Wellordering proofs for metapredicative Mahlo

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Abstract

In this article we provide wellordering proofs for metapredicative systems of explicit mathematics and admissible set theory featuring suitable axioms about the Mahloness of the underlying universe of discourse. In particular, it is shown that in the corresponding theories EMA of explicit mathematics and KPm^0 of admissible set theory, transfinite induction along initial segments of the ordinal $\varphi\omega00$, for φ being a ternary Veblen function, is derivable. This reveals that the upper bounds given for these two systems in the paper Jäger and Strahm [11] are indeed sharp.

1 Introduction

This paper is a companion to the article Jäger and Strahm [11], where systems of explicit mathematics and admissible set theory for metapredicative Mahlo are introduced. Whereas the main concern of [11] was to establish proof-theoretic upper bounds for these systems, in this article we provide the corresponding wellordering proofs, thus showing that the upper bounds derived in [11] are sharp.

The central systems of this article are the theories EMA and KPm^0 for metapredicative Mahlo in explicit mathematics and admissible set theory, respectively. EMA is based on Feferman's explicit mathematics with elementary comprehension and join (cf. Feferman [2, 3]). Crucial for its formulation are so-called *universes*: these are types of representations or names which are closed under elementary comprehension and join. The principal axiom of EMA claims that for each operation from names of types to names of types there exists a uniformly given universe that is closed under this operation. We note that EMA does not include inductive generation and that induction on the natural number is restricted to types. For more information concerning EMA plus inductive generation see Jäger and Studer [12].

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The theory KPm^0, on the other hand, is Rathjen's theory KPM (cf. Rathjen [17, 18]) with induction on the natural numbers restricted to sets and $\in$ induction omitted completely. The Mahlo axiom schema in KPm^0 features Π_2 reflection on admissible sets. It happens that the absence of $\in$ induction causes a dramatic collaps in proof-theoretic strength: whereas KPM is a highly impredicative theory exceeding $(\Delta^1_2\text{-}\mathsf{CA})+(\mathsf{BI})$ in proof strength by far, the strength of KPm^0 is between the Feferman-Schütte ordinal Γ_0 and the Bachmann-Howard ordinal.

The theories EMA and KPm^0 and their proof-theoretic analyses typically belong to the new area of so-called *metapredicative proof theory*. Metapredicativity is concerned with the study and analysis of formal systems whose proof-theoretic ordinal is beyond Γ_0, but which can nevertheless be given a proof-theoretic analysis that uses *methods* from predicative proof theory only. Quite recently, numerous interesting metapredicative systems have been identified, cf. e.g. [8, 10, 13, 16, 19, 22, 23].

The term *metapredicative* indeed also applies to the wellordering proofs for EMA and KPm^0 given in this paper. First of all, the notation system used is based on a *ternary Veblen or φ function $\varphi\alpha\beta\gamma$*, which is a straightforward generalization of the well-known binary φ function; in particular, no collapsing is used in this notation system. Secondly and most importantly, the general *methodology* of the wellordering proofs given below is very much in the spirit of the wellordering proofs for predicative systems due to Feferman and Schütte, cf. e.g. [4, 5, 21]. For example, instead of working in initial segments of the ramified analytic hierarchy or the ordinary jump hierarchy one considers hierarchies of universes, hyperuniverses, admissibles, hyperinaccessibles, and so on.

The plan of this paper is as follows. In the next section we review the system EMA introduced in [11] and we identify its crucial subsystems S_n ($n \in \mathbb{N}$). The principal universe generation axiom of S_n features the existence of so-called n-hyperuniverses, which can be seen as an analogue of n-(hyper)inaccessibles. Section 3 constitutes the heart of this article: after some ordinal-theoretic preliminaries we show that S_n derives transfinite induction along all initial segments of the ordinal $\varphi(n{+}1)00$, thus establishing $\varphi\omega 00$ as a lower bound of EMA. In Section 4 we indicate how the wellordering proofs given for EMA can be adapted to the framework of admissible set theory, namely the theory KPm^0. We will end our paper in Section 5 with some remarks concerning the strength of our theories in the presence of the full schema of complete induction on the natural numbers: it turns out that the methods of this paper readily yield that the ordinal $\varphi\varepsilon_0 00$ is a lower bound of EMA and KPm^0 augmented by formula induction.

2 The theory EMA and its subsystems

In this section we recapitulate the theory EMA for metapredicative Mahlo in explicit mathematics with universes, which has been introduced in Jäger and Strahm [11]. Universes are types of (names of) types which are closed under elementary comprehension and join (disjoint union). The principal type existence axiom of EMA claims that each total operation on types (names) can be (uniformly) reflected in a universe. Furthermore, we identify crucial subsystems S_n ($n \in \mathbb{N}$) of EMA which are suited for carrying through the wellordering proofs in the next section.

Large parts of the first paragraph of this section are very much like in related papers; nevertheless, we decided to include them in order to make our article self-contained and also accessible for a reader who is not a specialist in explicit mathematics.

2.1 Defining EMA

EMA is formulated in the second order language $\mathbb{L}$ for individuals and types. It comprises individual variables $a, b, c, f, g, h, u, v, w, x, y, z, \dots$ as well as type variables $U, V, W, X, Y, Z, \dots$ (both possibly with subscripts). $\mathbb{L}$ also includes the individual constants k, s (combinators), $\mathsf{p}, \mathsf{p_0}, \mathsf{p_1}$ (pairing and projections), 0 (zero), $\mathsf{s_N}$ (successor), $\mathsf{p_N}$ (predecessor), $\mathsf{d_N}$ (definition by numerical cases) and additional individual constants, called *generators*, which will be used for the uniform naming of types, namely nat (natural numbers), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image), j (join) and m (universe generator). There is one binary function symbol $\cdot$ for (partial) application of individuals to individuals. Further, $\mathbb{L}$ has unary relation symbols $\downarrow$ (defined) and N (natural numbers) as well as three binary relation symbols $\in$ (membership), $=$ (equality) and $\Re$ (naming, representation).

For a uniform definition of the notion of proof-theoretic ordinal (cf. Jäger and Strahm [11], Definition 1) it is convenient that $\mathbb{L}$ also includes an anonymous unary relation symbol Q and a corresponding generator q. The relation Q plays the role of an anonymous predicate on the natural numbers with no specific mathematical meaning.

The *individual terms* $(r, s, t, r_1, s_1, t_1, \dots)$ of $\mathbb{L}$ are built up from individual variables and individual constants by means of our function symbol $\cdot$ for application. In the following we often abbreviate $(s \cdot t)$ simply as (st), st or sometimes also $s(t)$; the context will always ensure that no confusion arises. We further adopt the convention of association to the left so that $s_1 s_2 \dots s_n$

stands for $(\dots(s_1 \cdot s_2)\dots s_n)$. We also set $t' := \mathsf{s_N} t$. Finally, we define general n tupling by induction on $n \geq 2$ as follows:

$$(s_1, s_2) := \mathsf{p} s_1 s_2, \qquad (s_1, \dots, s_{n+1}) := ((s_1, \dots, s_n), s_{n+1}).$$

The positive literals of $\mathbb{L}$ are of the form $\mathsf{N}(s)$, $s\downarrow$, $s = t$, $U = V$, $s \in U$ and $\Re(s, U)$. Since we work with a logic of partial terms, it is not guaranteed that all terms have values, and $s\downarrow$ is read as s *is defined*. Moreover, $\mathsf{N}(s)$ says that s is a natural number, and the formula $\Re(s, U)$ is used to express that the individual s *represents* the type U or is a *name* of U.

