universität bern
institut für informatik
und angewandte mathematik

Natural numbers and forms of weak induction in applicative theories

Reinhard Kahle

Technischer Bericht IAM-95-001

Januar 1995

Adresse:
Institut für Informatik und angewandte Mathematik
Universität Bern
Länggaßstr. 51
CH - 3012 Bern

Natural numbers and forms of weak induction in applicative theories

Reinhard Kahle

Abstract

In this paper we study the relationship between forms of weak induction in theories of operations and numbers. Therefore, we investigate the structure of the natural numbers. Introducing a concept of N-strictness, we give a natural extension of the theory BON which implies the equivalence of operation and N-induction. In addition, we show that in the presence of the non-constructive μ-operator the above equivalence is provable without this extension.

1 Introduction

Applicative theories go back to Feferman's systems of explicit mathematics introduced in [Fef75, Fef79]. They are based on the basic theory of operations and numbers BON which is introduced in [FJ93] as the classic version of Beeson's theory EON (cf. [Bee85]) without induction.

Combined with various induction principles, applicative theories provide a natural framework for constructive mathematics and functional programming. If they are strengthened by the so-called *non-constructive μ-operator*, a predicatively acceptable quantification operator over the natural numbers, we get an uniform reference theory for proof-theoretical investigations. Detailed results can be found in [FJ93, FJ9x, JS9xa, JS9xb, GS9x].

In this paper the logical relationship between three weak forms of induction on the natural numbers in BON is investigated: *set induction, operation induction,* and *N-induction.* It is shown that set induction is the weakest form and that operation and N induction are equivalent in a natural extension of BON. For this extension the axiomatization of the natural numbers becomes important. Introducing the concept of N-strictness — expressing that $f\,x \in N$ implies $x \in N$ — we show that the only N-strict function is the identity function. In particular, it is impossible to define a successor function *suc* in BON such that $suc\,x \in N \rightarrow x \in N$. In our extension of BON which is satisfied in all usual models of BON this problem is remedied. Nevertheless, we can establish the required results about the inductions in the presence of the μ-operator without this extension.

2 The Basic Theory of Operations and Numbers

As introduced in [FJ93] the basic theory of operations and numbers BON is formulated in L_p, the first order language of partial operations and numbers.

L_p comprises individual variables $x, y, z, u, v, f, g, h, \ldots$ (possibly with subscripts), individual constants $0, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{s}_N, \mathbf{p}_N, \mathbf{d}_N, \mathbf{r}_N, \mu$, a binary function symbol $\cdot$ for term application, and the relation symbols $\downarrow, =$ and N.

Terms $(r, s, t, r_1, s_1, t_1, \ldots)$ and formulae $(\varphi, \psi, \ldots)$, starting from the atomic formulae $t\downarrow$, $t = s$, and $N(t)$, are defined as usual.

In the following we write $s\,t$ for $(s \cdot t)$ with the convention of association to the left, and we use the following abbreviations.

$$
\begin{array}{rcl}
t \simeq s & := & (t\downarrow \vee\, s\downarrow) \rightarrow t = s \\
t \neq s & := & t\downarrow \wedge\, s\downarrow \wedge\, \neg t = s \\
1 & := & \mathbf{s}_N\, 0 \\
\overline{0} & := & 0 \\
\overline{n+1} & := & \mathbf{s}_N\, \overline{n} \\
t' & := & \mathbf{s}_N\, t \\
t \in N & := & N(t) \\
t \notin N & := & \neg N(t) \\
\forall x \in N.\varphi & := & \forall x.x \in N \rightarrow \varphi \\
(t : N \rightarrow N) & := & \forall x \in N.t\,x \in N \\
(t : N^{m+1} \rightarrow N) & := & \forall x \in N.(t\,x : N^m \rightarrow N) \\
t \in P(N) & := & \forall x \in N.t\,x = 0 \vee t\,x = 1
\end{array}
$$

Terms of the form $\overline{n}$ are called *numerals* of L_p, and the elements of $P(N)$ are called *sets*.

The logic of BON is the (classical) logic of partial terms (cf. [Bee85, Sec. VI.1]).

The non-logical axioms of BON include:

I. **Partial combinatory algebra.**

(1) $\mathbf{k}\,x\,y = x$,

(2) $\mathbf{s}\,x\,y\downarrow \wedge\, \mathbf{s}\,x\,y\,z \simeq x\,z\,(y\,z)$,

(3) $\mathbf{k} \neq \mathbf{s}$.

II. **Pairing and projection.**

(4) $\mathbf{p}\,x\,y\downarrow \wedge\, \mathbf{p}_0\,(\mathbf{p}\,x\,y) = x \wedge \mathbf{p}_1\,(\mathbf{p}\,x\,y) = y$,

(5) $\mathbf{p}\,x\,y \neq 0$.

