Justification logic with approximate conditional probabilities

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Abstract

The importance of logics with approximate conditional probabilities is reflected by the fact that they can model non-monotonic reasoning. We introduce a new logic of this kind, CPJ, which extends earlier work in two respects. First, our base logic includes justifications, thus we can formalize non-monotonic reasoning with and about evidences. Second, we introduce a novel inference rule, which makes it possible to work with the usual definition of a consistent set of formulas. Hence CPJ is the first logic of approximate conditional probabilities for which a traditional strong completeness result can be established.

1 Introduction

Justification logic [5] is a variant of modal logic that 'unfolds' the $\Box$-modality into justification terms, i.e., justification logics replaces modal formulas $\Box\alpha$ with formulas of the form $t{:}\alpha$ where t is a justification term. The first justification logic, the Logic of Proofs, was developed by Artemov [1] with the aim of giving a classical provability interpretation for the modal logic S4 and hence also for intuitionistic logic. The Logic of Proofs interprets justification terms as formal proofs (e.g., in Peano Arithmetic) and thus $t{:}\alpha$ is read as *t is a proof of α* [1, 24].

Fitting [16] provides a possible world semantics for justification logics. Based on this epistemic semantics, a much more general interpretation of $t{:}\alpha$ is possible, namely *t is a justification for the agent's belief (or knowledge) in α*. This interpretation of justification logic has many applications and has been successfully employed to analyze many different epistemic situations [2, 3, 6, 10, 12]. Also dynamic epistemic logics and certain forms of defeasible knowledge have been studied in justification logics [7, 8, 9, 11, 23, 33].

In a general setting, justifications need not to be certain. Milnikel [27] was the first to approach this problem with his logic of uncertain justifications. Kokkinis et al. [19, 20, 21] study probabilistic justification logic, which provides a very general framework for uncertain reasoning with justifications that subsumes Milnikel's system. Further work in this area includes Fan and Liau's [15] possibilistic justification logic and Ghari's [17] fuzzy justification logic. Recently, Artemov [4] has introduced a justification logic to handle aggregated probabilistic evidence.

In the present paper we extend probabilistic justification logic with operators for approximate conditional probabilities. Formally, we introduce formulas $\mathsf{CP}_{\approx r}(\alpha,\beta)$ meaning *the probability of α under the condition β is approximately r*. This makes it possible to express defeasible

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inferences for justification logic. For instance, we can express

if x justifies that Tweety is a bird, then *usually* $t(x)$ justifies that Tweety flies

as $\mathsf{CP}_{\approx 1}(t(x){:}\mathsf{flies}, x{:}\mathsf{bird})$.

Our paper builds on previous work on probabilistic logics and non-monotonic reasoning. Logics with probability operators are important in artificial intelligence and computer science in general [14, 13, 28]. They are interpreted over Kripke-style models with probability measures over possible worlds. Ognjanović and Rašković [29, 30] develop probability logics with infinitary rules to obtain strong completeness results. The recent [31] provides an overview over the topic of probability logics.

Kraus et al. [22] propose a hierarchy of non-monotonic reasoning systems. In particular, they introduce a core system P for default reasoning and establish that P is sound and complete with respect to preferential models. Lehmann and Magidor [25] propose a family of non-standard ($^*\mathbb{R}$) probabilistic models. A default $\alpha \rightarrowtail \beta$ holds in a model of this kind if either the probability of α is 0 or the conditional probability of β given α is infinitesimally close to 1. Using this interpretation, they show that system P also is sound and complete with respect to $^*\mathbb{R}$-probabilistic models. Rašković et al. [32] present a logic with approximate conditional probabilities, $\mathsf{LPP}^{\mathsf{S}}$, whose models are a subclass of non-standard $^*\mathbb{R}$-probabilistic models. They prove the following:

Theorem 1. *For any finite default base Δ and for any default $\alpha \rightarrowtail \beta$*

$$\Delta \vdash_{\mathsf{P}} \alpha \rightarrowtail \beta \quad \textit{iff} \quad \Delta \vdash_{\mathsf{LPP}^{\mathsf{S}}} \alpha \rightarrowtail \beta.$$

First-order variants of $\mathsf{LPP}^{\mathsf{S}}$ have recently been studied by Ikodinović et al. [18]. Marchioni and Godo [26] present a fuzzy logic with conditional probabilities that uses non-standard probabilities.

There are two main contributions of this paper. The first, obviously, is the introduction of operators for approximate conditional probabilities to justification logic. This makes it possible to formalize non-monotonic reasoning with justifications. We study some initial examples of this feature in Section 6. The second contribution is of technical nature. Our approach closely follows [32]. However, there is no traditional strong completeness result established in [32], that means they do not show a result of the form: for each set of formulas T and each formula θ,

$$T \models \theta \quad \text{implies} \quad T \vdash \theta. \tag{1}$$

In fact, they only establish that every consistent set is satisfiable. But since they do not use the usual definition of a consistent set, their result does not imply (1). We introduce a new inference rule, which makes it possible to employ the usual definition of a consistent set of formulas and, thus, also to establish strong completeness of our system (in particular, see Remark 10 and Corollary 20). This is an important technical advancement compared to the earlier [32].

