Justified Belief Change

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Abstract

Justification Logic is a framework for reasoning about evidence and justification. Public Announcement Logic is a framework for reasoning about belief changes caused by public announcements. This paper develops JPAL, a dynamic justification logic of public announcements that corresponds to the modal theory of public announcements due to Gerbrandy and Groeneveld. JPAL allows us to reason about evidence brought about by and changed by Gerbrandy–Groeneveld-style public announcements.

Keywords: justification logic, dynamic epistemic logic, public announcements, belief revision

1 Introduction

Dynamic Epistemic Logic is a modal-logic approach to reasoning about communication and belief change [21]. This approach originated with work on *public announcements* by Plaza [13] and by Gerbrandy and Groeneveld [11, 12]. As in everyday life, a public announcement of A informs its listeners about A. However, unlike everyday life, public announcements in Dynamic Epistemic Logic are traditionally assumed to be from a completely trustworthy source, so the agents accept the announcements to be true regardless of what the announcements may be. This assumption has proved itself useful for reasoning about public information update in a number of multi-agent situations; examples include the puzzles of the Muddy Children, Sum and Product, and Russian Cards [21].

Though Plaza and Gerbrandy–Groeneveld agree in making the assumption of complete trustworthiness of announcements, the two approaches disagree on whether what is announced must be true. According to Plaza, A can be announced only if it is true. So after the Plaza-announcement of a basic assertion p, the listeners

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not only come to believe p (due to complete trustworthiness) but they are also correct in doing so: since p was Plaza-announced, it must be true. In contrast, Gerbrandy–Groeneveld allow A to be announced no matter its truth value. So after the GG-announcement of p, the listeners come to believe p (again due to complete trustworthiness) but they may not be correct in doing so because p might be false. Note that it is not at all contradictory for the GG approach to allow the announcement of false information within a framework that assumes trusting listeners; after all, such a framework is just what is needed when we want to reason not only about how public announcements can inform trusting agents with truthful information but also how they can deceive trusting agents with misinformation.

Despite their differences, the Plaza and GG logics of public announcements, like all dynamic epistemic logics, describe the beliefs of agents using the language of modal logic: the formula $\Box A$ is assigned the reading "the agent believes A." While this language is useful for reasoning about the content of beliefs, it cannot be used for reasoning about the justification or evidence supporting these beliefs. In essence, the problem arises due to limitations in the expressiveness of the language: while the language can express that "the agent believes it is raining," it cannot express that "the agent believes it is raining *because she saw raindrops outside her window*." The language is simply unable to state the reasons supporting a given belief.

Justification Logic is a framework that provides a way to express these supporting reasons [3]. The basic language of Justification Logic extends that of propositional logic by adding structured syntactic objects called *terms* and by allowing us to take a term t and a formula A and form the formula $t : A$. The structure of terms in a given theory of justification logic lines up with the axiomatics of that theory so as to guarantee the property of *internalization*: for each derivation $\mathcal{D}$ of a theorem B of the logic in question, there is a step-by-step construction that transforms $\mathcal{D}$ into a term $t_{\mathcal{D}}$ in such a way that $t_{\mathcal{D}} : B$ is also a theorem of the logic. The term $t_{\mathcal{D}}$, therefore, describes the reasons, according to the logic, why B must hold. This suggests that we think of a term t in a formula $t : A$ as an explicit reason that justifies the assertion A, which leads us to read the formula $t : A$ as "the agent believes A for reason t." Fitting's Kripke-style semantics for Justification Logic [8] provides a semantic view of this reading that indicates the close connection between an assertion $t : A$ of justified belief and an assertion $\Box A$ of simple (modal) belief. This connection can be formalized [1] and in recent years has been extensively studied in the setting of multi-agent systems and formal epistemology [2, 3, 4, 6, 7, 10, 14, 22].

While changes in simple (modal) belief $\Box A$ have been extensively studied in Dynamic Epistemic Logic, work on the dynamics of justified belief $t : A$ has just begun. In particular, Renne has investigated joint systems of Dynamic Epistemic Logic and Justification Logic for reasoning about communications that introduce new evidence [14] or eliminate unreliable evidence [15] in addition to affecting agents' beliefs. Renne has also studied the effect Plaza-announcements that do not change evidence have on the expressivity of the language of various Justification Logics [16]. However, there is still no logic of public announcements with justified belief whose modal-logic counterpart is either of the Plaza or Gerbrandy–Groeneveld systems.

In this paper, we fill this gap by developing a dynamic justification logic of public announcements whose justified-belief dynamics corresponds to the modal-belief dynamics of the Gerbrandy–Groeneveld framework of public announcements. Our logic, called JPAL, can be used to reason about evidence that is provided and affected by Gerbrandy–Groeneveld-style announcements. After presenting the syntax, axiom system, and semantics of JPAL, we will highlight some key differences between justified-belief dynamics arising as a result of GG-announcements and modal-belief dynamics arising as a result of the same. In particular, we will see that the traditional "reduction" technique for proving completeness of modal logics of public announcements, wherein a modal formula that contains announcements can be "reduced to" a provably equivalent modal formula that is announcement-free, does not work quite as well in a justified-belief framework. Nevertheless, we will prove that our system is sound and complete, discuss the justified-belief variant of Moore sentences ("p is true but the agent does not believe it"), and describe our progress toward a formal proof that JPAL is an exact justification counterpart of Gerbrandy–Groeneveld modal logic of public announcements.

2 Justification Logic

We begin by introducing the justification logic J4 that is a justification counterpart of the modal logic K4. J4 replaces K4-claims $\Box A$ of (modal) belief with statements $t : A$ of justified belief.

Definition 1 (J4 Language)**.** We fix countable sets Cons of *constants*, Vars of *variables*, and Prop of *atomic propositions*. The *language of* J4 consists of the *terms* $t \in \mathsf{Tm}$ and the *formulas* $A \in \mathsf{Fml_J}$ formed by the following grammar:

$$\begin{array}{rcl} t & ::= & x \mid c \mid (t \cdot t) \mid (t + t) \mid !t \\ A & ::= & p \mid \neg A \mid (A \to A) \mid t : A \\ & & x \in \mathsf{Vars},\ c \in \mathsf{Cons},\ p \in \mathsf{Prop} \end{array}$$

We define the connectives $\wedge$, $\vee$, and $\leftrightarrow$ as usual. To say that a term $t \in \mathsf{Tm}$ is *ground* means that t does not contain variables.

Definition 2 (J4 Deductive System)**.** The *axioms of* J4 consist of all $\mathsf{Fml_J}$-instances of the following schemes:

1. All classical propositional tautologies
2. $t : (A \to B) \to (s : A \to t \cdot s : B)$ (application)
3. $t : A \to t + s : A, \quad s : A \to t + s : A$ (sum)
4. $t : A \to \, !t : t : A$ (introspection)

The *deductive system* J4 is the Hilbert system that consists of the above axioms of J4 and the following rules of *modus ponens* (MP) and *axiom necessitation* (AN):

$$\frac{A \quad A \to B}{B}\,(\mathrm{MP}) \ , \qquad \frac{c \in \mathsf{Cons} \quad C \text{ is a J4-axiom}}{c : C}\,(\mathrm{AN}) \ .$$

Our theory JPAL extends the theory $\mathsf{J4}$ by adding public announcement formulas $[A]B$ to the language. A new formula $[A]B$ is read, "after the public announcement of A, formula B is true."