III. **Natural numbers.**

(6) $0 \in N \wedge \forall x \in N. x' \in N$,

(7) $\forall x \in N. x' \neq 0 \wedge \mathbf{p}_N (x') = x$,

(8) $\forall x \in N. x \neq 0 \rightarrow \mathbf{p}_N\, x \in N \wedge (\mathbf{p}_N\, x)' = x$.

IV. **Definition by cases on N.**

(9) $v \in N \wedge w \in N \wedge v = w \rightarrow \mathbf{d}_N\, x\, y\, v\, w = x$,

(10) $v \in N \wedge w \in N \wedge v \neq w \rightarrow \mathbf{d}_N\, x\, y\, v\, w = y$.

V. **Primitive recursion on N.**

(11) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \rightarrow (\mathbf{r}_N\, f\, g : N^2 \rightarrow N)$,

(12) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \wedge x \in N \wedge y \in N \wedge h = \mathbf{r}_N\, f\, g \rightarrow$
$h\, x\, 0 = f\, x \wedge h\, x\, (y') = g\, x\, y\, (h\, x\, y)$.

It is well-known that we can introduce in BON a notion of λ-abstraction and prove the recursion theorem (cf. [Bee85]).

Proposition 1 For every variable x and every term t of L_p there exists a term $\lambda x.t$ of L_p whose free variables are those of t, excluding x, so that

$$\mathrm{BON} \vdash \lambda x.t \downarrow \wedge\, (\lambda x.t)\, x \simeq t.$$

Proposition 2 There is a term **rec** of L_p so that

$$\mathrm{BON} \vdash \mathbf{rec}\, f \downarrow \wedge\, \forall x. \mathbf{rec}\, f\, x \simeq f\, (\mathbf{rec}\, f)\, x.$$

In the following we need the fact that a certain term does not belong to N, provably in BON.

Proposition 3 There is a term $\mathbf{not}_N$ of L_p so that

$$\mathrm{BON} \vdash \mathbf{not}_N \notin N.$$

Proof: Choose the term $\mathbf{not}_N$ as $\mathbf{rec}\, (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0)\, 0$. The claim is straightforward by the recursion-theorem and by case-distinction on $\mathbf{not}_N \notin N$, $\mathbf{not}_N = 0$, and $\mathbf{not}_N \neq 0$. We show that the assumption $\mathbf{not}_N = 0$ leads to a contradiction.

$$\begin{aligned}
0 &= \mathbf{not}_N \\
&= \mathbf{rec}\, (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0)\, 0 \\
&= (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0)\, (\mathbf{rec}\, (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0))\, 0 \\
&= (\lambda y. \mathbf{d}_N\, 1\, 0\, ((\mathbf{rec}\, (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0))\, y)\, 0)\, 0 \\
&= \mathbf{d}_N\, 1\, 0\, ((\mathbf{rec}\, (\lambda x. \lambda y. \mathbf{d}_N\, 1\, 0\, (x\, y)\, 0))\, 0)\, 0 \\
&= \mathbf{d}_N\, 1\, 0\, \mathbf{not}_N\, 0 \\
&= \mathbf{d}_N\, 1\, 0\, 0\, 0 \\
&= 1
\end{aligned}$$

□

Remark 4

1. Note that we do not have $\mathbf{not}_N \downarrow$ in general. With respect to the recursion-theoretic models (cf. below), it is impossible to demand this property for $\mathbf{not}_N$. In particular, $\downarrow$ and N coincide in the recursion-theoretic models.

2. Schlüter [Sch9x] introduces an applicative theory based on a primitive-recursive application. In such a theory the recursion theorem no longer holds, and a term like $\mathbf{not}_N$ is not definable.

Now we sketch the two classes of natural models of BON, which are extensively discussed in the literature (e.g. [Bee85, FJ93, JS9xb]).

The first ones are the *recursion-theoretic models*, where the universe and the denotation of N are the set of natural numbers. We get the standard recursion-theoretic model of BON by interpreting application $x\,y$ as Kleene-application $\{x\}(y)$.

The second class includes different *term models*, of which the *normal term model* NT is the most elementary one is. In the following we need the closed variant CNT (both discussed in [Bee85, Sec. VI.6.1]).

