Systems of explicit mathematics with non-constructive μ-operator and join

Thomas Glaß and Thomas Strahm*

Abstract

The aim of this article is to give the proof-theoretic analysis of various subsystems of Feferman's theory T_1 for explicit mathematics which contain the non-constructive μ-operator and join.

We make use of standard proof-theoretic techniques such as cut-elimination of appropriate semi-formal systems and asymmetrical interpretations in standard structures for explicit mathematics.

1 Introduction

Systems of explicit mathematics were introduced in Feferman [5, 9]. Two families of theories were presented there, namely the theories T_0 and T_1 together with their various subsystems. T_1 results from T_0 by adding the so-called *non-constructive minimum operator*, a predicatively justified quantification operator over the natural numbers.

Complete proof-theoretic information about T_0 and its various subsystems is available since 1983 by the work of Feferman [5, 9], Feferman and Sieg [14], Jäger [20] and Jäger and Pohlers [22]. A crucial step in the proof-theoretic analysis of subsystems of T_1 was established only recently in the two papers by Feferman and Jäger [13, 12].[1] Whereas the first of these papers deals with pure applicative theories plus non-constructive μ-operator, the second paper is concerned about extensions of these systems with (variable) types and classes. More precisely, a theory $\mathsf{EET}(\mu)$ of elementary explicit type theory with non-constructive μ-operator is studied in [12] in the context of various induction principles on the natural numbers, namely set induction (S-$\mathsf{I_N}$), type induction (T-$\mathsf{I_N}$) and formula induction (F-$\mathsf{I_N}$) (cf. Section 2 for precise definitions), and the following proof-theoretic equivalences to well-known

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[1]Important work on corresponding systems with a *typed* application operation is due to Feferman [6, 7].

subsystems of analysis are established there:[2]

$$
\begin{aligned}
\mathsf{EET}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N}) &\equiv \mathsf{PA},\\
\mathsf{EET}(\mu) + (\mathsf{T}\text{-}\mathsf{I_N}) &\equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0},\\
\mathsf{EET}(\mu) + (\mathsf{F}\text{-}\mathsf{I_N}) &\equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_{\varepsilon_0}}.
\end{aligned}
$$

The crucial class existence principle of $\mathsf{EET}(\mu)$ is the axiom of elementary comprehension (ECA). However, there are two other class existence principles which are relevant in explicit mathematics, namely the join axiom (J) and inductive generation (IG), cf. Feferman [5, 9]. It is established in Section 8 that μ does not increase proof-theoretic strength if (IG) is present, since the applicative axioms with μ can be interpreted by making use of Π^1_1 comprehension, which in turn follows from (IG). As a consequence, the remaining interesting systems to be studied in the context of the non-constructive μ-operator contain join (J) but *not* inductive generation (IG). Accordingly, it is the aim of this paper to give an exact proof-theoretic analysis of the three systems $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{S}\text{-}\mathsf{I_N})$, $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F}\text{-}\mathsf{I_N})$.

The system $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F}\text{-}\mathsf{I_N})$ corresponds to the theory $\mathsf{T}_1^{(N)}$ in the terminology of Feferman [5], where the following conjecture is raised about the proof-theoretic strength of $\mathsf{T}_1^{(N)}$, cf. p. 123:

$$\mathsf{T}_1^{(N)} \textit{ is (proof-theoretically) reducible to predicative analysis } \bigcup_{\alpha<\Gamma_0} \mathsf{R}_\alpha. \tag{i}$$

Furthermore, Feferman writes, cf. ibidem:

$$\textit{It may even be that } \mathsf{T}_1^{(N)} \textit{ is of the same strength as predicative analysis.} \tag{ii}$$

In the following we prove (i) and disprove (ii). More precisely, the main results of this article can be stated as follows:

$$
\begin{aligned}
\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N}) &\equiv \mathsf{PA},\\
\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N}) &\equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0},\\
\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N}) &\equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}.
\end{aligned}
$$

Before we turn to the details of our presentation, let us briefly indicate the main lines of the proof-theoretic analysis of the above systems.

The treatment of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N})$ is straightforward by establishing that this system is a conservative extension of the theory $\mathsf{BON}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N})$ of Feferman and Jäger [13], so that the equivalence of the latter theory to PA yields the desired result.

Let us turn to the systems $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F}\text{-}\mathsf{I_N})$. As far as their lower bounds are concerned, we know from [13] that $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$ is contained

[2]Here '$\equiv$' denotes the usual notion of proof-theoretic equivalence as it is defined, e.g., in Feferman [11].

in $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$, and a generalization of the proof given in [13] yields an embedding of $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}$ into $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$.

The corresponding upper bounds are obtained by first providing Tait style reformulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$, respectively, where $\mathcal{T}_2$ includes a form of the ω rule in order to handle full formula induction on the natural numbers, $(\mathsf{F}\text{-}\mathsf{I_N})$. Partial cut elimination in $\mathcal{T}_1$ and $\mathcal{T}_2$ yields quasi normal derivations of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ of length less than ω and ε_0, respectively; those are subsequently used in order to provide *asymmetrical interpretations* into initial segments of standard structures for $\mathsf{EET}(\mu) + (\mathsf{J})$, namely $\mathfrak{S}_{\mathfrak{M}}(\omega)$ and $\mathfrak{S}_{\mathfrak{M}}(\varepsilon_0)$, respectively (cf. Section 4 for precise definitions).

The question arises where the standard structures $\mathfrak{S}_{\mathfrak{M}}(\omega)$ and $\mathfrak{S}_{\mathfrak{M}}(\varepsilon_0)$ can be formalized in order to yield the desired proof-theoretic upper bounds of $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$. As in Feferman and Jäger [13, 12], the proper treatment of the application operation again requires certain fixed point theories with ordinals, which were introduced in Jäger [21] and extended to second order theories with ordinals in Jäger and Strahm [24]. The theory $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ of [24] is in fact appropriate for formalizing the treatment of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ that we have sketched above, and it is shown in [21, 24] that $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ has the same strength as $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$; this yields the exact strength of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$.

For the description of $\mathfrak{S}_{\mathfrak{M}}(\varepsilon_0)$, finally, we introduce a new second order theory with ordinals plus elementary comprehension iterated through all levels below ε_0, namely the system $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, and we show that this theory has the same strength as $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}$ by complete ordinal analysis. This finishes the brief sketch of our treatment of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N})$, $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$.

