esting case is the one with a $\diamondast$-rule:

$$
\mathsf{d}\,\dfrac{\diamondast\,\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma\{[\diamondast A, \Diamond^k A, \Delta]_i\}\end{array}}{\Gamma\{[\diamondast A, \Delta]_i\}}}{\Gamma\{\diamondast A, [\Delta]_i\}}
\quad\rightsquigarrow\quad
\diamondast\,\dfrac{\mathsf{wk},\vee^*\,\dfrac{\mathsf{wk},\Diamond_i\,\dfrac{\mathsf{d}\,\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma\{[\diamondast A, \Diamond^k A, \Delta]_i\}\end{array}}{\Gamma\{\diamondast A, [\Diamond^k A, \Delta]_i\}}}{\Gamma\{\diamondast A, \Diamond_i \Diamond^k A, [\Delta]_i\}}}{\Gamma\{\diamondast A, \Diamond^{k+1} A, [\Delta]_i\}}}{\Gamma\{\diamondast A, [\Delta]_i\}}
\quad ,
$$

where the instance of d shown on the right is removed by induction hypothesis. ☐

Theorem 3.12 (Shallow into deep) If $\mathsf{G_C} \overset{\alpha}{\underset{\gamma}{\vdash}} \Gamma$ then $\mathsf{D_C} \overset{\omega\cdot\alpha}{\underset{\gamma}{\vdash}} \Gamma$.

Proof. By induction on α and a case analysis on the last rule in the proof. Each rule of $\mathsf{G_C}$ except for the $\Box_i$-rule is a special case of its respective rule in $\mathsf{D_C}$. For the $\Box_i$-rule we have the following transformation:

$$
\Box_i\,\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma, \diamondast\Delta, A\end{array}}{\Diamond_i\Gamma, \diamondast\Delta, \Box_i A, \Sigma}
\quad\rightsquigarrow\quad
\Box_i,\mathsf{wk}\,\dfrac{\mathsf{d}^*\,\dfrac{\mathsf{wk}^*,\Diamond_i^*\,\dfrac{\mathsf{nec}\,\dfrac{\begin{array}{c}\mathcal{P}'\\ \Gamma, \diamondast\Delta, A\end{array}}{[\Gamma, \diamondast\Delta, A]_i}}{\Diamond_i\Gamma, [\diamondast\Delta, A]_i}}{\Diamond_i\Gamma, \diamondast\Delta, [A]_i}}{\Diamond_i\Gamma, \diamondast\Delta, \Box_i A, \Sigma}
\quad ,
$$

where $\mathcal{P}'$ is obtained by induction hypothesis. ☐

3.3.2 Embedding Deep into Shallow

This is the harder direction, since we need to simulate deep applicability of rules in the shallow system. We use the invertibility of rules in the shallow system in order to do so. The $\Box_i$-rule is the only rule in $\mathsf{G_C}$ which is not invertible. However, a somewhat weaker property than invertibility holds, which is sufficient for our purposes, and which is stated in the upcoming lemma.

Example 3.13 To motivate the following definition consider the following three provable sequents to which the which the $\Box_i$-rule cannot be applied (upwards) in an invertible way:

$$\Box_i(a \wedge b), \Diamond_i\bar{a} \vee \Diamond_i\bar{b} \qquad \Box_i(a \wedge b), \diamondast\bar{a} \vee \diamondast\bar{b} \qquad \Box_i a, \diamondast\bar{a} \qquad .$$

Definition 3.14 (hiding formula, $\diamondast$-saturated sequent) A formula is *essentially* $\Diamond_i$ if 1) it is of the form $\Diamond_i A$ for any formula A or 2) it is of the form $A \vee B, B \vee A, A \wedge B$ or $B \wedge A$ where A is any formula and B is a formula which is essentially $\Diamond_i$. A formula is *hiding* $\Diamond_i$ in case 2). We define *essentially* $\diamondast$ and *hiding* $\diamondast$ formulas likewise. A formula is just *hiding* if it is either *hiding* $\Diamond_i$ for some i or *hiding* $\diamondast$. A sequent Γ is $\diamondast$*-saturated* if $\diamondast A \in \Gamma$ implies $\Diamond_i A \in \Gamma$, for each formula A and each i with $1 \leq i \leq h$.

