Second order theories with ordinals and elementary comprehension

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Abstract

We study elementary second order extensions of the theory $\mathbf{ID_1}$ of non-iterated inductive definitions and the theory $\mathbf{PA_\Omega}$ of Peano arithmetic with ordinals. We determine the exact proof-theoretic strength of those extensions and their natural subsystems, and we relate them to subsystems of analysis with arithmetic comprehension plus Π^1_1 comprehension and bar induction without set parameters.

1 Introduction

This paper grew out of the need to set up a proper framework for the proof-theoretic analysis of some extension of elementary explicit type theory with non-constructive minimum operator $\mathbf{EET}(\mu)$, which are presented in Feferman and Jäger [6]. During this research work it became clear, more or less immediately, that certain second order theories with ordinals and elementary comprehension are tailored for this purpose.

However, it also turned out that second order theories with ordinals and elementary comprehension are interesting for their own sake and that the concentration on those theories needed for the treatment of elementary explicit type theory with unbounded minimum operator is too narrow. One obtains a much clearer picture by studying second order theories with ordinals and elementary comprehension in a broader context, and their careful analysis yields some relevant proof-theoretic information.

Following a rather general principle it is possible to assign so-called Gödel-Bernays extensions to arbitrary first order theories $\mathbf{Th}$: One simply reformulates $\mathbf{Th}$ in a second order language and adds further principles about the existence of second order objects. Of course the whole process is very sensitive to these set or class existence axioms and to the way in which the interplay of first and second order objects is regulated. In particular, it makes a difference in most cases whether induction principles are available for first or second order properties.

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In the following we study the Gödel-Bernays extensions which are generated from the theory $\mathbf{ID_1}$ of non-iterated inductive definitions and the theory $\mathbf{PA_\Omega}$ of Peano arithmetic with ordinals (cf. Jäger [10]) by elementary comprehension. This process yields two second order theories $\mathbf{EID_1}$ and $\mathbf{E\Omega}$ of proof-theoretic strength greater than $\mathbf{ID_1}$ and $\mathbf{PA_\Omega}$. In addition, we consider natural subsystems of $\mathbf{EID_1}$ and $\mathbf{E\Omega}$ which are obtained by restricting induction (on the natural numbers, fixed points and ordinals).

It is a matter of routine to obtain similar results for elementary Gödel-Bernays extensions of variants of Kripke-Platek set theory. But instead of going into this direction here, we apply our results about $\mathbf{EID_1}$, $\mathbf{E\Omega}$ and their subsystems to systems of second order arithmetic with arithmetic comprehension plus Π^1_1 comprehension and bar induction without set parameters. This gives the exact proof-theoretic analysis of some second order theories introduced in Feferman [3].

2 The syntax of second order theories with ordinals and elementary comprehension

All systems considered in this article will be based on the usual language $\mathbb{L}$ of second order arithmetic or on one of the languages L_Ω and $\mathbb{L}_\Omega$ with ordinals. $\mathbb{L}$ contains the following basic symbols:

1. Countably many number variables $(v, w, x, y, z, v_0, w_0, x_0, y_0, z_0, \ldots)$ and countably many set variables $(V, W, X, Y, Z, V_0, W_0, X_0, Y_0, Z_0, \ldots)$.

2. Symbols for all primitive-recursive functions and relations and a further unary relation symbols U which will have no specific interpretation.

3. The symbol $\sim$ for forming the complements of these relation symbols.

4. The symbols $\in$ and $\notin$ for the membership relation and the nonmembership relation between numbers and sets.

5. The propositional connectives $\vee$ and $\wedge$ and the quantifies $\exists$ and $\forall$.

As auxiliary symbols we have parentheses and commas; primitive-recursive functions and relations are often identified with the corresponding function and relation symbols. Observe that there is no propositional connective $\neg$ for negation.

The *number terms* $(a, b, c, a_0, b_0, c_0, \ldots)$ of $\mathbb{L}$ are defined as usual. The *positive literals* of $\mathbb{L}$ are all expressions $R(a_1, \ldots, a_n)$ so that R is a symbol for an n-ary primitive-recursive relation and all expressions $U(a)$; the *negative literals* of $\mathbb{L}$ are all expressions $\sim E$ so that E is a positive literal of $\mathbb{L}$. The *formulas* of $\mathbb{L}$ are generated as follows:

1. All literals of $\mathbb{L}$ and $(a \in X)$ as well as $(a \notin X)$ are $\mathbb{L}$ formulas.

2. If A and B are $\mathbb{L}$ formulas, then $(A \vee B)$ and $(A \wedge B)$ are $\mathbb{L}$ formulas.

3. If A is an $\mathbb{L}$ formula, then $\exists x A$, $\forall x A$, $\exists X A$ and $\forall X A$ are $\mathbb{L}$ formulas.

The *arithmetic* formulas are the $\mathbb{L}$ formulas which do not contain bound set variables; however, they may contain free set variables. L is defined to be the *first order part* of $\mathbb{L}$, i.e. L is the sublanguage of $\mathbb{L}$ which we obtain by omitting the relation symbol $\in$ and all set variables.

An arithmetic formula is said to be *X-positive* if it has no subformulas of the form $(a \notin X)$. We call X-positive arithmetic formulas which contain no other free variables than x and X *inductive operator forms* and let $\mathcal{A}(X, x)$ range over such forms.

Now we extend $\mathbb{L}$ to a new second order language $\mathbb{L}_\Omega$ by adding a new sort of *ordinal variables* $(\sigma, \tau, \eta, \xi, \sigma_0, \tau_0, \eta_0, \xi_0, \ldots)$, new binary relation symbols $<$ and $=$ for the less relation and the equality relation on ordinals and a binary relation symbol $P_\mathcal{A}$ for each inductive operator form $\mathcal{A}(X, x)$.

The *positive literals* of $\mathbb{L}_\Omega$ are the positive literals of $\mathbb{L}$ as well as the expressions $(\sigma < \tau)$, $(\sigma = \tau)$ and $P_\mathcal{A}(\sigma, a)$; the *negative literals* of $\mathbb{L}_\Omega$ are all expressions $\sim E$ so that E is a positive literal of $\mathbb{L}_\Omega$.

The *formulas* $(A, B, C, A_0, B_0, C_0, \ldots)$ of $\mathbb{L}_\Omega$ are inductively generated as follows:

1. Each literal of $\mathbb{L}_\Omega$ as well as $(a \in X)$ and $(a \notin X)$ are $\mathbb{L}_\Omega$ formulas.

2. If A and B are $\mathbb{L}_\Omega$ formulas, then $(A \vee B)$ and $(A \wedge B)$ are $\mathbb{L}_\Omega$ formulas.

3. If A is an $\mathbb{L}_\Omega$ formula, then $\exists x A$, $\forall x A$, $\exists X A$ and $\forall X A$ are $\mathbb{L}_\Omega$ formulas.

4. If A is an $\mathbb{L}_\Omega$ formula, then $(\exists \xi < \sigma)A$, $(\forall \xi < \sigma)A$, $\exists \xi A$ and $\forall \xi A$ are $\mathbb{L}_\Omega$ formulas.

The *elementary* $\mathbb{L}_\Omega$ formulas are the $\mathbb{L}_\Omega$ formulas which do not contain bound set variables; as in the case of arithmetic formulas, however, they may contain free set variables. Similar to above, L_Ω is defined to be the *first order part* of $\mathbb{L}_\Omega$, i.e. L_Ω is the sublanguage of $\mathbb{L}_\Omega$ which we obtain by omitting the relation symbol $\in$ and all set variables.

The *negation* $\neg A$ of an arbitrary $\mathbb{L}_\Omega$ formula A is inductively defined as usual by making use of the law of double negation and de Morgan's laws. This means in particular that $\neg A$ is $\sim A$ if A is a positive literal and $\neg A$ is B if A is $\sim B$ for some positive literal B of $\mathbb{L}_\Omega$. Now we set $(A \to B) := (\neg A \vee B)$, and we define $(A \leftrightarrow B)$ as usual. Additional abbreviations are:

$$P^\sigma_\mathcal{A}(a) := P_\mathcal{A}(\sigma, a), \qquad P^{<\sigma}_\mathcal{A}(a) := (\exists \xi < \sigma) P^\xi_\mathcal{A}(a), \qquad P_\mathcal{A}(a) := \exists \xi P^\xi_\mathcal{A}(a).$$

Quantifiers of the form $(Q\xi < \sigma)$ are called bounded ordinal quantifiers. For every $\mathbb{L}_\Omega$ formula A we write A^σ to denote the $\mathbb{L}_\Omega$ formula which is obtained from A

by replacing all unbounded ordinal quantifiers $Q\xi$ in A by the bounded ordinal quantifier $(Q\xi < \sigma)$.

In the following we introduce several classes of L_Ω formulas which will be important for the ordinal part of the theories considered later. The Δ_0^Ω formulas are the L_Ω formulas which do not contain unbounded ordinal quantifiers, and the Σ^Ω $[\Pi^\Omega]$ formulas are the L_Ω formulas which do not contain unbounded universal [existential] ordinal quantifiers. The collection of all Σ^Ω and Π^Ω formulas is denoted by ∇^Ω. Please keep in mind that all ∇^Ω formulas do not contain set variables. Obviously a formula A is a Σ^Ω formula if and only if $\neg A$ is a Π^Ω formula. It is also clear that each arithmetic formula without set variables is a Δ_0^Ω formula.

Induction principles will play a major role, and we distinguish between induction on the natural numbers and induction on the ordinals. Let $\mathcal{F}$ be a collection of $\mathbb{L}_\Omega$ formulas. Then $\mathcal{F}$ induction on the natural numbers consists of all formulas

$$(\mathcal{F}\text{-}\mathrm{I}_\mathrm{N}) \qquad A(0) \,\wedge\, \forall x(A(x) \to A(x')) \;\to\; \forall x A(x)$$

so that A belongs to the collection $\mathcal{F}$. On the other hand, $\mathcal{F}$ induction on the ordinals comprises all formulas

$$(\mathcal{F}\text{-}\mathrm{I}_\Omega) \qquad \forall \xi[(\forall \eta < \xi)A(\eta) \to A(\xi)] \;\to\; \forall \xi A(\xi)$$

where A is in $\mathcal{F}$. In the following we consider the schemes (Δ_0^Ω-I_N), (Δ_0^Ω-I_Ω), (El-I_N), (El-I_Ω), ($\mathbb{L}_\Omega$-I_N) and ($\mathbb{L}_\Omega$-I_Ω) in which induction is available for all Δ_0^Ω formulas, elementary $\mathbb{L}_\Omega$ formulas and arbitrary $\mathbb{L}_\Omega$ formulas, respectively.

By an $\mathbb{L}_\Omega$ theory we mean a collection of $\mathbb{L}_\Omega$ formulas. Further, if **Th** is an $\mathbb{L}_\Omega$ theory, then we write $\mathbf{Th} \vdash A$ if the $\mathbb{L}_\Omega$ formula A is derivable from **Th** by the usual axioms and rules of the many sorted predicate calculus with equality.

Let **Th** be an $\mathbb{L}_\Omega$ theory which comprises the induction schemes ($\mathbb{L}_\Omega$-I_N) and ($\mathbb{L}_\Omega$-I_Ω). Then one can introduce the following interesting subtheories of **Th**:

(i) $\widetilde{\mathbf{Th}}$ is the subsystem which is obtained from **Th** by restricting induction on the ordinals to elementary $\mathbb{L}_\Omega$ formulas; $\widehat{\mathbf{Th}}$ on the other hand results from **Th** by restricting induction on the ordinals to Δ_0^Ω formulas.

(ii) W-**Th** is the subsystem which is obtained from **Th** by restricting induction on the natural numbers to elementary $\mathbb{L}_\Omega$ formulas; R-**Th** on the other hand results from **Th** by restricting induction on the natural numbers to Δ_0^Ω formulas.

It is also permitted to combine forms of restriction (i) and (ii), and the resulting theories are denoted in the obvious way (e.g. W-$\widetilde{\mathbf{Th}}$).

Now we are ready to introduce several second order theories with elementary comprehension and ordinals. The strongest of those is the theory $\mathbf{E\Omega}$ and contains the following groups of non-logical axioms.

I. Number-theoretic axioms. These comprise the axioms of Peano arithmetic **PA** with exception of complete induction on the natural numbers.

II. Inductive operator axioms. For all inductive operator forms $\mathcal{A}(X,x)$:

$$P^{\sigma}_{\mathcal{A}}(a) \leftrightarrow \mathcal{A}(P^{<\sigma}_{\mathcal{A}}, a).$$

III. Σ^{Ω} reflection axioms. For every Σ^{Ω} formula A:

$$(\Sigma^{\Omega}\text{-Ref}) \qquad A \rightarrow \exists \xi A^{\xi}.$$

IV. Linearity axioms

$$(\mathrm{LO}) \qquad \sigma \not< \sigma \ \wedge \ (\sigma < \tau \wedge \tau < \eta \rightarrow \sigma < \eta) \ \wedge \ (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

V. Elementary comprehension. For every elementary formula A of $\mathbb{L}_{\Omega}$:

$$(\mathrm{ECA}) \qquad \exists X \forall x (x \in X \leftrightarrow A(x)).$$

VI. Formula induction on the natural numbers and the ordinals. The above introduced schemes ($\mathbb{L}_{\Omega}$-I_{N}) and ($\mathbb{L}_{\Omega}$-I_{Ω}).

In this paper we carry through the proof-theoretic analysis of the following theories: R-$\widehat{\mathbf{E\Omega}}$, W-$\widehat{\mathbf{E\Omega}}$, W-$\widetilde{\mathbf{E\Omega}}$, $\widehat{\mathbf{E\Omega}}$, $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$. We determine their proof-theoretic ordinals and relate them to well-known (sub)systems of first and second order arithmetic.

Systems in which induction on the ordinals is allowed for a larger class of formulas than induction on the natural numbers are not considered since we expect that forms of I_{Ω} without counterpart in I_{N} cannot be exploited properly. In a different context Cantini also considered theories with induction on the natural numbers and the ordinals and made exactly the same observation (cf. e.g. [2]).

There exist several (more or less equivalent) methods to measure the proof-theoretic strength of formal systems, and one standard way is to assign a proof-theoretic ordinal to each theory. Given a binary primitive recursive relation $\prec$ and an arbitrary formula $A(x)$, we set as usual:

$$\begin{aligned} Prog(\prec, A) &:= \forall x(\forall y(y \prec x \rightarrow A(y)) \rightarrow A(x)), \\ TI(\prec, A) &:= Prog(\prec, A) \rightarrow \forall x A(x). \end{aligned}$$

Definition 1 Let **Th** be a theory which is formulated in a language containing the first order part of $\mathbb{L}$.