2 The formal framework for elementary explicit type theory

In this section we briefly recapitulate the elementary explicit type theory with non-constructive μ-operator $\mathsf{EET}(\mu)$ and its various extensions.

The language $\mathbb{L}_p$ of applicative theories with types is a two-sorted language with *individual variables* $a, b, c, x, y, z, f, g, h, \ldots$ and *type variables* $A, B, C, X, Y, Z, \ldots$ (both possibly with subscripts). In addition, $\mathbb{L}_p$ includes *individual constants* k, s (partial combinatory algebra), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and unpairing), 0 (zero), s_N (successor), p_N (predecessor), d_N (definition by numerical cases), r_N (primitive recursion), $\boldsymbol{\mu}$ (unbounded minimum operator), $(\mathsf{c}_e)_{e<\omega}$ (elementary comprehension) and j (join). $\mathbb{L}_p$ has a binary function symbol $\cdot$ for (partial) term application, unary relation symbols $\downarrow$ (defined) and N (natural numbers) as well as binary relation symbols $=, \in$ and $\Re$ (naming relation).

The *individual terms* $(r, s, t, \ldots)$ of $\mathbb{L}_p$ are inductively defined as follows:

1. The individual variables and individual constants are individual terms.

2. If s and t are individual terms, then so also is $(s \cdot t)$.

In the following we write (st) or just st instead of $(s \cdot t)$, and we adopt the convention of association to the left, i.e. $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 s_2)\dots s_n)$. Furthermore, we write (t_1, t_2) for $\mathsf{p}(t_1, t_2)$ and $(t_1, \dots, t_n)$ for $(t_1, (t_2, \dots, t_n))$.

The *formulas* $(F, G, H, \dots)$ of $\mathbb{L}_p$ are inductively defined as follows:

1. Each atomic formula $N(t)$, $t\downarrow$, $(s = t)$, $(s \in A)$, $(A = B)$ and $\Re(s, A)$ is a formula.

2. If F and G are formulas, then so also are $\neg F$, $(F \vee G)$ and $(F \wedge G)$.

3. If F is a formula, then so also are $\exists x F$, $\forall x F$, $\exists X F$ and $\forall X F$.

As usual we write $(F \to G)$ for $(\neg F \vee G)$. An $\mathbb{L}_p$ formula F is called *stratified* if the relation symbol $\Re$ does not occur in F; the *elementary* $\mathbb{L}_p$ formulas are the stratified $\mathbb{L}_p$ formulas which do not contain bound type variables. L_p is defined to be the *first order part* of $\mathbb{L}_p$, i.e. L_p is the sublanguage of $\mathbb{L}_p$ which we obtain by omitting the relation symbols $\in$ and $\Re$ as well as the constants $(\mathsf{c}_e)_{e<\omega}$ and j.

In the sequel we write t' for $\mathsf{s}_N t$ and 1 for $0'$. More generally, the *numerals* of L_p are inductively given by $\overline{0} = 0$ and $\overline{n+1} = \mathsf{s}_N \overline{n}$. In addition, we use the following abbreviations concerning the predicate N.

$$\begin{aligned}
t \in N &:= N(t), \\
(\exists x \in N)F &:= \exists x(x \in N \wedge F), \\
(\forall x \in N)F &:= \forall x(x \in N \to F), \\
(t : N \to N) &:= (\forall x \in N)(tx \in N), \\
(t : N^{m+1} \to N) &:= (\forall x \in N)(tx : N^m \to N), \\
t \in P(N) &:= (\forall x \in N)(tx = 0 \vee tx = 1), \\
A \subseteq N &:= \forall x(x \in A \to x \in N).
\end{aligned}$$

The quantifications $(\exists x \in A)F$ and $(\forall x \in A)F$ are understood analogously.

The logic of applicative theories with types is the (classical) *logic of partial terms*, cf. Beeson [1]. Accordingly, the *partial equality relation* $\simeq$ is introduced by

$$s \simeq t := (s\downarrow \vee\, t\downarrow) \to (s = t).$$

We have prepared the ground in order to state the axioms of EET, the first order part of which corresponds to the theory BON of basic operations and numbers of Feferman and Jäger [13]. Hence, the applicative part of EET contains the following axioms (1)–(10).

I. Partial Combinatory Algebra.

(1) $\mathsf{k}xy = x$,

(2) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq xz(yz)$.

II. Pairing and Projection.

(3) $\mathsf{p}_0(x,y) = x \wedge \mathsf{p}_1(x,y) = y$.

III. Natural Numbers.

(4) $0 \in N \wedge (\forall x \in N)(x' \in N)$,

(5) $(\forall x \in N)(x' \neq 0 \wedge \mathsf{p}_N(x') = x)$,

(6) $(\forall x \in N)(x \neq 0 \rightarrow \mathsf{p}_N x \in N \wedge (\mathsf{p}_N x)' = x)$.

IV. Definition by Cases on N.

(7) $a \in N \wedge b \in N \wedge a = b \rightarrow \mathsf{d}_N xyab = x$,

(8) $a \in N \wedge b \in N \wedge a \neq b \rightarrow \mathsf{d}_N xyab = y$.

V. Primitive Recursion on N.

(9) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \rightarrow (\mathsf{r}_N fg : N^2 \rightarrow N)$,

(10) $(f : N \rightarrow N) \wedge (g : N^3 \rightarrow N) \wedge x \in N \wedge y \in N \wedge h = \mathsf{r}_N fg \rightarrow$
$hx0 = fx \wedge hx(y') = gxy(hxy)$.

As usual the axioms of a partial combinatory algebra allow one to define lambda abstraction and to prove a recursion theorem (cf. e.g. [1, 5]).

As already mentioned above, the binary relation $\Re$ between objects and types acts as a *naming relation*, i.e. $\Re(s, A)$ means that s is a name of A or *s represents A*. While the naming of types must be understood intensionally, the types themselves are extensional in the usual set-theoretical sense.

VI. Extensionality.

$$\text{(EXT)} \qquad \forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B.$$

The axioms about explicit representation state that every type has a name, (E.1), and that there are no *homonyms*, i.e. different types have different names, (E.2).