This paper is organized as follows. The next section recalls a basic system of justification logic. Section 3 introduces syntax and semantics for our justification logic CPJ with approximate conditional probabilities. Section 4 presents an axiomatization for CPJ. Section 5 establishes soundness and completeness of that axiomatization. Section 6 discusses some examples and concludes the paper.

2 Basic Justification Logic J

In this section we recall the basic notions and properties of the justification logic J.

Let C be a countable set of constants and V a countable set of variables. The formal grammar for justification terms is the following:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t,$$

where $c \in C$, $x \in V$. The set of all terms will be denoted by Term. For any non-negative integer n, we define

$$!^0 t := t \quad \textit{and} \quad !^{n+1} t := !(!^n t).$$

As usual, ! has greater precedence than $\cdot$ and $+$, and $\cdot$ has greater precedence than $+$ and both operators $+$ and $\cdot$ are left-associative. So, for example

$$!x \cdot y + z = (((!x) \cdot y) + z),$$

therefore, when it is clear from the context, parentheses will be omitted.

By Prop we denote a countable set of atomic propositions. We build justification formulas as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid t : \alpha,$$

where $t \in \mathsf{Term}$, $p \in \mathsf{Prop}$. The set of all justification formulas will be denoted by Fml_J and its elements will be denoted by $\alpha, \beta, \ldots$. Other classical Boolean connectives, $\vee$, $\rightarrow$, $\leftrightarrow$, as well as $\perp$ and $\top$, are defined as usual.

The axioms of the logic J are following three axioms:

(P) α, for all propositional tautologies α

(J) $u : (\alpha \rightarrow \beta) \rightarrow (v : \alpha \rightarrow u \cdot v : \beta)$

(+) $u : \alpha \vee v : \alpha \rightarrow u + v : \alpha$

Axiom (J), or the application axiom, states that we can obtain a justification for β, combining justifications for $\alpha \rightarrow \beta$ and α, while Axiom (+) states that if any of u of v (or both) is a justification for α, then $u + v$ is as well.

A *constant specification* is any set CS with

$$\mathsf{CS} \subseteq \{(c, \alpha) \mid c \in C, \alpha \text{ is an instance of any axiom of } \mathsf{J}\}.$$

For a given constant specification CS, we define the Hilbert-style *deductive system* J_CS by adding the following two rules to axioms of J:

(AN!) For $(c, \alpha) \in \mathsf{CS}$, $n \in \mathbb{N}$, infer $!^n c :!^{n-1} c : \cdots :!c : c : \alpha$

(MP) From α and $\alpha \rightarrow \beta$ infer β

Note that we use the standard notation $T \vdash_L A$, meaning that A is deducible from T in logic L. Often, L will be omitted because it is clear from the context.

The rule (AN!) is called axiom necessitation and gives us the connection between the constant specification and proofs in J_CS, while rule (MP) is the usual modus ponens.

In order to present the semantics of the logic J, first we need an auxiliary definition.

Definition 2. *Consider two sets of formulas $X, Y \subseteq \mathsf{Fml}_\mathsf{J}$. The set $X \cdot Y$ is defined as follows:*

$$X \cdot Y := \{\alpha \in \mathsf{Fml}_\mathsf{J} \mid \beta \rightarrow \alpha \in X \textit{ and } \beta \in Y \textit{ for some } \beta \in \mathsf{Fml}_\mathsf{J}\}.$$

Definition 3. *A* basic evaluation *for* J_{CS}*, where* CS *is any constant specification, is a function* $*$ *such that* $* : \mathsf{Prop} \to \{true, false\}$ *and* $* : \mathsf{Term} \to \mathcal{P}(\mathsf{Fml}_{\mathsf{J}})$*, and for* $u, v \in \mathsf{Term}$*, any constant* c *and* $\alpha \in \mathsf{Fml}_{\mathsf{J}}$ *we have:*

(1) $u^* \cdot v^* \subseteq (u \cdot v)^*$

(2) $u^* \cup v^* \subseteq (u + v)^*$

(3) if $(c, \alpha) \in \mathsf{CS}$*, then*

  (a) $\alpha \in c^*$

  (b) for each $n \in \mathbb{N}$, $!^n c :!^{n-1} c : \cdots :!c : c : \alpha \in (!^{n+1} c)^*$.

Instead of writing $*(t)$ and $*(p)$, we write t^* and p^* respectively. Now, we are ready to define the notion of truth under a basic evaluation.

Definition 4. *Let* $\alpha \in \mathsf{Fml}_{\mathsf{J}}$*. The binary relation* $\Vdash$ *is defined by:*

- *If* $\alpha = p \in \mathsf{Prop}$, $* \Vdash \alpha$ *iff* $p^* = true$
- *If* $\alpha = \neg\beta$, $* \Vdash \alpha$ *iff it is not the case that* $* \Vdash \beta$
- *If* $\alpha = \beta \wedge \gamma$, $* \Vdash \alpha$ *iff* $* \Vdash \beta$ *and* $* \Vdash \gamma$
- *If* $\alpha = t : \beta$, $* \Vdash \alpha$ *iff* $\beta \in t^*$.

