From Mathesis Universalis to fixed points and related set-theoretic concepts

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Abstract

This article is about fixed point axioms and related principles in Kripke-Platek environments. We begin with surveying some principles and results of [5] and turn to more recent developments afterwards.

1 Introduction

The Humboldt-Kolleg *Proof Theory as Mathesis Universalis* (Villa Vigoni, July 24-28, 2017) was dedicated to modern approaches stemming from or being related to a sort of hypothetical universal science modelled on mathematics as envisaged by Descartes and Leibniz. A point of view that may be discussed is that set theory or – if one wants to be more modest – predicate logic provide a universal framework for mathematics and computer science that shares some of such universal characteristics.

However, recent developments made it very clear that there are interesting alternatives to systems like Zermelo-Fraenkel set theory and related approaches. Let us give you some examples:

- There is Martin-Löf type theory that provides a powerful framework for constructive mathematics and there are many other type-theoretic approaches.
- In recent years proof assistants and proof development systems such as Coq, Agda, Nuprl (to mention only a few) have started to play an increasingly important role for (formalized) mathematics.
- And there is the operational approach to mathematics and set theory propagated by Feferman in, for example, his survey article [3]. The three main representatives of this approach are explicit mathematics, operational set theory, and unfolding of theories. However, for extending the present systems of explicit mathematics and operational set theory, new forms of operational reflection are required, and these new reflections require new inductive model constructions of operational theories.

One of our motivations for pursuing the research that we will survey below was the desire to classify and characterize the strengths of these new and more powerful systems of explicit mathematics and operational set theory in terms of subsystems of set theory. It turned out that fixed point constructions play a major role in this enterprise.

Let us point out that we work in the context of classical logic and the models we have in mind are models of explicit mathematics and of operational set theory, both based on classical logic. It is an interesting question to find out whether the results of this paper can also find use in studying the intuitionistic systems of explicit mathematics and operational set theory. We also want to remark that this paper does not address the relationship between our fixed points and the inductive definitions and fixed point constructions in the recent type-theoretic approaches to constructive mathematics.

In addition – and that is the second main motivation for our approach – fixed point constructions and inductive definitions belong to the most general and universal principles in mathematics and computer science. Hence they may also be considered from the point of view of Mathesis Universalis.

If $\mathfrak{A}[R^+, x]$ is an R-positive arithmetic formula, then the theories ID_1 and KP provide simple environments for introducing and studying the least fixed point of the operator $\Gamma_{\mathfrak{A}}$ that maps a set S of natural numbers to the set

$$\Gamma_{\mathfrak{A}}(S) \ := \ \{n \in \omega : \mathfrak{A}[S, n]\}.$$

See, for example Buchholz, Feferman, Pohlers, and Sieg [2] and Jäger [4]. But what happens if we go up in the logical complexity of the operator forms and allow them to be Δ_1 definable? We may even replace positivity by a monotonicity condition.

Let a monotone Δ_1 operator be an operator that is defined by a pair consisting of a Σ_1 formula $C[x, u]$ and a Π_1 formula $D[x, u]$ such that

(i) $\forall x (\forall u \in a)(C[x, u] \leftrightarrow D[x, u])$,

(ii) $\forall x, y (\forall u \in a)(C[x, u] \wedge x \subseteq y \ \rightarrow \ C[y, u])$.

The we define

$$\Gamma_{(C,D)}(x) \ := \ \{u \in a : C[x, u]\}.$$

for all $x \subseteq a$. The fixed point axioms claim that or any pair (C, D) that satisfies (i) and (ii) above and any set a we we have

$$\text{(fixed point)} \qquad (\exists b \subseteq a)(\forall u \in a)(u \in b \ \leftrightarrow \ C[b, u])$$

and analogously for the least fixed point.

It is more convenient to work with what we call set-bounded Σ_1 operators. Take a monotone Δ_1 operator given by the pair (C, D) and define $A[x, y]$ to be the formula

$$y \ = \ \{u \in a : C[x, u]\}.$$

Then $A[x, y]$ is equivalent to a Σ_1 formula and we have

(iii) $\forall x \exists! y A[x, y] \ \wedge \ \forall x, y (A[x, y] \ \rightarrow \ y \subseteq a)$,

(iv) $\forall x_0, x_1, y_0, y_1 (A[x_0, y_0] \ \wedge \ A[x_1, y_1] \wedge x_0 \subseteq x_1 \ \rightarrow \ y_0 \subseteq y_1)$,

Clearly, the (least) fixed points of A are the (least) fixed points of the operator defined by (C, D).

On the other hand, assume that $A[x, y]$ is a Σ_1 formula such that (iii) and (iv) hold. Now we define

$$C[x, u] \ := \ \exists y (A[x, y] \ \wedge \ u \in y),$$

$$D[x, u] \ := \ \forall y (A[x, y] \ \rightarrow \ u \in y).$$

Then (C, D) defines a monotone Δ_1 operator with the same (least) fixed points as A.

In this article we do not discuss the new models of explicit mathematics and operational set theory. Instead, we will concentrate on fixed point axioms and related principles over Kripke-Platek set theory. We believe that these concepts are interesting in their own and worth to be analyzed further.

