Totality in applicative theories

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Abstract

In this paper we study applicative theories of operations and numbers with (and without) the non-constructive minimum operator in the context of a *total* application operation. We determine the proof-theoretic strength of such theories by relating them to well-known systems like Peano Arithmetic PA and the system $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$ of second order arithmetic. Essential use will be made of so-called fixed-point theories with ordinals, certain infinitary term models and Church Rosser properties.

1 Introduction

Partial and total applicative theories provide an elementary framework for many activities in (the foundations of) mathematics and computer science. They are discussed in a series of publications and studied from a proof-theoretic and model-theoretic point of view. Feferman [4, 5] introduced theories with self-application as a basis for his systems of explicit mathematics, e.g. the theory T_0, and those are broadly discussed in the textbooks Beeson [2] and Troelstra and van Dalen [16]. Applicative theories emphasizing on a *total* application operation are considered e.g. in Cantini [3], Gordeev [10] and Renardel de Lavalette [14].

This article is a direct companion of Feferman and Jäger [8]. It deals with applicative theories of operations and numbers and with the non-constructive minimum operator in this context. However, in contrast to the theories in [8], which are based on a partial form of term application, we now assume that term application is total. This modification of application has some drastic consequences, including the fact that elementary recursion-theoretic models are no longer permitted. Furthermore, theories with a total application operation have some important advantages compared to their partial analogues, e.g. as far as the role of substitutions is concerned. Questions concerning substitutions are discussed in Strahm [15].

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The natural and interesting models for total applicative theories are *term models* with suitable forms of term reduction and Church Rosser properties. As the recursion-theoretic models in a partial setting, term models are very attractive from a computational point of view. They provide an operational semantics of total applicative theories based on term reductions. We will make use of formalized versions of such constructions in order to show that the proof-theoretic strength of the theories studied in [8] does not change if the axiom of totality (*Tot*) is added. Moreover, term models provide the adequate tool in order to handle the axiom of extensionality (*Ext*). It turns out that extensionality does not raise the proof-theoretic strength of the theories studied in this paper.

Taking up the notation of Feferman and Jäger [8], we will establish in particular that the system $BON(\mu) + (Tot) + (Ext) + (Set\text{-}IND_N)$ is proof-theoretically equivalent to PA, and that $BON(\mu) + (Tot) + (Ext) + (Fmla\text{-}IND_N)$ is equivalent to $(\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$.

2 The general framework for total applicative theories

In this section we introduce the basic theory *TON* of *total* operations and numbers. We recapitulate various forms of induction on the natural numbers as well as the axioms for the non-constructive minimum operator.

The language L of the theory of total operations and numbers *TON* is a first order language with the individual variables $u, v, w, x, y, z, f, g, h, \ldots$ (possibly with subscripts). In addition, L includes the individual constants 0, k, s, p, p_0, p_1, s_N, p_N, d_N, r_N and $\boldsymbol{\mu}$, the meaning of which will be explained later. L has a binary function symbol $\cdot$ for term application and the relation symbols $=$ and N.

The *individual terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of L are generated as follows:

1. Each individual variable is an individual term.

2. Each individual constant is an individual term.

3. If s and t are individual terms, then so also is $(s \cdot t)$.

In the following we write (st) or just st instead of $(s \cdot t)$, and we adopt the convention of association to the left, i.e. $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2) \ldots s_n)$. Furthermore, we write $\vec{x}$ for a sequence $x_1, \ldots, x_n$ of individual variables.

The *formulas* $(\varphi, \chi, \psi, \varphi_1, \chi_1, \psi_1, \ldots)$ of L are generated as follows:

1. Each atomic formula $(s = t)$ and $N(t)$ is a formula.

2. If φ and ψ are formulas, then so also are $\neg\varphi$ and $(\varphi \vee \psi)$.

3. If φ is a formula, then so also is $(\exists x)\varphi$.

The underlying logic of *TON* is classical first order predicate calculus with equality. Hence the remaining logical connectives and the universal quantifier are defined as usual.

In the sequel we write t' for $\mathsf{s}_N t$. If $n \in \omega$ then $\overline{n}$ denotes the nth numeral of L, i.e. $\overline{0} = 0$ and $\overline{n+1} = \overline{n}'$. In addition, we will use the following abbreviations concerning the predicate N:

$$\begin{array}{rcl} t \in N & := & N(t), \\ (\exists x \in N)\varphi & := & (\exists x)(x \in N \wedge \varphi), \\ (\forall x \in N)\varphi & := & (\forall x)(x \in N \rightarrow \varphi), \\ (t : N \rightarrow N) & := & (\forall x \in N)(tx \in N), \\ (t : N^{m+1} \rightarrow N) & := & (\forall x \in N)(tx : N^m \rightarrow N), \\ t \in P(N) & := & (\forall x \in N)(tx = 0 \vee tx = 1). \end{array}$$

The non-logical axioms of *TON* are divided into the following five groups:

I. Combinatory Algebra

(1) $\mathsf{k}xy = x$,

(2) $\mathsf{s}xyz = xz(yz)$.

II. Pairing and Projection

(3) $\mathsf{p}_0(\mathsf{p}xy) = x \;\wedge\; \mathsf{p}_1(\mathsf{p}xy) = y$,

(4) $\mathsf{p}xy \neq 0$.

III. Natural Numbers

(5) $0 \in N \;\wedge\; (\forall x \in N)(x' \in N)$,

(6) $(\forall x \in N)(x' \neq 0 \wedge \mathsf{p}_N(x') = x)$,

(7) $(\forall x \in N)(x \neq 0 \;\rightarrow\; \mathsf{p}_N x \in N \wedge (\mathsf{p}_N x)' = x)$.

IV. Definition by Cases on N

(8) $v \in N \wedge w \in N \wedge v = w \;\rightarrow\; \mathsf{d}_N xyvw = x$,

(9) $v \in N \wedge w \in N \wedge v \neq w \;\rightarrow\; \mathsf{d}_N xyvw = y$.

V. Primitive Recursion on N

(10) $(f : N \to N) \ \wedge \ (g : N^3 \to N) \ \to \ (\mathsf{r}_N fg : N^2 \to N)$,

(11) $(f : N \to N) \ \wedge \ (g : N^3 \to N) \ \wedge \ x \in N \ \wedge \ y \in N \ \wedge \ h = \mathsf{r}_N fg \ \to$
$hx0 = fx \ \wedge \ hx(y') = gxy(hxy)$.