For $T \subseteq \mathsf{Fml}_{\mathsf{J}}$ and $\alpha \in \mathsf{Fml}_{\mathsf{J}}$, $* \Vdash T$ means that $* \Vdash \beta$, for every $\beta \in T$, while $T \Vdash_{\mathsf{CS}} \alpha$ means that for every J_{CS}-evaluation $*$, $* \Vdash T$ implies $* \Vdash \alpha$.

Finally, we mention two important results, which are fundamental properties of justification logic. First, we state the deduction theorem for J. Sometimes, we will write T, α instead of $T \cup \{\alpha\}$.

Theorem 5. *Let* $T \subseteq \mathsf{Fml}_{\mathsf{J}}$ *and* $\alpha, \beta \in \mathsf{Fml}_{\mathsf{J}}$*. For any* J_{CS} *we have:*

$$T, \alpha \vdash_{\mathsf{J}_{\mathsf{CS}}} \beta \quad \textit{iff} \quad T \vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha \to \beta.$$

Second, J_{CS} is strongly complete with respect to basic evaluations.

Theorem 6. *For any constant specification* CS*,* $T \subseteq \mathsf{Fml}_{\mathsf{J}}$ *and any* $\alpha \in \mathsf{Fml}_{\mathsf{J}}$*, we have:*

$$T \vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha \quad \textit{iff} \quad T \Vdash_{\mathsf{CS}} \alpha.$$

3 The Logic CPJ

In this section we present the logic CPJ which is the logic that speaks about conditional probabilities over the justification logic J. First we introduce its syntax and semantics.

3.1 Syntax

Consider a non-standard elementary extension ${}^*\mathbb{R}$ of the real numbers. An element ϵ of ${}^*\mathbb{R}$ is called an infinitesimal iff $|\epsilon| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Let S be the unit interval of the Hardy field $\mathbb{Q}[\epsilon]$, which contains all rational functions of a fixed positive infinitesimal ϵ of ${}^*\mathbb{R}$, for details see, e.g., [18]. The language of our logic contains all symbols needed for defining the language of basic justification logic plus the following lists of operators:

- $(\mathsf{CP}_{\geq s})_{s \in S}$,

- $(\mathsf{CP}_{\leq s})_{s\in S}$,
- $(\mathsf{CP}_{\approx r})_{r\in\mathbb{Q}\cap[0,1]}$.

The set of probabilistic formulas, denoted by Fml_P, is the smallest set that contains all the formulas of the form

- $\mathsf{CP}_{\geq s}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $s\in S$,
- $\mathsf{CP}_{\leq s}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $s\in S$,
- $\mathsf{CP}_{\approx r}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $r\in\mathbb{Q}\cap[0,1]$.

and is closed under negation and conjuction: if $\varphi,\psi\in\mathsf{Fml}_\mathsf{P}$, then $\neg\varphi\in\mathsf{Fml}_\mathsf{P}$ and $\varphi\wedge\psi\in\mathsf{Fml}_\mathsf{P}$. Formulas from Fml_P are denoted by $\varphi,\psi,\ldots$, possibly indexed. Again we use the standard abbreviations for other Boolean connectives. The set of all formulas, Fml, of the logic CPJ is defined by $\mathsf{Fml}=\mathsf{Fml}_\mathsf{J}\cup\mathsf{Fml}_\mathsf{P}$. Elements of Fml will be denoted by $\theta,\theta_1,\theta_2,\ldots$. Note that nested probabilistic operators and Boolean combinations of justification formulas with probabilistic formulas is not allowed, i.e., the following are *not* formulas of our language:

- $t{:}p\wedge\mathsf{CP}_{\geq\frac{1}{2}}(t{:}p,s{:}q)$,
- $\mathsf{CP}_{=\frac{1}{3}}\mathsf{CP}_{\approx 0}(t{:}p,s{:}q)$.

We use the following abbreviations as well:

- $\mathsf{CP}_{<s}(\alpha,\beta)$ stands for $\neg\mathsf{CP}_{\geq s}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $s\in S$,
- $\mathsf{CP}_{>s}(\alpha,\beta)$ stands for $\neg\mathsf{CP}_{\leq s}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $s\in S$,
- $\mathsf{CP}_{=s}(\alpha,\beta)$ stands for $\mathsf{CP}_{\geq s}(\alpha,\beta)\wedge\mathsf{CP}_{\leq s}(\alpha,\beta)$, for $\alpha,\beta\in\mathsf{Fml}_\mathsf{J}$, $s\in S$,
- $\mathsf{P}_{\rho s}\alpha$ stands for $\mathsf{CP}_{\rho s}(\alpha,\top)$, for $\alpha\in\mathsf{Fml}_\mathsf{J}$, $\rho\in\{\geq,\leq,>,<,=,\approx\}$.

We can not define $\mathsf{CP}_{\geq}$ using $\mathsf{CP}_{\leq}$, or vice versa, since the case when the probability of the condition is equal to 0 breaks down the appropriate equivalence, see the next section.

3.2 Semantics

The semantics for the logic CPJ is based on possible worlds models.

Definition 7. *Let* CS *be the constant specification. A* CPJ_CS*-model (or just model) is a tuple* $M=\langle W,H,\mu,*\rangle$ *where:*

- W *is a non-empty set of objects called worlds*
- H *is an algebra of subsets of* W*, i.e., a set of subsets of* W *such that:*
  - $W\in H$,
  - *if* $A,B\in H$*, then* $W\setminus A\in H$ *and* $A\cup B\in H$.