Definition 3.15 (canonical $\Box_i$-instance) An instance of the rule

$$\Box_i \; \frac{\Gamma, \diamondast\Delta, A}{\Diamond_i\Gamma, \diamondast\Delta, \Box_i A, \Sigma}$$

is *canonical* if no formulas of the form $\Diamond_i B$ or $\diamondast B$ are in Σ.

Lemma 3.16 (Quasi-invertibility of the $\Box_i$-rule) Let Γ be a $\diamondast$-saturated sequent without hiding formulas and let there be a proof of the sequent $\Box_i A, \Gamma$ in $\mathsf{G_C}$. Then there is a proof of the same depth in $\mathsf{G_C}$ either 1) of the sequent Γ or 2) of the sequent $\Box_i A, \Gamma$ where the last rule instance is a canonical instance of the $\Box_i$-rule applying to the shown formula $\Box_i A$.

Proof. By induction on the depth of the given proof and a case analysis on the last rule. If the endsequent is axiomatic then Γ is axiomatic and the first disjunct of our lemma applies. If the last rule is the $\boxast$-rule then the proof is of the form

$$\boxast \; \frac{\vdots \quad \begin{array}{c} \mathcal{P}_k \\ \Box_i A, \Gamma_1, \Box^k B \end{array} \quad \vdots \quad {}_{1 \leq k}}{\Box_i A, \Gamma_1, \boxast B}$$

We apply the induction hypothesis to each premise, with $\Gamma = \Gamma_1, \Box^k B$. Notice that Γ is $\diamondast$-saturated and does not contain hiding formulas. There are two cases. First, if for all premises the first disjunct of the induction hypothesis is true then for each k we have a proof $\mathcal{P}'_k$ such that the following shows the first disjunct of our lemma:

$$\boxast \; \frac{\vdots \quad \begin{array}{c} \mathcal{P}'_k \\ \Gamma_1, \Box^k B \end{array} \quad \vdots \quad {}_{1 \leq k}}{\Gamma_1, \boxast B} \quad .$$

Second, if for some premise the second disjunct of the induction hypothesis is true then for some k we have a proof of the form

$$\Box_i \; \frac{\begin{array}{c} \vdots \\ A, \Gamma' \end{array}}{\Box_i A, \Gamma_1, \Box^k B} \quad .$$

Notice that the $\Box_i$-rule can only introduce a formula of the form $\Box^k B$ in Σ, so we can easily turn this into a proof

$$
\dfrac{A, \Gamma'}{\Box_i A, \Gamma_1, \boxast B}\;\Box_i \quad ,
$$

and we have shown the second disjunct of our lemma. The cases for $\vee$ and $\wedge$ are similar.

If the last rule is the $\diamondast$-rule then the following transformation yields a shorter proof:

$$
\diamondast\,\dfrac{\overset{\mathcal{P}}{\Box_i A, \Gamma_1, \diamondast B, \Diamond B}}{\Box_i A, \Gamma_1, \diamondast B}
\quad\leadsto\quad
\mathsf{ctr}^*\,\dfrac{\bar{\nabla}^*\,\dfrac{\overset{\mathcal{P}}{\Box_i A, \Gamma_1, \diamondast B, \Diamond B}}{\Box_i A, \Gamma_1, \diamondast B, \Diamond_1 B, \ldots, \Diamond_n B}}{\Box_i A, \Gamma_1, \diamondast B} \quad ,
$$

where by assumption of $\diamondast$-saturation all the $\Diamond_i B$ are in Γ_1. To this proof we can now apply the induction hypothesis which yields our lemma.

If the last rule in the given proof is the $\Box_j$-rule, then we distinguish two cases. First, if $\Box_i A$ is the active formula then the second disjunct of our lemma is either immediate or obtained via weakening admissibility if the rule instance is not canonical.

