Probabilistic Justification Logic

Ioannis Kokkinis[a], Thomas Studer[a,*], Zoran Ognjanović[b]

[a] *Institute of Computer Science, University of Bern, Switzerland*
{kokkinis,tstuder}@inf.unibe.ch

[b] *Mathematical Institute SANU, Belgrade, Serbia*
zorano@mi.sanu.ac.rs

Abstract

We present a probabilistic justification logic, PPJ, as a framework for uncertain reasoning about rational belief, degrees of belief and justifications. We establish soundness and strong completeness for PPJ with respect to the class of so-called measurable Kripke-like models and show that the satisfiability problem is decidable. We discuss how PPJ provides insight into the well-known lottery paradox.

Keywords: Justification logic, probabilistic logic, strong completeness, decidability, lottery paradox

1. Introduction

In epistemic modal logic, we use formulas of the form $\Box A$ to express that A *is believed*. Justification logic unfolds the $\Box$-modality into a family of so-called *justification terms* to represent evidence for an agent's belief. That is in justification logic we use $t : A$ to state that A *is believed for reason* t.

Originally, Artemov developed the first justification logic, the Logic of Proofs, to give a classical provability semantics for intuitionistic logic [1, 2, 18]. Later, Fitting [8] introduced epistemic models for justification logic. As it turned out this interpretation provides a very successful approach to study many epistemic puzzles and problems [3, 6, 17].

In this paper, we extend justification logic with probability operators in order to accommodate the idea that

different kinds of evidence for A lead to different degrees of belief in A. (1)

In [13] we have introduced a first probabilistic justification logic PJ, which features formulas of the form $P_{\geq s}(t : A)$ to state that *the probability of* $t : A$ *is*

*Corresponding author

greater than or equal to s. The language of PJ, however, does neither include justification statements over probabilities (i.e. $t : (P_{\geq s} A)$) nor iterated probabilities (i.e. $P_{\geq r}(P_{\geq s} A)$).

In the present paper, we remedy these shortcomings and present the logic PPJ, which supports formulas of the form $t : (P_{\geq s} A)$ as well as $P_{\geq r}(P_{\geq s} A)$. This explains the name PPJ: the two P's refer to iterated P-operators. We introduce syntax and semantics for PPJ and establish soundness and strong completeness. We also show that satisfiability for PPJ is decidable. In the final part we present an application of PPJ to the lottery paradox.

Related work. The design of PPJ follows that of LPP_1, which is a probability logic over classical propositional logic [24, 25]. The proofs that we present for PPJ are extensions of the corresponding proofs for LPP_1. Note, however, that these extensions are non-trivial due to the presence of formulas of the form $t : (P_{\geq s} A)$.

As already mentioned, PJ [13] is the precursor of PPJ without iterations of probability operators. Kokkinis [12] shows that PJ has the same complexity as the underlying justification logic.

Our probability logics are not compact. Consider the set

$$T := \{\neg P_{=0} A\} \cup \{P_{<1/n} A \mid n \text{ is a positive integer}\}.$$

Although every finite subset of T is satisfiable, the set T is not. Hence in order to obtain a strong completeness result, we use an infinitary rule, which originates from [11, 24, 26].

Milnikel [22] proposes a logic with uncertain justifications. We thoroughly study the relationship between Milnikel's logic and our approach in [13] where we show that three of his four axioms are theorems in our logic and that the fourth axiom holds under an additional independence assumption.

In the preprint [10], Ghari presents fuzzy variants of justification logic, in which an agent can have a justification for a statement with certainty between 0 and 1. He introduces fuzzy Fitting models and establishes a graded completeness theorem. Ghari also shows that Milnikel's principles are valid in his fuzzy setting.

Fan and Liau [7] introduce a possibilistic justification logic, which is an explicit version of a graded modal logic. Their logic includes formulas $t :_r A$ to express that *according to evidence* t*,* A *is believed with certainty at least* r. However, the following principle holds in their logic:

$$s :_r A \wedge t :_q A \to s :_{\max(r,q)} A.$$

Hence all justifications for a belief yield the same (strongest) certainty, which is not in accordance with our guiding idea (1).

Artemov [5] studies a justification logic to formalize aggregated probabilistic evidence. His approach can handle conflicting and inconsistent data and as well as positive and negative evidence for the same proposition.

Funding

Ioannis Kokkinis and Thomas Studer are supported by the SNSF project 153169, *Structural Proof Theory and the Logic of Proofs*. Zoran Ognjanović is supported by the Serbian Ministry of Education, Science and Technological Development.

2. The Probabilistic Justification Logic PPJ

In this section we present syntax and semantics for the probabilistic justification logic PPJ.

Syntax

Justification terms are built from countably many constants and countably many variables according to the following grammar:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t$$

where c is a constant and x is a variable. Tm denotes the set of all terms and Con denotes the sets of all constants. For any term t and natural number n we define $!^0 t := t$ and $!^{n+1} t := !\,(!^n t)$.

Let Prop be a countable set of atomic propositions. We will represent the set of all rational numbers with the symbol $\mathbb{Q}$. We define $\mathbb{Q}[0,1] := \mathbb{Q} \cap [0,1]$, while $\mathbb{Q}[0,t)$ will denote the set $[0,t) \cap \mathbb{Q}[0,1]$.

The set of formulas $\mathcal{L}$ is defined by the following grammar[1]:

$$A ::= p \mid P_{\geq s} A \mid \neg A \mid A \wedge A \mid t : A$$

where $t \in \mathsf{Tm}$, $s \in \mathbb{Q}[0,1]$ and $p \in \mathsf{Prop}$. We define the following abbreviations:

$$\begin{aligned} A \vee B &\equiv \neg(\neg A \wedge \neg B) \\ A \to B &\equiv \neg A \vee B \\ A \leftrightarrow B &\equiv (A \to B) \wedge (B \to A) \\ \bot &\equiv A \wedge \neg A, \text{for some } A \in \mathcal{L} \\ \top &\equiv A \vee \neg A, \text{for some } A \in \mathcal{L} \end{aligned}$$

Additionally, we set

$$\begin{aligned} P_{<s} A &\equiv \neg P_{\geq s} A & \qquad P_{\leq s} A &\equiv P_{\geq 1-s} \neg A \\ P_{>s} A &\equiv \neg P_{\leq s} A & \qquad P_{=s} A &\equiv P_{\geq s} A \wedge P_{\leq s} A \end{aligned}$$

The axiom schemes of PPJ are presented in Figure 1.

[1] In order to have a countable language and obtain decidability, we restrict our probabilistic operators to the rational numbers.

(P)	finitely many schemes in the language of $\mathcal{L}$ axiomatizing classical propositional logic
(J)	$\vdash u : (A \to B) \to (v : A \to u \cdot v : B)$
(+)	$\vdash u : A \lor v : A \to u + v : A$
(PI)	$\vdash P_{\geq 0} A$
(WE)	$\vdash P_{\leq r} A \to P_{<s} A$, where $s > r$
(LE)	$\vdash P_{<s} A \to P_{\leq s} A$
(DIS)	$\vdash P_{\geq r} A \land P_{\geq s} B \land P_{\geq 1} \neg (A \land B) \to P_{\geq \min(1, r+s)}(A \lor B)$
(UN)	$\vdash P_{\leq r} A \land P_{<s} B \to P_{<r+s}(A \lor B)$, where $r + s \leq 1$

Figure 1: Axioms Schemes of PPJ

A *constant specification* is any set CS that satisfies

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an instance of some axiom of } \mathsf{PPJ}\} .$$

A constant specification CS is called:

axiomatically appropriate: if for every axiom instance A of PPJ, there exists a constant c such that $(c, A) \in \mathsf{CS}$;

schematic: if for every constant c, the set $\{A \mid (c, A) \in \mathsf{CS}\}$ consists of all instances of several (possibly zero) axiom schemes;

finite: if CS is a finite set;

almost schematic: if $\mathsf{CS} = \mathsf{CS}_1 \cup \mathsf{CS}_2$ where $\mathsf{CS}_1 \cap \mathsf{CS}_2 = \emptyset$, CS_1 is schematic and CS_2 is finite.

The notion of a schematic constant specification is crucial for establishing decidability of PPJ.

Let CS be any constant specification. The deductive system $\mathsf{PPJ}_{\mathsf{CS}}$ is the Hilbert system obtained by adding to the axioms of PPJ the rules (MP), (CE), (ST) and (AN!) as given in Figure 2. Note that (ST) is an infinitary rule, which we need to obtain strong completeness.

