Justification logic enjoys the strong finite model property

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Abstract

We observe that justification logic enjoys a form the strong finite model property (sometimes also called small model property). Thus we obtain decidability proofs for justification logic that do not rely on Post's theorem.

1 Introduction

Justification logics [4] are a family of logics that that, like modal logics, can express knowledge or provability of propositions. However, instead of an implicit $\Box$-operator justification logics include explicit modalities of the form $t:$ where t is a term representing a reason for an agent's knowledge or a proof a proposition.

Artemov developed the first justification logic [1, 2] to provide intuitionistic logic with a classical provability semantics. Later Fitting [9] introduced epistemic models for justification logic. In this semantics, justification terms represent evidence a very general sense. For instance, our belief in A may be justified by direct observation of A or by learning that a friend heard about A. This general reading of justification led to a big variety of epistemic justification logics for many different applications [3, 6, 7, 10, 12, 13].

There are many known decidability results for justification logics, see, for instance, [8, 11, 16]. However, many of these decidability proofs rely on completeness with respect to a recursively enumerable class of models and Post's theorem [15].

In the present note we show that justification logic enjoys a form of the strong finite model property (which sometimes is called small model property) [5]. Thus we obtain decidability proofs for justification logics that do not make use of Post's theorem.

This note makes heavy use of [7].

2 Justification Logics

Justification terms are built from countably many constants c_i and countably many variables x_i according to the following grammar:

$$t ::= c_i \mid x_i \mid (t \cdot t) \mid (t + t) \mid !t \quad .$$

We denote the set of terms by Tm. Formulae are built from countably many atomic propositions p_i according to the following grammar:

$$F ::= p_i \mid \neg F \mid (F \to F) \mid t : F \quad .$$

Prop denotes the set of atomic propositions and Fm denotes the set of formulae.

The axioms of J_{CS} consist of all instances of the following schemes:

A1 finitely many schemes axiomatizing classical propositional logic

A2 $t : (A \to B) \to (s : A \to t \cdot s : B)$

A3 $t : A \vee s : A \to t + s : A$

We will consider extension of J_{CS} by the following axioms schemes.

(jd) $t : \bot \to \bot$

(jt) $t : A \to A$

(j4) $t : A \to !t : t : A$

A *constant specification* CS for a logic L is any subset

$$\mathsf{CS} \subseteq \{c : A \mid c \text{ is a constant and } A \text{ is an axiom of } \mathsf{L}\}.$$

A constant specification CS for a logic L is called

1. *axiomatically appropriate* if for each axiom A of L there is a constant c such that $c : A \in \mathsf{CS}$
2. *schematic* if for each constant c the set $\{A \mid c : A \in \mathsf{CS}\}$ consists of one or several (possibly zero) axiom schemes, i.e., every constant justifies certain axiom schemes.

For a constant specification CS the deductive system J_{CS} is the Hilbert system given by the axioms A1–A3 and by the rules modus ponens and axiom necessitation:

$$\frac{A \quad A \to B}{B}\,(\text{MP}) \ , \qquad \frac{c : A \in \mathsf{CS}}{\underbrace{!! \cdots !}_{n} c : \underbrace{! \cdots !}_{n-1} c : \cdots :!!c :!c : c : A}\,(\text{AN!}) \ ,$$

where $n \geq 0$. In the presence of the j4 axiom a simplified axiom necessitation rule can be used:

$$\frac{c : A \in \mathsf{CS}}{c : A}\,(\text{AN}) \ .$$

Table 1 defines the various logics we consider. We now present the semantics for these logics

Definition 1 (Evidence relation)**.** Let (W, R) be a Kripke frame, i.e., $W \neq \varnothing$ and $R \subseteq W \times W$, and CS be a constant specification. An admissible evidence relation $\mathcal{E}$ for a logic L_{CS} is a subset of $\mathrm{Tm} \times \mathrm{Fm} \times W$ that satisfies the closure conditions:

1. if $(s, A, w) \in \mathcal{E}$ or $(t, A, w) \in \mathcal{E}$, then $(s + t, A, w) \in \mathcal{E}$
2. if $(s, A \to B, w) \in \mathcal{E}$ and $(t, A, w) \in \mathcal{E}$, then $(s \cdot t, B, w) \in \mathcal{E}$

Depending on whether or not the logic L_{CS} contains the j4 axiom, the evidence function has to satisfy one of the following two sets of closure conditions. If L_{CS} does not include the j4 axiom, then the additional requirement is:

	A1	A2	A3	jd	jt	j4	MP	AN!	AN
$\mathsf{J_{CS}}$	✓	✓	✓				✓	✓	
$\mathsf{JD_{CS}}$	✓	✓	✓	✓			✓	✓	
$\mathsf{JT_{CS}}$	✓	✓	✓		✓		✓	✓	
$\mathsf{JD4_{CS}}$	✓	✓	✓	✓		✓	✓		✓
$\mathsf{J4_{CS}}$	✓	✓	✓			✓	✓		✓
$\mathsf{LP_{CS}}$	✓	✓	✓		✓	✓	✓		✓

Table 1: Deductive Systems

3. if $c : A \in \mathsf{CS}$ and $w \in W$, then $(\underbrace{!! \cdots !}_{n} c, \underbrace{! \cdots !}_{n-1} c : \cdots :!!c :!c : c : A, w) \in \mathcal{E}$

If $\mathsf{L_{CS}}$ includes the j4 axiom, then the additional requirement is:

4. if $c : A \in \mathsf{CS}$ and $w \in W$, then $(c, A, w) \in \mathcal{E}$
5. if $(t, A, w) \in \mathcal{E}$, then $(!t, t : A, w) \in \mathcal{E}$
6. if $(t, A, w) \in \mathcal{E}$ and wRv, then $(t, A, v) \in \mathcal{E}$

If we drop condition 6, then we say $\mathcal{E}$ is a *t-evidence relation.* Sometimes we use $\mathcal{E}(s, A, w)$ for $(s, A, w) \in \mathcal{E}$.

Definition 2 (Evidence bases)**.**

1. An *evidence base* $\mathcal{B}$ is a subset of $\mathrm{Tm} \times \mathrm{Fm} \times W$.
2. An evidence relation $\mathcal{E}$ is *based on* $\mathcal{B}$, if $\mathcal{B} \subseteq \mathcal{E}$.

The closure conditions in the definition of admissible evidence function give rise to a monotone operator. The minimal evidence relation based on $\mathcal{B}$ is the least fixed point of that operator and thus always exists.

Definition 3 (Model)**.** Let CS be a constant specification. A *Fitting model* for a logic $\mathsf{L_{CS}}$ is a quadruple $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ where

- (W, R) is a Kripke frame such that
  - if $\mathsf{L_{CS}}$ includes the j4 axiom, then R is transitive;
  - if $\mathsf{L_{CS}}$ includes the jt axiom, then R is reflexive;
  - if $\mathsf{L_{CS}}$ includes the jd axiom, then R is serial.
- $\mathcal{E}$ is an admissible evidence relation for $\mathsf{L_{CS}}$ over the frame (W, R),
- $\nu : \mathrm{Prop} \to \mathcal{P}(W)$, called a valuation function.

Definition 4 (Satisfaction relation)**.** The relation of formula A being satisfied in a model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ at a world $w \in W$ is defined by induction on the structure of A by

- $\mathcal{M}, w \Vdash p_i$ if and only if $w \in \nu(p_i)$
- $\Vdash$ commutes with Boolean connectives

- $\mathcal{M}, w \Vdash t : B$ if and only if

  1) $\mathcal{M}, v \Vdash B$ for all $v \in W$ with wRv and
  2) $(t, B, w) \in \mathcal{E}$

We say a formula A is valid in a model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ if for all $w \in W$ we have $\mathcal{M}, w \Vdash A$. We say a formula A is valid for a logic L_{CS} if for all models $\mathcal{M}$ for L_{CS} we have that A is valid in $\mathcal{M}$.

The logics defined above are sound and complete (with a restriction in case of the logics containing the jd axiom). See [3, 9, 14] for the full proofs of the following results.

Theorem 5 (Soundness)**.** *Let* CS *be a constant specification. If a formula* A *is derivable in a logic* L_{CS}*, then* A *is valid for* L_{CS}*.*

Theorem 6 (Completeness)**.** *1. Let* CS *be a constant specification. If a formula* A *is not derivable in* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{J}_{\mathsf{CS}}, \mathsf{JT}_{\mathsf{CS}}, \mathsf{J4}_{\mathsf{CS}}, \mathsf{LP}_{\mathsf{CS}}\}$*, then there exists a model* $\mathcal{M}$ *for* L_{CS} *with* $\mathcal{M}, w \not\Vdash A$ *for some world* w *in* $\mathcal{M}$*.*

2. Let CS *be an axiomatically appropriate constant specification. If a formula* A *is not derivable in* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{JD}_{\mathsf{CS}}, \mathsf{JD4}_{\mathsf{CS}}\}$*, then there exists a model* $\mathcal{M}$ *for* L_{CS} *with* $\mathcal{M}, w \not\Vdash A$ *for some world* w *in* $\mathcal{M}$*.*

3 The Strong Finite Model Property and Decidability

In this section we define and establish the strong finitary model property for many justification logics. As a corollary we get decidability proofs for these logics.