The *formulas* $(A, B, C, A_1, B_1, C_1, \dots)$ of $\mathbb{L}$ are generated from the positive literals by closing against the usual propositional connectives, as we as existential and universal quantification for individuals and types. The following table contains a useful list of abbreviations:

$$\begin{array}{rcl}
s \simeq t & := & s\downarrow \vee t\downarrow \to s = t, \\
s \in \mathsf{N} & := & \mathsf{N}(s), \\
(\exists x \in \mathsf{N}) A(x) & := & (\exists x)(x \in \mathsf{N} \wedge A(x)), \\
(\forall x \in \mathsf{N}) A(x) & := & (\forall x)(x \in \mathsf{N} \to A(x)), \\
U \subset V & := & (\forall x)(x \in U \to x \in V), \\
s \,\dot{\in}\, t & := & (\exists X)(\Re(t, X) \wedge s \in X), \\
(\exists x \,\dot{\in}\, s) A(x) & := & (\exists x)(x \,\dot{\in}\, s \wedge A(x)), \\
(\forall x \,\dot{\in}\, s) A(x) & := & (\forall x)(x \,\dot{\in}\, s \to A(x)), \\
s \doteq t & := & (\exists X)[\Re(s, X) \wedge \Re(t, X)], \\
s \,\dot{\subset}\, t & := & (\exists X, Y)[\Re(s, X) \wedge \Re(t, Y) \wedge X \subset Y], \\
\Re(s) & := & (\exists X)\Re(s, X).
\end{array}$$

The vector notation $\vec{U}$ and $\vec{s}$ is sometimes used to denote finite sequences of type variables $U_1, \dots, U_m$ and individual terms $s_1, \dots, s_n$, respectively, whose length is given by the context.

The logic of EMA is Beeson's classical *logic of partial terms* (cf. Beeson [1] or Troelstra and Van Dalen [24]) for the individuals and classical logic with equality for the types. Observe that Beeson's formalization includes the usual strictness axioms.

Now let us first introduce the auxiliary theory EETJ, which provides a framework for explicit elementary types with join. The nonlogical axioms of EETJ can be divided into the following groups I–IV:

I. Applicative axioms. These axioms formalize that the individuals form a partial combinatory algebra, that we have pairing and projection and the usual closure conditions on the natural numbers plus definition by numerical cases.

(1) $\mathsf{k}uv = u$,

(2) $\mathsf{s}uv\downarrow \ \wedge \ \mathsf{s}uvw \simeq uw(vw)$,

(3) $\mathsf{p}_0(u, v) = u \ \wedge \ \mathsf{p}_1(u, v) = v$,

(4) $0 \in \mathsf{N} \ \wedge \ (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(x' \neq 0 \wedge \mathsf{p}_\mathsf{N}(x') = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \ \rightarrow \ \mathsf{p}_\mathsf{N}x \in \mathsf{N} \wedge (\mathsf{p}_\mathsf{N}x)' = x)$,

(7) $u \in \mathsf{N} \wedge v \in \mathsf{N} \ \wedge \ u = v \rightarrow \mathsf{d}_\mathsf{N}xyuv = x$,

(8) $u \in \mathsf{N} \wedge v \in \mathsf{N} \ \wedge \ u \neq v \rightarrow \mathsf{d}_\mathsf{N}xyuv = y$.

II. Explicit representation and extensionality. The following axioms state that each type has a name, that there are no homonyms and that equality of types is extensional.

(1) $(\exists x)\Re(x, U)$,

(2) $\Re(u, U) \wedge \Re(u, V) \rightarrow U = V$,

(3) $(\forall x)(x \in U \leftrightarrow x \in U) \rightarrow U = V$.

III. Basic type existence axioms. In the following we provide a finite axiomatization of uniform elementary comprehension plus join.

Natural numbers

$\Re(\mathsf{nat}) \ \wedge \ (\forall x)(x \,\dot{\in}\, \mathsf{nat} \leftrightarrow \mathsf{N}(x))$.

Representation of Q

$\Re(\mathsf{q}) \ \wedge \ (\forall x)(x \,\dot{\in}\, \mathsf{q} \leftrightarrow \mathsf{Q}(x)) \ \wedge \ \mathsf{q} \,\dot{\subset}\, \mathsf{nat}$.

Identity

$\Re(\mathsf{id}) \ \wedge \ (\forall x)(x \,\dot{\in}\, \mathsf{id} \leftrightarrow (\exists y)(x = (y, y)))$.

Complements

$\Re(u) \ \rightarrow \ \Re(\mathsf{co}(u)) \wedge (\forall x)(x \,\dot{\in}\, \mathsf{co}(u) \leftrightarrow x \,\dot{\notin}\, u)$.

Intersections

$\Re(u) \land \Re(v) \;\rightarrow\; \Re(\mathsf{int}(u, v)) \land (\forall x)(x \mathrel{\dot\in} \mathsf{int}(u, v) \leftrightarrow x \mathrel{\dot\in} u \land x \mathrel{\dot\in} v)$.

Domains

$\Re(u) \;\rightarrow\; \Re(\mathsf{dom}(u)) \land (\forall x)(x \mathrel{\dot\in} \mathsf{dom}(u) \leftrightarrow (\exists y)((x, y) \mathrel{\dot\in} u))$.

Inverse images

$\Re(u) \;\rightarrow\; \Re(\mathsf{inv}(u, f)) \land (\forall x)(x \mathrel{\dot\in} \mathsf{inv}(u, f) \leftrightarrow fx \mathrel{\dot\in} u)$.

Joins

$\Re(u) \land (\forall x \mathrel{\dot\in} u)\Re(fx) \;\rightarrow\; \Re(\mathsf{j}(u, f)) \land \Sigma(u, f, \mathsf{j}(u, f))$.

In this last axiom the formula $\Sigma(u, f, v)$ expresses that v names the disjoint union of f over u, i.e.

$$\Sigma(u, f, v) \;:=\; (\forall x)(x \mathrel{\dot\in} v \leftrightarrow (\exists y, z)(x = (y, z) \land y \mathrel{\dot\in} u \land z \mathrel{\dot\in} fy)).$$

IV. Uniqueness of generators. These axioms essentially guarantee that different generators create different names. To achieve this, we have for syntactically different generators r_0 and r_1 and arbitrary generators s and t:

(1) $r_0 \neq r_1$,

(2) $(\forall x)(sx \neq t)$,

(3) $(\forall x, y)(sx = ty \;\rightarrow\; s = t \land x = y)$.

As usual, the axioms of a partial combinatory algebra allow one to define λ abstraction and to prove a recursion or fixed point theorem. For proofs of these standard results the reader is referred to [1, 2].

Lemma 1 (Abstraction and recursion) *1. For each $\mathbb{L}$ term t and all variables x there exists an $\mathbb{L}$ term $(\lambda x.t)$ whose variables are those of t, excluding x, so that* EETJ *proves*

$$(\lambda x.t)\downarrow \;\land\; (\lambda x.t)x \simeq t.$$

2. There exists a closed $\mathbb{L}$ term rec *so that* EETJ *proves*

$$\mathsf{rec}f\downarrow \;\land\; \mathsf{rec}fx \simeq f(\mathsf{rec}f)x.$$

In the original formulation of explicit mathematics, elementary comprehension is not dealt with by a finite axiomatization but directly as an infinite axiom scheme. An $\mathbb{L}$ formula is called *elementary* if it contains neither the relation symbol $\Re$ nor bound type variables. The following result of Feferman and Jäger [6] shows that this scheme of uniform elementary comprehension is provable from our finite axiomatization. Join and uniqueness of generators are not needed for this argument.

Lemma 2 (Elementary comprehension) *For every elementary formula $A(u, \vec{v}, W_1, \dots, W_n)$ with at most the indicated free variables there exists a closed term t of $\mathbb{L}$ so that* EETJ *proves:*

1. $\bigwedge_{i=1}^{n} \Re(w_i, W_i) \;\rightarrow\; \Re(t(\vec{v}, w_1, \dots, w_n))$,
2. $\bigwedge_{i=1}^{n} \Re(w_i, W_i) \;\rightarrow\; (\forall x)(x \,\dot{\in}\, t(\vec{v}, w_1, \dots, w_n) \leftrightarrow A(x, \vec{v}, W_1, \dots, W_n))$.

