The μ quantification operator in explicit mathematics with universes and iterated fixed point theories with ordinals

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Abstract

This paper is about two topics: 1. systems of explicit mathematics with universes and a non-constructive quantification operator $\boldsymbol{\mu}$; 2. iterated fixed point theories with ordinals. We give a proof-theoretic treatment of both families of theories; in particular, ordinal theories are used to get upper bounds for explicit theories with finitely many universes.

1 Introduction

The two major frameworks for explicit mathematics that were introduced in Feferman [4, 5] are the theories T_0 and T_1. T_1 results from T_0 by strengthening the applicative axioms by the so-called non-constructive $\boldsymbol{\mu}$ operator. Although highly non-constructive, $\boldsymbol{\mu}$ is predicatively acceptable and makes quantification over the natural numbers explicit. While the proof theory of T_0 is well-known since the early eighties (cf. Feferman [4, 5], Feferman and Sieg [10], Jäger [14], Jäger and Pohlers [17]), the corresponding investigations of subystems of T_1 have been completed only recently by Feferman and Jäger [9, 8] and Glaß and Strahm [12].

Universes are a frequently studied concept in constructive mathematics at least since Martin-Löf (cf. e.g. [21]). In the context of predicative, explicit mathematics *without* $\boldsymbol{\mu}$ operator they have been proof-theoretically analyzed by Feferman [6] and Marzetta [23, 22]. The aim of this work is to clarify the role of the non-constructive $\boldsymbol{\mu}$ operator in this setting. One of our results shows that $\boldsymbol{\mu}$ does not raise the proof-theoretic strength of the main theory studied in [23, 22].

The crucial concept used in [9, 8, 12] in order to treat an application operation with $\boldsymbol{\mu}$ are fixed point theories with ordinals which were introduced by Jäger in [16]. They will be the key to the proof-theoretic analysis of the theories studied in this article, too. More precisely, we define suitable extensions $\widehat{\mathsf{ID}\Omega}_n$ of Peano arithmetic with ordinals refining the fixed point theories $\widehat{\mathsf{ID}}_n$ of Feferman [6]. They are not

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only needed for the adequate treatment of $\boldsymbol{\mu}$, but also facilitate the formalization of standard structures for explicit mathematics as well as the investigation of restricted induction principles on the natural numbers.

The detailed plan of this article is as follows. We first present the formal framework of the theory UTN of universes, types and names as it is introduced in Marzetta [23, 22]. UTN is too weak in order to prove the existence of universes, and therefore, we add the so-called limit axiom, (Lim), saying that every type has a name which belongs to a universe. The applicative fragment of the so-obtained theory $\mathsf{UTN}+(\mathsf{Lim})$ may be strengthened by the axioms for the non-constructive $\boldsymbol{\mu}$ operator, yielding the strongest theory studied in this article, $\mathsf{UTN}+(\mathsf{Lim})+(\boldsymbol{\mu})$.

In a next step, following Marzetta [23, 22], we establish that the limit axiom (Lim) can be eliminated in favor of finitely many universes. More precisely, the system $\mathsf{UTN}+(\mathsf{Lim})[+(\boldsymbol{\mu})]$ is reduced to the family of theories $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$ $(k<\omega)$ by an asymmetric interpretation argument. Here UTN_k denotes the extension of UTN which claims the existence of exactly k universes.

In order to proof-theoretically analyze the theories $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$, we then introduce a family of iterated fixed point theories with ordinals $\widehat{\mathsf{ID}\Omega}_n$ $(n<\omega)$. In contrast to the plain fixed point theories $\widehat{\mathsf{ID}}_n$ (cf. Feferman [6]), $\widehat{\mathsf{ID}\Omega}_n$ includes a very weak form of foundation, namely $\Delta_0^{\Omega,n}$ induction on the ordinals. Moreover, $\widehat{\mathsf{ID}\Omega}_n$ admits the formulation of very restricted forms of complete induction on the natural numbers, yielding natural subsystems $\widehat{\mathsf{ID}\Omega}^r_n$.

In a subsequent step we outline embeddings of UTN_k into $\widehat{\mathsf{ID}\Omega}^r_{k+1}$, and $\mathsf{UTN}_k+(\boldsymbol{\mu})$ into $\widehat{\mathsf{ID}\Omega}^r_{k+2}$, demonstrating that fixed point theories with ordinals provide a suitable framework for standard structures for explicit mathematics as well as formalized Π^1_1 recursion theory.

Finally, we give a complete proof-theoretic treatment of the theories $\widehat{\mathsf{ID}\Omega}_n$ and $\widehat{\mathsf{ID}\Omega}^r_n$. We first show that $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ is a conservative extension of $\widehat{\mathsf{ID}\Omega}_n$, thus generalizing a result of Jäger [16]. Then we provide an ordinal analysis of the theories $\widehat{\mathsf{ID}\Omega}_n$, which yields $|\widehat{\mathsf{ID}\Omega}_n|=\gamma_n$, where $\gamma_0=\varepsilon_0$ and $\gamma_{n+1}=\varphi\gamma_n 0$. The main techniques applied are partial cut elimination and asymmetric interpretation.

To sum up, the main results of this article can be stated as follows; here '$\equiv$' denotes the notion of proof-theoretic equivalence as it is defined, e.g., in Feferman [7].

$$\mathsf{UTN}_k \equiv \widehat{\mathsf{ID}\Omega}^r_{k+1} \equiv \widehat{\mathsf{ID}\Omega}_k \equiv \widehat{\mathsf{ID}}_k \equiv (\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_{k-1}};$$

$$\mathsf{UTN}_k+(\boldsymbol{\mu}) \equiv \widehat{\mathsf{ID}\Omega}^r_{k+2} \equiv \widehat{\mathsf{ID}\Omega}_{k+1} \equiv \widehat{\mathsf{ID}}_{k+1} \equiv (\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_k};$$

$$\mathsf{UTN}+(\mathsf{Lim}) \equiv \mathsf{UTN}+(\mathsf{Lim})+(\boldsymbol{\mu}) \equiv \bigcup_{k<\omega}\widehat{\mathsf{ID}\Omega}_k \equiv \bigcup_{k<\omega}\widehat{\mathsf{ID}}_k \equiv (\Pi^0_1\text{-}\mathsf{CA})_{<\Gamma_0}.$$

2 The theories $\mathsf{UTN} + (\mathsf{Lim})$ and UTN_k (with $\boldsymbol{\mu}$)

In this section we first introduce the formal framework for universes, types and names, together with a limit axiom and the non-constructive $\boldsymbol{\mu}$ operator. We subsequently eliminate the limit axiom in favor of finitely many universes.

2.1 The formal framework

In the sequel we present the exact syntax, axioms and rules of the theory $\mathsf{UTN}+(\mathsf{Lim})$ as it is introduced in Marzetta [23, 22], and we give the axioms of the quantification operator $\boldsymbol{\mu}$.

2.1.1 Syntax

In this section we introduce the language $\mathrm{L_U}$ of the theories for universes, types and names that we are going to consider in the following. It is a two-sorted language, with countable lists of *individual variables* $a, b, c, f, g, h, x, y, z, \ldots$ and *type variables* $A, B, C, X, Y, Z, \ldots$ (both possibly with subscripts). $\mathrm{L_U}$ includes also several *individual constants:* k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projection), 0 (zero), s_N (successor), p_N (predecessor), d_N (definition by numerical cases), $\boldsymbol{\mu}$ (non-constructive quantification operator), j (join), and constants c_n (comprehension) for every $n \in \mathbb{N}$. There is only one binary function symbol $\cdot$ for (partial) application of individuals to individuals.

The relation symbols of $\mathrm{L_U}$ include equality for both individuals and types, the unary predicate symbols $\downarrow$ (defined) and N (natural numbers) on individual terms, U (universes) on types, and the binary symbols $\in$ (membership), n (naming) between individuals and types. These predicate symbols are called *positive.* Since we will use a Tait calculus for our metamathematical investigations, we introduce a *negative* predicate symbol $\bar{P}$ for every positive one; $\bar{P}$ is intended to represent the complement of P, and therefore has the same arity as P. $\neq, \not<, \ldots$ stand for the rather cumbersome $\bar{=}, \bar{<}, \ldots$

The *individual terms* (denoted by $r, s, t, \ldots$) are built up from individual variables and individual constants by means of $\cdot$, with the usual conventions for application in combinatory logic or λ-calculus. We write (s, t) for $\mathsf{p}st$, s' for $\mathsf{s}_N s$, 1 instead of $0'$ and so on. The *type terms* (denoted by $R, S, T, \ldots$) are the type variables[1].

Positive atoms have the form: $s = t, S = T, s\downarrow, N(s), U(S), s \in T$ or $n(s, T)$; *negative atoms* are obtained in the same way, with the corresponding negative predicate symbols. *Formulae* $(E, F, G, H, \ldots, V)$ are built up from atoms, using the connectives $\wedge, \vee$ and quantifiers $\forall, \exists$ ranging over both sorts. Negation is an operation on formulae defined by $\neg P(\_) := \bar{P}(\_)$ and $\neg \bar{P}(\_) := P(\_)$ on atomic formulae, and by De

[1] Later we will consider extensions of $\mathrm{L_U}$ with type constants.

Morgan's laws on compound formulae; $(E \rightarrow F)$ will be considered as an abbreviation for $(\neg E \vee F)$, and, similarly, $(E \leftrightarrow F) := ((E \rightarrow F) \wedge (F \rightarrow E))$, $(\forall x \in S)\ E := \forall x(x \in S \rightarrow E)$, and $(\exists x \in S)\ E := \exists x(x \in S \wedge E)$. Further useful abbreviations are: Kleene's equality between individual terms, $s \simeq t := ((s\downarrow \vee t\downarrow) \rightarrow s = t)$; $s \in N := N(s)$.