VII. Explicit Representation.

$$\text{(E.1)} \qquad \exists x \Re(x, A),$$

$$\text{(E.2)} \qquad \Re(a, B) \wedge \Re(a, C) \rightarrow B = C.$$

As usual one introduces an element relation $\tilde{\in}$ between individual terms given by

$$s \mathrel{\tilde{\in}} t \;:=\; \exists X(\Re(t,X) \,\wedge\, s \in X).$$

In order to facilitate the naming process, we will adopt the conventions of [12]. In particular, let us assume that we have some fixed standard Gödel numbering of the formulas of $\mathbb{L}_p$. Furthermore, let $v0, v1, \ldots$ and $V0, V1, \ldots$ be an arbitrary but fixed enumeration of the individual and type variables, respectively. If F is an $\mathbb{L}_p$ formula with all its individual variables from the list $v0, \ldots, vm$ and all its type variables among $V0, \ldots, Vn$, and if $\vec{x} = x_0, \ldots, x_m$ and $\vec{Y} = Y_0, \ldots, Y_n$, then we write $F[\vec{x}, \vec{Y}]$ for the $\mathbb{L}_p$ formula which results from F by simultaneously replacing vi by x_i and Vj by Y_j $(0 \le i \le m,\ 0 \le j \le n)$. Finally, if $\vec{x} = x_1, \ldots, x_n$ and $\vec{X} = X_0, \ldots, X_n$, then we write $\Re(\vec{x}, \vec{X})$ instead of $\bigwedge_{i=0}^{n} \Re(x_i, X_i)$.

We are ready to state the axioms for elementary comprehension. ($\mathsf{ECA.1}$) assumes the existence of a type given by an (elementary) defining formula. The paradigm of explicitness tells us that we also need a name for this type and that this name is built uniformly in the individual and type variables of the defining formula. This is the content of ($\mathsf{ECA.2}$).

VIII. Elementary Comprehension. Let $F[x, \vec{y}, \vec{Z}]$ be an elementary $\mathbb{L}_p$ formula with Gödelnumber e; then we have:

($\mathsf{ECA.1}$) $$\exists X \forall x(x \in X \leftrightarrow F[x, \vec{a}, \vec{B}]),$$

($\mathsf{ECA.2}$) $$\Re(\vec{b}, \vec{B}) \wedge \forall x(x \in A \leftrightarrow F[x, \vec{a}, \vec{B}]) \;\rightarrow\; \Re(\mathsf{c}_e(\vec{a}, \vec{b}), A).$$

Observe that according to our conventions above, it is not necessary to encode information about the comprehension variable and the order of the parameters in the constants c_e.

This finishes the description of the axioms of elementary explicit type theory EET, which is defined to be the $\mathbb{L}_p$ theory consisting of the axiom groups I–VIII. Furthermore, let BON be the L_p theory containing the axiom groups I–V only.

One of our main concerns is the analysis of the *join axiom* in the presence of the non-constructive minimum operator. Let us write $A = \Sigma(B, f)$ for the statement

$$\forall x(x \in A \;\leftrightarrow\; x = (\mathsf{p}_0 x, \mathsf{p}_1 x) \,\wedge\, \mathsf{p}_0 x \in B \,\wedge\, \exists X(\Re(f(\mathsf{p}_0 x), X) \,\wedge\, \mathsf{p}_1 x \in X)),$$

i.e. A is the disjoint sum over all $x \in B$ of the types named by fx. Now the (uniform) axiom of join (J) has the form

(J) $$\Re(a, A) \wedge (\forall x \in A)\exists Y \Re(fx, Y) \;\rightarrow\; \exists Z(\Re(\mathsf{j}(a, f), Z) \wedge Z = \Sigma(A, f)).$$

In the following we are mainly interested in three forms of complete induction on the natural numbers N, namely set induction, type induction and formula induction. Sets of natural numbers are represented via their total characteristic functions.

Set Induction on N $(\mathsf{S\text{-}I_N})$

$$f \in P(N) \wedge f0 = 0 \wedge (\forall x \in N)(fx = 0 \to f(x') = 0) \;\to\; (\forall x \in N)(fx = 0),$$

Type Induction on N $(\mathsf{T\text{-}I_N})$

$$0 \in A \wedge (\forall x \in N)(x \in A \to x' \in A) \;\to\; (\forall x \in N)(x \in A),$$

Formula Induction on N $(\mathsf{F\text{-}I_N})$

$$F(0) \wedge (\forall x \in N)(F(x) \to F(x')) \;\to\; (\forall x \in N)F(x)$$

for all formulas $F(x)$ of $\mathbb{L}_p$.

We finish this section by giving the exact axiomatization of the non-constructive minimum operator $\boldsymbol{\mu}$.

The Unbounded Minimum Operator

$(\mu.1)$ $\quad (f : N \to N) \;\to\; \boldsymbol{\mu} f \in N,$

$(\mu.2)$ $\quad (f : N \to N) \wedge (\exists x \in N)(fx = 0) \;\to\; f(\boldsymbol{\mu} f) = 0.$

In the sequel we write $\mathsf{EET}(\mu)$ for $\mathsf{EET} + (\mu.1, \mu.2)$

Remark 1 Jäger and Strahm [23, 25] consider a slight strengthening of the axioms for $\boldsymbol{\mu}$. In contrast to the theories studied in [23], this extension is irrelevant for the theories studied in this article as far as proof-theoretic strength is concerned.

3 Predicative subsystems of analysis

For the sake of completeness, let us briefly recapitulate the definition of some well-known subsystems of analysis, which will be relevant in the sequel.

Let $\mathcal{L}_2$ be the usual language of second order arithmetic with number variables $x, y, z, \ldots$, set variables $X, Y, Z, \ldots$ (both possibly with subscripts), the constant 0 as well as function and relation symbols for all primitive recursive functions and relations. The *number terms* $(r, s, t, \ldots)$ of $\mathcal{L}_2$ and the *formulas* $(F, G, H, \ldots)$ are defined as usual. An $\mathcal{L}_2$ formula F is called *arithmetic*, if F does not contain bound set variables; let Π^0_∞ denote the class of all arithmetic $\mathcal{L}_2$ formulas.

In the following we presuppose some standard primitive recursive coding machinery: $\langle t_0, \ldots, t_{n-1}\rangle$ is the sequence number associated to the numbers $t_0, \ldots, t_{n-1}$ with related projections $(\cdot)_i$ so that $(\langle t_0, \ldots, t_{n-1}\rangle)_i = t_i$ for $0 \le i < n$.