  The elements of H *are called measurable sets of worlds.*
- μ *is a finitely additive probability measure defined on* H*, i.e.* $\mu:H\to S$ *and the following holds:*
  - $\mu(W)=1$,
  - $\mu(A\cup B)=\mu(A)+\mu(B)$*, whenever* $A\cap B=\emptyset$.
- $*$ *is a function from* W *to set of all basic* J_CS*-evaluations, i.e. for each* $w\in W$*,* $*(w)$ *is a basic* J_CS*-evaluation. Instead of* $*(w)$ *we will write* $*_w$.

Let $M = \langle W, H, \mu, * \rangle$. We define the set $[\alpha]_M$ as follows:

$$[\alpha]_M = \{w \in W \mid *_w \Vdash \alpha\}.$$

Whenever M is clear from the context, we will write $[\alpha]$ instead of $[\alpha]_M$.

Definition 8 (Measurable and Neat model)**.** *A* $\mathsf{CPJ_{CS}}$*-model* M *is* measurable *if and only if* $[\alpha]_M \in H$*, for every* $\alpha \in \mathsf{Fml_J}$*. A* $\mathsf{CPJ_{CS}}$*-model* M *is* neat *if and only if the empty set has the zero probability and no other set has. The class of all measurable and neat* $\mathsf{CPJ_{CS}}$ *models will be denoted by* $\mathsf{CPJ_{CS,Meas,Neat}}$.

Definition 9 (Satisfiability relation)**.** *Let* CS *be any constant specification. The satisfiability relation* $\models \subseteq \mathsf{CPJ_{CS,Meas,Neat}} \times \mathsf{Fml}$ *is defined, for any* $M \in \mathsf{CPJ_{CS,Meas,Neat}}$*, as follows:*

1. $M \models \alpha$ *if for every* $w \in W$, $*_w \Vdash \alpha$,
2. $M \models \mathsf{CP}_{\leq s}(\alpha, \beta)$ *if either* $\mu([\beta]) = 0$ *and* $s = 1$, *or* $\mu([\beta]) > 0$ *and* $\frac{\mu([\alpha \wedge \beta])}{\mu([\beta])} \leq s$,
3. $M \models \mathsf{CP}_{\geq s}(\alpha, \beta)$ *if either* $\mu([\beta]) = 0$, *or* $\mu([\beta]) > 0$ *and* $\frac{\mu([\alpha \wedge \beta])}{\mu([\beta])} \geq s$,
4. $M \models \mathsf{CP}_{\approx r}(\alpha, \beta)$ *if either* $\mu([\beta]) = 0$ *and* $r = 1$, *or* $\mu([\beta]) > 0$ *and for each* $n \in \mathbb{N}$, $\frac{\mu([\alpha \wedge \beta])}{\mu([\beta])} \in [\max\{0, r - \frac{1}{n}\}, min\{1, r + \frac{1}{n}\}]$,
5. $M \models \neg\varphi$ *iff it is not the case that* $M \models \varphi$,
6. $M \models \varphi \wedge \psi$ *iff* $M \models \varphi$ *and* $M \models \psi$.

We assume that the conditional probability is by default 1, whenever the condition has the probability 0, which explains the formulation of case 3 in the previous definition. Also, note that the case 4 is equivalent with: the conditional probability equals to $r - \epsilon$ (or $r + \epsilon$) for some infinitesimal $\epsilon \in S$. Defined operators $\mathsf{P}_{\rho s}$, for $\rho \in \{\geq, \leq, >, <, =, \approx\}$ behave as we expect. For example, let us prove that $M \models \mathsf{P}_{\geq s}\alpha$ iff $\mu([\alpha]) \geq s$.

$M \models \mathsf{P}_{\geq s}\alpha$ iff

$M \models \mathsf{CP}_{\geq s}(\alpha, \top)$ iff

$\mu([\top]) = 0$ or $\mu([\top]) > 0$ and $\frac{\mu([\alpha \wedge \top])}{\mu([\top])} \geq s$ iff (since $\mu([\top]) = 1 > 0$)

$\frac{\mu([\alpha \wedge \top])}{\mu([\top])} \geq s$ iff (using $\mu([\top]) = 1$ and $[\alpha \wedge \top] = [\alpha]$)

$\mu([\alpha]) \geq s$.

A formula $\theta \in \mathsf{Fml}$ is satisfiable iff there exists an $\mathsf{CPJ_{CS,Meas,Neat}}$-model M such that $M \models \theta$. Formula θ is valid if for every $\mathsf{CPJ_{CS,Meas,Neat}}$-model M we have $M \models \theta$. The set of formulas T is satisfiable iff there exists an $\mathsf{CPJ_{CS,Meas,Neat}}$-model M such that $M \models \theta$, for every $\theta \in T$. Moreover, $T \models_{\mathsf{CPJ_{CS,Meas,Neat}}} \theta$ means that for every model $M \in \mathsf{CPJ_{CS,Meas,Neat}}$, $M \models T$ implies $M \models \theta$.