For the analysis of the formal systems we aim at, we will not have to go into the structure of formulae belonging to the classes defined below; this fact also motivates the subsequent definition of *rank* of a formula.

Definition 1 An L_U formula is called $\Sigma+$ (respectively $\Pi-$), if it does not contain any universal (resp. existential) quantifier over types, nor any subformula of the form $\bar{n}(s,T)$ (resp. $n(s,T)$).

Definition 2 The *rank* $rn(E)$ of a L_U formula E is defined inductively by:

1. If E is a $\Sigma+$ or a $\Pi-$ formula, then $rn(E) := 0$.
2. If E has the form $(F \wedge G)$ or $(F \vee G)$, then $rn(E) := \max(rn(F), rn(G)) + 1$.
3. If E has one of the forms $\forall x E, \exists x E, \forall X E$, or $\exists X E$, then $rn(E) := rn(F) + 1$.

2.1.2 Axioms and rules

UTN is formulated as a Tait calculus for finite sets $(\Gamma, \Delta, \ldots)$ of L_U formulae which are interpreted disjunctively (see e.g. Schwichtenberg [26]). If E is an L_U formula, then Γ, E is a shorthand for $\Gamma \cup \{E\}$. We will present the axioms showing only a most general instance of their so-called *principal formulae*: therefore, it is understood that when we speak e.g. of a formula $E(a_1, \ldots, a_m, B_1, \ldots, B_n)$ as an axiom of UTN, or of some extension thereof, actually all finite sets of L_U formulae of the form

$$\Gamma, \neg s_1\downarrow, \ldots, \neg s_m\downarrow, E(s_1, \ldots, s_m, T_1, \ldots, T_n),$$

where the s_i's are arbitrary individual terms and the T_j's arbitrary type terms of L_U, are axioms of the formal system UTN. The logical axioms of UTN are the axioms of classical logic of partial terms by Beeson [1] for the individuals, and classical logic with equality for the types (see [22] for the details). The non-logical axioms of UTN are divided into the following groups:

I. Partial Combinatory Algebra

(1) $\mathsf{k}ab = a$,

(2) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$.

II. Pairing and Projection

(3) $\mathsf{p}_0(a,b) = a \wedge \mathsf{p}_1(a,b) = b.$

III. Natural Numbers

(4) $N(0) \wedge \forall x(N(x) \rightarrow N(x'))$,

(5) $(\forall x \in N)(\neg x' = 0 \wedge \mathsf{p}_N(x') = x)$,

(6) $(\forall x \in N)(\neg x = 0 \rightarrow (N(\mathsf{p}_N x) \wedge (\mathsf{p}_N x)' = x))$.

IV. Definition by Numerical Cases

(7) $N(a) \wedge N(b) \wedge a = b \rightarrow \mathsf{d}_N cdab = c$,

(8) $N(a) \wedge N(b) \wedge \neg a = b \rightarrow \mathsf{d}_N cdab = d$.

V. Extensionality

(9) $\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B$.

VI. Ontological Axioms

(10) $n(a,B) \wedge n(a,C) \rightarrow B = C$,

(11) $\exists x\, n(x,A)$.

Comprehension is restricted to so-called *elementary formulae*, i.e. formulae without type quantifiers and without occurrences of the predicates n and U, nor of their complementary predicates $\bar{n}, \bar{U}$.
For $\vec{s} = s_1, \ldots, s_m$ and $\vec{T} = T_1, \ldots, T_m$, let $n(\vec{s}, \vec{T}) := n(s_1, T_1) \wedge \ldots \wedge n(s_m, T_m)$.

VII. Elementary Comprehension
This schema contains an instance of the following axioms for every elementary formula $E(x, \vec{a}, \vec{B})$ (all free variables indicated) with Gödel number $m = \ulcorner E(x, \vec{a}, \vec{B}) \urcorner$ and 'fresh' variables C and Z:

(12) $\exists Z \forall x(x \in Z \leftrightarrow E(x, \vec{a}, \vec{B}))$,

(13) $\neg n(\vec{b}, \vec{B}), \neg\forall x(x \in C \leftrightarrow E(x, \vec{a}, \vec{B})), n(\mathsf{c}_m \vec{a}\vec{b}, C)$.[2]

[2]The meaning of axiom (13) is obviously $n(\vec{b}, \vec{B}) \wedge \forall x(x \in C \leftrightarrow E(x, \vec{a}, \vec{B})) \rightarrow n(\mathsf{c}_m \vec{a}\vec{b}, C)$; it has been split into three principal formulae in order to allow partial cut elimination up to $\Sigma+/\Pi-$ formulae.

Example 3 The formula $N(x)$ is elementary. Let i be its Gödel number. As a consequence of the axioms above there is a type whose elements are exactly the natural numbers, and which is represented by the constant c_i. With a slight abuse of notation we define $\mathsf{c}_N := \mathsf{c}_i$.

VIII. Join
is the second type formation scheme of UTN, and is stated as a pair of inference rules:

$$\frac{\Gamma, t\downarrow \wedge\, n(s,S) \wedge (\forall x \in S)\exists X\ n(tx,X)}{\Gamma, \exists Z(n(\mathbf{j}st,Z) \wedge Z{\subset}\Sigma(S,t))}\ (J_1)$$

$$\frac{\Gamma, t\downarrow \wedge\, n(s,S) \wedge (\forall x \in S)\exists X\ n(tx,X)}{\Gamma, \exists Z(n(\mathbf{j}st,Z) \wedge Z{\supset}\Sigma(S,t))}\ (J_2)$$

where:
$Z{\subset}\Sigma(S,t) := \forall z(z \in Z \to z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in S \wedge \exists X(n(t(\mathsf{p}_0 z), X) \wedge \mathsf{p}_1 z \in X))$
$Z{\supset}\Sigma(S,t) := \forall z(z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in S \wedge \forall X(n(t(\mathsf{p}_0 z), X) \to \mathsf{p}_1 z \in X) \to z \in Z)$.
The formulae shown in the conclusion are called the principal formulae of these inferences.

In the following we will use the abbreviation: $S \subseteq T := \forall x(x \in S \to x \in T)$. Since types have individuals as elements, and these can in turn be names of types, a type S can be considered as an element of a type T in the following sense:

$$S \,\dot{\in}\, T := \exists x(n(x,S) \wedge x \in T).$$

IX. Ontological Axioms for Universes

(14) $U(A) \wedge b \in A \to \exists X n(b,X)$,

(15) $U(A) \wedge U(B) \to (A \,\dot{\in}\, B \vee A = B \vee B \,\dot{\in}\, A)$,

(16) $U(A) \wedge U(B) \wedge A \,\dot{\in}\, B \to A \subseteq B$.

For $s = s_1, \dots, s_m$, let $\vec{s} \in T := s_1 \in T \wedge \dots \wedge s_m \in T$.

X. Closure Properties of Universes

(17) $U(C) \wedge \vec{b} \in C \to \mathsf{c}_m \vec{a}\vec{b} \in C$,
for all Gödel numbers m of elementary formulae $E(x, \vec{a}, \vec{B})$ (all free variables indicated, $\vec{b}$ and $\vec{B}$ of the same length).

(18) $U(C) \wedge a \in C \wedge n(a,A) \wedge (\forall x \in A)\ fx \in C \to \mathbf{j}af \in C$.

Remark 4 Axiom (17) entails $U(A) \to \mathsf{c}_N \in A$.

XI. Induction Axiom

(19) $0 \in A \wedge (\forall x \in N)(x \in A \rightarrow x' \in A) \rightarrow (\forall x \in N)\; x \in A.$

The inference rules for the logical connectors $(\wedge), (\vee)$ and for the quantifiers ranging over types $(\forall^1), (\exists^1)$, as well as the rule for the universal quantifier over individuals $(\forall^0)$ are as usual. The rule for the existential quantifier over individuals has the following form:

$$\frac{\Gamma, E(s) \quad \Gamma, s\downarrow}{\Gamma, \exists x E(x)} \; (\exists^0)$$

The only structural rule is the cut rule:

$$\frac{\Gamma, E \quad \Gamma, \neg E}{\Gamma} \; (cut)$$

(here E is called the *cut formula* of the inference, and $rn(E)$ is called the *cut rank*). So far the formal system UTN. Notice that $\mathsf{UTN} \nvdash \exists X U(X)$; the axiom that enforces the existence of universes is the following.

Limit Axiom

$(\mathsf{Lim}) \quad \exists Y(U(Y) \wedge A \,\dot{\in}\, Y)$

The axiomatization of the non-constructive $\boldsymbol{\mu}$ operator is that of Jäger and Strahm [20]. Let $r \in S \rightarrow T := r\downarrow \wedge (\forall x \in S)\; rx \in T$.

Non-constructive $\boldsymbol{\mu}$ Operator

$(\mu.1) \quad (f \in N \rightarrow N) \leftrightarrow \boldsymbol{\mu} f \in N,$

$(\mu.2) \quad (f \in N \rightarrow N) \wedge (\exists x \in N)\; fx = 0 \rightarrow f(\boldsymbol{\mu} f) = 0.$

For extensions Th of UTN, natural numbers n, r, and finite sets Γ of $\mathrm{L_U}$ formulae, the relations $\mathsf{Th} \vdash^{n}_{r} \Gamma$ ("Th proves Γ with length n and cut rank less than r"), $\mathsf{Th} \vdash_{r} \Gamma$, and $\mathsf{Th} \vdash \Gamma$ are defined as usual. If Γ is a singleton $\{E\}$, we omit the curly braces '$\{\}$'.
We are now going to give a proof-theoretic treatment of the systems $\mathsf{UTN} + (\mathsf{Lim})$ and $\mathsf{UTN} + (\mathsf{Lim}) + (\boldsymbol{\mu})$. ATR_0 can be used as a lower bound for the first (see [23]) and, a fortiori, for the second.