A formula A is deducible from a set T of formulas ($T \vdash A$) if there is an at most countable sequence of formulas $A_0, A_1, \ldots, A$ such that every A_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Rule (CE) can be applied on the theorems only. A formula A is a theorem ($\vdash A$) if it is deducible from the empty set.

We will use the following abbreviations when we present proofs.

	axioms of PPJ
	$+$
($\mathsf{AN!}$)	$\vdash\, !^{n+1}c : !^n c : \cdots : !c : c : A$, where $(c, A) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(MP)	if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(CE)	if $\vdash A$ then $\vdash P_{\geq 1} A$
(ST)	if $T \vdash A \to P_{\geq s - \frac{1}{k}} B$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to P_{\geq s} B$

Figure 2: System $\mathsf{PPJ}_{\mathsf{CS}}$

P.R. stands for *propositional reasoning*. E.g. when we have $\vdash A \to B$ we can claim that by **P.R.** we get $\vdash \neg B \to \neg A$. We can think of **P.R.** as an abbreviation of the phrase *by some applications of* (P) *and* (MP).

S.E. stands for *syntactical equivalence*. E.g., according to our syntactical conventions from the beginning of this section the formulas $P_{\geq 1-s}(A \vee B)$ and $P_{\leq s}(\neg A \wedge \neg B)$ are syntactically equivalent. We will transform our formulas to syntactically equivalent ones in order to increase readability of our proofs. We have to be very careful when we apply **S.E.** For example the formulas $P_{\geq s}(\neg A \vee B)$ and $P_{\geq s}(A \to B)$ are syntactically equivalent, whereas the formulas $P_{\geq s} A$ and $P_{\geq s} \neg\neg A$ are not.

Semantics

To introduce semantics for $\mathsf{PPJ}_{\mathsf{CS}}$, we begin with the notion of a basic evaluation, which is the cornerstone for many interpretations of justification logic [4, 16]. In the following we use $\mathcal{P}(X)$ to denote the power set of a set X. We will also use the symbols T and F to represent the truth values true and false respectively.

Definition 1 (Basic Evaluation)**.** Let CS be a constant specification. A *basic evaluation* for CS, or a basic CS-evaluation, is a function $*$ that maps atomic propositions to truth values and maps justification terms to subsets of $\mathcal{L}$, i.e.

$$* : \mathsf{Prop} \to \{\mathsf{T}, \mathsf{F}\} \quad \text{and} \quad * : \mathsf{Tm} \to \mathcal{P}(\mathcal{L}),$$

such that for $u, v \in \mathsf{Tm}$, for $c \in \mathsf{Con}$ and $A, B \in \mathcal{L}$ we have:

1. $\big(A \to B \in u^*$ and $A \in v^*\big) \Longrightarrow B \in (u \cdot v)^*$
2. $u^* \cup v^* \subseteq (u + v)^*$
3. if $(c, A) \in \mathsf{CS}$ then for all $n \in \mathbb{N}$ we have[2]:

$$!^{n-1}c : !^{n-2}c : \cdots :!c : c : A \in (!^n c)^*$$

[2] We agree to the convention that the formula $!^{n-1}c : !^{n-2}c : \cdots : !c : c : A$ represents the formula A for $n = 0$.

We usually write t^* and p^* instead of $*(t)$ and $*(p)$, respectively.

Definition 2 (Algebra over a Set)**.** Let W be a non-empty set and let H be a non-empty subset of $\mathcal{P}(W)$. We call H an *algebra over* W iff the following hold:

- $W \in H$;
- $U, V \in H \Longrightarrow U \cup V \in H$;
- $U \in H \Longrightarrow W \setminus U \in H$.

Definition 3 (Finitely Additive Measure)**.** Let H be an algebra over W and $\mu : H \to [0, 1]$. We call μ a *finitely additive measure* iff the following hold:

1. $\mu(W) = 1$;
2. for all $U, V \in H$:

$$U \cap V = \emptyset \quad \Longrightarrow \quad \mu(U \cup V) = \mu(U) + \mu(V) \ .$$

Definition 4 (Probability Space)**.** A *probability space* is a triple

$$\mathsf{Prob} = \langle W, H, \mu \rangle \ ,$$

where:

- W is a non-empty set;
- H is an algebra over W;
- $\mu : H \to [0, 1]$ is a finitely additive measure.

Definition 5 (Model)**.** Let CS be a constant specification. A $\mathsf{PPJ}_{\mathsf{CS}}$-*model* is a quintuple $M = \langle U, W, H, \mu, * \rangle$ where:

1. U is a non-empty set of objects called worlds;
2. W, H, μ and $*$ are functions, which have U as their domain, such that for every $w \in U$:
   - $\langle W(w), H(w), \mu(w) \rangle$ is a probability space with $W(w) \subseteq U$;
   - $*_w$ is a basic CS-evaluation[3].

The ternary satisfaction relation $\models$ is defined between models, worlds, and formulas.

[3]We will usually write $*_w$ instead of $*(w)$.

Definition 6 (Truth in a $\mathsf{PPJ_{CS}}$-model)**.** Let CS be a constant specification and let $M = \langle U, W, H, \mu, * \rangle$ be a $\mathsf{PPJ_{CS}}$-model. We define what it means for an $\mathcal{L}$-formula to hold in M at a world $w \in U$ inductively as follows:

$$
\begin{aligned}
M, w \models p \quad &:\Longleftrightarrow \quad p^*_w = \mathsf{T} \quad \text{for } p \in \mathsf{Prop} \\
M, w \models P_{\geq s} B \quad &:\Longleftrightarrow \quad \Big([B]_{M,w} \in H(w) \text{ and } \mu(w)\big([B]_{M,w}\big) \geq s\Big) \\
&\qquad\quad \text{where } [B]_{M,w} = \{x \in W(w) \mid M, x \models B\} \\
M, w \models \neg B \quad &:\Longleftrightarrow \quad M, w \not\models B \\
M, w \models B \wedge C \quad &:\Longleftrightarrow \quad \big(M, w \models B \text{ and } M, w \models C\big) \\
M, w \models t : B \quad &:\Longleftrightarrow \quad B \in t^*_w
\end{aligned}
$$

Definition 7 (Measurable Model)**.** Let CS be a constant specification and let $M = \langle U, W, H, \mu, * \rangle$ be a $\mathsf{PPJ_{CS}}$-model. M is called measurable iff for every $w \in U$ and for every $A \in \mathcal{L}$:

$$[A]_{M,w} \in H(w)$$

$\mathsf{PPJ_{CS,Meas}}$ denotes the class of $\mathsf{PPJ_{CS}}$-measurable models.

For a model $M = \langle U, W, H, \mu, * \rangle$, $M \models A$ means that $M, w \models A$ for all $w \in U$. Let $T \subseteq \mathcal{L}$. Then $M \models T$ means that $M \models A$ for all $A \in T$. Further $T \models A$ means that for all $M \in \mathsf{PPJ_{CS,Meas}}$, $M \models T$ implies $M \models A$.

To be precise we should write $T \vdash_{\mathsf{CS}} A$ and $T \models_{\mathsf{CS}} A$ instead of $T \vdash A$ and $T \models A$, respectively, since these two notions depend on a given constant specification CS. However, CS will always be clear from the context and thus can be omitted.

Definition 8 (Satisfiability)**.** We say a formula A of $\mathcal{L}$ is satisfiable if there exist a $\mathsf{PPJ_{CS,Meas}}$-model $M = \langle U, W, H, \mu, * \rangle$ and $w \in U$ with $M, w \models A$. A set of formulas T is satisfiable if there is a world w from a model M such that every formula from T holds in M at w.

The following lemma was proved for the probabilistic justification logic PJ in [13]. The same proof also works in PPJ.

Lemma 9. *For any* $\mathsf{PPJ_{CS}}$ *we have:*

$$\vdash A \to B \quad \Longrightarrow \quad \vdash P_{\geq s} A \to P_{\geq s} B$$

We established the Deduction Theorem for PJ in [13]. Now we present the version for PPJ, which can be proved in the same way.

Theorem 10 (Deduction Theorem)**.** *Let* $T \subseteq \mathcal{L}$ *and* $A, B \in \mathcal{L}$*. For any constant specification* CS *we have:*

$$T, A \vdash B \quad \Longleftrightarrow \quad T \vdash A \to B$$

3. Soundness and Completeness

As usual, we can establish soundness by induction on the depth of the derivation of a formula A.