1. We say that an ordinal α is *provable* in **Th**, if there exists a primitive recursive wellordering $\prec$ of ordertype α so that $\mathbf{Th} \vdash TI(\prec, U)$.

2. The *proof-theoretic ordinal* of **Th**, denoted by $|\mathbf{Th}|$, is the least ordinal which is not provable in **Th**.

In this definition we make use of the anonymous relation symbol U so that our formulation corresponds to the usual Π^1_1 definition of proof-theoretic ordinal (cf. e.g. Schütte [15]).

3 An ordinal notation system

In this section we briefly mention an ordinal notation system which is adequate for the treatment of all the theories considered in this article, and we introduce some conventions concerning the use of ordinal terms in this paper.

The notation system used in the following is the notation system $(T, <)$ which is developed in all details in Sections 23 and 24 of Pohlers [13]. Hence, ordinal terms in T are composed from 0, Ω (the first regular $> \omega$), $+$, φ (the Veblen functions), and ψ. In the sequel we presuppose that the reader is familiar with the system $(T, <)$ of [13] or a similar notation system. In particular, $K(\alpha)$ denotes the *set of components* and $SC(\alpha)$ the set of *strongly critical subterms* of an ordinal term $\alpha \in T$. We also write $M < \alpha$ if M is a subset of T so that $\beta < \alpha$ for all $\beta \in M$.

Furthermore, we have a *collapsing function* D defined from ψ with the property $D\alpha \in T$ for all $\alpha \in T$. Based on D one defines as usual the binary *essentially less relation* $\ll$ on T by setting

$$\alpha \ll \beta \quad :\Longleftrightarrow \quad \alpha < \beta \ \wedge \ D\alpha < D\beta$$

for all $\alpha, \beta \in T$. Then one easily verifies the fundamental characterization property of $\ll$ given by

$$\alpha \ll \beta \quad \Longleftrightarrow \quad \alpha < \beta \ \wedge \ SC(\alpha) \cap \Omega < D\beta.$$

Notice that for $\alpha, \beta \in T \cap \Omega$ one has $\alpha \ll \beta$ if and only if $\alpha < \beta$. For important closure properties of the $\ll$ relation the reader is referred to [13]. In the sequel we often write $\alpha \underline{\ll} \beta$ for $\alpha \ll \beta \vee \alpha = \beta$, and if $M \subset T$, then $M \underline{\ll} \alpha$ means that $\beta \underline{\ll} \alpha$ for all $\beta \in M$.

Finally, we use the following definition, which corresponds to a similar definition of [13]. The addition of k in clause (1) is not significant and postulated for technical reasons only (cf. e.g. the proof of Proposition 22).

Definition 2 Assume that $n \geq 1$ and $\alpha \in T$, and let f be an n-ary function from $T \cap \Omega$ to T. Then we write $f \ll \alpha$, if the following two conditions are satisfied:

(1) For all $\beta_1, \ldots, \beta_n \in T \cap \Omega$ and all $k < \omega$:

$$f(\beta_1, \ldots, \beta_n) + k < \alpha.$$

(2) If $\beta \in T$ and $\alpha \underline{\ll} \beta$, then for all $\gamma_1, \ldots, \gamma_n \in T \cap \Omega$:

$$\gamma_1, \ldots, \gamma_n \ll \beta \quad \Longrightarrow \quad f(\gamma_1, \ldots, \gamma_n) \ll \beta.$$

Let us finish this section by adopting some conventions concerning the encoding of $(T, <)$ into arithmetic, which will be used in the wellordering proofs below. In the following we identify T, $<$, K, SC, 0, Ω, $+$, φ and ψ with their primitive recursive arithmetizations. In particular, we freely use expressions like $a \in T$ and $a \in K(b)$ instead of their obvious formalization in arithmetic. Moreover, we assume that the ordinal term $0 \in T$ is coded by the numeral 0.

4 The proof-theoretic strength of $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$, $\mathrm{W}\text{-}\widehat{\mathbf{E\Omega}}$ and $\mathrm{W}\text{-}\widetilde{\mathbf{E\Omega}}$

The analysis of the theories $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$, $\mathrm{W}\text{-}\widehat{\mathbf{E\Omega}}$ and $\mathrm{W}\text{-}\widetilde{\mathbf{E\Omega}}$ is given by showing that they are conservative extensions of suitable fixed point theories over Peano arithmetic with ordinals. These first order theories $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$, $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ and $\mathbf{PA}_{\mathbf{\Omega}}$ have been introduced and studied in Jäger [10].

In our present formalism, $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$ is the theory which is formulated in the first order language L_Ω and contains all axioms of $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$ with the exception of elementary comprehension; i.e. $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$ is the restriction of $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$ to the first order language L_Ω. $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ is the extension of $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$ by the scheme of complete induction on the natural numbers for all L_Ω formulas. $\mathbf{PA}_{\mathbf{\Omega}}$, finally, results from $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ by adding the scheme of induction on the ordinals for all L_Ω formulas.

It is shown in [10] that these theories are closely related to Peano arithmetic $\mathbf{PA}$, to the fixed point theory $\widehat{\mathbf{ID}}_{\mathbf{1}}$ of Feferman [4] and to the well-known theory $\mathbf{ID_1}$ of non-iterated inductive definitions (cf. e.g. [1, 3]). More precisely, the exact connections between these theories are as follows.

Theorem 3 (Conservative extensions I) *We have:*

1. $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$ *is a conservative extension of* $\mathbf{PA}$ *with respect to* L,
2. $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ *is a conservative extension of* $\widehat{\mathbf{ID}}_{\mathbf{1}}$ *with respect to* L,
3. $\mathbf{PA}_{\mathbf{\Omega}}$ *is a conservative extension of* $\mathbf{ID_1}$ *with respect to* L.

It follows a simple model-theoretic observation which makes it very easy to relate the second order theories $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$, $\mathrm{W}\text{-}\widehat{\mathbf{E\Omega}}$ and $\mathrm{W}\text{-}\widetilde{\mathbf{E\Omega}}$ and the first order theories $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$, $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ and $\mathbf{PA}_{\mathbf{\Omega}}$. Its proof is standard. One simply expands the given first order structure $\mathcal{M}$ to $\mathcal{M}^*$ by interpreting the set variables to range over all subsets of the universe $|\mathcal{M}|$ of $\mathcal{M}$ which are definable by formulas of L_Ω with parameters from $|\mathcal{M}|$.

Lemma 4 *Let $\mathcal{M}$ be a model of $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$. Then there exists a model $\mathcal{M}^*$ of* $\mathrm{R}\text{-}\widehat{\mathbf{E\Omega}}$ *which has the following properties:*

1. *$\mathcal{M}^*$ is an extension of $\mathcal{M}$ in the sense that we have for all sentences A of* L_Ω*:*
$$\mathcal{M}^* \models A \quad \Longleftrightarrow \quad \mathcal{M} \models A.$$
2. *If $\mathcal{M}$ is a model of* $(\mathrm{L}_\Omega\text{-}\mathrm{I}_\mathrm{N})$*, then $\mathcal{M}^*$ is a model of* $(\mathrm{El}\text{-}\mathrm{I}_\mathrm{N})$.
3. *If $\mathcal{M}$ is a model of* $(\mathrm{L}_\Omega\text{-}\mathrm{I}_\Omega)$*, then $\mathcal{M}^*$ is a model of* $(\mathrm{El}\text{-}\mathrm{I}_\Omega)$.

This lemma is now immediately applied to obtain the following result which characterizes R-$\widehat{\mathbf{E\Omega}}$ and its extensions by elementary induction in terms of the theories $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$, $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ and $\mathbf{PA}_{\mathbf{\Omega}}$. A proof-theoretic proof of the following theorem would be possible as well but will not be given here.

Theorem 5 (Conservative extensions II) *We have:*

1. R-$\widehat{\mathbf{E\Omega}}$ *is a conservative extension of* $\mathbf{PA}^{\mathbf{r}}_{\mathbf{\Omega}}$ *with respect to* L_Ω*,*
2. W-$\widehat{\mathbf{E\Omega}}$ *is a conservative extension of* $\mathbf{PA}^{\mathbf{w}}_{\mathbf{\Omega}}$ *with respect to* L_Ω*,*
3. W-$\widetilde{\mathbf{E\Omega}}$ *is a conservative extension of* $\mathbf{PA}_{\mathbf{\Omega}}$ *with respect to* L_Ω*.*

By combining Theorem 3 and Theorem 5, and by making use of standard results about proof-theoretic ordinals, we obtain the following result about the proof-theoretic strength of the theories R-$\widehat{\mathbf{E\Omega}}$, W-$\widehat{\mathbf{E\Omega}}$ and W-$\widetilde{\mathbf{E\Omega}}$.

Corollary 6 *We have the following proof-theoretic equivalences:*

$$\text{R-}\widehat{\mathbf{E\Omega}} \equiv \mathbf{PA}, \qquad \text{W-}\widehat{\mathbf{E\Omega}} \equiv \widehat{\mathbf{ID}}_{\mathbf{1}}, \qquad \text{W-}\widetilde{\mathbf{E\Omega}} \equiv \mathbf{ID}_{\mathbf{1}}.$$

In addition, the proof-theoretic ordinals of the three theories R-$\widehat{\mathbf{E\Omega}}$*,* W-$\widehat{\mathbf{E\Omega}}$ *and* W-$\widetilde{\mathbf{E\Omega}}$ *are given by:*

$$\mid \text{R-}\widehat{\mathbf{E\Omega}} \mid = \varepsilon_0, \qquad \mid \text{W-}\widehat{\mathbf{E\Omega}} \mid = \varphi\varepsilon_0 0, \qquad \mid \text{W-}\widetilde{\mathbf{E\Omega}} \mid = \psi\varepsilon_{\Omega+1}.$$

5 Fixed point theories with elementary comprehension

Before we continue with the proof-theoretic analysis of second order theories with ordinals and elementary comprehension, we turn to fixed point theories with elementary comprehension. These systems are interesting for their own sake but will be used here mainly for establishing proof-theoretic lower bounds. The basic idea is to lift the fixed point theory $\widehat{\mathbf{ID}}_{\mathbf{1}}$ and the theory $\mathbf{ID}_{\mathbf{1}}$ of one inductive definition, which have both been mentioned before, to the language of second order arithmetic and to add the possibility of elementary comprehension.

Fixed point theories with elementary comprehension are formulated in the second order language $\mathbb{L}_{\mathrm{FP}}$ which extends $\mathbb{L}$ by adding fixed point constants $\mathcal{P}_{\mathcal{A}}$ for all inductive operator forms $\mathcal{A}(X, x)$. The *elementary* formulas of $\mathbb{L}_{\mathrm{FP}}$ are the $\mathbb{L}_{\mathrm{FP}}$ formulas which do not contain bound set variables; they may contain, however, free set variables and fixed point constants. $\mathcal{E}_{\mathrm{FP}}$ denotes the collection of all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas.

The theories $\widehat{\mathbf{EID}}_{\mathbf{1}}$, $\widetilde{\mathbf{EID}}_{\mathbf{1}}$ and $\mathbf{EID}_{\mathbf{1}}$ are formulated in the language $\mathbb{L}_{\mathrm{FP}}$ and comprise the axioms of Peano arithmetic $\mathbf{PA}$ with the scheme of complete induction on the natural numbers for all $\mathbb{L}_{\mathrm{FP}}$ formulas. In addition, there is the scheme of elementary comprehension

$$(\mathcal{E}_{\mathrm{FP}}\text{-CA}) \qquad\qquad \exists X \forall x(x \in X \leftrightarrow A(x))$$

for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas A. The further axioms of our three theories refer to the fixed point constants.

In $\widehat{\mathbf{EID}}_{\mathbf{1}}$ it is simply stated that the relation symbols $\mathcal{P}_{\mathcal{A}}$ represent fixed points of the inductive operator forms $\mathcal{A}(X, x)$; i.e. we have the fixed point axioms

$$(\text{FP-}\mathcal{P}_{\mathcal{A}}) \qquad\qquad \forall x(\mathcal{A}(\mathcal{P}_{\mathcal{A}}, x) \leftrightarrow \mathcal{P}_{\mathcal{A}}(x))$$

for all inductive operator forms $\mathcal{A}(X, x)$. In $\mathbf{EID_1}$ these fixed point axioms are replaced by the following closure and induction axioms:

$$(\mathcal{P}_{\mathcal{A}}.1) \qquad\qquad \forall x(\mathcal{A}(\mathcal{P}_{\mathcal{A}}, x) \rightarrow \mathcal{P}_{\mathcal{A}}(x)),$$

$$(\mathcal{P}_{\mathcal{A}}.2) \qquad\qquad \forall x(\mathcal{A}(B, x) \rightarrow B(x)) \;\rightarrow\; \forall x(\mathcal{P}_{\mathcal{A}}(x) \rightarrow B(x))$$

for all inductive operator forms $\mathcal{A}(X, x)$ and arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas B. $\widetilde{\mathbf{EID}}_{\mathbf{1}}$, finally, is a theory lying between $\widehat{\mathbf{EID}}_{\mathbf{1}}$ and $\mathbf{EID_1}$. It results from $\mathbf{EID_1}$ by restricting the induction scheme $(\mathcal{P}_{\mathcal{A}}.2)$ to elementary $\mathbb{L}_{\mathrm{FP}}$ formulas B.

It is obvious that the fixed point properties $(\text{FP-}\mathcal{P}_{\mathcal{A}})$ are provable in $\widetilde{\mathbf{EID}}_{\mathbf{1}}$ and in $\mathbf{EID_1}$. Hence, $\mathbf{EID_1}$ expresses that $\mathcal{P}_{\mathcal{A}}$ is a fixed point of the inductive operator form $\mathcal{A}(X, x)$ which is minimal with respect to all definable sets; in $\widetilde{\mathbf{EID}}_{\mathbf{1}}$ it is only minimal with respect to all elementary sets, and in $\widehat{\mathbf{EID}}_{\mathbf{1}}$ it may be any fixed point.

Remark 7 Let $\widehat{\mathbf{EID}}_{\mathbf{1}}\!\upharpoonright$ and $\widetilde{\mathbf{EID}}_{\mathbf{1}}\!\upharpoonright$ be the subsystems of $\widehat{\mathbf{EID}}_{\mathbf{1}}$ and $\widetilde{\mathbf{EID}}_{\mathbf{1}}$ in which the scheme of complete induction on the natural numbers is restricted to elementary $\mathbb{L}_{\mathrm{FP}}$ formulas. Then an easy model-theoretic argument shows that $\widehat{\mathbf{EID}}_{\mathbf{1}}\!\upharpoonright$ is a conservative extension of $\widehat{\mathbf{ID}}_{\mathbf{1}}$ and $\widetilde{\mathbf{EID}}_{\mathbf{1}}\!\upharpoonright$ a conservative extension of $\mathbf{ID_1}$.