4 An axiomatization for $\mathsf{CPJ_{CS}}$

In order to characterize the set of all valid formulas, we introduce an axiomatic system for the logic $\mathsf{CPJ_{CS}}$.

Axiom schemes

1. all $\mathsf{Fml_J}$-formulas derivable in $\mathsf{J_{CS}}$

2. all $\mathsf{Fml_P}$-instances of classical propositional tautologies
3. $\mathsf{CP}_{\geq 0}(\alpha, \beta)$
4. $\mathsf{CP}_{\leq s}(\alpha, \beta) \rightarrow \mathsf{CP}_{<t}(\alpha, \beta)$, $t > s$
5. $\mathsf{CP}_{<s}(\alpha, \beta) \rightarrow \mathsf{CP}_{\leq s}(\alpha, \beta)$
6. $\mathsf{P}_{\geq 1}(\alpha \leftrightarrow \beta) \rightarrow (\mathsf{P}_{=s}\alpha \rightarrow \mathsf{P}_{=s}\beta)$
7. $\mathsf{P}_{\leq s}\alpha \leftrightarrow \mathsf{P}_{\geq 1-s}\neg\alpha$
8. $(\mathsf{P}_{=s}\alpha \wedge \mathsf{P}_{=t}\beta \wedge \mathsf{P}_{\geq 1}\neg(\alpha \wedge \beta)) \rightarrow \mathsf{P}_{=min\{1,s+t\}}(\alpha \vee \beta)$
9. $\mathsf{P}_{=0}\beta \rightarrow \mathsf{CP}_{=1}(\alpha, \beta)$
10. $(\mathsf{P}_{=t}\beta \wedge \mathsf{P}_{=s}(\alpha \wedge \beta)) \rightarrow \mathsf{CP}_{=\frac{s}{t}}(\alpha, \beta)$, $t \neq 0$
11. $\mathsf{CP}_{\approx r}(\alpha, \beta) \rightarrow \mathsf{CP}_{\geq r_1}(\alpha, \beta)$, for rational $r_1 \in [0, r)$
12. $\mathsf{CP}_{\approx r}(\alpha, \beta) \rightarrow \mathsf{CP}_{\leq r_1}(\alpha, \beta)$, for rational $r_1 \in (r, 1]$.

Inference Rules

1. From θ_1 and $\theta_1 \rightarrow \theta_2$ infer θ_2.
2. From α infer $\mathsf{P}_{\geq 1}\alpha$, for $\alpha \in \mathsf{Fml_J}$.
3. From the set of premises $\{\varphi \rightarrow \mathsf{P}_{\neq s}\alpha \mid s \in S\}$ infer $\varphi \rightarrow \bot$.
4. Let $r \in \mathbb{Q} \cap [0, 1]$. From the two sets of premises $\{\varphi \rightarrow \mathsf{CP}_{\geq r-\frac{1}{n}}(\alpha, \beta) \mid n \geq \frac{1}{r}, n \in \mathbb{N}\}$ and $\{\varphi \rightarrow \mathsf{CP}_{\leq r+\frac{1}{n}}(\alpha, \beta) \mid n \geq \frac{1}{1-r}, n \in \mathbb{N}\}$ infer $\varphi \rightarrow \mathsf{CP}_{\approx r}(\alpha, \beta)$.
5. From $\mathsf{P}_{\leq 0}\alpha$ infer $\neg\alpha$, for $\alpha \in \mathsf{Fml_J}$.

Axiom 3, putting $\top$ instead of β, says that the probability of each formula being satisfied in some set of worlds is at least 0, and we can easily infer (using $\neg\alpha$ instead of α) that the upper bound is 1, i.e. $\mathsf{P}_{\leq 1}\alpha$. Axioms 4 and 5 say that we can weaken the degree of confidence of truth, while Axiom 6 says that equivalent formulas have the same probability. Axiom 8 corresponds to finite addivity of a measure. Axiom 9 ensures that the conditional probability is equal to 1 whenever the condition has probability 0. Axiom 10 is the formula that states the standard definition of the conditional probability. Finally, the Axioms 11 and 12 (together with Inference Rule 4) give us the relationship between the conditional probability infinitesimally close to the some rational number $r \in [0, 1]$ and the standard conditional probability.

Inference Rule 1 is modus ponens, while Rule 2 is the probabilistic necessitation and gives us the connection between justification formulas and probabilistic formulas. Rules 3 and 4 are infinitary rules of inference and Rule 3 states that the probability of any formula belongs to S. Rule 5 formalizes that in neat models only the empty set has probability zero. This rule makes it possible to infer non-probabilistic formulas from probabilistic ones.

Remark 10. *Our language includes two bottom elements $\bot_\mathsf{J} \in \mathsf{Fml_J}$ and $\bot_\mathsf{P} \in \mathsf{Fml_P}$. Because of Rules (2) and (5), they are interderivable. For any set of formulas T, we have*

$$T \vdash \bot_\mathsf{J} \quad \textit{if and only if} \quad T \vdash \bot_\mathsf{P}. \tag{2}$$

Therefore, it does often not matter which one we consider and we only write $\bot$.

Note that (2) only holds because $\mathsf{CPJ_{CS}}$ includes Rule 5. The system in [32] does not include that rule and thus (2) is not a valid principle for that system.