Let us now introduce the concept of a *universe* into explicit mathematics. To put it very simply, a universe is supposed to be a type which consists of names only and reflects the theory EETJ. For the detailed definition of a universe we introduce some auxiliary notation and let $\mathcal{C}(W, u)$ be the closure condition which is the disjunction of the following $\mathbb{L}$ formulas:

(1) $u = \mathsf{nat} \vee u = \mathsf{q} \vee u = \mathsf{id}$,

(2) $(\exists x)(u = \mathsf{co}(x) \wedge x \in W)$,

(3) $(\exists x, y)(u = \mathsf{int}(x, y) \wedge x \in W \wedge y \in W)$,

(4) $(\exists x)(u = \mathsf{dom}(x) \wedge x \in W)$,

(5) $(\exists x, f)(u = \mathsf{inv}(x, f) \wedge x \in W)$,

(6) $(\exists x, f)[u = \mathsf{j}(x, f) \wedge x \in W \wedge (\forall y \,\dot{\in}\, x)(fy \in W)]$.

Thus, the formula $(\forall x)(\mathcal{C}(W, x) \rightarrow x \in W)$ states that W is a type which is closed under the type constructions of EETJ, i.e. elementary comprehension and join. If, in addition, all elements of W are names, we call W a universe, in symbols, $\mathsf{U}(W)$. Moreover, we write $\mathcal{U}(u)$ to express that the individual u is the name of a universe.

$$\begin{array}{rcl} \mathsf{U}(W) & := & (\forall x)(\mathcal{C}(W, x) \rightarrow x \in W) \;\wedge\; (\forall x \in W)\Re(x), \\ \mathcal{U}(u) & := & (\exists X)(\Re(u, X) \;\wedge\; \mathsf{U}(X)). \end{array}$$

Based on (names of) universes we can now introduce the Mahlo axiom for explicit mathematics. Given a name x and an operation f from names to names one simply claims that there exists (a name of) a universe $\mathsf{m}(x, f)$ which contains x and reflects f. The following shorthand notations are useful for obtaining a compact form of our Mahlo axiom:

$$\begin{array}{rcl} (f : \Re \rightarrow \Re) & := & (\forall x)(\Re(x) \rightarrow \Re(fx)), \\ (f : s \rightarrow s) & := & (\forall x \,\dot{\in}\, s)(fx \,\dot{\in}\, s). \end{array}$$

Mahloneness in explicit mathematics is now expressed by the axioms

$$(\mathsf{M}.1) \qquad \Re(x) \land (f : \Re \to \Re) \;\to\; \mathcal{U}(\mathsf{m}(x,f)) \land x \,\dot{\in}\, \mathsf{m}(x,f),$$

$$(\mathsf{M}.2) \qquad \Re(x) \land (f : \Re \to \Re) \;\to\; (f : \mathsf{m}(x,f) \to \mathsf{m}(x,f)).$$

It is interesting to examine what kind of ordering principles for universes can be consistently added to the previous axioms. This question is discussed at full length in Jäger, Kahle and Studer [9], and it is shown there that one must not be too liberal. As a consequence of these considerations we do not claim linearity and connectivity for arbitrary universes, but only for so-called *normal universes*, i.e. universes which are named by means of the type generator m,

$$\mathcal{U}_{\mathsf{m}}(u) := (\exists x, f)[u = \mathsf{m}(x,f) \;\land\; \mathcal{U}(u)].$$

Linearity and connectivity of normal universes are then given by the following two axioms:

$$(\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Lin}) \qquad (\forall x, y)[\mathcal{U}_{\mathsf{m}}(x) \land \mathcal{U}_{\mathsf{m}}(y) \;\to\; x \,\dot{\in}\, y \lor x \doteq y \lor y \,\dot{\in}\, x],$$

$$(\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Con}) \qquad (\forall x, y)[\mathcal{U}_{\mathsf{m}}(x) \land \mathcal{U}_{\mathsf{m}}(y) \;\to\; x \,\dot{\subset}\, y \lor y \,\dot{\subset}\, x].$$

It is shown in [9] that connectivity of normal universes also implies transitivity of normal universes in its most general form. For the reader's convenience we briefly sketch the relevant argument.

Lemma 3 (Strong transitivity) *We have that* $\mathsf{EETJ} + (\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Con})$ *proves*

$$\mathcal{U}_{\mathsf{m}}(u) \land \mathcal{U}_{\mathsf{m}}(v) \land w \doteq u \land w \,\dot{\in}\, v \;\to\; u \,\dot{\subset}\, v.$$

Proof. Assume the premise of the implication to be proved. Then w is also a name of the universe named by u. Since universes never contain their names (cf. e.g. Marzetta [14]) we have $w \,\dot{\notin}\, u$, thus $v \,\dot{\not\subset}\, u$. But now connectivity of normal universes $(\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Con})$ yields $u \,\dot{\subset}\, v$ as desired. □

The last principle present in EMA is complete induction on the natural numbers for types. Accordingly, type induction $(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$ is the axiom

$$(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}}) \quad (\forall X)(0 \in X \land (\forall x \in \mathsf{N})(x \in X \to x' \in X) \;\to\; (\forall x \in \mathsf{N})(x \in X)).$$

To sum up, the theory EMA of explicit mathematics, whose universe is Mahlo, comprises the theory EETJ plus the Mahlo axioms $(\mathsf{M}.1)$ and $(\mathsf{M}.2)$, the ordering principles $(\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Lin})$ and $(\mathcal{U}_{\mathsf{m}}\text{-}\mathsf{Con})$ as well as type induction $(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$.

2.2 The subsystems S_n of EMA

The crucial type existence axiom of S_n claims the existence of n-hyperuniverses, which can be seen as an analogue of n-(hyper)inaccessible sets. We will see that the existence of n-hyperuniverses for each natural number n is an immediate consequence of the Mahlo axioms (M.1) and (M.2). Moreover, the wellordering proofs in the next section will reveal that the proof-theoretic strength of EMA is already exhausted by its subsystems S_n for each $n \in \mathbb{N}$.

For the formulation of S_n we augment our language $\mathbb{L}$ by a generator constant u_n for each natural number n. Below we define the notion of a *type W being an n-hyperuniverse*, n-$\mathsf{U}(W)$; accordingly, n-$\mathcal{U}(u)$ expresses that u is the name of an n-hyperuniverse,

$$\begin{aligned} 0\text{-}\mathsf{U}(W) &:= \mathsf{U}(W), \\ (n{+}1)\text{-}\mathsf{U}(W) &:= \mathsf{U}(W) \wedge (\forall x \in W)(\mathsf{u}_n(x) \in W), \\ n\text{-}\mathcal{U}(u) &:= (\exists X)(\Re(u, X) \wedge n\text{-}\mathsf{U}(X)). \end{aligned}$$

The defining axiom for the constant u_n claims for each name x that $\mathsf{u}_n(x)$ is the name of an n-hyperuniverse containing x,

$$\Re(x) \;\rightarrow\; n\text{-}\mathcal{U}(\mathsf{u}_n(x)) \wedge x \,\dot{\in}\, \mathsf{u}_n(x).$$

The theory S_n now extends elementary explicit type theory with join EETJ by (i) the defining axioms for the constants u_m $(m \leq n)$, (ii) linearity and connectivity axioms for universes which are normal with respect to the generators u_m $(m \leq n)$, and (iii) type induction (T-I_{N}) on the natural numbers.

We observe that due to the presence of the linearity and connectivity axioms for normal universes in S_n, we also have *strong transitivity* for such universes according to (the proof of) Lemma 3 above.

Lemma 4 (n-hyperuniverses in EMA) *We have that S_n is contained in EMA for each natural number n.*

Proof. The type generators u_n can be defined in EMA by means of m,

$$\mathsf{u}_0 \;=\; \lambda x.\mathsf{m}(x, \lambda y.y), \qquad \mathsf{u}_{n+1} \;=\; \lambda x.\mathsf{m}(x, \mathsf{u}_n).$$

One readily shows by induction on n and by making use of the Mahlo axioms (M.1) and (M.2) that the so-defined u_n's satisfy their defining axioms in S_n. In the case $n = 0$ we have that $\mathsf{u}_0(x)$ is a universe containing x for each name x, since trivially $(\lambda y.y)$ is a total operation from $\Re$ to $\Re$. For the induction step we assume that the defining axiom for u_n has been derived in EMA; in particular, this yields that $\mathsf{u}_n : \Re \to \Re$ and, hence, by the Mahlo axioms we have for each name x that (i) $\mathsf{u}_{n+1}(x)$ is a universe containing x and (ii) $\mathsf{u}_{n+1}(x)$ is closed under u_n, thus showing that indeed $(n+1)\text{-}\mathcal{U}(\mathsf{u}_{n+1}(x))$. This concludes our inductive argument.