Let PA denote the usual first order theory of Peano arithmetic formulated in the first order part $\mathcal{L}_1$ of $\mathcal{L}_2$. For the definition of theories with iterated arithmetical comprehension we refer to a primitive recursive standard wellordering $\prec$ of order type Γ_0 and field $\mathbb{N}$, and the reader is assumed to be familiar with the Veblen functions φ_α, cf. Pohlers [29] or Schütte [30]. Furthermore, let $\prec_n$ denote the restriction of $\prec$ to $\{m : m \prec n\}$, and let us write $TI(\alpha, F)$ for the formula

$$(\forall x \prec n)((\forall y \prec x)F(y) \to F(x)) \;\to\; (\forall x \prec n)F(x),$$

provided that the order type of $\prec_n$ is α. In addition, if $F(X, x)$ is an $\mathcal{L}_2$ formula (possibly with other free variables) and $n \in \mathbb{N}$, then $\mathcal{H}_F(X, n)$ denotes the $\mathcal{L}_2$ formula given by

$$\mathcal{H}_F(X, n) \;:=\; \forall y[y \in X \;\leftrightarrow\; y = \langle (y)_0, (y)_1\rangle \wedge (y)_0 \prec n \wedge F((X)^{(y)_0}, (y)_1)],$$

where $(X)^{(y)_0}$ is the set $\{\langle x, z\rangle \in X : x \prec (y)_0\}$. If α is an ordinal less than Γ_0, then let $(\Pi^0_\infty\text{-}\mathsf{CA})_\alpha$ denote the $\mathcal{L}_2$ theory comprising the axioms of PA, the formulas $TI(\prec_n, F)$ for all $\mathcal{L}_2$ formulas F, and the universal closure of

$$\exists X \mathcal{H}_G(X, n)$$

for every *arithmetic* $\mathcal{L}_2$ formula $G(X, x)$, where the order type of $\prec_n$ is α. Finally, let $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\alpha}$ denote the union of the theories $(\Pi^0_\infty\text{-}\mathsf{CA})_\beta$ for $\beta < \alpha$. For more details about theories with iterated comprehension the reader is referred to [4, 13, 15, 32].

In order to establish proof-theoretic lower bounds of systems of explicit mathematics, we will make use of two interpretations of the language $\mathcal{L}_2$ of second order arithmetic into the language $\mathbb{L}_p$ of elementary explicit type theory; they only differ in the interpretation of the set variables of $\mathcal{L}_2$. The first order part of $\mathcal{L}_2$ is translated in the obvious way by interpreting the number variables of $\mathcal{L}_2$ as ranging over the predicate N and assigning an L_p term t_f to each primitive recursive function f in a straightforward manner. According to the first interpretation, $(\cdot)^{\mathcal{N}}$, the set variables of $\mathcal{L}_2$ are interpreted as subtypes of N. This is in contrast to the second interpretation, $(\cdot)^N$, where the set variables are supposed to range over elements of $P(N)$. In particular, we have for all $\mathcal{L}_2$ formulas F that

$$\begin{aligned}(\exists X F(X))^{\mathcal{N}} &:= \exists X(X \subseteq N \;\wedge\; F^{\mathcal{N}}(X)),\\ (\exists X F(X))^{N} &:= \exists x(x \in P(N) \;\wedge\; F^{N}(x)),\end{aligned}$$

and similarly for universal set quantifiers. Observe that the interpretation $(\cdot)^N$ is actually a translation from $\mathcal{L}_2$ into L_p, the first order part of $\mathbb{L}_p$. For more details about these translations the reader is referred to [13, 12].

4 Standard structures

The purpose of this section is to define standard structures for $\mathsf{EET}(\mu) + (\mathsf{J})$. Initial segments of those will be used in Section 7, where proof-theoretic upper bounds of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ are obtained by partial reductions using so-called asymmetrical interpretations.

A standard structure for $\mathsf{EET}(\mu)+(\mathsf{J})$ is obtained by extending a model of $\mathsf{BON}(\mu)$ to an $\mathbb{L}_p$ structure which interprets the second order part of $\mathsf{EET}(\mu)+(\mathsf{J})$. The standard models for $\mathsf{BON}(\mu)$ have a recursion-theoretic flavor, but the interpretation of the second order part is done in a set-theoretical way. Standard structures for systems of explicit mathematics were introduced in Feferman [5, 8, 9]; refined versions thereof are studied in Takahashi [33] and Glaß [17].

Whereas the standard model for BON is given in terms of ordinary Σ^0_1 recursion theory, the standard model for $\mathsf{BON}(\mu)$ makes use of Π^1_1 recursion theory. It is obtained by taking the universe $\mathbb{N}$, interpreting N as $\mathbb{N}$ and $(x \cdot y)$ as $\{x\}^E(y)$, where $\{x\}^E$ is a standard enumeration of the functions which are partial recursive in E. Here E denotes the well-known type 2 equality functional given by

$$E(f) = \begin{cases} 0, & \text{if } \exists x f(x) = 0, \\ 1, & \text{else,} \end{cases}$$

cf. Hinman [18]. The functional E is interdefinable with μ, of course. A (formalized) version of this standard model of $\mathsf{BON}(\mu)$ is described in detail in [13].

In the following let us start off from an *arbitrary* model $\mathfrak{M}$ of $\mathsf{BON}(\mu)$,

$$\mathfrak{M} = (M, App, Nat, \hat{\mathsf{k}}, \hat{\mathsf{s}}, \ldots, \hat{\boldsymbol{\mu}}),$$

where M denotes the universe of $\mathfrak{M}$, $App \subset M^3$ interprets application, Nat is the interpretation of the predicate N and $\hat{\mathsf{k}}, \hat{\mathsf{s}}, \ldots, \hat{\boldsymbol{\mu}}$ are the denotations of the constants of L_p in $\mathfrak{M}$. We first interpret the constants $(\mathsf{c}_e)_{e<\omega}$ and j in some suitable way. For example, choose $\hat{\mathsf{c}}_e$ and $\hat{\mathsf{j}}$ in M so that $\hat{\mathsf{c}}_e v = (1, e, v)$ and $\hat{\mathsf{j}} v = (2, v)$ holds for all $v \in M$. Then we define the $\mathbb{L}_p$ structures $\mathfrak{S}_{\mathfrak{M}}(\alpha) = (\mathfrak{M}, T_\alpha, R_\alpha)$ by transfinite recursion on the ordinals, where $T_\alpha \subset Pow(M)$ is the range of the type variables of $\mathbb{L}_p$ and $R_\alpha \subset M \times T_\alpha$ interprets the naming relation $\Re$. The $\in$ relation is interpreted as the restriction of the usual element relation to $M \times T_\alpha$. In the following we inductively define representations $\rho_\alpha \subset M$ and collections $\tau(w) \subset M$ for each $w \in \rho_\alpha$; then R_α and T_α are obtained by setting

$$R_\alpha := \{(w, \tau(w)) : w \in \rho_\alpha\}, \qquad T_\alpha := \{\tau(w) : w \in \rho_\alpha\}.$$

(1) If $\alpha = 0$, then we set $\rho_0 = \emptyset$.