This survey article begins with a sketch of the standard constructions of least fixed points of monotone operators a la Knaster and Tarski. Then we introduce a class version KP^c of Kripke-Platek set theory as the framework for our further discussions. Afterwards we turn to the axioms and principles that form the core of this survey and discuss their mutual relationships.

2 Traditional approaches to least fixed points

A lattice $(\mathbb{L}, \leq_{\mathbb{L}})$ is a collection equipped with a partial order in which every two elements have a unique supremum and a unique infimum. A lattice is complete if all its subcollections have both a supremum and an infimum. Assume that we are given a complete lattice $\mathbb{L}$ and a monotone function F on it. According to Knaster and Tarski [7, 13], there exists the least fixed point, where an element x of $\mathbb{L}$ is a fixed point of F if $F(x) =_{\mathbb{L}} x$. In the following we sketch two standard arguments to show the existence of the least fixed point for monotone functions.

The first argument exploits the fact that the least fixed point is the infimum of the collection of all elements of $\mathbb{L}$ which are closed under F, where we say that an element x is closed under F if $F(x) \leq_{\mathbb{L}} x$. Indeed, define

$$z := \bigwedge_{\mathbb{L}} \{x \in \mathbb{L} : F(x) \leq_{\mathbb{L}} x\}$$

and show as usual that z is closed under F. By definition of z we thus have $z \leq_{\mathbb{L}} F(z)$, which yields that z is the least fixed point of F.

The second argument provides a construction of the least fixed point by using the property that the least fixed point is the supremum of a special chain of elements of $\mathbb{L}$. Let $0_{\mathbb{L}}$ be the infimum of $\mathbb{L}$. Then define a chain c of elements of $\mathbb{L}$ as follows:

$$0_{\mathbb{L}} \leq_{\mathbb{L}} F(0_{\mathbb{L}}) \leq_{\mathbb{L}} F(F(0_{\mathbb{L}})) \leq_{\mathbb{L}} \dots$$

Note that it is a $\leq_{\mathbb{L}}$-increasing chain by both monotonicity of F and $0_{\mathbb{L}} \leq_{\mathbb{L}} F(0_{\mathbb{L}})$, as $0_{\mathbb{L}}$ is the infimum of $\mathbb{L}$. Then define z to be the supremum of this chain: i.e., $z = \bigvee_{\mathbb{L}} c$. A straightforward cardinality argument shows that c cannot provide new arguments forever. And so that z belongs to c and $F(z) =_{\mathbb{L}} z$. Leastness follows from the fact that we can prove by induction that every element of the chain is $\mathbb{L}$-smaller than every closed set.

In this paper we are interested in the particular case that the given lattice is the collection[1] of the subsets of some set a, denoted as $(\mathcal{P}(a), \subseteq)$. Observe that if we work in strong set theories as ZFC, then $\mathcal{P}(a)$ is a set and $(\mathcal{P}(a), \subseteq)$ is a complete lattice. This guarantees that the first argument can be performed. A cardinality argument on a provides also the success of the second approach over ZFC. If we work over weaker set theories which do not comprise the axiom of powerset, the situation is completely different.

In this article our general framework is a class version of Kripke-Platek set theory KP with infinity. Kripke-Platek is a truely interesting subsystem of Zermelo-Fraenkel set theory that plays an important role in the interaction between set theory, recursion theory, model theory, and proof theory. For more on Kripke-Platek set theory consult, e.g., Barwise [1].

[1] It is on purpose that we do not assume it to be a set.

3 A class version of Kripke-Platek set theory

KP^c is a conservative class extension of KP in which, for example, $\mathcal{P}(a)$ is a class for any set a. Our main reason for moving to this extension is that it provides a natural framework for speaking about (monotone) operators from $\mathcal{P}(a)$ to itself. Here we provide a brief description of KP^c, for more details see [5].

Let $\mathcal{L}$ be the standard language of set theory containing $\in$ as the only non-logical symbol besides $=$ and countably many set variables $a, b, c, , \ldots$ (possibly with subscripts). The language $\mathcal{L}^c$ is the extension of $\mathcal{L}$ by countably many class variables $F, G, U, V \ldots$ (possibly with subscripts). The atomic formulas of $\mathcal{L}^c$ comprise the atomic formulas of $\mathcal{L}$ and all expressions of the form $(a \in U)$. The formulas of $\mathcal{L}^c$ are built up from these atomic formulas by use of the propositional connectives and quantification over sets and classes. Equality of classes is defined by

$$(U = V) \ := \ \forall x(x \in U \leftrightarrow x \in V)$$

and not treated as an atomic formula.

We say that an $\mathcal{L}^c$ formula is elementary iff it contains no class quantifiers. The Δ_0^c, Σ^c, Π^c, Σ_n^c and Π_n^c formulas of $\mathcal{L}^c$ are defined according to the usual Levy hierarchy but now permitting subformulas of the form $(a \in U)$.

