Having a look again at some theories of proof-theoretic strengths around Γ_0

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Abstract

In the first part of this article we introduce generalizations of ATR_0 and FP_0: arithmetical properties are replaced by Δ^1_1 properties, and the positivity requirement of the definition clauses of FP_0 is liberalized to monotonicity. Afterwards we turn to systems of admissible sets above the natural numbers as urelements. In this context we look, in particular, at principles of Σ and Π reduction.

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1 Introduction

One of the big challenges in proof theory of the sixties of the previous century was the determination of the precise ordinal bound for predicative mathematics. This problem was solved independently by Feferman [5] and Schütte [15, 16]. Feferman and Schütte showed that the least non-predicatively provable ordinal is the ordinal Γ_0 that can be characterized in terms of the Veblen functions $(\varphi_\alpha : \alpha \in On)$ as the least α such that $\varphi_\alpha(0) = \alpha$. For establishing this ordinal bound, Feferman and Schütte made use of an intricate interplay between systems of ramified analysis, well-ordering proofs, cut elimination and a bootstrapping process. See the original articles mentioned above or Schütte's text book [17] for further details.

In parallel to the formal characterization of predicative mathematics an issue to be dealt with was the question of how much mathematics could be developed predicatively. This was pursued theoretically by identifying equivalent formal systems and alternative approaches to predicativity and by means of detailed case studies.

Fresh blood was brought into the whole discussion in connection with the Friedman-Simpson program of reverse mathematics. In this program five subsystems of second order arithmetic play a prominent role, and the fourth of these Big Five is the theory ATR_0 of arithmetical transfinite recursion. The proof-theoretic ordinal of ATR_0 is the Feferman-Schütte ordinal Γ_0, and ATR_0

is equivalent, for example, to the fixed point theory FP_0 and Σ^1_1 separation Σ^1_1-Sep_0; cf. Avigad [1] an Simpson [18].

The first part of this article introduces generalizations of ATR_0 and FP_0: arithmetical properties are replaced by Δ^1_1 properties, and the positivity requirement of the definition clauses of FP_0 is liberalized to monotonicity. We show that the corresponding theories are equivalent to ATR_0, thus answering a question of Feferman. The proof of these equivalences goes via a system for weak transfinite dependent choice, which is interesting on its own.

Afterwards we turn to systems of admissible sets above the natural numbers as urelements. In this context we look, in particular, at principles of Σ and Π reduction. For establishing their upper proof-theoretic bounds we make us of an adaptation of Simpson's suitable trees. For dealing with the system BS^0 of basic set theory without $\in$-induction and complete induction restricted to sets, we need a preparatory step consisting of partial cut elimination and asymmetric interpretations.

2 Subsystems of second order arithmetic

As mentioned in the introduction, the subsystems of second order arithmetic that interest us most in this article are variants of the theory ATR_0 of arithmetical transfinite recursion and of the fixed point theory FP_0. We begin this section with describing the general context of these systems. Afterwards we introduce the principles (Δ^1_1-TR), ($\mathsf{M}\Delta^1_1$-FP), and (weak-Σ^1_1-TDC) and prove their equivalence over ACA_0.

2.1 The general context

Let $\mathcal{L}_1$ be a standard first order language with (i) a countably infinite supply of first order variables, (ii) a numeral $\overline{n}$ for every natural number n, (iii) function and relation symbols for all primitive recursive functions and relations. All theories considered in this section are formulated in the second order language $\mathcal{L}_2$ that extends $\mathcal{L}_1$ by adding countably many second order variables and the relation symbol $\in$ for elementhood between first and second order objects. In $\mathcal{L}_2$ the first order objects are, of course, supposed to range over natural numbers and the second order objects to range over sets of natural numbers. The number terms and formulas of $\mathcal{L}_2$ are built up as usual; in general, we simply write n for the numeral $\overline{n}$.

We use the following categories of letters (possibly with subscripts) as metavariables:

- $a, b, c, u, v, w, x, y, z$ for first order variables;
- $R, S, T, U, V, W, X, Y, Z$ for second order variables;
- r, s, t for number terms;
- A, B, C for formulas.

A formula without bound set variables is called *arithmetical*. The Σ^1_1 and Π^1_1 formulas are those of the form $\exists X A$ and $\forall X A$, respectively, where A is arithmetical. A formula is a Σ^1 formula iff its negation normal form does not contain a universal set quantifier, and a Π^1 formula iff its negation normal form does not contain an existential set quantifier.

Equality is only taken as basic symbol between numbers. The subset relation on sets and equality of sets are defined by

$$\begin{aligned} S \subseteq T &:= \forall x(x \in S \to x \in T), \\ S = T &:= S \subseteq T \wedge T \subseteq S. \end{aligned}$$

In the following, we will make use of the standard primitive recursive coding machinery in $\mathcal{L}_2$: (a, b) stands for the primitive recursively formed ordered pair of a, b. For the corresponding projection functions we write $(.)_0$ and $(.)_1$ such that $a = ((a)_0, (a)_1)$ iff a codes an ordered pair.

Sets of natural numbers can be considered as codes of binary relations via this pairing function. For every set R we write

$$\begin{aligned} a \preceq_R b &:= (a, b) \in R, \\ a \prec_R b &:= (a, b) \in R \wedge (b, a) \notin R. \end{aligned}$$

We say that a belongs to the *field* of R – in symbols $Fd[R, a]$ – iff there exists a b such that $a \preceq_R b$ or $b \preceq_R a$, and we write $LO[R]$ to express that R is a *linear ordering* of its field, i.e., R is a set of ordered pairs satisfying

(LO.1) $a \preceq_R b \wedge b \preceq_R c \to a \preceq_R c$,

(LO.2) $a \preceq_R b \wedge b \preceq_R a \to a = b$,

(LO.3) $a \preceq_R b \vee b \preceq_R a$

for all a, b, c from the field of R. We also make use of the following auxiliary notations:

$$\begin{aligned} b \in (S)_a &:= (a, b) \in S, \\ b \in (S)_{Ra} &:= b = ((b)_0, (b)_1) \wedge (b)_0 \prec_R a \wedge b \in S, \\ WO[R] &:= LO[R] \wedge \forall X(X \neq \varnothing \to (\exists x \in X)\forall y(y \prec_R x \to y \notin X)), \\ \mathcal{H}_A[R, S] &:= LO[R] \wedge \forall x(Fd[R, x] \to \forall y(y \in (S)_x \leftrightarrow A[(S)_{Rx}, y])). \end{aligned}$$

where $A[X, y]$ is an arithmetical formula with distinguished free variables X and y; it may contain other free variables than those displayed. The formula $\mathcal{H}_A[R, S]$ says that S describes the hierarchy obtained by iterating comprehension with respect to $A[X, y]$ along R.

If Th is a theory (i.e. a collection of formulas) then $\mathsf{Th} \vdash A$ means that A can be derived from Th in classical logic with equality. Given two theories

Th_1 and Th_2, we write $\mathsf{Th}_1 \subseteq \mathsf{Th}_2$ iff the theorems of Th_1 are also provable in Th_2. The theories Th_1 and Th_2 are called proof-theoretically equivalent – in symbols $\mathsf{Th}_1 \equiv \mathsf{Th}_2$ – iff they prove the same arithmetical sentences. The proof-theoretic ordinal of the theory Th is denoted by $|\mathsf{Th}|$ and clearly $\mathsf{Th}_1 \equiv \mathsf{Th}_2$ implies $|\mathsf{Th}_1| = |\mathsf{Th}_2|$.

Our basic system is the theory ACA_0 that comprises the defining axioms for all primitive recursive functions and relations, the axiom schema of *arithmetical comprehension*

$$\exists Y \forall x (x \in Y \leftrightarrow A[x])$$

for all arithmetical formulas $A[x]$, and the *induction axiom*

$$0 \in X \wedge \forall x (x \in X \rightarrow x+1 \in X) \rightarrow \forall x (x \in X).$$

In ACA_0 we thus can prove the schema of complete induction

$$(*) \qquad A[0] \wedge \forall x (A[x] \rightarrow A[x+1]) \rightarrow \forall x A[x]$$

for all arithmetical formulas $A[x]$. ACA is the extension of ACA_0 obtained by adding (*) for arbitrary formulas of $\mathcal{L}_2$.

Now we introduce a series of axiom schemas. Given such a schema (Sch) we shall write Sch_0 for the theory $\mathsf{ACA}_0 + (\mathsf{Sch})$ and Sch for $\mathsf{ACA} + (\mathsf{Sch})$. All formulas in the following schemas may contain other free variables than those displayed.

Δ^1_1 comprehension. For all Σ^1_1 formulas $A[x]$ and Π^1_1 formulas $B[x]$:

$$(\Delta^1_1\text{-}\mathsf{CA}) \qquad \forall x (A[x] \leftrightarrow B[x]) \rightarrow \exists Y \forall x (x \in Y \leftrightarrow A[x]).$$

Σ^1_1 axiom of choice. For all Σ^1_1 formulas $A[x, Y]$:

$$(\Sigma^1_1\text{-}\mathsf{AC}) \qquad \forall x \exists Y A[x, Y] \rightarrow \exists Y \forall x A[x, (Y)_x].$$

Fixed points of positive arithmetical clauses. For all X-positive arithmetical formulas $A[X, y]$:

$$(\mathsf{FP}) \qquad \exists X \forall y (y \in X \leftrightarrow A[X, y]).$$

Arithmetical transfinite recursion. For all arithmetical formulas $A[X, y]$:

$$(\mathsf{ATR}) \qquad WO[R] \rightarrow \exists X \mathcal{H}_A[R, X].$$

Π^1_1 and Σ^1_1 reduction. For all Σ^1_1 formulas $A[x]$ and Π^1_1 formulas $B[x]$:

$$(\Pi^1_1\text{-}\mathsf{Red}) \qquad \forall x (A[x] \rightarrow B[x]) \rightarrow \exists Y (\forall x (A[x] \rightarrow x \in Y) \wedge \forall x (x \in Y \rightarrow B[x])).$$

$$(\Sigma^1_1\text{-}\mathsf{Red}) \qquad \forall x (B[x] \rightarrow A[x]) \rightarrow \exists Y (\forall x (B[x] \rightarrow x \in Y) \wedge \forall x (x \in Y \rightarrow A[x])).$$

These forms (Π^1_1-Red) and (Σ^1_1-Red) of Π^1_1 reduction and Σ^1_1 reduction are simply reformulations of the respective schemas of Σ^1_1 separation and Π^1_1 separation in Simpson [18]. We prefer our terminology in order to avoid any confusion with the separation principles used in the context of set theory in the next section.