(2) If $\alpha = \beta + 1$, then $\rho_\beta \subset \rho_\alpha$. In addition, we distinguish the following two cases:

(i) If $F[x, \vec{y}, \vec{Z}]$ is an elementary $\mathbb{L}_p$ formula with Gödelnumber e, then we have $\hat{\mathsf{c}}_e(\vec{v}, \vec{w}) \in \rho_\alpha$ for all $\vec{v} \in M$ and $\vec{w} \in \rho_\beta$, and $\tau(\hat{\mathsf{c}}_e(\vec{v}, \vec{w}))$ is the set

$$\{m \in M : \mathfrak{S}_{\mathfrak{M}}(\beta) \models F[m, \vec{v}, \tau(\vec{w})]\},$$

where $\tau(\vec{w})$ is the sequence $\tau(w_1), \ldots, \tau(w_n)$ for $\vec{w} = w_1, \ldots, w_n$.

(ii) If $v \in \rho_\beta$ and $f \in M$ so that

$$\mathfrak{S}_{\mathfrak{M}}(\beta) \models (\forall x \in \tau(v)) \exists Y \Re(fx, Y),$$

then we have $\hat{\mathbf{j}}(v, f) \in \rho_\alpha$, and $\tau(\hat{\mathbf{j}}(v, f))$ is the set

$$\{m \in M : \mathfrak{S}_{\mathfrak{M}}(\beta) \models m = (\mathsf{p}_0 m, \mathsf{p}_1 m) \wedge \mathsf{p}_0 m \in \tau(v) \wedge \mathsf{p}_1 m \mathrel{\tilde{\in}} f(\mathsf{p}_0 m)\}.$$

(3) If α is a limit ordinal, then we set $\rho_\alpha := \bigcup_{\beta<\alpha} \rho_\beta$.

This finishes the description of the $\mathbb{L}_p$ structures $\mathfrak{S}_{\mathfrak{M}}(\alpha) = (\mathfrak{M}, T_\alpha, R_\alpha)$. Finally, let us put

$$\mathfrak{S}_{\mathfrak{M}} = (\mathfrak{M}, T, R),$$

where $T = \bigcup_\alpha T_\alpha$ and $R = \bigcup_\alpha R_\alpha$. It is easy to see that $\mathfrak{S}_{\mathfrak{M}}(\kappa) = \mathfrak{S}_{\mathfrak{M}}$ for a regular cardinal $\kappa > \mathit{card}(\mathfrak{M})$. Moreover, it is straightforward to check that the axioms of $\mathsf{EET}(\mu) + (\mathsf{J})$ are true in $\mathfrak{S}_{\mathfrak{M}}$.

Proposition 2 *$\mathfrak{S}_{\mathfrak{M}}$ is a model of $\mathsf{EET}(\mu) + (\mathsf{J})$. More precisely, if $\kappa = \mathit{card}(M)$, then it is $\mathfrak{S}_{\mathfrak{M}} = \mathfrak{S}_{\mathfrak{M}}(\kappa^+)$.*

5 Lower bounds

In this section let us briefly address the lower bounds for $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I}_\mathsf{N})$, $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N})$. As the methods are very similar to those in Feferman and Jäger [13, 12], we will only concentrate on the main differences.

For the systems $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I}_\mathsf{N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$ the desired lower bounds are already available by Feferman and Jäger [12], namely Peano arithmetic PA and the subsystem of second order arithmetic $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$, respectively. Therefore, we will only consider the system $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N})$ for the rest of this section, and we will show that it contains the theory $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}$.

The system $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$ is contained in $\mathsf{EET} + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N})$ via the translation $(\cdot)^{\mathcal{N}}$ of Section 3 (cf. e.g. Glaß [17]). This is due to the fact that $\mathsf{EET} + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N})$ proves transfinite induction up to each $\alpha < \varepsilon_0$ w.r.t. *arbitrary* $\mathbb{L}_p$ formulas, so that arithmetic comprehension can be iterated below ε_0 in the presence of the uniform axiom of join (J).

Proposition 3 *We have for all $\mathcal{L}_2$ sentences F:*

$$(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0} \vdash F \implies \mathsf{EET} + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N}) \vdash F^{\mathcal{N}}.$$

By methods of Schütte [30] it is well-know that $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$ proves $(\forall X)\mathit{TI}(\alpha, X)$ for each $\alpha < \varphi\varepsilon_0 0$. Hence, we have the following corollary.

Corollary 4 $\mathsf{EET} + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$ *proves* $TI^N(\alpha, F)$ *for all elementary* $\mathbb{L}_p$ *formulas* F *and all* $\alpha < \varphi\varepsilon_0 0$.

We have prepared the grounds in order to embed the theory $(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}$ into $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$ via our (second) translation $(\cdot)^N$.

Theorem 5 *We have for all* $\mathcal{L}_2$ *sentences* F*:*

$$(\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0} \vdash F \quad \Longrightarrow \quad \mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N}) \vdash F^N.$$

Proof The argument runs in exactly the same way as the proof of Theorem 9 in Feferman and Jäger [13], with the only exception that $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$ proves transfinite induction below $\varphi\varepsilon_0 0$ for arbitrary L_p formulas, whereas in the proof of Theorem 9 in [13] it is only available for each $\alpha < \varepsilon_0$. More precisely, if α is less than $\varphi\varepsilon_0 0$ and $n \in \mathbb{N}$ so that the order type of $\prec_n$ is α, then for each elementary $\mathcal{L}_2$ formula $F(X, x)$ with additional parameters $\vec{Y}, \vec{z}$ there exists an L_p term h so that

$$\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N}) \vdash \vec{y} \in P(N) \wedge \vec{z} \in N \;\rightarrow\; h\vec{y}\vec{z} \in P(N) \wedge \mathcal{H}^N_F(h\vec{y}\vec{z}, n),$$

where essential use is made of the axioms for the unbounded μ operator, the recursion theorem and the previous corollary. For the details of this argument the reader is referred to [13]. $\quad\Box$

6 Second order theories with ordinals and iterated elementary comprehension

In this section we introduce certain fixed point theories with ordinals and (iterated) elementary comprehension which will be crucial in order to determine the proof-theoretic upper bounds of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T\text{-}I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$.