Definition 5 The binary reduction relation ϱ on the closed terms of L_p is defined as follows, where we assume that t_0, t_1, t_2 and s are in normal form (with respect to ϱ) and r_0 is not normal.

$$\begin{array}{rcll}
\mathbf{k}\,t_0\,t_1 & \varrho & t_0, & \\
\mathbf{s}\,t_0\,t_1\,t_2 & \varrho & t_0\,t_2\,(t_1\,t_2), & \\
\mathbf{p}_1\,(\mathbf{p}\,t_0\,t_1) & \varrho & t_0, & \\
\mathbf{p}_2\,(\mathbf{p}\,t_0\,t_1) & \varrho & t_1, & \\
\mathbf{p}_N\,(\mathbf{s}_N\,\overline{m}) & \varrho & \overline{m}, & \\
\mathbf{d}_N\,t_0\,t_1\,\overline{m}\,\overline{m} & \varrho & t_0, & \\
\mathbf{d}_N\,t_0\,t_1\,\overline{m}\,\overline{n} & \varrho & t_1, & (n \neq m) \\
\mathbf{r}_N\,t_0\,t_1\,s\,\overline{0} & \varrho & t_0\,s, & \\
\mathbf{r}_N\,t_0\,t_1\,s\,\overline{m+1} & \varrho & t_1\,s\,\overline{m}\,(\mathbf{r}_N\,t_0\,t_1\,s\,\overline{m}); &
\end{array}$$

$$\text{If} \quad r_0 \;\varrho\; r_1 \quad \text{then} \quad t_0\,r_0 \;\varrho\; t_0\,r_1 \quad \text{and} \quad r_0\,r_2 \;\varrho\; r_1\,r_2.$$

The universe of CNT is the set of all closed L_p-terms in normal form with respect to ϱ. For arbitrary terms t and s of L_p, $\mathit{InFirst}(t,s)$ is defined as the (uniquely determined) normal term r, which we get by reducing $t\,s$ using ϱ if it exist, undefined otherwise. So we interpret the constants by themselves and $(t \cdot s)$ as $\mathit{InFirst}(t,s)$. Equality is interpreted as literal identity, and N as the set of numerals.

Fact 6 CNT $\models$ BON.

If one replaces the universe by the set of all terms and allows arbitrary reductions we obtain a *total term model* TT (and also CTT for closed terms) as described in [JS9xb]. In this model application is trivial, but equality becomes complicated since it is defined via common reducts. In the following we will be cautious that possible strengthenings of BON do not essentially affect these models, since totality is an interesting and important concept in the context of BON.

3 Weak forms of induction

As printed out we aim at discussing the relationship between the following three forms of complete induction on the natural numbers:

1. set induction on N (S-I_N)

$$f \in P(N) \wedge f\,0 = 0 \wedge (\forall x \in N. f\,x = 0 \to f\,(x') = 0) \to \forall x \in N. f\,x = 0,$$

2. operation induction on N (O-I_N)

$$f\,0 = 0 \wedge (\forall x \in N. f\,x = 0 \to f\,(x') = 0) \to \forall x \in N. f\,x = 0,$$

3. N induction on N (N-I_N)

$$f\,0 \in N \wedge (\forall x \in N. f\,x \in N \to f\,(x') \in N) \to \forall x \in N. f\,x \in N.$$

It is known from [FJ93] that BON plus any of these induction principles is proof-theoretically equivalent to PRA. However, we are interested in the logical relationship between these induction principles. We first show that (S-I_N) is the weakest of these three forms.

Fact 7 BON $\vdash$ (O-I_N) $\to$ (S-I_N)

Using the term $\mathbf{not}_N$ it is easy to deduce that $t = 0$ is expressible by a conjunction of tests for membership in N.

Lemma 8 BON $\vdash t = 0 \leftrightarrow t \in N \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,t\,0\,0 \in N$

Note that we use a form of strong definition by cases because of strictness: since it is not guaranteed that $\mathbf{not}_N$ is defined, we cannot use $\mathbf{d}_N\,0\,\mathbf{not}_N\,t\,0$ in the second conjunct.

Since the set-property in (S-I_N) already guarantees the first conjunct this lemma implies that (S-I_N) is derivable from (N-I_N).

Proposition 9 $\mathrm{BON} \vdash (N\text{-I}_N) \to (\text{S-I}_N)$

Proof: Arguing in BON assume the antecedent of (S-I$_N$):

$$f \in P(N) \wedge f\,0 = 0 \wedge \forall x \in N.f\,x = 0 \to f\,(x') = 0$$

By the previous lemma this is equivalent to

$$\begin{aligned} & f \in P(N) \wedge (f\,0 \in N \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,0)\,0\,0 \in N) \\ & \qquad \wedge \Big(\forall x \in N.(f\,x \in N \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,x)\,0\,0 \in N) \\ & \qquad\qquad \to (f\,(x') \in N \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,(x'))\,0\,0 \in N)\Big). \end{aligned}$$

Since $f \in P(N)$ is $\forall x \in N.f\,x \in N$, we obtain:

$$\begin{aligned} & f \in P(N) \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,0)\,0\,0 \in N \\ & \qquad \wedge \Big(\forall x \in N.\mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,x)\,0\,0 \in N \\ & \qquad\qquad \to \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,(x'))\,0\,0 \in N\Big). \end{aligned}$$

Now an application of (N-I$_N$) yields

$$f \in P(N) \wedge \forall x \in N.\mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,x)\,0\,0 \in N$$

that is

$$\forall x \in N.f\,x \in N \wedge \mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,x)\,0\,0 \in N.$$

Hence $\forall x \in N.f\,x = 0$ by the previous lemma, concluding the proof. $\square$

4 N-strict definition by cases

Once the formulae $f\,x \in N$ and $f\,x = 0$ can be translated into each other, the equivalence of (N-I$_N$) and (O-I$_N$) is an immediate consequence. In order to get such a translation we need a new form of definition by cases, based on the notion of N-strictness.