Definition 3 (JPAL Language)**.** The *language of* JPAL consists of the *terms* $t \in \mathsf{Tm}$ and the *formulas* $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ formed by the following grammar:

$$\begin{aligned} t &::= x \mid c \mid (t \cdot t) \mid (t + t) \mid\, !t \\ A &::= p \mid \neg A \mid (A \to A) \mid t : A \mid [A]A \\ &\qquad x \in \mathsf{Vars},\ c \in \mathsf{Cons},\ p \in \mathsf{Prop} \end{aligned}$$

When the word "formula" is used without qualification, we mean an $\mathsf{Fml}_{\mathsf{J},[\cdot]}$-formula.

Notation 4 (σ-Sequences)**.** The lowercase Greek letters σ and τ (with and without subscripts) denote finite sequences of formulas. ε denotes the empty sequence. Given a finite sequence $\sigma = (A_1, \dots, A_n)$ of formulas and a formula B, the formula $[\sigma]B$ is defined as follows:

$$[\sigma]B := \begin{cases} [A_1] \dots [A_n]B & \text{if } n > 0, \\ B & \text{if } n = 0. \end{cases}$$

Further, sequences $\sigma, B := (A_1, \dots, A_n, B)$ and $B, \sigma := (B, A_1, \dots, A_n)$ are obtained by appending B to σ from the right and the left, respectively. If $\tau = (C_1, \dots, C_m)$ is another finite sequence of formulas, then $\tau, \sigma := (C_1, \dots, C_m, A_1, \dots, A_n)$ is the concatenation of the two sequences. Naturally, we have $\sigma, \varepsilon = \sigma = \varepsilon, \sigma$.

Definition 5 (JPAL Deductive System)**.** The *axioms of* JPAL consist of all $\mathsf{Fml}_{\mathsf{J},[\cdot]}$-instances of the following schemes:

1. $[\sigma]A$, where A is a classical propositional tautology
2. $[\sigma](t : (A \to B) \to (s : A \to t \cdot s : B))$ (application)
3. $[\sigma](t : A \to t + s : A)$, $\quad [\sigma](s : A \to t + s : A)$ (sum)
4. $[\sigma](t : A \to\, !t : t : A)$ (introspection)
5. $[\sigma]p \leftrightarrow p$ (independence)
6. $[\sigma](B \to C) \leftrightarrow ([\sigma]B \to [\sigma]C)$ (normality)
7. $[\sigma]\neg B \leftrightarrow \neg[\sigma]B$ (functionality)
8. $[\sigma][A]t : B \leftrightarrow [\sigma]t : (A \to [A]B)$ (update)
9. $[\sigma][A][B]C \leftrightarrow [\sigma][A \wedge [A]B]C$ (iteration)

The *deductive system* JPAL is the Hilbert system that consists of the above axioms of JPAL and the following rules of *modus ponens* (MP) and *axiom necessitation* (AN):

$$\frac{A \quad A \to B}{B}\,(\text{MP}) \;, \qquad \frac{c_1, \dots, c_n \in \mathsf{Cons} \quad C \text{ is a } \mathsf{JPAL}\text{-axiom}}{[\sigma_1]c_1 : \dots : [\sigma_n]c_n : C}\,(\text{AN}) \;,$$

where σ_i's are (possibly empty) finite sequences of formulas. Though we use the same name "axiom necessitation (AN)" for two different rules, it will always be clear from the context whether the rule from $\mathsf{J4}$ or the one from JPAL is meant at each moment. Given a set Δ of formulas, we write $\Delta \vdash A$ to state that A is derivable from Δ in the deductive system of JPAL. The negation of $\Delta \vdash A$ is written $\Delta \nvdash A$.

The following example gives some intuition as to how the proof system works and what it can achieve.

Example 6. For any $p \in \mathsf{Prop}$ and any $c_1, c_2 \in \mathsf{Cons}$, we have $\vdash [p](c_1 \cdot c_2) : p$.

Proof. We use PR to denote the use of propositional reasoning.

1.	$c_1 : \big(([p]p \leftrightarrow p) \to (p \to [p]p)\big)$	AN for the tautology $([p]p \leftrightarrow p) \to (p \to [p]p)$
2.	$c_2 : ([p]p \leftrightarrow p)$	AN for the independence axiom $[p]p \leftrightarrow p$
3.	$(c_1 \cdot c_2) : (p \to [p]p)$	from 1 and 2 by application and PR
4.	$[p](c_1 \cdot c_2) : p$	from 3 by update and PR $\square$

This example shows that, independent of the truth value of an atomic proposition $p \in \mathsf{Prop}$, after p is announced, the term $c_1 \cdot c_2$ becomes a reason to believe that p. We will return to this example after we define the semantics.

The following lemma states a standard property of justification logics; it is proved by an easy induction on the length of derivation.

Lemma 7 (Lifting)**.** *If*

$$s_1 : B_1, \dots, s_m : B_m, C_1, \dots, C_n \vdash A \ ,$$

then there is a term $t(s_1, \dots, s_m, y_1, \dots, y_n)$ *such that*

$$s_1 : B_1, \dots, s_m : B_m, y_1 : C_1, \dots, y_n : C_n \vdash t(s_1, \dots, s_m, y_1, \dots, y_n) : A$$

for fresh variables $y_1, \dots, y_n$.

Corollary 8 (Constructive Necessitation)**.** *For any formula A, if $\vdash A$, then there is a ground term t such that $\vdash t : A$.*

The axioms of independence, normality, functionality, update, and iteration are called the *announcement axioms*. They are all formulated as equivalences, which provides a way for going back and forth in reasoning about the states before and after an announcement (see Lemma 20, where this back-and-forth is made precise in the context of a modal theory of public announcements). On the level of justifications, the possibility of this back-and-forth reasoning is reflected in the update axiom: the evidence term t is the same on both sides of the equivalence.

But now suppose that we were to formulate the language of JPAL differently, say by introducing additional terms $\mathsf{up}(t)$ and $\mathsf{down}(t)$ for each term t and then replacing the update axiom with the following two axioms:

$$\begin{aligned} t : (A \to [A]B) \quad &\to \quad [A]\mathsf{up}(t) : B \ , \\ [A]t : B \quad &\to \quad \mathsf{down}(t) : (A \to [A]B) \ . \end{aligned}$$

In this modified version of JPAL, we would be able to prove that

$$[A]t : B \vdash [A]\mathsf{up}(\mathsf{down}(t)) : B \ .$$

In other words, whenever t is evidence for B after the public announcement of A, so would be $\mathsf{up}(\mathsf{down}(t))$. This would suggest an introduction of an equivalence relation on terms so that $\mathsf{up}(\mathsf{down}(t)) \simeq t$ to provide a formal link between these two naturally connected pieces of evidence. While this could be an interesting variation of JPAL, we do not pursue this path here to minimize the set of operations on terms.