Theorem 5 ATR_0 *can be embedded in* $\mathsf{UTN} + (\mathsf{Lim})$

Hence $\Gamma_0 \leq |\mathsf{UTN} + (\mathsf{Lim})| \leq |\mathsf{UTN} + (\mathsf{Lim}) + (\boldsymbol{\mu})|$. Therefore we can concentrate on the upper bounds.

2.2 Reduction to finitely many universes

Let $k \in \mathbb{N}$. The language $\mathrm{L}_{\mathrm{U},k}$ is defined as the extension of L_{U} by *type constants* D_i $(i < k)$ for universes. The formal system UTN_k is obtained by extending UTN to the language $\mathrm{L}_{\mathrm{U},k}$ and adding the axioms

$$\begin{array}{ll} U(\mathsf{D}_i) & (0 \leq i < k), \\ \mathsf{D}_i \,\dot{\in}\, \mathsf{D}_{i+1} & (0 \leq i < k-1). \end{array}$$

This section is concerned with the reduction of $\mathsf{UTN} + (\mathsf{Lim})[+(\boldsymbol{\mu})]$ to $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$ for some $k \geq 0$. Intuitively speaking this means that the number of universes we really "need" in order to prove $\Sigma+$ sentences is finite. The main result of this section is the following

Theorem 6 *Let E be a $\Sigma+$-sentence of L_{U}. Assume $\mathsf{UTN} + (\mathsf{Lim})[+(\boldsymbol{\mu})] \vdash E$. Then there is $k \in \mathbb{N}$ such that $\mathsf{UTN}_k[+(\boldsymbol{\mu})] \vdash E$.*

To prove this we proceed in two steps: first, we prove partial cut elimination up to $\Sigma+$ and $\Pi-$ cuts, and then, after an asymmetric interpretation, we prove a theorem similar to the above, which uses quasi cut-free proofs.

Theorem 7 (Partial cut elimination) *Assume $\mathsf{UTN} + (\mathsf{Lim})[+(\boldsymbol{\mu})] \vdash \Gamma$. Then $\mathsf{UTN} + (\mathsf{Lim})[+(\boldsymbol{\mu})] \vdash_{\overline{1}} \Gamma$.*

Proof Observe that the principal formulae of the axioms and of the join rules (J_1) and (J_2) are all $\Sigma+$ or $\Pi-$. Then the proof proceeds straightforwardly. $\Box$

For natural numbers m, n and for every formula E of L_{U} we define its asymmetric interpretation $E^{(m,n)}$ essentially by bounding the universal type quantifiers by the universe D_m, and the existential ones by D_n. The aim is to get control over the level of the universe where the witnesses for the existential quantifiers of a theorem of the theory are to be found; this is done by looking at its proof. Let us write $(\forall X \,\dot{\in}\, S)E$ for $\forall X(X \,\dot{\in}\, S \rightarrow E)$ and $(\exists X \,\dot{\in}\, S)E$ for $\exists X(X \,\dot{\in}\, S \wedge E)$.

Definition 8 Let E be a L_{U} formula and $m, n \in \mathbb{N}$. Then $E^{(m,n)}$ is a $\mathrm{L}_{\mathrm{U},\max(m,n)}$ formula defined by the following clauses:

1. $n(s,T)^{(m,n)} := n(s,T) \wedge s \in \mathsf{D}_n$,
2. $\bar{n}(s,T)^{(m,n)} := \neg(n(s,T) \wedge s \in \mathsf{D}_m)$,
3. $F^{(m,n)} := F$ for all other atomic formulae of L_{U},
4. $(\forall X\ F)^{(m,n)} := (\forall X \,\dot{\in}\, \mathsf{D}_m)F^{(m,n)}$,
5. $(\exists X\ F)^{(m,n)} := (\exists X \,\dot{\in}\, \mathsf{D}_n)F^{(m,n)}$,

6. homomorphically for propositional connectives and quantifiers ranging over individuals.

For finite sets Γ of $\mathrm{L_U}$ formulae, $\Gamma^{(m,n)}$ is defined by lifting the operation on formulae.

The asymmetric interpretation does not change elementary formulae; moreover for $\Sigma+$ formulae E and $\Pi-$ formulae F the following properties hold for all $m,n,p<k$:

$$\neg(E^{(m,p)}) = (\neg E)^{(p,n)} \quad \text{and} \quad \neg(F^{(p,n)}) = (\neg F)^{(m,p)} \tag{1}$$

Lemma 9 (Persistency) *Let $m_0, m_1, n_0, n_1 < k$ be natural numbers such that $m_1 \leq m_0$ and $n_0 \leq n_1$, and let E be an $\mathrm{L_U}$ formula. Then $\mathsf{UTN}_k \vdash E^{(m_0,n_0)} \to E^{(m_1,n_1)}$.*

Proof The claim follows by logic from $\mathsf{D}_{m_1} \subseteq \mathsf{D}_{m_0}$ and $\mathsf{D}_{n_0} \subseteq \mathsf{D}_{n_1}$. □

Theorem 10 (Asymmetric interpretation) *Let $\Gamma(\vec{a}, B_1, \ldots, B_i)$ be a finite set of $\Sigma+$ and $\Pi-$ formulae of $\mathrm{L_U}$ with all free variables shown in the list. Let l, m be natural numbers and $n := m + 2^l$. Assume $\mathsf{UTN} + (\mathsf{Lim})[+(\boldsymbol{\mu})] \,\vert\!\frac{l}{1}\, \Gamma$. Then*

$$\mathsf{UTN}_k[+(\boldsymbol{\mu})] \vdash \neg B_1 \,\dot{\in}\, \mathsf{D}_m, \ldots, \neg B_i \,\dot{\in}\, \mathsf{D}_m, \Gamma(\vec{a}, B_1, \ldots, B_i)^{(m,n)}$$

holds for all $k > n$.

Proof By induction on l. If Γ is an instance of the limit axiom, we essentially have to show in UTN_k that $A \,\dot{\in}\, \mathsf{D}_m$ implies:

$$(\exists Y \,\dot{\in}\, \mathsf{D}_n)(U(Y) \wedge \exists x(n(x,A) \wedge x \in \mathsf{D}_n \wedge x \in Y));$$

since $m < n$, $Y := \mathsf{D}_m$ is a witness for this claim. The properties (1) and the persistency lemma handle the case of the cut. For the remaining details see [22]. □

Let us finish this section by shortly addressing the lower bound of $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$. Combining methods of Feferman [6], Feferman and Jäger [9], and Marzetta [23, 22], it is routine to obtain the following lower bounds in terms of theories with iterated elementary comprehension.

Theorem 11 *1. $(\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_{k-1}}$ can be embedded in UTN_k.*

2. $(\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_k}$ can be embedded in $\mathsf{UTN}_k + (\boldsymbol{\mu})$.

Therefore, $\gamma_k \leq |\mathsf{UTN}_k|$ and $\gamma_{k+1} \leq |\mathsf{UTN}_k + (\boldsymbol{\mu})|$.

3 The theories $\widehat{\mathsf{ID}\Omega}_n$ and their application to UTN_k

In this section we first define new iterated fixed point theories with ordinals, which we subsequently use in order to provide embeddings of theories with finitely many universes with (and without) the non-constructive $\boldsymbol{\mu}$ operator.

3.1 The formal framework

Fixed point theories over Peano arithmetic with ordinals have been introduced in Jäger [16], and extended in Jäger and Strahm [19]. They have been used in the proof-theoretic analysis of systems of explicit mathematics with the non-constructive $\boldsymbol{\mu}$ operator in an essential way, cf. Feferman and Jäger [9, 8], Glaß and Strahm [12], and Jäger and Strahm [20].

In the sequel we define suitable extensions $\widehat{\mathsf{ID}\Omega}_n$ of Peano arithmetic with ordinals, refining the well-known fixed point theories $\widehat{\mathsf{ID}}_n$ of Feferman [6]. In contrast to $\widehat{\mathsf{ID}}_n$, the theories $\widehat{\mathsf{ID}\Omega}_n$ include a very weak form of induction on the ordinals, which will be crucial for a proper treatment of the $\boldsymbol{\mu}$ operator, and in addition, considerably facilitates the formulation of standard structures for explicit mathematics. Moreover, we define natural restrictions $\widehat{\mathsf{ID}\Omega}^r_n$ of $\widehat{\mathsf{ID}\Omega}_n$.

3.1.1 Syntax

In this subsection we describe the formal framework for the iterated fixed point theories with ordinals $\widehat{\mathsf{ID}\Omega}_n$ to be introduced below. Let L be the usual first order language of arithmetic with a countable list of *number variables* $a, b, c, x, y, z, \ldots$, the constant 0 as well as function and relation symbols for all primitive recursive functions and relations. The *number terms* $r, s, t, \ldots$ of L are defined as usual.

The languages $\mathrm{L}_{\Omega,n}$ $(n > 0)$ are extensions of L be n sorts of *ordinal variables* $\sigma^i, \tau^i, \eta^i, \xi^i, \ldots$ $(0 < i \leq n)$, a binary relation symbol $<$ for the less relation on the ordinals as well as certain relation symbols $P_{F,i}$ $(0 < i \leq n)$ which represent stages of (iterated) positive inductive definitions.

In the following let P be a fresh unary[3] relation symbol. The (*P-positive*) $\mathrm{L}_{\Omega,n}(P)$ *formulas* $F, G, H, \ldots$ $(n \geq 0)$ and the binary *relation symbols* $P_{F,n}$ $(n > 0)$ are given by the following simultaneous inductive definition; here an $\mathrm{L}_{\Omega,n}$ formula is an $\mathrm{L}_{\Omega,n}(P)$ formula which does not contain P.