The theory KP^c is formulated in the language $\mathcal{L}^c$ and based on classical logic for sets and classes. Its non-logical axioms are:

- Extensionality, pair, union and infinity for sets as defined in KP.
- Δ_0^c separation (Δ_0^c-Sep): for every Δ_0^c formula A in which x is not free and for any set a,
$$\exists x(x = \{y \in a : A[y]\}).$$
- Δ_0^c collection (Δ_0^c-Col): for every Δ_0^c formula A and any set a,
$$\forall x \in a \exists y A[x, y] \rightarrow \exists b \forall x \in a \exists y \in b A[x, y].$$
- Δ_1^c comprehension (Δ_1^c-CA): for every Σ_1^c formula A and every Π_1^c-formula B,
$$\forall x(A[x] \leftrightarrow B[x]) \rightarrow \exists X \forall x(x \in X \leftrightarrow A[x]).$$
- Elementary $\in$-induction (El-$\mathrm{I}_\in$): for every elementary formula A,
$$\forall x((\forall y \in x A[y]) \rightarrow A[x]) \rightarrow \forall x A[x].$$

It is straightforward to check that core properties of KP can be carried over to KP^c; among others, the proof that every Σ^c-formula (resp. Π^c-formula) is equivalent to a Σ_1^c-formula (resp. Π_1^c-formula), Σ^c-reflection and Σ^c-recursion. Moreover KP^c is a conservative extension of KP. For the proof we refer to [5, Theorem 1].

Theorem 1. *Every $\mathcal{L}$ sentence provable in KP^c is already provable in KP.*

We follow the standard set theoretic terminology and the notation introduced in [5]. In particular, we use lower case Greek letters to range over ordinals and write On for the class of all ordinals. Also, $\mathrm{Fun}[f, a]$ is a short for saying that f is a function whose domain is the set a.

4 Operators and fixed point statements

Note that (Δ_1^c-CA) yields that the collection of all subsets of a given set a,

$$\mathcal{P}(a) := \{x : x \subseteq a\},$$

is a class. However, keep in mind that in general it is a class and not a set. It is routine work to show that $(\mathcal{P}(a), \subseteq)$ is a lattice, though it might not be complete in KP^c, since our theory lacks of Σ_1-separation.

Following the notation of [5], we call a class U an operator if all its elements are ordered pairs and it is right-unique (i.e., functional). Formally:

$$\mathrm{Op}[U] := \forall x \in U \exists y, z(x = \langle y, z\rangle) \wedge \forall y, z_0, z_1(\langle y, z_0\rangle \in U \wedge \langle y, z_1\rangle \in U \rightarrow z_0 = z_1).$$

In this paper we use F, G and H to denote operators. We define the domain of the operator F as the collection of all sets a for which there exists a set x such that $\langle a, x\rangle \in F$. If a belongs to the domain of the operator F, then $F(a)$ is the unique set x such that $\langle a, x\rangle \in F$.

If F is an operator, x and y are sets, U is a class and x belongs to the domain of F all the following abbreviations correspond to Δ_1^c formulae: $F(x) = y$, $F(x) \in y$, $y \in F(x)$, $F(x) \subseteq y$, $y \subseteq F(x)$, $F(x) = F(y)$, $F(x) \subseteq F(y)$, $F(x) \in U$. When introducing new operators later, we will freely make use of the following remark.

Remark 2. For every Σ_1^c formula A and class U such that $\forall x \in U \exists! y A[x, y]$, ($\Delta_1^c$-CA) tells us that there exists an operator F such that $\forall x, y(F(x) = y \leftrightarrow x \in U \wedge A[x, y])$. For details see [5].

Let us now consider total operators which are monotone and which map sets to subsets of a given set a:

$$\mathrm{Mon}[F, a] := \forall x(F(x) \subseteq a) \wedge \forall x, y(x \subseteq y \rightarrow F(x) \subseteq F(y)).$$

Then there are two obvious fixed point principles which come with such an operator. The first stating the existence of a fixed point, the second claiming the existence of a least fixed point:

(FP^c) $$\mathrm{Mon}[F, a] \rightarrow \exists x(F(x) = x),$$

(LFP^c) $$\mathrm{Mon}[F, a] \rightarrow \exists x(F(x) = x \wedge \forall y(F(y) = y \rightarrow x \subseteq y)).$$

5 Two principles to prove the existence of least fixed points

To perform the two standard arguments of the Knaster-Tarski theorem it is enough to require that given a set a, the (class)lattice $(\mathcal{P}(a), \subseteq)$ is class-complete.[2] In fact, both the collection of all sets which are closed under F and the collection of the elements of the increasing chain provided by the second argument are subclasses of $\mathcal{P}(a)$, as they are Δ_1^c-definable. To conclude the second proof we just need Σ-Reflection, which is valid in KP^c. Class-completeness is a direct consequence of the well-known separation principle (Σ_1^c-Sep):

(Σ_1^c-Sep) $$\exists y \forall x(x \in y \leftrightarrow x \in a \wedge A[x])$$

[2] We call $(\mathcal{P}(a), \subseteq)$ class-complete if for every subclass $X \subseteq \mathcal{P}(a)$ the union of X is a subset of a.

for arbitrary Σ_1^c formulae $A[u]$. In fact, given a subclass $X \subseteq \mathcal{P}(a)$, ($\Sigma_1^c$-$\mathsf{Sep}$) yields the existence of the set

$$\bigcup X = \{x \in a : \exists y(y \in X \wedge x \in y)\} .$$

So a first very natural question which arises is: does (LFP^c) imply (Σ_1^c-Sep) over KP^c? The results in [5] provide a negative answer to this question. Hence we may wonder whether there are theories weaker (in term of consistency strength or proof-theoretically) than KP^c + (Σ_1^c-Sep) in which the two standard constructions of the least fixed point go through. To this aim we consider two principles which generalize the two constructions.