The theories $\widetilde{\mathbf{EID}}_{\mathbf{1}}$ and $\mathbf{EID_1}$ are closely related to systems of second order arithmetic based on certain axioms which are presented in detail for example in Feferman [3]. As usual we write $\mathbf{ACA}$ for the theory, formulated in $\mathbb{L}$, which contains $\mathbf{PA}$, the scheme of complete induction on the natural numbers for all $\mathbb{L}$ formulas plus the scheme of *arithmetic comprehension* for all arithmetic $\mathbb{L}$ formulas A:

$$(\Pi^0_\infty\text{-CA}) \qquad\qquad \exists X \forall x(x \in X \leftrightarrow A(x)).$$

The theory $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$ is the extension of $\mathbf{ACA}$ in which Π^1_1 *comprehension without set parameters* is permitted as well. More precisely, this means that in addition to the axioms of $\mathbf{ACA}$ we have

$$(\Pi^1_1\text{-CA})^- \qquad\qquad \exists X \forall x(x \in X \leftrightarrow \forall Y B(Y, x))$$

for all arithmetic formulas B so that $\forall Y B(Y, x)$ does not contain free set variables. This theory, which played some role in the discussion of theories for non-iterated inductive definitions (cf. e.g. [1, 3]), can be shown to be equivalent to $\widehat{\mathbf{EID}}_{\mathbf{1}}$.

Theorem 8 *We have for all $\mathbb{L}$ formulas A:*

$$\widetilde{\mathbf{EID_1}} \vdash A \quad\Longleftrightarrow\quad \mathbf{ACA} + (\Pi^1_1\text{-CA})^- \vdash A.$$

PROOF. The direction from left to right is obvious. Given an inductive operator form $\mathcal{A}(X,x)$, one interprets the formula $\mathcal{P}_\mathcal{A}(v)$ of $\mathbb{L}_{\mathrm{FP}}$ as the following Π^1_1 formula $\mathcal{I}_\mathcal{A}(v)$ of $\mathbb{L}$; obviously $\mathcal{I}_\mathcal{A}(v)$ does not contain set-parameters.

$$\mathcal{I}_\mathcal{A}(v) \;:=\; \forall X[\forall x(\mathcal{A}(X,x) \to x \in X) \to v \in X].$$

Then the translation of $(\mathcal{P}_\mathcal{A}.1)$ is logically valid. In order to deal with the other axioms of $\widetilde{\mathbf{EID_1}}$, one only has to observe that the translations of all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas define sets in $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$, since parameter-free Π^1_1 comprehension and arithmetic comprehension with parameters are available there.

The direction from right to left is more or less obvious. One first observes that one obtains a theory equivalent to $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$ if Π^1_1 comprehension is restricted to Π^1_1 formulas without set and number parameters. In a second step one proceeds similar to Feferman [5] and makes use of the following consequence of the Π^1_1 normal form theorem: Let $B(v)$ be a Π^1_1 formula with v as its only free variable. Then there exist an inductive operator form $\mathcal{A}(X,x)$ and a primitive recursive function f so that

$$\widetilde{\mathbf{EID_1}} \vdash B(v) \leftrightarrow \mathcal{P}_\mathcal{A}(f(v)).$$

Since the relations $\mathcal{P}_\mathcal{A}$ define sets in $\widetilde{\mathbf{EID_1}}$, parameter-free Π^1_1 comprehension is provable there; arithmetic comprehension is taken care of in $\widetilde{\mathbf{EID_1}}$ by $(\mathcal{E}_{\mathrm{FP}}\text{-CA})$. Hence $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$ is contained in $\widetilde{\mathbf{EID_1}}$. ☐

We obtain a natural system of second order arithmetic which is equivalent to $\mathbf{EID_1}$ if we add to $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$ the scheme of *primitive recursive bar induction without set parameters*. Let R be a ternary primitive recursive relation and let $A(x)$ be an arbitrary $\mathbb{L}$ formula. Given a number parameter a, the progressiveness of R_a, the *transfinite induction* along R_a with respect to $A(x)$ and the *wellfoundedness* of R_a are expressed by the following formulas:

$$\begin{aligned} \mathit{Prog}(R_a, A) \;&:=\; \forall x(\forall y(R(a,y,x) \to A(y)) \to A(x)), \\ \mathit{TI}(R_a, A) \;&:=\; \mathit{Prog}(R_a, A) \to \forall x A(x), \\ \mathit{WF}(R_a) \;&:=\; \forall X\, \mathit{TI}(R_a, X). \end{aligned}$$

Now the scheme of primitive recursive bar induction without set parameters consists of all formulas of the form

$$(\mathrm{BI_{pr}})^- \qquad\qquad \mathit{WF}(R_a) \to \mathit{TI}(R_a, A)$$

so that R is a ternary primitive recursive relation and $A(x)$ an arbitrary $\mathbb{L}$ formula. The presence of the number parameter a is crucial; see Remark 10 below.

Theorem 9 *We have for all $\mathbb{L}$ formulas A:*

$$\mathbf{EID_1} \vdash A \quad \Longleftrightarrow \quad \mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^- \vdash A.$$

Proof. In order to show the direction from left to right we proceed as in the proof of Theorem 8 and translate $\mathcal{P}_\mathcal{A}(v)$ by $\mathcal{I}_\mathcal{A}(v)$ for all inductive operator forms $\mathcal{A}(X,x)$. As before, this settles the axioms $(\mathcal{P}_\mathcal{A}.1)$ and $(\mathcal{E}_{\mathrm{FP}}\text{-CA})$. To prove the translation of the induction principles $(\mathcal{P}_\mathcal{A}.2)$ we only have to employ the result of Feferman [3] that

$$\mathbf{ACA} + (\mathrm{BI_{pr}})^- \vdash \forall X B(X) \to B(C)$$

for all arithmetic formulas $B(X)$ and arbitrary $\mathbb{L}$ formulas C, provided that $\forall X B(X)$ does not contain set parameters. This is the case for the formula $\mathcal{I}_\mathcal{A}(v)$.

Now we show the direction from right to left. In the following argument we presuppose some standard primitive recursive coding machinery: $\langle a_0, \ldots, a_{n-1}\rangle$ is the sequence number associated to the numbers $a_0, \ldots, a_{n-1}$ with the related projections $(\cdot)_i$ so that $(\langle a_0, \ldots, a_{n-1}\rangle)_i = a_i$ for $1 \le i < n$. $Seq_n(v)$ means that v is a sequence number of length n. Given the symbol R for a ternary primitive recursive relation we set

$$\mathcal{A}(X,x) \;:=\; Seq_2(x) \;\wedge\; \forall y(Seq_2(y) \wedge (y)_0 = (x)_0 \wedge R((x)_0, (y)_1, (x)_1) \;\to\; x \in X).$$

$\mathcal{A}(X,x)$ is an inductive operator form and, therefore, there exists a relation symbol $\mathcal{P}_\mathcal{A}$ so that the axioms $(\mathcal{P}_\mathcal{A}.1)$ and $(\mathcal{P}_\mathcal{A}.2)$ are satisfied in $\mathbf{EID_1}$. It is easy to see that

$$\mathbf{EID_1} \vdash \; WF(R_a) \to \forall x \mathcal{P}_\mathcal{A}(\langle a, x\rangle). \tag{1}$$

Further, to every $\mathbb{L}$ formula $B(v)$ and number parameter a we associate the formula $B_a(v)$ which is defined as follows:

$$B_a(v) \;:=\; Seq_2(v) \;\wedge\; [(v)_0 = a \to B((v)_1)].$$

Some trivial transformations then imply that

$$\mathbf{EID_1} \vdash \forall x[\forall y(R(a,y,x) \to B(y)) \to B(x)] \;\to\; \forall x[\mathcal{A}(B_a, x) \to B_a(x)]. \tag{2}$$

In view of (1) and (2), the definition of $B_a(v)$ and by applying $(\mathcal{P}_\mathcal{A}.2)$ we can conclude that

$$\mathbf{EID_1} \vdash \; WF(R_a) \to \; TI(R_a, B).$$

Hence, primitive recursive bar induction without set parameters is provable in $\mathbf{EID_1}$ and, therefore, $\mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^-$ is contained in $\mathbf{EID_1}$. □

Remark 10 Our form of primitive recursive bar induction without set parameters is formulated for a ternary primitive recursive relation in order to keep the notation as simple as possible. It does not change the strength of the theory if we allow arbitrary primitive recursive relations or even arithmetic relations with number but without set parameters. By a result of Rathjen [14] the presence of number parameters is important: He showed that $\mathbf{ACA}$ plus primitive recursive bar induction without number and set parameters is significantly weaker than $\mathbf{ACA} + (\mathrm{BI_{pr}})^-$.

We end this section with the straightforward observation that the theories $\widehat{\mathbf{EID_1}}$, $\widetilde{\mathbf{EID_1}}$ and $\mathbf{EID_1}$ are contained in $\widehat{\mathbf{E\Omega}}$, $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$, respectively. For each relation symbol $\mathcal{P}_\mathcal{A}$ of $\mathbb{L}_{\mathrm{FP}}$, translate the $\mathbb{L}_{\mathrm{FP}}$ formula $\mathcal{P}_\mathcal{A}(v)$ by the Σ^Ω formula $P_\mathcal{A}(v)$ of $\mathbb{L}_\Omega$. This determines a translation of $\mathbb{L}_{\mathrm{FP}}$ formulas A in $\mathbb{L}_\Omega$ formulas A° which leaves $\mathbb{L}$ unchanged and interprets elementary $\mathbb{L}_{\mathrm{FP}}$ formulas as elementary $\mathbb{L}_\Omega$ formulas.

Lemma 11 *We have for all $\mathbb{L}_{\mathrm{FP}}$ formulas A:*

1. $\widehat{\mathbf{EID_1}} \vdash A \quad \Longrightarrow \quad \widehat{\mathbf{E\Omega}} \vdash A^\circ$,
2. $\widetilde{\mathbf{EID_1}} \vdash A \quad \Longrightarrow \quad \widetilde{\mathbf{E\Omega}} \vdash A^\circ$,
3. $\mathbf{EID_1} \vdash A \quad \Longrightarrow \quad \mathbf{E\Omega} \vdash A^\circ$.

The proof of this lemma is standard. From the inductive operator and Σ^Ω reflection axioms we deduce that each $P_\mathcal{A}$ describes a fixed point of the inductive operator form $\mathcal{A}(X,x)$; the induction principles of $\widehat{\mathbf{EID_1}}$, $\widetilde{\mathbf{EID_1}}$ and $\mathbf{EID_1}$ go over into the induction principles available in $\widehat{\mathbf{E\Omega}}$, $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$.

6 Three semiformal systems

The purpose of this section is to introduce three semiformal systems $\mathbf{E\Omega_1^*}$, $\mathbf{E\Omega_2^*}$ and $\mathbf{Z\Omega}$, which will be used in the following sections for establishing the upper bounds of $\widehat{\mathbf{E\Omega}}$, $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$. They combine aspects of Schütte's $\mathbf{RA}^*$ with extensions of infinitary systems for the treatment of $\mathbf{ID_1}$. The application of methods of predicative proof theory and (the first bits) of impredicative proof theory is a matter of routine.

6.1 The systems $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$

The systems $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$ as well as the system $\mathbf{Z\Omega}$ of the next subsection are based on the language $\mathbb{L}_\infty$ which extends $\mathbb{L}_\Omega$ by adding constants $\bar{\alpha}$ for all ordinals $\alpha \in T \cap \Omega$ and by permitting the explicit formation of set terms. The *ordinal terms* $(\theta, \theta_0, \theta_1, \ldots)$ of $\mathbb{L}_\infty$ are the ordinal variables and the ordinal constants of $\mathbb{L}_\infty$. The literals of $\mathbb{L}_\infty$ are the literals of $\mathbb{L}_\Omega$ plus all expressions which result from the literals of $\mathbb{L}_\Omega$ by replacing some ordinal variables by ordinal constants. To simplify the notation we often write $A(\alpha)$ instead of $A(\bar{\alpha})$ if α is an element of $T \cap \Omega$.

The *set terms* $(S, T, S_0, T_0, \ldots)$ and *elementary formulas* of $\mathbb{L}_\infty$ are introduced by a simultaneous inductive definition:

1. Each set variable is a set term.
2. If A is an elementary formula of $\mathbb{L}_\infty$, then $\{x : A(x)\}$ is a set term of $\mathbb{L}_\infty$.
3. Every literal of $\mathbb{L}_\infty$ is an elementary formula of $\mathbb{L}_\infty$.

4. If a is a number term and S a set term of $\mathbb{L}_\infty$, then $(a \in S)$ and $(a \notin S)$ are elementary formulas of $\mathbb{L}_\infty$.

5. If A and B are elementary formulas of $\mathbb{L}_\infty$, then $(A \vee B)$ and $(A \wedge B)$ are elementary formulas of $\mathbb{L}_\infty$.

6. If A is an elementary formula of $\mathbb{L}_\infty$, then $\exists x A$, $\forall x A$, $\exists \xi A$, $\forall \xi A$, $(\exists \xi < \theta) A$ and $(\forall \xi < \theta) A$ are elementary formulas of $\mathbb{L}_\infty$.

Finally, the *formulas* of $\mathbb{L}_\infty$ are generated from the elementary formulas of $\mathbb{L}_\infty$ as follows:

1. Every elementary formula of $\mathbb{L}_\infty$ is an $\mathbb{L}_\infty$ formula.

2. If A and B are $\mathbb{L}_\infty$ formulas, then $(A \vee B)$ and $(A \wedge B)$ are $\mathbb{L}_\infty$ formulas.

3. If A is an $\mathbb{L}_\infty$ formula, then $\exists x A$, $\forall x A$, $\exists \xi A$, $\forall \xi A$, $(\exists \xi < \theta) A$, $(\forall \xi < \theta) A$, $\exists X A$ and $\forall X A$ are $\mathbb{L}_\infty$ formulas.

$\mathbb{L}_\infty$ formulas in which the symbols $\in$ and $\notin$ do not occur are called first order formulas of $\mathbb{L}_\infty$. The Δ_0^Ω formulas of $\mathbb{L}_\infty$ are the first order formulas of $\mathbb{L}_\infty$ which do not contain unbounded ordinal quantifiers. The Σ^Ω, Π^Ω and ∇^Ω formulas of $\mathbb{L}_\infty$ are defined in an analogous way.