Definition 12 1. $P(s)$ is an atomic $\mathrm{L}_{\Omega,0}(P)$ formula.

2. $R(\vec{s})$ and $\bar{R}(\vec{s})$ are atomic $\mathrm{L}_{\Omega,0}$ formulas for every positive relation symbol R of L.

[3]We assume that P is a unary for notational simplicity only. However, our approach works for operator forms of arbitrary arities.

3. $(\sigma^n = \tau^n)$, $(\sigma^n \neq \tau^n)$, $(\sigma^n < \tau^n)$ and $(\sigma^n \not< \tau^n)$ are atomic $L_{\Omega,n}$ formulas for every $n > 0$.

4. If $F(P, x)$ is a $L_{\Omega,m}(P)$ formula with at most x free and $m < n$, then $P_{F,n}$ and $\bar{P}_{F,n}$ are binary relation symbols of $L_{\Omega,n}$; $P_{F,n}(\sigma^n, s)$ and $\bar{P}_{F,n}(\sigma^n, s)$ are atomic formulas of $L_{\Omega,n}$.

5. If F and G are $L_{\Omega,n}(P)$ formulas, then $(F \vee G)$, $(F \wedge G)$, $\exists x F$ and $\forall x F$ are $L_{\Omega,n}$ formulas.

6. If F is an $L_{\Omega,n}(P)$ formula and $0 < m \leq n$, then $(\exists \xi^m < \sigma^m)F$, $(\forall \xi^m < \sigma^m)F$, $\exists \xi^m F$ and $\forall \xi^m F$ are $L_{\Omega,n}(P)$ formulas.

7. If F is an $L_{\Omega,m}(P)$ formula and $m < n$, then F is an $L_{\Omega,n}(P)$ formula.

Observe that the $L_{\Omega,0}$ formulas are exactly the L formulas.

The negation $\neg F$ of an $L_{\Omega,n}$ formula F is defined as usual by making use of the law of double negation and de Morgan's laws, and the remaining logical connectives are defined in the obvious way. Furthermore, if $F(P)$ is an $L_{\Omega,n}(P)$ formula and $G(x)$ and $L_{\Omega,n}$ formula, then $F(G)$ denotes the $L_{\Omega,n}$ formula which is obtained from F by replacing each atom $P(s)$ by $G(s)$. In addition, for every $L_{\Omega,n}$ formula F we write F^{σ^n} to denote the $L_{\Omega,n}$ formula which is obtained from F by replacing each unbounded ordinal quantifier $Q\xi^n$ by $(Q\xi^n < \sigma^n)$. Finally, we use the following abbreviations concerning the relation symbols $P_{F,n}$:

$$
\begin{aligned}
P^{\sigma^n}_{F,n}(s) &:= P_{F,n}(\sigma^n, s), \\
P^{<\sigma^n}_{F,n}(s) &:= (\exists \xi^n < \sigma^n) P^{\xi^n}_{F,n}(s), \\
P_{F,n}(s) &:= \exists \xi^n P^{\xi^n}_{F,n}(s).
\end{aligned}
$$

Definition 13 We define the following classes of $L_{\Omega,n}$ formulas.

1. A $\Delta_0^{\Omega,n}$ formula is an $L_{\Omega,n}$ formula which does not contain unbounded ordinal quantifiers of the form $\forall \xi^n$ and $\exists \xi^n$.

2. A $\Sigma^{\Omega,n}$ [$\Pi^{\Omega,n}$] formula is an $L_{\Omega,n}$ which does not contain unbounded ordinal quantifiers of the form $\forall \xi^n$ [$\exists \xi^n$].

Observe that a $\Delta_0^{\Omega,n}$ (and a fortiori $\Sigma^{\Omega,n}$) formula can have unbounded ordinal quantifiers of the form $\exists \xi^m$ and $\forall \xi^m$ for $0 < m < n$. $\Sigma^{\Omega,n}$ formulas, in addition, may have unbounded existential ordinal quantifiers $\exists \xi^n$ of level n.

3.1.2 Axioms and rules

In this subsection we introduce a family of fixed point theories with ordinals $\widehat{\mathsf{ID}\Omega}_n$ together with their natural restrictions $\widehat{\mathsf{ID}\Omega}^r_n$ $(n > 0)$. $\widehat{\mathsf{ID}\Omega}_n$ is formulated in the language $\mathrm{L}_{\Omega,n}$, and it is presented in a Tait style manner in order to facilitate a subsequent proof-theoretic analysis; we adopt the same conventions for displaying axioms as in the previous section. The logic of $\widehat{\mathsf{ID}\Omega}_n$ is $(n+1)$ sorted predicate logic with one sort for numbers and n sorts for ordinals together with equality axioms for all $(n+1)$ sorts; the *bounded* ordinal quantifiers are treated by separate rules.

In a next step we define $\widehat{\mathsf{ID}\Omega}_n$ by induction on $n > 0$. All theories $\widehat{\mathsf{ID}\Omega}_n$ include the axioms of Peano arithmetic PA with the exception of complete induction on the natural numbers. The crucial non-logical axioms and rules of inference of $\widehat{\mathsf{ID}\Omega}_n$ are those of $\widehat{\mathsf{ID}\Omega}_m$ for $0 < m < n$ plus the following.

I. Linearity of $<$ on the Ordinals

$$\sigma^n \not< \sigma^n \;\wedge\; (\sigma^n < \tau^n \wedge \tau^n < \eta^n \rightarrow \sigma^n < \eta^n) \;\wedge\; (\sigma^n < \tau^n \vee \sigma^n = \tau^n \vee \tau^n < \sigma^n)$$

II. Formula Induction on the Natural Numbers
For all $\mathrm{L}_{\Omega,n}$ formulas $F(x)$:

$$\neg F(0),\ \neg\forall x(F(x) \rightarrow F(x')),\ \forall x F(x)$$

III. $\Delta_0^{\Omega,n}$ induction on the ordinals
For all $\Delta_0^{\Omega,n}$ formulas $F(\sigma^n)$:

$$\neg\forall\xi^n((\forall\eta^n < \xi^n)F(\eta^n) \rightarrow F(\xi^n)),\ \forall\xi^n F(\xi^n)$$

IV. Inductive Operator Rules
For all $m < n$ and all P-positive $\mathrm{L}_{\Omega,m}(P)$ formulas $F(P,x)$ with at most x free:

$$\frac{\Gamma,\ F(P^{<\sigma^n}_{F,n}, s)}{\Gamma,\ P^{\sigma^n}_{F,n}(s)} \qquad\qquad \frac{\Gamma,\ \neg F(P^{<\sigma^n}_{F,n}, s)}{\Gamma,\ \neg P^{\sigma^n}_{F,n}(s)}$$

V. $\Sigma^{\Omega,n}$ Reflection Rules
For all $\Sigma^{\Omega,n}$ formulas F:

$$\frac{\Gamma,\ F}{\Gamma,\ \exists\xi^n F^{\xi^n}}$$

This finishes the description of $\widehat{\mathsf{ID}\Omega}_n$. The theory $\widehat{\mathsf{ID}\Omega}^r_n$ is defined in the same way as $\widehat{\mathsf{ID}\Omega}_n$ with the only exception that induction on the natural numbers for $\mathrm{L}_{\Omega,n}$ formulas II. is restricted to $\Delta_0^{\Omega,n}$ formulas.

It is an immediate consequence of the inductive operator and $\Sigma^{\Omega,n}$ reflection rules that the $\Sigma^{\Omega,n}$ formula $P_{F,n}$ determines a fixed point of the $\mathrm{L}_{\Omega,m}$ operator $F(P,x)$ $(m < n)$, provably in $\widehat{\mathsf{ID}\Omega}^r_n$. This is the content of the following theorem.

Theorem 14 *Let $m, n \in \omega$ so that $m < n$. Then we have for all P-positive $\mathrm{L}_{\Omega,m}(P)$ formulas $F(P,x)$ which contain at most x free:*

$$\widehat{\mathsf{ID}\Omega}^r_n \vdash \forall x(F(P_{F,n},x) \leftrightarrow P_{F,n}(x)).$$

As an immediate consequence of this theorem we obtain that the fixed point theories $\widehat{\mathsf{ID}}_n$ of Feferman [6] can be embedded into $\widehat{\mathsf{ID}\Omega}_n$.

Corollary 15 $\widehat{\mathsf{ID}}_n$ *is contained in* $\widehat{\mathsf{ID}\Omega}_n$ *for each* $n > 0$.

There is one point which needs mentioning here. The reader may wonder whether it is possible to embed, e.g., ID_1 into $\widehat{\mathsf{ID}\Omega}_2$, since the latter theory allows induction on the ordinals for arbitrary $\mathrm{L}_{\Omega,1}$ formulas. Such an embedding, however, does not work, due to the strict ramification of the ordinal variables and the relation symbols $P_{F,i}$. Observe that $\Delta_0^{\Omega,n}$ induction on the ordinals is formulated with respect to ordinal variables of level n in $\widehat{\mathsf{ID}\Omega}_n$.

We will show in Section 4 that $\widehat{\mathsf{ID}\Omega}_n$ in fact has the same proof-theoretic strength as $\widehat{\mathsf{ID}}_n$. Moreover, we will establish that the restricted theories $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ are a conservative extension of $\widehat{\mathsf{ID}\Omega}_n$.

3.2 Modelling finitely many universes in $\widehat{\mathsf{ID}\Omega}^r_n$

The theories $\widehat{\mathsf{ID}\Omega}_n$ introduced in the previous section can be used to give a proof-theoretical treatment of the formal systems $\mathsf{UTN}_k[+\boldsymbol{\mu}]$; more precisely, $\widehat{\mathsf{ID}\Omega}^r_{k+1}$ is an upper bound for UTN_k, and $\widehat{\mathsf{ID}\Omega}^r_{k+2}$ for $\mathsf{UTN}_k + (\boldsymbol{\mu})$. In the following we will show how this can be achieved.