Second, if $\Box_i A$ is not the active formula then the proof is of the form

$$
\Box_j\,\dfrac{\overset{\mathcal{P}}{\Gamma'}}{\Box_i A, \Gamma_1, \Box_j B} \quad ,
$$

where the formula $\Box_i A$ has been introduced inside Σ. We can thus change it into a proof

$$
\Box_j\,\dfrac{\overset{\mathcal{P}}{\Gamma'}}{\Gamma_1, \Box_j B} \quad ,
$$

which shows the first disjunct of our lemma. $\Box$

In order to translate a derivation with deep rule applications into a derivation where only shallow rules are allowed we need a way of simulating the deep applicability. It turns out that, for certain shallow rules, if they are admissible for the shallow system, then their "deep version" is also admissible.

Definition 3.17 (Make a shallow rule deep) Let $C\{\ \}$ be a formula context. Given a rule ρ we define a rule *rule* $C\{\rho\}$ as follows: an instance of the rule ρ is shown on the left iff an instance of the *rule* $C\{\rho\}$ is shown on the right:

$$\rho\,\frac{\Gamma, A_1 \ldots \Gamma, A_i \ldots}{\Gamma, A} \qquad\qquad C\{\rho\}\,\frac{\Gamma, C\{A_1\} \ldots \Gamma, C\{A_i\} \ldots}{\Gamma, C\{A\}} \quad .$$

Given a rule ρ we define the rule *rule* $\check{\rho}$ as follows: its set of instances is the union of all sets of instances of $C\{\rho\}$ where $C\{\ \}$ ranges over formula contexts which only contains connectives from $\{\vee, \Box_1, \ldots, \Box_h\}$.

Lemma 3.18 (Deep applicability preserves finite admissibility) Let $C\{\ \}$ be a formula context which only contains connectives from $\{\vee, \Box_1 \ldots, \Box_h\}$.
(i) There is an n such that for all Γ we have $\mathsf{G_C} \overset{n}{\underset{0}{\vdash}} \Gamma, C\{p \vee \bar{p}\}$.
(ii) If a rule ρ is finitely admissible for $\mathsf{G_C}$ then $C\{\rho\}$ is also finitely admissible for system $\mathsf{G_C}$.
(iii) If a rule ρ is finitely admissible for $\mathsf{G_C}$ then $\check{\rho}$ is also finitely admissible for system $\mathsf{G_C}$.

Proof. Statement (iii) is immediate from (ii). Both (i) and (ii) are proved by induction on $C\{\ \}$. The case with $C\{\ \} = C_1\{\ \} \vee C_2$ is of course analogous to the case with $C\{\ \} = C_1 \vee C_2\{\ \}$ and is omitted. We first prove (i). The case that $C\{\ \}$ is empty is handled by an application of the $\vee$-rule. If $C\{\ \} = C_1 \vee C_2\{\ \}$ or $C\{\ \} = \Box_i C_1\{\ \}$ then we obtain a proof respectively as follows:

$$\vee\,\frac{\overset{\mathcal{P}}{\Gamma, C_1, C_2\{p \vee \bar{p}\}}}{\Gamma, C_1 \vee C_2\{p \vee \bar{p}\}} \qquad \text{or} \qquad \Box_i\,\frac{\overset{\mathcal{P}}{C_1\{p \vee \bar{p}\}}}{\Gamma, \Box_i C_1\{p \vee \bar{p}\}}$$

where in both cases $\mathcal{P}$ exists by induction hypothesis. For statement (ii) the case that $C\{\ \}$ is empty is clear, so we assume that it is non-empty. If $C\{\ \} = C_1 \vee C_2\{\ \}$ then the following transformation proves our claim:

$$C_1 \vee C_2\{\rho\}\,\frac{\vdots \quad \overset{\mathcal{P}_k}{\Gamma, C_1 \vee C_2\{A_k\}} \quad \vdots}{\Gamma, C_1 \vee C_2\{A\}} \quad \rightsquigarrow \quad \vee\,\frac{C_2\{\rho\}\,\dfrac{\vdots \quad \bar{\vee}\,\dfrac{\overset{\mathcal{P}_k}{\Gamma, C_1 \vee C_2\{A_k\}}}{\Gamma, C_1, C_2\{A_k\}} \quad \vdots}{\Gamma, C_1, C_2\{A\}}}{\Gamma, C_1 \vee C_2\{A\}}$$