The following theorem recalls some well-known results, and we refer to Simpson [18] and Avigad [1] for all details.

Theorem 1.

1. (ATR), (FP), *and* (Π^1_1-Red) *are equivalent over* ACA_0.
2. ATR_0 *proves all instances of* (Δ^1_1-CA) *and* (Σ^1_1-AC).
3. Σ^1_1-AC_0 *proves all instances of* (Σ^1_1-Red).
4. *For any* Σ^1_1 *formula* $A[X]$, ACA_0 *proves*

$$\neg\forall X(A[X] \leftrightarrow WO[X]).$$

It will have some technical advantages later to work with the following extensions of (Π^1_1-Red) and (Σ^1_1-Red) to Π^1 and Σ^1 formulas.

Π^1 and Σ^1 reduction. For all Σ^1 formulas $A[x]$ and Π^1 formulas $B[x]$:

(Π^1-Red)
$$\forall x(A[x] \to B[x]) \to \exists Y(\forall x(A[x] \to x \in Y) \wedge \forall x(x \in Y \to B[x])).$$

(Σ^1-Red)
$$\forall x(B[x] \to A[x]) \to \exists Y(\forall x(B[x] \to x \in Y) \wedge \forall x(x \in Y \to A[x])).$$

However, as stated in the following corollary, Π^1 and Σ^1 reduction are equivalent to Π^1_1 and Σ^1_1 reduction in all interesting cases.

Corollary 2. *In* Σ^1_1-AC_0 *and therefore also in* ATR_0, *every* Σ^1 *formula is provably equivalent to a* Σ^1_1 *formula and every* Π^1 *formula to a* Π^1_1 *formula. In particular, we have the following equivalences:*

1. (Π^1-Red) *is equivalent to* (Π^1_1-Red) *over* ACA_0.
2. (Σ^1-Red) *is equivalent to* (Σ^1_1-Red) *over* Σ^1_1-AC_0.

We conclude this subsection with a further remark summarizing some obvious facts that will play a role later.

Remark 3. *Let* $A[X, y]$ *be an arithmetical formula with the displayed free variables and possibly further parameters and set*

$$\mathcal{H}^{\circ}_A[a, S] \ := \ (\forall x < a)\forall y(y \in (S)_x \leftrightarrow A[\bigcup_{z<x}(S)_z, y]).$$

Then we have:

1. $\mathsf{ACA}_0 \vdash \exists X \mathcal{H}^{\circ}_{A}[n, X]$ *for all natural numbers* n.

2. $\mathsf{ACA} \vdash \forall x \exists X \mathcal{H}^{\circ}_{A}[x, X]$.

3. $\mathsf{ATR}_0 \vdash \forall x \exists X \mathcal{H}^{\circ}_{A}[x, X]$.

The first assertion is easily obtained by complete induction from outside, whereas the second follows by complete induction inside ACA for the Σ^1_1 formula $\exists X \mathcal{H}^{\circ}_{A}[x, X]$. To see why the third assertion is true, we work in ATR_0 and introduce the set $R := \{(x, y) : x \leq y\}$ by arithmetical comprehension. Then the induction axiom implies $WO[R]$, and with the schema (ATR) we obtain what we want.

2.2 Δ^1_1 transfinite recursion and its relatives

The schemas (ATR) and (FP) are formulated for arithmetical formulas. Now we generalize these two schemas to Δ^1_1 formulas and, in the case of the fixed point axioms, only ask for monotonicity instead of positivity.

Δ^1_1 transfinite recursion. For all Σ^1_1 formulas $A[X, y]$ and Π^1_1 formulas $B[X, y]$:

$$(\Delta^1_1\text{-}\mathsf{TR}) \qquad \forall X \forall y (A[X, y] \leftrightarrow B[X, y]) \wedge WO[R] \; \rightarrow \; \exists X \mathcal{H}_A[R, X].$$

Fixed points of monotone Δ^1_1 clauses. For all Σ^1_1 formulas $A[X, y]$ and Π^1_1 formulas $B[X, y]$:

$$(\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}) \qquad C_{A,B} \; \rightarrow \; \exists X \forall y (y \in X \; \leftrightarrow \; A[X, y]),$$

where $C_{A,B}$ stands for the formula

$$\forall X \forall y (A[X, y] \leftrightarrow B[X, y]) \; \wedge \; \forall X, Y (X \subseteq Y \; \rightarrow \; \forall z (A[X, z] \; \rightarrow \; A[Y, z]))$$

that states that $A[X, y]$ is Δ^1_1 and monotone in X.

The following remark tells us that in $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0$ all monotone Σ^1_1 definable functions have fixed points. We leave it to the reader to check that also the converse is true: From the existence of fixed points of monotone Σ^1_1 definable functions, the principle ($\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}$) can be derived.

Remark 4. Let $A[X, Y]$ be a Σ^1_1 formula. Then $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0$ proves that from

(1) $\forall X \exists ! Y A[X, Y]$,

(2) $\forall X_0, X_1, Y_0, Y_1 (A[X_0, Y_0] \; \wedge \; A[X_1, Y_1] \; \wedge \; X_0 \subseteq X_1 \; \rightarrow \; Y_0 \subseteq Y_1)$

we obtain that $\exists X A[X, X]$.

Proof. Indeed, assume (1) and (2) and define

$$B_0[Z,z] \ := \ \exists X(A[Z,X] \wedge z \in X),$$
$$B_1[Z,z] \ := \ \forall X(A[Z,X] \rightarrow z \in X).$$

Then $B_0[Z,z]$ is equivalent to a Σ^1_1 formula, $B_1[Z,z]$ to a Π^1_1 formula. We also have $\forall X \forall y(B_0[X,y] \leftrightarrow B_1[X,y])$ and

$$\forall X, Y(X \subseteq Y \rightarrow \forall z(B_0[X,z] \rightarrow B_0[Y,z])).$$

Hence $(\mathsf{M}\Delta^1_1\text{-}\mathsf{FP})$ provides a set S such that $\forall x(x \in S \leftrightarrow B_0[S,x])$. But this implies $A[S,S]$. □

Now we make use of a straightforward extension of the proof in Simpson [18] that $\Pi^1_1\text{-}\mathsf{Red}_0$ – $\Sigma^1_1\text{-}\mathsf{Sep}_0$ in Simpson's terminology – proves all instances of (ATR) in order to show that $\Pi^1_1\text{-}\mathsf{Red}_0$ even proves all instances of $(\Delta^1_1\text{-}\mathsf{TR})$. For this end we define for any formula $A[X,y]$ with the distinguished free variables X and y (possibly containing further parameters) the following auxiliary formulas:

$$\mathcal{H}_A[a,R,S] \ := \ LO[R] \wedge (\forall x \prec_R a)\forall y(y \in (S)_x \leftrightarrow A[(S)_{Rx},y]),$$
$$\mathcal{H}^+_A[a,R,S] \ := \ LO[R] \wedge (\forall x \preceq_R a)\forall y(y \in (S)_x \leftrightarrow A[(S)_{Rx},y]).$$

Then it is obvious that $\mathcal{H}_A[R,S]$ is equivalent to $\forall a \mathcal{H}^+_A[a,R,S]$. Moreover, these hierarchies satisfy the following uniqueness property.

Lemma 5. *For any formula $A[X,y]$ we can prove in ACA_0 that*

$$WO[R] \wedge \mathcal{H}_A[a,R,S] \wedge \mathcal{H}_A[a,R,T] \rightarrow (\forall b \prec_R a)((S)_b = (T)_b).$$

Indeed, it is straightforward that from the given assumptions the arithmetical assertion

$$C[z] \ := \ z \prec_R a \rightarrow (S)_z = (T)_z$$

can be proved by induction on $\prec$.

Theorem 6. $\Delta^1_1\text{-}\mathsf{TR}_0 \subseteq \Pi^1_1\text{-}\mathsf{Red}_0$.