Two $\mathbb{L}_\infty$ formulas A and B are called *numerically equivalent* if they differ in closed number terms with identical value only. Furthermore, a closed literal of $\mathbb{L}_\infty$ is called *primitive* if it is not of the form $U(a)$, $\sim U(a)$, $P^\alpha_{\mathcal{A}}(a)$ or $\sim P^\alpha_{\mathcal{A}}(a)$. Obviously, every primitive literal of $\mathbb{L}_\infty$ is either true or false, and in the following we write TRUE for the set of true primitive literals.

For the formulas of $\mathbb{L}_\infty$ we introduce a specific measure of complexity, which will be used for proving partial cut elimination for $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$.

Definition 12 The *degree* $dg(A)$ of an $\mathbb{L}_\infty$ formula A is inductively defined as follows:

1. If A is a ∇^Ω formula of $\mathbb{L}_\infty$, then $dg(A) := 0$.

2. If A is a formula $(a \in X)$ or $(a \notin X)$, then $dg(A) := 1$.

3. If A is a formula $(a \in \{x : B(x)\})$ or $(a \notin \{x : B(x)\})$ so that $dg(B) = \alpha$, then $dg(A) := \alpha + 1$.

4. If A is a formula $(B \vee C)$ or $(B \wedge C)$ so that $dg(B) = \alpha$, $dg(C) = \beta$ and B or C is not a ∇^Ω formula of $\mathbb{L}_\infty$, then $dg(A) := max(\alpha, \beta) + 1$.

5. If A is a formula $\exists x B$ or $\forall x B$ so that $dg(B) = \alpha$ and B is not a ∇^Ω formula of $\mathbb{L}_\infty$, then $dg(A) := \alpha + 1$.

6. If A is a formula $\exists XB$ or $\forall XB$ so that $dg(B) = \alpha$, then
$$dg(A) := max(\omega, \alpha + 1).$$
7. If A is a formula $\exists \xi B$ or $\forall \xi B$ so that $dg(B) = \alpha$ and B is not a ∇^{Ω} formula of $\mathbb{L}_\infty$, then $dg(A) := \alpha + 1$.
8. If A is a formula $(\exists \xi < \theta)B$ or $(\forall \xi < \theta)B$ and B is not a ∇^{Ω} formula of $\mathbb{L}_\infty$, then $dg(A) := \alpha + 2$.

Hence, the degree of an $\mathbb{L}_\infty$ formula becomes infinite as soon as bound set variables occur. The following lemma, whose proof is straightforward, collects some important observations.

Lemma 13 *We have for all $\mathbb{L}_\infty$ formulas A:*

1. $dg(A) = 0$ *if and only if A is a ∇^{Ω} formula of $\mathbb{L}_\infty$.*
2. $0 < dg(A) < \omega$ *if and only if A is an elementary formula of $\mathbb{L}_\infty$ but not a ∇^{Ω} formula of $\mathbb{L}_\infty$.*
3. $\omega \leq dg(A) < \omega + \omega$ *if and only if A is not elementary.*

Derivations in $\mathbf{E\Omega_1^*}$, $\mathbf{E\Omega_2^*}$ and $\mathbf{Z\Omega}$ are presented in a Tait-style manner. Accordingly, their axioms and rules of inference are formulated for finite sets of formulas which have to be interpreted disjunctively. The capital greek letters $\Gamma, \Lambda, \Phi, \Psi, \ldots$ (possibly with subscripts) denote finite sets of formulas, and we write (for example) Γ, Φ, A, B for the union of Γ, Φ and $\{A, B\}$.

The following basic axioms and rules are the usual axioms and rules of the (many-sorted) Tait calculus with ω-rule. The principal axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$ correspond to the ordinal-theoretic axioms of $\widehat{\mathbf{E\Omega}}$ and $\widetilde{\mathbf{E\Omega}}$.

In $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$ we restrict ourselves to *simple* $\mathbb{L}_\infty$ formulas, i.e. $\mathbb{L}_\infty$ formulas containing neither free number variables nor constants for ordinals. Such formulas will be called $\mathsf{S}\mathbb{L}_\infty$ formulas in the following.

I. Basic axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas, all numerically equivalent $\mathsf{S}\mathbb{L}_\infty$ formulas A_1 and A_2, and all literals B of L which belong to TRUE:
$$\Gamma,\ \neg A_1,\ A_2 \quad \text{and} \quad \Gamma,\ B$$

II. Ordinal equality axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all ∇^{Ω} formulas A of $\mathsf{S}\mathbb{L}_\infty$:
$$\Gamma,\ \sigma \neq \tau,\ \neg A(\sigma),\ A(\tau)$$

III. Principal axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 1. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas, all inductive operator axioms or linearity axioms A of $\mathbf{E\Omega}$ without free number variables:
$$\Gamma,\ A$$

IV. Principal axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 2. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all simple Σ^Ω formulas A:

$$\Gamma,\ \neg A,\ \exists\xi A^\xi$$

V. Principal axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 3. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all simple Δ_0^Ω formulas B:

$$\Gamma,\ \exists\xi\,[(\forall\eta<\xi)B(\eta)\wedge\neg B(\xi)],\ \forall\xi B(\xi)$$

In $\mathbf{E\Omega_2^*}$ these axioms are extended to the stronger form in which B is permitted to be an *elementary* formula of $\mathsf{S}\mathbb{L}_\infty$.

VI. Basic rules of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 1. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all $\mathsf{S}\mathbb{L}_\infty$ formulas A, B and $C(a)$:

$$\frac{\Gamma,\ A}{\Gamma,\ A\vee B}\qquad\qquad\frac{\Gamma,\ B}{\Gamma,\ A\vee B}\qquad\qquad\frac{\Gamma,\ A\qquad\Gamma,\ B}{\Gamma,\ A\wedge B}$$

$$\frac{\Gamma,\ C(a)}{\Gamma,\ \exists xC(x)}\qquad\qquad\frac{\Gamma,\ C(b)\text{ for all closed number terms }b}{\Gamma,\ \forall xC(x)}\quad(\forall^*)$$

VII. Basic rules of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 2. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas, all elementary $\mathsf{S}\mathbb{L}_\infty$ formulas $A(a)$, all $\mathsf{S}\mathbb{L}_\infty$ formulas $B(S)$ and all free set variables Y which do not occur in $\Gamma, \forall X B(X)$:

$$\frac{\Gamma,\ A(a)}{\Gamma,\ a\in\{x:A(x)\}}\qquad\qquad\frac{\Gamma,\ \neg A(a)}{\Gamma,\ a\notin\{x:A(x)\}}$$

$$\frac{\Gamma,\ B(S)}{\Gamma,\ \exists XB(X)}\qquad\qquad\frac{\Gamma,\ B(Y)}{\Gamma,\ \forall XB(X)}$$

VIII. Basic rules of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$, part 3. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas, all $\mathsf{S}\mathbb{L}_\infty$ formulas A and all ordinal variables σ and τ so that the usual variable conditions are satisfied:

$$\frac{\Gamma,\ A(\sigma)}{\Gamma,\ \exists\xi A(\xi)}\qquad\qquad\frac{\Gamma,\ A(\sigma)}{\Gamma,\ \forall\xi A(\xi)}$$

$$\frac{\Gamma,\ \sigma<\tau\wedge A(\sigma)}{\Gamma,\ (\exists\xi<\tau)A(\xi)}\qquad\qquad\frac{\Gamma,\ \sigma<\tau\to A(\sigma)}{\Gamma,\ (\forall\xi<\tau)A(\xi)}$$

IX. Cut rules of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$. For all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all $\mathsf{S}\mathbb{L}_\infty$ formulas A:

$$\frac{\Gamma,\ A\qquad\Gamma,\ \neg A}{\Gamma}$$

The formulas A and $\neg A$ are called the cut formulas of this cut; the degree of a cut is the degree of its cut formulas.

The rule $(\forall^*)$ is the usual ω-rule. It will be used to deal with the scheme of complete induction on the natural numbers, which is available in $\widehat{\mathbf{E\Omega}}$. Part 2 and 3 of the principal axioms of $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$ correspond to the Σ^Ω reflection axioms and the ordinal induction for Δ_0^Ω formulas and elementary formulas of $\mathsf{S}\mathbb{L}_\infty$, respectively.

Based on these axioms and rules of inference derivability in the systems $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$ is defined in the standard way. For $\ell = 1, 2$ the notation $\mathbf{E\Omega}_\ell^* \vdash_\rho^\alpha \Gamma$ expresses that Γ is provable in $\mathbf{E\Omega}_\ell^*$ by a proof whose depth is bounded by α and whose cut degree is bounded by ρ.

Definition 14 Assume that $\ell = 1$ or $\ell = 2$ and let Γ be a finite set of $\mathsf{S}\mathbb{L}_\infty$ formulas. Then we define $\mathbf{E\Omega}_\ell^* \vdash_\rho^\alpha \Gamma$ for all ordinals α and ρ in T by induction on α.

1. If Γ is basic or principal axiom of $\mathbf{E\Omega}_\ell^*$, then we have $\mathbf{E\Omega}_\ell^* \vdash_\rho^\alpha$ for all ordinals α and ρ in T.
2. If $\mathbf{E\Omega}_\ell^* \vdash_\rho^{\alpha_i} \Gamma_i$ and $\alpha_i < \alpha$ for every premise Γ_i of a basic rule, a principal rule or a cut of $\mathbf{E\Omega}_\ell^*$ whose degree is less than ρ, then we have $\mathbf{E\Omega}_\ell^* \vdash_\rho^\alpha \Gamma$ for the conclusion Γ of this rule.

Because of the ordinal equality axioms and the principal axioms it is not possible to obtain complete cut elimination for $\mathbf{E\Omega_1^*}$ and $\mathbf{E\Omega_2^*}$. However, the main formulas of all ordinal equality axioms and all principal axioms of $\mathbf{E\Omega_1^*}$ are of degree 0 and the main formulas of all principal axioms of $\mathbf{E\Omega_2^*}$ are of degree less than ω. Hence, partial cut elimination can be proved by standard techniques as presented, for example, in Schütte [15].

Theorem 15 (Partial cut elimination) *We have for all finite sets Γ of $\mathsf{S}\mathbb{L}_\infty$ formulas and all ordinals $\alpha, \beta, \rho \in T$:*

1. $\mathbf{E\Omega_1^*} \vdash_{\beta+\omega^\rho}^\alpha \Gamma$ *and* $\beta \geq 1 \quad \Longrightarrow \quad \mathbf{E\Omega_1^*} \vdash_\beta^{\varphi\rho\alpha} \Gamma$,
2. $\mathbf{E\Omega_2^*} \vdash_{\beta+\omega^\rho}^\alpha \Gamma$ *and* $\beta \geq \omega \quad \Longrightarrow \quad \mathbf{E\Omega_2^*} \vdash_\beta^{\varphi\rho\alpha} \Gamma$.

6.2 The system $\mathbf{Z\Omega}$

For the introduction of the system $\mathbf{Z\Omega}$ we also start off from the language $\mathbb{L}_\infty$. However, for $\mathbf{Z\Omega}$ only those $\mathbb{L}_\infty$ formulas are relevant which do not contain free number and free ordinal variables. Such formulas, which we call $\mathsf{C}\mathbb{L}_\infty$ formulas, may contain free set variables and are therefore closed only with respect to the first order part of $\mathbb{L}_\infty$.

In order to measure the complexity of cuts in $\mathbf{Z\Omega}$ we assign a rank to each $\mathsf{C}\mathbb{L}_\infty$ formula. This definition is tailored so that the process of building up stages of an inductive definition is reflected by the rank of the formulas $P_{\mathcal{A}}^\alpha(a)$.

Definition 16 The *rank* $rn(A)$ of a $\mathsf{C}\mathbb{L}_\infty$ formula A is inductively defined as follows:

1. If A is a literal of $\mathbb{L}$ or a literal $(\alpha < \beta)$, $\sim(\alpha < \beta)$, $(\alpha = \beta)$ or $\sim(\alpha = \beta)$ for some ordinals α and β, then $rn(A) := 0$.

2. If A is a literal $P^\alpha_{\mathcal{A}}(a)$ or $\sim P^\alpha_{\mathcal{A}}(a)$ for some ordinal α, then $rn(A) := \omega(\alpha + 1)$.

3. If A is a formula $(a \in X)$ or $(a \notin X)$, then $rn(A) := \Omega + 1$.

4. If A is a formula $(a \in \{x : B(x)\})$ or $(a \notin \{x : B(x)\})$ so that $rn(B(a)) = \alpha$, then $rn(A) := \alpha + 1$.

5. If A is a formula $(B \vee C)$ or $(B \wedge C)$ so that $rn(B) = \beta$ and $rn(C) = \gamma$, then $rn(A) := max(\beta, \gamma) + 1$.

6. If A is a formula $\exists x B(x)$ or $\forall x B(x)$ so that $rn(B(0)) = \alpha$, then $rn(A) := \alpha+1$.

7. If A is a formula $\exists X B$ or $\forall X B$ so that $rn(B) = \alpha$, then
$$rn(A) := max(\Omega + \omega, \alpha + 1).$$

8. If A is a formula $(\exists \xi < \alpha)B(\xi)$ or $(\forall \xi < \alpha)B(\xi)$ for some ordinal α, then
$$rn(A) := sup\{rn(B(\beta)) + 1 : \beta < \alpha\}.$$

9. If A is a formula $\exists \xi B(\xi)$ or $\forall \xi B(\xi)$, then
$$rn(A) := sup(\{\Omega\} \cup \{rn(B(\beta)) + 1 : \beta \in T \cap \Omega\}).$$

If A is an $\mathbb{L}_\infty$ formula without free number variables, then we write $A[\sigma_1, \ldots, \sigma_n]$ in order to indicate that all free ordinal variables of A come from the list $\sigma_1, \ldots, \sigma_n$. In addition, we write $oc(B)$ for the set of ordinal constants which occur in the $\mathbb{L}_\infty$ formula B. Accordingly, if Γ is a finite set of $\mathbb{L}_\infty$ formulas, then $oc(\Gamma)$ denotes the union of the sets $oc(B)$, where B is a formula in Γ.

The proof of the following lemma is a matter of routine.

Lemma 17 *Let A be a Δ^Ω_0 formula of $\mathsf{C}\mathbb{L}_\infty$, $B[\sigma]$ a Δ^Ω_0 formula of $\mathbb{L}_\infty$, and C an arbitrary formula of $\mathsf{C}\mathbb{L}_\infty$. Assume further that α is an element of $T \cap \Omega$ and that a is a closed number term. Then we have:*

1. *$rn(\mathcal{A}(P^{<\alpha}_{\mathcal{A}}, a)) < rn(P^\alpha_{\mathcal{A}}(a))$.*

2. *If $oc(A) < \alpha$, then $rn(A) < \omega\alpha + \omega$.*

3. *$rn(\exists \xi B[\xi]) = \Omega$.*

4. *If C is an elementary but not a Δ^Ω_0 formula of $\mathsf{C}\mathbb{L}_\infty$, then $\Omega \leq rn(C) < \Omega+\omega$.*

5. If C is not elementary, then $\Omega + \omega \leq rn(C) < \Omega + \omega + \omega$.

The system $\mathbf{Z\Omega}$ contains the following axioms and rules of inference.