Theorem 11 (Soundness)**.** *For any constant specification* CS*,* $\mathsf{PPJ}_{\mathsf{CS}}$ *is sound with respect to the class of* $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$*-models. I.e. for any* $A \in \mathcal{L}$ *and* $T \subseteq \mathcal{L}$ *we have:*

$$T \vdash A \quad \Longrightarrow \quad T \models A.$$

The completeness proof for $\mathsf{PPJ}_{\mathsf{CS}}$ is a combination of the completeness proof for LPP_1 [25] and the completeness proof for PJ [13]. In the rest of the section we are going to present a series of definitions and lemmata that lead to the strong completeness theorem for PPJ. When the proofs are very similar to the ones for PJ we will simply provide a reference to [13].

First we need the notion of a $\mathsf{PPJ}_{\mathsf{CS}}$-consistent set.

Definition 12 ($\mathsf{PPJ}_{\mathsf{CS}}$-consistent Set)**.** Let CS be a constant specification and let T be a set of $\mathcal{L}$-formulas.

- T is said to be $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent* iff $T \nvdash \bot$. Otherwise T is said to be $\mathsf{PPJ}_{\mathsf{CS}}$*-inconsistent*.
- T is said to be *maximal* iff for every $A \in \mathcal{L}$ either $A \in T$ or $\neg A \in T$.
- T is said to be *maximal* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent* iff it is maximal and $\mathsf{PPJ}_{\mathsf{CS}}$-consistent.

The next two lemmata state some standard properties of $\mathsf{PPJ}_{\mathsf{CS}}$-consistent sets. Both lemmata have been proved in [13] for the logic PJ. The proofs for PPJ are similar. Lemma 13 is essential for the construction of maximal consistent sets. Its second claim is needed to deal with the rule (ST).

Lemma 13 (Properties of $\mathsf{PPJ}_{\mathsf{CS}}$-Consistent Sets)**.** *Let* CS *be a constant specification and let* T *be a* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set of* $\mathcal{L}$*-formulas.*

(1) For any formula $A \in \mathcal{L}$ *we have that* T, A *is* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent or* $T, \neg A$ *is* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent.*

(2) If $\neg(A \to P_{\geq s}B) \in T$ *for* $s > 0$*, then there is some integer* $n \geq \frac{1}{s}$ *such that* $T, \neg(A \to P_{\geq s-\frac{1}{n}}B)$ *is* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent.*

Lemma 14 (Properties of Maximal $\mathsf{PPJ}_{\mathsf{CS}}$-Consistent Sets)**.** *Let* CS *be a constant specification and let* $\mathcal{T}$ *be a maximal* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set. Then the following hold:*

(1) For any formula $A \in \mathcal{L}$*, exactly one member of* $\{A, \neg A\}$ *is in* $\mathcal{T}$*.*

(2) For any formula $A \in \mathcal{L}$*:*

$$\mathcal{T} \vdash A \Longleftrightarrow A \in \mathcal{T}$$

(3) For all formulas $A, B \in \mathcal{L}$ we have:

$$A \vee B \in \mathcal{T} \Longleftrightarrow A \in \mathcal{T} \text{ or } B \in \mathcal{T}$$

(4) For all formulas $A, B \in \mathcal{L}$ we have:

$$A \wedge B \in \mathcal{T} \Longleftrightarrow \{A, B\} \subseteq \mathcal{T}$$

(5) For all formulas $A, B \in \mathcal{L}$ we have:

$$\{A, A \to B\} \subseteq \mathcal{T} \Longrightarrow B \in \mathcal{T}$$

(6) Let $A \in \mathcal{L}$, $X = \{s \mid P_{\geq s}A \in \mathcal{T}\}$ and $t = \sup(X)$. Then:

(i) For all $r \in \mathbb{Q}[0, t)$ we have that $P_{>r}A \in \mathcal{T}$

(ii) For all $r \in \mathbb{Q}[0, t)$ we have that $P_{\geq r}A \in \mathcal{T}$

(iii) If $t \in \mathbb{Q}[0, 1]$ then $P_{\geq t}A \in \mathcal{T}$.

(iv) For any $r \in \mathbb{Q}[0, 1]$:

$$t \geq r \Longleftrightarrow P_{\geq r}A \in \mathcal{T}$$

As the next step we have to show that Lindenbaum lemma holds for the logic PPJ.

Lemma 15 (Lindenbaum)**.** *Let* CS *be a constant specification. Every* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set can be extended to a maximal* $\mathsf{PPJ}_{\mathsf{CS}}$*-consistent set.*

Proof. This lemma has been proved for the logic PJ in [13] and the proof for PPJ is similar. However, it is worth highlighting that the proof of the Lindenbaum lemma for the logics PJ and PPJ is much more complex than usual. The reason is that the logics PJ and PPJ can have proofs of infinite depth because of the rule (ST). We briefly explain the amendments that need to be done to the usual proof for Lindenbaum lemma.

The typical construction of the maximal consistent set in Lindenbaum lemma is as follows:

Let T be a $\mathsf{PPJ}_{\mathsf{CS}}$-consistent set. Let $A_0,\ A_1,\ A_2,\ \ldots$ be an enumeration of all formulas in $\mathcal{L}$. We define a sequence of sets $\{T_i\}_{i \in \mathbb{N}}$ such that:

(1) $T_0 := T$

(2) for every $i \geq 0$:

(a) if $T_i \cup \{A_i\}$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent, then we set $T_{i+1} := T_i \cup \{A_i\}$, otherwise

(b) we set $T_{i+1} := T_i \cup \{\neg A_i\}$

(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$

It is then straightforward to show that $\mathcal{T}$ is $\mathsf{PPJ}_{\mathsf{CS}}$-maximal and that every T_i is $\mathsf{PPJ}_{\mathsf{CS}}$-consistent. However the consistency of every T_i does not imply the consistency of $\mathcal{T}$, since proofs in PJ and PPJ may have infinite depth. We can tackle this problem by adding the formula $\neg(B \to P_{\geq s-\frac{1}{n}}C)$ to T_{i+1} for a suitable n, when A_i is of the form $B \to P_{\geq s}C$ and $T_i \cup \{A_i\}$ is not $\mathsf{PJ}_{\mathsf{CS}}$-consistent. Lemma 13 guarantees the existence of such an n. □

Now we define a canonical model for any maximal $\mathsf{PPJ}_{\mathsf{CS}}$-consistent set of formulas.

Definition 16 (Canonical Model)**.** Let CS be a constant specification. The *canonical model* for $\mathsf{PPJ}_{\mathsf{CS}}$ is given by the quintuple $M = \langle U, W, H, \mu, * \rangle$, defined as follows:

- $U = \{w \mid w \text{ is a maximal } \mathsf{PPJ}_{\mathsf{CS}}\text{-consistent set of } \mathcal{L}\text{-formulas}\}$
- for every $w \in U$ the probability space $\langle W(w), H(w), \mu(w) \rangle$ is defined as follows:
  1. $W(w) = U$
  2. $H(w) = \{(A)_M \mid A \in \mathcal{L}\}$
     where $(A)_M = \{x \mid x \in U, A \in x\}$. If M is clear from the context, we may simply write (A) instead of $(A)_M$.
  3. for all $A \in \mathcal{L}$, $\mu(w)\big((A)_M\big) = \sup_s \{P_{\geq s}A \in w\}$
- for every $w \in W$ the basic CS-evaluation $*_w$ is defined as follows:
  1. for all $p \in \mathsf{Prop}$:
     $$p^*_w = \begin{cases} \mathsf{T} & \text{if } p \in w \\ \mathsf{F} & \text{if } \neg p \in w \end{cases}$$
  2. for all $t \in \mathsf{Tm}$:
     $$t^*_w = \{A \mid t : A \in w\}$$

The following properties of the set $(A)_M$ are direct consequences of Lemma 14.

Lemma 17. *Let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for some $\mathsf{PPJ}_{\mathsf{CS}}$ and let $A, B \in \mathcal{L}$. Then the following hold:*

(i) $(\neg A)_M = U \setminus (A)_M$

(ii) $(A)_M \cap (B)_M = (A \wedge B)_M$

(iii) $(A)_M \cup (B)_M = (A \vee B)_M$

Now we will prove that the canonical model for $\mathsf{PPJ}_{\mathsf{CS}}$ is a $\mathsf{PPJ}_{\mathsf{CS}}$-model.

Lemma 18. *Let CS be a constant specification. The canonical model for $\mathsf{PPJ}_{\mathsf{CS}}$ is a $\mathsf{PPJ}_{\mathsf{CS}}$-model.*

Proof. Let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for $\mathsf{PPJ}_{\mathsf{CS}}$. In order for M to be a $\mathsf{PPJ}_{\mathsf{CS}}$-model we have to prove the following:

U is a non-empty set:

There exists a $\mathsf{PPJ}_{\mathsf{CS}}$-maximal consistent set. Thus $U \neq \emptyset$.