In a first step we formalize a model construction for the applicative axioms I. – IV. $+\boldsymbol{\mu}$ in $\widehat{\mathsf{ID}\Omega}^r_1$, following Feferman and Jäger [9]. In a second step we formalize a model construction for UTN_k parametrized by an application relation, in such a way that the formalization can be carried out in $\widehat{\mathsf{ID}\Omega}^r_{k+1}$ if the application is arithmetical, and in $\widehat{\mathsf{ID}\Omega}^r_{k+2}$ if the application is represented by an $\mathrm{L}_{\Omega,1}$ fixed-point relation symbol.

The applicative structure. The embedding of I. – IV. $+(\boldsymbol{\mu})$ in $\widehat{\mathsf{ID}\Omega}^r_1$ is essentially the same as in [9], where the theory $\mathsf{BON}(\boldsymbol{\mu})$, which is very similar to the first, is embedded into $\widehat{\mathsf{ID}\Omega}^r_1$, which in turn is called there PA^r_Ω as in the original paper by Jäger [16].

The main idea is to define an application relation $App(x,y,z)$, which is supposed to interpret the relation $x \cdot y = z$, by an $\mathrm{L}_{\Omega,0}$-inductive definition $F(P,x,y,z)$[4]. To start with, some numerals $\hat{\mathsf{k}}, \hat{\mathsf{s}}, \hat{\mathsf{p}}, \ldots$ of L are chosen as unique codes for the

[4]Actually, the three parameters x, y, z must be coded in a single one.

individual constants $\mathsf{k}, \mathsf{s}, \mathsf{p}, \mathsf{p}_0, \mathsf{p}_1, \mathsf{s}_N, \mathsf{p}_N, \mathsf{d}_N, \boldsymbol{\mu}$ of $\mathrm{L_U}$. At every step of the inductive definition, those applications that cannot be simplified according to some axiom, such as $\mathsf{k}x$, are left "unevaluated" i.e. coded into the recursion-theoretic pair $\langle\hat{\mathsf{k}}, x\rangle$ by defining $App(\hat{\mathsf{k}}, x, \langle\mathsf{k}, x\rangle)$, while the others are simplified by defining e.g. $App(\hat{\mathsf{p}}_0, \langle\langle\hat{\mathsf{p}}, x\rangle, y\rangle, x)$. Since $F(P, x, y, z)$ is a positive $\mathrm{L}_{\Omega,0}(P)$ formula, $\widehat{\mathsf{ID}\Omega}_1^r$ proves that P_F, which we choose as interpretation for App, is a fixed point. Moreover, we can show by $\Delta_0^{\Omega,1}$ induction in $\widehat{\mathsf{ID}\Omega}_1^r$ that App is functional in the third argument.

Then $\mathrm{L}_{\Omega,1}$ formulae $V_t(x)$, expressing "the $\mathrm{L_U}$ individual term t has value x", can be defined by induction on the build-up of t. Now the atomic formulae without type variables can be translated according to

$$
\begin{aligned}
(s\downarrow)^{\hat{}} := (N(s))^{\hat{}} &:= \exists x V_s(x) \quad \text{and} \\
(s = t)^{\hat{}} &:= \exists x (V_s(x) \wedge V_t(x))
\end{aligned}
$$

For every axiom F of the logic of partial terms or of I. – IV. $+\boldsymbol{\mu}$ we yield: $\widehat{\mathsf{ID}\Omega}_1^r \vdash F^{\hat{}}$. In absence of $\boldsymbol{\mu}$ we can choose for $App(x, y, z)$ the standard recursion-theoretic relation $\{x\}(y) = z$, instead of building it in an inductive process.

The explicit type structure. Let us now turn to the embedding of $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$ into $\widehat{\mathsf{ID}\Omega}_n^r$, where $n = k + 1[+1]$. (In fact we embed only a fragment of the first theory, characterized by finitely many instances Φ of the comprehension scheme.)

The strategy is to formalize an inductive model construction similar to that of Feferman [4] in $\widehat{\mathsf{ID}\Omega}_n^r$. To this extent we use (non unique) codes for the types

- $\langle 0, m, \vec{a}, \vec{b}\rangle$ if the type is obtained by elementary comprehension with respect to x on a formula $E(x, \vec{a}, \vec{B})$ with Gödel number m, individual parameters $\vec{a}$ and type parameters $\vec{B}$ coded by $\vec{b}$;
- $\langle 1, a, f\rangle$ for the join of f over a type coded by a;
- $\langle 2, i\rangle$ for the universe D_i.

The interpretations of c_m and j can easily be chosen to accommodate with this encoding. We interprete the naming relation as equality of extensions.

Assume a fresh unary relation symbol QT, which identifies those numbers that are codes of types, and a binary, positive relation symbol $Q\epsilon$, which expresses the fact that the first argument belongs to the type coded by the second argument. Then the interpretation of the other relations involving types is determined as follows:

$$
\begin{aligned}
\text{“}y_1 = y_2\text{”} &:\Longleftrightarrow QT(y_1) \wedge QT(y_2) \wedge \forall x (Q\epsilon(x, y_1) \leftrightarrow Q\epsilon(x, y_2)), \\
\text{“}x \text{ names } y\text{”} &:\Longleftrightarrow \text{“}x = y\text{”}, \\
\text{“}y \text{ is a universe”} &:\Longleftrightarrow \text{“}y = \langle 2, 0\rangle\text{”} \vee \ldots \vee \text{“}y = \langle 2, k-1\rangle\text{”}.
\end{aligned}
$$

Finally, interpreting the type quantifiers of $\mathrm{L}_{\mathrm{U},k}$ as number quantifiers of $\mathrm{L}_{\Omega,n}$ bounded by QT, gives us a formalized $\mathrm{L}_{\mathrm{U},k}$ structure.

Unfortunately, in this setting Feferman's inductive model construction cannot be carried out in $\widehat{\mathsf{ID}\Omega}_n$ because of the negative occurrences of $Q\epsilon$ in the defining clauses. We can use the standard workaround, i.e., the introduction of one more (positive) binary relation symbol $Q\bar{\epsilon}$, with the intended meaning: "the first argument does not belong to the type coded by the second". To be more precise, we can define for every elementary $\mathrm{L}_{\mathrm{U},k}$ formula $E(a_0, \vec{a}, \vec{B})$ (all free variables indicated), and for $\mathrm{L}_{\Omega,n}$ variables $x_0, \vec{x}, \vec{y}$ two $\mathrm{L}_{\Omega,n-1}(Q\epsilon, Q\bar{\epsilon})$ formulae

$$\langle Q\epsilon, Q\bar{\epsilon}\rangle \Vdash^{\pm} E[a_0 := x_0, \vec{a} := \vec{x}, \vec{B} := \vec{y}]$$

(one marked with '+', one with '−') in such a way that

1. both coincide with the translation $E^{\hat{}}$ given above, if no type variable appears in E;
2. $\langle Q\epsilon, Q\bar{\epsilon}\rangle \Vdash^{+} a \in B[a := x, B := y]$ is the formula $Q\epsilon(x_0, y)$;
3. $\langle Q\epsilon, Q\bar{\epsilon}\rangle \Vdash^{-} a \in B[a := x, B := y]$ is the formula $\neg Q\bar{\epsilon}(x_0, y)$;
4. both depend only on $Q\epsilon(\cdot, y_i)$ and $Q\bar{\epsilon}(\cdot, y_i)$ for those y_i that are among $\vec{y}$;
5. $\Vdash^{+}$ and $\Vdash^{-}$ coincide and behave as satisfaction relations on elementary formulae under the assumption that $Q\epsilon(\cdot, y)$ and $Q\bar{\epsilon}(\cdot, y)$ are complementary for all type parameters y_i;
6. both $Q\epsilon$ and $Q\bar{\epsilon}$ appear only positively (negatively) in $\Vdash^{+}$ (respectively $\Vdash^{-}$).

Now it is possible to give a simultaneous inductive definition for $QT, Q\epsilon$ and $Q\bar{\epsilon}$, which is easy to encode into a single one, in order to fit into $\mathrm{L}_{\Omega,n}$. Let us assume that we have already fixed points $QT^{k-1}, Q\epsilon^{k-1}$ and $Q\bar{\epsilon}^{k-1}$ from the embedding of $\mathsf{UTN}_{k-1}[+(\boldsymbol{\mu})]$ into $\widehat{\mathsf{ID}\Omega}^{r}_{n-1}$, which is contained in $\widehat{\mathsf{ID}\Omega}^{r}_{n}$, and let us proceed by induction on k. The defining formula for $QT(y_0)$ is the disjunction of the following clauses:

1. $QT^{k-1}(y_0)$
2. $\bigvee_{m=\ulcorner E \urcorner \in \Phi} \exists\vec{x}\exists\vec{y}(y_0 = \langle 0, m, \vec{x}, \vec{y}\rangle \wedge QT(\vec{y}))$
3. $\exists y \exists f(y_0 = \langle 1, y, f\rangle \wedge QT(y) \wedge \forall x(\neg Q\bar{\epsilon}(x, y) \rightarrow QT(f \cdot x)))$[5]
4. $y_0 = \langle 2, k-1\rangle$

[5] "fx" in the this clause should in fact be replaced by a z such that $App(f, x, z)$, which is an $\mathrm{L}_{\Omega,0[+1]}$ formula.