Given the previous remark, we can define the notion of a consistent set as usual. This will later be used to establish completeness of $\mathsf{CPJ_{CS}}$ (see Corollary 20), which is an important progress from the earlier [32].

Definition 11 (Inference relation, consistent set)**.**

- *$T \vdash \theta$ (θ is derivable from T) if there is an at most denumerable sequence of formulas $\theta_1, \theta_2, \dots, \theta$, such that every θ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule;*
- *$\vdash \theta$ (θ is a theorem) iff $\emptyset \vdash \theta$;*
- *T is consistent iff $T \nvdash \bot$. Otherwise T is inconsistent;*
- *T is maximally consistent set iff it is consistent and:*
  - *For every $\alpha \in \mathsf{Fml_J}$, if $T \vdash \alpha$, then $\alpha \in T$,*
  - *for every $\varphi \in \mathsf{Fml_P}$, either $\varphi \in T$ or $\neg\varphi \in T$.*
- *T is deductively closed if for every $\theta \in \mathsf{Fml}$, if $T \vdash \theta$, then $\theta \in T$.*

Note that if T is maximally consistent set and $\alpha \in T$, then also $\mathsf{P}_{\geq 1}\alpha \in T$ as well, because of the Inference Rule 2.

The deduction theorem holds for $\mathsf{CPJ_{CS}}$. The proof is similar to the corresponding proof for $\mathsf{LPP^S}$, see [32, Theorem 4], and therefore omitted.

Theorem 12 (Deduction theorem for the logic CPJ)**.** *Let T be a set of formulas and CS any constant specification. Then:*

$$T \cup \{\theta_1\} \vdash_{\mathsf{CPJ_{CS}}} \theta_2 \quad \textit{iff} \quad T \vdash_{\mathsf{CPJ_{CS}}} \theta_1 \to \theta_2,$$

where either both $\theta_1, \theta_2 \in \mathsf{Fml_J}$, or $\theta_1, \theta_2 \in \mathsf{Fml_P}$.

We have the following lemma, which is straightforward.

Lemma 13. *Let $\alpha, \beta \in \mathsf{Fml_J}$. Then the following holds:*

$$\begin{array}{ll}
\vdash \mathsf{CP}_{\geq t}(\alpha,\beta) \to \mathsf{CP}_{\geq s}(\alpha,\beta), t > s & \vdash \mathsf{CP}_{\leq t}(\alpha,\beta) \to \mathsf{CP}_{\leq s}(\alpha,\beta), t < s \\
\vdash \mathsf{CP}_{=t}(\alpha,\beta) \to \neg\mathsf{CP}_{=s}(\alpha,\beta), t \neq s & \vdash \mathsf{CP}_{=t}(\alpha,\beta) \to \neg\mathsf{CP}_{\geq s}(\alpha,\beta), t < s \\
\vdash \mathsf{CP}_{=t}(\alpha,\beta) \to \neg\mathsf{CP}_{\leq s}(\alpha,\beta), t > s & \vdash \mathsf{CP}_{=r}(\alpha,\beta) \to \mathsf{CP}_{\approx r}(\alpha,\beta), r \in \mathbb{Q} \cap [0,1] \\
\vdash \mathsf{P}_{=0}\beta \to \neg\mathsf{CP}_{\leq s}(\alpha,\beta), s < 1 & \vdash \mathsf{P}_{\leq 1}\alpha \\
\multicolumn{2}{l}{\vdash \mathsf{CP}_{\approx r_1}(\alpha,\beta) \to \neg\mathsf{CP}_{\approx r_2}(\alpha,\beta),\ r_1, r_2 \in \mathbb{Q} \cap [0,1], r_1 \neq r_2}
\end{array}$$

By setting $\beta = \top$ we obtain the analogous statements for unconditional probabilities.

5 Soundness and Completeness

The proofs for soundness and completeness of CPJ follow the patterns of the corresponding proofs for $\mathsf{LPP^S}$ given in [32]. For showing completeness, the tricky part is to establish a Lindenbaum lemma for $\mathsf{CPJ_{CS}}$.

Theorem 14 (Soundness)**.** *Let CS be any constant specification. The axiomatic system $\mathsf{CPJ_{CS}}$ is sound with respect to the class of $\mathsf{CPJ_{CS,Meas,Neat}}$-models.*

Proof. As usual, we show, by induction on the depth of $\mathsf{CPJ}_{\mathsf{CS}}$-derivations, that for any formula θ,

$$T \vdash_{\mathsf{CPJ}_{\mathsf{CS}}} \theta \quad \text{implies} \quad T \models_{\mathsf{CPJ}_{\mathsf{CS,Meas,Neat}}} \theta.$$

Let us only consider the case of Rule 5, where $M = \langle W, H, \mu, * \rangle$ is a $\mathsf{CPJ}_{\mathsf{CS,Meas,Neat}}$-model. Suppose $M \models \mathsf{P}_{\leq 0}\alpha$. We find that $\mu([\alpha]) = 0$. Since M is neat, we obtain $[\alpha] = \emptyset$ and thus $[\neg\alpha] = W$, which implies $M \models \neg\alpha$. □

In order to prove the strong completeness theorem, we first establish Lindenbaum's Lemma, i.e., how to extend a consistent set of formulas T to a maximal consistent set of formulas $T^{\clubsuit}$. Then, we construct the canonical model using the set $T^{\clubsuit}$ such that $\mathcal{M}_{T^{\clubsuit}} \models \theta$ iff $\theta \in T^{\clubsuit}$.