I. Axioms of $\mathbf{Z\Omega}$. For all finite sets Γ of CL_∞ formulas, all numerically equivalent CL_∞ formulas A_1 and A_2, all numerically equivalent Σ^Ω formulas B_1 and B_2 of CL_∞, all ordinals $\alpha \leq \beta \in T \cap \Omega$, and all literals C in $\textsc{True}$:

$$\Gamma,\, \neg A_1,\, A_2 \quad \text{and} \quad \Gamma,\, \neg B_1^\alpha,\, B_2^\beta \quad \text{and} \quad \Gamma,\, C$$

II. Predicative rules, part 1. For all finite sets Γ of CL_∞ formulas, all CL_∞ formulas A, B and $C(a)$:

$$\frac{\Gamma,\, A}{\Gamma,\, A \vee B} \qquad \frac{\Gamma,\, B}{\Gamma,\, A \vee B} \qquad \frac{\Gamma,\, A \qquad \Gamma,\, B}{\Gamma,\, A \wedge B}$$

$$\frac{\Gamma,\, C(a)}{\Gamma,\, \exists x C(x)} \qquad \frac{\Gamma,\, C(b) \text{ for all closed number terms } b}{\Gamma,\, \forall x C(x)} \quad (\forall^*)$$

III. Predicative rules, part 2. For all finite sets Γ of CL_∞ formulas, all elementary CL_∞ formulas $A(a)$, all CL_∞ formulas $B(S)$ and all free set variables Y which do not occur in $\Gamma,\, \forall X B(X)$:

$$\frac{\Gamma,\, A(a)}{\Gamma,\, a \in \{x : A(x)\}} \qquad \frac{\Gamma,\, \neg A(a)}{\Gamma,\, a \notin \{x : A(x)\}}$$

$$\frac{\Gamma,\, B(S)}{\Gamma,\, \exists X B(X)} \qquad \frac{\Gamma,\, B(Y)}{\Gamma,\, \forall X B(X)}$$

IV. Predicative rules, part 3. For all finite sets Γ of CL_∞ formulas, all inductive operator forms $\mathcal{A}(X, x)$, all closed number terms a and all ordinals $\alpha \in T \cap \Omega$:

$$\frac{\Gamma,\, \mathcal{A}(P_\mathcal{A}^{<\alpha}, a)}{\Gamma,\, P_\mathcal{A}^\alpha(a)} \qquad \frac{\Gamma,\, \neg\mathcal{A}(P_\mathcal{A}^{<\alpha}, a)}{\Gamma,\, \neg P_\mathcal{A}^\alpha(a)}$$

V. Predicative rules, part 4. For all finite sets Γ of CL_∞ formulas, all $\mathbb{L}_\infty$ formulas $A[\sigma]$ and all ordinals $\alpha < \beta \in T \cap \Omega$:

$$\frac{\Gamma,\, A[\alpha]}{\Gamma,\, \exists \xi A[\xi]} \qquad \frac{\Gamma,\, A[\alpha]}{\Gamma,\, (\exists \xi < \beta) A[\xi]} \qquad \frac{\Gamma,\, A[\gamma] \text{ for all } \gamma < \beta}{\Gamma,\, (\forall \xi < \beta) A[\xi]}$$

VI. $\forall^\Omega$-rules. For all finite sets Γ of CL_∞ formulas, all natural numbers $m > 0$ and all $\mathbb{L}_\infty$ formulas $A[\sigma_1, \ldots, \sigma_m]$ and $B[\sigma_1, \ldots, \sigma_m, \tau]$:

$$\frac{\Gamma,\, A[\alpha_1, \ldots, \alpha_m] \text{ for all } \alpha_1, \ldots, \alpha_m \in T \cap \Omega}{\Gamma,\, \forall \xi_1, \ldots, \xi_m A[\xi_1, \ldots, \xi_m]} \quad (\forall^\Omega.1)$$

$$\frac{\Gamma,\, A[\alpha_1, \ldots, \alpha_m],\, B[\alpha_1, \ldots, \alpha_m, \beta] \text{ for all } \alpha_1, \ldots, \alpha_m, \beta \in T \cap \Omega}{\Gamma,\, \forall \xi_1, \ldots, \xi_m (A[\xi_1, \ldots, \xi_m] \ \vee\ \forall \eta B[\xi_1, \ldots, \xi_m, \eta])} \quad (\forall^\Omega.2)$$

VII. Impredicative rules. For all finite sets Γ of $\mathsf{C}\mathbb{L}_\infty$ formulas and all Σ^Ω formulas A of $\mathsf{C}\mathbb{L}_\infty$:

$$\frac{\Gamma,\ A}{\Gamma,\ \exists\xi A^\xi}$$

VIII. Cuts of $\mathbf{Z\Omega}$. For all finite sets Γ of $\mathsf{C}\mathbb{L}_\infty$ formulas and all $\mathsf{C}\mathbb{L}_\infty$ formulas A:

$$\frac{\Gamma,\ A \qquad \Gamma,\ \neg A}{\Gamma}$$

The formulas A and $\neg A$ are the cut formulas of this cut, the rank of a cut is the rank of its cut formulas.

This fairly unusual form of the $\forall^\Omega$-rules is necessary in our approach in order to obtain a sufficiently short embedding of $\mathbf{E\Omega}_\mathbf{2}^*$ into $\mathbf{Z\Omega}$. In particular, we do not see how Lemma 51 and Theorem 53 can be proved with the traditional form

$$\frac{\Gamma,\ A(\alpha) \text{ for all } \alpha \in T\cap\Omega}{\Gamma,\ \forall\xi A(\xi)}$$

only. Then too many applications of this rule would be needed, and our ordinal bounds would be spoilt.

The impredicative rules are impredicative in the sense that the rank of the main formula of the premise of such a rule is in general greater than the rank of the main formula of the corresponding conclusion. Hence, derivations must be measured in a rather complex way in order to permit a proper proof-theoretic treatment. Here we follow Jäger [9], Jäger and Pohlers [11] and Pohlers [13].

Definition 18 Let Γ be a finite set of $\mathsf{C}\mathbb{L}_\infty$ formulas. Then we define $\mathbf{Z\Omega} \vdash^\alpha_\rho \Gamma$ for all ordinals $\alpha, \rho \in T$ by induction on α.

1. If Γ is an axiom of $\mathbf{Z\Omega}$, then we have $\mathbf{Z\Omega} \vdash^\alpha_\rho \Gamma$ for all $\alpha, \rho \in T$ so that $oc(\Gamma) \underline{\ll} \alpha$.

2. Let $(\Phi_i : i \in I)$ be the family of the premises and let Φ be the corresponding conclusion of a predicative rule, an impredicative rule or a cut of $\mathbf{Z\Omega}$ whose rank is less than ρ. Assume that

   (1) $\mathbf{Z\Omega} \vdash^{\alpha_i}_\rho \Phi_i$ and $\alpha_i \ll \alpha$ for all $i \in I$,

   (2) $\Phi \subset \Gamma$ and $oc(\Gamma) \underline{\ll} \alpha$.

   Then we have $\mathbf{Z\Omega} \vdash^\alpha_\rho \Gamma$.

3. Let m be a natural number greater than 0, let f be a m-ary function from $T \cap \Omega$ to T, and let $A[\sigma_1, \ldots, \sigma_m]$ be an $\mathbb{L}_\infty$ formula. Assume that

   (1) $\mathbf{Z\Omega} \vdash^{f(\beta_1,\ldots,\beta_m)}_\rho \Phi, A[\beta_1, \ldots, \beta_m]$ for all $\beta_1, \ldots, \beta_m \in T \cap \Omega$,

   (2) $f \ll \alpha$,

   (3) $\Phi, \forall\xi_1, \ldots, \forall\xi_m A[\xi_1, \ldots, \xi_m] \subset \Gamma$ and $oc(\Gamma) \underline{\ll} \alpha$.

   Then we have $\mathbf{Z\Omega} \vdash^\alpha_\rho \Gamma$.

4. Let m be a natural number greater than 0, let f be a $(m+1)$-ary function from $T \cap \Omega$ to T, and let $A[\sigma_1, \ldots, \sigma_m]$ and $B[\sigma_1, \ldots, \sigma_m, \tau]$ be $\mathbb{L}_\infty$ formulas. Assume that

   (1) $\mathbf{Z\Omega} \vdash^{f(\beta_1,\ldots,\beta_m,\gamma)}_{\rho} \Phi, A[\beta_1, \ldots, \beta_m],\ B[\beta_1, \ldots, \beta_m, \gamma]$
   for all $\beta_1, \ldots, \beta_m, \gamma \in T \cap \Omega$,

   (2) $f \ll \alpha$,

   (3) $\Phi, \forall \xi_1, \ldots, \forall \xi_m (A[\xi_1, \ldots, \xi_m] \ \vee \ \forall \eta B[\xi_1, \ldots, \xi_m, \eta]) \subset \Gamma$ and $oc(\Gamma) \underline{\ll} \alpha$.

   Then we have $\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma$.

Now we list a series of propositions similar to those in Jäger [9], Jäger and Pohlers [11] and Pohlers [13]. We omit their proofs since they can easily be reconstructed from the proofs in [9, 11, 13]. The general form of our $(\forall^\Omega)$-rules and the fact that $\mathbf{Z\Omega}$ comprises identity axioms $\Gamma, \neg A_1, A_2$ for arbitrary numerically equivalent $\mathsf{C}\mathbb{L}_\infty$ formulas A_1 and A_2 do not cause serious problems.

Proposition 19 (Collapsing) We have for all finite sets Γ of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$ and all ordinals $\alpha, \rho \in T$:

$$\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma \text{ and } \rho \leq \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{D\alpha}_{\rho} \Gamma.$$

Proposition 20 (Persistency) We have for all finite sets Γ of $\mathsf{C}\mathbb{L}_\infty$ formulas, all Σ^Ω formulas A of $\mathsf{C}\mathbb{L}_\infty$, all Π^Ω formulas B of $\mathsf{C}\mathbb{L}_\infty$, and all ordinals $\alpha, \beta, \gamma, \rho \in T$ so that $\gamma \leq \beta \leq \alpha < \Omega$:

1. $\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ A^\gamma \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ A^\beta.$
2. $\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ B^\beta \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ B^\gamma.$

Proposition 21 (Boundedness) We have for all finite sets Γ of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$, all Σ^Ω formulas A of $\mathsf{C}\mathbb{L}_\infty$ and all ordinals $\alpha, \rho \in T$:

$$\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ A \text{ and } \alpha < \Omega \text{ and } \rho \leq \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ A^\alpha.$$

Proposition 22 (Inversion) We have for all finite sets Γ of $\mathsf{C}\mathbb{L}_\infty$ formulas, all $\mathsf{C}\mathbb{L}_\infty$ formulas A and all ordinals $\alpha, \beta, \rho \in T$:

1. $\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ \forall \xi A(\xi) \text{ and } \beta < \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha \# \beta \# 2}_{\rho} \Gamma,\ A(\beta).$
2. $\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ \forall \xi A(\xi) \text{ and } \beta < \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha \# \beta \# 2}_{\rho} \Gamma,\ (\forall \xi < \beta) A(\xi).$

Proposition 23 (Weak elimination) We have for all finite sets Γ and Φ of $\mathsf{C}\mathbb{L}_\infty$ formulas, all $\mathsf{C}\mathbb{L}_\infty$ formulas A and all ordinals $\alpha, \beta, \rho \in T$:

$$\mathbf{Z\Omega} \vdash^{\alpha}_{\rho} \Gamma,\ A \text{ and } \mathbf{Z\Omega} \vdash^{\beta}_{\rho} \Phi, \neg A \text{ and } rn(A) \leq \rho \neq \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \vdash^{\alpha \# \beta \# 2}_{\rho} \Gamma,\ \Phi.$$

Proposition 24 (Strong elimination) Let Γ be a finite set of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$, α and β elements of T, and assume that

(A1) $\quad \mathbf{Z\Omega} \mid\!\frac{\alpha}{\Omega}\, \Gamma, \forall \xi A_1(\xi), \ldots, \forall \xi A_n(\xi), \exists \xi B(\xi)$,

(A2) $\quad \mathbf{Z\Omega} \mid\!\frac{\beta}{\Omega}\, \Gamma, \forall \xi A_1(\xi), \ldots, \forall \xi A_n(\xi), \forall \xi \neg B(\xi)$,

(A3) $\quad A_1[\sigma], \ldots, A_n[\sigma]$ and $B[\sigma]$ are Δ_0^Ω formulas of $\mathbb{L}_\infty$.

Then we have $\mathbf{Z\Omega} \mid\!\frac{\alpha\#\beta\#\Omega(n+1)}{\Omega}\, \Gamma, \forall \xi A_1(\xi), \ldots, \forall \xi A_n(\xi)$.

These propositions provide the instruments for deducing the following three cut elimination theorems. Predicative cut elimination is as usual (cf. [15]) and does not make use of collapsing techniques, of course.

Theorem 25 (Predicative cut elimination) *We have for all finite sets Γ of $\mathsf{C}\mathbb{L}_\infty$ formulas and all ordinals $\alpha, \beta, \rho \in T$:*

$$\mathbf{Z\Omega} \mid\!\frac{\alpha}{\beta+\omega^\rho}\, \Gamma \text{ and } \beta \neq \Omega \text{ and } \rho < \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \mid\!\frac{\varphi\rho\alpha}{\beta}\, \Gamma.$$

The following theorem about complete elimination of level 0 cuts is proved by induction on α and using intermediate collapsing and the fact that the rank of a cut formula can be controlled in the sense of the $\ll$ relation by α. For details see [11] and [13].

Theorem 26 (Complete elimination of Δ_0^Ω cuts) *We have for all finite sets Γ of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$ and all ordinals $\alpha \in T$:*

$$\mathbf{Z\Omega} \mid\!\frac{\alpha}{\Omega}\, \Gamma \quad \Longrightarrow \quad \mathbf{Z\Omega} \mid\!\frac{\Omega+\alpha}{0}\, \Gamma.$$

The impredicative cut elimination theorem is an immediate consequence of strong elimination; it is proved by induction on α (cf. [9, 11, 13]). By $rn(\Phi) \leq \Omega$ we mean that the rank of all formulas in Φ is less than or equal to Ω.