The defining formula for $Q\epsilon(x_0, y_0)$ is the disjunction of the following clauses:

5. $QT^{k-1}(y_0) \wedge Q\epsilon^{k-1}(x_0, y_0)$

6. $$\bigvee_{m=\ulcorner E \urcorner \in \Phi} \exists \vec{x} \exists \vec{y} (y_0 = \langle 0, m, \vec{x}, \vec{y} \rangle \wedge QT(\vec{y}) \\ \wedge \langle Q\epsilon, Q\bar{\epsilon} \rangle \Vdash^+ E[a_0 := x_0, \vec{a} := \vec{x}, \vec{B} := \vec{y}])$$

7. $$\exists y \exists f (y_0 = \langle 1, y, f \rangle \wedge QT(y) \wedge \forall x (\neg Q\bar{\epsilon}(x, y) \rightarrow QT(f \cdot x)) \\ \wedge \exists z_0, z_1 (x_0 = \langle z_0, z_1 \rangle \wedge Q\epsilon(z_0, y) \wedge Q\epsilon(z_1, f z_0)))$$

8. $y_0 = \langle 2, k-1 \rangle \wedge QT^{k-1}(x_0)$

The clauses for $Q\bar{\epsilon}$ are straightforward, $(QT, Q\epsilon, Q\bar{\epsilon})$ positive adaptations of 5.–8. Let us have a look at the defining formulae. Consider first the case $k = 0$: in presence of $\boldsymbol{\mu}$ they are in $L_{\Omega,1}$, because App is obtained from the inductive model construction shown above, while in absence of $\boldsymbol{\mu}$ they are in $L_{\Omega,0}$. For $k > 0$ observe that QT^{k-1} appears in the defining formulae (even negatively, in the clauses for $Q\bar{\epsilon}$), which are therefore truly in $L_{\Omega,n-1}$. Since all clauses are positive in QT, $Q\epsilon$ and $Q\bar{\epsilon}$, $\widehat{\mathsf{ID}\Omega}^r_n$ proves the existence of fixed points QT^k, $Q\epsilon^k$ and $Q\bar{\epsilon}^k$.
Unlike in the case of the embedding in $\widehat{\mathsf{ID}}_n$, we do not have to use Aczel's trick to eliminate from QT^k the type codes where $Q\epsilon^k$ and $Q\bar{\epsilon}^k$ are not complementary, because we can show this property by induction on the ordinals. But first notice that the extension of a type code is fully determined at the first stage it gets into the fixed point QT^k. (In the following we omit the ordinal term superscripts ^{n}, if there is no danger of confusion.)

Lemma 16 *1.* $\widehat{\mathsf{ID}\Omega}^r_n \vdash QT^k(\sigma, y) \wedge \sigma < \tau \rightarrow (\forall x (Q\epsilon^k(\sigma, x, y) \leftrightarrow Q\epsilon^k(\tau, x, y)))$,

2. $\widehat{\mathsf{ID}\Omega}^r_n \vdash QT^k(\sigma, y) \wedge \sigma < \tau \rightarrow (\forall x (Q\bar{\epsilon}^k(\sigma, x, y) \leftrightarrow Q\bar{\epsilon}^k(\tau, x, y)))$.

Proof Both claims follow from

$$(\forall \sigma < \tau) \forall y (QT^k(\sigma, y) \rightarrow \forall x [(Q\epsilon^k(\sigma, x, y) \leftrightarrow Q\epsilon^k(\tau, x, y)) \\ \wedge (Q\bar{\epsilon}^k(\sigma, x, y) \leftrightarrow Q\bar{\epsilon}^k(\tau, x, y))])$$

which can be proved by $\Delta_0^{\Omega,n}$ induction on the ordinals. □

Lemma 17 $\widehat{\mathsf{ID}\Omega}^r_n \vdash QT^k(y) \rightarrow (\forall x (Q\epsilon^k(x, y) \leftrightarrow \neg Q\bar{\epsilon}^k(x, y)))$.

Proof A simple $\Delta_0^{\Omega,n}$ induction on the ordinals shows

$$\forall y (QT^k(\sigma, y) \rightarrow (\forall x (Q\epsilon^k(\sigma, x, y) \leftrightarrow \neg Q\bar{\epsilon}^k(\sigma, x, y))))$$

and an application of Lemma 16 gives the claim. □

Theorem 18 $\mathsf{UTN}_k[+(\boldsymbol{\mu})]$ *can be embedded in* $\widehat{\mathsf{ID}\Omega}^r_{k+1[+1]}$.

PROOF I.–IV. and $(\boldsymbol{\mu})$ have already been checked in the applicative structure. The verification of the other axioms is straightforward. Let us just have a look at the induction axiom: we must prove

$$QT^k(a) \land Q\epsilon^k(0,a) \land \forall x(Q\epsilon^k(x,a) \to Q\epsilon^k(x',a)) \to \forall x(Q\epsilon^k(x,a)).$$

Assume $QT^k(a)$. Hence, there is σ such that $QT^k(\sigma,a)$. By Lemma 16, we can replace $Q\epsilon^k(\cdot,a)$ by $Q\epsilon^k(\sigma,\cdot,a)$ in the formula above, which becomes an instance of $\Delta_0^{\Omega,n}$ induction on the natural numbers. □

4 Proof-theoretic analysis of $\widehat{\mathsf{ID}\Omega}_n$ and $\widehat{\mathsf{ID}\Omega}^r_n$

This section contains a proof-theoretic analysis of the fixed point theories with ordinals $\widehat{\mathsf{ID}\Omega}_n$ and their restrictions $\widehat{\mathsf{ID}\Omega}^r_n$. First we sketch that $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ is a conservative extension of $\widehat{\mathsf{ID}\Omega}_n$, thus generalizing a result of Jäger [16]. In a second step we give a complete ordinal analysis of the theories $\widehat{\mathsf{ID}\Omega}_n$; in particular, we show that $|\widehat{\mathsf{ID}\Omega}_n| = \gamma_n$, where $\gamma_0 = \varepsilon_0$ and $\gamma_{n+1} = \varphi\gamma_n 0$, and hence, $\widehat{\mathsf{ID}\Omega}_n$ is proof-theoretically equivalent to $\widehat{\mathsf{ID}}_n$. The major proof-theoretic techniques used in the sequel are (partial) cut elimination and asymmetric interpretation. Our treatment of $\widehat{\mathsf{ID}\Omega}_n$ is similar in spirit to the analysis of the predicative systems presented in Cantini [2] or Jäger [15].

4.1 $\Delta_0^{\Omega,n}$ induction on the natural numbers

In the following let us briefly address the proof-theoretic strength of the theories $\widehat{\mathsf{ID}\Omega}^r_n$. Recall that in $\widehat{\mathsf{ID}\Omega}^r_n$ induction on the natural numbers is available for arbitrary $\mathrm{L}_{\Omega,m}$ formulas $(m < n)$, and in addition, for $\Delta_0^{\Omega,n}$ formulas. In particular, induction for the $\Sigma^{\Omega,n}$ formulas $P_{F,n}(s)$ is not permitted in $\widehat{\mathsf{ID}\Omega}^r_n$.

The theory $\widehat{\mathsf{ID}\Omega}^r_1$ corresponds to the theory PA^r_Ω of Jäger [16], where it is shown that PA^r_Ω is a conservative extension of Peano arithmetic PA. The proof of the fact that $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ is a conservative extension of $\widehat{\mathsf{ID}\Omega}^r_n$ runs in the very same way, and hence, we only sketch the main lines of the argument.

The *first step* in the treatment of $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ consists in a *partial cut elimination* argument. Observe that the principal formulas of the axioms, the inductive operator and the $\Sigma^{\Omega,n+1}$ reflection rules of $\widehat{\mathsf{ID}\Omega}_{n+1}$ are $\Sigma^{\Omega,n+1}$ or $\Pi^{\Omega,n+1}$ formulas. Hence, each $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ derivation can be transformed into a quasi normal $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ derivation, so that each cut formula is either $\Sigma^{\Omega,n+1}$ or $\Pi^{\Omega,n+1}$; this is established by the usual (finite) cut elimination argument, cf. [11, 24, 25] for details.

In a *second step*, the $\Sigma^{\Omega,n+1}$-$\Pi^{\Omega,n+1}$ fragment of $\widehat{\mathsf{ID}\Omega}^r_{n+1}$ is reduced to $\widehat{\mathsf{ID}\Omega}_n$ by an *asymmetric interpretation*; ordinal variables of level $n+1$ are replaced by finite ordinals, so that a formula $P_{F,n+1}(m,s)$ with $m<\omega$ translates into an $\mathrm{L}_{\Omega,n}$ formula that describes the build up in stages of the corresponding inductive definition. The so-obtained asymmetric interpretation validates $\Sigma^{\Omega,n+1}$ and $\Pi^{\Omega,n+1}$ cuts as well as $\Sigma^{\Omega,n+1}$ reflection. In addition, $\Delta_0^{\Omega,n+1}$ induction on the natural numbers translates into complete induction for arbitrary $\mathrm{L}_{\Omega,n}$ formulas. For the details of this argument, the reader is referred to Jäger [16].

We are ready to state our theorem. For $n=0$ it corresponds to Corollary 13 of [16], where $\widehat{\mathsf{ID}\Omega}_0$ stands for PA.

Theorem 19 *$\widehat{\mathsf{ID}\Omega}^r_{n+1}$ is a conservative extension of $\widehat{\mathsf{ID}\Omega}_n$ for each $n \geq 0$.*

4.2 Full induction on the natural numbers

This subsection contains a complete ordinal analysis of the theories $\widehat{\mathsf{ID}\Omega}_n$. We introduce infinitary systems $\mathsf{Z}\Omega_n$ and provide (partial) cut elimination theorems for them. In particular, $\mathsf{Z}\Omega_1$ enjoys *full* cut elimination. Furthermore, we show that $\mathsf{Z}\Omega_{n+1}$ is reducible to $\mathsf{Z}\Omega_n$ via an asymmetric interpretation. A straightforward combination of these arguments yields the upper bound γ_n for the length of cut free $\mathsf{Z}\Omega_1$ derivations of arithmetic theorems of $\widehat{\mathsf{ID}\Omega}_n$, and hence, γ_n is an upper bound for the proof-theoretic ordinal of $\widehat{\mathsf{ID}\Omega}_n$.