5.1 An ingredient to perform the first argument.

The first argument to prove the Knaster-Tarski theorem characterizes the least fixed point as the infimum of the collection of all closed sets. Since our lattice is the powerclass of some set a, this yields that the least fixed point is the intersections of all subsets x of a which are closed under F, i.e., $F(x) \subseteq x$. Therefore it requires that the following collection is a set:

$$\{y \in a : \forall x \subseteq a(F(x) \subseteq x \to y \in x)\} .$$

This is a special case of the new principle of *subset-bounded separation* that was introduced in [5]:

(SBS^c) $$\exists z \forall x(x \in z \leftrightarrow x \in a \wedge (\forall y \subseteq b)A[x, y])$$

for arbitrary Δ_0^c formulae $A[u, v]$. Do not underestimate the strength of (SBS^c), by (Δ_1^c-CA) it is straightforward to show that (SBS^c) yields:

$$\forall x, y(A[x, y] \leftrightarrow B[x, y]) \to \exists z \forall x(x \in z \leftrightarrow x \in a \wedge (\forall y \subseteq b)A[x, y])$$

for arbitrary Σ_1^c formulae $A[u, v]$ and Π_1^c formulae $B[u, v]$. Therefore, given an operator F, our principle (SBS^c) guarantees the existence of the intersection of all sets closed under F. The proof that this is the least fixed point works over KP^c as in the standard argument. Furthermore, it is straightforward to show that (SBS^c) follows from (Σ_1^c-Sep).

There is also an interesting largeness property that follows from (SBS^c), namely *bounded proper injection*.[3] It guarantees that the universe is so large that it cannot be injected into a set,

(BPI^c) $$\forall x(F(x) \in a) \to \exists x, y(x \neq y \wedge F(x) = F(y)).$$

Indeed, given F an injective operator from the universe to any set a, (SBS^c) allows us to define the range of F restricted to $\mathcal{P}(a)$. The fact that this is a set is enough to show that $\mathcal{P}(a)$ is a set. By applying a usual Cantor-style argument, the existence of an injection from the $\mathcal{P}(a)$ to a provides a contradiction. See [5] for details.

There is even an important strengthening of (BPI^c) which claims that the class of all ordinals cannot be injected into any set,

($\mathsf{BPI}^c_{\mathrm{On}}$) $$\forall \alpha(F(\alpha) \in a) \to \exists \alpha, \beta(\alpha \neq \beta \wedge F(\alpha) = F(\beta)).$$

We will come back to the relationship between (BPI^c) and ($\mathsf{BPI}^c_{\mathrm{On}}$) later.

[3] Maybe this is a misname since this principle claims that such injections do not exist. But we decided to follow the terminology of [5].

5.2 An ingredient to perform the second argument.

In the second approach the least fixed point is constructed as the supremum of the $\leq_{\mathbb{L}}$-increasing chain $0_{\mathbb{L}} \leq_{\mathbb{L}} F(0_{\mathbb{L}}) \leq_{\mathbb{L}} F(F(0_{\mathbb{L}})) \leq_{\mathbb{L}} \dots$. To guarantee the existence of such fixed points of monotone operators over lattices $(\mathcal{P}(a), \subseteq)$ in KP^c we introduced a *maximal iterations principle*:

$$(\mathsf{MI}^c) \qquad \forall x(F(x) \subseteq a) \to \exists \alpha, f(\mathrm{Hier}[F, f, \alpha] \wedge f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi)),$$

where $\mathrm{Hier}[F, f, \alpha]$ is the formula

$$\mathrm{Fun}[f, \alpha + 1] \wedge \forall \beta \leq \alpha(f(\beta) = F(\bigcup_{\xi<\beta} f(\xi))).$$

Note that this principle makes sense and is formulated for arbitrary operators mapping into a class $\mathcal{P}(a)$. If F is monotone, then it is trivial to see that

$$\mathrm{Hier}[F, f, \alpha] \wedge f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi)$$

yields that $f(\alpha)$ is the least fixed point of F.

In [5, Theorem 4 and Theorem 6] it is shown that (MI^c) is provable in $\mathsf{KP}^c + (\Sigma^c_1\text{-}\mathsf{Sep})$. Another well-know consequence of $(\Sigma^c_1\text{-}\mathsf{Sep})$ is axiom (Beta). Informally, axiom (Beta) states that every well-founded relation has a collapsing function. If we write $\mathrm{WF}[r, a]$ for

$$r \subseteq a \times a \wedge \forall b \subseteq a(b \neq \emptyset \to \exists x \in b \forall y \in b(\langle y, x\rangle \notin r)),$$

then axiom (Beta) is the following implication:

$$(\mathsf{Beta}) \qquad \mathrm{WF}[r, a] \to \exists f(\mathrm{Fun}[f, a] \wedge \forall x \in a(f(x) = \{f(y) : \langle y, x\rangle \in r\})).$$

It is shown in [5, Theorem 6] that axiom (Beta) is a consequence of (MI^c) by applying (MI^c) to the operator defined as

$$F(x) = \{y \in a : \forall z \in a(\langle z, y\rangle \in r \to z \in x)\},$$

where r is the given well-founded relation on a, and then following the same argument as in Barwise [1, Chapter I.9].

