \\ 0 & \text{if } n < i \end{cases} \end{array}$$

Clearly, $\alpha[i] < \omega\delta$ and $0\{i\} = 0[i] = 0$. Further, we have the following important property.

Ordinal-theoretic preliminaries (ctd.)

Lemma

We have that $\alpha\{i+1\} < \beta\{i+1\}$ *if and only if*

$$\alpha\{i\} < \beta\{i\} \vee (\alpha\{i\} = \beta\{i\} \wedge \alpha[i+1] < \beta[i+1]).$$

The crucial property

The crucial step in Tait's argument is to define a function $\mu : \omega^2 \to \omega$ such that (writing $\mu_i(j)$ for $\mu(i,j)$)

The property ($\star\star$)

$$f(j + \mu_i(j)) = 0 \vee f(j + \mu_i(j))\{i\} < f(j)\{i\} \qquad (\star\star)$$

It will then suffice to choose $\mu_0(0)$ as a root of f, since according to ($\star\star$), $f(\mu_0(0)) = 0$.

Defining $\mu_i(j)$

The definition of $\mu_i(j)$ will be by nested recursion on $f(j)[i+1] < \omega\delta$.

Defining $\mu_i(j)$

The definition of $\mu_i(j)$ will be by nested recursion on $f(j)[i+1] < \omega\delta$.

- Let n be the number of summands in the normal form of $f(j)$. If $i \geq n$, we may simply set $\mu_i(j) = 1$; then $(\star\star)$ holds due to property $(\star)$ of our given function f.
- So assume $0 \leq i < n$. Because $f(j)[i+2] < f(j)[i+1]$, we can use $\mu_{i+1}(j) = \bar{\mu}$ in the definition of $\mu_i(j)$. Hence, according to $(\star\star)$ we have for $\bar{\mu}$ that either (1) or (2) holds:

$$f(j+\bar{\mu}) = 0 \tag{1}$$

$$f(j+\bar{\mu})\{i+1\} < f(j)\{i+1\} \tag{2}$$

If (1) holds, we set $\mu_i(j) = \bar{\mu}$.

Defining $\mu_i(j)$ (ctd.)

- In case of (2), we use the lemma above to obtain one of the following properties (3) or (4):

$$f(j+\bar{\mu})\{i\} < f(j)\{i\} \tag{3}$$

$$f(j+\bar{\mu})\{i\} = f(j)\{i\} \wedge f(j+\bar{\mu})[i+1] < f(j)[i+1] \tag{4}$$

  In case of (3), we again set $\mu_i(j) = \bar{\mu}$.

- If (4) holds, then clearly $\mu_i(j+\bar{\mu}) = \bar{\bar{\mu}}$ is defined. In this case we set $\mu_i(j) = \bar{\mu} + \bar{\bar{\mu}}$. Then we can verify, using property $(\star\star)$ for $\bar{\bar{\mu}}$, that one of the following conditions (5) or (6) holds:

$$f(j+\mu_i(j)) = f((j+\bar{\mu})+\bar{\bar{\mu}}) = 0 \tag{5}$$

$$f(j+\mu_i(j))\{i\} < f(j+\bar{\mu})\{i\} = f(j)\{i\} \tag{6}$$

  This is as desired and concludes the definition of $\mu_i(j)$.

Summarizing...

Summarizing, $\mu_i(j)$ has been defined to satisfy the following equation:

The recursive definition of $\mu_i(j)$

$$\mu_i(j) = \begin{cases} 1 & \text{if } i \geq n \\ \mu_{i+1}(j) & \text{if } f(j + \mu_{i+1}(j)) = 0 \text{ or} \\ & \quad f(j + \mu_{i+1}(j))\{i\} < f(j)\{i\} \\ \mu_{i+1}(j) + \mu_i(j + \mu_{i+1}(j)) & \text{else} \end{cases}$$

It is now easy to explicitely express the definition of $\mu_i(j)$ as a nested recursion on $\omega\delta$.

The end

Thank you for your attention!