For every $w \in U$ the triple $\langle W(w), H(w), \mu(w) \rangle$ is a probability space:

We have to prove the following:

(1) **$W(w)$ is a non-empty subset of U**

It is obvious since $W(w) = U$ and $U \neq \emptyset$.

(2) **$H(w)$ is an algebra over $W(w)$**

It holds that:

$$(\top)_M = \{x \mid x \in U, \top \in x\} = U = W(w) \ .$$

Thus $W(w) \in H(w)$.

Let $(A)_M \in H(w)$ for some $A \in \mathcal{L}$. It holds that:

$$(A)_M = \{x \mid x \in U, A \in x\} \subseteq U = W(w) \ .$$

Thus $H(w) \subseteq \mathcal{P}(W(w))$.

Let $(A), (B) \in H(w)$ for some $A, B \in \mathcal{L}$. By Lemma 17 we have that $W(w) \setminus (A) = U \setminus (A) = (\neg A) \in H(w)$ and $(A) \cup (B) = (A \vee B) \in H(w)$. So, according to Definition 2, $H(w)$ is an algebra over $W(w)$.

(3) **$\mu(w)$ is a function from $H(w)$ to $[0,1]$:**

We have to prove the following:

(a) **the domain of $\mu(w)$ is $H(w)$ and the codomain of $\mu(w)$ is $[0,1]$:**

Let $(A) \in H(w)$. We have that $P_{\geq 0}A$ is an axiom of PPJ, thus $P_{\geq 0}A \in w$. Hence the set $\{s \in S \mid P_{\geq s}A \in w\}$ is not empty which means that it has a supremum. So $\mu(w)((A))$ is defined. Thus the domain of $\mu(w)$ is $H(w)$.

Let $(A) \in H(w)$. We have that $\mu(w)((A)) = \sup_s\{P_{\geq s}A \in w\} \geq 0$. In $\sup_s\{P_{\geq s}A \in w\}$ we have by definition that $s \in \mathbb{Q}[0,1]$, i.e. $s \leq 1$. Thus $\sup_s\{P_{\geq s}A \in w\} \leq 1$, i.e. $\mu(w)\big((A)\big) \leq 1$. So the codomain of $\mu(w)$ is $[0,1]$.

(b) **for every $V \in H(w)$, $\mu(w)(V)$ is unique:**

Let $V \in H(w)$ and assume that $V = (A) = (B)$ for some $A, B \in \mathcal{L}$. We will prove that $\mu(w)\big((A)\big) = \mu(w)\big((B)\big)$. Of course it suffices to prove that:

$$(A) \subseteq (B) \Longrightarrow \mu(w)\big((A)\big) \leq \mu(w)\big((B)\big) \qquad (2)$$

We have:

$$
\begin{array}{lr}
\quad (A) \subseteq (B) & \text{implies} \\
(\forall x \in U)\big[x \in (A) \Longrightarrow x \in (B)\big] & \text{implies} \\
(\forall x \in U)\big[A \in x \Longrightarrow B \in x\big] & \text{implies} \\
(\forall x \in U)\big[A \notin x \text{ or } B \in x\big] & \text{implies by L. 14(1)} \\
(\forall x \in U)\big[\neg A \in x \text{ or } B \in x\big] & \text{implies by L. 14(3)} \\
(\forall x \in U)\big[\neg A \vee B \in x\big] & \text{implies by } \textbf{S.E.}
\end{array}
$$

$$
(\forall x)\big[x \text{ is a maximal } \mathsf{PPJ_{CS}}\text{-consistent set} \Longrightarrow A \to B \in x\big] \qquad (3)
$$

Assume that $\nvdash A \to B$. By **P.R.** we get $\nvdash \neg(A \to B) \to \bot$, which implies that the set $\{\neg(A \to B)\}$ is $\mathsf{PPJ_{CS}}$-consistent. By Lemma 15 we have that there exists a maximal $\mathsf{PPJ_{CS}}$-consistent set $\mathcal{T}$ such that $\mathcal{T} \supseteq \{\neg(A \to B)\}$. However by (3) we have that $A \to B \in \mathcal{T}$ which contradicts the fact that $\mathcal{T}$ is $\mathsf{PPJ_{CS}}$-consistent. Thus $\vdash A \to B$. As a consequence, by Lemma 9, we have that

$$
(\forall s \in \mathbb{Q}[0,1])\big[\vdash P_{\geq s}A \to P_{\geq s}B\big] \ .
$$

Hence, since w is a maximal $\mathsf{PPJ_{CS}}$-consistent set, we get the following:

$$
\begin{array}{lr}
(\forall s \in \mathbb{Q}[0,1])\big[P_{\geq s}A \to P_{\geq s}B \in w\big] & \text{implies by L. 14(5)} \\
(\forall s \in \mathbb{Q}[0,1])\big[P_{\geq s}A \in w \Longrightarrow P_{\geq s}B \in w\big] & \text{implies} \\
\{s \in \mathbb{Q}[0,1] \mid P_{\geq s}A \in w\} \subseteq \{s \in \mathbb{Q}[0,1] \mid P_{\geq s}B \in w\} & \text{implies} \\
\sup_s\{P_{\geq s}A \in w\} \leq \sup_s\{P_{\geq s}B \in w\} & \text{i.e.} \\
\mu(w)\big((A)\big) \leq \mu(w)\big((B)\big) &
\end{array}
$$

Hence (2) holds, which proves that $\mu(w)(V)$ is unique.

(4) **$\mu(w)$ is a finitely additive measure:**

This case has been proved in [13] for the logic PJ. This proof also works for PPJ.

For every $w \in W$ $*_w$ is a basic CS-evaluation:

This holds by the construction of the canonical model ☐

The fact that the canonical model is measurable will be a corollary of the next lemma.

Lemma 19. *Let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for $\mathsf{PPJ_{CS}}$. Then we have*

$$
(\forall A \in \mathcal{L})(\forall w \in U)\big[[A]_{M,w} = (A)_M\big].
$$

Proof. Let $w \in U$ and let $A \in \mathcal{L}$. We will prove the claim by induction on the complexity of A. We distinguish the following cases:

1. $A \equiv p \in \mathsf{Prop}$. It holds that:

$$\begin{aligned}[A]_{M,w} = [p]_{M,w} &= \{x \in W(w) \mid M, x \models p\} = \{x \in U \mid p^*_x = \mathsf{T}\} \\ &= \{x \in U \mid p \in x\} = (p)_M = (A)_M \ .\end{aligned}$$

2. $A \equiv t : B$. It holds that:

$$\begin{aligned}[A]_{M,w} = [t:B]_{M,w} &= \{x \in W(w) \mid M, x \models t : B\} = \{x \in U \mid B \in t^*_x\} \\ &= \{x \in U \mid t : B \in x\} = (t:B)_M = (A)_M \ .\end{aligned}$$

3. $A \equiv P_{\geq s}B$. By i.h. we have that for all $x \in U$, $[B]_{M,x} = (B)_M \in H(x)$. Thus, we have:

$$\begin{aligned}[A]_{M,w} = [P_{\geq s}B]_{M,w} =& \{x \in W(w) \mid M, x \models P_{\geq s}B\} \\ =& \{x \in W(w) \mid \mu(x)\big([B]_{M,x}\big) \geq s\} \\ =& \{x \in W(w) \mid \mu(x)\big((B)_M\big) \geq s\} \\ =& \{x \in U \mid \sup_r \{P_{\geq r}B \in x\} \geq s\}\end{aligned}$$

   By Lemma 14(6)(iv) we get:

$$[A]_{M,w} = \{x \in U \mid P_{\geq s}B \in x\} = (P_{\geq s}B)_M = (A)_M \ .$$

4. $A \equiv B \wedge C$. It holds that:

$$\begin{aligned}[A]_{M,w} = [B \wedge C]_{M,w} = [B]_{M,w} \cap [C]_{M,w} &\overset{\text{i.h.}}{=} (B)_{M,w} \cap (C)_{M,w} \\ &\overset{L.\ 17(ii)}{=} (B \wedge C)_{M,w} = (A)_M \ .\end{aligned}$$

5. $A \equiv \neg B$. It holds that:

$$\begin{aligned}[A]_{M,w} = [\neg B]_{M,w} = W(w) \setminus [B]_{M,w} &\overset{\text{i.h.}}{=} U \setminus (B)_{M,w} \\ &\overset{L.\ 17(i)}{=} (\neg B)_{M,w} = (A)_M \ .\end{aligned}$$

□

From Lemmata 18 and 19 we get the following corollary.