The basis of the theory *TON* are the axioms of a *total* combinatory algebra, which are well-known to be of an enormous expressive power. p is a pairing operation on the universe, which has p_0 and p_1 as its inverse functions. s_N and p_N provide the usual successor and predecessor function on the natural numbers N. Furthermore, we have definition by integer cases, which is accomplished by the d_N operation. r_N guarantees closure under primitive recursion.

It is an immediate consequence of the standard work in combinatory logic (cf. e.g. Barendregt [1]) that *TON* proves a theorem about λ abstraction and the recursion theorem.

Remark 1 If (*Tot*) denotes the *axiom of totality* saying that application is always defined, $(\forall x, y)(xy\downarrow)$, then it is obvious that *TON* is the total version of the theory *BON* of Feferman and Jäger [8], i.e. *TON* is equivalent to *BON*+ (*Tot*).

In the following we are mainly interested in two forms of complete induction on the natural numbers N, namely set and formula induction. Sets of natural numbers are represented via their total characteristic functions.

Set Induction on N $\quad$ (*Set-IND*$_N$)

$$f \in P(N) \ \wedge \ f0 = 0 \ \wedge \ (\forall x \in N)(fx = 0 \to f(x') = 0) \ \to \ (\forall x \in N)(fx = 0),$$

Formula Induction on N $\quad$ (*Fmla-IND*$_N$)

$$\varphi(0) \ \wedge \ (\forall x \in N)(\varphi(x) \to \varphi(x')) \ \to \ (\forall x \in N)\varphi(x)$$

for all formulas φ of L.

Below we give the axioms for the *non-constructive unbounded minimum operator* $\boldsymbol{\mu}$. The present formulation of the axiom $(\mu.1)$ is a strengthening of the axiom $(\mu.1)$ in Feferman and Jäger [8] in the following sense: $\boldsymbol{\mu}$ is not only a functional on $(N \to N)$ which assigns to each f with $(f : N \to N)$ an $x \in N$ with $fx = 0$, if there is any such x, and any y in N otherwise, but $\boldsymbol{\mu}$ also has the property that $\boldsymbol{\mu} f \in N$ already implies that f is an operation from N to N, i.e. $(f : N \to N)$. This modification is irrelevant for the systems studied in this paper, however, it will be of great importance for systems with forms of induction between set and formula induction.

$(\mu.1)$ $\quad (f : N \to N) \;\leftrightarrow\; \boldsymbol{\mu} f \in N,$

$(\mu.2)$ $\quad (f : N \to N) \;\wedge\; (\exists x \in N)(fx = 0) \;\to\; f(\boldsymbol{\mu} f) = 0.$

In the following we write $TON(\mu)$ for $TON+(\mu.1, \mu.2)$.

Finally, we are also interested in the *axiom of extensionality* (*Ext*), which has the following form:

$$(Ext) \qquad (\forall x)(fx = gx) \;\to\; (f = g).$$

This finishes the description of the framework for total applicative theories with natural numbers.

3 Term models of total applicative theories

In this section we outline the general idea of a term model of a total applicative theory. The kernel of a term model is a specific reduction relation on closed terms of L, which leads to a translation of the language L into a language of arithmetic. Formalized term models will be the essential tool in order to determine proof-theoretic upper bounds of our systems in the following two sections.

In the following let L_1 be the usual first order language of arithmetic with number variables $u, v, w, x, y, z, \ldots$ (possibly with subscripts), the constant 0, as well as function and relation symbols for all primitive recursive functions and relations. The number terms of L_1 $(r, s, t, r_1, s_1, t_1, \ldots)$ are defined as usual.

We will use standard notation for coding sequences of natural numbers: $\langle \ldots \rangle$ is a primitive recursive function for forming n tuples $\langle t_0, \ldots, t_{n-1} \rangle$; *Seq* denotes the primitive recursive set of sequence numbers; $lh(t)$ gives the length of the sequence coded by t, i.e. if $t = \langle t_0, \ldots, t_{n-1} \rangle$ then $lh(t) = n$; $(t)_i$ denotes the ith component of the sequence coded by t if $i < lh(t)$. Furthermore, $\dot{-}$ is the usual primitive recursive cut-off difference on the naturals.

In order to formalize term models we need a Gödelnumbering of the closed terms of the language L. Therefore, let us assign to each constant c of L and the application symbol $\cdot$ natural numbers $\ulcorner c \urcorner$ and $\ulcorner \cdot \urcorner$ in some appropriate way. In particular, $\ulcorner c \urcorner$ and $\ulcorner \cdot \urcorner$ must not be elements of *Seq*. The Gödelnumber of a compound term (st) is then given in the obvious way by

$$\ulcorner st \urcorner = \langle \ulcorner \cdot \urcorner, \ulcorner s \urcorner, \ulcorner t \urcorner \rangle\,.$$

In the following $CTer(x)$ denotes the primitive recursive predicate expressing that x is the Gödelnumber of a closed term of L. If $\vec{x} = x_1, \ldots, x_n$ then we often write

$CTer(\vec{x})$ instead of $CTer(x_1) \wedge \cdots \wedge CTer(x_n)$. Furthermore, let $Num : \omega \to \omega$ be the primitive recursive function satisfying $Num(x) = \ulcorner \overline{x} \urcorner$, i.e. $Num(x)$ is the Gödelnumber of the xth numeral of L.

We are ready to describe term models by giving a translation of the language L. The translation depends on a formula $Red(x, y)$, which has the intended meaning that the term with Gödelnumber x reduces to the term with Gödelnumber y.