Theorem 27 (Impredicative cut elimination) *We have for all finite sets Γ of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$, all finite sets Φ of $\mathsf{C}\mathbb{L}_\infty$ formulas and all ordinals $\alpha \in T$:*

$$\mathbf{Z\Omega} \mid\!\frac{\alpha}{\Omega+1}\, \Gamma, \Phi \text{ and } rn(\Phi) \leq \Omega \quad \Longrightarrow \quad \mathbf{Z\Omega} \mid\!\frac{\omega^{\Omega+\alpha}}{\Omega}\, \Gamma, \Phi.$$

Now we combine the previous results and obtain the following corollary. Together with the embedding theorems in Sections 8.2 and 9.2 it is the crucial step in determining the upper bounds of the proof-theoretic strength of $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$.

Corollary 28 *Let Γ be a finite set of Σ^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$. Then we have:*

1. *If $\mathbf{Z\Omega} \mid\!\frac{\Omega+\alpha}{\Omega+\omega}\, \Gamma$ for some ordinal $\alpha < \varepsilon_0$, then there exists an ordinal $\beta < \psi\varepsilon_{\Omega+\varepsilon_0}$ so that $\mathbf{Z\Omega} \mid\!\frac{\beta}{0}\, \Gamma$.*

2. If $\mathbf{Z\Omega} \frac{\;\Omega+\omega^\omega\;}{\Omega+\omega+n} \Gamma$ for some $n < \omega$, then there exists an ordinal $\beta < \psi\varepsilon_{\varepsilon_{\Omega+1}}$ so that $\mathbf{Z\Omega} \frac{\;\beta\;}{0} \Gamma$.

We conclude this section with stating a well-known result, which says that cut-free derivations provide upper bounds for the order type of provable wellorderings. Similar results are proved in detail in [13, 15].

Theorem 29 *Let $\prec$ be a primitive recursive wellordering so that $\mathbf{Z\Omega} \frac{\;\alpha\;}{0} TI(\prec, U)$ for some $\alpha \in T \cap \Omega$. Then the order type of $\prec$ is bounded by $\omega\alpha$.*

7 The proof-theoretic strength of $\widehat{\mathbf{E\Omega}}$ and $\widehat{\mathbf{EID}}_1$

In the remaining three sections of this article we determine the proof-theoretic strength of the theories mentioned in Lemma 11. The general strategy is always the same: lower bounds for $\widehat{\mathbf{EID}}_\mathbf{1}$, $\widetilde{\mathbf{EID}}_\mathbf{1}$ and $\mathbf{EID_1}$, and upper bounds for $\widehat{\mathbf{E\Omega}}$, $\widetilde{\mathbf{E\Omega}}$ and $\mathbf{E\Omega}$.

We begin with the systems $\widehat{\mathbf{EID}}_\mathbf{1}$ and $\widehat{\mathbf{E\Omega}}$, and we show that they have the same proof-theoretic strength as the second order system $\mathbf{(\Pi^0_1\text{-}CA)}_{<\varepsilon_{\varepsilon_0}}$. In particular, both theories have the proof-theoretic ordinal $\varphi\varepsilon_{\varepsilon_0}0$.

7.1 Lower bounds

Lower bounds of $\widehat{\mathbf{EID}}_\mathbf{1}$, $\widetilde{\mathbf{EID}}_\mathbf{1}$ and $\mathbf{EID_1}$ are established in this article by proving wellorderings with respect to the primitive recursive notation system $(T, <)$ of Section 3. Therefore, in the wellordering proofs of Sections 7.1, 8.1 and 9.1 the symbols $<$ and $\leq$ generally stand for the less and less or equal relation on T and neither for the less relation on the natural numbers nor the less relation on the ordinals, as it is available in $\mathbb{L}_\Omega$. In this context we need the following notation with respect to $(T, <)$: If $\lhd$ is a subrelation of the less relation $<$ on T, then we set

$$TI(\lhd, a, A) \; := \; Prog(\lhd, A) \rightarrow (\forall x \lhd a)A(x).$$

In this section we show that each ordinal less than $\varphi\varepsilon_{\varepsilon_0}0$ is provable in $\widehat{\mathbf{EID}}_\mathbf{1}$. Since $\widehat{\mathbf{EID}}_\mathbf{1}$ contains $\mathbf{ACA}$, it follows from standard proof theory (cf. e.g. Schütte [15]) that $TI(<, a, X)$ is derivable in $\widehat{\mathbf{EID}}_\mathbf{1}$ for each $a < \varepsilon_{\varepsilon_0}$. By ($\mathcal{E}_{\mathrm{FP}}$-CA) this yields $TI(<, a, A)$ for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas A.

Lemma 30 *We have for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas $A(x)$ and all $a < \varepsilon_{\varepsilon_0}$:*

$$\widehat{\mathbf{EID}}_\mathbf{1} \vdash TI(<, a, A).$$

Furthermore, let us assume that we have primitive recursive auxiliary functions h and e from T to T satisfying

- $h(0) = e(0) = 0$; $h(\omega^a) = 0$ and $e(\omega^a) = a$;

- if $a = \omega^{a_1} + \cdots + \omega^{a_n}$ for more than one summand so that $a_n \leq \cdots \leq a_1$, then $h(a) = \omega^{a_1} + \cdots + \omega^{a_{n-1}}$ and $e(a) = a_n$.

In the sequel we will make us of some sort of jump operator J. Since we use J in an inductive operator form below, we define it as a pair (J^+, J^-) as follows:

$$\begin{aligned} J^+(X, Y, a) &:= \forall y\, [(\exists x < y)(x \in Y) \ \vee \ (\forall x < y + a)(x \in X)], \\ J^-(X, Y, a) &:= \exists y\, [(\forall x < y)(x \in X) \ \wedge \ (\exists x < y + a)(x \in Y)]. \end{aligned}$$

In order to prove $TI(<, a, U)$ for each $a < \varphi\varepsilon_{\varepsilon_0}0$, we build up a hierarchy of sets $(H_b)_{b<a}$ with initial set U, for each $a < \varepsilon_{\varepsilon_0}$. The definition of the hierarchy corresponds to the formula $\mathcal{R}(P, Q, t)$ of Schütte [15], p. 184 ff., formalized in the framework of $\widehat{\mathbf{EID}}_1$. The idea is to define the hierarchy along $<$ by means of a fixed point $\mathcal{P}_\mathcal{A}$ of a certain inductive operator form $\mathcal{A}(X, x)$. The elements of the fixed point will be triples $\langle a, i, x\rangle$, where $a \in T$ and i equals 0 or 1, depending on whether x belongs to the ath stage of the hierarchy.

In the definition below we will use the abbreviation $c \in (X)_{a,b}$ for $\langle a, b, c\rangle \in X$. The inductive operator form $\mathcal{A}(X, x)$ is defined to be the disjunction of the following formulas (1)–(4):

(1) $Seq_3(x) \ \wedge \ (x)_0 = 0 \ \wedge \ (x)_1 = 0 \ \wedge \ U((x)_2)$

(2) $Seq_3(x) \ \wedge \ (x)_0 = 0 \ \wedge \ (x)_1 = 1 \ \wedge \ \neg U((x)_2)$

(3) $Seq_3(x) \ \wedge \ 0 < (x)_0 \ \wedge \ (x)_1 = 0 \ \wedge$

$\forall z\, [h((x)_0) \leq z < (x)_0 \ \rightarrow \ J^+((X)_{z,0}, (X)_{z,1}, \varphi(e((x)_0), (x)_2))]$

(4) $Seq_3(x) \ \wedge \ 0 < (x)_0 \ \wedge \ (x)_1 = 1 \ \wedge$

$\exists z\, [h((x)_0) \leq z < (x)_0 \ \wedge \ J^-((X)_{z,0}, (X)_{z,1}, \varphi(e((x)_0), (x)_2))]$.

From the fixed point axioms alone it is not possible to prove the fact that the membership and non-membership relation defined above are *complementary*, i.e. that

$$Comp(b) \ := \ b \in T \rightarrow \forall y(\mathcal{P}_\mathcal{A}(\langle b, 0, y\rangle) \leftrightarrow \neg\mathcal{P}_\mathcal{A}(\langle b, 1, y\rangle))$$

is derivable in $\widehat{\mathbf{EID}}_1$ for all sets (coded by) b.

Lemma 31 *We have for all $a < \varepsilon_{\varepsilon_0}$:*

$$\widehat{\mathbf{EID}}_1 \vdash (\forall x < a)\ Comp(x).$$

PROOF. First observe that $Comp(x)$ is an elementary $\mathbb{L}_{\mathrm{FP}}$ formula. Furthermore, it is easily verified that

$$\widehat{\mathbf{EID}}_1 \vdash Prog(<, Comp),$$

where essential use is made of the fixed point axioms (FP-$\mathcal{P}_\mathcal{A}$) for $\mathcal{P}_\mathcal{A}$. Now the claim immediately follows from Lemma 30. □

In the following we write $H_b(x)$ instead of $\mathcal{P}_{\mathcal{A}}(\langle b, 0, x\rangle)$. According to the previous lemma, $(H_b)_{b<a}$ is well-defined for each $a < \varepsilon_{\varepsilon_0}$ in the sense that $\neg H_b(x)$ is equivalent to $\mathcal{P}_{\mathcal{A}}(\langle b, 1, x\rangle)$ for each $b < a$.

Remark that the ordinal $\psi 0$ of our notation system is the ordinal Γ_0 of Schütte [15] and up to this ordinal both notation systems coincide. Let us define a form of restricted progressivness $Prg_{\psi 0}(A)$ by setting

$$Prg_{\psi 0}(A) \ :=\ (\forall x < \psi 0)((\forall y < x)A(y) \rightarrow A(x)).$$

The next lemma is essential in the wellordering proof for $\widehat{\mathbf{EID}}_{\mathbf{1}}$. It corresponds to Lemma 9 of Schütte [15], p. 186, and its proof is very similar to the proof of Lemma 9. Therefore, we omit it.

Lemma 32 *Assume that $a < \varepsilon_{\varepsilon_0}$. Then we have that*

$$\widehat{\mathbf{EID}}_{\mathbf{1}} \vdash 0 < x < a \wedge (\forall y < x)Prg_{\psi 0}(H_y) \ \rightarrow\ Prg_{\psi 0}(H_x).$$

We have prepared the ground in order to show that $\varphi a0$ is provable in $\widehat{\mathbf{EID}}_{\mathbf{1}}$ for each $a < \varepsilon_{\varepsilon_0}$. This will immediately imply the desired lower bound.

Theorem 33 *We have for all $a < \varepsilon_{\varepsilon_0}$:*

$$\widehat{\mathbf{EID}}_{\mathbf{1}} \vdash TI(<, \varphi a0, U).$$

Proof. In the following we work informally in $\widehat{\mathbf{EID}}_{\mathbf{1}}$. Let us assume that $a < \varepsilon_{\varepsilon_0}$ and choose $b := \omega^a + 1$. Then we have $a < b < \varepsilon_{\varepsilon_0}$. By Lemma 31 we know that $(H_x)_{x<b}$ is a well-defined hierarchy of sets, and by definition $H_0 = U$. Hence, we have

$$Prog(<, U) \ \rightarrow\ Prg_{\psi 0}(H_0). \tag{1}$$

From (1) we can conclude by the previous lemma

$$Prog(<, U) \wedge x < b \wedge (\forall y < x)Prg_{\psi 0}(H_y) \ \rightarrow\ Prg_{\psi 0}(H_x). \tag{2}$$

If we put $A(x) \ :=\ x < b \rightarrow Prg_{\psi 0}(H_x)$, then (2) amounts to

$$Prog(<, U) \ \rightarrow\ Prog(A). \tag{3}$$

Since $A(x)$ is an elementary $\mathbb{L}_{\mathrm{FP}}$ formula and $b < \varepsilon_{\varepsilon_0}$, we know by Lemma 30 that $TI(<, b, A)$ holds in $\widehat{\mathbf{EID}}_{\mathbf{1}}$. Hence, we can derive from (3) that

$$Prog(<, U) \ \rightarrow\ Prg_{\psi 0}(H_{\omega^a}). \tag{4}$$

Furthermore, we trivially have

$$Prg_{\psi 0}(H_{\omega^a}) \ \rightarrow\ H_{\omega^a}(0). \tag{5}$$

Since $h(\omega^a) = 0$ and $e(\omega^a) = a$, we get by the definition of H_{ω^a} that

$$H_{\omega^a}(0) \;\rightarrow\; J^+(H_0, \neg H_0, \varphi a0). \tag{6}$$

In addition, the definition of J^+ immediately yields

$$J^+(H_0, \neg H_0, \varphi a0) \;\rightarrow\; (\forall x < \varphi a0) H_0(x). \tag{7}$$

From (4)–(7) we now obtain

$$\mathit{Prog}(<, U) \;\rightarrow\; (\forall x < \varphi a0) H_0(x). \tag{8}$$

Since $H_0 = U$ this amounts to $\mathit{TI}(<, \varphi a0, U)$. $\square$

Corollary 34 $\varphi\varepsilon_{\varepsilon_0}0 \leq |\widehat{\mathbf{EID_1}}| \leq |\widehat{\mathbf{E\Omega}}|$.

Proof. Let $a < \varphi\varepsilon_{\varepsilon_0}0$. Then there is a $b < \varepsilon_{\varepsilon_0}$ so that $a < \varphi b0$. By the theorem we have that

$$\widehat{\mathbf{EID_1}} \vdash \mathit{TI}(<, \varphi b0, U),$$

yielding the claim of the corollary. $\square$

Remark 35 Notice that the anonymous relation symbol U occurs positively and negatively in the inductive operator form $\mathcal{A}(X, x)$ used in the proof above, and this fact is crucial: If we do not allow U in inductive operator forms, then the corresponding modification of $\widehat{\mathbf{EID_1}}$ has proof-theoretic ordinal $\varepsilon_{\varepsilon_0}$ (in the sense of Definition 1 or similar definitions). See Jäger and Primo [12] for this and related results.

Let us mention that in Feferman and Jäger [6] a lower bound of $\widehat{\mathbf{E\Omega}}$ is obtained by embedding a certain system of explicit mathematics with non-constructive minimum operator, which in turn is shown to contain $(\mathbf{\Pi^0_1}\textbf{-CA})_{<\varepsilon\varepsilon_0}$. However, the methods used there do not yield an interpretation of $(\mathbf{\Pi^0_1}\textbf{-CA})_{<\varepsilon\varepsilon_0}$ in $\widehat{\mathbf{EID_1}}$ via the corresponding system of explicit mathematics, since the reduction of the latter to $\widehat{\mathbf{E\Omega}}$ makes essential use of the stages of an inductive definition and Δ^Ω_0 induction on the ordinals, which are not available in $\widehat{\mathbf{EID_1}}$.