If $C\{\ \} = \Box_i C_1\{\ \}$ then we have the following situation:

$$\Box_i C_1\{\rho\}\,\frac{\vdots \quad \overset{\mathcal{P}_k}{\Gamma, \Box_i C_1\{A_k\}} \quad \vdots}{\Gamma, \Box_i C_1\{A\}} \quad .$$

In order to apply quasi-invertibility of $\Box_i$, Lemma 3.16, we first need to replace the shown instance of the rule $\Box_i C_1\{\rho\}$ by several instances of it which are applied in a context which is $\diamond$-saturated and free of hiding formulas. We apply conjunction invertibility, disjunction invertibility and weakening admissibility to each $\mathcal{P}_k$ to obtain a sequence of proofs $\mathcal{P}_{k1} \ldots \mathcal{P}_{km}$ such that for each k there is a proof of the form

$$
\dfrac{\dfrac{\mathcal{P}_{k1}}{\Gamma_1, \Box_i C_1\{A_k\}} \quad \ldots \quad \dfrac{\mathcal{P}_{km}}{\Gamma_m, \Box_i C_1\{A_k\}}}{\Gamma, \Box_i C_1\{A_k\}}\ {\scriptstyle \wedge,\vee,\diamond} \quad ,
$$

where each Γ_j is $\diamond$-saturated and free of hiding formulas.

Fix some j. For all k apply quasi-invertibility of $\Box_i$, Lemma 3.16, to the proof $\mathcal{P}_{kj}$. Either this yields some proof $\mathcal{P}$ of Γ_j or for each k it yields a proof $\mathcal{P}'_{kj}$ of some sequent $\Gamma'_j, C_1\{A_k\}$. Then we can build either

$$
\mathsf{wk}\,\dfrac{\dfrac{\mathcal{P}}{\Gamma}}{\Gamma_j, \Box_i C_1\{A\}} \qquad \text{or} \qquad \Box_i\,\dfrac{C_1\{\rho\}\,\dfrac{\vdots \quad \dfrac{\mathcal{P}'_{kj}}{\Gamma'_j, C_1\{A_k\}} \quad \vdots}{\Gamma'_j, C_1\{A\}}}{\Gamma_j, \Box_i C_1\{A\}} \quad ,
$$

where in the second case $C_1\{\rho\}$ is finitely admissible by induction hypothesis. Repeat this argument for each j with $1 \leq j \leq m$, which for each j yields a proof $\mathcal{P}''_j$ in $\mathsf{G_C}$. From those we build

$$
\dfrac{\dfrac{\mathcal{P}''_1}{\Gamma_1, \Box_i C_1\{A\}} \quad \ldots \quad \dfrac{\mathcal{P}''_m}{\Gamma_m, \Box_i C_1\{A\}}}{\Gamma, \Box_i C_1\{A\}}\ {\scriptstyle \wedge,\vee,\diamond} \quad ,
$$

which shows our lemma. $\Box$

Lemma 3.19 (Some glue) The rules in Figure 3.6 are finitely admissible for system $\mathsf{G_C}$.