3 Semantics

We adapt the Kripke-style semantics for Justification Logic due to Fitting [8]. This semantics uses Kripke models augmented by a function called an *evidence function* that relates each world–term pair (w, t) to a set of formulas $\mathcal{E}(w, t)$ that the term t can justify at the world w.

Definition 9 ($\mathsf{K4}$-frame)**.** A $\mathsf{K4}$-*frame* is a pair (W, R) that consists of a nonempty set $W \neq \emptyset$ of *(possible) worlds* and of a transitive *accessibility relation* $R \subseteq W \times W$.

Definition 10 (Evidence Function)**.** An *evidence function* on a $\mathsf{K4}$-frame (W, R) is a function

$$\mathcal{E} \colon W \times \mathsf{Tm} \to \mathcal{P}\left(\mathsf{Fml}_{\mathsf{J},[\cdot]}\right)$$

that satisfies the following closure conditions:

1. *Monotonicity:* if $R(w, v)$, then $\mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$ for any $t \in \mathsf{Tm}$.
2. *Axioms:* if a formula A has the form $[\sigma_1]c_1 : \cdots : [\sigma_n]c_n : C$ and is derivable by the AN-rule, then $A \in \mathcal{E}(w, c)$ for any $c \in \mathsf{Cons}$ and any $w \in W$.
3. *Application:* if $(A \to B) \in \mathcal{E}(w, t)$ and $A \in \mathcal{E}(w, s)$, then $B \in \mathcal{E}(w, t \cdot s)$.
4. *Sum:* $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$ for any $s, t \in \mathsf{Tm}$ and any $w \in W$.
5. *Introspection:* if $A \in \mathcal{E}(w, t)$, then $t : A \in \mathcal{E}(w, !t)$.

A model of JPAL uses a family of such evidence functions, one for each finite sequence σ of formulas. The idea is that the evidence function $\mathcal{E}^\sigma$ that corresponds to sequence σ represents the "evidential situation" that arises after the announcements in σ have been made.

Definition 11 (JPAL Model)**.** A *model* is a structure $\mathcal{M} = (W, R, \mathcal{E}, \nu)$, where

1. (W, R) is a $\mathsf{K4}$-frame;
2. $\nu : \mathsf{Prop} \to \mathcal{P}(W)$ is a *(truth) valuation*;
3. $\mathcal{E}$ is a function that maps finite sequences σ of formulas to evidence functions $\mathcal{E}^\sigma$ on (W, R) in such a way that

$$A \to [A]B \in \mathcal{E}^\sigma(w, t) \text{ if and only if } B \in \mathcal{E}^{\sigma, A}(w, t) \tag{1}$$

and

$$\mathcal{E}^{\sigma, A, B}(w, t) = \mathcal{E}^{\sigma, A \wedge [A]B}(w, t) \ . \tag{2}$$

Conditions (1) and (2) correspond to the update axiom and the iteration axiom respectively.

Definition 12 (Truth in JPAL Models)**.** A ternary relation $\mathcal{M}, w \Vdash A$ for *formula* A *being satisfied at a world* $w \in W$ *in a model* $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ is defined by induction on the structure of A:

- $\mathcal{M}, w \Vdash p$ if and only if $w \in \nu(p)$.

- $\mathcal{M}, w \Vdash \neg A$ if and only if $\mathcal{M}, w \nVdash A$.
- $\mathcal{M}, w \Vdash A \to B$ if and only if $\mathcal{M}, w \nVdash A$ or $\mathcal{M}, w \Vdash B$.
- $\mathcal{M}, w \Vdash t\!:\!A$ if and only if 1) $A \in \mathcal{E}^\varepsilon(w,t)$ and 2) $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $R(w,v)$.
- $\mathcal{M}, w \Vdash [A]B$ if and only if $\mathcal{M}_A, w \Vdash B$, where the model $\mathcal{M}_A = (W_A, R_A, \mathcal{E}_A, \nu_A)$ is defined as follows:

$$\begin{array}{lcl} W_A & := & W \ , \\ R_A & := & \{(s,t) \mid R(s,t) \text{ and } \mathcal{M}, t \Vdash A\} \ , \\ (\mathcal{E}_A)^\sigma & := & \mathcal{E}^{A,\sigma} \ , \\ \nu_A & := & \nu \ . \end{array}$$

$\mathcal{M}_A$ is indeed a model: R_A is transitive and $\mathcal{E}_A$ satisfies both (1) and (2).

We write $\mathcal{M} \Vdash A$ to mean that $\mathcal{M}, w \Vdash A$ for all $w \in W$. We say that formula A is *valid*, written $\Vdash A$, to mean that $\mathcal{M} \Vdash A$ for all models $\mathcal{M}$. The negation of $\Vdash A$ is written $\nVdash A$. Further, for a finite sequence of formulas $\tau = (A_1, \ldots, A_n)$ we write $\mathcal{M}_\tau = (W_\tau, R_\tau, \mathcal{E}_\tau, \nu_\tau)$ to denote the model $(\cdots((\mathcal{M}_{A_1})_{A_2})\cdots)_{A_n}$. Note that we have $(\mathcal{E}_\tau)^\sigma = \mathcal{E}^{\tau,\sigma}$; in particular, $(\mathcal{E}_\tau)^\varepsilon = \mathcal{E}^\tau$.

The following example shows that our notion of model is not empty.

Example 13. Define the structure $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ as follows:

$$\begin{array}{lcl} W & := & \{w, v\} \ , \\ R & := & \{(w,w), (w,v), (v,v)\} \ , \\ \mathcal{E}^\sigma(x,t) & := & \mathsf{Fml}_{\mathsf{J},[\cdot]} \text{ for all } \sigma, \text{ all } x \in W, \text{ and all } t \in \mathsf{Tm} \ , \\ \nu(p) & := & \{w\} \text{ for all } p \in \mathsf{Prop} \ . \end{array}$$

It is easy to see that $\mathcal{M}$ is a model.

To illustrate how the semantics works, we prove a semantic version of the result from Example 6.

Example 14. For any $p \in \mathsf{Prop}$ and any $c_1, c_2 \in \mathsf{Cons}$, we have $\Vdash [p](c_1 \cdot c_2)\!:\!p$.

Proof. Let $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ be an arbitrary model and let $w \in W$. By Def. 10.2, we have that $([p]p \leftrightarrow p) \to (p \to [p]p) \in \mathcal{E}^\varepsilon(w, c_1)$ and $([p]p \leftrightarrow p) \in \mathcal{E}^\varepsilon(w, c_2)$. Thus, $(p \to [p]p) \in \mathcal{E}^\varepsilon(w, c_1 \cdot c_2)$ by Def. 10.3. So, by (1), we have that $p \in \mathcal{E}^p(w, c_1 \cdot c_2)$. Since $R_p(w,v)$ implies $\mathcal{M}, v \Vdash p$, i.e., $v \in \nu(p) = \nu_p(p)$, we have by Def. 12 that $\mathcal{M}_p, w \Vdash (c_1 \cdot c_2)\!:\!p$ and, hence, $\mathcal{M}, w \Vdash [p](c_1 \cdot c_2)\!:\!p$. □

4 Modal Public Announcement Logic

In this section, we recall some of the basic definitions and facts concerning the Gerbrandy–Groeneveld modal logic of public announcements [11, 12, 21].