Further, the linearity and connectivity axioms ($\mathcal{U}_{\mathsf{m}}$-Lin) and ($\mathcal{U}_{\mathsf{m}}$-Con) of EMA entail the corresponding axioms of S_n. We have established that S_n is a subsystem of EMA for each natural number n. $\Box$

3 The wellordering proof for EMA

In this section we will show that S_n proves transfinite induction with respect to all types along each initial segment of the ordinal $\varphi(n+1)00$. This will immediately yield the desired lower bound $\varphi\omega 00$ for EMA. We assume that the reader is familiar with wellordering proofs below the Feferman-Schütte ordinal Γ_0 as they are presented, for example, in Feferman [4, 5] or Schütte [21]. Moreover, we presuppose the recent wellordering proofs in the context of metapredicativity in the two papers Jäger, Kahle, Setzer and Strahm [8] as well as Strahm [23].

3.1 Ordinal-theoretic preliminaries

The ordinals which are relevant in the wellordering proofs below are most easily expressed by making use of a *ternary* Veblen or φ function which we are going to define now. The usual Veblen hierarchy generated by the *binary* function φ, starting off with the function $\varphi 0\beta = \omega^\beta$ is well known from the literature, cf. Pohlers [15] or Schütte [21]. The *ternary* φ function is obtained as a straightforward generalization of the binary case by defining $\varphi\alpha\beta\gamma$ inductively as follows:

(i) $\varphi 0\beta\gamma$ is just $\varphi\beta\gamma$;

(ii) if $\alpha > 0$, then $\varphi\alpha 0\gamma$ denotes the γth ordinal which is strongly critical with respect to all functions $\lambda\xi,\eta.\varphi\delta\xi\eta$ for $\delta < \alpha$.

(iii) if $\alpha > 0$ and $\beta > 0$, then $\varphi\alpha\beta\gamma$ denotes the γth common fixed point of the functions $\lambda\xi.\varphi\alpha\delta\xi$ for $\delta < \beta$.

For example, $\varphi 10\alpha$ is Γ_α, and more generally, $\varphi 1\alpha\beta$ denotes a Veblen hierarchy over $\lambda\alpha.\Gamma_\alpha$. It is straightforward how to extend these ideas in order to obtain φ functions of all finite arities, and even further to Schütte's Klammersymbole [20].

We let Λ_3 denote the least ordinal greater than 0 which is closed under the ternary φ function. In the following we confine ourselves to the standard notation system which is based on this function. Since the exact definition of such a system is a straightforward generalization of the notation system for Γ_0 (cf. [15, 21]), we do not go into details here. We write $\prec$ for the corresponding primitive recursive wellordering of order type Λ_3 and assume without loss of generality that 0 is the least element with respect to $\prec$. Further, we let Lim denote the primitive recursive set of limit notations and we presuppose a primitive recursively given fundamental sequence $(\ell[n] : n \in \mathbb{N})$ for each limit notation ℓ; we will assume that $\ell[0] > 0$. As the definition of fundamental sequences is easy in the setting of φ functions we do not give it here and refer the reader to the relevant proofs in the next paragraph.

There exist primitive recursive functions acting on the codes of our notation system which correspond to the usual operations on ordinals. In the sequel it is often convenient in order to simplify notation to use ordinals and ordinal operations instead of their codes and primitive recursive analogues. Then (for example) ω and $\omega + \omega$ stand for the natural numbers whose order type with respect to $\prec$ are ω and $\omega + \omega$.

By making use of the recursion theorem and a little amount of complete induction on the natural numbers one can easily represent all the above primitive recursive notions in EMA and each of its subsystems S_n. *When working in the systems S_n in this section, we let $a, b, c, d, e, \dots$ range over the field of $\prec$ and ℓ denote limit notations.* Finally, let us put as usual for each $\mathbb{L}$ formula $A(x)$:

$$\begin{array}{rcl} \mathsf{Prog}(A) & := & (\forall a)[(\forall b \prec a)A(b) \to A(a)], \\ \mathsf{TI}(A, a) & := & \mathsf{Prog}(A) \to (\forall b \prec a)A(b). \end{array}$$

If we want to stress the relevant induction variable of a formula A, we sometimes write $\mathsf{Prog}(\lambda x.A(x))$ and $\mathsf{TI}(\lambda x.A(x), a)$ instead of $\mathsf{Prog}(A)$ and $\mathsf{TI}(A, a)$, respectively. Moreover, we let $\mathsf{Prog}(U)$ and $\mathsf{Prog}(u)$ stand for $\mathsf{Prog}(\lambda x.x \in U)$ and $\mathsf{Prog}(\lambda x.x \mathrel{\dot{\in}} u)$, respectively; $\mathsf{TI}(U, a)$ and $\mathsf{TI}(u, a)$ are understood analogously.

3.2 Deriving transfinite inductions in S_n

In this paragraph we will establish that S_n proves $(\forall X)\mathsf{TI}(X, \alpha)$ for each ordinal α less than $\varphi(n{+}1)00$. This shows in particular that $\varphi\omega 00$ is a lower bound for the proof-theoretic ordinal of EMA. The key lemma to be proved in the sequel says that if x is a name and we know that transfinite induction holds below a with respect to all types (names) in $\mathsf{u}_n(\mathsf{u}_n(x))$ (i.e. a universe containing a universe that contains x), then transfinite induction holds even below $\varphi na0$ for all types (names) in $\mathsf{u}_n(x)$.

Main Lemma 5 *We have for all natural numbers n that S_n proves:*

$$(1) \qquad \Re(x) \land (\forall y \,\dot{\in}\, \mathsf{u}_n(\mathsf{u}_n(x)))\mathsf{TI}(y,a) \;\rightarrow\; (\forall y \,\dot{\in}\, \mathsf{u}_n(x))\mathsf{TI}(y,\varphi na0).$$

The proof of the main lemma is by (meta) induction on n. The case $n = 0$ is immediate from the work of Feferman and Schütte on wellordering proofs below Γ_0, cf. e.g. Feferman [4, 5] and Schütte [21]. The key steps are as follows: given a name x and assuming $(\forall y \,\dot{\in}\, \mathsf{u}_0(\mathsf{u}_0(x)))\mathsf{TI}(y,a)$, we also have $(\forall y \,\dot{\in}\, \mathsf{u}_0(\mathsf{u}_0(x)))\mathsf{TI}(y,\omega^{a+1})$, due to the fact that universes are closed under elementary (and hence arithmetical) comprehension. Further, given an arbitrary name y in $\mathsf{u}_0(x)$ we can now set up the ordinary (arithmetical) jump hierarchy starting with y below ω^{a+1} in $\mathsf{u}_0(x)$; this hierarchy can be described by making use of the recursion theorem and using join at limit stages. The fact that the hierarchy is total or well-defined in $\mathsf{u}_0(x)$ is shown by induction up to ω^{a+1} and indeed this is possible since the relevant statement to be established defines a type in $\mathsf{u}_0(\mathsf{u}_0(x))$, a universe above $\mathsf{u}_0(x)$, cf. Lemma 6 below for a similar argument. But the existence of the jump hierarchy starting from y below ω^{a+1} immediately entails $\mathsf{TI}(y,\varphi a0)$, for example by Lemma 5.3.1 in Feferman [5] or Lemma 10 on p. 187 in Schütte [21]. This ends our brief sketch of the (well-known) assertion of our main lemma in the case $n = 0$.