Definition 10 A term t is called N*-strict* in the theory T if we have

$$T \vdash t\,x \in N \to x \in N.$$

Now we extend BON by an axiom which expresses that $\mathbf{d}_N$ is N-strict in its last two arguments.

Definition 11 Let $\widehat{\mathrm{BON}}$ be the theory BON with the additional axiom:

(10a) $\mathbf{d}_N\,x\,y\,v\,w \in N \to v \in N \wedge w \in N.$

To illustrate the intension of this axiom, consider its contraposition.

$$v \notin N \vee w \notin N \to \mathbf{d}_N\, x\, y\, v\, w \notin N$$

Since $\mathbf{d}_N$ is a definition by cases operator *on* N, we can claim that $\mathbf{d}_N\, x\, y\, v\, w$ must be a term without a specific meaning whenever v or w does not belong to N. With respect to the total version of our theory it is impossible to express this fact by $\neg\, \mathbf{d}_N\, x\, y\, v\, w \downarrow$. Moreover, in the total models we have no possibility to distinguish any term not belonging to N, so the statement $\mathbf{d}_N\, x\, y\, v\, w \notin N$ is the only way to express "meaninglessness".

It is easy to see that (10a) is satisfied in the natural models: in a recursion-theoretic one it follows immediately by strictness. In the term models the axiom is satisfied, because all reductions of $\mathbf{d}_N\, t\, s\, r_1\, r_2$ require that r_1 and r_2 are numerals.

Remark 12 For uniformity reasons it might be adequate to introduce N-strictness also for the other constants that require arguments in N, namely $\mathbf{s}_N, \mathbf{p}_N$ and $\mathbf{r}_N$.

$$\begin{aligned} \mathbf{s}_N\, x \in N &\quad\to\quad x \in N, \\ \mathbf{p}_N\, x \in N &\quad\to\quad x \in N, \\ \mathbf{r}_N\, f\, g\, x\, y \in N &\quad\to\quad x \in N \wedge y \in N. \end{aligned}$$

But terms with this properties are easy definable using the new $\mathbf{d}_N$ (cf. Prop. 20).

Using $\mathbf{d}_N$ we are able to *semidecide* the elements of N, i.e. we can define a term $\mathbf{sd}_N$ so that $\mathbf{sd}_N\, x$ is 0 if and only if x belongs to N. But for $x \notin N$ we cannot demand $\mathbf{sd}_N\, x = 1$, so $\mathbf{sd}_N$ only *semi*decides N. The reason is that the total version of BON (where all terms are defined) is inconsistent with decidable natural numbers.

Lemma 13 There is a term $\mathbf{sd}_N$ of L_p so that

$$\widehat{\mathrm{BON}} \vdash \mathbf{sd}_N\, x = 0 \leftrightarrow x \in N.$$

Proof: Choose $\mathbf{sd}_N$ as $\lambda x.\mathbf{d}_N\, 0\, 1\, x\, x$. $\square$

So we can express $f\, x \in N$ by $(\lambda y.\mathbf{sd}_N\, (f\, y))\, x = 0$ and get

Proposition 14 $\widehat{\mathrm{BON}} \vdash (\mathrm{O\text{-}I}_N) \to (N\text{-}\mathrm{I}_N)$.

To establish the converse we go back to Lemma 8 where $t = 0$ is expressed by a conjunction of N-memberships. The following shows that $\widehat{\mathrm{BON}}$ is strong enough to express such a conjunction by a single test of membership in N.

Lemma 15 There is a term $\mathbf{and}_N$ of L_p so that

$$\widehat{\mathrm{BON}} \vdash \mathbf{and}_N\, x\, y \in N \leftrightarrow x \in N \wedge y \in N.$$

Proof: Choose $\mathbf{and}_N$ as $\lambda x, y.\mathbf{d}_N\, 0\, 0\, x\, y$. $\Box$

So $f\,x = 0$ is expressible by $(\lambda y.\mathbf{and}_N\,(f\,y)\,(\mathbf{d}_N\,(\lambda z.0)\,(\lambda z.\mathbf{not}_N)\,(f\,y)\,0\,0))\,x \in N$ following Lemma 8, concluding the converse of Proposition 14.

Proposition 16 $\widehat{\mathrm{BON}} \vdash (N\text{-I}_N) \rightarrow (\text{O-I}_N)$.