The systems $\mathsf{Z}\Omega_n$ are based on the language $\mathrm{L}_{\infty,n}$ which results from $\mathrm{L}_{\Omega,n}$ by adding constants $\bar{\alpha}$ for all ordinals $\alpha<\Gamma_0$. In order to simplify notation we often use α instead of $\bar{\alpha}$ in formal expressions. Formulas of $\mathrm{L}_{\infty,n}$ are built analogously to the formulas of $\mathrm{L}_{\Omega,n}$; the classes $\Delta_0^{\Omega,n}$, $\Sigma^{\Omega,n}$ and $\Pi^{\Omega,n}$ are defined similarly to $\mathrm{L}_{\Omega,n}$.

An $\mathrm{L}_{\infty,n}$ formula is called *simple* if it contains neither free number variables nor constants for ordinals; we write $\mathrm{SL}_{\infty,n}$ for the class of simple $\mathrm{L}_{\infty,n}$ formulas. In the system $\mathsf{Z}\Omega_n$ we derive a subclass of the $\mathrm{L}_{\infty,n}$ formulas, so-called $\mathrm{CL}_{\infty,n}$ *formulas*. A $\mathrm{CL}_{\infty,n}$ formula is an $\mathrm{SL}_{\infty,n}$ formula without unbounded ordinal quantifiers $\exists\xi^n$ and $\forall\xi^n$, where each free ordinal variable σ^n of level n is replaced by an ordinal constant. Observe that a $\mathrm{CL}_{\infty,n}$ formula may contain unbounded quantifiers $\exists\xi^m$ or $\forall\xi^m$ as well as free ordinal variables σ^m of level $0<m<n$; ordinal constants, however, do occur on level n only.

In the sequel let us write $\textsc{True}$ for the set of true $\mathrm{CL}_{\infty,n}$ formulas of the form $R(\vec{s})$, $\bar{R}(\vec{s})$, $(\alpha=\beta)$, $(\alpha\neq\beta)$, $(\alpha<\beta)$ or $(\alpha\not<\beta)$. Furthermore, two $\mathrm{L}_{\infty,n}$ formulas are called *numerically equivalent*, if they differ in closed number terms with identical value only.

In order to measure the complexity of cuts in $\mathsf{Z}\Omega_n$, we assign an *n rank $rk_n(E)$* to each $\mathrm{CL}_{\infty,n}$ formula E. The definition is tailored so that the process of building up stages of an inductive definition is reflected by the rank of the formulas $P^{\alpha}_{F,n}(s)$.

Definition 20 The *n rank $rk_n(E)$* of a $\mathrm{CL}_{\infty,n}$ formula F is inductively defined as follows.

1. If $n \geq 2$ and E is a $\Sigma^{\Omega,n-1}$ or $\Pi^{\Omega,n-1}$ formula of $\mathrm{SL}_{\infty,n-1}$, then $rk_n(E) := 0$. Below we assume that 1. does not apply.

2. If E has one of the forms $R(\vec{s})$, $\bar{R}(\vec{s})$, $(\alpha = \beta)$, $(\alpha \neq \beta)$, $(\alpha < \beta)$, or $(\alpha \not< \beta)$, then $rk_n(E) := 0$ if $n = 1$, and $rk_n(E) := 1$ if $n \geq 2$.

3. If E has the form $P^{\alpha}_{F,n}(s)$ or $\neg P^{\alpha}_{F,n}(s)$, then $rk_n(E) := \omega(\alpha + 1)$.

4. If E has the form $(F \vee G)$ or $(F \wedge G)$, then $rk_n(E) := \max(rk_n(F), rk_n(G)) + 1$.

5. If E has the form $\exists x F$ or $\forall x F$, then $rk_n(E) := rk_n(F(0)) + 1$.

6. If $0 < i < n$ and E has the form $\exists \xi^i F$ or $\forall \xi^i F$, then $rk_n(E) := rk_n(F) + 1$.

7. If $0 < i < n$ and E has the form $(\exists \xi^i < \sigma^i)F$ or $(\forall \xi^i < \sigma^i)F$, then $rk_n(E) := rk_n(F) + 2$.

8. If E has the form $(\exists \xi^n < \alpha)F(\xi^n)$ or $(\forall \xi^n < \alpha)F(\xi^n)$, then

$$rk_n(E) \ := \ \sup\{rk_n(F(\beta)) + 1 : \beta < \alpha\}.$$

Let us write $oc(F)$ for the set of ordinal constants which occur in the $\mathrm{CL}_{\infty,n}$ formula F. Then the proof of the following lemma is a matter of routine (cf. [19, 18]).

Lemma 21 *Let $m, n \in \omega$ so that $m < n$. Then we have for all P-positive $\mathrm{L}_{\Omega,m}(P)$ formulas $F(P,x)$ with at most x free, all $\mathrm{CL}_{\infty,n}$ formulas G and all ordinals $\alpha < \Gamma_0$:*

1. *$rk_n(F(P^{<\alpha}_{F,n}, s)) < rk_n(P^{\alpha}_{F,n}(s))$.*

2. *If $\beta < \alpha$ for all $\beta \in oc(G)$, then $rk_n(G) < \omega\alpha + \omega$.*

The systems $\mathsf{Z}\Omega_n$ $(n > 0)$ are formulated in the language $\mathrm{CL}_{\infty,n}$ and contain the following axioms and rules of inference.

I. Basic axioms. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas, all numerically equivalent $\Delta^{\Omega,n-1}_0$ formulas F_1 and F_2 of $\mathrm{SL}_{\infty,n-1}$ $(n \geq 2)$ and all formulas G in True:

$$\Gamma,\ \neg F_1,\ F_2 \quad \text{and} \quad \Gamma,\ G$$

II. Axioms and Rules of $\widehat{\mathsf{ID}\Omega}_m$ $(0 < m < n)$. These include the non-logical axioms and rules of inference of $\widehat{\mathsf{ID}\Omega}_m$ for every $0 < m < n$, restricted to $\mathrm{SL}_{\infty,m}$, with the exception of complete induction on the natural numbers. Observe that the main formulas of these axioms and rules do *not* contain constants for ordinals.

III. Propositional rules. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas and all $\mathrm{CL}_{\infty,n}$ formulas F and G:

$$\frac{\Gamma,\ F}{\Gamma,\ F \vee G} \qquad \frac{\Gamma,\ G}{\Gamma,\ F \vee G} \qquad \frac{\Gamma,\ F \qquad \Gamma,\ G}{\Gamma,\ F \wedge G}$$

IV. Number quantifier rules. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas and all $\mathrm{CL}_{\infty,n}$ formulas $F(s)$:

$$\frac{\Gamma,\ F(s)}{\Gamma,\ \exists x F(x)} \qquad \frac{\Gamma,\ F(t) \text{ for all closed number terms } t}{\Gamma,\ \forall x F(x)}\ (\omega)$$

V. Ordinal quantifier rules of level $0 < i < n$. For every $0 < i < n$, all $\mathrm{CL}_{\infty,n}$ formulas F and all ordinal variables σ^i and τ^i so that the usual variable conditions are satisfied:

$$\frac{\Gamma,\ \sigma^i < \tau^i \wedge F(\sigma^i)}{\Gamma,\ (\exists \xi^i < \tau^i) F(\xi^i)} \qquad \frac{\Gamma,\ \sigma^i < \tau^i \rightarrow F(\sigma^i)}{\Gamma,\ (\forall \xi^i < \tau^i) F(\xi^i)}$$

$$\frac{\Gamma,\ F(\sigma^i)}{\Gamma,\ \exists \xi^i F(\xi^i)} \qquad \frac{\Gamma,\ F(\sigma^i)}{\Gamma,\ \forall \xi^i F(\xi^i)}$$

VI. Ordinal quantifier rules of level n. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas, all $\mathrm{CL}_{\infty,n}$ formulas $F(\alpha)$ and all ordinals β with $\alpha < \beta < \Gamma_0$:

$$\frac{\Gamma,\ F(\alpha)}{\Gamma,\ (\exists \xi^n < \beta) F(\xi^n)} \qquad \frac{\Gamma,\ F(\gamma) \text{ for all } \gamma < \beta}{\Gamma,\ (\forall \xi^n < \beta) F(\xi^n)}$$

VII. Inductive operator rules of level n. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas, all $m < n$, all P-positive $\mathrm{L}_{\Omega,m}(P)$ formulas $F(P, x)$ with at most x free, all closed number terms s and all ordinals $\alpha < \Gamma_0$:

$$\frac{\Gamma,\ F(P^{<\alpha}_{F,n}, s)}{\Gamma,\ P^{\alpha}_{F,n}(s)} \qquad \frac{\Gamma,\ \neg F(P^{<\alpha}_{F,n}, s)}{\Gamma,\ \neg P^{\alpha}_{F,n}(s)}$$

VIII. Cut rules. For all finite sets Γ of $\mathrm{CL}_{\infty,n}$ formulas and all $\mathrm{CL}_{\infty,n}$ formulas F:

$$\frac{\Gamma,\ F \qquad \Gamma,\ \neg F}{\Gamma}$$

The formulas F and $\neg F$ are the cut formulas of this cut; the rank of a cut is the rank of its cut formulas.