Theorem 3. $\mathsf{KP}^c + (\mathsf{MI}^c)$ *proves axiom* (Beta).

From Theorem 5 of [5] we know that (BPI^c) holds in $\mathsf{KP}^c + (\mathsf{MI}^c)$. But because of Theorem 3 we can actually do better.

Theorem 4. $\mathsf{KP}^c + (\mathsf{MI}^c)$ *proves* $(\mathsf{BPI}^c_{\mathrm{On}})$.

Proof. Assume by contradiction that there exists an injective operator F from the ordinals to some set a. For any α we introduce the auxiliary sets

$$F_<[\alpha] := \{\langle F(\eta), F(\xi)\rangle : \eta < \xi \leq \alpha\}.$$

It follows immediately that $F_<[\alpha] = \bigcup_{\xi<\alpha} F_<[\xi + 1]$ for all ordinals α.

Given a binary relation r on a we say that r is a well-order - in symbols $\mathrm{WO}[r,a]$ - if r is an order[4] on a which is well-founded. We also write $\mathrm{Height}[r,\alpha]$ for the formula

$$\exists f(\mathrm{Fun}[f,a] \wedge \forall x \in a(f(x) = \{f(y) : \langle y,x\rangle \in r\}) \wedge \alpha = \bigcup_{x\in a} f(x)).$$

Please observe that $\mathrm{Height}[F_<[\alpha],\alpha]$. Axiom (Beta), which holds under (MI^c), guarantees that

$$\forall r \subseteq a\times a(\mathrm{WO}[r,a] \to \exists\alpha(\mathrm{Height}[r,\alpha]).$$

Indeed, since every well-order is transitive, the range of the collapsing function provided by axiom (Beta) is some ordinal. We define a surjective operator G from $\mathcal{P}(a\times a)$ to On as follows:

$$G(x) = \begin{cases} 0 & \text{if } \neg\, \mathrm{WO}[x,a], \\ 0 & \text{if } \exists\alpha(\mathrm{Height}[x,\alpha] \wedge x \neq F_<[\alpha]), \\ \alpha & \text{if } x = F_<[\alpha]. \end{cases}$$

Depending on G then define an operator H from $\mathcal{P}(a\times a)$ to $\mathcal{P}(a\times a)$ by:

$$H(x) = \begin{cases} \emptyset & \text{if } G(x) = 0 \wedge x \neq \emptyset, \\ F_<[G(x)+1] & \text{if } G(x) \neq 0 \vee x = \emptyset. \end{cases}$$

By definition we have $H(F_<[\alpha]) = F_<[\alpha+1]$ for all α and $H(x) = \emptyset$ for all x that are not of the form $F_<[\alpha]$ for some α. Furthermore, $\emptyset = F_<[0]$, hence $H(\emptyset) = F_<[1]$. Given any f and α such that $\mathrm{Hier}[H,f,\alpha]$, it is easy to see that $f(\beta) = F_<[\beta+1]$ for all $\beta \leq \alpha$. Since F is injective there is no α such that

$$F_<[\alpha+1] = f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi) = \bigcup_{\xi<\alpha} F_<[\xi+1] = F_<[\alpha].$$

This contradicts our principle (MI^c). □

Note that we really need to use axiom (Beta) to define G and H. Since the direct definition (i.e., $H(x) = F_<[\alpha+1]$ if exists α such that $x = F_<[\alpha]$ and $\emptyset$ otherwise), would not define an operator, as this formula is not Σ_1^c.

If we have the axiom of choice for sets (AC), then we can also prove that also the opposite implication holds.

Theorem 5. KP^c + (AC) + $(\mathsf{BPI}^c_{\mathrm{On}})$ *proves* (MI^c).

Proof. Assume by contradiction that there exists an operator F such that $\forall x(F(x) \subseteq a)$ and

$$\forall\alpha, f(\mathrm{Hier}[F,f,\alpha] \to f(\alpha) \nsubseteq \bigcup_{\xi<\alpha} f(\xi)).$$

Fix a well-order $<_a$ of a provided by (AC) and define the operator G as

$$G(\alpha) = x \leftrightarrow \exists f(\mathrm{Hier}[F,f,\alpha] \to (x \in f(\alpha) \setminus \bigcup_{\xi<\alpha} f(\xi) \wedge \forall y <_a x(y \notin f(\alpha) \setminus \bigcup_{\xi<\alpha} f(\xi))))$$

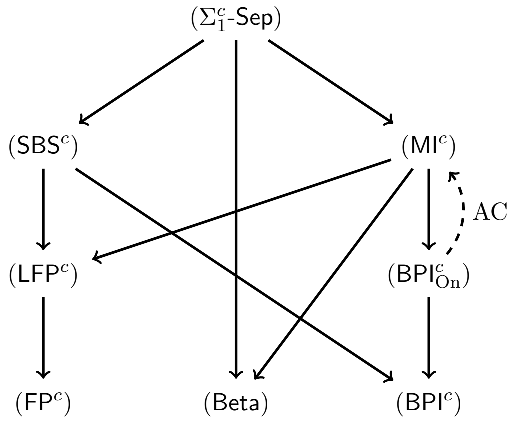
Then $G(\alpha)$ is an injection from the ordinals to a. Contradiction. □

The axiom of choice here is used for the set a. If we restrict our principles to ω then the use of choice can be clearly avoided and the two principles are equivalent.