Theorem 15 (Lindenbaum)**.** *Let* CS *be any constant specification. Every* $\mathsf{CPJ}_{\mathsf{CS}}$*-consistent set can be extended to a maximal* $\mathsf{CPJ}_{\mathsf{CS}}$*-consistent set.*

Proof. Consider a consistent set T. We will only show how to construct a maximal consistent set that extends T. For the proofs of its consistency and maximality, we again refer to [32].

By $\mathsf{Cn}_{\mathsf{J}}(T)$, we will denote the set of all $\mathsf{Fml}_{\mathsf{J}}$ formulas that are J_{CS}-consequences of the set T. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all formulas from $\mathsf{Fml}_{\mathsf{J}}$ and let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all formulas from $\mathsf{Fml}_{\mathsf{P}}$. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ as follows:

1. $T_0 = T \cup \mathsf{Cn}_{\mathsf{J}}(T)$
2. for every $i \geq 0$,
   (a) if $T_{2i} \cup \{\varphi_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{\varphi_i\}$, otherwise
   (b) if φ_i is of the form $\varphi \to \mathsf{CP}_{\approx r}(\alpha, \beta)$, then $T_{2i+1} = T_{2i} \cup \{\neg\varphi_i, \varphi \to \neg\mathsf{CP}_{\geq r-\frac{1}{n}}(\alpha, \beta)\}$, or $T_{2i+1} = T_{2i} \cup \{\neg\varphi_i, \varphi \to \neg\mathsf{CP}_{\leq r+\frac{1}{n}}(\alpha, \beta)\}$, for some positive integer n, such that T_{2i+1} is consistent (below we will prove that such n exists), otherwise,
   (c) $T_{2i+1} = T_{2i} \cup \{\neg\varphi_i\}$.
3. for every $i \geq 0$, $T_{2i+2} = T_{2i+1} \cup \{\mathsf{P}_{=s}\alpha_i\}$, for some $s \in S$, such that T_{2i+2} is consistent (below we will prove that this is possible),
4. for every $i \geq 0$, if $\mathsf{P}_{=0}\alpha$ is added to the set T_i, also add $\neg\alpha$ to $T_i \cup \{P_{=0}\alpha\}$.

Let $T^{\clubsuit} = \bigcup_{i \geq 0} T_i$. One can show that $T^{\clubsuit}$ is a maximal consistent set extending T. □

$T^{\clubsuit}$ has the expected properties of a maximal consistent set.

Lemma 16. *Let* $\varphi, \psi \in \mathsf{Fml}_{\mathsf{P}}$ *and* $\alpha, \beta \in \mathsf{Fml}_{\mathsf{J}}$*. Then the following holds:*

1. $T^{\clubsuit}$ *contains all theorems*
2. $\varphi \in T^{\clubsuit}$ *implies* $\neg\varphi \notin T^{\clubsuit}$
3. $\varphi \wedge \psi \in T^{\clubsuit}$ *if and only if* $\varphi \in T^{\clubsuit}$ *and* $\psi \in T^{\clubsuit}$
4. $\varphi, \varphi \to \psi \in T^{\clubsuit}$ *implies* $\psi \in T^{\clubsuit}$
5. *There is the unique* $s \in S$ *such that* $\mathsf{P}_{=s}\alpha \in T^{\clubsuit}$
6. *There is the unique* $s \in S$ *such that* $\mathsf{CP}_{=s}(\alpha, \beta) \in T^{\clubsuit}$
7. *If* $\mathsf{CP}_{\geq s}(\alpha, \beta) \in T^{\clubsuit}$*, then exists* $r \in S$ *such that* $r \geq s$ *and* $\mathsf{CP}_{=r}(\alpha, \beta) \in T^{\clubsuit}$
8. *If* $\mathsf{CP}_{\leq s}(\alpha, \beta) \in T^{\clubsuit}$*, then exists* $r \in S$ *such that* $r \leq s$ *and* $\mathsf{CP}_{=r}(\alpha, \beta) \in T^{\clubsuit}$
9. *If* $\mathsf{CP}_{\approx r_1}(\alpha, \beta) \in T^{\clubsuit}$ *and* $r_2 \in (\mathbb{Q} \cap [0, 1]) \setminus \{r_1\}$*, then* $\mathsf{CP}_{\approx r_2}(\alpha, \beta) \notin T^{\clubsuit}$*.*

Now, we are ready to define the canonical model.

