A finitary cut-free axiomatization for stratified modal fixed point logic

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The modal μ-calculus [5] is the extension of propositional modal logic with least and greatest fixed point operators. The μ-calculus is an important tool for specifying and verifying properties of programs and it has been thoroughly investigated. However, the deductive systems that are complete all contain a cut-rule [8, 9] and it is not known how to eliminate these cuts. At present, there is no cut-free axiomatization available. Decidability of the μ-calculus has only been established via a reduction to $S\omega S$ or via a reduction to the emptiness problem of certain automata [7, 4].

In this paper we study the stratified modal fixed point logic FPL for which we present a finitary cut-free axiomatic system. FPL corresponds to the first level of the variable hierarchy of the μ-calculus [2, 3], that is the fragment of the modal μ-calculus with only one fixed point variable. In this fragment, it is still possible to formulate nested fixed point definitions by reusing the single fixed point variable. Indeed, FPL captures many logics such as PDL, CTL, and the logic of common knowledge.

FPL is a *stratified* fixed point logic in the following sense: Consider a formula $\mu X.A(X)$ where $A(X)$ is X positive. $A(X)$ may contain a subformula $\nu Y.B(Y)$ only if $B(Y)$ does not contain X free. This allows us to compute the meaning of $\nu Y.B(Y)$ and then use this result to determine the interpretation of $\mu X.A(X)$. Stratification guarantees that inner fixed points do not depend on outer fixed points. Hence, one can easily determine the interpretation of a formula by induction on its structure.

This is not possible if interleaving of fixed points is allowed. Consider the formula $\mu X.\nu Y.A(X, Y)$. The interpretation of the inner fixed point $\nu Y.A(X, Y)$ depends on the interpretation of X which is given by the outer fixed point $\mu X.\nu Y.A(X, Y)$ which in turn depends on the inner fixed point $\nu Y.A(X, Y)$. Hence, interleaving of fixed points has the effect that the interpretations of fixed points cannot be computed one after the other. Instead, the they mutually depend on each other. In order to avoid this problem, we exclude such constructions from FPL.

We will prove soundness and completeness of our axiomatization of FPL. Completeness is shown by means of a canonical countermodel construction where saturated sets take the role of possible worlds. Here, stratification is essential in order to show by induction that we indeed do have a countermodel. The main observation to prove soundness is the following: a greatest fixed point, say $\nu X.A(X)$ is valid if already some finite iteration of $A(X)$, that is $A(A(\ldots(\top)\ldots))$, is valid. This fact is a consequence of the finite model property for the μ-calculus and standard results about monotone operators. The technique used to prove soundness of a finitary rule which derives a greatest fixed point is very general and applies also to other calculi. For instance, it becomes possible to finitize the infinitary calculus for common knowledge studied in [1, 6]. In that system, common knowledge is derived by an infinitary rule which has all iterations of 'everybody knows' as premises. We can cut off this rule at a certain stage and still have soundness since the logic of common knowledge enjoys the finite model property. Completeness of the system is not affected for the resulting rule has less premises.

References

[1] Luca Alberucci and Gerhard Jäger. About cut elimination for logics of common knowledge. To appear in *Annals of Pure and Applied Logic.*

[2] Dietmar Berwanger, Erich Grädel, and Giacomo Lenzi. On the variable hierarchy of the modal μ-calculus. In *CSL '02: Proceedings of the 16th International Workshop and 11th Annual Conference of the EACSL on Computer Science Logic*, pages 352–366. Springer, 2002.

[3] Dietmar Berwanger and Giacomo Lenzi. The variable hierarchy of the μ-calculus is strict. In *STACS 2005, Proceedings of the 22nd Symposium on Theoretical Aspects of Computer Science*, LNCS. Springer, 2005. To appear.

[4] E. Allen Emerson and Charanjit S. Jutla. Tree automata, mu-calculus and determinacy (extended abstract). In *Proceedings of the 32th Annual*

Symposium on Foundations of Computer Science, pages 368–3377. IEEE Computer Society Press, 1991.

[5] Dexter Kozen. Results on the propositional μ-calculus. *Theoretical Computer Science*, 27:333–354, 1983.

[6] Mathis Kretz and Thomas Studer. Deduction chains for common knowledge. To appear in *Journal of Applied Logic*.

[7] Robert S. Streett and E. Allen Emerson. An automata theoretic decision procedure for the propositional mu-calculus. *Information and Computation*, 81(3):249–264, 1989.

[8] Igor Walukiewicz. A complete deductive system for the μ-calculus. In *Proceedings 8th Annual IEEE Symp. on Logic in Computer Science, LICS'93*, pages 136–146. IEEE Computer Society Press, 1993.

[9] Igor Walukiewicz. Completeness of Kozen's axiomatization of the propositional μ-calculus. In *Proceedings 10th Annual IEEE Symp. on Logic in Computer Science, LICS'95*, pages 14–24. IEEE Computer Society Press, 1995.