[4] A binary relation r on a is an order if it is irreflexive, antisymmetric, and transitive

6 Under strong hypotheses

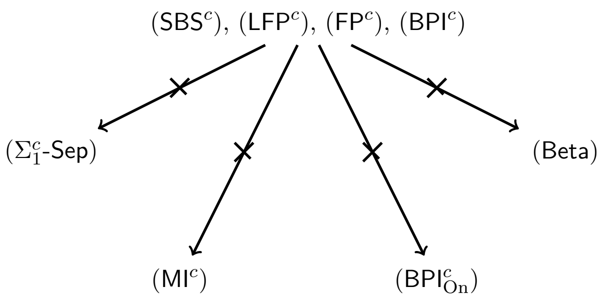
The present situation over KP^c is summarized in the following diagram:



From results of Mathias [9] and Rathjen [11] we conclude that there exists a model of KP + (Pow) + (AC) in which axiom (Beta) fails, where

$$\text{(Pow)} \qquad \forall a \exists p \forall x (x \subseteq a \leftrightarrow x \in p).$$

It follows that every statement S which can be proved in KP^c + (Pow) + (AC) does not imply (Beta).[5] Note that (Pow) implies (SBS^c) over KP^c. Moreover we know that both (MI^c) and (BPI$^c_{\mathrm{On}}$) imply axiom Beta over KP^c + (AC). Therefore we have the the following non-implication results over KP^c + (AC).



Hence a natural question arises: Is it the case that (SBS^c), (LFP^c), (FP^c) or (BPI^c) are provable over KP^c? The results in this section provide a negative answer to this question.

6.1 Under the axiom of constructibility $(V{=}L)$

The axiom of constructibility states that every set belongs to some level L_α of Gödel's hierarchy of constructible sets:

$$(V{=}L) \qquad \forall x \exists \alpha (x \in L_\alpha).$$

In addition, we write $(a <_L b)$ to express that a is smaller than b according to the well-order $<_L$ of the constructible universe. It is well known that $(a \in L_\alpha)$ and $(a <_L b)$ are Δ over KP. For more on the constructible universe see, e.g., Barwise [1] or Kunen [8].

In KP^c + $(V{=}L)$ it is possible to show that there exists an operator that maps all sets one-to-one into the ordinals. Making use of this injection it is then easy to prove that in KP^c + $(V{=}L)$ every instance of (BPI$^c_{\mathrm{On}}$) follows from (BPI^c).

[5] This means, of course, that S also cannot imply (Σ^c_1-Sep) or (MI^c).

Theorem 6. KP^c + $(V{=}L)$ + (BPI^c) *proves* $(\mathsf{BPI}^c_{\mathrm{On}})$.

On the other hand, $(\mathsf{BPI}^c_{\mathrm{On}})$ together with $(V{=}L)$ proves every instance of $(\Sigma^c_1\text{-}\mathsf{Sep})$. The idea of the proof is simple: Suppose for contradiction that there exist a set a and a Δ^c_0 formula $A[u,v]$ such that the collection

$$\mathcal{R} := \{x \in a : \exists y A[x,y]\}$$

is not a set. For every ordinal α we set $G(\alpha) := \{x \in a : \exists y \in L_\alpha A[x,y]\}$ and observe that $\mathcal{R} = \bigcup_\alpha G(\alpha)$. Then we use induction on the ordinals to define an operator F from On to $\mathcal{R}$: If F has been defined for all ordinals $\xi < \alpha$, then $\{F(\xi) : \xi < \alpha\}$ is a set. However, since $\mathcal{R}$ is not a set there exists a least β such that

$$\{F(\xi) \in a : \xi < \alpha\} \subsetneq G(\beta).$$

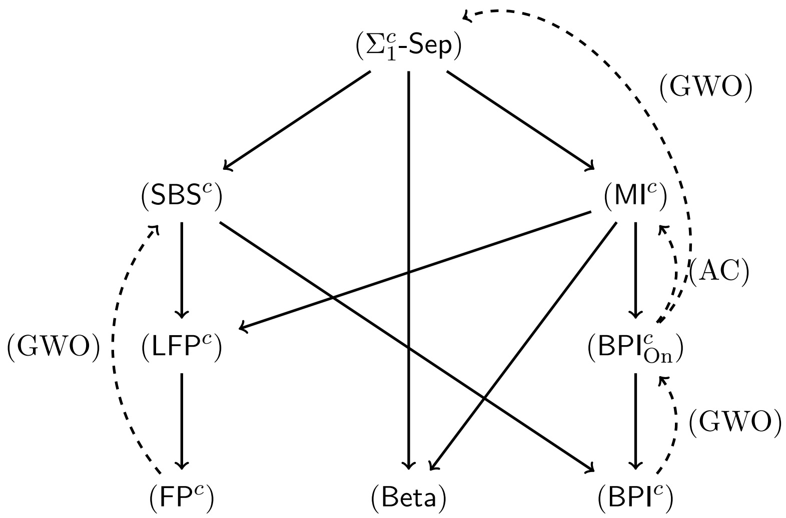
and we define $F(\alpha)$ to be the $<_L$-least element of $G(\beta) \setminus \{F(\xi) : \xi < \alpha\}$. This F is a one-to-one operator from the ordinals to a, thus violating $(\mathsf{BPI}^c_{\mathrm{On}})$.