Remark 36 Instead of giving a direct wellordering proof up to $\varphi\varepsilon_{\varepsilon_0}0$, it is also possible to interpret the second order system $(\mathbf{\Pi^0_1}\textbf{-CA})_{<\varepsilon\varepsilon_0}$ into $\widehat{\mathbf{EID_1}}$. This approach makes use of formalized recursion theory.

7.2 Upper bounds

We obtain the upper bound for $\widehat{\mathbf{E\Omega}}$ by the following two step reduction: First $\widehat{\mathbf{E\Omega}}$ is embedded into the semiformal system $\mathbf{E\Omega}_1^*$; afterwards the ∇^Ω part of $\mathbf{E\Omega}_1^*$ is reduced to $\mathbf{Z\Omega}$ via a so-called asymmetric interpretation.

The following embedding theorem is proved in a standard way. Since ($\mathbb{L}_\Omega$-$\mathrm{I_N}$) is handled by means of the rule ($\forall^\infty$), we have to deal with infinitary derivations. For a detailed presentation of similar results we refer, for example, to Schütte [15].

For notational convenience we call $\mathbb{L}_\Omega$ formulas which do not contain free number variables numerically closed. Hence each numerically closed formula A is an $\mathsf{S}\mathbb{L}_\infty$ formula.

Theorem 37 (Embedding of $\widehat{\mathbf{E\Omega}}$) *Let A be a numerically closed $\mathbb{L}_\Omega$ formula which is provable in $\widehat{\mathbf{E\Omega}}$. Then there exists ordinals $\alpha, \beta < \omega + \omega$ so that*

$$\mathbf{E\Omega}_1^* \mathrel{|}\!\frac{\alpha}{\beta}\, A.$$

A combination of Theorem 15 and Theorem 37 yields the following corollary. It means that for every numerically closed $\mathbb{L}_\Omega$ formula which is provable in $\widehat{\mathbf{E\Omega}}$ there exists a proof in $\mathbf{E\Omega}_1^*$ of depth less than $\varepsilon_{\varepsilon_0}$ so that all cut formulas which occur in this proof are of degree 0, i.e. ∇^Ω formulas of $\mathsf{S}\mathbb{L}_\infty$.

Corollary 38 *Let A be a numerically closed $\mathbb{L}_\Omega$ formula which is provable in $\widehat{\mathbf{E\Omega}}$. Then there exists an ordinal $\alpha < \varepsilon_{\varepsilon_0}$ so that*

$$\mathbf{E\Omega}_1^* \mathrel{|}\!\frac{\alpha}{1}\, A.$$

This corollary implies, in particular, that a numerically closed ∇^Ω formula which is provable in $\widehat{\mathbf{E\Omega}}$ has a $\mathbf{E\Omega}_1^*$ proof of depth less than $\varepsilon_{\varepsilon_0}$ which consists of ∇^Ω formulas only. This observation is important for the reduction of the ∇^Ω part of $\mathbf{E\Omega}_1^*$ to $\mathbf{Z\Omega}$.

Let Γ be a finite set of numerically closed L_Ω formulas and suppose that α and β are ordinals. Then a finite set Φ of Δ_0^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$ is called a (β, α)-instance of Γ if it results from Γ by replacing

(i) each ordinal variable in the formulas of Γ by an ordinal less than β;

(ii) each universal ordinal quantifier $\forall\xi$ in the formulas of Γ by $(\forall\xi < \beta)$;

(iii) each existential ordinal quantifier $\exists\xi$ in the formulas of Γ by $(\exists\xi < \alpha)$.

Then the reduction of the ∇^Ω part of $\mathbf{E\Omega}_1^*$ to $\mathbf{Z\Omega}$ is provided by the following asymmetric interpretation.

Theorem 39 (Asymmetric interpretation) *Let Γ be a finite set of numerically closed ∇^Ω formulas so that $\mathbf{E\Omega}_1^* \mathrel{|}\!\frac{\alpha}{1}\, \Gamma$ for some ordinal $\alpha < \varepsilon_{\varepsilon_0}$. Then we have for all ordinals $\beta < \varepsilon_{\varepsilon_0}$ and all finite sets Φ of Δ_0^Ω formulas of $\mathsf{C}\mathbb{L}_\infty$ that*

$$\Phi \textit{ is a } (\beta, \beta + 2^\alpha) \textit{ instance of } \Gamma \quad \Longrightarrow \quad \mathbf{Z\Omega} \mathrel{|}\!\frac{\omega^{\beta+\omega^\alpha}}{\omega(\beta+2^\alpha+1)}\, \Phi.$$

This theorem is proved by induction on α. One simply has to follow the pattern of similar asymmetric interpretations, for example Jäger [8, 10] and Schütte [15]. For the case of Δ_0^Ω induction on the ordinals see also Lemma 51 below.

Because of Theorem 25 and Corollary 38 we may therefore conclude that the closed Σ^Ω fragment of $\widehat{\mathbf{E\Omega}}$ can be embedded into $\mathbf{Z\Omega}$ as follows.

Theorem 40 *Let A be a closed Σ^Ω formula which is provable in $\widehat{\mathbf{E\Omega}}$. Then there exists an $\alpha < \varphi\varepsilon_{\varepsilon_0}0$ and a $\beta < \varepsilon_{\varepsilon_0}$ so that $\mathbf{Z\Omega} \mid\!\frac{\alpha}{0}\, A^\beta$.*

By Theorem 29 we obtain that $\varphi\varepsilon_{\varepsilon_0}0$ is an upper bound for the provable ordinals of $\widehat{\mathbf{E\Omega}}$. Together with Lemma 11 and Corollary 34 this completes the computation of the proof-theoretic ordinal of $\widehat{\mathbf{EID_1}}$ and $\widehat{\mathbf{E\Omega}}$. In addition, a careful formalization of the previous arguments also provides the proof-theoretic equivalence of these theories to $(\mathbf{\Pi_1^0\text{-}CA})_{<\varepsilon\varepsilon_0}$.

Corollary 41 *The proof-theoretic strength of the two theories $\widehat{\mathbf{EID_1}}$ and $\widehat{\mathbf{E\Omega}}$ can be characterized as follows:*

1. $|\widehat{\mathbf{E\Omega}}| = |\widehat{\mathbf{EID_1}}| = \varphi\varepsilon_{\varepsilon_0}0$.
2. $\widehat{\mathbf{E\Omega}} \equiv \widehat{\mathbf{EID_1}} \equiv (\mathbf{\Pi_1^0\text{-}CA})_{<\varepsilon\varepsilon_0}$.

8 The proof-theoretic strength of $\widetilde{\mathbf{E\Omega}}$, $\widetilde{\mathbf{EID_1}}$ and $\mathbf{ACA} + (\Pi_1^1\text{-CA})^-$

It is a big step from $\widehat{\mathbf{EID_1}}$ and $\widehat{\mathbf{E\Omega}}$ to $\widetilde{\mathbf{EID_1}}$ and $\widetilde{\mathbf{E\Omega}}$; it leads from predicative theories to impredicative systems which are slightly stronger than $\mathbf{ID_1}$. More precisely, we will show in this section that $\widetilde{\mathbf{EID_1}}$ and $\widetilde{\mathbf{E\Omega}}$ have the proof-theoretic ordinal $\psi\varepsilon_{\Omega+\varepsilon_0}$. As a consequence of previous considerations this is also the proof-theoretic ordinal of $\mathbf{ACA} + (\Pi_1^1\text{-CA})^-$.

8.1 Lower bounds

In this section we show that $\widetilde{\mathbf{EID_1}}$ proves the wellfoundedness of each initial segment of $\psi\varepsilon_{\Omega+\varepsilon_0}$. We presuppose the impredicative notation system $(T, <)$ of Section 3, and the reader is assumed to be familiar with the wellordering proof for $\mathbf{ID_1}$. Nevertheless, let us repeat the basic definitions used in this wellordering proof. The relation $<_0$ between elements of T is given by

$$a <_0 b \ :=\ a < b \wedge b < \Omega.$$

Furthermore, let Acc be the *accessible part* of $<_0$, i.e. Acc denotes the fixed point constant $\mathcal{P}_\mathcal{A}$ of the positive operator form $\mathcal{A}(X, x)$ given by

$$\mathcal{A}(X, x) \ :=\ x < \Omega \wedge \forall y(y <_0 x \rightarrow y \in X).$$

In addition, the set $\mathfrak{M}$ and the relation $<_\Omega$ on T are defined by

$$\begin{aligned} \mathfrak{M}(a) &:= a \in T \wedge \forall y (y \in SC(a) \wedge y < \Omega \rightarrow Acc(y)), \\ a <_\Omega b &:= \mathfrak{M}(a) \wedge a < b. \end{aligned}$$

The following lemma is an immediate consequence of the induction axiom for Acc in $\widetilde{\mathbf{EID_1}}$ and $\mathbf{EID_1}$, respectively.

Lemma 42 *We have for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas $A(x)$ and arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas $B(x)$:*

1. $\widetilde{\mathbf{EID_1}} \vdash \mathit{Prog}(<, A) \rightarrow \forall x (Acc(x) \rightarrow A(x))$.
2. $\mathbf{EID_1} \vdash \mathit{Prog}(<, B) \rightarrow \forall x (Acc(x) \rightarrow B(x))$.

The next two results are standard and, therefore, their proofs are omitted. Detailed proofs can be found, e.g., in Pohlers [13].

Proposition 43 The set Acc is closed under ordinal addition and the φ-function, provably in $\widetilde{\mathbf{EID_1}}$.

Proposition 44 (Condensation) Let $\mathbf{Th}$ be $\widetilde{\mathbf{EID_1}}$ or $\mathbf{EID_1}$. Then we have for all $a \in T$:

$$\mathbf{Th} \vdash \mathit{TI}(<_\Omega, a, X) \wedge Ka < a \wedge \mathfrak{M}(a) \quad \Longrightarrow \quad \mathbf{Th} \vdash \mathit{TI}(<, \psi a, X).$$

The definition of the $<_\Omega$ relation is tailored so that transfinite induction up to $\Omega+1$ is trivially provable in $\widetilde{\mathbf{EID_1}}$ for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas; in the presence of full induction on the accessible part it is even derivable for arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas.

Lemma 45 *We have for all elementary $\mathbb{L}_{\mathrm{FP}}$ formulas $A(x)$ and arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas $B(x)$:*

1. $\widetilde{\mathbf{EID_1}} \vdash \mathit{TI}(<_\Omega, \Omega+1, A)$.
2. $\mathbf{EID_1} \vdash \mathit{TI}(<_\Omega, \Omega+1, B)$.

Proof. For the proof of the first part of our assertion we work informally in $\widetilde{\mathbf{EID_1}}$. Let $A(x)$ be an elementary $\mathbb{L}_{\mathrm{FP}}$ formula and assume $\mathit{Prog}(<_\Omega, A)$. We have to show $(\forall x \leq_\Omega \Omega) A(x)$. Let $x <_\Omega \Omega$, i.e. $\mathfrak{M}(x) \wedge x < \Omega$. From $\mathfrak{M}(x)$ and by Proposition 43 we can conclude $Acc(x)$. This immediately yields $A(x)$ by Lemma 42, since $\mathit{Prog}(<_\Omega, A)$ implies $\mathit{Prog}(<, A)$. Hence, we have established $(\forall x <_\Omega \Omega) A(x)$, from which we derive $A(\Omega)$ by $\mathit{Prog}(<_\Omega, A)$. We have shown $(\forall x \leq_\Omega \Omega) A(x)$, as desired. The proof of the second part runs in exactly the same way except that we use the second part of Lemma 42. □

Let us now turn to the specific wellordering proof for $\widetilde{\mathbf{EID_1}}$. The following lemma is crucial.

Lemma 46 *We have for all $a < \varepsilon_0$:*

$$\widetilde{\mathbf{EID_1}} \vdash TI(<_\Omega, \varepsilon_{\Omega+a}, X).$$

Proof. Let $a < \varepsilon_0$ be given. We first claim that

$$\widetilde{\mathbf{EID_1}} \vdash x \in T \rightarrow [(\forall y{<_\Omega}x)\forall X\, TI(<_\Omega, \varepsilon_{\Omega+y}, X) \rightarrow \forall X\, TI(<_\Omega, \varepsilon_{\Omega+x}, X)]\,. \qquad (1)$$

If $x = 0$, then $\varepsilon_{\Omega+x} = \Omega$ and, therefore, (1) is a consequence of the previous lemma. In the case $x > 0$ one proceeds exactly as in the wellordering proof for **ACA**. For the detailed argument the reader is referred to Schütte [15], p. 183 ff. Therefore, by setting

$$A(x) \;:=\; x \in T \rightarrow \forall X\, TI(<_\Omega, \varepsilon_{\Omega+x}, X),$$

we have established $Prog(<_\Omega, A)$ in $\widetilde{\mathbf{EID_1}}$. Furthermore, from standard proof theory one knows that $\widetilde{\mathbf{EID_1}} \vdash TI(<_\Omega, a, B)$ for arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas $B(x)$. Hence, we can conclude from (1) that

$$\widetilde{\mathbf{EID_1}} \vdash (\forall x{<_\Omega}a)A(x), \qquad (2)$$

yielding $\widetilde{\mathbf{EID_1}} \vdash A(a)$ by (1) again. This is our claim. $\square$

Theorem 47 *We have for all $a < \varepsilon_0$:*

$$\widetilde{\mathbf{EID_1}} \vdash TI(<, \psi\varepsilon_{\Omega+a}, X).$$

Proof. The claim of the theorem is an immediate consequence of the previous lemma and condensation, since $\mathfrak{M}(\varepsilon_{\Omega+a})$ and $K(\varepsilon_{\Omega+a}) < \varepsilon_{\Omega+a}$ are trivially provable in $\widetilde{\mathbf{EID_1}}$ for each $a < \varepsilon_0$. $\square$

Corollary 48 $\quad \psi\varepsilon_{\Omega+\varepsilon_0} \leq |\widetilde{\mathbf{EID_1}}| \leq |\widetilde{\mathbf{E\Omega}}|$.

8.2 Upper bounds

In order to establish the upper bound for $\widetilde{\mathbf{E\Omega}}$ we first embed $\widetilde{\mathbf{E\Omega}}$ into the semiformal system $\mathbf{E\Omega_2^*}$. The following theorem is the analogue of Theorem 37 with the only difference that the scheme of elementary induction on the ordinals, which is available in $\widetilde{\mathbf{E\Omega}}$ is taken care of by the corresponding axioms of $\mathbf{E\Omega_2^*}$.