Proof. Let $A[X,y]$ be a Σ^1_1 formula and $B[X,y]$ a Π^1_1 formula. Working in $\Pi^1_1\text{-}\mathsf{Red}_0$ we assume

(1) $$\forall X \forall y(A[X,y] \leftrightarrow B[X,y]) \wedge WO[R].$$

Depending on $A[Z,z]$ and R we now set

$$C_0[z] \ := \ \exists b, x \exists Y(Fd[R,b] \wedge z = (b,x) \wedge \mathcal{H}_A[b,R,(Y)_{Rb}] \wedge A[(Y)_{Rb},x]),$$

$$C_1[z] \ := \ \forall b, x \forall Y(Fd[R,b] \wedge z = (b,x) \wedge \mathcal{H}_A[b,R,(Y)_{Rb}] \rightarrow A[(Y)_{Rb},x]).$$

In view of (1) it is easy to see that $C_0[z]$ is equivalent to a Σ^1 formula and $C_1[z]$ to a Π^1 formula, provable in $\Pi^1_1\text{-}\mathsf{Red}_0$. Furthermore, from Lemma 5 we deduce that

$$\forall z(C_0[z] \rightarrow C_1[z]).$$

By Π^1_1 reduction, cf. Theorem 1 and Corollary 2, we thus obtain a set S such that

$$\forall z(C_0[z] \rightarrow z \in S), \tag{2}$$

$$\forall z(z \in S \rightarrow C_1[z]). \tag{3}$$

Now we want to show that

$$\forall a \mathcal{H}^+_A[a, R, S]. \tag{4}$$

Since $\mathcal{H}^+_A[a, R, S]$ is equivalent to a Σ^1_1 and a Π^1_1 formula, R is a well-ordering, and $\Pi^1_1\text{-}\mathsf{Red}_0$ comprises $\Delta^1_1\text{-}\mathsf{CA}$, this can be done by transfinite induction on R. So let us assume

$$(\forall b \prec_R a)\mathcal{H}^+_A[b, R, S]. \tag{5}$$

Then we clearly have

$$\mathcal{H}_A[a, R, S], \tag{6}$$

$$\mathcal{H}_A[a, R, (S)_{Ra}]. \tag{7}$$

Thus we have, for any x,

$$A[(S)_{Ra}, x] \rightarrow C_0[(a, x)].$$

Hence, because of (2), also

$$A[(S)_{Ra}, x] \rightarrow x \in (S)_a. \tag{8}$$

On the other hand, from (3) we obtain

$$x \in (S)_a \rightarrow C_1[(a, x)].$$

Together with $\mathcal{H}_A[a, R, (S)_{Ra}]$ and the uniqueness assertion of Lemma 5 we thus have

$$x \in (S)_a \rightarrow A[(S)_{Ra}, x]. \tag{9}$$

The assertions (6), (8), and (9) give us $\mathcal{H}^+_A[a, R, S]$, finishing the proof of (4). As remarked earlier, (4) immediately implies $\mathcal{H}_A[R, S]$, and so S is a suitable witness for our Δ^1_1 transfinite recursion. □

We will show that both schemas, (Δ^1_1-TR) and ($\mathsf{M}\Delta^1_1$-FP), are equivalent to (ATR) over ACA_0. For doing that, we introduce a further principle, which is interesting on its own.

Weak Σ^1_1 transfinite dependent choice. For all Σ^1_1 formulas $A[X,Y]$:

$$(\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}) \qquad \begin{array}{l} \forall X \exists! Y A[X,Y] \wedge WO[R] \;\rightarrow \\ \qquad\qquad \exists Z \forall x (Fd[R,x] \;\rightarrow\; A[(Z)_{Rx},(Z)_x]). \end{array}$$

Our next step in establishing the desired equivalences is to show that all instances of weak Σ^1_1 transfinite dependent choice are provable in the theory Δ^1_1-TR_0.

Theorem 7. weak-Σ^1_1-$\mathsf{TDC}_0 \subseteq \Delta^1_1$-$\mathsf{TR}_0$.

Proof. We work in Δ^1_1-TR_0 and assume that

$$(*) \qquad \forall X \exists! Y A[X,Y]$$

for some Σ^1_1 formula $A[X,Y]$ and that $WO[R]$. Now we define

$$\begin{aligned} B[Z,z] &:= \exists Y (A[Z,Y] \wedge z \in Y), \\ C[Z,z] &:= \forall Y (A[Z,Y] \rightarrow z \in Y) \end{aligned}$$

and observe that (*) yields

$$\forall X \forall y (B[X,y] \leftrightarrow C[X,y]).$$

Hence we can apply (Δ^1_1-TR) and obtain a set S such that $\mathcal{H}_B[R,S]$, i.e.,

$$\forall x (Fd[R,x] \;\rightarrow\; \forall y (y \in (S)_x \;\leftrightarrow\; B[(S)_{Rx}, y])).$$

In view of (*) and the definition of $B[Z,z]$ we thus have $A[(S)_{Rx},(S)_x]$ for all x from the field of R, as was required for validating (weak-Σ^1_1-TDC). $\square$

A further and easy observation tells us that weak Σ^1_1 transfinite dependent choice proves all instances of (ATR). From this proof it is clear that instances of (weak-Σ^1_1-TDC) with arithmetical matrices $A[X,Y]$ are sufficient.

Lemma 8. $\mathsf{ATR}_0 \subseteq \mathsf{weak}$-$\Sigma^1_1$-$\mathsf{TDC}_0$.

Proof. Given an arithmetical formula $A[X,y]$ we set

$$B[X,Y] \;:=\; \forall z (z \in Y \;\leftrightarrow\; A[X,z])$$

and observe that $\forall X \exists! Y B[X,Y]$. Given any R with $WO[R]$, weak Σ^1_1 transfinite dependent choice provides the existence of a set S such that

$$\forall x (Fd[R,x] \;\rightarrow\; B[(S)_{Rx},(S)_x]).$$

However, this implies

$$\forall x(Fd[R,x] \to \forall y(y \in (S)_x \leftrightarrow A[(S)_{Rx}, y])),$$

yielding $\mathcal{H}_A[R,S]$ and finishing our proof. $\Box$

Remark 9. It is an immediate consequence of Theorem 1 and the previous theorem that in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$ every Σ^1 formula is equivalent to a Σ^1_1 formula and every Π^1 formula is equivalent to a Π^1_1 formula.

In order to show that every instance of $(\mathsf{M}\Delta^1_1\text{-}\mathsf{FP})$ can be proved in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$ we only have to adapt Avigad's proof that FP_0 is contained in ATR_0, see Avigad [1].

Lemma 10. *Assume that $A[X,y]$ is Σ^1_1 and that $B[X,y]$ is Π^1_1. Working in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$ we assume that*

(1) $\forall X \forall y(A[X,y] \leftrightarrow B[X,y])$,

(2) $\forall X, Y(X \subseteq Y \to \forall z(A[X,z] \to A[Y,z]))$.

Also, we let $C[R,Z]$ be the conjunction of the following (C1) – (C4):

(C1) $LO[R]$,

(C2) $\forall a(Fd[R,a] \to (Z)_a = \{x : A[\bigcup\{(Z)_b : b \prec_R a\}, x]\})$,

(C3) $\forall a, b(a \prec_R b \to (Z)_a \subseteq (Z)_b)$,

(C4) $\forall a, x(Fd[R,a] \wedge x \in (Z)_a \to$
$\exists b(Fd[R,b] \wedge x \in (Z)_b \wedge x \notin \bigcup\{(Z)_c : c \prec_R b\}))$.

Then we have for any set R that

$$WO[R] \to \exists Z C[R,Z].$$

Proof. To see why this is the case, set

$$D[Z_0, Z_1] \;:=\; \exists X(X = \{x : \exists a((a,x) \in Z_0)\} \wedge Z_1 = \{x : A[Z_0, x]\})$$

and recall from Remark 9 that $D[Z_0, Z_1]$ is equivalent to a Σ^1_1 formula. Furthermore, since Δ^1_1 comprehension is available in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$ according to Theorem 1 and Lemma 8, we see that $\forall X \exists! Y D[X,Y]$. Given any R with $WO[R]$, we can, therefore, apply $(\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC})$ and obtain a set S such that

$$(*) \qquad \forall x(Fd[R,x] \to D[(S)_{Rx}, (S)_x]).$$

It remains to check that $C[R,S]$: Condition (C1) follows from $WO[R]$. The definition of D plus (*) immediately give us (C2) and – because of the monotonicity of A – also (C3). Finally, (C4) is a direct consequence of $WO[R]$. $\Box$

Theorem 11. $\mathsf{M}\Delta^1_1\text{-}\mathsf{FP}_0 \subseteq \mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$.

Proof. As in the previous lemma we assume that $A[X,y]$ is Σ^1_1, that $B[X,y]$ is Π^1_1, and that

(1) $\forall X \forall y(A[X,y] \leftrightarrow B[X,y])$,

(2) $\forall X, Y(X \subseteq Y \to \forall z(A[X,z] \to A[Y,z]))$

are provable in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$. According to the previous lemma we then have, always working in $\mathsf{weak}\text{-}\Sigma^1_1\text{-}\mathsf{TDC}_0$, that

$$\forall X(WO[X] \to \exists Z C[X,Z]),$$

where $C[X,Z]$ is defined as in the previous lemma. In view of (1) and Remark 9 we know that $\exists Z C[X,Z]$ is equivalent to a Σ^1_1 formula. Hence Theorem 1 implies that there exist a linear ordering R and a set S such that

$$C[R,S] \wedge \neg WO[R].$$

Since R is not a well-ordering, there exists a non-empty subset W of its field with no R-minimal element; without loss of generality we may also assume that W is upwards closed. Furthermore, define (by arithmetical comprehension)

$$V := \{x : (\forall y \in W)(x \prec_R y)\}.$$

We make use of arithmetical comprehension once more and introduce the sets

$$T_0 := \bigcap_{a \in W} (S)_a \qquad \text{and} \qquad T_1 := \bigcup_{b \in V} (S)_b.$$

We claim that $T_0 = T_1$. Indeed, $T_1 \subseteq T_0$ is immediate from (C3). For the converse inclusion, assume that $x \in (S)_a$ for all $a \in W \neq \varnothing$. Because of (C4) there exists a b such that $x \in (S)_b$ and $x \notin \bigcup\{(S)_c : c \prec_R b\}$. Since W does not have an R-minimal element, this implies $b \in V$, hence $x \in T_1$, and our claim is proved. Next we will show:

(3) $\forall x(A[T_0,x] \to x \in T_0)$,

(4) $\forall x(x \in T_1 \to A[T_1,x])$.