Theorem 7. KP^c + $(V{=}L)$ + $(\mathsf{BPI}^c_{\mathrm{On}})$ *proves* $(\Sigma^c_1\text{-}\mathsf{Sep})$.

Detailed proofs of Theorem 6 and Theorem 7 are given in [5]. There you find also the proof of the following theorem. This proof is very technical, but checking it carefully reveals that we do not really need $(V{=}L)$ but only the existence of a suitable global well-order of the universe.

Theorem 8. KP^c + $(V{=}L)$ + (FP^c) *proves* (SBS^c).

This result closes the circle, and we can easily check that all our principles are equivalent over KP^c + $(V{=}L)$.[6] Since $L_{\omega_1^{CK}}$ is a model of KP + $(V{=}L)$ in which $(\Sigma^c_1\text{-}\mathsf{Sep})$ does not hold, we conclude that all our principles are not provable in KP^c + $(V{=}L)$.



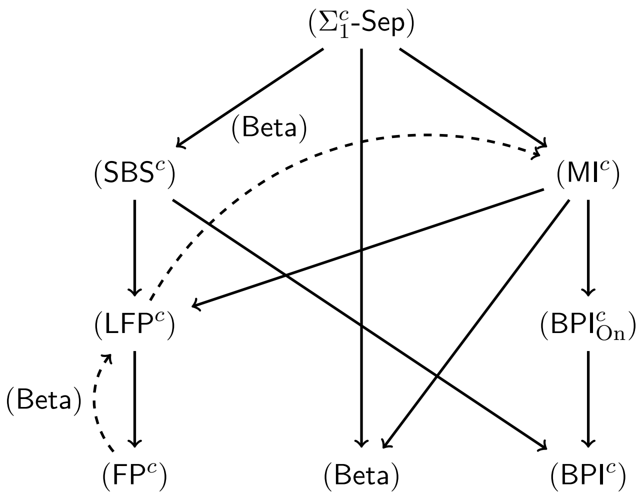
[6]Keep in mind that (AC) is a consequence of $(V{=}L)$

6.2 Under axiom (Beta)

Another property which has a significant impact on the relationship between our principles is axiom (Beta). We already noticed that in KP^c the principle (MI^c) does follow from (FP^c). However, in the presence of axiom (Beta), we can modify the Stage Comparison Theorem of Moschovakis [10] to prove that if F is a monotone operator to $\mathcal{P}(a)$, then there exists an operator F' such that, if F' has a fixed point, then F has a least fixed point [6]. [7] As a consequence we show that the least fixed point under axiom (Beta) is Δ^0_1 definable, and this can be used to prove (MI^c) from (LFP^c). Since the proofs are quite technical we refer to [6] for details.

Theorem 9.

- KP^c + (Beta) + (FP^c) *proves* (LFP^c)
- KP^c + (Beta) + (LFP^c) *proves* (MI^c)



As indicated in the diagram above, these results do not allow us to prove that all our principles are equivalent, as we have no implication between for instance (MI^c) and (SBS^c) and from (BPI^c) to any other principle.

7 Consistency strength

To conclude our survey we make a few remarks about the consistency strength of (some of) the principles we have been discussing. To this end we introduce the subform of (Σ^c_1-Sep) in which the unbounded existential quantifier ranges over the ordinals only: For all Δ^c_0 formulas $A[x, \xi]$,

(Σ^c_1(On)-Sep) $$\exists y \forall x (x \in y \leftrightarrow x \in a \land \exists \xi A[x, \xi])$$

A first and straightforward observation states that in KP^c + $(V{=}L)$ this restricted form of Σ^c_1 separation implies (Σ^c_1-Sep).

Proposition 10. KP^c + $(V{=}L)$ + (Σ^c_1(On)-Sep) *proves* (Σ^c_1-Sep).

[7] A similar argument but in a completely different context has been used by Sato, cf. [12].

Proof. Let $A[x,y]$ be an arbitrary Δ_0^c formula. Then, working in $\mathsf{KP}^c + (V{=}L)$, we clearly have that

$$\exists y A[x,y] \leftrightarrow \exists \xi \exists y \in L_\xi A[x,y].$$

Hence $\{x \in a : \exists y A[x,y]\}$ is a set because of $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$. □

It is also easy to see that L is an inner model of $\mathsf{KP}^c + (\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$. In view of [1, Theorem 5.5] and Theorem 1 of this article we only have to establish the following observation.

Proposition 11. $\mathsf{KP}^c + (\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$ *proves the L-relativization A^L of every instance A of* $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$.