Definition 15 (PAL and Modal Languages)**.** The *modal language* consists of the *formulas* $A \in \mathsf{Fml}_\Box$ formed by the grammar

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box A \qquad (p \in \mathsf{Prop})$$

The *language of* PAL consists of the *formulas* $A \in \mathsf{Fml}_{\Box,[\cdot]}$ formed by the grammar

$$A ::= p \mid \neg A \mid (A \to A) \mid \Box A \mid [A]A \qquad (p \in \mathsf{Prop})$$

The Gerbrandy–Groeneveld theory PAL of Public Announcement Logic uses the language $\mathsf{Fml}_{\Box,[\cdot]}$ to reason about belief change caused by public announcements.

Definition 16 (PAL Deductive System)**.** The *axioms of* PAL consist of all $\mathsf{Fml}_{\Box,[\cdot]}$-instances of the following schemes:

1. Axioms schemes for the modal logic $\mathsf{K4}$
2. $[A]p \leftrightarrow p$ (independence)
3. $[A](B \to C) \leftrightarrow ([A]B \to [A]C)$ (normality)
4. $[A]\neg B \leftrightarrow \neg[A]B$ (functionality)
5. $[A]\Box B \leftrightarrow \Box(A \to [A]B)$ (update)
6. $[A][B]C \leftrightarrow [A \land [A]B]C$ (iteration)

The *deductive system* PAL is the Hilbert system that consists of the above axioms of PAL and the following rules of *modus ponens* (MP) and *necessitation* (N):

$$\frac{A \quad A \to B}{B}\,(\mathrm{MP}) \ , \qquad \frac{A}{\Box A}\,(\mathrm{N}) \ .$$

We write $\mathsf{PAL} \vdash A$ to state that $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is a theorem in the deductive system of PAL.

Again, we use some of the same names for both axioms of JPAL and axioms of PAL because it will always be clear from the context which of the two is meant. As before, the axioms of independence, normality, functionality, update, and iteration are called the *announcement axioms*.

Definition 17 (PAL Model)**.** A *modal model* is a structure $\mathcal{M} = (W, R, \nu)$, where (W, R) is a $\mathsf{K4}$-frame and $\nu : \mathsf{Prop} \to \mathcal{P}(W)$ is a *(truth) valuation.*

Definition 18 (Truth in PAL Models)**.** A ternary relation $\mathcal{M}, w \Vdash A$ for *formula* $A \in \mathsf{Fml}_{\Box,[\cdot]}$ *being satisfied at a world* $w \in W$ *in a modal model* $\mathcal{M} = (W, R, \nu)$ is defined by induction on the structure of A:

- $\mathcal{M}, w \Vdash p$ if and only if $w \in \nu(p)$.
- $\mathcal{M}, w \Vdash \neg A$ if and only if $\mathcal{M}, w \nVdash A$.
- $\mathcal{M}, w \Vdash A \to B$ if and only if $\mathcal{M}, w \nVdash A$ or $\mathcal{M}, w \Vdash B$.
- $\mathcal{M}, w \Vdash \Box A$ if and only if $\mathcal{M}, v \Vdash A$ for all $v \in W$ with $R(w, v)$.

- $\mathcal{M}, w \Vdash [A]B$ if and only if $\mathcal{M}_A, w \Vdash B$, where the modal model $\mathcal{M}_A = (W_A, R_A, \nu_A)$ is defined as follows:

$$
\begin{aligned}
W_A &:= W \ , \\
R_A &:= \{(s,t) \mid R(s,t) \text{ and } \mathcal{M}, t \Vdash A\} \ , \\
\nu_A &:= \nu \ .
\end{aligned}
$$

  It is easy to see that $\mathcal{M}_A$ is indeed a modal model.

We write $\mathcal{M} \Vdash A$ to mean that $\mathcal{M}, w \Vdash A$ for all $w \in W$. We say that formula $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is *valid*, written $\Vdash A$, to mean that $\mathcal{M} \Vdash A$ for all modal models $\mathcal{M}$. The negation of $\Vdash A$ is written $\nVdash A$.

One of the essential features of PAL is that $\mathsf{Fml}_{\Box,[\cdot]}$-formulas with announcements can be "reduced to" provably equivalent $\mathsf{Fml}_{\Box}$-formulas without announcements [11, 12, 21]. This implies that we can express what is the case after an announcement by saying what was the case before the announcement. The following lemma describes this reduction procedure.

Definition 19 (Reduction)**.** Define the reduction function $\mathsf{red} : \mathsf{Fml}_{\Box,[\cdot]} \to \mathsf{Fml}_{\Box}$ as follows:

- $\mathsf{red}(p) = p$.
- red commutes with the connectives $\neg$, $\to$, and $\Box$.
- $\mathsf{red}([A]p) = p$.
- $\mathsf{red}([A]\neg B) = \mathsf{red}(\neg[A]B)$.
- $\mathsf{red}([A](B \to C)) = \mathsf{red}([A]B \to [A]C)$.
- $\mathsf{red}([A]\Box B) = \mathsf{red}(\Box(A \to [A]B))$.
- $\mathsf{red}([A][B]C) = \mathsf{red}([A \wedge [A]B]C)$.

Lemma 20 (Provable Equivalence of Reductions)**.** *For all formulas $A \in \mathsf{Fml}_{\Box,[\cdot]}$,*

$$\mathsf{PAL} \vdash A \leftrightarrow \mathsf{red}(A) \ .$$

Proof sketch. In this proof, "formula" means $\mathsf{Fml}_{\Box,[\cdot]}$-formula. There is a complexity measure on formulas [21] (see Def. 25 for the case with justifications) such that each reduction step results in a "simpler" formula. The claim of the lemma is shown by induction on this formula complexity. The cases where A is of the form $[B]C$ are easily dealt with by the announcement axioms of PAL. Of interest for us is the case where A is of the form $\Box B$. By the induction hypothesis we have

$$\mathsf{PAL} \vdash B \leftrightarrow \mathsf{red}(B) \ . \tag{3}$$

Therefore, by modal reasoning, we obtain

$$\mathsf{PAL} \vdash \Box B \leftrightarrow \Box\mathsf{red}(B) \ , \tag{4}$$

which is the same as $\mathsf{PAL} \vdash \Box B \leftrightarrow \mathsf{red}(\Box B)$. $\Box$

Since every formula $A \in \mathsf{Fml}_{\Box,[\cdot]}$ has a PAL-provably equivalent formula $\mathsf{red}(A) \in \mathsf{Fml}_{\Box}$, the soundness of PAL and the completeness of $\mathsf{K4}$ together imply the completeness of PAL. To see why, suppose that $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is valid. Then $\mathsf{red}(A)$ is also valid by the soundness of PAL and Lemma 20. Since $\mathsf{red}(A)$ is a formula of $\mathsf{Fml}_{\Box}$, we get by the completeness of $\mathsf{K4}$ that $\mathsf{K4} \vdash \mathsf{red}(A)$ and, hence, that $\mathsf{PAL} \vdash \mathsf{red}(A)$ because PAL extends $\mathsf{K4}$. Applying Lemma 20, we conclude that $\mathsf{PAL} \vdash A$.