Let us turn to the induction step. For that purpose we fix a natural number n and assume that (1) is true for n, aiming at a proof of the assertion of our main lemma for $n{+}1$. I.e. we want to show in S_{n+1} that for all names x,

$$(2) \qquad (\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x)))\mathsf{TI}(y,a) \;\rightarrow\; (\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(x))\mathsf{TI}(y,\varphi(n{+}1)a0).$$

A crucial ingredient in the proof of (2) are (uniform) *transfinite hierarchies of n-hyperuniverses* within an $(n{+}1)$-hyperuniverse. Such hierarchies are introduced via the recursion or fixed point theorem. In particular, we let h_n be a closed term of $\mathbb{L}$ so that we have provably in EETJ:

$$\begin{aligned}
\mathsf{h}_n x0 &\simeq \mathsf{u}_n(x), \\
\mathsf{h}_n x(a{+}1) &\simeq \mathsf{u}_n(\mathsf{h}_n xa), \\
\mathsf{h}_n x\ell &\simeq \mathsf{u}_n(\mathsf{j}(\{a : a \prec \ell\}, \mathsf{h}_n x)).
\end{aligned}$$

Hence, the hierarchy starts with a n-hyperuniverse containing (the name) x, at successor stages one puts an n-hyperuniverse on top of the hierarchy defined so far, and at limit stages a universe above the disjoint union of the previously defined hierarchy is taken. Of course, in general, one needs some amount of transfinite induction in order to show that h_n is well-defined in an $(n+1)$-hyperuniverse y. Therefore, in order to express the well-definedness of h_n below a in y, we let $\mathsf{Hier}_n(y, a)$ denote the conjunction of the following three formulas:

(i) $(\forall x \mathrel{\dot\in} y)(\forall b \prec a)(\mathsf{h}_n xb \mathrel{\dot\in} y)$,

(ii) $(\forall x \mathrel{\dot\in} y)(\forall b \prec a)n\text{-}\mathcal{U}(\mathsf{h}_n xb)$,

(iii) $(\forall x \mathrel{\dot\in} y)(\forall b \prec a)(\forall c \prec b)(\mathsf{h}_n xc \mathrel{\dot\in} \mathsf{h}_n xb)$.

The following lemma expresses that h_n is well-defined below a in an $(n+1)$-hyperuniverse $\mathsf{u}_{n+1}(x)$ provided that transfinite induction below a is available with respect to all types (names) in $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$.

Lemma 6 *We have that* S_{n+1} *proves:*

$$\Re(x) \wedge (\forall y \mathrel{\dot\in} \mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))\mathsf{TI}(y, a) \;\rightarrow\; \mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a).$$

Proof. Reasoning in S_{n+1} we assume that x is a name and for all types (names) y in $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$ transfinite induction is available below a. We have to show $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$, i.e. for all $z \mathrel{\dot\in} \mathsf{u}_{n+1}(x)$,

$$(\forall b \prec a)(\mathsf{h}_n zb \mathrel{\dot\in} \mathsf{u}_{n+1}(x)), \tag{3}$$

$$(\forall b \prec a)n\text{-}\mathcal{U}(\mathsf{h}_n zb), \tag{4}$$

$$(\forall b \prec a)(\forall c \prec b)(\mathsf{h}_n zc \mathrel{\dot\in} \mathsf{h}_n zb). \tag{5}$$

Since $\{b \prec a : \mathsf{h}_n zb \mathrel{\dot\in} \mathsf{u}_{n+1}(x)\}$ defines a type in $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$ by elementary comprehension, (3) follows by a straightforward transfinite induction. Moreover, (4) is immediate from (3) by the definition of h_n, the fact that universes consist of names only, and the defining axioms for the u_n's.

As to (5), we first observe that $\{b \prec a : (\forall c \prec b)(\mathsf{h}_n zc \mathrel{\dot\in} \mathsf{h}_n zb)\}$ defines a type in $\mathsf{u}_{n+1}(x)$ (and hence in $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$ by transitivity): to see this one basically applies join to (3) and subsequently uses an obvious instance of elementary comprehension. Given our general assumption, we can now derive (5) by an inductive argument. To show that the above type is progressive with respect to $\prec$ one proceeds straightforwardly in case b is not a limit ordinal. If b is limit and $c \prec b$, then also $c{+}1 \prec b$ and $\mathsf{h}_n zc \mathrel{\dot\in} \mathsf{h}_n z(c{+}1)$. On the other hand, one easily sees that there is a name of the universe denoted by $\mathsf{h}_n z(c{+}1)$ which belongs to $\mathsf{h}_n zb$, since we have by definition

$\mathsf{j}(\{c : c \prec b\}, \mathsf{h}_n z) \mathrel{\dot{\in}} \mathsf{h}_n zb$. But then $\mathsf{h}_n zc \mathrel{\dot{\in}} \mathsf{h}_n zb$ is immediate by strong transitivity (Lemma 3), which also holds for normal universes in S_{n+1}. $\Box$

Crucial for the wellordering proof below is the notion $n\mathsf{I}_x^c(a)$ of *transfinite induction up to a for all types (respectively names) belonging to a n-hyper-universe* $\mathsf{h}_n xb$ *for* $b \prec c$, which is given as follows:

$$n\mathsf{I}_x^c(a) \;:=\; (\forall b \prec c)(\forall y \mathrel{\dot{\in}} \mathsf{h}_n xb)\mathsf{TI}(y, a).$$

The next lemma tells us that $n\mathsf{I}_x^\ell(a)$ can be represented by a type in $\mathsf{h}_n x\ell$.

Lemma 7 *We have that* S_{n+1} *proves:*

$$\Re(x) \wedge \mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a) \;\rightarrow\; (\forall y \mathrel{\dot{\in}} \mathsf{u}_{n+1}(x))(\forall \ell \prec a)(\exists z \mathrel{\dot{\in}} \mathsf{h}_n y\ell)(\forall b)[b \mathrel{\dot{\in}} z \leftrightarrow n\mathsf{I}_y^\ell(b)].$$

Proof. Working in S_{n+1}, let x be a name and assume $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$. In addition, fix a name y in $\mathsf{u}_{n+1}(x)$ and a limit notation $\ell \prec a$. By the definition of $\mathsf{h}_n y\ell$ we have that $\mathsf{j}(\{c : c \prec \ell\}, \mathsf{h}_n y) \mathrel{\dot{\in}} \mathsf{h}_n y\ell$. By closure of $\mathsf{h}_n y\ell$ under join this readily entails that also (a name of) the type

$$\{(c, u, v) : c \prec \ell \;\wedge\; u \mathrel{\dot{\in}} \mathsf{h}_n yc \;\wedge\; v \mathrel{\dot{\in}} u\}$$

belongs to $\mathsf{h}_n y\ell$. Therefore, by closure of $\mathsf{h}_n y\ell$ under elementary comprehension, there exists a type (name) z in $\mathsf{h}_n y\ell$ which satisfies the condition claimed by the lemma. $\Box$

The following lemma makes crucial use of our general induction hypothesis, i.e. the claim (1) of our Main Lemma 5 for n.

Lemma 8 *We have that* S_{n+1} *proves:*

$$\Re(x) \wedge \mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a) \;\rightarrow\; (\forall y \mathrel{\dot{\in}} \mathsf{u}_{n+1}(x))(\forall \ell \prec a)\mathsf{Prog}(\lambda b.n\mathsf{I}_y^\ell(\varphi(n{+}1)0b)).$$

Proof. Assuming that x is a name and $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$, we aim at showing that $(\lambda b.n\mathsf{I}_y^\ell(\varphi(n{+}1)0b))$ is progressive for arbitrary $y \mathrel{\dot{\in}} \mathsf{u}_{n+1}(x)$ and limit notations $\ell \prec a$. This claim is immediate by an easy inductive argument from

$$(\forall c)[n\mathsf{I}_y^\ell(c) \;\rightarrow\; n\mathsf{I}_y^\ell(\varphi nc0)]. \tag{6}$$

Towards a proof of (6) assume $n\mathsf{I}_y^\ell(c)$ and fix a $d \prec \ell$. We have to show $(\forall z \mathrel{\dot{\in}} \mathsf{h}_n yd)\mathsf{TI}(z, \varphi nc0)$. Since ℓ is limit we also have $d{+}1 \prec \ell$ and, hence, our assumption yields

$$(\forall z \mathrel{\dot{\in}} \mathsf{h}_n y(d{+}1))\mathsf{TI}(y, c).$$

Further, since $\mathsf{h}_n y(d{+}1) = \mathsf{u}_n(\mathsf{h}_n yd)$ and $\mathsf{h}_n yd = \mathsf{u}_n(w)$ for a suitable name w in the universe $\mathsf{u}_{n+1}(x)$, we are now in a position to apply our general assumption (1) for n and obtain

$$(\forall z \dot{\in} \mathsf{h}_n yd)\mathsf{TI}(z, \varphi nc0).$$

Since d was an arbitrary notation less than ℓ we thus have shown $n\mathsf{I}_y^\ell(\varphi nc0)$. This ends our proof of (6).