Assume that $\mathcal{L}$ is a first order language containing L_1 and let $Red(x, y)$ be an $\mathcal{L}$ formula having exactly the free variables x and y. The translation $*$ of L into $\mathcal{L}$ depending on Red is given by the following clauses 1–8:

The $*$ translation t^* of an individual term t of L is given as follows:

1. If t is an individual variable, then t^* is t.

2. If t is an individual constant, then t^* is $\ulcorner t \urcorner$.

3. If t is the individual term (rs), then t^* is $\langle \ulcorner \cdot \urcorner, r^*, s^* \rangle$.

The $*$ translation φ^* of an L formula φ is given as follows:

4. If φ is the formula $(s = t)$, then φ^* is
$$(\exists x)(Red(s^*, x) \ \wedge \ Red(t^*, x)).$$

5. If φ is the formula $N(t)$, then φ^* is
$$(\exists x)\, Red(t^*, Num(x)).$$

6. If φ is the formula $\neg\psi$ then φ^* is $\neg(\psi^*)$.

7. If φ is the formula $(\psi \vee \chi)$, then φ^* is the formula $(\psi^* \vee \chi^*)$.

8. If φ is the formula $(\exists x)\psi$, then φ^* is $(\exists x)(CTer(x) \ \wedge \ \psi^*)$.

Summarizing, s and t are equal in the term model if they have a common reduct w.r.t. Red, and s is a natural number if s reduces to a numeral w.r.t. Red. The connectives have their standard meaning and the quantifiers are supposed to range over the closed terms of L. Furthermore, observe that application has a trivial interpretation via a suitable Gödelnumbering.

For a specific choice of the reduction relation Red it will be essential to verify the *Church Rosser property* $CR(Red)$, i.e. the statement
$$(\forall x)(\forall y_1)(\forall y_2)[Red(x, y_1) \wedge Red(x, y_2) \ \to \ (\exists z)(Red(y_1, z) \wedge Red(y_2, z))].$$

$CR(Red)$ is needed e.g. in order to establish the transitivity of the equality relation.

This has to be contrasted to the treatment of a *partial* application relation in Feferman and Jäger [8]. There recursion-theoretic models are used in order to establish proof-theoretic upper bounds of partial applicative theories. The main step in finding a recursion-theoretic interpretation is to find a suitable formula $App(x, y, z)$, which interprets the formula $xy \simeq z$ (cf. Section 5.2 of [8]). It is then essential to prove that App is *functional*, i.e.

$$(\forall x)(\forall y)(\forall z_1)(\forall z_2)(App(x, y, z_1) \wedge App(x, y, z_2) \ \rightarrow \ z_1 = z_2),$$

(cf. e.g. Lemma 19 of [8]). Summing up, the specification of a functional application relation App in the partial setting corresponds to finding a reduction relation Red enjoying the Church Rosser property in the context of a total application operation.

In the next two sections we will specify two different reduction relations, which will enable us to determine proof-theoretic upper bounds of $TON + (\textit{Set-IND}_N)$, $TON + (\textit{Fmla-IND}_N)$ and $TON(\mu) + (\textit{Set-IND}_N)$, $TON(\mu) + (\textit{Fmla-IND}_N)$, respectively, by formalizing the corresponding term models in appropriate systems of arithmetic.

4 The proof-theoretic strength of *TON* with set and with formula induction

Although the main emphasis of this paper is put on the systems with the unbounded minimum operator, we will give the proof-theoretic analysis of the systems without the minimum operator for the sake of completeness. In the following we establish the proof-theoretic strength of TON with set and with formula induction respectively as:

$$\begin{aligned} TON + (\textit{Set-IND}_N) &\equiv PRA, \\ TON + (\textit{Fmla-IND}_N) &\equiv PA\,. \end{aligned}$$

Here '$\equiv$' denotes the usual notion of proof-theoretic equivalence as it is defined e.g. in Feferman [7]. It has to be mentioned that the first of these equivalences follows from independent work of Cantini [3], whereas the second equivalence is well-known from the literature, cf. e.g. Beeson [2].

As usual PA denotes the system of Peano arithmetic formulated in L_1; PA includes defining axioms for all primitive recursive functions and relations as well as all instances of complete induction on the natural numbers. PRA is the system of primitive recursive arithmetic and is obtained from PA by restricting induction to quantifier-free L_1 formulas. It is well-known from Parsons [13][1] that PRA is proof-theoretically equivalent to $PRA + (\Sigma^0_1\text{-}IND_N)$, i.e. the subsystem of PA with induction restricted to Σ^0_1 formulas; as usual φ is a Σ^0_1 formula if φ has the form $(\exists x)\psi$, where ψ is a quantifier-free L_1 formula.

From the work in [8] it is immediate that PRA is contained in $TON + (Set\text{-}IND_N)$ and that $TON + (Fmla\text{-}IND_N)$ contains PA. Hence, it suffices to show that these lower bounds are sharp. This will be established by formalizing a term model of $TON + (Set\text{-}IND_N)$ and $TON + (Fmla\text{-}IND_N)$ in $PRA + (\Sigma^0_1\text{-}IND_N)$ and PA, respectively.

In the following we will introduce some general notions concerning reduction relations. We will adopt the notation from Barendregt [1] pages 50ff. with the only exception that all our reduction relations are defined on *closed* L terms only.

A *notion of reduction* is just a binary relation R on the closed L terms. If R_1 and R_2 are notions of reduction, then R_1R_2 denotes $R_1 \cup R_2$. A notion of reduction R induces the binary relation $\rightarrow_R$ of one step R reduction (the compatible closure of R) and the binary relation $\twoheadrightarrow_R$ of R reduction (the reflexive, transitive closure of $\rightarrow_R$).

In the sequel we will need formalized versions of R, $\rightarrow_R$ and $\twoheadrightarrow_R$ respectively on the Gödelnumbers of closed terms of L. Therefore, let $\mathcal{L}$ be a first order language containing L_1 and let $RedCon_R(x, y)$ be an $\mathcal{L}$ formula formalizing R. Then the formalized version $Red1_R(x, y)$ of $\rightarrow_R$ can be described by the following primitive recursive (in $RedCon_R$) definition:

$$Red1_R(x, y) \ := \ CTer(x) \wedge CTer(y) \wedge Red1^*_R(x, y),$$

where $Red1^*_R(x, y)$ is the disjunction of the following formulas:

(1) $RedCon_R(x, y)$,

(2) $x = \langle \ulcorner \cdot \urcorner, (x)_1, (x)_2 \rangle \ \wedge y = \langle \ulcorner \cdot \urcorner, (x)_1, (y)_2 \rangle \ \wedge \ Red1_R((x)_2, (y)_2)$,

(3) $x = \langle \ulcorner \cdot \urcorner, (x)_1, (x)_2 \rangle \ \wedge y = \langle \ulcorner \cdot \urcorner, (y)_1, (x)_2 \rangle \ \wedge \ Red1_R((x)_1, (y)_1)$.