5 Soundness and Completeness for JPAL

Lemma 21 (Soundness)**.** *For all formulas A, we have that $\vdash A$ implies $\Vdash A$.*

Proof. As usual the proof is by induction on the length of the derivation of A in JPAL. We only show the cases for the axioms that relate announcements and justifications.

1. Independence. $\mathcal{M}, w \Vdash [\sigma]p$ iff $\mathcal{M}_\sigma, w \Vdash p$ iff $w \in \nu_\sigma(p)$ iff $w \in \nu(p)$ iff $\mathcal{M}, w \Vdash p$.

2. Normality. $\mathcal{M}, w \Vdash [\sigma](B \to C)$ iff $\mathcal{M}_\sigma, w \Vdash B \to C$ iff $\mathcal{M}_\sigma, w \nVdash B$ or $\mathcal{M}_\sigma, w \Vdash C$ iff $\mathcal{M}, w \nVdash [\sigma]B$ or $\mathcal{M}, w \Vdash [\sigma]C$ iff $\mathcal{M}, w \Vdash [\sigma]B \to [\sigma]C$.

3. Functionality. $\mathcal{M}, w \Vdash [\sigma]\neg B$ iff $\mathcal{M}_\sigma, w \Vdash \neg B$ iff $\mathcal{M}_\sigma, w \nVdash B$ iff $\mathcal{M}, w \nVdash [\sigma]B$ iff $\mathcal{M}, w \Vdash \neg[\sigma]B$.

4. Update. $\mathcal{M}, w \Vdash [\sigma]t : (A \to [A]B)$ is equivalent to the conjunction of

$$A \to [A]B \in \mathcal{E}^\sigma(w, t) \tag{5}$$

and

$$\mathcal{M}_\sigma, v \Vdash A \to [A]B \text{ for all } v \text{ with } R_\sigma(w, v) \ . \tag{6}$$

By the condition (1) on $\mathcal{E}$ from Def. 11, we obtain that (5) if and only if

$$B \in \mathcal{E}^{\sigma,A}(w, t) \ . \tag{7}$$

Moreover, (6) is equivalent to

$$\mathcal{M}_\sigma, v \Vdash A \text{ implies } \mathcal{M}_\sigma, v \Vdash [A]B \text{ for all } v \text{ with } R_\sigma(w, v) \ .$$

This is equivalent to

$$\mathcal{M}_\sigma, v \Vdash A \text{ implies } \mathcal{M}_{\sigma,A}, v \Vdash B \text{ for all } v \text{ with } R_\sigma(w, v) \ ,$$

which, in turn, is equivalent to

$$\mathcal{M}_{\sigma,A}, v \Vdash B \text{ for all } v \text{ with } R_{\sigma,A}(w, v) \ .$$

The conjunction of this and (7) is equivalent to

$$\mathcal{M}_{\sigma,A}, w \Vdash t : B \ ,$$

or equivalently

$$\mathcal{M}, w \Vdash [\sigma][A]t : B \ .$$

5. Iteration. First we show that

$$R_{\sigma,A,B} = R_{\sigma,A\wedge[A]B} \quad . \tag{8}$$

$R_{\sigma,A,B}(u,v)$ is equivalent to

$$R_\sigma(u,v) \text{ and } \mathcal{M}_\sigma, v \Vdash A \text{ and } \mathcal{M}_{\sigma,A}, v \Vdash B \quad .$$

This is equivalent to

$$R_\sigma(u,v) \text{ and } \mathcal{M}_\sigma, v \Vdash A \wedge [A]B \quad ,$$

which, in turn, is equivalent to $R_{\sigma,A\wedge[A]B}(u,v)$ and thus (8) is established.

The case for iteration is now as follows:

$$\mathcal{M}, w \Vdash [\sigma][A][B]C$$

if and only if

$$\mathcal{M}_{\sigma,A,B}, w \Vdash C \quad .$$

By condition (2) on $\mathcal{E}$ from Def. 11 and by (8), this is equivalent to

$$\mathcal{M}_{\sigma,A\wedge[A]B}, w \Vdash C$$

which, in turn, is equivalent to

$$\mathcal{M}, w \Vdash [\sigma][A \wedge [A]B]C \quad . \qquad \square$$

In the presence of justifications it is not possible to follow the modal-logic reduction approach to establish completeness. The reason is that the replacement property does not hold in Justification Logics. In particular, it is not the case that $\vdash t\!:\!A \leftrightarrow t\!:\!B$ whenever $\vdash A \leftrightarrow B$. (See [9, Sect. 6] for a detailed discussion of the replacement property in Justification Logic.) Therefore, we cannot perform the step from (3) to (4) to prove the JPAL-analogue of Lemma 20. Thus, it is not possible to transfer the completeness of $\mathsf{J4}$ to JPAL. We shall therefore provide a canonical model construction to prove the completeness of JPAL.

Definition 22 (Maximal Consistent Sets)**.** A set Φ of formulas is called *consistent* if $\Phi \nvdash \phi$ for some formula ϕ. A set Φ is called *maximal consistent* if it is consistent but has no consistent proper extensions.

It can be easily shown that maximal consistent sets contain all axioms of JPAL and are closed under modus ponens and axiom necessitation.

Definition 23 (Canonical Model)**.** The *canonical model* $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ is defined as follows:

1. $W := \{w \subseteq \mathsf{Fml}_{\mathsf{J},[\cdot]} \mid w \text{ is a maximal consistent set}\}$,
2. $R(w,v)$ iff for all finite sequences σ and all $t \in \mathsf{Tm}$, we have $[\sigma]t\!:\!A \in w$ implies $[\sigma]A \in v$,

3. $\mathcal{E}^\sigma(w,t) := \{A \in \mathsf{Fml}_{\mathsf{J},[\cdot]} \ : \ [\sigma]t : A \in w\}$,

4. $\nu(p) := \{w \in W \ : \ p \in w\}$.

Lemma 24 (Correctness of the Canonical Model)**.** *The canonical model is a model.*

Proof. First, we observe that the set W is nonempty: the set of all formulas that are true at world w of the model from Example 13 is maximally consistent. We next show that $\mathcal{E}^\sigma$ is an evidence function for each σ.

- Monotonicity. Assume $A \in \mathcal{E}^\sigma(w,t)$ and $R(w,v)$. We have $[\sigma]t : A \in w$. By introspection and normality we get $[\sigma]!t : t : A \in w$. By $R(w,v)$ we find $[\sigma]t : A \in v$. Thus $A \in \mathcal{E}^\sigma(v,t)$.
- Axioms. Follows by rule AN.
- Application. Assume $A \to B \in \mathcal{E}^\sigma(w,t)$ and $A \in \mathcal{E}^\sigma(w,s)$. We have $[\sigma]t : (A \to B) \in w$ and $[\sigma]s : A \in w$. By application and normality we get $[\sigma]t \cdot s : B \in w$. Thus $B \in \mathcal{E}^\sigma(w, t \cdot s)$.
- Sum and Introspection are shown as in the previous case, though using the axioms of sum and introspection respectively.