Definition 7 (Finitary model)**.** A model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ is called *finitary* if

1. W is finite,
2. there exists a finite base $\mathcal{B}$ such that $\mathcal{E}$ is the minimal evidence relation based on $\mathcal{B}$, and
3. the set $\{(w, p) \in W \times \mathrm{Prop} \mid w \in \nu(p)\}$ is finite.

If $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ is a finitary model for L_{CS}, then will sometimes specify this model by the tuple $(W, R, \mathcal{B}, \nu)$ where $\mathcal{B}$ is the finite base for $\mathcal{E}$.

Making use of filtrations for justification logics, we obtain the following theorem [8].

Lemma 8 (Completeness w.r.t. finitary models)**.**

1. *Let* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{J}_{\mathsf{CS}}, \mathsf{JT}_{\mathsf{CS}}, \mathsf{J4}_{\mathsf{CS}}, \mathsf{LP}_{\mathsf{CS}}\}$ *and* CS *be a constant specification for* L*. If a formula* A *is not derivable in* L_{CS}*, then there exists a finitary model* $\mathcal{M}$ *for* L_{CS} *with* $\mathcal{M}, w \not\Vdash A$ *for some world* w *in* $\mathcal{M}$*.*

2. *Let* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{JD}_{\mathsf{CS}}, \mathsf{JD4}_{\mathsf{CS}}\}$ *and* CS *be an axiomatically appropriate constant specification for* L*. If a formula* A *is not derivable in* L_{CS}*, then there exists a finitary model* $\mathcal{M}$ *for* L_{CS} *with* $\mathcal{M}, w \not\Vdash A$ *for some world* w *in* $\mathcal{M}$*.*

Definition 9.

1. Let A be a formula. We denote the length of A (i.e. the number symbols in A) by $|A|$.

2. Let Γ be a set. We denote the cardinality of Γ (i.e. the number of elements of Γ) by $|\Gamma|$.

Definition 10 (Strong finitary model property)**.** A justification logic L_{CS} has the *strong finitary model property* if there are computable functions f, g, h such that for any formula A that is not satisfiable, there exists a finitary model $\mathcal{M} = (W, R, \mathcal{B}, \nu)$ for L_{CS} with

1. $\mathcal{M}, w \not\Vdash A$ for some $w \in W$

2. $|W| \leq f(|A|)$,

3. $|\mathcal{B}| \leq g(|A|)$,

4. $|\nu| \leq h(|A|)$.

Given the proof of Lemma 8 in [8] it is easy to see that we can effectively compute bounds on the size of the finitary model. Thus we get the strong finitary model property as a corollary of Lemma 8.

Corollary 11 (Strong finitary model property)**.**

1. *Let* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{J}_{\mathsf{CS}}, \mathsf{JT}_{\mathsf{CS}}, \mathsf{J4}_{\mathsf{CS}}, \mathsf{LP}_{\mathsf{CS}}\}$ *and* CS *be a constant specification for* L*. Then* L_{CS} *has the strong finitary model property.*

2. *Let* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{JD}_{\mathsf{CS}}, \mathsf{JD4}_{\mathsf{CS}}\}$ *and* CS *be an axiomatically appropriate constant specification for* L*. Then* L_{CS} *has the strong finitary model property.*

For a proof of the following lemma see [11, Lemma 4.4.6].

Lemma 12. *Let* CS *be a decidable schematic constant specification and* $\mathsf{L}_{\mathsf{CS}} \in \{\mathsf{J}_{\mathsf{CS}}, \mathsf{JD}_{\mathsf{CS}}, \mathsf{JD4}_{\mathsf{CS}}, \mathsf{JT}_{\mathsf{CS}}, \mathsf{J4}_{\mathsf{CS}}, \mathsf{LP}_{\mathsf{CS}}\}$*. Let* $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ *be a finitary model for* L_{CS}*. Then the relation* $\mathcal{M}, w \Vdash A$ *between worlds* $w \in W$ *and formulae* A *is decidable.*

Corollary 13 (Decidability)**.**

1. *Any justification logic in* $\{\mathsf{J}_{\mathsf{CS}}, \mathsf{JT}_{\mathsf{CS}}, \mathsf{J4}_{\mathsf{CS}}, \mathsf{LP}_{\mathsf{CS}}\}$ *with a decidable schematic* CS *is decidable.*