In order to prove (3) assume $A[T_0,x]$ and let c be an element of W. Then we have $T_0 \subseteq \bigcup\{(S)_b : b \prec_R c\}$, and the monotonicity of A and (C2) yield $x \in (S)_c$. Since this is true for all $c \in W$ and all x we have (3). It remains to show (4). Pick an arbitrary $c \in V$. Then $\bigcup\{(S)_b : b \prec_R c\} \subseteq T_1$. Hence the monotonicity of A and (C2) give us $(S)_c \subseteq \{x : A[T_1,x]\}$. Therefore, also (4) is established. Together with $T_0 = T_1$ we obtain from (3) and (4) that

$$\forall x(x \in T_0 \leftrightarrow A[T_0,x]).$$

Hence T_0 is the required fixed point, and our theorem is proved. □

Summarizing Theorem 1, Theorem 6, Theorem 7, and Theorem 11 we obtain the following intermediate result.

Corollary 12. *The theories* ATR_0, FP_0, Π^1_1-Red_0, Δ^1_1-TR_0, $\mathsf{M}\Delta^1_1$-FP_0, *and* weak-Σ^1_1-TDC_0 *prove the same* $\mathcal{L}_2$ *formulas.*

3 Subsystems of set theory

In this section we consider several subsystems of set theory above the natural numbers as urelements, related to the theories of the previous section. Since we are interested in the relationship between subsystems of second order arithmetic and set theory, this is a very natural choice. It has the further advantage that we can distinguish in a natural way between two forms of induction: complete induction on the natural numbers and $\in$-induction. This offers a further possibility to calibrate subsystems by restricting these forms of induction separately.

3.1 Basic set theory BS

$\mathcal{L}^* = \mathcal{L}_1(\in, N, S, Ad)$ is the first order language that is obtained from $\mathcal{L}_1$ by adding the membership symbol $\in$, the constant N for the set of natural numbers as well as the unary relation symbols S and Ad to express that an object is a set and an admissible set, respectively. For notational simplicity, we drop all symbols for primitive recursive functions.

The definitions of terms and formulas of $\mathcal{L}^*$ are standard. $\mathcal{L}^*$ and $\mathcal{L}_2$ diverge in the intended interpretations of the first order variables: in $\mathcal{L}^*$ they are supposed to range over natural numbers and sets, not just over natural numbers as in $\mathcal{L}_2$.

We extend the use of metavariables and allow r, s, t and A, B, C also to stand for terms and formulas of $\mathcal{L}^*$. It will always be clear from the context whether we speak about $\mathcal{L}_2$ or $\mathcal{L}^*$ terms and formulas.

The Δ_0, Σ_n, Π_n, Σ, and Π formulas of $\mathcal{L}^*$ are defined as usual. Also, we write A^r for the result of replacing all unbounded quantifiers $\exists x(\ldots)$ and $\forall x(\ldots)$ in A by $(\exists x \in r)(\ldots)$ and $(\forall x \in r)(\ldots)$, respectively. Equality of objects is defined as

$$r = s \;:=\; \begin{cases} (r \in N \,\wedge\, s \in N \,\wedge\, r =_N s) \,\vee \\ (S(r) \,\wedge\, S(s) \,\wedge\, r \subseteq s \,\wedge\, s \subseteq r), \end{cases}$$

where $=_N$ is the relation symbol for the primitive recursive equality relation and $r \subseteq s$ is short for $(\forall x \in r)(x \in s)$. Hence two objects are equal iff they are primitive recursively equal natural numbers or sets which contain the same elements.

We begin with fixing the theory BS of basic set theory. It is formulated in classical first order logic (with equality axioms), and its non-logical axioms are grouped as follows:

Number-theoretic axioms

(PA) A^N for all closed axioms of Peano arithmetic PA.

Complete induction ($\mathcal{L}^*$-I_N) on N. For any $\mathcal{L}^*$ formula A and with $R_{\mathcal{S}}$ being the relation symbol for the primitive recursive successor relation $\mathcal{S}$,

$$A[0] \wedge (\forall x, y \in N)(A[x] \wedge R_{\mathcal{S}}(x,y) \rightarrow A[y]) \rightarrow (\forall x \in N)A[x].$$

Ontological axioms

(O.1) $a = b \rightarrow (A[a] \rightarrow A[b])$ for all atomic formulas A.

(O.2) $S(a) \leftrightarrow a \notin N$.

(O.3) $a \in N \rightarrow b \notin a$.

(O.4) $n \in N$ for all natural numbers n.

(O.5) $R_{\mathcal{Z}}(a_1, \ldots, a_n) \rightarrow a_1, \ldots, a_n \in N$, if $R_{\mathcal{Z}}$ is the relation symbol for the n-ary primitive recursive relation $\mathcal{Z}$.

Set-theoretic axioms

Pairing. $\exists x(a \in x \wedge b \in x)$.

Union. $\exists x(\forall y \in a)(\forall z \in y)(z \in x)$.

Δ_0 Separation (Δ_0-Sep). For any Δ_0 formula A,

$$\exists x(S(x) \wedge (\forall y \in x \rightarrow y \in a \wedge A[y]) \wedge (\forall y \in a)(A[y] \rightarrow y \in x)).$$

Full $\in$-induction ($\mathcal{L}^*$-$\mathsf{I}_\in$). For any $\mathcal{L}^*$ formula A,

$$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x].$$

In the following, subsystems of BS without $\in$-induction will play a central role: BS^0 is obtained from BS by restricting complete induction on N to Δ_0 formulas – denoted (Δ_0-I_N) – and deleting $\in$-induction, whereas BS^1 simply is BS without $\in$-induction. Of course, more variations of restricting inductions would be possible – for example restricting both forms of induction to Δ_0 formulas – but such systems are not relevant for us later.

Remark 13. *Please observe that all non-logical axioms of* BS^0 *and all those of* BS^1 *– apart from induction – are* Σ *formulas.*

Clearly, there is a canonical embedding of the language $\mathcal{L}_2$ into the language $\mathcal{L}^*$. Let $u_0, u_1, \dots$ and $U_0, U_1, \dots$ be enumerations of the first and second order variables of $\mathcal{L}_2$, and let $v_0, v_1, \dots$ be an enumeration of the variables of $\mathcal{L}^*$. Now we set $\hat{u}_i := v_{2i}$ and $\hat{U}_i := v_{2i+1}$. The translation of the $\mathcal{L}_2$ formulas A into the $\mathcal{L}^*$ formulas $\hat{A}$ is then as follows: (i) Replace all first order variables x and X of A by $\hat{x}$ and $\hat{X}$, respectively; (ii) replace all number quantifiers $\exists x(\dots)$ and $\forall x(\dots)$ by $(\exists \hat{x} \in N)(\dots)$ and $(\forall \hat{x} \in N)(\dots)$, respectively; (iii) replace all set quantifiers $\exists X(\dots)$ and $\forall X(\dots)$ by $(\exists \hat{X} \subseteq N)(\dots)$ and $(\forall \hat{X} \subseteq N)(\dots)$, respectively.

It is obvious that the translation of every arithmetical $\mathcal{L}_2$ formula is a Δ_0 formula of $\mathcal{L}^*$; furthermore, if A is Σ^1_1, then $\hat{A}$ is Σ_1, and if A is Σ^1, then $\hat{A}$ is Σ; analogously for Π^1_1 and Π^1 formulas.

From now on we often identify $\mathcal{L}_2$ formulas A with their translations $\hat{A}$ and consider $\mathcal{L}_2$ as a sublanguage of $\mathcal{L}^*$. With this convention in mind, the following lemma is easily verified.

Lemma 14. $\mathsf{ACA}_0 \subseteq \mathsf{BS}^0$ *and* $\mathsf{ACA} \subseteq \mathsf{BS}^1$.

3.2 Theories of admissible sets

We enter the field of admissible set theory if we strengthen basic set theory by adding the schema of Δ_0 collection, i.e., for any Δ_0 formula A,

$$(\Delta_0\text{-}\mathsf{Col}) \qquad (\forall x \in a)\exists y A[x,y] \;\to\; \exists z(\forall x \in a)(\exists y \in z)A[x,y].$$

The textbook Barwise [2] provides an excellent introduction to Kripke-Platek set theory with and without urelements. The theories for admissible sets above the natural numbers as urelements, which play a role now, are presented in detail in, for example, Jäger [7, 8, 9, 11, 12, 14].

Kripke-Platek set theory KPu above the natural numbers as urelements is $\mathsf{BS} + (\Delta_0\text{-}\mathsf{Col})$. Its canonical models are the structures $L(\alpha)_{\mathbb{N}}$ where α is an admissible ordinal greater than ω; see Barwise [2] for further reading.

As in the case of BS we obtain natural subsystems of KPu by restricting induction on N and eliminating $\in$-induction:

$$\mathsf{KPu}^0 \;:=\; \mathsf{BS}^0 + (\Delta_0\text{-}\mathsf{Col}) \quad \text{and} \quad \mathsf{KPu}^1 \;:=\; \mathsf{BS}^1 + (\Delta_0\text{-}\mathsf{Col}).$$

Such restrictions have dramatic consequences. For example, in KPu^0 we cannot prove the existence of ω. The following characterizations of the proof-theoretic strengths of KPu and its two subsystems follow from Jäger [7, 9, 12].

Theorem 15. *We have the following proof-theoretic equivalences:*

$$\mathsf{KPu}^0 \;\equiv\; \Delta^1_1\text{-}\mathsf{CA}_0, \quad \mathsf{KPu}^1 \;\equiv\; \Delta^1_1\text{-}\mathsf{CA}, \quad \mathsf{KPu} \;\equiv\; \mathsf{ID}_1.$$

Thus far the predicate Ad did not play a role. It will be used now to speak about admissible sets and its meaning is given by the following Ad-axioms.