Next we show condition (1) on $\mathcal{E}$ from Def. 11. We have $A \to [A]B \in \mathcal{E}^\sigma(w,t)$ if and only if $[\sigma]t : (A \to [A]B) \in w$. By the update axiom, the latter is equivalent to $[\sigma][A]t : B \in w$ which is equivalent to $B \in \mathcal{E}^{\sigma,A}(w,t)$.

Condition (2) holds too: $C \in \mathcal{E}^{\sigma,A,B}(w,t)$ if and only if $[\sigma][A][B]t : C \in w$. By the iteration axiom, the latter is equivalent to $[\sigma][A \wedge [A]B]t : C \in w$ which is equivalent to $C \in \mathcal{E}^{\sigma,A\wedge[A]B}(w,t)$.

Finally, we show that R is transitive. Let $R(w,v)$, $R(v,u)$ and $[\sigma]t : A \in w$. By introspection and normality we have $[\sigma]!t : t : A \in w$. By $R(w,v)$ we find $[\sigma]t : A \in v$ and by $R(v,u)$ we get $[\sigma]A \in u$. Thus, we conclude $R(w,u)$. $\square$

Definition 25 (Rank)**.** The *rank* $\mathsf{rk}(A)$ of a formula A is defined as follows.

- $\mathsf{rk}(p) := 1$ for each $p \in \mathsf{Prop}$,
- $\mathsf{rk}(\neg A) := \mathsf{rk}(A) + 1$,
- $\mathsf{rk}(A \to B) := \max(\mathsf{rk}(A), \mathsf{rk}(B)) + 1$,
- $\mathsf{rk}(t : A) := \mathsf{rk}(A) + 1$,
- $\mathsf{rk}([A]B) := (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B)$.

Lemma 26 (Reductions Reduce Rank)**.** *For all formulas A, B, C and all terms t we have the following:*

1. $\mathsf{rk}(A) > \mathsf{rk}(B)$ *if B is a proper subformula of A,*

2. $\mathsf{rk}([A]p) > \mathsf{rk}(p)$ *for each $p \in \mathsf{Prop}$.*

3. $\mathsf{rk}([A]\neg B) > \mathsf{rk}(\neg[A]B)$.

4. $\mathsf{rk}([A](B \to C)) > \mathsf{rk}([A]B \to [A]C)$.

5. $\mathsf{rk}([A]t : B) > \mathsf{rk}(t : (A \to [A]B))$.

6. $\mathsf{rk}([A][B]C) > \mathsf{rk}([A \wedge [A]B]C)$.

Proof. Let us only show the last two cases. First, case 5.

$$\begin{aligned}
\mathsf{rk}([A]t : B) &= (2 + \mathsf{rk}(A)) \cdot (\mathsf{rk}(B) + 1) \\
&= (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B) + 2 + \mathsf{rk}(A) \\
&> (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B) + 1 + 1 \\
&= \mathsf{rk}(t : (A \to [A]B)) \ .
\end{aligned}$$

And now case 6.

$$\begin{aligned}
\mathsf{rk}([A][B]C) &= (2 + \mathsf{rk}(A)) \cdot (2 + \mathsf{rk}(B)) \cdot \mathsf{rk}(C) \\
&= (4 + 2\mathsf{rk}(A) + 2\mathsf{rk}(B) + \mathsf{rk}(A)\mathsf{rk}(B)) \cdot \mathsf{rk}(C) \\
&\geq (6 + 2\mathsf{rk}(B) + \mathsf{rk}(A)\mathsf{rk}(B)) \cdot \mathsf{rk}(C) \\
&> (2 + 3 + (2 + \mathsf{rk}(A)) \cdot \mathsf{rk}(B)) \cdot \mathsf{rk}(C) \\
&= (2 + \mathsf{rk}(\neg(A \to \neg[A]B))) \cdot \mathsf{rk}(C) \\
&= \mathsf{rk}([A \wedge [A]B]C) \ .
\end{aligned}$$

□

Lemma 27 (Truth Lemma)**.** *Let $\mathcal{M}$ be the canonical model. For all formulas D and all worlds w in $\mathcal{M}$,*

$$D \in w \text{ if and only if } \mathcal{M}, w \Vdash D \ .$$

Proof. Proof by induction on $\mathsf{rk}(D)$ and a case distinction on the structure of D. Let us only show the cases where D is of the form $[A]B$. The other cases are standard and follow easily from the closure conditions on the evidence function.

1. $D = [A]p$. Suppose $[A]p \in w$. This is equivalent to $p \in w$ by the independence axiom. By the induction hypothesis this is equivalent to $\mathcal{M}, w \Vdash p$ which is equivalent to $\mathcal{M}, w \Vdash [A]p$ by the soundness of the independence axiom.

2. $D = [A]\neg B$. Suppose $[A]\neg B \in w$. This is equivalent to $\neg[A]B \in w$ by the functionality axiom. By the induction hypothesis this is equivalent to $\mathcal{M}, w \Vdash \neg[A]B$ which is equivalent to $\mathcal{M}, w \Vdash [A]\neg B$ by the soundness of the functionality axiom.

3. $D = [A](B \to C)$. Suppose $[A](B \to C) \in w$. By the normality axiom this is equivalent to $[A]B \to [A]C \in w$. By the induction hypothesis this is equivalent to $\mathcal{M}, w \Vdash [A]B \to [A]C$ which is equivalent to $\mathcal{M}, w \Vdash [A](B \to C)$ by the soundness of the normality axiom.

4. $D = [A]t : B$. Suppose $[A]t : B \in w$. By the update axiom this is equivalent to $t : (A \to [A]B) \in w$. By the induction hypothesis this is equivalent to $\mathcal{M}, w \Vdash t : (A \to [A]B)$ which by the soundness of the update axiom is equivalent to $\mathcal{M}, w \Vdash [A]t : B$.

5. $D = [A][B]C$. Suppose $[A][B]C \in w$. By the iteration axiom this is equivalent to $[A \wedge [A]B]C \in w$. By the induction hypothesis this, in turn, is equivalent to $\mathcal{M}, w \Vdash [A \wedge [A]B]C$ which by the soundness of the iteration axiom is equivalent to $\mathcal{M}, w \Vdash [A][B]C$. $\square$

Theorem 28 (Completeness)**.** JPAL *is sound and complete; that is, for all formulas* $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$, *we have*

$$\vdash A \text{ if and only if } \Vdash A \ .$$

Proof. Soundness has already been shown in Lemma 21. For completeness, consider the canonical model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ and assume that $\nvdash A$. Then $\{\neg A\}$ is consistent and, hence, contained in some maximal consistent set $w \in W$. By Lemma 27, it follows that $\mathcal{M}, w \Vdash \neg A$ and, hence, that $\mathcal{M}, w \nVdash A$. Since $\mathcal{M}$ is a model (Lemma 24), we have shown that $\nvdash A$ implies $\nVdash A$. Completeness follows by contraposition. $\square$

Corollary 29 (Announcement Necessitation)**.** *Announcement necessitation is admissible; that is, for all formulas* $A, B \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$, *we have that*

$$\vdash A \quad \text{implies} \quad \vdash [B]A \ .$$

Proof. Assume $\vdash A$. By soundness, $\Vdash A$. Therefore, $\mathcal{M} \Vdash A$ for all models $\mathcal{M}$. In particular, $\mathcal{M}_B \Vdash A$ for all models of the form $\mathcal{M}_B$. Thus, we obtain $\mathcal{M} \Vdash [B]A$ for all models $\mathcal{M}$. By completeness, we conclude $\vdash [B]A$. $\square$

6 Successful Updates

The notion of "successful formula" has been studied by various authors; see [20] and the references therein for details. Intuitively, a formula is successful if its announcement causes its listeners to believe in its truth. In the case of justified belief, what we want is the following: a formula is successful if its announcement brings about evidence of its truth.