Corollary 20. *Let* CS *be any constant specification. The canonical model for* $\mathsf{PPJ}_{\mathsf{CS}}$ *is a* $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$*-model.*

Proof. Let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for $\mathsf{PPJ}_{\mathsf{CS}}$ and let $A \in \mathcal{L}$. By Lemma 18 we have that M is a $\mathsf{PPJ}_{\mathsf{CS}}$-model. We also have that for any $w \in U$, by Lemma 19, $[A]_{M,w} = (A)_M \in H(w)$. Thus, $M \in \mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$. □

Making use of the properties of maximal consistent sets, we can now establish the Truth Lemma.

Lemma 21 (Truth Lemma)**.** *Let* CS *be some constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be the canonical model for* $\mathsf{PPJ_{CS}}$*. For every* $A \in \mathcal{L}$ *and any* $w \in U$ *we have:*

$$A \in w \iff M, w \models A.$$

Proof. We prove the claim by induction on the structure of A. Let us only show the case for $A = P_{\geq s}B$. It holds:

$$\begin{aligned}
&M, w \models A && \iff \\
&M, w \models P_{\geq s}B && \iff \\
&\mu(w)\big([B]_{M,w}\big) \geq s && \overset{L.\ 19}{\iff} \\
&\mu(w)\big((B)_M\big) \geq s && \overset{D.\ 16}{\iff} \\
&\sup_r \{P_{\geq r}B \in w\} \geq s && \overset{L.\ 14(6)(iv)}{\iff} \\
&P_{\geq s}B \in w && \iff \\
&A \in w
\end{aligned}$$

□

Finally, we get the strong completeness theorem as usual.

Theorem 22 (Strong Completeness for PPJ)**.** *Let* CS *be a constant specification, let* $T \subseteq \mathcal{L}$ *and let* $A \in \mathcal{L}$*. Then we have:*

$$T \models A \implies T \vdash A.$$

Proof. We prove the claim by contraposition. Assume that $T \nvdash A$. This means that $T \nvdash (\neg A) \to \bot$, By Theorem 10 we get $T, \neg A \nvdash \bot$, i.e. the set $T \cup \{\neg A\}$ is $\mathsf{PPJ_{CS}}$-consistent. By Lemma 15 there exists a maximal $\mathsf{PPJ_{CS}}$-consistent set w, such that $w \supseteq T \cup \{\neg A\}$. Let M be the canonical model for $\mathsf{PPJ_{CS}}$. By Corollary 20 we have that $M \in \mathsf{PPJ_{CS,Meas}}$. By Lemma 21 we have that $M, w \models T$ and $M, w \models \neg A$. Hence $T \not\models A$. □

4. Decidability for a fragment of $\mathcal{L}$

Before we can show that satisfiability is decidable for all $\mathcal{L}$-formulas, we have to show that satisfiability is decidable for a language $\mathcal{L}^{\mathsf{r}} \subseteq \mathcal{L}$ that is given by the following grammar:

$$A ::= p \mid \neg A \mid A \wedge A \mid t : B$$

where $t \in \mathsf{Tm}, p \in \mathsf{Prop}$, and $B \in \mathcal{L}$.

The key fact about $\mathcal{L}^{\mathsf{r}}$-formulas is that the probability operators may appear only in the scope of justification operators. Therefore, $\mathcal{L}^{\mathsf{r}}$-formulas practically behave as formulas of the language of justification logic [1, 2]. The truth of an $\mathcal{L}^{\mathsf{r}}$-formula A in a $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-model $M = \langle U, W, H, \mu, * \rangle$ at a world $w \in U$ only depends on the basic CS-evaluation $*_w$. Hence it makes sense to use the notation $* \models A$ for $A \in \mathcal{L}^{\mathsf{r}}$ and $*$ being a CS-evaluation. The next lemma can be proved by straightforward induction on the complexity of the formula.

Lemma 23. *Let $A \in \mathcal{L}^{\mathsf{r}}$ and let CS be any constant specification. It holds:*

A is $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiable if and only if there exists a basic CS-evaluation that satisfies A

Therefore, we can use an extension of the usual decision procedure for the basic justification logic J [23, 14, 15] to decide satisfiability for formulas of $\mathcal{L}^{\mathsf{r}}$.

Let A be a formula of justification logic and let CS be a constant specification. The key point in the satisfiability algorithm for the logic J is to decide whether a given term $t \in \mathsf{Tm}$ justifies A. In general, this problem is undecidable since a given term may justify infinitely many formulas.

This problem can be solved by restricting the constant specification to be schematic. If we use schematic variables for formulas, then we can represent a schematic constant specification in a finite way, which also makes the number of (schematic) formulas that are justified by a term finite. The decision algorithm for the logic J also requires finding all formulas that are instances of two formula schemes, i.e. unifying two formula schemes. This question is naturally answered by finding the most general unifier of the two schemes.

Theorem 24. *Let CS be a decidable almost schematic constant specification. For any formula $A \in \mathcal{L}^{\mathsf{r}}$, it is decidable whether A is satisfiable.*

Proof. As we mentioned earlier, due to the structural similarities between the $\mathcal{L}^{\mathsf{r}}$-formulas and the justification logic formulas and because of Lemma 23 we can obtain decidability for $\mathcal{L}^{\mathsf{r}}$-formulas by extending the decidability algorithm for justification logic J. Most of the algorithm can be easily adapted to our probabilistic setting. The only step of the algorithm that needs major adaptations is the representation of schematic formulas and therefore the unification algorithm.

In our setting we need three kinds of schematic variables: for terms, formulas and rational numbers. Because of the side conditions that come with the axioms (WE) and (UN) our schematic formulas should be paired with systems of linear inequalities. For example, the scheme (WE) should be represented by the schematic formula $P_{\leq r} A \to P_{<s} A$ (with the schematic variables r, s, and A) together with the inequality $r < s$. We should not forget that the rational variables belong to $\mathbb{Q}[0,1]$. So we have to add constraints like $0 \leq r \leq 1$.

Hence in addition to constructing a substitution, the unification algorithm also has to take care of the linear constraints. For instance, in order to unify the schemes $P_{\geq r} A$ and $P_{\geq s} B$ the algorithm has to unify A and B, and to equate

r and s, i.e. it adds $r = s$ to the linear system. In the end, the constructed substitution only is a most general unifier if the linear system is satisfiable.

Another complication are constraints of the form

$$l = \min(1, r + s) \tag{4}$$

that originate from the scheme (DIS). Obviously, (4) is not linear. However, for a system C of linear inequalities, we find that

$$C \cup \{l = \min(1, r + s)\}$$

has a solution if and only if

$$C \cup \{l = r + s, r + s \leq 1\} \text{ or } C \cup \{l = 1, r + s > 1\}$$

has a solution. Thus we can reduce solving a system involving (4) to solving several linear systems. □

5. Decidability of PPJ

The decidability proof for $\mathsf{PPJ}_{\mathsf{CS}}$ is based on a small model property, decidability for $\mathcal{L}^{\mathsf{r}}$-formulas, and decidability for systems of linear inequalities.

Definition 25 (Subformulas)**.** The set of subformulas $\mathsf{subf}(A)$ of an $\mathcal{L}$-formula is recursively defined by:

$$\begin{aligned} \mathsf{subf}(p) &:= \{p\} \qquad \text{for } p \in \mathsf{Prop} \\ \mathsf{subf}(P_{\geq s}A) &:= \{P_{\geq s}A\} \cup \mathsf{subf}(A) \\ \mathsf{subf}(\neg A) &:= \{\neg A\} \cup \mathsf{subf}(A) \\ \mathsf{subf}(A \wedge B) &:= \{A \wedge B\} \cup \mathsf{subf}(A) \cup \mathsf{subf}(B) \\ \mathsf{subf}(t : A) &:= \{t : A\} \cup \mathsf{subf}(A) \end{aligned}$$

Definition 26. Let $A \in \mathcal{L}$ and assume that $\mathsf{subf}(A) = \{A_1, \dots, A_k\}$. The set $\mathsf{subfCon}(A)$ contains all sets of the form $\{\pm A_1, \dots, \pm A_k\}$, where $\pm A_i$ is either A_i or $\neg A_i$.

Elements of $\mathsf{subfCon}(A)$ are interpreted conjunctively, i.e. for $C \in \mathsf{subfCon}(A)$, we simply write $M, w \models C$ instead of $M, w \models \bigwedge C$. Hence $M, w \models C$ means that all elements of C are true at w in M. Accordingly, we say that C is satisfiable if the formula $\bigwedge C$ is so.