All in all we have thus established the equivalence of (O-I_N) and $(N\text{-I}_N)$ in $\widehat{\mathrm{BON}}$.

Theorem 17 $\widehat{\mathrm{BON}} \vdash (\text{O-I}_N) \leftrightarrow (N\text{-I}_N)$.

In the following we give two further applications of the new axiom for $\mathbf{d}_N$.

By the presence of (10a) the usual notion of representability of a number-theoretic function can be strengthened as follows.

Definition 18 A closed term t_F *strongly represents* a number-theoretic function $F : \mathbb{N}^k \rightarrow \mathbb{N}$ in a theory T if

1. t_F is N-strict in T and
2. t_F *provably represents* F in T, i.e.
   (a) For all $m_1, \ldots, m_k, n \in \mathbb{N}$,
   $$F(m_1, \ldots, m_k) = n \Leftrightarrow T \vdash t_F\,\overline{m_1}\,\ldots\,\overline{m_k} = \overline{n}.$$
   (b) $T \vdash (t_F : N^k \rightarrow N)$.

Lemma 19 There is a term $\mathbf{if}_N$ of L_p so that

$$\widehat{\mathrm{BON}} \vdash (\mathbf{if}_N\,x\,y \in N \rightarrow x \in N) \wedge (x \in N \rightarrow \mathbf{if}_N\,x\,y = y).$$

Proof: Choose $\mathbf{if}_N$ as $\lambda x, y.\mathbf{d}_N\,y\,0\,x\,x$. $\Box$

With the new $\mathbf{d}_N$-axiom the following proposition is easily verified.

Proposition 20 If t_F provably represents $F : \mathbb{N}^k \rightarrow \mathbb{N}$ then

$$\lambda x_1.\ldots\lambda x_k.\mathbf{if}_N\,x_1\,(\ldots(\mathbf{if}_N\,x_k\,(t_F\,x_1\,\ldots\,x_k))\cdots)$$

strongly represents F.

Remark 21 This result yields another proof of both Lemma 13 and 15 if we choose $\mathbf{sd}_N$ and $\mathbf{and}_N$ as strong representations of the constant-zero function and an arbitrary primitive-recursive, binary function. Moreover, we get an N-strict successor function as the strong representation of the number-theoretic successor.

Remark 22 As a second consequence of Theorem 17, in $\widehat{\mathrm{BON}} + (\text{O-I}_N)$ we can dispense with the constant $\mathbf{r}_N$ without any loss. It is well-known that the primitive recursive functions are provably representable in $\mathrm{BON} + (N\text{-I}_N)$ without using $\mathbf{r}_N$. In $\mathrm{BON} + (\text{O-I}_N)$, however, we do not know how to establish such a result and so we still need the explicit axiomatization of $\mathbf{r}_N$. With the equivalence of $(N\text{-I}_N)$ and (O-I_N) this is no longer necessary in $\widehat{\mathrm{BON}}$.

5 The non-constructive μ-operator

In [FJ93, JS9xb] extensions of BON are treated concerning the following non-constructive μ-operator.

Definition 23 Let BON(μ) be the theory BON plus the following axioms for the non-constructive μ-operator.
(μ.1) $\quad (f : N \to N) \leftrightarrow \mu f \in N$,
(μ.2) $\quad (f : N \to N) \wedge \exists x \in N. f\, x = 0 \to f\,(\mu f) = 0$.

So μf yields a root in N of the total function f, if f has a root in N, but — in contrast to the recursion-theoretic μ — also an element of N if not. Moreover, the implication from right to left in (μ.1) allows one to test the totality of f by $\mu f \in N$. Hence this form of μ is closely related to Kleene's E^2.
In [FJ93] the μ-operator was introduced without the implication from right to left in (μ.1). But in the following we need this stronger form in an essential way (cf. also [JS9xa]).

Since the μ-operator implicitly involves an universal quantification, we can use it to define a term $\mathbf{and}_N$ in BON(μ) in an easy way.

Lemma 24 There is a term $\mathbf{and}_N$ of L_p so that

$$\mathrm{BON}(\mu) \vdash \mathbf{and}_N\, x\, y \in N \leftrightarrow x \in N \wedge y \in N.$$

Proof: Choose $\mathbf{and}_N$ as $\lambda x, y.\mu\,(\lambda z.\mathbf{d}_N\, x\, y\, z\, 0)$. $\square$

The definition of a term $\mathbf{sd}_N$ is more tricky.

Lemma 25 There is a term $\mathbf{sd}_N$ of L_p satisfying

$$\mathrm{BON}(\mu) \vdash \mathbf{sd}_N\, x = 0 \leftrightarrow x \in N.$$

Proof: Choose $\mathbf{sd}_N$ as $\lambda x.\mu\,(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(\lambda z.\mathbf{d}_N\ (x')\,(\mathbf{d}_N\, 0\, x\, z\, 1)\, z\, 0))\, y\, 0)$. Let f be $\lambda x.\lambda z.\mathbf{d}_N\ (x')\,(\mathbf{d}_N\, 0\, x\, z\, 1)\, z\, 0$, so we can write $\mathbf{sd}_N$ as $\lambda x.\mu\,(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, y\, 0)$.