Proof. The rules $\mathsf{g_c}, \mathsf{g_a}$ and $\mathsf{g_{ctr}}$ are easily seen to be finitely admissible by using invertibility of the $\vee$-rule. For the $\mathsf{g}_\diamond$-rule we proceed by induction on the given proof of the premise and make a case analysis on the last rule in this proof.

$$\mathsf{g_c}\;\dfrac{\Gamma, A \vee B}{\Gamma, B \vee A} \qquad\qquad \mathsf{g_a}\;\dfrac{\Gamma, (A \vee B) \vee C}{\Gamma, A \vee (B \vee C)}$$

$$\mathsf{g_{ctr}}\;\dfrac{\Gamma, A \vee A}{\Gamma, A} \qquad \mathsf{g}_\Diamond\;\dfrac{\Gamma, \Box_i(A \vee B)}{\Gamma, \Diamond_i A, \Box_i B} \qquad \mathsf{g}_\circledast\;\dfrac{\Gamma, \Diamond^k A}{\Gamma, \circledast A}\ \text{where } k \geq 1$$

Figure 3.6: Some glue

All cases are trivial except when this is the $\Box_i$-rule. We distinguish two cases: either 1) $\Box_i(A \vee B)$ is the active formula or 2) it is not. In the first case we have:

$$\mathsf{g}_\Diamond\;\dfrac{\Box_i\;\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ \circledast\Delta, \Lambda, A \vee B\end{matrix}}{\Sigma, \circledast\Delta, \Diamond_i\Lambda, \Box_i(A \vee B)}}{\Sigma, \circledast\Delta, \Diamond_i\Lambda, \Diamond_i A, \Box_i B} \quad\rightsquigarrow\quad \Box_i\;\dfrac{\overline{\vee}\;\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ \circledast\Delta, \Lambda, A \vee B\end{matrix}}{\circledast\Delta, \Lambda, A, B}}{\Sigma, \circledast\Delta, \Diamond_i\Lambda, \Diamond_i A, \Box_i B}$$

and in the second case we have the following:

$$\mathsf{g}_\Diamond\;\dfrac{\Box_i\;\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ C, \Gamma''\end{matrix}}{\Box_i C, \Gamma', \Box_i(A \vee B)}}{\Box_i C, \Gamma', \Diamond_i A, \Box_i B} \quad\rightsquigarrow\quad \Box_i\;\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ C, \Gamma''\end{matrix}}{\Box_i C, \Gamma', \Diamond_i A, \Box_i B} \quad .$$

For the $\mathsf{g}_\circledast$-rule we proceed by induction on k and a subinduction on the depth of the given proof of the premise. For $k = 1$ the $\mathsf{g}_\circledast$-rule coincides with the $\circledast$-rule plus a weakening, so we assume that we have a proof of $\Gamma, \Diamond^{k+1} A$. By invertibility of the $\vee$-rule we obtain a proof

$$\begin{matrix}\mathcal{P}\\ \vdots\\ \Gamma, \Diamond_1 \Diamond^k A, \ldots, \Diamond_h \Diamond^k A\end{matrix}$$

of the same depth. By induction on the depth of $\mathcal{P}$ and a case analysis on the last rule in $\mathcal{P}$ we now show that we have a proof of the same depth of $\Gamma, \circledast A$. All cases are trivial except when the last rule is $\Box_i$. Then the following transformation:

$$\Box_i\;\dfrac{\begin{matrix}\mathcal{P}'\\ \vdots\\ B, \Delta, \circledast\Lambda, \Diamond^k A\end{matrix}}{\Box_i B, \Diamond_i\Delta, \circledast\Lambda, \Sigma, \Diamond_1\Diamond^k A, \ldots, \Diamond_h \Diamond^k A} \quad\rightsquigarrow\quad \Box_i\;\dfrac{\mathsf{g}_\circledast\;\dfrac{\begin{matrix}\mathcal{P}'\\ \vdots\\ B, \Delta, \circledast\Lambda, \Diamond^k A\end{matrix}}{B, \Delta, \circledast\Lambda, \circledast A}}{\Box_i B, \Diamond_i\Delta, \circledast\Lambda, \Sigma, \circledast A}$$

proves our claim, where the instance of the $\mathsf{g}_{\diamondplus}$-rule on the right is finitely admissible by the outer induction hypothesis. $\square$