As usual $\mathsf{Z}\Omega_n \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ means that Γ is provable in $\mathsf{Z}\Omega_n$ by a proof of depth less than or equal to α so that all cuts in this proof have rank *less* than ρ. We write $\mathsf{Z}\Omega_n \overset{<\alpha}{\underset{<\rho}{\vdash}} \Gamma$, if there exists $\alpha' < \alpha$ and $\rho' < \rho$ so that $\mathsf{Z}\Omega_n \overset{\alpha'}{\underset{\rho'}{\vdash}} \Gamma$. The notation $\mathsf{Z}\Omega_n \overset{<\alpha}{\underset{\rho}{\vdash}} \Gamma$ reads similarly.

One readily verifies that the assignment of ranks (cf. Lemma 21) and the rules of inference of $\mathsf{Z}\Omega_n$ are tailored in such a way that all but $\Sigma^{\Omega,n-1}/\Pi^{\Omega,n-1}$ cuts can be eliminated in $\mathsf{Z}\Omega_n$ ($n \geq 2$), and *full* cut elimination holds for $\mathsf{Z}\Omega_1$. Hence, we can state the following well-known (partial) cut elimination theorem of predicative proof theory, cf. Pohlers [24] or Schütte [25] for proofs.

Theorem 22 ((Partial) cut elimination for $\mathsf{Z}\Omega_n$) *Let $n \geq 1$ and Γ be a finite set of $\mathrm{CL}_{\infty,n}$ formulas. Assume further that $\mathsf{Z}\Omega_n \vdash^{\alpha}_{\beta+\omega^\rho} \Gamma$ for ordinals $\alpha, \beta, \rho < \Gamma_0$ so that $\beta > 0$ if $n \geq 2$. Then we have $\mathsf{Z}\Omega_n \vdash^{\varphi\rho\alpha}_{\beta} \Gamma$.*

In a next step we provide asymmetric interpretations of $\mathsf{Z}\Omega_{n+1}$ into $\mathsf{Z}\Omega_n$ for each $n \geq 1$. For this purpose we need the notion of a (β, α) instance of a set of formulas. Let Γ be a finite set of $\mathrm{SL}_{\infty,n}$ formulas, Λ a finite set of $\mathrm{CL}_{\infty,n}$ formulas and $\alpha, \beta < \Gamma_0$. Then Λ is called a (β, α) *instance* of Γ if it results from Γ by replacing

(i) each free ordinal variable σ^n of level n in the formulas of Γ by an ordinal less than β;

(ii) each universal ordinal quantifier $\forall\xi^n$ in the formulas of Γ by $(\forall\xi^n < \beta)$;

(iii) each existential ordinal quantifier $\exists\xi^n$ in the formulas of Γ by $(\exists\xi^n < \alpha)$.

Observe again that Λ may still contain unbounded ordinal quantifiers $\exists\xi^m$ and $\forall\xi^m$ as well as free ordinal variables σ^m of level $0 < m < n$.

We are in a position to state our asymmetric interpretation theorem. It provides a reduction of the $\Sigma^{\Omega,n}/\Pi^{\Omega,n}$ fragment of $\mathsf{Z}\Omega_{n+1}$ to $\mathsf{Z}\Omega_n$.

Theorem 23 (Asymmetric interpretation of $\mathsf{Z}\Omega_{n+1}$ into $\mathsf{Z}\Omega_n$) *Let $n \geq 1$ and Γ be a finite set of $\Sigma^{\Omega,n}$ and $\Pi^{\Omega,n}$ formulas of $\mathrm{SL}_{\infty,n}$ so that $\mathsf{Z}\Omega_{n+1} \vdash^{\alpha}_{1} \Gamma$ for some ordinal $\alpha < \Gamma_0$. Then we have for all ordinals $\beta < \Gamma_0$ and all $(\beta, \beta + 2^\alpha)$ instances Λ of Γ:*

$$\mathsf{Z}\Omega_n \vdash^{\omega^{\beta+\omega^\alpha}}_{\omega(\beta+2^\alpha+1)} \Lambda.$$

This theorem is proved by induction on α. One simply follows the pattern of similar asymmetric interpretations, cf. Cantini [3], Jäger [13, 16] or Schütte [25]. In particular, the asymmetric interpretation of $\Sigma^{\Omega,n}$ reflection and $\Delta_0^{\Omega,n}$ induction on the ordinals become provable in $\mathsf{Z}\Omega_n$, and the asymmetric interpretation of cuts is as usual.

Finally, we get the following embedding of $\widehat{\mathsf{ID}}\Omega_n$ into $\mathsf{Z}\Omega_{n+1}$, where complete induction on the natural numbers is proved in the usual way by making use of the ω rule. Observe that one actually establishes an embedding into the $\mathrm{SL}_{\infty,n}$ fragment of $\mathsf{Z}\Omega_{n+1}$.

Theorem 24 (Embedding of $\widehat{\mathsf{ID}}\Omega_n$ into $\mathsf{Z}\Omega_{n+1}$) *Let Γ be a finite set of $\mathrm{SL}_{\infty,n}$ formulas so that $\widehat{\mathsf{ID}}\Omega_n \vdash \Gamma$. Then there exist an $\alpha < \omega + \omega$ and a $k < \omega$ so that $\mathsf{Z}\Omega_{n+1} \vdash^{\alpha}_{k} \Gamma$.*

Now we have all necessary tools available which are needed for computing the exact proof-theoretic strength of the theories $\widehat{\mathsf{ID}}\Omega_n$.

Theorem 25 (Reduction of $\widehat{\mathsf{ID}\Omega}_n$ to $\mathsf{Z}\Omega_1$) *Let F be an arithmetic sentence so that $\widehat{\mathsf{ID}\Omega}_n \vdash F$. Then there exists an $\alpha < \gamma_n$ so that $\mathsf{Z}\Omega_1 \vdash_0^\alpha F$.*

Proof Let us assume that $\widehat{\mathsf{ID}\Omega}_n \vdash F$ for an arithmetic sentence F. By the embedding theorem and partial cut elimination for $\mathsf{Z}\Omega_{n+1}$ we obtain $\mathsf{Z}\Omega_{n+1} \vdash_1^{<\varepsilon_0} F$. From this we we get by the asymmetric interpretation theorem $\mathsf{Z}\Omega_n \vdash_{<\varepsilon_0}^{<\varepsilon_0} F$. If $n = 1$ this yields $\mathsf{Z}\Omega_1 \vdash_0^{<\varphi\varepsilon_0 0} F$ by full cut elimination for $\mathsf{Z}\Omega_1$ as claimed. If $n > 1$ we derive by iterated application of the partial cut elimination and the asymmetric interpretation theorem that $\mathsf{Z}\Omega_1 \vdash_{<\gamma_{n-1}}^{<\gamma_{n-1}} F$. From this we obtain $\mathsf{Z}\Omega_1 \vdash_0^{<\gamma_n}$ by full cut elimination for $\mathsf{Z}\Omega_1$ as desired. $\Box$

From this theorem one derives as usual an upper bound for the proof-theoretic ordinal of $\widehat{\mathsf{ID}\Omega}_n$, cf. Pohlers [24] or Schütte [25] for details.

Corollary 26 $|\widehat{\mathsf{ID}\Omega}_n| \leq \gamma_n$.

By Theorem 19, Corollary 15, careful formalization of the above arguments, and the results of Feferman [6], we are now able to state the following theorem about the proof-theoretic strength of $\widehat{\mathsf{ID}\Omega}_n$.

Theorem 27 *The theories $\widehat{\mathsf{ID}\Omega}_n$, $\widehat{\mathsf{ID}\Omega}^r_{n+1}$, $\widehat{\mathsf{ID}}_n$, and $(\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_{n-1}}$ are all proof-theoretically equivalent and have proof-theoretic ordinal γ_n.*

Corollary 28 *The theories $\bigcup_{n<\omega} \widehat{\mathsf{ID}\Omega}_n$, $\bigcup_{n<\omega} \widehat{\mathsf{ID}}_n$ and $(\Pi^0_1\text{-}\mathsf{CA})_{<\Gamma_0}$ are all proof-theoretically equivalent and have proof-theoretic ordinal Γ_0.*

5 Conclusions

In the following let us briefly summarize what we have achieved in this paper. Combining Theorems 11, 18 and 27 we yield:

Theorem 29 *We have the following proof-theoretic equivalences:*

$$\begin{array}{ccccccccc} \mathsf{UTN}_k & \equiv & \widehat{\mathsf{ID}\Omega}^r_{k+1} & \equiv & \widehat{\mathsf{ID}\Omega}_k & \equiv & \widehat{\mathsf{ID}}_k & \equiv & (\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_{k-1}}; \\ \mathsf{UTN}_k + (\boldsymbol{\mu}) & \equiv & \widehat{\mathsf{ID}\Omega}^r_{k+2} & \equiv & \widehat{\mathsf{ID}\Omega}_{k+1} & \equiv & \widehat{\mathsf{ID}}_{k+1} & \equiv & (\Pi^0_1\text{-}\mathsf{CA})_{<\gamma_k}. \end{array}$$

On the first line the proof-theoretic ordinal is γ_k, on the second γ_{k+1}.

From the above and Theorems 5 and 6 we finally get:

Theorem 30 *We have the following proof-theoretic equivalences:*

$$\mathsf{ATR}_0 \equiv \mathsf{UTN} + (\mathsf{Lim}) \equiv \mathsf{UTN} + (\mathsf{Lim}) + (\boldsymbol{\mu}) \equiv \bigcup_{k<\omega} \widehat{\mathsf{ID}\Omega}_k \equiv \bigcup_{k<\omega} \widehat{\mathsf{ID}}_k \equiv (\Pi^0_1\text{-}\mathsf{CA})_{<\Gamma_0}.$$

The proof-theoretic ordinal of all these theories is Γ_0.

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