In order to formalize the reflexive, transitive closure $\twoheadrightarrow_R$ of $\rightarrow_R$ one defines an intermediate predicate $RedSeq_R(x, y, z)$ with the intended meaning that x codes a reduction sequence from the closed term with Gödelnumber y to the closed term with Gödelnumber z w.r.t. R:

$$RedSeq_R(x, y, z) \ := \ Seq(x) \wedge CTer(y) \wedge CTer(z) \wedge RedSeq^*_R(x, y, z),$$

where $RedSeq^*_R(x, y, z)$ is the disjunction of the following formulas:

[1] Parsons showed that $PRA + (\Sigma^0_1\text{-}IND_N)$ is conservative over PRA for Π^0_2 statements.

(1) $lh(x) = 1 \;\wedge\; x = \langle y \rangle \;\wedge y = z$,

(2) $lh(x) > 1 \;\wedge\; y = (x)_0 \;\wedge\; z = (x)_{lh(x) \dot{-} 1} \;\wedge\; (\forall i < lh(x) \dot{-} 1)\; \mathit{Red1}_R((x)_i, (x)_{i+1})$.

The formalization Red_R of $\twoheadrightarrow_R$ is then given in a straightforward manner as follows:

$$\mathit{Red}_R(x, y) \;:=\; (\exists z)\, \mathit{RedSeq}_R(z, x, y).$$

This finishes our general considerations concerning reduction relations. The notion of reduction ρ that is appropriate for the proof-theoretic analysis of the systems $\mathit{TON} + (\mathit{Set}\text{-}\mathit{IND}_N)$ and $\mathit{TON} + (\mathit{Fmla}\text{-}\mathit{IND}_N)$ is just the usual notion of reduction for combinatory logic (cf. e.g. Barendregt [1]) extended by reduction rules for the constants $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{s}_N$, $\mathbf{p}_N$, $\mathbf{d}_N$ and $\mathbf{r}_N$. The relation ρ is given by the following redex-contractum pairs, where t_0, t_1, t_2, s are closed L terms and $m, n \in \omega$ with $m \neq n$:

$$\begin{array}{rcl}
\mathbf{k} t_0 t_1 & \rho & t_0, \\
\mathbf{s} t_0 t_1 t_2 & \rho & t_0 t_2 (t_1 t_2), \\
\mathbf{p}_0 (\mathbf{p} t_0 t_1) & \rho & t_0, \\
\mathbf{p}_1 (\mathbf{p} t_0 t_1) & \rho & t_1, \\
\mathbf{p}_N (\mathbf{s}_N \overline{m}) & \rho & \overline{m}, \\
\mathbf{d}_N t_0 t_1 \overline{m}\, \overline{m} & \rho & t_0, \\
\mathbf{d}_N t_0 t_1 \overline{m}\, \overline{n} & \rho & t_1, \\
\mathbf{r}_N t_0 t_1 s\, \overline{0} & \rho & t_0 s, \\
\mathbf{r}_N t_0 t_1 s\, \overline{m+1} & \rho & t_1 s\, \overline{m} (\mathbf{r}_N t_0 t_1 s\, \overline{m}).
\end{array}$$

It is obvious that the formalization RedCon_ρ of ρ is primitive recursive. Hence we have the following observation.

Remark 2 The formula $\mathit{Red}_\rho(x, y)$ is (equivalent in PRA to) a Σ^0_1 formula.

Using the standard method of parallelization it is straightforward to prove that $\twoheadrightarrow_\rho$ has the Church Rosser property (cf. e.g. Barendregt [1]). Furthermore, it is well-known that such a proof can be formalized in PRA (cf. e.g. Girard [9]).

Theorem 3 *$\mathit{PRA} \vdash \mathit{CR}(\mathit{Red}_\rho)$.*

In the sequel we will work with the translation $*$ of L into L_1 depending on Red_ρ which we have discussed in Section 3. Before we state the final proof-theoretic reduction, we want to mention an important lemma, which is an immediate consequence of the (formalized) Church Rosser theorem.

Lemma 4 *We have for all L formulas $\varphi(x)$:*

$$PRA \vdash Red_\rho(x,y) \;\rightarrow\; (\varphi^*(x) \;\leftrightarrow\; \varphi^*(y)).$$

It is easy to verify that the $*$ translations of the axioms of TON are provable in $PRA + (\Sigma^0_1\text{-}IND_N)$. Furthermore, $(Set\text{-}IND_N)$ in L translates into $(\Sigma^0_1\text{-}IND_N)$ in L_1 according to Remark 2, and $(Fmla\text{-}IND_N)$ in L translates into full induction in L_1. Thus we have established the reduction of $TON + (Set\text{-}IND_N)$ to $PRA + (\Sigma^0_1\text{-}IND_N)$ and of $TON + (Fmla\text{-}IND_N)$ to PA. Using the result of Parsons mentioned above this yields the final proof-theoretic equivalences stated at the beginning of this section.

Theorem 5 *We have for all L formulas $\varphi(\vec{x})$ with at most $\vec{x}$ free:*

1. $TON + (Set\text{-}IND_N) \vdash \varphi(\vec{x}) \quad \Longrightarrow \quad PRA + (\Sigma^0_1\text{-}IND_N) \vdash CTer(\vec{x}) \;\rightarrow\; \varphi^*(\vec{x})$.
2. $TON + (Fmla\text{-}IND_N) \vdash \varphi(\vec{x}) \quad \Longrightarrow \quad PA \vdash CTer(\vec{x}) \;\rightarrow\; \varphi^*(\vec{x})$.

Corollary 6 *We have the following proof-theoretic equivalences:*

1. $TON + (Set\text{-}IND_N) \;\equiv\; PRA$.
2. $TON + (Fmla\text{-}IND_N) \;\equiv\; PA$.

Remark 7

1. The theories $TON + (Fmla\text{-}IND_N)$ and PA are not only proof-theoretically equivalent, but they are even mutually relatively interpretable in the standard sense of Tarski.
2. Using the translation $*$ it is easy to see that one can strengthen set induction $(Set\text{-}IND_N)$ without going beyond PRA; e.g. one can allow complete induction on N for Σ^+_1 formulas, i.e. positive existential formulas, and it is even possible to admit bounded universal quantifiers in a careful way. Then it is obvious that $TON + (\Sigma^+_1\text{-}IND_N)$ and $PRA + (\Sigma^0_1\text{-}IND_N)$ are mutually relatively interpretable. But we do not know whether this already holds for the pair $TON + (Set\text{-}IND_N)$ and $PRA + (\Sigma^0_1\text{-}IND_N)$.