Proof. Working in $\mathsf{KP}^c + (\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$, pick a set a and a Δ_0^c formula $A[x,\xi]$. By $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$ there exists the set

$$b = \{x \in a : \exists \xi A[x,\xi]\}.$$

Now we apply Δ_0^c collection and conclude that there exists an ordinal α such that

$$b = \{x \in a : \exists \xi < \alpha A[x,\xi]\}.$$

Hence b is an element of L and $b = \{x \in a : \exists \xi \in L A[x,\xi]\}$. □

Following the pattern of the proof of Theorem 7, we can easily convince ourselves that $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$ follows from $(\mathsf{BPI}^c_{\mathrm{On}})$, provided that a little bit of choice is available.

Proposition 12. $\mathsf{KP}^c + (\mathsf{AC}) + (\mathsf{BPI}^c_{\mathrm{On}})$ *proves* $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$.

Proof. Now we work in $\mathsf{KP}^c + (\mathrm{AC}) + (\mathsf{BPI}^c_{\mathrm{On}})$, pick a set a and *a* Δ_0^c formula $A[x,\xi]$. By choice on a we first fix a well-order $<_a$ of a. Now we proceed following the idea of the proof of Theorem 7 where $G(\alpha)$ is defined to be the set $\{x \in a : \exists \xi < \alpha A[x,\xi]\}$ and $F(\alpha)$ is defined as the $<_a$-least element of $G(\beta)$ that does not belong to $\{F(\xi) : \xi < \alpha\}$. □

Consider the following restrictions of the principles $(\Sigma_1^c\text{-}\mathsf{Sep})$, $(\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$ and $(\mathsf{BPI}^c_{\mathrm{On}})$ to ω:

$(\omega\text{-}\Sigma_1^c\text{-}\mathsf{Sep})$	$\exists y \forall x (x \in y \leftrightarrow x \in \omega \wedge \exists y A[x,y])$,
$(\omega\text{-}\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$	$\exists y \forall x (x \in y \leftrightarrow x \in \omega \wedge \exists \xi A[x,\xi])$,
$(\omega\text{-}\mathsf{BPI}^c_{\mathrm{On}})$	$\forall \alpha (F(\alpha) \in \omega) \rightarrow \exists \alpha, \beta (\alpha \neq \beta \wedge F(\alpha) = F(\beta))$,

where A ranges again over all Δ_0^c formulas. Then it is clear from the previous that Proposition 10 and Proposition 11 hold for $(\omega\text{-}\Sigma_1^c\text{-}\mathsf{Sep})$ and $(\omega\text{-}\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$, respectively, and that $\mathsf{KP}^c + (\omega\text{-}\mathsf{BPI}^c_{\mathrm{On}})$ proves all instances of $(\omega\text{-}\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep})$.

It is also well-known that $\mathsf{KP}^c + (\Sigma_1^c\text{-}\mathsf{Sep})$ and its subsystem with $(\Sigma_1^c\text{-}\mathsf{Sep})$ restricted to ω in the sense above are equiconsistent; both are proof-theoretically equivalent to the theory $\Pi^1_2\text{-CA} + (\mathrm{BI})$ of second order arithmetic.

Summing up everything and recalling that (MI^c) implies $(\mathsf{BPI}^c_{\mathrm{On}})$ over KP^c according to Theorem 4, we can deduce that several of our principles have the same consistency strength.

Theorem 13. *Added to KP^c, the following principles lead to theories of the same consistency strength:*

$$(\mathsf{BPI}^c_{\mathrm{On}}),\ (\mathsf{MI}^c),\ (\Sigma_1^c(\mathrm{On})\text{-}\mathsf{Sep}),\ (\Sigma_1^c\text{-}\mathsf{Sep}).$$

8 Future Work

In this paper we discussed the behaviour of the fixed point statements introduced in [5]. In our analysis there are still open questions to be solved. The most puzzling principle from the author's point of view is (SBS^c). Under $(V{=}L)$ the principle (SBS^c) implies $(\Sigma^c_1\text{-}\mathsf{Sep})$ and under axiom (Beta) it has the same consistency strength of $(\Sigma^c_1\text{-}\mathsf{Sep})$. In both the cases (SBS^c) implies Π^1_2-CA. Hence a first question is: Does (SBS^c) imply Π^1_2-CA over KP^c?

As mentioned in [5, Remark 7 and Remark 21], notions similar to our subset-bounded formulae have already considered by Mathias [9] and Rathjen [11]. They defined the class of $\Delta^{\mathcal{P}}_0$ formulae as the collection of formulae whose quantifiers all range over sets or powerclasses of some set. $(\Delta^{\mathcal{P}}_0\text{-Sep})$ is proof theoretically stronger than (SBS^c) since it implies full arithmetical comprehension. On the other hand, it is still not clear whether (SBS^c) can be proved from $(\Delta^{\mathcal{P}}_0\text{-Sep})$.

The authors are working to address these and further problems from different perspectives. In particular, we want to make clear what the relationship is between our principles over pure KP^c, without the axiom of constructibility $(V{=}L)$ and without axiom (Beta).

Acknowledgment. This publication was supported by a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

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