If $\mu\,(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, y\, 0) = 0$ we know by the implication from right to left of the axiom (μ.1) that $(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, y\, 0 : N \to N)$, so $\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, 1\, 0 \in N$. This implies $\mu\,(f\, x) \in N$, and hence $(f\, x : N \to N)$. So $f\, x\, \overline{2} \in N$, and we get with $f\, x\, \overline{2} = \mathbf{d}_N\ (x')\,(\mathbf{d}_N\, 0\, x\, \overline{2}\, 1)\, \overline{2}\, 0 = \mathbf{d}_N\, 0\, x\, \overline{2}\, 1 = x$ that $x \in N$.

Now assume $x \in N$. First we observe that $(f\, x : N \to N)$ and $f\, x\, 1 = 0 \wedge f\, x\, 0 = x'$, i.e. $f\, x$ has a root in N, but 0 is not a root. Hence $\mu\,(f\, x) \in N \wedge \mu\,(f\, x) \neq 0$. Therefore, 0 is the only root in N of $(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, y\, 0 : N \to N)$ and we get $\mu\,(\lambda y.\mathbf{d}_N\, 0\,(\mu\,(f\, x))\, y\, 0) = 0$. $\square$

The existence of $\mathbf{and}_N$ and $\mathbf{sd}_N$ implies the equivalence of (N-I$_N$) and (O-I$_N$) in BON(μ). Hence, in the presence of μ we do no longer need the stronger form of $\mathbf{d}_N$ in order to show the equivalence of these induction principles.

Theorem 26 $\mathrm{BON}(\mu) \vdash (N\text{-I}_N) \leftrightarrow (\text{O-I}_N)$.

Remark 27 The proof-theoretic equivalence of $\mathrm{BON}(\mu) + (\text{S-I}_N)$ and PA is known by [FJ93, Cor. 24]. Using the above Prop. 9, Jäger and Strahm prove in [JS9xa] that $|\mathrm{BON}(\mu) + (N\text{-I}_N)| = \varphi\,\omega\,0$, and hence $|\mathrm{BON}(\mu) + (\text{O-I}_N)| = \varphi\,\omega\,0$ by the theorem above.

6 The undefinability of $\mathbf{sd}_N$ and $\mathbf{and}_N$ in BON

Finally, we will show by a model-theoretic argument that terms with the property of $\mathbf{sd}_N$ and $\mathbf{and}_N$ are undefinable in BON.

The crucial point in this argument is that we are not able to say anything about the result of a function whenever a defined argument which does not belong to N appears in a argument position of $\mathbf{d}_N, \mathbf{s}_N$ or $\mathbf{r}_N$ where the axioms require elements of N. Obviously, such a case cannot arise if all defined terms are elements of N (like in the recursion-theoretic model), however, we are able to construct a corresponding term model.
Let us add to the term model CNT a new constant $\mathbf{a}$ which does not belongs to L_p nor to N, but behaves as 1, more precisely:

Definition 28 Let CNT$\mathbf{a}$ result from CNT by

1. extending the universe by a new constant $\mathbf{a}$ which does not belong to the denotation of N, i.e. is not a numeral,
2. extending the reduction relation ϱ to $L_p \cup \{\mathbf{a}\}$ and adding the following clauses, where t, t_0, t_1 and s in normal form:

$$\begin{array}{rcll}
\mathbf{d}_N\, t\, s\, \mathbf{a}\, \mathbf{a} & \varrho & t & \\
\mathbf{d}_N\, t\, s\, \mathbf{a}\, 1 & \varrho & t & \\
\mathbf{d}_N\, t\, s\, 1\, \mathbf{a} & \varrho & t & \\
\mathbf{d}_N\, t\, s\, \mathbf{a}\, \overline{n} & \varrho & s & (n \neq 1) \\
\mathbf{d}_N\, t\, s\, \overline{n}\, \mathbf{a} & \varrho & s & (n \neq 1) \\
\mathbf{s}_N\, \mathbf{a} & \varrho & \overline{2} & \\
\mathbf{p}_N\, \mathbf{a} & \varrho & 0 & \\
\mathbf{r}_N\, t_0\, t_1\, s\, \mathbf{a} & \varrho & t_1\, s\, 0\, (\mathbf{r}_N\, t_0\, t_1\, s\, 0). &
\end{array}$$

Since $\mathbf{a}$ does not belongs to L_p, it is easy to see:

Fact 29 CNT$\mathbf{a} \models$ BON.