Theories of ordinals over PA have been introduced in Jäger [21], and they have played an essential role in Feferman and Jäger [13] in order to analyze (first order) applicative theories with the non-constructive minimum operator. They have recently been extended in Jäger and Strahm [24] to second order systems with elementary comprehension, which in turn were used in Feferman and Jäger [12] in order to determine the proof-theoretic upper bound of the system $\mathsf{EET}(\mu) + (\mathsf{F\text{-}I_N})$. In order to prove an upper bound for $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$, we need extensions of the theories in [24], where elementary comprehension is iterated through every ordinal less than ε_0.

In the first paragraph of this section we recapitulate the system $\widehat{\mathsf{E}\Omega}$ and its restrictions of [24], and we introduce the system $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$. The second paragraph contains the proof-theoretic analysis of $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$.

6.1 The theories $\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$

We first introduce the notion of an inductive operator form. Let P be an n-ary relation symbol which does not belong to the language $\mathcal{L}_2$, and let $\mathcal{L}_2(P)$ denote the extension of $\mathcal{L}_2$ by P. An $\mathcal{L}_2(P)$ formula F is called *P-positive* if each occurrence of P in F is positive. We call P-positive formulas without free or bound set variables which contain at most $\vec{x} = x_1, \ldots, x_n$ free *inductive operator forms*; we let $\mathcal{A}(P, \vec{x})$ range over such forms.

Now we extend $\mathcal{L}_2$ to a new second order language $\mathcal{L}_\Omega$ by adding a new sort of *ordinal variables* $(\sigma, \tau, \eta, \xi, \ldots)$, new binary relation symbols $<$ and $=$ for the less relation and the equality relation on the ordinals[3] and an $(n+1)$-ary relation symbol $P_\mathcal{A}$ for each inductive operator form $\mathcal{A}(P, \vec{x})$ for which P is n-ary.

The *number terms* of $\mathcal{L}_\Omega$ are the number terms of $\mathcal{L}_2$; the *ordinal terms* of $\mathcal{L}_\Omega$ are the ordinal variables of $\mathcal{L}_\Omega$. The *formulas* of $\mathcal{L}_\Omega$ $(F, G, H, \ldots)$ are defined inductively as follows:

1. Each atomic formula of $\mathcal{L}_2$ is an atomic formula of $\mathcal{L}_\Omega$.
2. The formulas $(\sigma < \tau)$, $(\sigma = \tau)$ and $P_\mathcal{A}(\sigma, t)$ are atomic formulas of $\mathcal{L}_\Omega$.
3. If F and G are $\mathcal{L}_\Omega$ formulas, then so also are $\neg F$, $(F \vee G)$ and $(F \wedge G)$.
4. If F is an $\mathcal{L}_\Omega$ formula, then so also are $\exists x F$, $\forall x F$, $\exists X F$ and $\forall X F$.
5. If F is an $\mathcal{L}_\Omega$ formula, then so also are $(\exists \xi < \sigma) F$, $(\forall \xi < \sigma) F$, $\exists \xi F$ and $\forall \xi F$.

For every $\mathcal{L}_\Omega$ formula F we write F^σ to denote the $\mathcal{L}_\Omega$ formula which is obtained by replacing all unbounded ordinal quantifiers $(Q\xi)$ in F by $(Q\xi < \sigma)$. Additional abbreviations are:

$$\begin{aligned}
P^\sigma_\mathcal{A}(\vec{s}) &:= P_\mathcal{A}(\sigma, \vec{s}), \\
P^{<\sigma}_\mathcal{A}(\vec{s}) &:= (\exists \xi < \sigma) P^\xi_\mathcal{A}(\vec{s}), \\
P_\mathcal{A}(\vec{s}) &:= \exists \xi P^\xi_\mathcal{A}(\vec{s}).
\end{aligned}$$

An $\mathcal{L}_\Omega$ formula without free or bound set variables is called Δ^Ω_0 if all its ordinal quantifiers are bounded; it is called Σ^Ω $[\Pi^\Omega]$ if all positive [negative] universal ordinal quantifiers and all negative [positive] existential ordinal quantifiers are bounded. Finally, the *elementary* $\mathcal{L}_\Omega$ formulas are the $\mathcal{L}_\Omega$ formulas without bound set variables.

We are ready to give the exact axiomatization of the theory $\widehat{\mathsf{E}\Omega}$, which is based on the usual many-sorted predicate calculus with equality and classical logic.

[3] In general it will be clear from the context whether $<$ and $=$ denote the less and equality relation on the nonnegative integers or on the ordinals.

I. Number-theoretic Axioms. These include the usual axioms of PA except for complete induction on the natural numbers.

II. Inductive Operator Axioms. For every inductive operator form $\mathcal{A}(X, x)$:

$$P^{\sigma}_{\mathcal{A}}(\vec{s}) \leftrightarrow \mathcal{A}(P^{<\sigma}_{\mathcal{A}}, \vec{s}).$$

III. Σ^{Ω} Reflection Axioms. For every Σ^{Ω} formula F:

$$(\Sigma^{\Omega}\text{-}\mathsf{Ref}) \qquad F \rightarrow \exists \xi F^{\xi}.$$

IV. Linearity of $<$ on the Ordinals.

$$(\mathsf{LO}) \qquad \sigma \not< \sigma \wedge (\sigma < \tau \wedge \tau < \eta \rightarrow \sigma < \eta) \wedge (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

V. Elementary Comprehension. For every elementary formula $F(x)$ of $\mathcal{L}_{\Omega}$:

$$(\mathsf{ECA}) \qquad \exists X \forall y (y \in X \leftrightarrow F(y)).$$

VI. Formula Induction on the Natural Numbers. For all $\mathcal{L}_{\Omega}$ formulas $F(x)$:

$$(\mathcal{L}_{\Omega}\text{-}\mathsf{I}_{\mathsf{N}}) \qquad F(0) \wedge \forall x (F(x) \rightarrow F(x+1)) \rightarrow \forall x F(x).$$

VII. Δ^{Ω}_0 Induction on the Ordinals. For all Δ^{Ω}_0 formulas $F(\sigma)$:

$$(\Delta^{\Omega}_0\text{-}\mathsf{I}_{\Omega}) \qquad \forall \xi ((\forall \eta < \xi) F(\eta) \rightarrow F(\xi)) \rightarrow \forall \xi F(\xi).$$

This finishes the description of $\widehat{\mathsf{E}\Omega}$. Let $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\mathsf{R}\text{-}\widehat{\mathsf{E}\Omega}$ denote those subsystems of $\widehat{\mathsf{E}\Omega}$, where complete induction on the natural numbers is restricted to elementary and Δ^{Ω}_0 formulas, respectively. The proof-theoretic strength of $\mathsf{R}\text{-}\widehat{\mathsf{E}\Omega}$, $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}$ is established in Jäger and Strahm [24].