Ad-axioms

$(Ad.1)$ $Ad(d) \;\rightarrow\; N \in d \,\wedge\, d$ is transitive.

$(Ad.2)$ $Ad(d) \;\rightarrow\; A^d$ for any closed instance A of an axiom of KPu^0.

$(Ad.3)$ $Ad(d_1) \wedge Ad(d_2) \;\rightarrow\; d_1 \in d_2 \,\vee\, d_1 = d_2 \,\vee\, d_2 \in d_1$.

These Ad-axioms determine closure properties of admissible sets; however, they do not provide for the existence of admissible sets. One possibility to overcome this "problem" is to add the limit axiom

$$(\mathsf{Lim}) \qquad \forall x \exists y (x \in y \wedge Ad(y)).$$

This axiom pins down a universe that is a limit of admissible sets. We add it to basic set theory, to Kripke-Platek set theory, and to their subsystems without ($\in$-induction) and obtain the following theories:

$$\begin{array}{lll} \mathsf{KPl}^0 := \mathsf{BS}^0 + (\mathsf{Lim}), & & \mathsf{KPi}^0 := \mathsf{KPu}^0 + (\mathsf{Lim}), \\ \mathsf{KPl}^1 := \mathsf{BS}^1 + (\mathsf{Lim}), & & \mathsf{KPi}^1 := \mathsf{KPu}^1 + (\mathsf{Lim}), \\ \mathsf{KPl} := \mathsf{BS} + (\mathsf{Lim}), & & \mathsf{KPi} := \mathsf{KPu} + (\mathsf{Lim}). \end{array}$$

The full systems KPl and KPi belong to a realm of proof theory that is not relevant for what we study in this article. To see why, we simply mention a result that characterizes their proof-theoretic strengths; for proofs see Jäger [10, 11, 12] and Jäger and Pohlers [14].

Theorem 16. *We have the following proof-theoretic equivalences:*

$$\mathsf{KPl} \equiv \Pi^1_1\text{-}\mathsf{CA} + (\mathsf{BI}) \quad \textit{and} \quad \mathsf{KPi} \equiv \Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI}).$$

However, as soon as $\in$-induction is dropped, we are in the range of predicativity/metapredicativity. For a discussion of the general context and the relationship between our theories and transfinitely iterated fixed point theories we refer, for example, to Jäger [11], Jäger, Kahle, Setzer, and Strahm [13], and Strahm [19].

Theorem 17. *We have the following proof-theoretic equivalences:*

1. $\mathsf{KPl}^0 \equiv \mathsf{KPi}^0 \equiv \mathsf{ATR}_0, \quad |\mathsf{KPl}^0| = |\mathsf{KPi}^0| = \Gamma_0.$
2. $\mathsf{KPl}^1 \equiv \mathsf{ATR} \equiv \widehat{\mathsf{ID}}_\omega, \quad |\mathsf{KPl}^1| = \Gamma_{\varepsilon_0}.$
3. $\mathsf{KPi}^1 \equiv \widehat{\mathsf{ID}}_{<\varepsilon_0}, \quad |\mathsf{KPi}^1| = \varphi 1 \varepsilon_0 0.$

In the articles mentioned above you also find all the necessary information about the ordinal notations based on the binary and ternary Veblen functions occurring in this theorem.

3.3 Σ and Π reduction

The set-theoretic siblings of Π^1 and Σ^1 reduction are the schemas (Π-Red) and (Σ-Red), and it is clear that (modulo our embedding of $\mathcal{L}_2$ into $\mathcal{L}^*$) the instances of (Π^1-Red) and (Σ^1-Red) are special cases of (Π-Red) and (Σ-Red).

Π and Σ reduction. For all Σ formulas $A[x]$ and Π formulas $B[x]$:

(Π-Red)
$$(\forall x \in a)(A[x] \to B[x]) \to \exists y((\forall x \in a)(A[x] \to x \in y) \wedge (\forall x \in y)(x \in a \wedge B[x])).$$

(Σ-Red)
$$(\forall x \in a)(B[x] \to A[x]) \to \exists y((\forall x \in a)(B[x] \to x \in y) \wedge (\forall x \in y)(x \in a \wedge A[x])).$$

It is also straightforward that (Π-Red) and (Σ-Red) yield Δ separation, i.e., for all Σ formulas $A[x]$ and Π formulas $B[x]$,

$$(\forall x \in a)(A[x] \leftrightarrow B[x]) \to \exists y \forall x(x \in y \leftrightarrow x \in a \wedge A[x]).$$

From that lower bounds for (Σ-Red) over BS^0 and BS^1 can be read off immediately.

Lemma 18. *We have the following inclusions:*

$$\Delta^1_1\text{-}\mathsf{CA}_0 \subseteq \mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red}) \quad \textit{and} \quad \Delta^1_1\text{-}\mathsf{CA} \subseteq \mathsf{BS}^1 + (\Sigma\text{-}\mathsf{Red}).$$

In the next subsection, we will show that these bounds are sharp as far as proof-theoretic strength is concerned. If (Π-Red) is available, we can do more.

Lemma 19. *We have the following inclusions:*

$$\mathsf{ATR}_0 \subseteq \mathsf{BS}^0 + (\Pi\text{-}\mathsf{Red}) \quad \textit{and} \quad \mathsf{ATR} \subseteq \mathsf{BS}^1 + (\Pi\text{-}\mathsf{Red}).$$

This is an immediate consequence of Theorem 1 since (Π-Red) yields (Π^1_1-Red). In the next subsection we will also show that the respective theories are equivalent in a strong sense.

Turning to Kripke-Platek set theory, the first observation is that (Σ-Red) is not interesting in this context.

Theorem 20. KPu^0 *proves all instances of* (Σ-Red).

Proof. We work in KPu^0 and suppose that $(\forall x \in a)(A[x] \to B[x])$ for some Π formula $A[x]$ and Σ formula $B[x]$. By applying Σ reflection we obtain a set b such that

$$(\star) \qquad (\forall x \in a)(A^b[x] \to B^b[x]).$$

Now we introduce the set $c := \{x \in a : B^b[x]\}$ by Δ_0 separation. We claim that c is a suitable witness for $(\Sigma\text{-}\mathsf{Red})$.

Indeed, from $x \in a$ and $A[x]$ we obtain $A^b[x]$ from the downward persistence of Π formulas. Therefore, we have $B^b[x]$ because of $(\star)$, and thus $x \in c$. On the other hand, if $x \in c$, then $x \in a$ and $B^b[x]$. So we also have $B[x]$ because of the upward persistence of Σ formulas. $\square$

The situation is different for $(\Pi\text{-}\mathsf{Red})$. Clearly, $(\Pi\text{-}\mathsf{Red})$ is provable in $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Sep})$ and thus also in $\mathsf{KPu}^0 + (\Sigma_1\text{-}\mathsf{Sep})$. However, the exact proof-theoretic analysis of $(\Pi\text{-}\mathsf{Red})$ will not be given here and is postponed to a later publication.

3.4 Reducing $\mathsf{BS} + (\Sigma\text{-}\mathsf{Red})$, $\mathsf{BS}^0 + (\Pi\text{-}\mathsf{Red})$, and $\mathsf{BS} + (\Pi\text{-}\mathsf{Red})$

In this subsection we provide the proof-theoretic upper bounds for the theories $\mathsf{BS}+(\Sigma\text{-}\mathsf{Red})$, $\mathsf{BS}^0+(\Pi\text{-}\mathsf{Red})$, and $\mathsf{BS}+(\Pi\text{-}\mathsf{Red})$. This will be achieved by reducing these set theories to the theories $\Sigma^1_1\text{-}\mathsf{AC}$, ATR_0, and ATR. For these reductions we follow the pattern of Simpson's reduction of his theory $\mathsf{ATR}_0^{\mathsf{set}}$ to ATR_0 in Simpson [18]. Some additional considerations are necessary for dealing with the natural numbers as urelements and for obtaining reductions to $\Sigma^1_1\text{-}\mathsf{AC}_0$ and $\Sigma^1_1\text{-}\mathsf{AC}$.

We adopt the terminology of [18], II.2, and take over Simpson's definition of *finite sequence of natural numbers.* We write *Seq* – called the set of sequence numbers – for the set of all codes of finite sequences of natural numbers. From now on we let σ, τ (possibly with indices) range over elements of *Seq*.

Also, we write $lh(\sigma)$ for the length of σ and use notations such as

$$\sigma = \langle s_0, \dots, s_{lh(\sigma)-1} \rangle$$

and then $(\sigma)_x$ for s_x if $x < lh(\sigma)$. Accordingly, $\langle\rangle$ stands for the code of the empty sequence.

We shall identify a finite sequence of natural numbers with its code whenever convenient. If σ is $\langle s_0, \dots s_{x-1}, \rangle$ and τ is $\langle t_0, \dots, t_{y-1}\rangle$ we denote the concatenation of the sequence numbers σ and τ by $\sigma * \tau$,

$$\sigma * \tau \;=\; \langle s_0, \dots s_{x-1}, t_0, \dots, t_{y-1}\rangle.$$

Hence, $lh(\sigma * \tau) = lh(\sigma) + lh(\tau)$. In particular,

$$\sigma * \langle s\rangle = \langle (\sigma)_0, \dots, (\sigma)_{lh(\sigma)-1}, s\rangle$$

and $lh(\sigma * \langle s\rangle) = lh(\sigma) + 1$. We write $\sigma \subseteq \tau$ to mean that σ is an initial segment of τ, i.e., $lh(\sigma) \leq lh(\tau)$ and $(\forall x < lh(\sigma))((\sigma)_x = (\tau)_x)$.