The following lemma will essentially establish the undefinability of $\mathbf{sd}_N$ and an N-strict successor function in BON.

Lemma 30 For all closed terms t of L_p,

$$\mathrm{CNT}\mathbf{a} \models t\,1 \in N \to t\,\mathbf{a} = t\,1 \vee t\,1 = 1$$

Proof: In this proof let $t[s/r]$ denote the result of substituting the term s for some occurrences of the subterm r in t. Furthermore, we write $t \to_n s$ if t reduces in CNT**a** to s in n steps with respect to ϱ, and $t = s$ means that t reduces to s or s to t. Finally, $\mathbf{s}_N^k\, t$ stands for $\mathbf{s}_N\,(\ldots \mathbf{s}_N\,(\mathbf{s}_N\, t)\ldots)$ with k occurrences of $\mathbf{s}_N$.
Let us assume in the sequel that $m \in \mathbb{N}$ and $m \neq 1$. We prove the following, more general statement by induction on n:

$$\text{For all terms } t \text{ and } t[\mathbf{a}/1] \text{ of } L_p\text{: } t \to_n \overline{m} \Rightarrow t[\mathbf{a}/1] = \overline{m}.$$

Base case. For $n = 0$ the term t is already $\mathbf{s}_N^m\, 0$. If $t[\mathbf{a}/1] = t$ the statement is trivially true, otherwise $t[\mathbf{a}/1] = \mathbf{s}_N^{m-1}\,\mathbf{a}$ and hence $t[\mathbf{a}/1] = \mathbf{s}_N^{m-1}\,\mathbf{a} = \mathbf{s}_N^{m-2}\,\overline{2} = \overline{m}$ by the extension of ϱ above.

Step case. The induction hypothesis reads

$$\text{For all terms } s \text{ and } s[\mathbf{a}/1] \text{ of } L_p\text{: } s \to_n \overline{m} \Rightarrow s[\mathbf{a}/1] = \overline{m}$$

and we are to show

$$\text{For all terms } t \text{ and } t[\mathbf{a}/1] \text{ of } L_p\text{: } t \to_{n+1} \overline{m} \Rightarrow t[\mathbf{a}/1] = \overline{m}.$$

So assume an arbitrary term t such that $t \to_1 s \to_n \overline{m}$ for some s. Hence by the induction hypothesis for s it suffices to show that an arbitrary $t[\mathbf{a}/1]$ reduces to some $s[\mathbf{a}/1]$. If $t \to_1 s$ there is a subterm $\tilde{t}$ of t such that we have a clause $\tilde{t}\ \varrho\ \tilde{s}$ in the definition of ϱ. So s is t, where the first occurrence of $\tilde{t}$ is replaced by $\tilde{s}$. Since a substitution of the term 1 by $\mathbf{a}$ cannot chance the reduction strategy, it is enough to show that for all $\tilde{t}[\mathbf{a}/1]$ there is some term $\tilde{s}[\mathbf{a}/1]$ such that $\tilde{t}[\mathbf{a}/1] \to \tilde{s}[\mathbf{a}/1]$. We prove this by case-distinction on $\tilde{t}\ \varrho\ \tilde{s}$:

1. $\tilde{t} \equiv \mathbf{k}\, t_0\, t_1\ \varrho\ t_0 \equiv \tilde{s}$. So we have $\tilde{t}[\mathbf{a}/1] \equiv \mathbf{k}\,(t_0[\mathbf{a}/1])\,(t_1[\mathbf{a}/1]) \to t_0[\mathbf{a}/1] \equiv \tilde{s}[\mathbf{a}/1]$.

2. $\tilde{t} \equiv \mathbf{s}\, t_0\, t_1\, t_2$, $\tilde{t} \equiv \mathbf{p}_0\,(\mathbf{p}\, t_0\, t_1)$ and $\tilde{t} \equiv \mathbf{p}_1\,(\mathbf{p}\, t_0\, t_1)$ are analogous cases.

3. $\tilde{t} \equiv \mathbf{p}_N\,(\mathbf{s}_N\, \overline{k})\ \varrho\ \overline{k} \equiv \tilde{s}$. Only the case $k = 0$ is interesting. If $\tilde{t}[\mathbf{a}/1] \equiv \tilde{t}$ we are done. If $\tilde{t}[\mathbf{a}/1] \equiv \mathbf{p}_N\,\mathbf{a}$ one of the new reductions yields $\tilde{t}[\mathbf{a}/1] \to 0 \equiv \tilde{s}$.