Proposition 6 *We have the following proof-theoretic equivalences:*

1. $\mathsf{R}\text{-}\widehat{\mathsf{E}\Omega} \equiv \mathsf{PA}$.
2. $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega} \equiv (\Pi^0_{\infty}\text{-}\mathsf{CA})_{<\varepsilon_0}$.
3. $\widehat{\mathsf{E}\Omega} \equiv (\Pi^0_{\infty}\text{-}\mathsf{CA})_{<\varepsilon_{\varepsilon_0}}$.

We finish this paragraph by introducing the system $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, which will be analyzed in the following subsection.

The theory $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$ is the extension of $\widehat{\mathsf{E}\Omega}$, where elementary comprehension can be iterated through each ordinal less than ε_0. More precisely, if α is an ordinal less than ε_0, then $\widehat{\mathsf{E}\Omega}_{\alpha}$ denotes the $\mathcal{L}_{\Omega}$ theory which is obtained from $\widehat{\mathsf{E}\Omega}$ by replacing the axioms of elementary comprehension (ECA) by the universal closure of

$$(\mathsf{ECA}_{\alpha}) \qquad \exists X \mathcal{H}_F(X, n)$$

for every *elementary* $\mathcal{L}_{\Omega}$ formula $F(X, x)$, where the order type of $\prec_n$ is α. Finally, $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$ is defined to be the union of the theories $\widehat{\mathsf{E}\Omega}_{\beta}$ for $\beta < \varepsilon_0$.

6.2 The proof-theoretic strength of $\widehat{\mathsf{E\Omega}}_{<\varepsilon_0}$

In this subsection we will analyze the proof-theoretic strength of $\widehat{\mathsf{E\Omega}}_{<\varepsilon_0}$. For that purpose we will introduce a semi-formal system $\mathsf{E\Omega}^*$ which combines features of the systems $\mathsf{E\Omega}^*_1$ of Jäger and Strahm [24] and RA^* of Schütte [30].

The language $\mathcal{L}^*$ of $\mathsf{E\Omega}^*$ is similar to $\mathcal{L}_\Omega$, but set variables $X, Y, \ldots$ are replaced by $X^\alpha, Y^\alpha, \ldots$ with $\alpha < \Gamma_0$. Number and ordinal terms of $\mathcal{L}^*$ are those of $\mathcal{L}_\Omega$. The set terms of $\mathcal{L}^*$ are defined simultaneously with the formulas of $\mathcal{L}^*$ (notice that $\mathcal{L}^*$ will only support formulas in negation normal form):

1. X^α is a set term.
2. If F is an $\mathcal{L}^*$ formula, then $\{x : F\}$ is a set term.
3. $R(t_1, \ldots, t_n)$ is an $\mathcal{L}^*$ formula for n-ary primitive recursive relation symbols R and number terms $t_1, \ldots, t_n$.
4. $(\sigma < \tau), (\sigma \not< \tau), (\sigma = \tau), (\sigma \neq \tau)$ are $\mathcal{L}^*$ formulas for ordinal terms σ, τ.
5. $P_{\mathcal{A}}(\sigma, t), \neg P_{\mathcal{A}}(\sigma, t)$ are $\mathcal{L}^*$ formulas for number terms t and ordinal terms σ.
6. $(t \in S), (t \notin S)$ are $\mathcal{L}^*$ formulas for number terms t and set terms S.
7. Formulas are closed under $\vee, \wedge, \exists x, \forall x, \exists \xi, \forall \xi, \exists \xi < \sigma, \forall \xi < \sigma, \exists X^\alpha, \forall X^\alpha$ for $\alpha \neq 0$.

The negation $\neg F$ of an $\mathcal{L}^*$ formula F is defined as usual by applying the de Morgan's rules. Δ_0^Ω, Σ^Ω, Π^Ω formulas of $\mathcal{L}^*$ are defined similarly to $\mathcal{L}_\Omega$.

The level of a set term S is defined by

$$lev(S) = max\{\alpha : \text{a set variable } X^\alpha \text{ occurs in } S\}.$$

The level of an $\mathcal{L}^*$ formula is defined analogously.

Now we define the cut rank $rk(F)$ for $\mathcal{L}^*$ formulas F:

1. If F is a Σ^Ω or a Π^Ω formula, then $rk(F) = 0$. Below we define the rank function for formulas which are not Σ^Ω or Π^Ω formulas.
2. $rk(t \in X^\alpha) = rk(t \notin X^\alpha) = \omega\alpha$, $rk(t \in \{x : F\}) = rk(t \notin \{x : F\}) = rk(F(t)) + 1$.
3. $rk(F \vee G) = rk(F \wedge G) = max\{rk(F), rk(G)\} + 1$.
4. $rk(\exists x F) = rk(\forall x F) = rk(\exists \xi F) = rk(\forall \xi F) = rk(F) + 1$.
5. $rk((\exists \xi < \sigma)F) = rk((\forall \xi < \sigma)F) = rk(F) + 2$.
6. $rk(\exists X^\alpha F(X^\alpha)) = rk(\forall X^\alpha F(X^\alpha)) = max\{\omega \cdot lev(\forall X^\alpha F(X^\alpha)), rk(F(X^0)) + 1\}$.

Notice that $rk(F) = rk(\neg F)$. We make the following observations:

1. If $lev(F) = \alpha$, then $\omega\alpha \leq rk(F) < \omega(\alpha + 1)$.
2. If $lev(S) < \alpha$, then $rk(F(S)) < rk(\exists X^\alpha F(X^\alpha))$.

In $\mathsf{E}\Omega^*$ we restrict $\mathcal{L}^*$ formulas to simple $\mathcal{L}^*$ formulas, i.e. to formulas which do not contain free number variables. Derivations are denoted in a Tait-style manner; Γ denotes a finite set of simple $\mathcal{L}^*$ formulas. The axioms of $\mathsf{E}\Omega^*$ are given as follows:

I. Equality Axioms for Numbers.

$$\Gamma,\ \neg F,\ G$$

if F and G are numerically equivalent, i.e. they differ only in (closed) number terms which have same value.