2. *Any justification logic in* $\{\mathsf{JD}_{\mathsf{CS}}, \mathsf{JD4}_{\mathsf{CS}}\}$ *with a decidable, schematic and axiomatically appropriate* CS *is decidable.*

Proof. Let L_{CS} be one of the above justification logics. Given a formula A we can generate all finitary models $\mathcal{M} = (W, R, \mathcal{B}, \nu)$ for L_{CS} with

1. $|W| \leq f(|A|)$,

2. $|\mathcal{B}| \leq g(|A|)$,

3. $|\nu| \leq h(|A|)$,

for the functions f, g, h from Definition 10. Note that we can decide whether a structure $\mathcal{M} = (W, R, \mathcal{B}, \nu)$ is a model for L_{CS} since the required conditions on the accessibility relation, some combination of transitivity, reflexivity, and seriality can be effectively verified.

By Lemma 12 we can decide for each of these finitary models, whether $\mathcal{M}, w \Vdash A$ for all $w \in W$.

Making use of Corollary 11 we know that if A is not L_{CS}-satisfiable, then the above procedure will generate a finitary model $\mathcal{M} = (W, R, \mathcal{B}, \nu)$ such that $\mathcal{M}, w \nVdash A$ for some $w \in W$. Therefore, we conclude that satisfiability for L_{CS} is decidable. □

4 Conclusion

We observed that justification logic enjoys a form of the strong finite model property (sometimes also called small model property). Thus we obtain decidability proofs for justification logics that do not rely on Post's theorem.

References

[1] Sergei N. Artemov. Operational modal logic. Technical Report MSI 95–29, Cornell University, December 1995.

[2] Sergei N. Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, March 2001.

[3] Sergei [N.] Artemov. The logic of justification. *The Review of Symbolic Logic*, 1(4):477–513, December 2008.

[4] Sergei [N.] Artemov and Melvin Fitting. Justification logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Fall 2012 edition, 2012.

[5] P. Blackburn, M. Rijke, and Y. Venema. *Modal logic.* Cambridge tracts in theoretical computer science. Cambridge University Press, 2002.

[6] Samuel Bucheli, Roman Kuznets, and Thomas Studer. Justifications for common knowledge. *Journal of Applied Non-Classical Logics*, 21(1):35–60, January–March 2011.

[7] Samuel Bucheli, Roman Kuznets, and Thomas Studer. Partial realization in dynamic justification logic. In Lev D. Beklemishev and Ruy de Queiroz, editors, *Logic, Language, Information and Computation, 18th International Workshop, WoLLIC 2011, Philadelphia, PA, USA, May 18–20, 2011, Proceedings*, volume 6642 of *Lecture Notes in Artificial Intelligence*, pages 35–51. Springer, 2011.

[8] Samuel Bucheli, Roman Kuznets, and Thomas Studer. Decidability for justification logics revisited. In Guram Bezhanishvili, Sebastian Löbner, Vincenzo Marra, and Frank Richter, editors, *Logic, Language, and Computation, 9th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2011, Kutaisi, Georgia, September 26-30, 2011, Revised*

Selected Papers, volume 7758 of *Lecture Notes in Computer Science*, pages 166–181. Springer, 2013.

[9] Melvin Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, February 2005.

[10] Ioannis Kokkinis, Petar Maksimović, Zoran Ognjanović, and Thomas Studer. First steps towards probabilistic justification logic. *Logic Journal of IGLP*, 2015.

[11] Roman Kuznets. *Complexity Issues in Justification Logic.* PhD thesis, City University of New York, May 2008.

[12] Roman Kuznets and Thomas Studer. Justifications, ontology, and conservativity. In Thomas Bolander, Torben Braüner, Silvio Ghilardi, and Lawrence Moss, editors, *Advances in Modal Logic, Volume 9*, pages 437–458. College Publications, 2012.

[13] Roman Kuznets and Thomas Studer. Update as evidence: Belief expansion. In Sergei [N.] Artemov and Anil Nerode, editors, *Logical Foundations of Computer Science, International Symposium, LFCS 2013, San Diego, CA, USA, January 6–8, 2013, Proceedings*, volume 7734 of *Lecture Notes in Computer Science*, pages 266–279. Springer, 2013.

[14] Eric Pacuit. A note on some explicit modal logics. In *Proceedings of the 5th Panhellenic Logic Symposium*, pages 117–125, Athens, Greece, July 25–28, 2005. University of Athens.

[15] Emil L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bull. Am. Math. Soc.*, 50:284–316, 1944.

[16] Thomas Studer. Decidability for some justification logics with negative introspection. *Journal of Symbolic Logic*, 78(2):388–402, June 2013.