We define the mapping j on sets C of $\mathcal{L}$-formulas by:

$$\mathsf{j}(C) := C \cap \mathcal{L}^{\mathsf{r}}.$$

For $C \in \mathsf{subfCon}(A)$, elements of $\mathsf{j}(C)$ are interpreted conjunctively too.

Before proving that $\mathsf{PPJ_{CS}}$ is decidable we need to establish some auxiliary lemmata.

Lemma 27. *Let $M = \langle U, W, H, \mu, * \rangle \in \mathsf{PPJ_{CS,Meas}}$ and let $A \in \mathcal{L}$. Further, let $B \in \mathsf{subf}(A)$, let $C \in \mathsf{subfCon}(A)$, and let $w \in U$. Assume that $M, w \models C$. Then we have:*

$$M, w \models B \quad \Longleftrightarrow \quad B \in C.$$

Proof. We prove the two directions of the lemma separately:

$\Longleftarrow$: From $B \in C$ and $M, w \models C$ we immediately get $M, w \models B$.

$\Longrightarrow$: Since B is a subformula of A, we have either $B \in C$ or $\neg B \in C$. If $\neg B \in C$, then we would have $M, w \models \neg B$, i.e. $M, w \not\models B$, which contradicts the fact that $M, w \models B$. Thus, we conclude $B \in C$. □

The next lemma is the key for proving decidability of $\mathsf{PPJ_{CS}}$. As it will be clear from the proof, the lemma essentially states the small model property.

Lemma 28. *Let CS be a constant specification and let A be an $\mathcal{L}$-formula. Then A is satisfiable if and only if there exists a set $Y = \{B_1, \ldots, B_n\} \subseteq \mathsf{subfCon}(A)$ such that all of the following conditions hold:*

1. *for some $i \in \{1, \ldots, n\}$, $A \in B_i$.*
2. *for every $1 \leq i \leq n$, there exists a basic CS-evaluation that satisfies $\mathsf{j}(B_i)$.*
3. *for every $1 \leq i \leq n$, there are variables x_{ij} with $1 \leq j \leq n$, such that the following system of linear inequalities is satisfiable:*

$$\sum_{j=1}^{n} x_{ij} = 1$$

$$(\forall 1 \leq j \leq n)\big[x_{ij} \geq 0\big]$$

$$\textit{for every } P_{\geq s}C \in B_i, \sum_{\{j \mid C \in B_j\}} x_{ij} \geq s$$

$$\textit{for every } \neg P_{\geq s}C \in B_i, \sum_{\{j \mid C \in B_j\}} x_{ij} < s$$

Proof. Let CS be a constant specification and let $A \in \mathcal{L}$. We prove the two directions of the lemma separately:

$\Longrightarrow$: Let $M = \langle U, W, H, \mu, * \rangle \in \mathsf{PPJ_{CS,Meas}}$. Assume that A is satisfiable in some world of M.

Let $\approx$ denote a binary relation over U such that for all $w, x \in U$ we have:

$$w \approx x \quad \text{if and only if} \quad \big(\forall B \in \mathsf{subf}(A)\big)\big[M, w \models B \Leftrightarrow M, x \models B\big].$$

It is easy to see that $\approx$ is an equivalence relation. Let $K_1, \dots, K_n$ be the equivalence classes of $\approx$ over U. For every $i \in \{1, \dots, n\}$ we choose some $w_i \in K_i$. For every $i \in \{1, \dots, n\}$ some subformulas of A hold in the world w_i and some do not. So for every $i \in \{1, \dots, n\}$ there exists a $B_i \in \mathsf{subfCon}(A)$ such that $M, w_i \models B_i$. For $i \neq j$ we have $B_i \neq B_j$ since w_i and w_j belong to different equivalence classes. Let $Y = \{B_1, \dots, B_n\}$. It remains to show that the conditions in the statement of the lemma hold:

1. Let $w \in U$ be such that $M, w \models A$. The world w belongs to some equivalence class of $\approx$ that is represented by w_i. Thus $M, w_i \models A$. By Lemma 27 we find $A \in B_i$, i.e. condition 1 holds.

2. For every $1 \leq i \leq n$ we have $M, w_i \models B_i$. Because of $\mathsf{j}(B_i) \subseteq B_i$ we immediately get $M, w_i \models \mathsf{j}(B_i)$. Since $\mathsf{j}(B_i) \subseteq \mathcal{L}^{\mathsf{r}}$ by Lemma 23 we have that there exists a basic CS-evaluation that satisfies $\mathsf{j}(B_i)$. Hence condition 2 holds.

3. Let $i \in \{1, \dots, n\}$. We set

$$y_{ij} = \mu(w_i)(K_j \cap W(w_i)), \text{ for every } 1 \leq j \leq n \ . \tag{5}$$

We are going to do some calculations to show that these values y_{ij} satisfy the linear system in condition 3. We have

$$\sum_{1 \leq j \leq n} y_{ij} = \sum_{1 \leq j \leq n} \mu(w_i)(K_j \cap W(w_i)) \overset{\text{the } K_j\text{'s are mutually disjoint}}{=} \mu(w_i)\Big(\bigcup_{1 \leq j \leq n} \big(K_j \cap W(w_i)\big)\Big) \overset{\bigcup_{j=1}^{n} K_j = U}{=} \mu(w_i)\big(W(w_i)\big)$$

And since $\mu(w_i)$ is a finitely additive measure over $W(w_i)$ we get:

$$\sum_{1 \leq j \leq n} y_{ij} = 1 \ . \tag{6}$$

By definition we also have

$$(\forall 1 \leq j \leq n)\big[y_{ij} \geq 0\big] \ . \tag{7}$$

Let $P_{\geq s}C \in B_i$. Since $M, w_i \models B_i$ it also holds that $M, w_i \models P_{\geq s}C$, i.e.

$$\mu(w_i)([C]_{M,w_i}) \geq s \ . \tag{8}$$

We will prove that:

$$\bigcup_{\{j | C \in B_j\}} \big(K_j \cap W(w_i)\big) = [C]_{M,w_i} \ . \tag{9}$$

Let $w \in [C]_{M,w_i}$. We have $w \in W(w_i)$ and $M, w \models C$. w must belong to some K_j. We also have that $M, w_j \models C$ and $M, w_j \models B_j$, which by Lemma 27 implies $C \in B_j$. Thus, we proved that there exists some j such that $C \in B_j$ and $w \in K_j \cap W(w_i)$. Thus

$$w \in \bigcup_{\{j \mid C \in B_j\}} \big(K_j \cap W(w_i)\big) \ .$$

On the other hand let $w \in \bigcup_{\{j \mid C \in B_j\}} \big(K_j \cap W(w_i)\big)$. So, there exists some j, such that $C \in B_j$ and $w \in K_j \cap W(w_i)$. It holds that $M, w_j \models B_j$ and since $w \in K_j$ we have that $M, w \models B_j$ which implies that $M, w \models C$. So, since $w \in W(w_i)$, we have that $w \in [C]_{M,w_i}$.

Therefore (9) holds.

By (8) and (9) we get:

$$\mu(w_i)\Big(\bigcup_{\{j \mid C \in B_j\}} \big(K_j \cap W(w_i)\big)\Big) \geq s \ .$$

Since the K_j's are mutually disjoint and $\mu(w_i)$ is a finitely additive measure we have:

$$\sum_{\{j \mid C \in B_j\}} \mu(w_i)\big(K_j \cap W(w_i)\big) \geq s$$

and by (5):

$$\sum_{\{j \mid C \in B_j\}} y_{ij} \geq s$$

So we proved that

$$\text{for every } P_{\geq s}C \in B_i, \sum_{\{j \mid C \in B_j\}} y_{ij} \geq s \tag{10}$$

By a similar reasoning we can prove that

$$\text{for every } \neg P_{\geq s}C \in B_i, \sum_{\{j \mid C \in B_j\}} y_{ij} < s \tag{11}$$

By (6), (7), (10) and (11) we have that the y_{ij}'s satisfy the linear system in condition 3.

$\Longleftarrow$: Assume that there exists some $Y = \{B_1, \ldots, B_n\} \subseteq \mathsf{subfCon}(A)$ such that conditions 1–3 in the lemma's statement hold. For every $1 \leq i \leq n$, let $*_i$ be a basic CS-evaluation such that $*_i \models \mathsf{j}(B_i)$ (by condition 2 we know that such a basic CS-evaluation exists). We define the quintuple $M = \langle U, W, H, \mu, * \rangle$ as follows:

- $U = \{w_1, \dots, w_n\}$ for some $w_1, \dots, w_n$.
- For all $1 \le i \le n$ we set:
  1. $W(w_i) = U$;
  2. $H(w_i) = \mathcal{P}(W(w_i))$;
  3. $\mu(w_i)(V) = \sum_{\{j | w_j \in V\}} x_{ij}$ for every $V \in H(w_i)$;
  4. $*_{w_i} = *_i$.