Definition 17. *Let* CS *be any constant specification and* $T^{\clubsuit}$ *be a maximally consistent set of formulas. Then, the canonical model* $M_{T^{\clubsuit}} = \langle W, H, \mu, * \rangle$ *is defined as follows:*

- $W = \{w \mid w$ *is a basic* $\mathsf{J_{CS}}$*-evaluation and* $w \Vdash J(T^{\clubsuit})\}$*, where* $J(T^{\clubsuit})$ *is the set of all* $\mathsf{Fml_J}$*-formulas in* $T^{\clubsuit}$
- $H = \{[\alpha] \mid \alpha \in \mathsf{Fml_J}\}$
- $\mu([\alpha]) = s$ *iff* $\mathsf{P}_{=s}\alpha \in T^{\clubsuit}$
- *For* $p \in \mathsf{Prop}$*,* $*_w(p) = true$ *iff* $w \Vdash p$*. For* $t \in \mathsf{Term}$*,* $*_w(t) = \{\alpha \mid w \Vdash t : \alpha\}$*.*

Theorem 18 (Canonical model is a well-defined $\mathsf{CPJ_{CS,Meas,Neat}}$-model)**.** *Let* $M_{T^{\clubsuit}}$ *be defined as above. Then the following holds:*

1. *H is an algerba of subsets of W*
2. μ *is well defined finitely additive probability measure, i.e.:*
   (a) *if* $[\alpha] = [\beta]$ *then* $\mu([\alpha]) = \mu([\beta])$
   (b) $\mu(W) = 1$
   (c) $[\alpha] \cap [\beta] = \emptyset$ *implies* $\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$
3. *for every* $\alpha \in \mathsf{Fml_J}$*,* $\mu([\alpha]) = 0$ *iff* $[\alpha] = \emptyset$*.*

Theorem 19. *Let* CS *be any constant specification. Every consistent* $\mathsf{CPJ_{CS}}$*-set* T *is satisfiable.*

As a corollary, we obtain that the axiomatic system $\mathsf{CPJ_{CS}}$ is strongly complete with respect to the class of $\mathsf{CPJ_{CS,Meas,Neat}}$-models.

Corollary 20 (Strong completeness)**.** *Let* CS *be any constant specification. For any set of formulas* T *and any formula* θ*,*

$$T \models_{\mathsf{CPJ_{CS,Meas,Neat}}} \theta \quad \textit{implies} \quad T \vdash_{\mathsf{CPJ_{CS}}} \theta.$$

6 Conclusion

We extended probabilistic justification logic by operators for approximate conditional probabilities, which makes it possible to express defaults in justification logic. In particular:

$$\mathsf{CP}_{\approx 1}(t(x){:}\mathsf{flies}, x{:}\mathsf{bird}) \tag{3}$$

means *if x justifies that Tweety is a bird, then usually t(x) justifies that Tweety flies*;

$$\mathsf{CP}_{\approx 1}(\neg t(x){:}\mathsf{flies}, x{:}\mathsf{penguin}) \tag{4}$$

means *if x justifies that Tweety is a penguin, then usually it is not the case that t(x) justifies that Tweety flies*;

$$\mathsf{CP}_{\approx 1}(x{:}\mathsf{bird}, x{:}\mathsf{penguin}) \tag{5}$$

means *if x justifies that Tweety is a penguin, then usually x also justifies that Tweety is a bird.*

Similar to [25, 32], it is possible to show that (the corresponding translations) of the axioms and rules of system P are sound with respect to CPJ. In particular we can apply the rule of cautious monotonicity to (4) and (5) in order to infer

$$\mathsf{CP}_{\approx 1}(\neg t(x){:}\mathsf{flies}, x{:}\mathsf{penguin} \wedge x{:}\mathsf{bird}),$$

which is consistent with (3).

Besides the possibility of expressing defaults, CPJ also features non-monotonic versions of classical operations on justifications. Let us consider the sum operator with its defining axiom

$$u : \alpha \vee v : \alpha \to u + v : \alpha. \tag{6}$$

This axiom states that justifications are monotone: if u justifies α, then the combination of u with v still justifies α. Often the sum operation is motivated as follows. Think of u and v as two volumes of book collection and $u + v$ as the set of those two volumes. Imagine that volume u contains a justification for a proposition α, i.e., $u : \alpha$ is the case. Then the larger set $u + v$ also contains a justifiction for α, i.e., $u + v : \alpha$. This idea is reflected in the provability semantics of justification logic where the sum operation is interpreted as proof concatenation, which, of course, is monotone.

This motivational example can also be read in another way. It is possible that the second volume v contains a retraction of α, i.e., it could withdraw the justification given for α in volume u. To model situations of this kind, one could introduce a non-monotonic sum operation, $\tilde{+}$, such that

$$\mathsf{CP}_{\approx 1}(u \tilde{+} v{:}\alpha, u{:}\alpha) \quad \text{and} \quad \mathsf{CP}_{\approx 1}(u \tilde{+} v{:}\alpha, v{:}\alpha).$$

Using the (Or) rule of system P we get

$$\mathsf{CP}_{\approx 1}(u \tilde{+} v{:}\alpha, u{:}\alpha \vee v{:}\alpha),$$

which is a non-monotonic version of (6).

We have introduced the system CPJ of justification logic with approximate conditional probabilities. On the conceptual level, the main contribution is that CPJ makes it possible to model non-monotonic reasoning with and about justifications. On a technical level, our main contribution is that CPJ includes a novel inference rule that earlier system with approximate conditional probability operators lacked. Hence CPJ is the first system of this kind for which a traditional strong completeness result can be established (see Remark 10 and Corollary 20). Future work includes the study of computational aspects of CPJ. It will also be interesting to examine the relationship of CPJ and other forms of defeasible reasoning with justifications.

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