Definition 30 (Successful Formulas)**.** To say that a formula $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ is *successful* means that there exists a term $t \in \mathsf{Tm}$ such that $\Vdash [A]t : A$.

Remark 31. Van Ditmarsch and Kooi [20] have studied a modal notion of success in the context of Plaza-announcements: a formula $A \in \mathsf{Fml}_{\Box,[\cdot]}$ is *vDK-successful* means that $\Vdash_{\text{Plaza}} [A]A$, where the validity is given with respect to Plaza notion of announcements. Since JPAL is based on the Gerbrandy–Groeneveld notion of announcements (in which the announced formula need not be true), the van Ditmarsch–Kooi definition of success will not work. In particular, $\nVdash [p]p$ in JPAL.

Definition 32 (Preserving Formulas)**.** A formula A is *preserving* if $\Vdash A \to [B]A$ for all formulas B.

Definition 33 (Grammar for Preserving Formulas)**.** Pres is the set of formulas defined by the following grammar:

$$A ::= p \mid \neg p \mid (A \wedge A) \mid (A \vee A) \qquad (p \in \mathsf{Prop})$$

Theorem 34 (Preserving Formulas in JPAL)**.** *Every formula A of* Pres *is preserving.*

Proof. Proof by induction on the structure of A.

1. $A = p$. This case follows immediately from the independence axiom and soundness.

2. $A = \neg p$. This case follows immediately from the independence and functionality axioms and soundness.

3. $A = B \wedge C$ or $A = B \vee C$. These cases follow from the induction hypothesis, the normality and functionality axioms, and soundness. $\square$

Remark 35. In the modal case we also have that if $A \in \mathsf{Pres}$ is preserving, so is $\Box A$. For JPAL we only get the following: for any preserving formula $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ and term t, we have that for each formula $B \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ there exists a term s_B such that $\Vdash t\!:\!A \to [B]s_B\!:\!A$.

Proof. Take an arbitrary formula B. Then $\Vdash A \to [B]A$ because A is preserving. By completeness, $\vdash A \to [B]A$. By constructive necessitation, there is a term q with $\vdash q\!:\!(A \to [B]A)$. Thus, we obtain $\vdash t\!:\!A \to q \cdot t\!:\![B]A$. By using the propositional tautology $[B]A \to (B \to [B]A)$ we get $\vdash t\!:\!A \to c \cdot (q \cdot t)\!:\!(B \to [B]A)$ for any constant c. By the update axiom and soundness, $\Vdash t\!:\!A \to [B]c \cdot (q \cdot t)\!:\!A$. $\square$

Theorem 36 (Successful Formulas in JPAL)**.** *Every preserving formula is successful.*

Proof. Let $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ be a preserving formula. In particular, $\Vdash A \to [A]A$. By completeness and constructive necessitation, there exists a term t such that $\vdash t\!:\!(A \to [A]A)$. Applying the update axiom, we find $\vdash [A]t\!:\!A$. Hence, $\Vdash [A]t\!:\!A$ by soundness. $\square$

Remark 37. Like in the modal case, not every formula is successful in JPAL. Consider a justified version $A = p \wedge \neg t\!:\!p$ of Moore sentence and the model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ from Example 13. We have $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, v \nVdash A$. The accessibility relation after the announcement of A is $R_A = \{(w, w)\}$. We find $\mathcal{M}_A, w \nVdash A$ and therefore that $\mathcal{M}_A, w \nVdash s\!:\!A$ for all terms s. Thus, we obtain $\mathcal{M}, w \nVdash [A]s\!:\!A$ for all terms s. So A is not successful.

Still, there is an important difference between the original modal Moore sentence $B = p \wedge \neg\Box p$ and its justified version $A = p \wedge \neg t\!:\!p$. In the modal case, believing B after it is announced leads to inconsistent beliefs: if $\mathcal{M}_B, w \Vdash \Box B$ for some modal model $\mathcal{M} = (W, R, \nu)$ and some $w \in W$, then there can be no v that would satisfy $R(w, v)$. Indeed, such a v would have to satisfy $\mathcal{M}_B, v \Vdash \Box p \wedge \neg\Box p$. But if no world is considered possible at w as the result of the announcement, everything is believed: $\mathcal{M}_B, w \Vdash \Box C$ for all formulas $C \in \mathsf{Fml}_{\Box,[\cdot]}$.

In JPAL the situation is different. Consider a formula $D = s\!:\!(p \wedge \neg t\!:\!p)$ and some model $\mathcal{M} = (W, R, \mathcal{E}, \nu)$ that satisfies $W = \{w\}$, $R = \{(w, w)\}$, $p \wedge \neg t\!:\!p \in \mathcal{E}^\sigma(w, s)$ and $p \notin \mathcal{E}^\sigma(w, t)$ for all σ, and $\nu(p) = \{w\}$. In such a model $\mathcal{M}_D, w \Vdash \neg t\!:\!p$ and $\mathcal{M}_D, w \Vdash s\!:\!(p \wedge \neg t\!:\!p)$. Hence, for any constant c we have $\mathcal{M}_D, w \Vdash (c \cdot s)\!:\!p$. However, this does not lead to an immediate contradiction with $\mathcal{M}_D, w \Vdash \neg t\!:\!p$ because terms $c \cdot s$ and t may be different.

7 Forgetful Projection and Realization

Definition 38 (Forgetful projection)**.** The mapping $\circ : \mathsf{Fml}_{\mathsf{J},[\cdot]} \to \mathsf{Fml}_{\Box,[\cdot]}$ is defined as follows:

1. $p^\circ = p$ for all $p \in \mathsf{Prop}$,
2. $\circ$ commutes with the propositional connectives $\neg$ and $\to$,
3. $(t : A)^\circ = \Box A^\circ$,
4. $([A]B)^\circ = [A^\circ]B^\circ$.

Theorem 39 (Forgetful Projection of JPAL)**.** *We have for all formulas $A \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$*

$$\mathsf{JPAL} \vdash A \quad \Longrightarrow \quad \mathsf{PAL} \vdash A^\circ .$$

Proof. The proof is by induction on the length of the derivation of A in JPAL. For the base case simply observe that the forgetful projection of each axiom of JPAL is derivable in PAL. The rest is straightforward. □

Definition 40 (Realization)**.**

1. An $\mathsf{Fml}_\Box$ realization is a mapping $r : \mathsf{Fml}_\Box \to \mathsf{Fml}_\mathsf{J}$ such that for all $A \in \mathsf{Fml}_\Box$ we have $(r(A))^\circ = A$.
2. An $\mathsf{Fml}_{\Box,[\cdot]}$ realization is a mapping $r : \mathsf{Fml}_{\Box,[\cdot]} \to \mathsf{Fml}_{\mathsf{J},[\cdot]}$ such that for all $A \in \mathsf{Fml}_{\Box,[\cdot]}$ we have $(r(A))^\circ = A$.