First we show that $M \in \mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$. Let $1 \le i \le n$. It holds that:

(i) $H(w_i)$ is an algebra over $W(w_i)$, since $H(w_i)$ is the powerset of $W(w_i)$.

(ii) For every $A \in \mathcal{L}$ we have that $[A]_{M,w_i} \in \mathcal{P}(W(w_i))$, i.e. $[A]_{M,w_i} \in H(w_i)$.

(iii) $\mu(w_i)$ is defined for all $V \in H(w_i)$ and by the first two lines of the linear system in condition 3 it holds that the codomain of $\mu(w_i)$ is $[0, 1]$.

We also have that:

$$\mu(w_i)(W(w_i)) = \mu(w_i)(U) = \sum_{\{j | w_j \in U\}} x_{ij} = \sum_{1 \le j \le n} x_{ij} = 1 \ .$$

Let $U, V \in H(w_i)$ such that $U \cap V = \emptyset$. It holds

$$\begin{aligned} \mu(w_i)(U \cup V) = & \sum_{\{j | w_j \in U \cup V\}} x_{ij} \\ = & \sum_{\{j | w_j \in U\}} x_{ij} + \sum_{\{j | w_j \in V\}} x_{ij} \\ = & \mu(w_i)(U) + \mu(w_i)(V) \ . \end{aligned}$$

Thus, $\mu(w_i)$ is a finitely additive measure over $H(w_i)$.

(iv) $*_{w_i}$ is a basic CS-evaluation since $*_i$ is also one.

From **(i)** - **(iv)** we conclude that $M \in \mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$. It remains to show $M, w_i \models A$ for some i.

We will now prove the following statement:

$$(\forall D \in \mathsf{j}(\mathsf{subf}(A)))(\forall 1 \le i \le n)\big[D \in \mathsf{j}(B_i) \iff *_i \Vdash D\big] \ . \tag{12}$$

Assume that $D \in \mathsf{j}(\mathsf{subf}(A))$ and let $1 \le i \le n$. We prove the two directions of (12) separately.

$\Longrightarrow$: Let $D \in j(B_i)$. Since $*_i$ satisfies $\mathsf{j}(B_i)$ we also have $*_i \Vdash D$.

$\Longleftarrow$: Let $*_i \Vdash D$. Since $D \in \mathsf{j}(\mathsf{subf}(A))$, we also find that either $D \in \mathsf{j}(B_i)$ or $\neg D \in \mathsf{j}(B_i)$. If $D \notin \mathsf{j}(B_i)$ then it is $\neg D \in \mathsf{j}(B_i)$, which implies that $*_i \Vdash \neg D$, i.e. $*_i \nVdash D$, which contradicts the fact that $*_i \Vdash D$. Thus, $D \in \mathsf{j}(B_i)$.

We conclude that (12) holds.

Now we show the following statement:

$$(\forall D \in \mathsf{subf}(A))(\forall 1 \leq i \leq n)\big[D \in B_i \Longleftrightarrow M, w_i \models D\big] \ . \tag{13}$$

Let $D \in B_i$. We will prove (13) by induction on the structure of D. Let $1 \leq i \leq n$. We distinguish the following cases:

$D \equiv p \in \mathsf{Prop}$**:** It holds:

$$\begin{aligned}
D \in B_i \quad & \Longleftrightarrow \\
p \in B_i \quad & \Longleftrightarrow \\
p \in j(B_i) \quad & \overset{(12)}{\Longleftrightarrow} \\
*_i \Vdash p \quad & \Longleftrightarrow \\
p^{*_i} = \mathsf{T} \quad & \Longleftrightarrow \\
p^*_{w_i} = \mathsf{T} \quad & \Longleftrightarrow \\
M, w_i \models p \quad & \Longleftrightarrow \\
M, w_i \models D &
\end{aligned}$$

$D \equiv t : C$**:** We have:

$$\begin{aligned}
D \in B_i \quad & \Longleftrightarrow \\
t : C \in B_i \quad & \Longleftrightarrow \\
t : C \in \mathsf{j}(B_i) \quad & \overset{(12)}{\Longleftrightarrow} \\
*_i \Vdash t : C \quad & \Longleftrightarrow \\
C \in t^{*_i} \quad & \Longleftrightarrow \\
C \in t^*_{w_i} \quad & \Longleftrightarrow \\
M, w_i \models t : C \quad & \Longleftrightarrow \\
M, w_i \models D &
\end{aligned}$$

$D \equiv P_{\geq s}C$. We prove the two directions of the claim separately.

$\Longrightarrow$: Assume that $D \in B_i$, i.e. $P_{\geq s}C \in B_i$. By the third line of the linear system in condition 3 we have:

$$\sum_{\{j | C \in B_j\}} x_{ij} \geq s \ .$$

By the inductive hypothesis we have:

$$\sum_{\{j | M, w_j \models C\}} x_{ij} \geq s \ . \tag{14}$$

It holds that

$$[C]_{M,w_i} = \{w_j \in W(w_i) \mid M, w_j \models C\} = \{w_j \mid M, w_j \models C\} \ . \qquad (15)$$

By the definition of M we have:

$$\mu(w_i)([C]_{M,w_i}) \quad = \quad \sum_{\{j|w_j \in [C]_{M,w_i}\}} x_{ij} \quad \overset{(15)}{=} \quad \sum_{\{j|M,w_j \models C\}} x_{ij} \ .$$

And by (14) we have that

$$\mu(w_i)([C]_{M,w_i}) \geq s$$

i.e.

$$M, w_i \models P_{\geq s} C$$

i.e.

$$M, w_i \models D$$

$\Longleftarrow$: Let $M, w_i \models D$. Assume that $D \notin B_i$, i.e. $\neg D \in B_i$, that is $\neg P_{\geq s} C \in B_i$. By the last line of the linear system in condition 3 we have that

$$\sum_{j:C \in B_j} x_{ij} < s$$

By using a similar argument as before we can prove that

$$\mu(w_i)([C]_{M,w_i}) < s$$

i.e.

$$M, w_i \not\models D$$

which is absurd. Therefore $D \in B_i$.

$D \equiv D_1 \wedge D_2$**:** Here $D \in B_i$ means

$$D_1 \wedge D_2 \in B_i \qquad (16)$$

We know that $*_i \Vdash B_i$. Assume that $D_1 \notin B_i$ or $D_2 \notin B_i$. Then it should be $*_i \nVdash D_1$ or $*_i \nVdash D_2$. But this is absurd since we have that $*_i \Vdash D_1 \wedge D_2$. So, both D_1 and D_2 belong to B_i. Hence (16) is equivalent to the following statements.

$$\begin{array}{cc} D_1 \in B_i \text{ and } D_2 \in B_i & \overset{\text{i.h.}}{\Longleftrightarrow} \\ M, w_i \models D_1 \text{ and } M, w_i \models D_2 & \Longleftrightarrow \\ M, w_i \models D_1 \wedge D_2 & \Longleftrightarrow \\ M, w_i \models D & \end{array}$$

$D \equiv \neg D'$**:** We have:

$$\begin{aligned}
D \in B_i &\iff \\
\neg D' \in B_i &\iff \\
D' \notin B_i &\overset{\text{i.h.}}{\iff} \\
M, w_i \not\models D' &\iff \\
M, w_i \models \neg D' &\iff \\
M, w_i \models D &
\end{aligned}$$

We conclude that (13) holds.

We have $A \in \mathsf{subf}(A)$. Thus, by (13) we find:

$$(\forall 1 \leq i \leq n)\big[A \in B_i \iff M, w_i \models A\big].$$

By condition 1, there exists an i such that $A \in B_i$. Thus, there exists an i such that $M, w_i \models A$. Hence, A is $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiable. □

In the proof of Lemma 28 we construct a model with at most $2^{|\mathsf{subf}(A)|}$ worlds that satisfies A. Hence a corollary of Lemma 28 is that any $A \in \mathcal{L}$ is $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiable if and only if it is satisfiable in a $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-model with at most $2^{|\mathsf{subf}(A)|}$ worlds. In other words, Lemma 28 implies a small model property for $\mathsf{PPJ}_{\mathsf{CS}}$. Moreover, Lemma 28 dictates a procedure to decide the satisfiability problem for $\mathsf{PPJ}_{\mathsf{CS}}$.