A *tree* is a non-empty subset T of *Seq* such that $\sigma \subseteq \tau$ and $\tau \in T$ implies $\sigma \in T$. Our next definition introduces the trees that we will use for the interpretation of the objects, i.e., urelements and sets, of our set theories.

Definition 21. *In* ACA_0 *we introduce the notions of* u-tree *and* representation tree.

1. *T is a u-tree, in symbols $Tree[u,T]$, iff*

   (i) T *is a tree,*

   (ii) $(\forall \sigma \in T)(lh(\sigma) \leq u)$,

   (iii) $(\forall \sigma, x, y)(\sigma * \langle 2x+1 \rangle \in T \wedge y \neq 2x+1 \rightarrow \sigma * \langle y \rangle \notin T)$,

   (iv) $(\forall \sigma, x)(\sigma * \langle x \rangle \in T \rightarrow (\forall y < lh(\sigma))((\sigma)_y \text{ is even}))$.

2. *T is a representation tree if it is a u-tree for some natural number u,*

$$Rep[T] := \exists u \, Tree[u,T].$$

3. *If T is a representation tree and $\sigma \in T$, we put $T^\sigma := \{\tau : \sigma * \tau \in T\}$.*

Obviously, every u-tree is well-founded. The u-trees will represent the objects of our set theories. The basic idea is that the natural number x is represented by the tree $\{\langle\rangle, \langle 2x+1 \rangle\}$ and that the elements of a set represented by the tree T are the sets represented by the immediate subtrees of T of the form $T^{\langle 2y \rangle}$ with $\langle 2y \rangle \in T$. However, in order to validate extensionality, we have to close this treatment of elementhood under isomorphisms; see Definition 23 below.

Definition 22. *If T is a u-tree for some u, we write $Iso[X,T]$ to state that $X \subseteq T \times T$ and, for all $\sigma, \tau \in T$, $(\sigma, \tau) \in X$ iff each of the following four properties is satisfied:*

(i) $\forall y(\sigma * \langle 2y \rangle \in T \rightarrow \exists z((\sigma * \langle 2y \rangle, \tau * \langle 2z \rangle) \in X))$,

(ii) $\forall z(\tau * \langle 2z \rangle \in T \rightarrow \exists y((\sigma * \langle 2y \rangle, \tau * \langle 2z \rangle) \in X))$,

(iii) $\forall y(\sigma * \langle 2y+1 \rangle \in T \rightarrow (\sigma * \langle 2y+1 \rangle, \tau * \langle 2y+1 \rangle) \in X)$,

(iv) $\forall y(\tau * \langle 2y+1 \rangle \in T \rightarrow (\sigma * \langle 2y+1 \rangle, \tau * \langle 2y+1 \rangle) \in X)$.

Following Simpson [18] this notion of isomorphism is now directly incorporated into the definitions of the equality and epsilon relations on representation trees.

Definition 23. *In* ACA_0 *we define for all representation trees S and T:*

1. $S \oplus T := \{\langle\rangle\} \cup \{\langle 0 \rangle * \sigma : \sigma \in S\} \cup \{\langle 2 \rangle * \tau : \tau \in T\}$,

2. $S =^\star T := \exists X(Iso[X, S \oplus T] \wedge (\langle 0 \rangle, \langle 2 \rangle) \in X)$,

3. $S \in^\star T := \exists X(Iso[X, S \oplus T] \wedge \exists x((\langle 0 \rangle, \langle 2, x \rangle) \in X))$.

These definitions of $=^\star$ and $\in^\star$ are so that the following properties of suitable trees can be proved in ACA_0.

Lemma 24. *The following is provable in* ACA_0*. Let* R, S, T *be representation trees and assume that* $Iso[X,T]$*,* $Iso[Y, S \oplus T]$*, and* $\sigma, \tau \in T$*. Then we have:*

1. $T^\sigma =^\star T^\tau \ \leftrightarrow\ (\sigma,\tau) \in X$.
2. $T^\sigma \in^\star T^\tau \ \leftrightarrow\ \exists x((\sigma, \tau * \langle 2x\rangle) \in X)$.
3. $S =^\star T \ \leftrightarrow\ \forall Z(Z \in^\star S \ \leftrightarrow\ Z \in^\star T)$.
4. $S =^\star T \wedge S \in^\star R \ \rightarrow\ T \in^\star R$.
5. $Tree[u,S] \wedge Tree[v,T] \wedge S \in^\star T \ \rightarrow\ u < v$.

For the following interpretability results we have to translate the $\mathcal{L}^*$ formulas into $\mathcal{L}_2$ formulas. To this end we fix enumerations $v_0, v_1, \ldots$ of the variables of $\mathcal{L}^*$ and $V_0, V_1, \ldots$ of the set variables of $\mathcal{L}_2$.

Definition 25. *We begin with translating the terms* t *of* $\mathcal{L}^*$ *into* $\mathcal{L}_2$*:*

$$
\begin{aligned}
v_i^\star &:= V_i \text{ for all indices } i,\\
n^\star &:= \{\langle\rangle, \langle 2n+1\rangle\} \text{ for all natural numbers } n,\\
N^\star &:= \{\langle\rangle\} \cup \{\sigma : \exists x(\sigma = \langle 2x\rangle \vee \sigma = \langle 2x, 2x+1\rangle)\}.
\end{aligned}
$$

Then the atomic formulas of $\mathcal{L}^*$ *are dealt with as follows:*

$$
(s \in t)^\star := s^\star \in^\star t^\star, \qquad S(t)^\star := t^\star \notin^\star N^\star,
$$

$$
R_{\mathcal{Z}}(t)^\star := \exists x(t^\star = \{\langle\rangle, \langle 2x+1\rangle\} \wedge R_{\mathcal{Z}}(x))
$$

if $R_{\mathcal{Z}}$ *is the relation symbol for the unary primitive recursive relation* $\mathcal{Z}$*;* n*-ary primitive recursive relations are treated accordingly. The propositional connectives commute with* $\star$*, and for the quantifiers we set*

$$
(\exists v_i A)^\star := \exists V_i(Rep[V_i] \wedge A^\star) \quad \text{and} \quad (\forall v_i A)^\star := \forall V_i(Rep[V_i] \rightarrow A^\star).
$$

Bounded quantifiers are treated accordingly.

The following lemma shows that we have Σ^1-Red on the collection of representation trees.

Lemma 26. *Assume that* $A[X]$ *is a* Σ^1 *formula of* $\mathcal{L}_2$ *and* $B[X]$ *a* Π^1 *formula of* $\mathcal{L}_2$*. Working in* Σ^1_1-AC_0*, we assume that* S *is a representation tree such that*

$$
\forall x(S^{\langle 2x\rangle} \in^\star S \wedge B[S^{\langle 2x\rangle}] \ \rightarrow\ A[S^{\langle 2x\rangle}]).
$$

Then there exists a representation tree T *such that:*

(i) $\forall x(\langle 2x\rangle \in S \wedge B[S^{\langle 2x\rangle}] \rightarrow \langle 2x\rangle \in T \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle})$,

(ii) $\forall x(\langle 2x\rangle \in T \rightarrow \langle 2x\rangle \in S \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle} \wedge A[T^{\langle 2x\rangle}])$.

Proof. Since we work in Σ^1_1-AC_0, all instances of (Σ^1-Red) are, according to Theorem 1 and Corollary 2, at our disposal. So we only have to apply Σ^1 reduction to our assumption and are rewarded with a set Y which satisfies:

- $\forall x(S^{\langle 2x\rangle} \in^\star S \wedge B[S^{\langle 2x\rangle}] \rightarrow x \in Y)$,
- $\forall x(x \in Y \rightarrow A[S^{\langle 2x\rangle}])$.

Now it is easy to check that

$$T := \{\langle\rangle\} \cup \{\sigma \in S : 1 \leq lh(\sigma) \wedge (\sigma)_0 \in Y\}$$

is a representation tree with the required properties. □

Our next task is to show that the translation $\in^\star$ of the $\in$-relation does not increase the complexity too much. To achieve this aim in the context of all representation trees, we have to work in ACA or ATR_0, and we show that every representation tree has a unique isomorphism then. In the next subjection we will see how to proceed in the context of Σ^1_1-AC_0.

Lemma 27 (Existence and uniqueness of isomorphisms). ACA *and* ATR_0 *prove that*

$$Rep[T] \rightarrow \exists! X\, Iso[X,T].$$

Proof. The existence of the isomorphism X on the representation tree T is proved by means of the hierarchy building principles mentioned in point two and three of Remark 3. The uniqueness of X follows by complete induction starting at the leaves of X. □

The proof of this lemma requires complete induction for Σ^1_1 formulas or the principle (ATR); we see no possibility to carry it through in Σ^1_1-AC_0. Lemma 32 below presents what we can do in Σ^1_1-AC_0.

The previous lemma is the decisive step in showing that $\in^\star$ is Δ^1_1 on all representation trees. It immediately implies the following equivalence.