Now in order to establish $\mathsf{Prog}(\lambda b.n\mathsf{I}_y^\ell(\varphi(n{+}1)0b)$, it is clearly enough to show the three claims,

(7) $\quad n\mathsf{I}_y^\ell(\varphi(n{+}1)00)$,

(8) $\quad n\mathsf{I}_y^\ell(\varphi(n{+}1)0b) \;\rightarrow\; n\mathsf{I}_y^\ell(\varphi(n{+}1)0(b{+}1))$,

(9) $\quad \mathsf{Lim}(b) \wedge (\forall b' \prec b)n\mathsf{I}_y^\ell(\varphi(n{+}1)0b') \;\rightarrow\; n\mathsf{I}_y^\ell(\varphi(n{+}1)0b)$.

For (7), observe that we are given a fundamental sequence $z_v = \varphi(n{+}1)00[v]$ for $\varphi(n{+}1)00$, where $z_0 = 1$ and $z_{v+1} = \varphi n z_v 0$. Hence, (7) follows from (6) by ordinary (type) induction. The argument for (8) is completely analogous by using the fundamental sequence $z_v = \varphi(n{+}1)0(b{+}1)[v]$ for $\varphi(n{+}1)0(b{+}1)$ with $z_0 = \varphi(n{+}1)0b + 1$ and $z_{v+1} = \varphi n z_v 0$. Finally, for (9) just observe that if $\mathsf{Lim}(b)$, then $\varphi(n{+}1)0b$ is the supremum over $b' \prec b$ of $\varphi(n{+}1)0b'$, so that the claim is immediate in this case. All together this completes the proof of our lemma. □

An important tool in the proof of Lemma 10 below is the formula $n\mathsf{Main}_a^x(b)$. It is the natural adaptation to our setting of similar formulas employed in a wellordering proof below Γ_0 in Feferman [5] and the metapredicative wellordering proofs in Jäger, Kahle, Setzer and Strahm [8] as well as Strahm [23]. Its definition makes use of the binary relation $\uparrow$ on the field of $\prec$,

$$a \uparrow b \;:=\; (\exists c, \ell)(b = c + a \cdot \ell).$$

Here of course $+$ and $\cdot$ are the primitive recursive operations corresponding to ordinal addition and multiplication on the field of $\prec$. The formula $n\mathsf{Main}_a^x(b)$ now has the following definition,

$$n\mathsf{Main}_a^x(b) \;:=\; (\forall y \dot{\in} x)(\forall c, d)[d \preceq a \;\wedge\; \omega^{1+b} \uparrow d \;\wedge\; n\mathsf{I}_y^d(c) \;\rightarrow\; n\mathsf{I}_y^d(\varphi(n{+}1)bc)]$$

Given a name x and assuming $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$, the following lemma says that the formula $n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b)$ defines a type in the universe $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$. The proof of the lemma is straightforward and very similar in spirit to the proof of Lemma 7.

Lemma 9 *We have that* S_{n+1} *proves:*

$$\Re(x) \wedge \mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a) \;\rightarrow$$
$$(\exists y \,\dot{\in}\, \mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x)))(\forall b)[b \,\dot{\in}\, y \leftrightarrow n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b)].$$

Proof. Reason in S_{n+1} and assume that x is a name so that $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$ holds. In particular, we have for each $z \,\dot{\in}\, \mathsf{u}_{n+1}(x)$,

$$(\forall c \prec a)\mathsf{h}_n zc \,\dot{\in}\, \mathsf{u}_{n+1}(x). \tag{10}$$

Applying join twice to (10) allows us to conclude that (a name of) the type

$$\{(c, u, v) : c \prec a \;\wedge\; u \,\dot{\in}\, \mathsf{h}_n zc \wedge v \,\dot{\in}\, u\} \tag{11}$$

belongs to the universe $\mathsf{u}_{n+1}(x)$ (and hence also to $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$). Since the name of the type (11) is *uniformly* given in each $z \,\dot{\in}\, \mathsf{u}_{n+1}(x)$ we can apply join in the universe $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$ in order to obtain a name of the type

$$\{(z, c, u, v) : z \,\dot{\in}\, \mathsf{u}_{n+1}(x) \;\wedge\; c \prec a \;\wedge\; u \,\dot{\in}\, \mathsf{h}_n zc \wedge v \,\dot{\in}\, u\} \tag{12}$$

in the universe $\mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x))$. But now, clearly, $\{b : n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b)\}$ is given elementarily in the type (12) and, hence, the claim of our lemma is established. □

We are now ready to turn to the crucial lemma concerning $n\mathsf{Main}_a^x(b)$. It is a natural generalization of Main Lemma I in [8] and [23]. Much of the proof is analogous to the proof in [8] and, therefore, we only want to concentrate on the main new points below.

Lemma 10 *We have that* S_{n+1} *proves:*

$$\Re(x) \wedge \mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a) \;\rightarrow\; \mathsf{Prog}(\lambda b.n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b)).$$

Proof. Let us work informally in S_{n+1} and assume that x is a name and $\mathsf{Hier}_n(\mathsf{u}_{n+1}(x), a)$. In order to show $\mathsf{Prog}(\lambda b.n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b))$ it is enough to verify the following three claims (13)–(15):

$$n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(0), \tag{13}$$
$$n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b) \;\rightarrow\; n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b{+}1), \tag{14}$$
$$\mathsf{Lim}(b) \wedge (\forall v \in \mathsf{N})n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b[v]) \;\rightarrow\; n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b). \tag{15}$$

In the following we only elaborate on (13), since (14) and (15) are proved in literally the same manner as (b) and (c) in the proof of Main Lemma I in [8], except for using the function $\lambda b, c.\varphi(n{+}1)bc$ instead of $\lambda b, c.\varphi 1bc$ in [8].

Towards the proof of $n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(0)$ we assume that a name y in the universe $\mathsf{u}_{n+1}(x)$ and a notation $d \preceq a$ with $\omega \uparrow d$ are given. We have to show

$$(\forall c)[n\mathsf{I}_y^d(c) \rightarrow n\mathsf{I}_y^d(\varphi(n{+}1)0c)]. \tag{16}$$

So let us assume $n\mathsf{I}_y^d(c)$. Since $\omega \uparrow d$ we know that d has the form $d_0 + \omega \cdot \ell$ for a limit notation ℓ. Hence, in order to derive $n\mathsf{I}_y^d(\varphi(n{+}1)0c)$ it is sufficient to establish for each natural number v,

$$n\mathsf{I}_y^{d_0+\omega\cdot\ell[v]}(\varphi(n{+}1)0c). \tag{17}$$

Since $\ell[v] > 0$ we have that $d_0 + \omega \cdot \ell[v]$ is always limit and, hence, by means of Lemma 8, we are in a position to conclude

$$\mathsf{Prog}(\lambda b.n\mathsf{I}_y^{d_0+\omega\cdot\ell[v]}(\varphi(n{+}1)0b)) \tag{18}$$

for each natural number v. Furthermore, we obtain from Lemma 7 that

$$\{b : n\mathsf{I}_y^{d_0+\omega\cdot\ell[v]}(\varphi(n{+}1)0b)\}$$

forms a type in the universe $\mathsf{h}_n y(d_0 + \omega \cdot \ell[v])$ for each natural number v. But this means in particular that we can now immediately derive assertion (17) from our assumption $n\mathsf{I}_y^d(c)$ and (18). Hence, we have shown (16) and, therefore, also (13). $\Box$

This concludes our preparatory work towards a proof of (2) in S_{n+1}, which is now immediate. Let x be a name and suppose

$$(\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(\mathsf{u}_{n+1}(x)))\mathsf{TI}(y, a). \tag{19}$$

Given this assumption, it is our aim to derive

$$(\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(x))\mathsf{TI}(y, \varphi(n{+}1)a0). \tag{20}$$