Theorem 29. *Let* CS *be a decidable almost schematic constant specification. The* $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$*-satisfiability problem is decidable.*

Proof. Let CS be a decidable almost schematic constant specification and let $A \in \mathcal{L}$. The formula A is satisfiable if and only if for some $Y \subseteq \mathsf{subfCon}(A)$ all conditions in the statement of Lemma 28 hold. Since $\mathsf{subfCon}(A)$ is finite, it suffices to show that for every $Y \subseteq \mathsf{subfCon}(A)$ the conditions 1–3 in the statement of Lemma 28 can be effectively checked:

- Decidability of condition 1 is trivial.
- Decidability of condition 2 follows from Lemma 24.
- In condition 3 we have to check for the satisfiability of a set of linear inequalities. There are several decision procedures available for this problem (see, for example, [21]).

We conclude that the $\mathsf{PPJ}_{\mathsf{CS},\mathsf{Meas}}$-satisfiability problem is decidable. □

6. Application to the Lottery Paradox

Kyburg's famous lottery paradox [19] goes as follows. Consider a fair lottery with 1000 tickets that has exactly one winning ticket. Now assume a proposition is believed if and only if its degree of belief is greater than 0.99. In this setting it is rational to believe that ticket 1 does not win, it is rational to believe that ticket 2 does not win, and so on. However, this entails that it is rational to believe that no ticket wins because rational belief is closed under conjunction. Hence it is rational to believe that no ticket wins and that one ticket wins.

$\mathsf{PPJ}_{\mathsf{CS}}$ makes the following analysis of the lottery paradox possible. First we need a principle to move from degrees of belief to rational belief (this formalizes what Foley [9] calls *the Lockean thesis*): we suppose that for each term t, there exists a term $\mathsf{pb}(t)$ such that

$$t : (P_{>0.99}A) \rightarrow \mathsf{pb}(t) : A \tag{17}$$

(where pb stands for probabilistic belief). Let w_i be the proposition *ticket i wins*. For each $1 \le i \le 1000$, there is a term t_i such that $t_i : (P_{=\frac{999}{1000}} \neg w_i)$ holds. Hence by (17) we get

$$\mathsf{pb}(t_i) : \neg w_i \quad \text{ for each } 1 \le i \le 1000. \tag{18}$$

Now if CS is axiomatically appropriate, then

$$s_1 : A \wedge s_2 : B \rightarrow \mathsf{con}(s_1, s_2) : (A \wedge B) \tag{19}$$

is a valid principle (for a suitable term $\mathsf{con}(s_1, s_2)$). Hence by (18) we conclude that

$$\text{there exists a term } t \text{ with } t : (\neg w_1 \wedge \cdots \wedge \neg w_{1000}), \tag{20}$$

which leads to a paradoxical situation since it is also believed that one of the tickets wins.

In $\mathsf{PPJ}_{\mathsf{CS}}$ we can resolve this problem by restricting the constant specification such that (19) is valid only if $\mathsf{con}(s_1, s_2)$ does not contain two different subterms of the form $\mathsf{pb}(t)$. Then the step from (18) to (20) is no longer possible and we can avoid the paradoxical belief.

This analysis is inspired by Leitgeb's [20] solution to the lottery paradox and his *Stability Theory of Belief* according to which *it is not permissible to apply the conjunction rule for beliefs across different contexts*. Our proposed restriction of (19) is one way to achieve this in a formal system. A related and very interesting question is whether one can interpret the above justifications t_i as stable sets in Leitgeb's sense. Of course, our discussion of the lottery paradox is very sketchy but we think that probabilistic justification logic provides a promising approach to it that is worth further investigations.

7. Conclusion

Extending the work of [13] we defined a probabilistic justification, PPJ, to study rational belief, degrees of belief and justifications. The logic PPJ is an extension of the probabilistic justification logic from [13] in the sense that PPJ allows iterations of the probability operators and justification operators over probability operators. In the framework of PPJ it is also possible to analyze a famous paradox from epistemology, i.e. the lottery paradox [19]. A natural open problem about the logic PPJ is establishing complexity bounds. It is worth mentioning that heuristics have been applied for attacking the satisfiability problem for probabilistic logics with classical base [25]. It could be of practical interest to apply these heuristics to the satisfiability problem of probabilistic justification logic, too.

References

[1] Artemov, S.: Operational modal logic. Tech. Rep. MSI 95–29, Cornell University (1995)

[2] Artemov, S.: Explicit provability and constructive semantics. Bulletin of Symbolic Logic 7(1), 1–36 (2001)

[3] Artemov, S.: The logic of justification. The Review of Symbolic Logic 1(4), 477–513 (2008)

[4] Artemov, S.: The ontology of justifications in the logical setting. Studia Logica 100(1–2), 17–30 (2012)

[5] Artemov, S.: On Aggregating Probabilistic Evidence. In: Artemov, S., Nerode, A. (eds.) Proc. LFCS 2016, LNCS, vol. 9537, pp. 27–42. Springer (2016)

[6] Bucheli, S., Kuznets, R., Studer, T.: Partial realization in dynamic justification logic. In: Beklemishev, L.D., de Queiroz, R. (eds.) Proc. WoLLIC 11, LNAI, vol. 6642, pp. 35–51. Springer (2011)

[7] Fan, T., Liau, C.: A logic for reasoning about justified uncertain beliefs. In: Yang, Q., Wooldridge, M. (eds.) Proc. IJCAI 2015. pp. 2948–2954. AAAI Press (2015)

[8] Fitting, M.: The logic of proofs, semantically. Annals of Pure and Applied Logic 132(1), 1–25 (2005)

[9] Foley, R.: Beliefs, Degrees of Belief, and the Lockean Thesis. In: Huber, F., Schmidt-Petri, C. (eds.) Degrees of Belief, pp. 37–47. Springer (2009)

[10] Ghari, M.: Justification logics in a fuzzy setting. ArXiv e-prints (2014)

[11] Ilić Stepić, A. and Ognjanović, Z.: Complex Valued Probability Logics. Publications de l'Institut Mathématique (N.S.) 95(109), 73–86 (2014)

[12] Kokkinis, I.: The Complexity of Non-Iterated Probabilistic Justification Logic. In: Simari, G., Gyssens, M. (eds.) FoIKS 2016, to appear

[13] Kokkinis, I., Maksimović, P., Ognjanović, Z., Studer, T.: First steps towards probabilistic justification logic. Logic Journal of IGPL 23(4), 662–687 (2015)

[14] Kuznets, R.: On the complexity of explicit modal logics. In: Clote, P.G., Schwichtenberg, H. (eds.) Proc. CSL 2000, LNCS, vol. 1862, pp. 371–383. Springer (2000)

[15] Kuznets, R.: Complexity Issues in Justification Logic. Ph.D. thesis, City University of New York (2008), http://gradworks.umi.com/33/10/3310747.html

[16] Kuznets, R., Studer, T.: Justifications, ontology, and conservativity. In: Bolander, T., Braüner, T., Ghilardi, S., Moss, L. (eds.) AiML 9, pp. 437–458. College Publications (2012)

[17] Kuznets, R., Studer, T.: Update as evidence: Belief expansion. In: Artemov, S., Nerode, A. (eds.) Proc. LFCS 2013, LNCS, vol. 7734, pp. 266–279. Springer (2013)

[18] Kuznets, R., Studer, T.: Weak arithmetical interpretations for the logic of proofs. Logic Journal of IGPL (to appear)

[19] Kyburg, H.E.J.: Probability and the Logic of Rational Belief. Wesleyan University Press (1961)

[20] Leitgeb, H.: The stability theory of belief. Philosophical Review 123(2), 131–171 (2014)

[21] Luenberger, D.G., Ye, Y.: Linear and Nonlinear Programming, International Series in Operations Research and Management Science, vol. 116. Springer (2008)

[22] Milnikel, R.S.: The logic of uncertain justifications. Annals of Pure and Applied Logic 165(1), 305–315 (2014)

[23] Mkrtychev, A.: Models for the logic of proofs. In: Adian, S., Nerode, A. (eds.) Proc. LFCS 1997, LNCS, vol. 1234, pp. 266–275. Springer (1997)

[24] Ognjanović, Z., Rašković, M.: Some first order probability logics. Theoretical Computer Science 247, 191–212 (2000)

[25] Ognjanović, Z., Rašković, M., Marković, Z.: Probability logics. Zbornik radova, subseries "Logic in Computer Science" 12(20), 35–111 (2009)

[26] Rašković, M., Ognjanović, Z.: A first order probability logic, LP_Q. Puplications de L'Institut Mathèmatique 65(79), 1–7 (1999)