Lemma 28. ACA *and* ATR_0 *prove*

$$\begin{aligned}&Rep[S] \wedge Rep[T] \rightarrow\\ &\qquad (S \in^\star T \leftrightarrow \forall X(Iso[X, S \oplus T] \rightarrow \forall x((\langle 0\rangle, \langle 2, x\rangle) \in X))).\end{aligned}$$

From now on we will freely deal with the correspondence between the variables of $\mathcal{L}^*$ and the set variables of $\mathcal{L}_2$. It should always be clear from the context what we have in mind. Besides that, if $\vec{U}$ is the list of variables $U_0, \ldots, U_n$ we use $\vec{U} \in Rep$ as shorthand notation for

$$Rep[U_0] \wedge \cdots \wedge Rep[U_n].$$

If A is a formula of $\mathcal{L}^*$ with at most the variables $\vec{u}$ free, we first translate it into the formula $A^\star$ with $\vec{U}$ listing the set variables linked to $\vec{u}$ and then set

$$|A| \;:=\; (\vec{U} \in \mathit{Rep} \;\rightarrow\; A^\star).$$

Remark 29. *From the definition of $\in^\star$ and Lemma 28 we conclude that in Σ^1_1-AC and in ATR_0 the translation $|A|$ of a Δ_0 formula A is provably equivalent to a Σ^1 and a Π^1 formula. In view of Corollary 2, A is even provably equivalent in these systems to a Σ^1_1 and a Π^1_1 formula.*

Keep in mind that this remark does not hold for the weaker system Σ^1_1-AC_0. Now we are ready for our first reduction theorem.

Theorem 30. *We have for all formulas A of $\mathcal{L}^*$:*

1. $\mathsf{BS} + (\Sigma\text{-}\mathsf{Red}) \vdash A \quad \Longrightarrow \quad \Sigma^1_1\text{-}\mathsf{AC} \vdash |A|$.
2. $\mathsf{BS}^0 + (\Pi\text{-}\mathsf{Red}) \vdash A \quad \Longrightarrow \quad \mathsf{ATR}_0 \vdash |A|$.
3. $\mathsf{BS} + (\Pi\text{-}\mathsf{Red}) \vdash A \quad \Longrightarrow \quad \mathsf{ATR} \vdash |A|$.

Proof. We first show that Σ^1_1-AC proves $|A|$ for all axioms A of BS. This is clear for all logical axioms, the number-theoretic axioms and the ontological axioms (O.2) – (O.5). The proofs of the translations of the (O.1)-axioms follow from Lemma 24. To handle pairing, let S and T be the representation trees that translate the sets a and b. Then $R := S \oplus T$ is a representation tree with $S \in^\star R$ and $T \in^\star R$. Similarly for (Union).

Now let $A[x]$ be a Δ_0 formula. Then (Δ_0-Sep) claims in BS the existence of the set $b := \{x \in a : A[x]\}$; for simplicity of notation we suppress all further parameters. We represent set a by the representation tree S and use (Δ^1_1-CA) to introduce the set

$$Z \;:=\; \{x : S^{\langle 2x \rangle} \in^\star S \;\wedge\; A^\star[S^{\langle 2x \rangle}]\}.$$

This is possible since $(S^{\langle 2x \rangle} \in^\star S \;\wedge\; A^\star[S^{\langle 2x \rangle}])$ is (provably in Σ^1_1-AC) equivalent to a Σ^1_1 and a Π^1_1 formula. Now it only remains to define the representation tree

$$T \;:=\; \{\langle\rangle\} \cup \{\sigma \in S : 0 < \mathit{lh}(\sigma) \;\wedge\; (\sigma)_0 \in Z\}$$

and to verify that it is a suitable representation of b.

Every instance of ($\mathcal{L}^*$-I_N) in BS clearly translates into an instance of the schema of complete induction of Σ^1_1-AC. Moreover, to see why the translations of the instances of ($\mathcal{L}^*$-$\mathsf{I}_\in$) hold in Σ^1_1-AC, recall from Lemma 24 that, for any u-tree S and v-tree T,

$$S \in^\star T \;\rightarrow\; u < v.$$

Hence the translation of any instance of ($\mathcal{L}^*$-$\mathsf{I}_\in$) is provable in Σ^1_1-AC since complete induction is available there for arbitrary $\mathcal{L}_2$ formulas.

For completing the proof of the first assertion of our theorem, it remains to handle (Σ-Red). But this is exactly what Lemma 26 does.

The proof of the third assertion of our theorem is practically the same, with the only difference that the translations of the instances of (Π-Red) are taken care of by Π^1-Red, which is available in ATR in view of Theorem 1 and Corollary 2.

The second assertion is different since there we have to embed into a second order system that does not provide complete induction for arbitrary formulas. But in BS^0 complete induction on N is restricted to Δ_0 formulas. Therefore, according to Remark 29, each such instance of complete induction translates into a formula that is equivalent to a Σ^1_1 and a Π^1_1 formula. Moreover, ATR_0 proves all instances of (Δ^1_1-CA), as stated in Theorem 1, and thus provides for complete induction with respect to all provably Δ^1_1 predicates. □

Corollary 31. *We have the following proof-theoretic equivalences:*

1. $\Sigma^1_1\text{-}\mathsf{AC} \;\equiv\; \mathsf{BS}^1 + (\Sigma\text{-}\mathsf{Red}) \;\equiv\; \mathsf{BS} + (\Sigma\text{-}\mathsf{Red})$.
2. $\mathsf{ATR}_0 \;\equiv\; \mathsf{BS}^0 + (\Pi\text{-}\mathsf{Red})$.
3. $\mathsf{ATR} \;\equiv\; \mathsf{BS}^1 + (\Pi\text{-}\mathsf{Red}) \;\equiv\; \mathsf{BS} + (\Pi\text{-}\mathsf{Red})$.

This corollary is an immediate consequence of Lemma 18, Lemma 19, and our previous reduction theorem.

3.5 The case $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$

It remains to analyze the theory $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$. As mentioned earlier, the important point is that in Σ^1_1-AC_0 we do not have the existence of isomorphisms for arbitrary representation trees. However, by simply replacing assertions two and three of Remark 3 by assertion one, we obtain the following weaker existence and uniqueness property.

Lemma 32 (Existence and uniqueness of isomorphisms on k-trees)**.** ACA_0 *proves for every standard natural number k that*

$$Tree[k, T] \;\rightarrow\; \exists! X\, Iso[X, T].$$

Consequently, $\in^\star$ restricted to k-trees for any fixed standard natural number k is Δ^1_1.

Lemma 33. ACA_0 *proves for all standard natural numbers k that*

$$\begin{aligned} &Tree[k, S] \wedge Tree[k, T] \;\rightarrow \\ &\qquad (S \in^\star T \;\leftrightarrow\; \forall X(Iso[X, S \oplus T] \;\rightarrow\; \forall x((\langle 0\rangle, \langle 2, x\rangle) \in X))). \end{aligned}$$

What we will show now is that suitable fragments of $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$ can be reduced to the substructures that are provided in $\Sigma^1_1\text{-}\mathsf{AC}_0$ by the k-trees for fixed natural numbers k. This is achieved by a combination of partial cut elimination and a technique that is often called the *method of asymmetric interpretations*; cf. Cantini [4].

For partial cut elimination and the asymetric interpretations we work with a reformulation of $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$ as a Tait-style one sided sequent calculus. This auxiliary system, we call it H, derives finite sets $(\Gamma, \Delta, \dots)$ of $\mathcal{L}^*$ formulas in negation normal form. If A is such an $\mathcal{L}^*$ formula then Γ, Δ, A is short for $\Gamma \cup \Delta \cup \{A\}$, and similar for expressions of the form Γ, A, B.

The axioms of H are all the sets Γ, A where A is the negation normal form of an axiom of BS^0. To simplify the notation we assume from now on that all $\mathcal{L}^*$ formulas are in negation normal form and write $\overline{A}$ for the negation normal form of $\neg A$.

The rules of H include the standard rules for the propositional connectives and quantifiers plus the cut rules

$$(\text{cut}) \qquad \frac{\Gamma,\, A \qquad \Gamma,\, \overline{A}}{\Gamma}$$

for all Γ and A; the formulas A and $\overline{A}$ are the cut formulas of this cut. In addition, we have rules, for all Σ formulas A and Π formulas B, of the form

$$(\Sigma\text{-}\mathsf{Red})^r \qquad \frac{\Gamma,\, (\forall x \in a)(\overline{B}[x] \,\vee\, A[x])}{\Gamma,\, \exists y((\forall x \in a)(\overline{B}[x] \,\vee\, x \in y) \,\wedge\, (\forall x \in y)(x \in a \,\wedge\, A[x]))}.$$

It should be clear that $(\Sigma\text{-}\mathsf{Red})^r$ is the reformulation of the schema $(\Sigma\text{-}\mathsf{Red})$ as a rule.

Definition 34. *We recursively assign a* rank $rk(A)$ *to all formulas (in negation normal form) as follows:*

1. *If A is a Σ or a Π formula, then $rk(A) := 0$,*
2. *If A is neither a Σ nor a Π formula, then*
   - *(a) $rk(A) := \max(rk(B), rk(C)) + 1$ if A is of the form $B \vee C$ or $B \wedge C$.*
   - *(b) $rk(A) := rk(B) + 1$ if A is of the form $\exists x B$, $\forall x B$, $(\exists x \in s)B$, or $(\forall x \in s)B$.*

By $H \vdash^m_n \Gamma$ we mean that there exists a derivation of Γ in H whose depth is bounded by m such that all cut formulas occurring in this derivation have rank less than n. Hence $\mathsf{H} \vdash^m_1 \Gamma$ says that there is a derivation of Γ in H of depth bounded by m whose cut formulas are Σ or Π formulas.

It is easy to check that the theory $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$ can be embedded into its Tait-style version H.

Lemma 35 (Embedding). *If* $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$ *proves* A, *then there exist natural numbers* m *and* n *such that* $\mathsf{H} \vdash^m_n A$.

The idea now is to perform a partial cut elimination for H that only leaves us with cuts on Σ and Π formulas. Please observe that the main formulas of all axioms of H and the main formula of the conclusion of any instance of $(\Sigma\text{-}\mathsf{Red})^r$ are Σ formulas. By standard proof-theoretic techniques we can therefore prove a partial cut elimination theorem, as desired.