Let us finish this section by sketching a proof of the fact that Corollary 6 still holds if the extensionality axiom (*Ext*) is added to the systems $TON + (Set\text{-}IND_N)$ and $TON + (Fmla\text{-}IND_N)$ respectively. Obviously, the term model induced by $*$ is not extensional. However, the proof-theoretic upper bounds can be established by formalizing the term model of the $\lambda\eta$ calculus (extended by reduction rules for the additional constants of L) using the standard translation of combinatory logic into λ calculus. It is easy to verify that the Church Rosser theorem for the $\lambda\eta$ calculus (cf. e.g. [1]) can be formalized in PRA. In the context of extensionality, the universe of a term model consists of *all* terms of L and not only the closed L terms, of course.

Theorem 8 *We have the following proof-theoretic equivalences:*

1. $TON + (Ext) + (Set\text{-}IND_N) \;\equiv\; PRA.$

2. $TON + (Ext) + (Fmla\text{-}IND_N) \;\equiv\; PA.$

It would also be possible to provide an extensional version of the combinatory reduction relation $\twoheadrightarrow_\rho$ (cf. e.g. Hindley and Seldin [11]), but since the Church Rosser theorem for such a reduction relation is proved using the confluence for the $\lambda\eta$ calculus, this does not seem very natural to us.

5 The proof-theoretic strength of $\boldsymbol{TON(\mu)}$ with set and with formula induction

In the following we determine the proof-theoretic strength of $TON(\mu)$ with set and with formula induction as:

$$\begin{aligned} TON(\mu) + (Set\text{-}IND_N) &\equiv PA, \\ TON(\mu) + (Fmla\text{-}IND_N) &\equiv (\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}. \end{aligned}$$

Again the lower bounds of the above equivalences are established in Feferman and Jäger [8]. Also as in [8], the corresponding upper bounds are computed using the fixed point theories with ordinals PA^r_Ω and PA^w_Ω whose proof-theoretic analysis has been carried through in Jäger [12]. In particular, the following equivalences have been established there:

$$PA^r_\Omega \equiv PA, \qquad PA^w_\Omega \equiv \widehat{ID}_1 .$$

In contrast to Feferman and Jäger [8] we will not make use of a (partial) recursion-theoretic model, but a specific *infinitary* term model, which can be formalized in the fixed point theories PA^r_Ω and PA^w_Ω respectively.

In the sequel we will give an informal description of the reduction relation $\twoheadrightarrow_{\rho\mu}$ before we discuss its formalization in the corresponding fixed point theories.

The stages μ_α of the μ redex-contractum pairs are defined by transfinite recursion on the ordinals and generated by the following two clauses (1) and (2), where t is a closed L term and $k, l, m \in \omega$:

(1) If $t\overline{m}\; r_\alpha\; \overline{0}$ and $(\forall k)(\exists l)[t\overline{k}\; r_\alpha\; \overline{l} \;\wedge\; (k < m \rightarrow l > 0)]$, then $\boldsymbol{\mu} t\; \mu_\alpha\; \overline{m}$,

(2) If $(\forall k)(\exists l > 0)(t\overline{k}\; r_\alpha\; \overline{l})$, then $\boldsymbol{\mu} t\; \mu_\alpha\; \overline{0}$,

where

$$r_\alpha = \bigcup_{\beta<\alpha} \twoheadrightarrow_{\rho\mu_\beta} .$$

This finishes our specification of $\twoheadrightarrow_{\rho\mu}$ by taking μ as $\bigcup_\alpha \mu_\alpha$.

In the following the reader is assumed to be familiar with the fixed point theories PA^r_Ω and PA^w_Ω of Jäger [12]. We will give a sketchy description of the language L_Ω for reasons of completeness only.

If P is a new n-ary relation symbol then $L_1(P)$ denotes the extension of L_1 by P. An *inductive operator form* $A(P,\vec{x})$ is an $L_1(P)$ formula in which each occurrence of P is positive and which contains at most $\vec{x}$ free. The first order language L_Ω is an extension of L_1 by a new sort of *ordinal variables* $\alpha, \beta, \gamma, \ldots$, a new relation symbol $<$ for the less relation on the ordinals[1] and an $(n+1)$-ary relation symbol P_A for each inductive operator form $A(P,\vec{x})$ for which P is n-ary.

The atomic formulas of L_Ω are the atomic formulas of L_1 plus formulas of the form $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_A(\alpha, \vec{s})$, the latter of which have the intended meaning that $\vec{s}$ belongs to the αth stage of the positive inductive definition induced by the inductive operator form $A(P,\vec{x})$. We will often write $P^\alpha_A(\vec{s})$ instead of $P_A(\alpha,\vec{s})$. The L_Ω formulas are obtained by closing the atomic formulas under negation, disjunction and quantification over both number and ordinal variables. As in Jäger [12] we will use the following abbreviations:

$$\begin{aligned} P^{<\alpha}_A(\vec{s}) &:= (\exists\beta < \alpha) P^\beta_A(\vec{s}), \\ P_A(\vec{s}) &:= (\exists\alpha) P^\alpha_A(\vec{s}). \end{aligned}$$

For the definition of Δ^Ω_0 and Σ^Ω formulas as well as the exact formulation of the L_Ω theories PA^r_Ω and PA^w_Ω the reader is referred to Jäger [12].

We are ready to describe the formalization of $\twoheadrightarrow_{\rho\mu}$ in the language L_Ω. In particular, $\twoheadrightarrow_{\rho\mu}$ will be represented as a fixed point of a positive inductive definition.