We can assume without loss of generality that a is an ε number, since universes are closed under arithmetical comprehension. Thus, it is enough to establish

$$(\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(x))\mathsf{TI}(y, \varphi(n{+}1)b0) \tag{21}$$

for each $b \prec a$. We fix such a b and observe that we also have $\omega^{1+b} \cdot \omega \prec a$. Further, by our assumption (19), Lemma 6 and Lemma 10 we have

$$\mathsf{Prog}(\lambda e.n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(e)). \tag{22}$$

But (22) together with (19), Lemma 6 and Lemma 9 immediately show that we have $n\mathsf{Main}_a^{\mathsf{u}_{n+1}(x)}(b)$, i.e. spelled out

$$(23)\quad (\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(x))(\forall c, d)[d \preceq a \wedge \omega^{1+b} \uparrow d \wedge n\mathsf{I}_y^d(c) \;\rightarrow\; n\mathsf{I}_y^d(\varphi(n{+}1)bc)].$$

By choosing $c = 0$ and $d = \omega^{1+b} \cdot \omega$ in (23) we get

$$(24)\qquad (\forall y \,\dot{\in}\, \mathsf{u}_{n+1}(x)) n\mathsf{I}_y^{\omega^{1+b}\cdot\omega}(\varphi(n{+}1)b0).$$

But now one immediately realizes that (24) entails (21). Since $b \prec a$ was arbitrary, we have thus shown (20). This is as desired and ends our proof of (2), given that the assumption (1) of our main lemma holds for n. All together this concludes our proof of Main Lemma 5.

A straightforward iterated application of Main Lemma 5 yields the following crucial theorem about the proof-theoretic lower bound of the theories S_n.

Theorem 11 *We have for all natural numbers n and all ordinals α less than $\varphi(n{+}1)00$ that S_n proves $(\forall X)\mathsf{TI}(X, \alpha)$.*

Proof. We fix a natural number n and inductively define the fundamental sequence $(\alpha_j : j \in \mathbb{N})$ for $\varphi(n{+}1)00$ by $\alpha_0 := 1$ and $\alpha_{j+1} := \varphi n \alpha_j 0$. We further use the notation $\mathsf{u}_n^{(j)}(x)$ for the j-fold application of u_n to x, i.e. $\mathsf{u}_n^{(0)}(x) := x$ and $\mathsf{u}_n^{(j+1)}(x) := \mathsf{u}_n(\mathsf{u}_n^{(j)}(x))$. We have to show that S_n proves $(\forall X)\mathsf{TI}(X, \alpha_k)$ for each natural number k. Towards that aim one makes straightforward use of Main Lemma 5 in order to show by induction on $j \leq k$ that S_n proves

$$(25)\qquad \Re(x) \;\rightarrow\; (\forall y \,\dot{\in}\, \mathsf{u}_n^{(k+1-j)}(x))\mathsf{TI}(y, \alpha_j).$$

If we choose $j = k$ in assertion (25), then we obtain that S_n derives

$$(26)\qquad \Re(x) \;\rightarrow\; (\forall y \,\dot{\in}\, \mathsf{u}_n(x))\mathsf{TI}(y, \alpha_k).$$

In particular, (26) entails that $(\forall x)(\Re(x) \rightarrow \mathsf{TI}(x, \alpha_k)$ is provable in S_n. Since we have an axiom saying that each type has a name, we have thus shown in S_n the assertion $(\forall X)\mathsf{TI}(X, \alpha_k)$. This is as desired and ends our proof. □

Using the definition of proof-theoretic ordinal from [11] we thus obtain:

Corollary 12 *We have for all natural numbers n that $\varphi(n{+}1)00 \leq |\mathsf{S}_n|$.*

Due to Lemma 4 we have now established the desired lower bound for EMA.

Corollary 13 $\varphi\omega 00 \leq |\mathsf{EMA}|$.

4 The wellordering proof for KPm^0

In the following let us quickly indicate how the wellordering proofs given in the previous section can be adapted to the context of admissible set theory, namely the theory KPm^0. As the procedure is very analogous to EMA we only sketch the main new points and do not spell out the wellordering proof for KPm^0 in all details.

4.1 The theory KPm^0 and its subsystems

In this paragraph we briefly review the theory KPm^0, and we identify its crucial subsystems T_n corresponding to the subsystems S_n of EMA. We refer to Jäger and Strahm [11] for a more detailed exposition of KPm^0.

Our version of KPm^0 is formulated with the natural numbers as *urelements*. Accordingly, we let $\mathcal{L}_1$ denote the usual language of arithmetic with function and relation symbols for all primitive recursive functions and relations. We further assume that $\mathcal{L}_1$ also includes the free anonymous relation symbol Q. The theory KPm^0 is now formulated in the extension $\mathcal{L}^* = \mathcal{L}_1(\in, \mathsf{N}, \mathsf{S}, \mathsf{Ad})$ of $\mathcal{L}_1$ by the membership relation symbol $\in$, the set constant N for the set of natural numbers and the unary relation symbols S and Ad for sets and admissibles, respectively. Terms and formulas of $\mathcal{L}^*$ are defined in the standard way; in particular, Δ_0, Σ, Π, Σ_n and Π_n denote the obvious classes of $\mathcal{L}^*$ formulas. Further, equality between objects is regarded as a defined notion in the expected manner, cf. [11].

The $\mathcal{L}^*$ theory KPm^0 is based on classical first order logic with equality. Its non-logical axioms comprise:

(i) the expected ontological axioms regarding the set constant N and the unary predicates S and Ad for sets and admissibles, respectively; in particular, it is claimed that admissibles are linearly ordered, i.e.,

$$\mathsf{Ad}(x) \wedge \mathsf{Ad}(y) \;\rightarrow\; x \in y \vee x = y \vee y \in x.$$

(ii) the axioms of Peano arithmetic PA.

(iii) the Kripke Platek axioms, namely pairing, union, separation for Δ_0 formulas and collection for Δ_0 formulas.

(iv) the Mahlo axioms; these include for all Δ_0 formulas $A(x, y, \vec{u})$ whose parameters belong to the list $x, y, \vec{u}$:

$$(\forall x)(\exists y)A(x, y, \vec{u}) \;\rightarrow\; (\exists z)[\mathsf{Ad}(z) \wedge \vec{u} \in z \wedge (\forall x \in z)(\exists y \in z)A(x, y, \vec{u})]$$

(v) complete induction on the natural numbers for Δ_0 formulas.

Let us now turn to the subsystems T_n of KPm^0. In complete analogy to the systems S_n, the principal set existence axiom of T_n claims that each set is contained in an n-(hyper)inaccessible set. For each natural number n we use n-$\mathsf{Ia}(z)$ in order to express that the set z is n-inaccessible; n-$\mathsf{Ia}(z)$ is a Δ_0 formula of $\mathcal{L}^*$ and is inductively given as follows:

$$\begin{aligned} 0\text{-}\mathsf{Ia}(z) &:= \mathsf{Ad}(z); \\ (n{+}1)\text{-}\mathsf{Ia}(z) &:= \mathsf{Ad}(z) \wedge (\forall x \in z)(\exists y \in z)(x \in y \wedge n\text{-}\mathsf{Ia}(y)). \end{aligned}$$

For each natural number n, the $\mathcal{L}^*$ theory T_n is now defined to be KPm^0 without the Mahlo axioms plus the following limit axiom,

$$(\forall x)(\exists y)(x \in y \wedge n\text{-}\mathsf{Ia}(y)).$$

Hence, T_n formalizes an $(n{+}1)$-inaccessible universe of sets (without foundation). Thus T_0 is exactly Jäger's well-known set theory KPi^0 (cf. [7]) and T_1 is the system KPh^0 considered in Strahm [22].