Lemma 36 (Partial cut elimination). *Assume that* $\mathsf{H} \vdash^m_n \Gamma$ *for some natural numbers* m *and* n. *Then there exists a natural number* k *such that* $\mathsf{H} \vdash^k_1 \Gamma$

The next step is to carry through an asymmetric interpretation of the $(\Sigma\cup\Pi)$-fragment of H into $\Sigma^1_1\text{-}\mathsf{AC}_0$. To this end we formulate, for all standard natural numbers k, translations $A^{[k]}$ of the formulas of $\mathcal{L}^*$.

Definition 37. *We translate the terms* t *of* $\mathcal{L}^*$ *into terms* $t^\star$ *of* $\mathcal{L}_2$ *as in Definition 25. For any standard natural number* k *and* $\mathcal{L}^*$ *formula* A *we recursively define its translation* $A^{[k]}$ *by:*

(k.1) The atomic formulas of $\mathcal{L}^*$ *are dealt with as follows:*

$$(s \in t)^{[k]} \;:=\; s^\star \in^\star t^\star, \qquad S(t)^{[k]} \;:=\; t^\star \notin^\star N^\star,$$

$$R_{\mathcal{Z}}(t)^{[k]} \;:=\; \exists x(t^\star = \{\langle\rangle, \langle 2x+1\rangle\} \wedge R_{\mathcal{Z}}(x))$$

if $R_{\mathcal{Z}}$ *is the relation symbol for the unary primitive recursive relation* $\mathcal{Z}$*; n-ary primitive recursive relations are treated accordingly.*

(k.2) The propositional connectives commute with $[k]$*, and for the quantifiers we set*

$$(\exists v_i A)^{[k]} \;:=\; \exists V_i(\mathit{Tree}[k, V_i] \wedge A^{[k]}),$$

$$(\forall v_i A)^{[k]} \;:=\; \forall V_i(\mathit{Tree}[k, V_i] \to A^{[k]}).$$

Bounded quantifiers are treated accordingly.

These translations are as the translation introduced in Definition 25 with the only difference that the sets are no longer interpreted as arbitrary representation trees but as k-trees. Also, following the pattern of the proof of Lemma 26, we see that $\Sigma^1\text{-}\mathsf{Red}$ holds in $\Sigma^1_1\text{-}\mathsf{AC}_0$ on the collections of k-trees.

Lemma 38. *Assume that* $A[X]$ *is a* Σ^1 *formula of* $\mathcal{L}_2$ *and* $B[X]$ *a* Π^1 *formula of* $\mathcal{L}_2$. *Let* k *be a natural number. Working in* $\Sigma^1_1\text{-}\mathsf{AC}_0$, *we assume that* S *is a* k*-tree such that*

$$\forall x(S^{\langle 2x\rangle} \in^\star S \;\wedge\; B[S^{\langle 2x\rangle}] \;\to\; A[S^{\langle 2x\rangle}]).$$

Then there exists a k*-tree* T *such that:*

(i) $\forall x(\langle 2x\rangle \in S \wedge B[S^{\langle 2x\rangle}] \rightarrow \langle 2x\rangle \in T \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle})$,

(ii) $\forall x(\langle 2x\rangle \in T \rightarrow \langle 2x\rangle \in S \wedge S^{\langle 2x\rangle} = T^{\langle 2x\rangle} \wedge A[T^{\langle 2x\rangle}])$.

It is clear from these definitions and Lemma 33 that the k-translation of every Σ and Π formula is provably equivalent in Σ^1_1-AC_0 to a Σ^1_1 and Π^1_1 formula, respectively. In particular, the translation $A^{[k]}$ of a Δ_0 formula A is Δ^1_1 in Σ^1_1-AC_0.

The following notations are useful for the formulation and the proof of Theorem 41.

Definition 39. *Let Γ be the finite set $\{A_1, \ldots, A_n\}$ of $\mathcal{L}^*$ formulas in negation normal form and let k be a natural number.*

1. *The* ($\geq k$)-instances *of Γ are all sets of $\mathcal{L}_2$ formulas of the form*

$$\{A_1^{[k_1]}, \ldots, A_n^{[k_n]}\}$$

   with $k \leq k_1, \ldots, k_n$.

2. *The* ($\leq k$)-instances *of Γ are all sets of $\mathcal{L}_2$ formulas of the form*

$$\{A_1^{[k_1]}, \ldots, A_n^{[k_n]}\}$$

   with $k_1, \ldots, k_n \leq k$.

Remark 40. *For all finite sets Γ of $\mathcal{L}^*$ formulas and all natural numbers k_1, k_2 with $k_1 \leq k_2$ we have:*

1. *If Φ is a ($\geq k_2$)-instance of Γ, then it is also a ($\geq k_1$)-instance of Γ.*
2. *If Φ is a ($\leq k_1$)-instance of Γ, then it is also a ($\leq k_2$)-instance of Γ.*

Theorem 41 (Asymmetric interpretation)**.** *We assume:*

(i) *Φ_1 is a finite set of Π formulas of $\mathcal{L}^*$ in negation normal form.*

(ii) *Φ_2 is a finite set of Σ formulas of $\mathcal{L}^*$ in negation normal form.*

(iii) *At most the variables $u_1, \ldots, u_n$ are free in Φ_1, Φ_2.*

(iv) $\mathsf{H} \vdash^m_1 \Phi_1, \Phi_2$ *for some natural number m.*

Then we have for all natural numbers k, all ($\leq k$)-instances Ψ_1 of Φ_1, and all ($\geq k + 2^m$)-instances Ψ_2 of Φ_2 that

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \vdash \bigvee(\neg \mathit{Tree}[k, U_1], \ldots, \neg \mathit{Tree}[k, U_n], \Psi_1, \Psi_2).$$

Proof. We show this assertion by induction on m and distinguish the following cases:

(C1) Φ_1, Φ_2 is an axiom of H. Then the assertion is easily verified.

(C2) Φ_1, Φ_2 is the conclusion of an inference rule for a propositional connective or a quantifier. Then the assertion follows more or less directly from the induction hypothesis.

(C3) Φ_1, Φ_2 is the conclusion of a cut with cut formulas A and $\overline{A}$. Without loss of generality we can thus assume that A is a Σ formula and $\overline{A}$ a Π formula. Also, there exist natural numbers $m_0, m_1 < m$ such that

$$\mathsf{H} \vdash_1^{m_0} \Phi_1,\, \Phi_2,\, A, \tag{1}$$

$$\mathsf{H} \vdash_1^{m_1} \Phi_1,\, \Phi_2,\, \overline{A}. \tag{2}$$

Since Ψ_2 is a $(\geq k + 2^m)$-instance of Φ_2, it follows that $\Psi_2,\, A^{[k+2^{m_0}]}$ is a $(\geq k + 2^{m_0})$-instance of Φ_2, A. Therefore, the induction hypothesis implies that

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \vdash \bigvee(\neg\, \mathit{Tree}[k, U_1],\, \ldots,\, \neg\, \mathit{Tree}[k, U_n],\, \Psi_1,\, \Psi_2,\, A^{[k+2^{m_0}]}). \tag{3}$$

Furthermore, simple calculations show that $\Psi_1, \overline{A}^{[k+2^{m_0}]}$ is a $(\leq k + 2^{m_0})$-instance of $\Phi, \overline{A}$ and Ψ_2 a $(\geq k + 2^{m_0} + 2^{m_1})$-instance of Φ_2. Hence the induction hypothesis applied to (2) – with k replaced by $k + 2^{m_0}$ – yields

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \vdash \bigvee(\neg\, \mathit{Tree}[k, U_1],\, \ldots,\, \neg\, \mathit{Tree}[k, U_n],\, \Psi_1,\, \Psi_2,\, \overline{A}^{[k+2^{m_0}]}). \tag{4}$$

Now the assertion follows from (3) and (4) by simple logical reasoning within Σ^1_1-AC_0.

(C4) Φ_1, Φ_2 is the conclusion of the rule $(\Sigma\text{-}\mathsf{Red})^r$. In this case we obtain our assertion directly with the help of Lemma 38. □

To finish this subsection we only have to collect what we have proved so far. The embedding lemma, partial cut elimination and the asymmetric interpretation of H yield the following theorem.

Theorem 42. *If A is a Σ sentence that is provable in $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$, then there exists a natural number m such that Σ^1_1-AC_0 proves $A^{[m]}$.*

Since Δ^1_1-CA_0 and Σ^1_1-AC_0 are proof-theoretically equivalent to Peano arithmetic according to, for example, Barwise and Schlipf [3] and Feferman and Sieg [6], and since $\mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red})$ contains Δ^1_1-CA_0 according to Lemma 18, this theorem yields the following proof-theoretic equivalences.

Corollary 43. *We have the following proof-theoretic equivalences:*

$$\mathsf{PA} \;\equiv\; \Delta^1_1\text{-}\mathsf{CA}_0 \;\equiv\; \Sigma^1_1\text{-}\mathsf{AC}_0 \;\equiv\; \mathsf{BS}^0 + (\Sigma\text{-}\mathsf{Red}).$$

Bärtschi is also interested in uniform versions of the second order systems studied in this paper. For example, the uniform version UFP_0 is obtained by adding for every X-positive arithmetical formula $A[X, Y, x, y]$ with the indicated free variables a fresh functional $\mathcal{F}_A$. Then the uniform version of the fixed point axiom for the operator form $A[X, Y, x, y]$ is

$$\forall Y \forall x, y(x \in \mathcal{F}_A(Y, y) \leftrightarrow A[\mathcal{F}(Y, y), Y, x, y]).$$

Similarly for ATR_0. This leads to some interesting new relationships; it is work in preparation and will be presented in full details elsewhere.

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