Let P be a new binary relation symbol. Then the μ redex-contractum pairs w.r.t. P are given as follows:

$$RedCon_\mu(P,x,y) := CTer(x) \wedge CTer(y) \wedge RedCon^*_\mu(P,x,y),$$

where $RedCon^*_\mu(P,x,y)$ is the disjunction of the following formulas:

(1) $x = \langle \ulcorner\cdot\urcorner, \ulcorner\boldsymbol{\mu}\urcorner, (x)_2\rangle \;\wedge$

$\quad (\exists z)[Num(z) = y \;\wedge\; P(\langle\ulcorner\cdot\urcorner, (x)_2, y\rangle, \ulcorner 0\urcorner) \;\wedge$

$\quad\quad (\forall u)(\exists v)(P(\langle\ulcorner\cdot\urcorner, (x)_2, Num(u)\rangle, Num(v)) \;\wedge\; (u < z \rightarrow v > 0))]$,

[1] It will always be clear from the context whether $<$ denotes the less relation on the nonnegative integers or on the ordinals.

(2) $x = \langle \ulcorner \cdot \urcorner, \ulcorner \boldsymbol{\mu} \urcorner, (x)_2 \rangle \ \wedge\ y = \ulcorner 0 \urcorner \ \wedge\ (\forall u)(\exists v > 0) P(\langle \ulcorner \cdot \urcorner, (x)_2, Num(u) \rangle, Num(v))$.

The following formula describes the $\rho\mu$ redex-contractum pairs w.r.t. P:

$$RedCon_{\rho\mu}(P, x, y) \ := \ RedCon_{\rho}(x, y) \ \vee \ RedCon_{\mu}(P, x, y).$$

Once we have given the formula $RedCon_{\rho\mu}(P, x, y)$ the formulas $Red1_{\rho\mu}(P, x, y)$, $RedSeq_{\rho\mu}(P, x, y, z)$ and finally $Red_{\rho\mu}(P, x, y)$ are defined exactly as in Section 4 with the only difference of containing the parameter P.

Remark 9 The formula $Red_{\rho\mu}(P, x, y)$ is an inductive operator form.

We are ready to put down the formal representation of $\twoheadrightarrow_{\rho\mu}$ in L_Ω as a fixed point $P_{Red_{\rho\mu}}(x, y)$ of $Red_{\rho\mu}(P, x, y)$, i.e. as the formula

$$(\exists \alpha)\ P^{\alpha}_{Red_{\rho\mu}}(x, y).$$

Again it is essential to verify that $\twoheadrightarrow_{\rho\mu}$ enjoys the Church Rosser property and that such a proof can be carried through in the system PA^r_Ω for the formalization $P_{Red_{\rho\mu}}$ of $\twoheadrightarrow_{\rho\mu}$. The detailed proof of this fact is given in the appendix of this paper.

Theorem 10 $PA^r_\Omega \vdash CR(P_{Red_{\rho\mu}})$.

We have prepared the grounds in order to work with the translation $*$ of L into L_Ω depending on $P_{Red_{\rho\mu}}$. It is easy to see that the analogue of Lemma 4 holds for $P_{Red_{\rho\mu}}$, too.

Lemma 11 *We have for all L formulas $\varphi(x)$:*

$$PA^r_\Omega \vdash P_{Red_{\rho\mu}}(x, y) \ \rightarrow\ (\varphi^*(x) \ \leftrightarrow\ \varphi^*(y)).$$

Since the system PA^w_Ω incorporates full induction on the natural numbers, it is straightforward to check the $*$ translation of (*Fmla-IND*$_N$) in PA^w_Ω. The treatment of (*Set-IND*$_N$) in the weaker theory PA^r_Ω is given in the following lemma.

Lemma 12 *The $*$ translation of (Set-IND$_N$) is provable in PA^r_Ω, i.e. PA^r_Ω proves*

$$[f \in P(N) \wedge f0 = 0 \wedge (\forall x \in N)(fx = 0 \rightarrow f(x') = 0) \ \rightarrow\ (\forall x \in N)(fx = 0)]^*.$$

Proof In the following we work informally in PA^r_Ω. Assume $(f \in P(N))^*$, $(f0 = 0)^*$ and $[(\forall x \in N)(fx = 0 \rightarrow f(x') = 0)]^*$. From the first premise and Theorem 10 we conclude

$$(\forall x)(\exists! y)\ P_{Red_{\rho\mu}}(\langle \ulcorner \cdot \urcorner, f, Num(x) \rangle, Num(y)). \tag{1}$$

The other two premises yield

$$P_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, \ulcorner 0\urcorner\rangle, \ulcorner 0\urcorner), \tag{2}$$

$$(\forall x)(P_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x)\rangle, \ulcorner 0\urcorner) \rightarrow P_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x+1)\rangle, \ulcorner 0\urcorner)). \tag{3}$$

From (1) we get by Σ^Ω reflection the existence of an ordinal α so that we have

$$(\forall x, y)(P_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x)\rangle, Num(y)) \leftrightarrow P^{<\alpha}_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x)\rangle, Num(y))). \tag{4}$$

Combining (2), (3) and (4) this amounts to

$$P^{<\alpha}_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, \ulcorner 0\urcorner\rangle, \ulcorner 0\urcorner), \tag{5}$$

$$(\forall x)(P^{<\alpha}_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x)\rangle, \ulcorner 0\urcorner) \rightarrow P^{<\alpha}_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x+1)\rangle, \ulcorner 0\urcorner)). \tag{6}$$

Now recall that we have Δ^Ω_0 induction on the natural numbers available in the system PA^r_Ω and, therefore, (5) and (6) imply

$$(\forall x)\, P^{<\alpha}_{Red_{\rho\mu}}(\langle\ulcorner\cdot\urcorner, f, Num(x)\rangle, \ulcorner 0\urcorner). \tag{7}$$

But from (7) we immediately obtain $[(\forall x \in N)(fx = 0)]^*$. This finishes our treatment of $(Set\text{-}IND_N)$ in PA^r_Ω. □

It is not difficult either to verify that the $*$ translation of each axiom of $TON(\mu)$ is provable in PA^r_Ω, in particular PA^r_Ω proves $(\mu.1)$ and $(\mu.2)$. Therefore, we have established the proof-theoretic reduction of $TON(\mu) + (Set\text{-}IND_N)$ to PA^r_Ω and of $TON(\mu) + (Fmla\text{-}IND_N)$ to PA^w_Ω.

Theorem 13 *We have for all L formulas $\varphi(\vec{x})$ with at most $\vec{x}$ free:*

1. $TON(\mu) + (Set\text{-}IND_N) \vdash \varphi(\vec{x}) \quad \Longrightarrow \quad PA^r_\Omega \vdash CTer(\vec{x}) \rightarrow \varphi^*(\vec{x})$.
2. $TON(\mu) + (Fmla\text{-}IND_N) \vdash \varphi(\vec{x}) \quad \Longrightarrow \quad PA^w_\Omega \vdash CTer(\vec{x}) \rightarrow \varphi^*(\vec{x})$.