For our translation from deep into shallow we translate nested sequents into formulas and thus fix an arbitrary order and association among elements of a sequent. The arbitrariness of this translation gets in the way, and we work around it as follows: we write

$$\mathsf{ac}\,\frac{A}{B}$$

if the formula B can be derived from the formula A in $\{\breve{\mathsf{g}}_{\mathsf{c}}, \breve{\mathsf{g}}_{\mathsf{a}}\}$. Clearly, in that case A and B are equal modulo commutativity and associativity of disjunction. The converse is not the case. For example $\diamondplus(C \vee D)$ can not be derived from $\diamondplus(D \vee C)$ by ac, in general. Note that since $\breve{\mathsf{g}}_{\mathsf{c}}$ and $\breve{\mathsf{g}}_{\mathsf{a}}$ are finitely admissible for system $\mathsf{G_C}$, so is the rule ac.

Theorem 3.20 (Deep into shallow)
If $\mathsf{D_C} \,\Big|\!\frac{\alpha}{0}\, \Gamma$ then we have $\mathsf{G_C} \,\Big|\!\frac{\omega\cdot(\alpha+1)}{0}\, \underline{\Gamma}_{\mathsf{F}}$.

Proof. By induction on α. If the endsequent of the given proof is of the form $\Gamma\{p, \bar{p}\}$, then we have

$$\Gamma\{p,\bar{p}\} \quad \rightsquigarrow \quad \mathsf{ac}\,\cfrac{\begin{array}{c}\mathcal{P}\\ \underline{\Gamma}_{\mathsf{F}}\{p \vee \bar{p}\}\end{array}}{\underline{\Gamma\{p,\bar{p}\}}_{\mathsf{F}}}$$

where $\mathcal{P}$ is of finite depth by Lemma 3.18 and ac is finitely admissible by Lemma 3.19 and Lemma 3.18. If the last rule is the $\vee$-rule then an application of ac proves our claim. The case of the $\square_i$-rule is trivial since the corresponding formula for the premise is the corresponding formula of the conclusion. For the $\boxast$-rule we apply the following transformation, where the $\mathcal{P}'_k$ are obtained by induction hypothesis:

$$\boxast\,\cfrac{\cdots\quad \begin{array}{c}\mathcal{P}_k\\ \Gamma\{\square^k A\}\end{array} \quad\cdots}{\Gamma\{\boxast A\}}\;1{\leq}k{<}\omega \qquad \rightsquigarrow \qquad \mathsf{ac}\,\cfrac{\underline{\Gamma}_{\mathsf{F}}\{\boxast\}\,\cfrac{\cdots\quad \mathsf{ac}\,\cfrac{\begin{array}{c}\mathcal{P}'_k\\ \underline{\Gamma\{\square^k A\}}_{\mathsf{F}}\end{array}}{\underline{\Gamma}_{\mathsf{F}}\{\square^k A\}} \quad\cdots}{\underline{\Gamma}_{\mathsf{F}}\{\boxast A\}}\;1{\leq}k{<}\omega}{\underline{\Gamma\{\boxast A\}}_{\mathsf{F}}}$$

Let the depth of the proof on the left be β with $\beta \leq \alpha$ and the depth of a proof $\mathcal{P}_k$ be β_k. Note that the depth of the ac-derivations both below and above the infinitary rule is bounded by a finite ordinal m because the context $\Gamma\{\ \}$ is finite. Then, by finite admissibility of the rule $\underline{\Gamma}_{\mathsf{F}}\{\boxast\}$ (Lemma 3.18) there is a finite

$$
\begin{array}{|c|}
\hline
\\
\text{(K)}\quad \Box_i A \wedge \Box_i (A \supset B) \supset \Box_i B \qquad \text{(CCL)}\quad \boxast A \supset (\Box A \wedge \Box \boxast A) \\[1em]
\text{(IND)}\ \dfrac{B \supset (\Box A \wedge \Box B)}{B \supset \boxast A} \qquad \text{(MP)}\ \dfrac{A \quad A \supset B}{B} \qquad \text{(NEC)}\ \dfrac{A}{\Box_i A} \\
\\
\hline
\end{array}
$$