4. $\tilde{t} \equiv \mathbf{d}_N\, t_0\, t_1\, \overline{k}\, \overline{l}$. We only treat the example case $k = 2$ and $l = 1$, where $\tilde{t}[\mathbf{a}/1]$ is $\mathbf{d}_N\,(t_0[\mathbf{a}/1])\,(t_1[\mathbf{a}/1])\,(\mathbf{s}_N\,\mathbf{a})\,\mathbf{a}$. So we have to reduce the third argument and then use the clause for $\mathbf{d}_N$: $\tilde{t}[\mathbf{a}/1] \to \mathbf{d}_N\,(t_0[\mathbf{a}/1])\,(t_1[\mathbf{a}/1])\,\overline{2}\,\mathbf{a} \to t_1[\mathbf{a}/1] \equiv \tilde{s}[\mathbf{a}/1]$.

5. $\tilde{t} \equiv \mathbf{r}_N\, t_0\, t_1\, t_2\, \overline{k}$. This case is analogous to the previous one. $\square$

Thus we get the undefinability of $\mathbf{sd}_N$ in BON.

Proposition 31 There is *no* term $\mathbf{sd}_N$ of L_p so that BON proves

$$\mathbf{sd}_N\, x = 0 \leftrightarrow x \in N.$$

Proof: Assume that BON proves $\mathbf{sd}_N\, x = 0 \leftrightarrow x \in N$ for some term $\mathbf{sd}_N$. In particular $\mathbf{sd}_N\, 1 = 0$, hence in CNT**a** by the previous lemma $\mathbf{sd}_N\, \mathbf{a} = \mathbf{sd}_N\, 1 = 0$. But since $\mathbf{a}$ does not belongs to N, this contradicts our assumption. □

Analogously one gets the undefinability of a N-strict successor function in BON.

Proposition 32 There is *no* term $\mathbf{suc}_N$ of L_p so that BON proves

$$\forall x \in N.\mathbf{suc}_N\, x = \mathbf{s}_N\, x \wedge (\mathbf{suc}_N\, x \in N \rightarrow x \in N).$$

For the undefinability of $\mathbf{and}_N$ we repeat the procedure above with a new constant $\mathbf{b}$, i.e. we add to CNT**a** a further constant $\mathbf{b}$ with the same behavior as $\overline{2}$. So we get in the same way a model CNT**ab** of BON and the following lemma.

Lemma 33 For all terms t of L_p,

$$\begin{aligned} \mathrm{CNT}\mathbf{ab} &\models t\,1 \in N \rightarrow t\,\mathbf{a} = t\,1 \vee t\,1 = 1, \\ \mathrm{CNT}\mathbf{ab} &\models t\,\overline{2} \in N \rightarrow t\,\mathbf{b} = t\,\overline{2} \vee t\,\overline{2} = \overline{2}. \end{aligned}$$

Proposition 34 There is *no* term $\mathbf{and}_N$ of L_p so that BON proves

$$\mathbf{and}_N\, x\, y \in N \leftrightarrow x \in N \wedge y \in N.$$

Proof: Assume that BON proves $\mathbf{and}_N\, x\, y \in N \leftrightarrow x \in N \wedge y \in N$ for some term $\mathbf{and}_N$. Hence by the previous lemma in CNT**ab**: $(\mathbf{and}_N\, \overline{2})\, 1 = 1$ and $(\lambda x.\mathbf{and}_N\, x\, 1)\, \overline{2} = \overline{2}$. So $1 = \mathbf{and}_N\, \overline{2}\, 1 = \overline{2}$ which is impossible. □

References

[Bee85] M. Beeson. *Foundations of Constructive Mathematics.* Ergebnisse der Mathematik und ihrer Grenzgebiete; 3.Folge, Bd. 6. Springer, 1985.

[Fef75] S. Feferman. A language and axioms for explicit mathematics. In J. Crossley, editor, *Algebra and Logic*, volume 450 of *Lecture Notes in Mathematics*, pages 87–139. Springer, 1975.

[Fef79] S. Feferman. Constructive theories of functions and classes. In M. Boffa, D. van Dalen, and K. McAloon, editors, *Logic Colloquium 78*, pages 159–224. North–Holland, Amsterdam, 1979.

[FJ93] S. Feferman and G. Jäger. Systems of explicit mathematics with non-constructive μ-operator. Part I. *Annals of pure and applied logic*, 65(3):243–263, December 1993.

[FJ9x] S. Feferman and G. Jäger. Systems of explicit mathematics with non-constructive μ-operator. Part II, 199x. Preprint November, 1994.

[GS9x] Th. Glaß and Th. Strahm. Systems of explicit mathematics with non-constructive μ-operator and join, 199x. In Preparation.

[JS9xa] G. Jäger and Th. Strahm. Some theories with positive induction of ordinal strength $\varphi\omega 0$, 199x. In Preparation.

[JS9xb] G. Jäger and Th. Strahm. Totality in applicative theories. *to appear in Annals of Pure and Applied Logic*, 199x. Preprint June 10, 1994.

[Sch9x] A. Schlüter. A theory of rules for enumerated classes of functions. *to appear in Archive for mathematical logic*, 199x.