For the logic K4 we have the following realization result due to Brezhnev [5]:

Theorem 41 (Realization for K4)**.** *There exists an $\mathsf{Fml}_\Box$ realization r_{K4} such that*

$$\mathsf{K4} \vdash A \quad \textit{implies} \quad \mathsf{J4} \vdash r_{\mathsf{K4}}(A) \qquad \textit{for all } A \in \mathsf{Fml}_\Box \ .$$

The problem of $\mathsf{Fml}_{\Box,[\cdot]}$ realization is whether there exists an $\mathsf{Fml}_{\Box,[\cdot]}$ realization such that the JPAL analog of Theorem 41 can be shown. At present, we need an additional assumption to establish such a realization result. To formulate this assumption we use the following convention: Assume $D(q)$ is a formula. Then $D(A)$ is the formula that is given by replacing every occurrence of the proposition q in D with the formula A.

Definition 42 (Update Replacement Property)**.** Let $D(q)$ be a formula from $\mathsf{Fml}_{\mathsf{J},[\cdot]}$ with at most one occurrence of the proposition q. We say that JPAL satisfies the *update replacement property* if the following implications hold:

1. if $\vdash D(p)$ then $\exists \hat{D}$ with $\vdash \hat{D}([A]p)$,
2. if $\vdash D([A]B \to [A]C)$ then $\exists \hat{D}$ with $\vdash \hat{D}([A](B \to C))$,
3. if $\vdash D(\neg[A]B)$ then $\exists \hat{D}$ with $\vdash \hat{D}([A]\neg B)$,
4. if $\vdash D(t : (A \to [A]B))$ then $\exists \hat{D}$ with $\vdash \hat{D}([A]t : B)$,

5. if $\vdash D([A \wedge [A]B]C)$ then $\exists \hat{D}$ with $\vdash \hat{D}([A][B]C)$

where in all cases $\hat{D} \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ is such that $D^\circ = \hat{D}^\circ$.

Theorem 43 (Conditional Realization for JPAL)**.** *If* JPAL *satisfies the update replacement property, then there exists an* $\mathsf{Fml}_{\Box,[\cdot]}$ *realization* r *such that*

$$\mathsf{PAL} \vdash A \quad \textit{implies} \quad \mathsf{JPAL} \vdash r(A) \quad \textit{for all } A \in \mathsf{Fml}_{\Box,[\cdot]} \ .$$

Proof. The idea of the proof is shown in the following diagram. We start with a formula $A \in \mathsf{Fml}_{\Box,[\cdot]}$. Using reduction, $\mathsf{Fml}_{\Box}$ realization from Theorem 41, and the update replacement property we construct a formula $r(A) \in \mathsf{Fml}_{\mathsf{J},[\cdot]}$ such that the forgetful projection of $r(A)$ is A.

$$\begin{tikzcd}
A \arrow[d, "\text{Reduction}"'] & & r(A) \arrow[ll, "\text{Forgetful projection}"'] \\
\mathsf{red}(A) \arrow[rr, "\mathsf{Fml}_{\Box}\text{ realization}"] & & r_{\mathsf{K4}}(\mathsf{red}(A)) \arrow[u, "\text{Update replacement property}"']
\end{tikzcd}$$

We will now give the details of this approach. Let A be a formula of $\mathsf{Fml}_{\Box,[\cdot]}$ such that $\mathsf{PAL} \vdash A$. By Lemma 20, we find $\mathsf{PAL} \vdash \mathsf{red}(A)$. By the soundness of PAL, $\mathsf{red}(A)$ is a valid formula of $\mathsf{Fml}_{\Box}$ and by completeness of $\mathsf{K4}$, we obtain $\mathsf{K4} \vdash \mathsf{red}(A)$. By Theorem 41 we have $\mathsf{J4} \vdash r_{\mathsf{K4}}(\mathsf{red}(A))$. Since $\mathsf{J4}$ is a subsystem of JPAL we also have $\mathsf{JPAL} \vdash r_{\mathsf{K4}}(\mathsf{red}(A))$. Now we make use of iterated applications of the update replacement property to 'invert' the reduction steps performed by red. Hence, there is a realization r such that $\mathsf{JPAL} \vdash r(A)$. □

We believe that by adapting techniques from [9] to logics with announcements, it will be possible to show a replacement theorem that is powerful enough for establishing $\mathsf{Fml}_{\Box,[\cdot]}$ realization.

Conjecture 44. *There exists an* $\mathsf{Fml}_{\Box,[\cdot]}$ *realization* r *such that*

$$\mathsf{PAL} \vdash A \quad \textit{implies} \quad \mathsf{JPAL} \vdash r(A) \quad \textit{for all } A \in \mathsf{Fml}_{\Box,[\cdot]} \ .$$

8 Conclusion

This paper offers a simple and direct combination of Public Announcement Logic and Justification Logic that describes how Gerbrandy–Groeneveld-style announcements affect the justification of beliefs. Our framework, JPAL, brings together the basic ingredients of these two areas: the justification formulas $t : A$ of Justification Logic with the public announcement formulas $[B]A$ of Public Announcement Logic, all in a single-agent framework. Syntactically, the combined language is just a synthesis of these components in which justification terms replace the doxastic modalities. This allows us to express the relationship between justifications before and after announcements using our update axiom:

$$[\sigma][A]t : B \leftrightarrow [\sigma]t : (A \to [A]B) \ .$$

In Example 6, we used this axiom to prove that $\vdash [p](c_1 \cdot c_2) : p$. While constant c_1 is used to justify a simple propositional tautology, c_2 is responsible for the independence axiom about the announcement of p. In this way, term $c_1 \cdot c_2$ (indirectly) relates the belief in p after the announcement of p with the fact that the announcement was made.

Were we to define the logic in which justifications directly reflect the number of announcements made, something that is implicitly done in, say, the Muddy Children puzzle, we would have to define a semantics that keeps some sort of record of the past or announcement order. This would invalidate the iteration axiom, and would most probably require additional operations on terms to account for the history-storing structures. Furthermore, such a semantics might use features of temporal extensions of dynamic epistemic logic that do not include the iteration axiom [17, 18, 19]. Instead, we have chosen a simpler semantics, which need not record such histories, and hence have been able to capture the dynamics of belief revision resulting from public announcements with a minimal evidence-handling instrumentary.

The semantics we have chosen, although simpler than a full history-based semantics, is still more than just the sum of the semantics of the logics it combines. From Public Announcement Logic, we adopt the method used by Gerbrandy and Groeneveld [11, 12], where trustful agents reject as impossible the worlds inconsistent with the announcement made, including perhaps the actual world in case of a false announcement. From Justification Logic, we use evidence functions to model reasons for belief, but instead of just one evidence function, the dynamics of announcements forces us to have a whole family of evidence functions: one for each potential sequence of announcements. The way beliefs change as the result of an announcement is governed by the relationship between the pre- and post-announcement evidence functions. Thus, our semantics allows for justifications to be shaped by announcements.

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