Using a result due to Aczel (cf. Feferman [6]) concerning the strength of $\widehat{ID}_1$ we have thus determined the equivalences mentioned at the beginning of this section.

Corollary 14 *We have the following proof-theoretic equivalences:*

1. $TON(\mu) + (Set\text{-}IND_N) \ \equiv\ PA^r_\Omega \ \equiv\ PA$.
2. $TON(\mu) + (Fmla\text{-}IND_N) \ \equiv\ PA^w_\Omega \ \equiv\ \widehat{ID}_1 \ \equiv\ (\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}$.

Again Corollary 14 can be strengthened so as to include the extensionality axiom (Ext). The reduction relation $\twoheadrightarrow_{\beta\eta\mu}$ on λ terms is defined in the same way as $\twoheadrightarrow_{\rho\mu}$ except that $\beta\eta$ is used instead of ρ at each step of the corresponding inductive definition. The proof of the Church Rosser theorem for $\twoheadrightarrow_{\rho\mu}$ (see the appendix) is then easily extended to $\twoheadrightarrow_{\beta\eta\mu}$. However, a few additional considerations have to be taken into account. It is also not difficult to see that the corresponding arguments can be formalized in the system PA^r_Ω.

Theorem 15 *We have the following proof-theoretic equivalences:*

1. $TON(\mu) + (Ext) + (Set\text{-}IND_N) \ \equiv\ PA^r_\Omega \ \equiv\ PA.$
2. $TON(\mu) + (Ext) + (Fmla\text{-}IND_N) \ \equiv\ PA^w_\Omega \ \equiv\ \widehat{ID}_1 \ \equiv\ (\Pi^0_\infty\text{-}CA)_{<\varepsilon_0}.$

Appendix

In this appendix we sketch a proof of Theorem 10. In particular, we show that $\twoheadrightarrow_{\rho\mu}$ has the Church Rosser property, and we argue that our proof can be carried through in the system PA^r_Ω for the formalization $P_{Red_{\rho\mu}}$ of $\twoheadrightarrow_{\rho\mu}$.

The main idea is to prove that each stage $\twoheadrightarrow_{\rho\mu_\alpha}$ of $\twoheadrightarrow_{\rho\mu}$ is confluent. This is achieved by combining the ρ and μ_α reductions using the well-known Lemma of Hindley and Rosen (see below).

As in Section 5 we put

$$r_\alpha = \bigcup_{\beta<\alpha} \twoheadrightarrow_{\rho\mu_\beta} .$$

Furthermore, let us call a closed L term t *N singular w.r.t. r_α*, if there is no $n \in \omega$ so that $t\ r_\alpha\ \overline{n}$. The following lemma states an important property of terms $\boldsymbol{\mu}t$, which are N singular w.r.t. r_α. We do not give the proof of the lemma here, but it is important to mention that the (formalized) proof only uses Δ^Ω_0 induction on the ordinals, which is available in the system PA^r_Ω.

Lemma 16 *Let $s(x)$ be an L term with at most x free, and let $\boldsymbol{\mu}t$ be a closed L term which is N singular w.r.t. r_α. Furthermore, assume that $s(\boldsymbol{\mu}t)\ r_\alpha\ \overline{m}$ for some $m \in \omega$. Then we have $s(t')\ r_\alpha\ \overline{m}$ for all closed L terms t'.*

We will also need the following observation, the proof of which is straightforward and, therefore, we omit it.

Lemma 17 *We have for all closed L terms t and all $m \in \omega$: If $\boldsymbol{\mu}t\ r_\alpha\ \overline{m}$, then $\boldsymbol{\mu}t\ \mu_\alpha\ \overline{m}$.*

The next lemma tells us that a μ_α stage has the Church Rosser property provided that $\twoheadrightarrow_{\rho\mu_\beta}$ is confluent for all $\beta < \alpha$. Again it is easy to see that the proof of this lemma can be formalized in the system PA^r_Ω.

Lemma 18 $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta}) \implies CR(\twoheadrightarrow_{\mu_\alpha})$.

Proof Let us assume $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta})$, which immediately implies $CR(r_\alpha)$, of course. Since $CR(\twoheadrightarrow_{\mu_\alpha})$ follows from $CR(\overset{=}{\rightarrow}_{\mu_\alpha})$ by an easy diagram chase, it is enough to show $CR(\overset{=}{\rightarrow}_{\mu_\alpha})$. Here $\overset{=}{\rightarrow}_{\mu_\alpha}$ denotes the reflexive closure of $\rightarrow_{\mu_\alpha}$. First of all it is an easy consequence of $CR(r_\alpha)$ that the following holds for all closed L terms t and all $m, n \in \omega$:

$$\boldsymbol{\mu} t\ \mu_\alpha\ \overline{m}\ \wedge\ \boldsymbol{\mu} t\ \mu_\alpha\ \overline{n} \implies m = n. \tag{1}$$

The second critical case comes up if we have terms $s(x), t$ and $m, n \in \omega$ so that

$$\boldsymbol{\mu} s(\boldsymbol{\mu} t)\ \mu_\alpha\ \overline{m}, \qquad \boldsymbol{\mu} s(\boldsymbol{\mu} t) \to_{\mu_\alpha} \boldsymbol{\mu} s(\overline{n}), \tag{2}$$

where $\boldsymbol{\mu} t\ \mu_\alpha\ \overline{n}$. Then we have to show that $\boldsymbol{\mu} s(\overline{n})\ \mu_\alpha\ \overline{m}$. Assume that $\boldsymbol{\mu} s(\boldsymbol{\mu} t)\ \mu_\alpha\ \overline{m}$ holds because of clause (1) of the definition of μ_α on page 11. Then we have

$$s(\boldsymbol{\mu} t)\overline{m}\ r_\alpha\ \overline{0}, \tag{3}$$

and for each k there exists a k' so that

$$s(\boldsymbol{\mu} t)\overline{k}\ r_\alpha\ \overline{k'}, \tag{4}$$

where $k' > 0$ if $k < m$. Let us first assume that the term $\boldsymbol{\mu} t$ is N singular w.r.t. r_α. Then we can conclude from (3), (4) and Lemma 16 that