Theorem 49 (Embedding of $\widetilde{\mathbf{E\Omega}}$) *Let A be a numerically closed $\mathbb{L}_\Omega$ formula which is provable in $\widetilde{\mathbf{E\Omega}}$. Then there exists ordinals $\alpha, \beta < \omega + \omega$ so that*

$$\mathbf{E\Omega_2^*} \vdash^{\alpha}_{\beta} A.$$

Now we apply Theorem 15 and eliminate all cuts of degree greater than or equal to ω. Observe, however, that we cannot do better since the main formulas of the principal axioms of $\mathbf{E\Omega_2^*}$ can be arbitrary elementary formulas.

Corollary 50 *Let A be a numerically closed $\mathbb{L}_\Omega$ formula which is provable in $\widetilde{\mathbf{E\Omega}}$. Then there exists an ordinal $\alpha < \varepsilon_0$ so that*

$$\mathbf{E\Omega_2^*} \mid\!\frac{\alpha}{\omega}\; A.$$

Now we turn to the interpretation of $\mathbf{E\Omega_2^*}$ into $\mathbf{Z\Omega}$. In a preliminary step we deal with Σ^Ω reflection and induction on the ordinals.

Lemma 51 *Assume that A is a Σ^Ω formula of SL_∞ and B an arbitrary SL_∞ formula. Assume further that C is the universal closure of one of the following two formulas:*

$$(i) \quad \neg A \vee \exists \xi A^\xi, \qquad (ii) \quad \exists \xi\, [(\forall \eta < \xi) B(\eta) \wedge \neg B(\xi)] \;\vee\; \forall \xi B(\xi).$$

Then we have $\mathbf{Z\Omega} \mid\!\frac{\Omega}{0}\; C$.

Proof. Let us first assume that C is the universal closure of (i). Then the claim is immediate by the identity axioms, the rule for Σ^Ω reflection and the $(\forall^\Omega.1)$ rules, which are all available in $\mathbf{Z\Omega}$. Secondly, let C be the universal closure of (ii) and let $\sigma_1, \ldots, \sigma_n$ be all the free variables of $\forall \xi B(\xi)$. If $\alpha_1, \ldots, \alpha_n \in T \cap \Omega$, then let $B'(\xi)$ denote the formula $B(\xi)$, where $\sigma_1, \ldots, \sigma_n$ are replaced by $\alpha_1, \ldots, \alpha_n$. Then one proves by an easy induction on β that

$$\mathbf{Z\Omega} \mid\!\frac{\alpha \# \omega^{\beta+1}}{0}\; \exists \xi\, [(\forall \eta < \xi) B'(\eta) \wedge \neg B'(\xi)],\; B'(\beta)$$

holds for all $\beta \in T \cap \Omega$, where $\alpha = \alpha_1 \# \cdots \# \alpha_n$. Now the claim is immediate by an application of the $(\forall^\Omega.2)$ rule. ☐

For formulating the interpretation theorem in compact form it is convenient to introduce the following abbreviation: If Γ is a finite set of SL_∞ formulas, then we write $\forall(\Gamma)$ for the set of all universal closures of some disjunction of the formulas in the set Γ.

Lemma 52 *Let Γ and Φ be finite sets of SL_∞ formulas so that $\Gamma \subset \Phi$. Then we have for all CL_∞ formulas A and B:*

$$A \in \forall(\Gamma) \text{ and } B \in \forall(\Phi) \quad \Longrightarrow \quad \mathbf{Z\Omega} \mid\!\frac{\Omega}{0}\; \neg A, B.$$

The proof of this lemma is straightforward, since the identity axioms are formulated for arbitrary formulas and the $(\forall^\Omega.1)$ rule is available in $\mathbf{Z\Omega}$.

Theorem 53 *Let Γ be a finite set of elementary SL_∞ formulas and A an elementary CL_∞ formula. Then we have for all ordinals $\alpha < \varepsilon_0$:*

$$\mathbf{E\Omega_2^*} \mid\!\frac{\alpha}{\omega}\; \Gamma \text{ and } A \in \forall(\Gamma) \quad \Longrightarrow \quad \mathbf{Z\Omega} \mid\!\frac{\Omega + \omega^\alpha}{\Omega + \omega}\; A.$$

PROOF. The theorem is proved by induction on α. If Γ is an axiom for Σ^Ω reflection or induction on the ordinals in $\mathbf{E\Omega_2^*}$, then the claim is immediate from Lemma 51 and the previous lemma by a cut. Further, the induction operator axioms of $\mathbf{E\Omega_2^*}$ are easily handled by the corresponding rules in $\mathbf{Z\Omega}$. The remaining basic and principal axioms of $\mathbf{E\Omega_2^*}$ follow from the axioms of $\mathbf{Z\Omega}$. If Γ is the conclusion of a basic rule of $\mathbf{E\Omega_2^*}$, then the claim essentially follows from the induction hypothesis. Consider e.g. the case where Γ is the conclusion of the introduction of an universal ordinal quantifier. Hence, Γ has the form $\Phi,\ \forall\xi B(\xi)$, and there is a $\beta < \alpha$ so that

$$\mathbf{E\Omega_2^*} \overset{\beta}{\underset{\omega}{\vdash}} \Phi,\ \forall\xi B(\xi),\ B(\sigma). \tag{1}$$

Let us assume that C is some fixed disjunction of the formulas in Φ. Furthermore, let D be the universal closure of $C \vee \forall\xi B(\xi)$ and let D' be the universal closure of $(C \vee \forall\xi B(\xi)) \vee B$. Then it is easy to establish that

$$\mathbf{Z\Omega} \overset{\Omega}{\underset{0}{\vdash}} \neg D',\ D, \tag{2}$$

where again essential use is made of the $\forall^\Omega$-rules. Furthermore, by (1) and the induction hypothesis we know that $\mathbf{Z\Omega} \overset{\Omega+\omega^\beta}{\underset{\Omega+\omega}{\vdash}} D'$. By a cut with (2) this amounts to $\mathbf{Z\Omega} \overset{\Omega+\omega^\beta+1}{\underset{\Omega+\omega}{\vdash}} D$. Now if $A \in \forall(\Gamma)$, then an application of Lemma 52 yields

$$\mathbf{Z\Omega} \overset{\Omega+\omega^\alpha}{\underset{\Omega+\omega}{\vdash}} A \tag{3}$$

by a cut as desired. This finishes the discussion of the basic rules of $\mathbf{E\Omega_2^*}$. Furthermore, if Γ is the conclusion of a cut rule, then the claim immediately follows from the induction hypothesis. $\square$

The proof-theoretic reduction of the Σ^Ω part of $\widetilde{\mathbf{E\Omega}}$ is now straightforward from Corollary 50, Theorem 53 and Corollary 28.

Theorem 54 *Let A be a closed Σ^Ω formula which is provable in $\widetilde{\mathbf{E\Omega}}$. Then there exists an $\alpha < \psi\varepsilon_{\Omega+\varepsilon_0}$ so that $\mathbf{Z\Omega} \overset{\alpha}{\underset{0}{\vdash}} A$.*

By Theorem 29, Lemma 11, Corollary 48 and formalization of the previous theorem we have established the following corollary.

Corollary 55 *The proof-theoretic strength of the three theories $\widetilde{\mathbf{E\Omega}}$, $\widetilde{\mathbf{EID_1}}$ and $\mathbf{ACA} + (\Pi^1_1\text{-CA})^-$ can be characterized as follows:*

1. $|\widetilde{\mathbf{E\Omega}}| = |\widetilde{\mathbf{EID_1}}| = |\mathbf{ACA} + (\Pi^1_1\text{-CA})^-| = \psi\varepsilon_{\Omega+\varepsilon_0}$.
2. $\widetilde{\mathbf{E\Omega}} \equiv \widetilde{\mathbf{EID_1}} \equiv \mathbf{ACA} + (\Pi^1_1\text{-CA})^-$.

9 The proof-theoretic strength of $\mathbf{E\Omega}$, $\mathbf{EID_1}$ and $\mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^-$

It remains to provide the proof-theoretic treatment of the elementary Gödel-Bernays extensions of $\mathbf{ID_1}$ and $\mathbf{PA_\Omega}$ with full induction on the natural numbers and full induction on the fixed points and ordinals, respectively. The treatment of the theories $\mathbf{EID_1}$ and $\mathbf{E\Omega}$ is simpler than that of the previous theories in the sense that the wellordering proof for $\mathbf{EID_1}$ is a straightforward combination of the wellordering proofs of $\mathbf{ACA}$ and $\mathbf{ID_1}$ and the upper bound for $\mathbf{E\Omega}$ can be established by a direct interpretation into $\mathbf{Z\Omega}$ without a previous partial cut elimination argument.

9.1 Lower bounds

In the sequel we show that $\mathbf{EID_1}$ proves the wellfoundedness of each initial segment of $\psi\varepsilon_{\varepsilon_{\Omega+1}}$. We will make use of the framework which we have developed in Section 8.1 for the wellordering proof for $\widetilde{\mathbf{EID}}_\mathbf{1}$. In addition, we need the following standard definition ($n < \omega$):

$$\omega_0(\Omega+1) \;:=\; \Omega+1; \quad \omega_{n+1}(\Omega+1) \;:=\; \omega^{\omega_n(\Omega+1)}.$$

Lemma 56 *We have for all $\mathbb{L}_{\mathrm{FP}}$ formulas $A(x)$ and for all $n < \omega$:*

$$\mathbf{EID_1} \vdash \mathit{TI}(<_\Omega, \omega_n(\Omega+1), A).$$

PROOF. The claim is proved by induction on n. For $n = 0$ we are done by the second part of Lemma 45. If $n > 0$, then we use the induction hypothesis and proceed as in the wellordering proof for $\mathbf{PA}$ (cf. e.g. Schütte [15]). This is possible, since we have complete induction on the natural numbers available in $\mathbf{EID_1}$ for arbitrary $\mathbb{L}_{\mathrm{FP}}$ formulas. $\square$

Lemma 57 *We have for all $n < \omega$:*

$$\mathbf{EID_1} \vdash \mathit{TI}(<_\Omega, \varepsilon_{\omega_n(\Omega+1)}, X).$$

PROOF. Let us fix $n < \omega$. The proof of this assertion runs in a similar way as the proof of Lemma 46. Again one proceeds as in the wellordering proof for $\mathbf{ACA}$ and establishes that

$$\mathbf{EID_1} \vdash x \in T \to [(\forall y <_\Omega x)\forall X\, \mathit{TI}(<_\Omega, \varepsilon_y, X) \to \forall X\, \mathit{TI}(<_\Omega, \varepsilon_x, X)]\,, \qquad (1)$$

i.e. we have that $\mathbf{EID_1} \vdash \mathit{Prog}(<_\Omega, A)$, where A is given by

$$A(x) \;:=\; x \in T \to \forall X\, \mathit{TI}(<_\Omega, \varepsilon_x, X).$$

Using the previous lemma this immediately yields

$$\mathbf{EID_1} \vdash (\forall x <_\Omega \omega_n(\Omega+1))A(x). \qquad (2)$$

Again by (1) we get $\mathbf{EID_1} \vdash A(\omega_n(\Omega+1))$, and we are done. $\square$

Theorem 58 *We have for all $n \in \mathbb{N}$:*

$$\mathbf{EID_1} \vdash TI(<, \psi\varepsilon_{\omega_n(\Omega+1)}, X).$$

PROOF. We first observe that for each $n \in \mathbb{N}$, we trivially have $\mathfrak{M}(\varepsilon_{\omega_n(\Omega+1)})$ and $K(\varepsilon_{\omega_n(\Omega+1)}) < \varepsilon_{\omega_n(\Omega+1)}$, provably in $\mathbf{EID_1}$. Then the theorem follows from the previous lemma by condensation. □

We have established the following lower bound for $\mathbf{EID_1}$ and $\mathbf{E\Omega}$, since one has $\sup_{n<\omega} \psi\varepsilon_{\omega_n(\Omega+1)} = \psi\varepsilon_{\varepsilon_{\Omega+1}}$.

Corollary 59 $\psi\varepsilon_{\varepsilon_{\Omega+1}} \leq |\mathbf{EID_1}| \leq |\mathbf{E\Omega}|$.

9.2 Upper bounds

The computation of the upper bound for $\mathbf{E\Omega}$ is a matter of routine. One simply establishes a direct interpretation of $\mathbf{E\Omega}$ into $\mathbf{Z\Omega}$ in a completely standard way. In particular, Σ^Ω reflection and full induction on the ordinals follow from Lemma 51, and complete induction on the natural numbers is proved in the usual way by making use of the ω rule, which is available in $\mathbf{Z\Omega}$. Summarizing, one easily proves the following interpretation theorem.

Theorem 60 *Let A be a closed $\mathbb{L}_\Omega$ formula which is provable in $\mathbf{E\Omega}$. Then there exists an $n < \omega$ so that*

$$\mathbf{Z\Omega} \mathrel{\vert}\!\frac{\Omega+\omega^\omega}{\Omega+\omega+n}\, A.$$

As a consequence of this theorem and Corollary 28 we immediately obtain the following reduction theorem for $\mathbf{E\Omega}$.

Theorem 61 *Let A be a closed Σ^Ω formula which is provable in $\mathbf{E\Omega}$. Then there exists an $\alpha < \psi\varepsilon_{\varepsilon_{\Omega+1}}$ so that $\mathbf{Z\Omega} \mathrel{\vert}\!\frac{\alpha}{0}\, A$.*

The final proof-theoretic equivalences are now available by Theorem 29, Lemma 11, Corollary 48 and some standard formalization arguments.

Corollary 62 *The proof-theoretic strength of the three theories $\mathbf{E\Omega}$, $\mathbf{EID_1}$ and $\mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^-$ can be characterized as follows:*

1. $|\mathbf{E\Omega}| = |\mathbf{EID_1}| = |\mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^-| = \psi\varepsilon_{\varepsilon_{\Omega+1}}$.
2. $\mathbf{E\Omega} \equiv \mathbf{EID_1} \equiv \mathbf{ACA} + (\Pi^1_1\text{-CA})^- + (\mathrm{BI_{pr}})^-$.

On the previous pages we have studied the Gödel-Bernays extension $\mathbf{EID_1}$ of $\mathbf{ID_1}$, the Gödel-Bernays extension $\mathbf{E\Omega}$ of $\mathbf{PA_\Omega}$ and a series of natural subsystems of $\mathbf{EID_1}$ and $\mathbf{E\Omega}$. Research in this direction can be generalized for example in the following two ways:

(i) Consider Gödel-Bernays extensions of $\mathbf{ID_1}$ and $\mathbf{PA_\Omega}$ generated by stronger comprehension principles.

(ii) Start off from other first order theories and analyze their Gödel-Bernays extension.

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