Lemma 14 (n-inaccessibles in KPm^0) *We have that T_n is contained in KPm^0 for each natural number n.*

Proof. We verify the assertion of this lemma by induction on n. In the case $n = 0$ we have to show that each set u is contained in an admissible set. This is immediate by applying Π_2 reflection on admissibles to the Π_2 formula $(\forall x)(\exists y)(u = u)$. For the induction step let us assume that KPm^0 includes T_n, i.e., in particular, KPm^0 proves $(\forall x)(\exists y)(x \in y \wedge n\text{-}\mathsf{Ia}(y))$. But then one makes use of pairing and the fact that admissibles are transitive to show that we also have, provably in KPm^0,

$$(\forall x)(\exists y)[x \in y \wedge u \in y \wedge n\text{-}\mathsf{Ia}(y)].$$

If we apply Π_2 reflection on Ad to this last assertion, then we obtain in KPm^0,

$$(\exists z)[\mathsf{Ad}(z) \wedge u \in z \wedge (\forall x \in z)(\exists y \in z)(x \in y \wedge n\text{-}\mathsf{Ia}(y))].$$

This reveals that KPm^0 derives $(\forall u)(\exists z)(u \in z \wedge (n{+}1)\text{-}\mathsf{Ia}(z))$. We have thus shown that T_{n+1} is contained in KPm^0. $\Box$

4.2 Remarks on deriving transfinite inductions in T_n

As mentioned above, the wellordering proofs given for the subsystems S_n of EMA directly carry over to the subsystems T_n of KPm^0. In the sequel we only want to address some delicate points which are characteristic for the set-theoretic framework, without redoing the whole wellordering proof.

We have seen in the previous section that one of the essential tools in the wellordering proof for EMA are transfinite hierarchies of n-hyperuniverses. Correspondingly, we have to consider transfinite hierarchies of n-inaccessibles in the framework of admissible set theory. For the construction of such hierarchies we need a uniform means for picking an n-inaccessible set above any given set. More precisely, we want a Σ_1 operation $(\cdot)^{+_n}$ of T_n such that provably in T_n we have that for each u, the set u^{+_n} is an n-inaccessible set containing u.

Not long ago Gerhard Jäger proved that in KPi^0 there exists a Σ_1 operation picking the next admissible set above any set u, i.e., the least admissible set containing u. It happens that Jäger's proof readily generalizes in order to provide the required operation $(\cdot)^{+_n}$ in T_n: one defines u^{+_n} to be the least n-inaccessible set above u, i.e.

$$u^{+_n} := \bigcap\{x : u \in x \wedge n\text{-}\mathsf{Ia}(x)\}.$$

For completeness we give the (adaptation of the) proof of Jägers theorem. It appears that linearity of admissibles is crucial in the argument below.

Lemma 15 *1. T_n proves that u^{+_n} is a set and, in addition, $n\text{-}\mathsf{Ia}(u^{+_n})$. Moreover, the operation $u \mapsto u^{+_n}$ is Σ_1 definable in T_n.*

2. We have that 1. relativizes to any $(n+1)$-inaccessible set.

Proof. In the following we prove the first part of the lemma only; the second part is immediate by relativization. Let us work informally in T_n. Given a set u, the limit axiom of T_n guarantees the existence of a set y such that $n\text{-}\mathsf{Ia}(y)$ and $u \in y$ and, hence, we have that

$$u^{+_n} = \bigcap\{x \in y \cup \{y\} : u \in x \wedge n\text{-}\mathsf{Ia}(x)\}$$

by linearity of admissibles. This proves that u^{+_n} is indeed a set and one readily sees that the operation $u \mapsto u^{+_n}$ is Σ_1 definable.

It remains to show that u^{+_n} is n-inaccessible, i.e. $n\text{-}\mathsf{Ia}(u^{+_n})$. For that purpose we define $u^{++_n} := (u^{+_n})^{+_n}$ and first convince ourselves that

$$u^{+_n} \neq u^{++_n}. \tag{27}$$

For a contradiction, assume $u^{+_n} = u^{++_n}$. By Δ_0 separation, we have that $r := \{x \in u^{+_n} : x \notin x\}$ is a set and, moreover, $r \in z$ for each n-inaccessible set z such that $u^{+_n} \in z$, i.e. $r \in u^{++_n}$ by definition. But then $r \in u^{+_n}$ since we have assumed $u^{+_n} = u^{++_n}$. This yields a contradiction since

$$r \in r \;\leftrightarrow\; r \in u^{+_n} \wedge r \notin r \;\leftrightarrow\; r \notin r.$$

Using (27), there exists a set z such that $n\text{-}\mathsf{Ia}(z)$, $u \in z$ and $u^{+_n} \notin z$, and indeed we now show that $z = u^{+_n}$. The inclusion $u^{+_n} \subset z$ is obvious. In order to establish that $z \subset u^{+_n}$ we pick an arbitrary set x with $n\text{-}\mathsf{Ia}(x)$ and $u \in x$ and verify $z \subset x$. By linearity we have $z \in x \vee z = x \vee x \in z$. In case of $z \in x$ or $z = x$, $z \subset z$ is obvious. But $x \in z$ is impossible since this would imply $u^{+_n} \in z$, a contradiction to the choice of z. All together we have shown $z = u^{+_n}$, which entails $n\text{-}\mathsf{Ia}(u^{+_n})$ as desired. This finishes our argument. We observe that Δ_0 collection was not used in this proof. $\Box$

Once we have the operations $(\cdot)^{+_n}$ at hand we are able to build transfinite hierarchies of n-inaccessibles within an $(n+1)$-inaccessible set. It is now a matter of routine to translate the wellordering proofs given in the framework of explicit mathematics for the systems S_n to the language of set theory and the systems T_n. Indeed, some points which needed special attention in explicit mathematics are even simpler in the set-theoretic framework. Hence, we are in a position to state the following lower bound for the proof-theoretic ordinal of the system T_n.

Theorem 16 *We have for all natural numbers n that $\varphi(n+1)00 \leq |\mathsf{T}_n|$.*

Lemma 14 now immediately entails the desired lower bound for KPm^0.

Corollary 17 *$\varphi\omega00 \leq |\mathsf{KPm}^0|$.*

5 Concluding remarks

Let us summarize our results concerning the proof-theoretic ordinals of the theories EMA, KPm^0, S_n, and T_n. The lower bounds for these theories have been established according to Corollary 12, Corollary 13, Theorem 16, and Corollary 17. The corresponding upper bounds are proved in the paper Jäger and Strahm [11]. Hence, we have the following main result.

Theorem 18 *We have the following proof-theoretic ordinals:*

1. $|\mathsf{EMA}| = |\mathsf{KPm}^0| = \varphi\omega00$;
2. $|\mathsf{S}_n| = |\mathsf{T}_n| = \varphi(n+1)00$.

If we denote by ($\mathbb{L}$-$\mathsf{I_N}$) and ($\mathcal{L}^*$-$\mathsf{I_N}$) the schema of complete induction on the natural numbers for all formulas in the language $\mathbb{L}$ and $\mathcal{L}^*$, respectively, then the lower bound computations given in this article can be extended in a rather straightforward manner in order to yield $\varphi\varepsilon_0 00$ as a proof-theoretic lower bound of the systems $\mathsf{EMA} + (\mathbb{L}\text{-}\mathsf{I_N})$ and $\mathsf{KPm}^0 + (\mathcal{L}^*\text{-}\mathsf{I_N})$. The principal benefit of the induction schema compared to the induction axiom is that one has available α-hyperuniverses and α-(hyper)inaccessibles for α less than ε_0 instead of n-hyperuniverses and n-(hyper)inaccessibles for n less than ω, respectively. The so-obtained lower bounds are sharp according to [11]. Moreover, one establishes the expected ordinals for the subsystems S_n and T_n augmented by the induction schema.

Theorem 19 *We have the following proof-theoretic ordinals:*

1. $|\mathsf{EMA} + (\mathbb{L}\text{-}\mathsf{I_N})| = |\mathsf{KPm}^0 + (\mathcal{L}^*\text{-}\mathsf{I_N})| = \varphi\varepsilon_0 00$;
2. $|\mathsf{S}_n + (\mathbb{L}\text{-}\mathsf{I_N})| = |\mathsf{T}_n + (\mathcal{L}^*\text{-}\mathsf{I_N})| = \varphi(n{+}1)\varepsilon_0 0$.

We finish this article by mentioning very recent work of Christian Rüede, who in his forthcoming PhD thesis [19] considers metapredicative subsystems of second order arithmetic which have the same strength as EMA and KPm^0. The key principles introduced and analyzed by Rüede are forms of ω model reflection and transfinite dependent choice.

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