$$s(\overline{n})\overline{m}\ r_\alpha\ \overline{0}, \qquad s(\overline{n})\overline{k}\ r_\alpha\ \overline{k'} \tag{5}$$

for all $k \in \omega$, which immediately implies $\boldsymbol{\mu} s(\overline{n})\ \mu_\alpha\ \overline{m}$ by the very definition of μ_α. If $\boldsymbol{\mu} t$ is not N singular w.r.t. r_α, then there exists an $l \in \omega$ with $\boldsymbol{\mu} t\ r_\alpha\ \overline{l}$. Using the previous lemma this implies $\boldsymbol{\mu} t\ \mu_\alpha\ \overline{l}$. Since $\boldsymbol{\mu} t\ \mu_\alpha\ \overline{n}$ holds by hypothesis, this amounts to $l = n$ according to (1). We have shown $\boldsymbol{\mu} t\ r_\alpha\ \overline{n}$. From this we conclude for all $k \in \omega$:

$$s(\boldsymbol{\mu} t)\overline{m}\ r_\alpha\ s(\overline{n})\overline{m}, \qquad s(\boldsymbol{\mu} t)\overline{k}\ r_\alpha\ s(\overline{n})\overline{k}. \tag{6}$$

Using (3), (4), (6) and $CR(r_\alpha)$ we can immediately derive

$$s(\overline{n})\overline{m}\ r_\alpha\ \overline{0}, \qquad s(\overline{n})\overline{k}\ r_\alpha\ \overline{k'} \tag{7}$$

for all $k \in \omega$. But (7) implies $\boldsymbol{\mu} s(\overline{n})\ \mu_\alpha\ \overline{m}$ by the definition of μ_α as desired. The case where $\boldsymbol{\mu} s(\boldsymbol{\mu} t)\ \mu_\alpha\ \overline{m}$ has been derived by clause (2) of the definition of μ_α is treated in a similar way. This finishes the proof of the lemma. $\square$

In order to apply the Lemma of Hindley and Rosen below we have to introduce the following terminology. Let R_1 and R_2 be two binary relations on a set X. Then R_1 and R_2 *commute*, if

$$(\forall x, x_1, x_2 \in X)[x\ R_1\ x_1\ \wedge\ x\ R_2\ x_2\ \to\ (\exists x_3 \in X)(x_1\ R_2\ x_3\ \wedge\ x_2\ R_1\ x_3)].$$

The next lemma is *the* essential step towards the use of the lemma of Hindley and Rosen. Again its proof can easily be formalized in the system PA^r_Ω.

Lemma 19 *Assume that* $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta})$. *Then the reduction relations* $\twoheadrightarrow_\rho$ *and* $\twoheadrightarrow_{\mu_\alpha}$ *commute.*

Proof From $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta})$ we immediately get $CR(r_\alpha)$. We show that the following holds for all closed L terms t, t_1 and t_2: If $t \to_\rho t_1$ and $t \to_{\mu_\alpha} t_2$, then there exists a closed L term t_3 so that

$$t_2 \overset{=}{\to}_\rho t_3, \qquad t_1 \twoheadrightarrow_{\mu_\alpha} t_3. \tag{1}$$

From this the claim of the lemma follows by an easy diagram chase. In the sequel we will discuss the only critical case, namely where we have terms $s(x), t$ and an $n \in \omega$ so that

$$s(\boldsymbol{\mu} t) \to_{\mu_\alpha} s(\overline{n}), \qquad s(\boldsymbol{\mu} t) \to_\rho s(\boldsymbol{\mu} t'), \tag{2}$$

where $\boldsymbol{\mu} t\ \mu_\alpha\ \overline{n}$ and $t \to_\rho t'$. Then it is easy to check that $\boldsymbol{\mu} t'\ \mu_\alpha\ \overline{n}$ also holds, since we know $CR(r_\alpha)$. Hence, we can derive

$$s(\boldsymbol{\mu} t') \to_{\mu_\alpha} s(\overline{n}), \tag{3}$$

and we are done. This finishes the sketch of the proof of this lemma. $\Box$

We have prepared the grounds in order to apply the Lemma of Hindley and Rosen. For reasons of completeness, we give its detailed formulation below. For a proof the reader is referred to Barendregt [1], where one easily sees that the proof there only uses finitary arguments.

Lemma 20 (Hindley and Rosen) *Let* R_1 *and* R_2 *be two notions of reduction and suppose that*

(1) $\twoheadrightarrow_{R_1}$ *and* $\twoheadrightarrow_{R_2}$ *are Church Rosser,*

(2) $\twoheadrightarrow_{R_1}$ *commutes with* $\twoheadrightarrow_{R_2}$.

Then $\twoheadrightarrow_{R_1R_2}$ *has the Church Rosser property, too.*

Taking R_1 as ρ and R_2 as μ_α and assuming $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta})$ the assumptions (1) and (2) of the Lemma of Hindley and Rosen are satisfied by Lemma 18, Lemma 19 and the fact that $\twoheadrightarrow_\rho$ is Church Rosser. Hence, we can state the following lemma.

Lemma 21 $(\forall\beta < \alpha)\, CR(\twoheadrightarrow_{\rho\mu_\beta}) \implies CR(\twoheadrightarrow_{\rho\mu_\alpha})$.

We have shown that $CR(\twoheadrightarrow_{\rho\mu_\beta})$ is progressive and hence $(\forall\alpha)\, CR(\twoheadrightarrow_{\rho\mu_\alpha})$ follows by transfinite induction. Furthermore, $(\forall\alpha)\, CR(\twoheadrightarrow_{\rho\mu_\alpha})$ implies $CR(\twoheadrightarrow_{\rho\mu})$.

Corollary 22 $CR(\twoheadrightarrow_{\rho\mu})$.

This finishes our proof that $\twoheadrightarrow_{\rho\mu}$ is confluent. Notice that the formalization of $CR(\twoheadrightarrow_{\rho\mu_\beta})$ in L_Ω is a Δ_0^Ω formula and, therefore, only transfinite induction for Δ_0^Ω statements is used in the argument above. Together with our previous remarks concerning the formalization of our Church Rosser proof, we have sketched that all arguments can be carried through in the system PA_Ω^r. This finishes the considerations of this appendix.

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