Figure 3.7: System $\mathsf{H_C}$

ordinal n such that the proof on the right has the depth

$$
\begin{array}{rl}
& \sup_k(|\mathcal{P}'_k| + m + 1) + n + m \ < \ \sup_k(|\mathcal{P}'_k|) + \omega \\
\leq & \sup_k(\omega \cdot (\beta_k + 1)) + \omega \ = \ \omega \cdot \sup_k(\beta_k + 1) + \omega \\
= & \omega \cdot \beta + \omega \ = \ \omega \cdot (\beta + 1) \ \leq \ \omega \cdot (\alpha + 1) \qquad .
\end{array}
$$

The case for the $\wedge$-rule is similar. For the $\Diamond_i$-rule we apply the following transformation, where $\mathcal{P}'$ is obtained by induction hypothesis and the bound on the depth is easy to check:

$$
\Diamond_i\,\dfrac{\begin{array}{c}\mathcal{P}\\ \Gamma\{\Diamond_i A, [A, \Delta]_i\}\end{array}}{\Gamma\{\Diamond_i A, [\Delta]_i\}}
\quad\rightsquigarrow\quad
\begin{array}{rc}
& \mathcal{P}' \\
& \underline{\Gamma\{\Diamond_i A, [A, \Delta]_i\}}_{\mathsf{F}} \\
\mathsf{ac} & \overline{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_i A \vee \Box_i (A \vee \underline{\Delta}_{\mathsf{F}})\}} \\
\underline{\Gamma}_{\mathsf{F}}\{\Diamond_i A \vee \mathsf{g}_\Diamond\} & \overline{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_i A \vee (\Diamond_i A \vee \Box_i \underline{\Delta}_{\mathsf{F}})\}} \\
\mathsf{ac} & \overline{\underline{\Gamma}_{\mathsf{F}}\{(\Diamond_i A \vee \Diamond_i A) \vee \Box_i \underline{\Delta}_{\mathsf{F}}\}} \\
\underline{\Gamma}_{\mathsf{F}}\{\mathsf{g_{ctr}} \vee \Box_i \underline{\Delta}_{\mathsf{F}}\} & \overline{\underline{\Gamma}_{\mathsf{F}}\{\Diamond_i A \vee \Box_i \underline{\Delta}_{\mathsf{F}}\}} \\
\mathsf{ac} & \overline{\underline{\Gamma\{\Diamond_i A, [\Delta]_i\}}_{\mathsf{F}}}
\end{array}
\quad .
$$

Note that here a rule like $C\{\rho \vee A\}$ means rule ρ applied in the context $C\{\{\ \} \vee A\}$, and is finitely admissible for $\mathsf{G_C}$ if is ρ is finitely admissible for $\mathsf{G_C}$, by Lemma 3.18.

The case for the $\Diamond\!\!\!\ast$-rule is similar. $\Box$

We can now state the cut-elimination theorem for the shallow system.

Theorem 3.21 (Cut-elimination for the shallow system)
If $\mathsf{G_C} \vdash^{\alpha}_{\omega \cdot n} \Gamma$ then $\mathsf{G_C} \vdash^{\omega \cdot (\varphi_1^n(\omega \cdot \alpha) + 1)}_{0} \Gamma$

3.4 An Upper Bound on the Depth of Proofs

The Hilbert system $\mathsf{H_C}$ is obtained from some Hilbert system for classical propositional logic by adding the axioms and rules shown in Figure 3.7. It is essentially the same as system K_h^C from the book [19], where also soundness and completeness are shown. We will now embed $\mathsf{H_C}$ into $\mathsf{D_C} + \mathsf{cut}$, keeping track of the proof depth and thus, via cut-elimination for $\mathsf{D_C}$, establish an upper bound for proofs in $\mathsf{D_C}$. Via the embedding of the deep system into the shallow system, this bound also holds for the shallow system.