II. Number-theoretic Axioms.

$$\Gamma,\ R(t_1, \ldots, t_n)$$

if R is a primitive recursive relation symbol and $R(t_1, \ldots, t_n)$ is true.

III. Equality Axioms for Ordinals.

$$\Gamma,\ \sigma \neq \tau,\ \neg F(\sigma),\ F(\tau)$$

for all Δ_0^Ω formulas F.

IV. Inductive Operator Axioms and Linearity of $<$.

$$\Gamma,\ F$$

if F is an instance of an inductive operator axiom or of the linearity axiom for $<$.

V. Σ^Ω Reflection Axioms.

$$\Gamma,\ \neg F,\ \exists\xi F^\xi$$

for Σ^Ω formulas F.

VI. Δ_0^Ω Induction on the Ordinals.

$$\Gamma,\ \exists\xi((\forall\eta < \xi)F(\eta) \wedge \neg F(\xi)),\ \forall\xi F(\xi)$$

for Δ_0^Ω formulas F.

The rules of $\mathsf{E}\Omega^*$ are divided into four groups:

VII. Logical Rules.

$$\frac{\Gamma,\ F}{\Gamma,\ F \vee G} \qquad \frac{\Gamma,\ G}{\Gamma,\ F \vee G} \qquad \frac{\Gamma,\ F \quad \Gamma,\ G}{\Gamma,\ F \wedge G}$$

VIII. SET TERM RULES.

$$\frac{\Gamma,\, F(t)}{\Gamma,\, t \in \{x : F\}} \qquad\qquad \frac{\Gamma,\, \neg F(t)}{\Gamma,\, t \notin \{x : F\}}$$

IX. QUANTIFIER RULES.

$$\frac{\Gamma,\, F(s)}{\Gamma,\, \exists x F(x)} \qquad\qquad \frac{\ldots \Gamma,\, F(s) \ldots \quad \text{for all number terms } s}{\Gamma,\, \forall x F(x)}$$

$$\frac{\Gamma,\, F(S)}{\Gamma,\, \exists X^\alpha F(X^\alpha)}\ lev(S) < \alpha \qquad\qquad \frac{\ldots \Gamma,\, F(S) \ldots \quad \text{for all } S,\ lev(S) < \alpha}{\Gamma,\, \forall X^\alpha F(X^\alpha)}$$

$$\frac{\Gamma,\, F(\tau)}{\Gamma,\, \exists \xi F(\xi)} \qquad\qquad \frac{\Gamma,\, F(\tau)}{\Gamma,\, \forall \xi F(\xi)}\ (vc)$$

$$\frac{\Gamma,\, \tau < \sigma \wedge F(\tau)}{\Gamma,\, (\exists \xi < \sigma) F(\xi)} \qquad\qquad \frac{\Gamma,\, \tau < \sigma \rightarrow F(\tau)}{\Gamma,\, (\forall \xi < \sigma) F(\xi)}\ (vc)$$

X. CUT RULES.

$$\frac{\Gamma,\, F \quad \Gamma,\, \neg F}{\Gamma}$$

Here we marked rules with (vc) if they have to respect the usual variable conditions. By $\mathsf{E}\Omega^* \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma$ we denote that there is a derivation of Γ in $\mathsf{E}\Omega^*$ such that α is an upper bound for the proof length and ρ is a strict upper bound for cut ranks which occur in the derivation.

Standard proof-theoretic techniques can be applied here to obtain partial cut elimination, cf. Pohlers [29] or Schütte [30]. Due to the presence of the axiom groups III–VI, $\mathsf{E}\Omega^*$ does not enjoy full cut elimination; this is reflected by the requirement $\beta \neq 0$ below.

Proposition 7 (Partial cut elimination)

$$\mathsf{E}\Omega^* \,\frac{\ \alpha\ }{\beta + \omega^\rho}\!\!\!\!\vdash \Gamma \quad \Longrightarrow \quad \mathsf{E}\Omega^* \,\frac{\ \varphi\rho\alpha\ }{\beta}\!\!\!\!\vdash \Gamma \qquad \textit{for } \beta \neq 0.$$

In the next step we embed $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$ into $\mathsf{E}\Omega^*$. For that we have to introduce:

An $\mathcal{L}^*$ formula F' is an α-instance of an $\mathcal{L}_\Omega$ formula F if F' is obtained from F by

- replacing free number variables by arbitrary closed number terms.
- free set variables are replaced by set terms of $\mathcal{L}^*$ with level $< \alpha$.
- bound set variables get the superscript α.

Notice if F' is an α-instance of an $\mathcal{L}_\Omega$ formula, then $rk(F') < \omega(\alpha + 1)$. We obtain the following embedding similarly to Jäger and Strahm [24], Theorem 37.

Proposition 8 *If $\widehat{\mathsf{E}\Omega} \vdash F$ and F' is an α-instance of F, then $\mathsf{E}\Omega^* \mathrel{\vert}\!\frac{\omega\cdot 2}{\omega(\alpha+1)}\, F'$.*

Thus for an embedding of $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$ into $\mathsf{E}\Omega^*$ we have to interpret only the axioms (ECA_α) for all $\alpha < \varepsilon_0$ in $\mathsf{E}\Omega^*$.

Proposition 9 *Let α be the order type of $\prec_n$. Then for elementary formulas F:*

$$\mathsf{E}\Omega^* \mathrel{\vert}\!\frac{\omega(\gamma(n)+\omega)}{\omega(\gamma(n)+1)}\, \exists X^{\gamma(n)+1} \mathcal{H}_F(X^{\gamma(n)+1}, n), \tag{1}$$

where $\gamma(n) = max\{lev(S) : S$ occurs in $F\} + \alpha$.

Proof We will only sketch this roughly without paying attention to ordinal bounds; the reader may convince her- or himself that the ordinal bounds noted in (1) are sufficient.

Define the set term

$$R(n) = \{\langle x, y\rangle : x \prec n \wedge \exists X^{\gamma(n)}(\mathcal{H}_F(X^{\gamma(n)}, x) \wedge F(X^{\gamma(n)}, y))\}. \tag{2}$$

Notice that $lev(R(n)) < \gamma(n) + 1$. We prove

$$\mathsf{E}\Omega^* \vdash \mathcal{H}_F(R(n), n) \tag{3}$$

simultaneously with (1) by induction on $\prec$. We have the induction hypothesis

$$\mathsf{E}\Omega^* \vdash (\forall x \prec n)\mathcal{H}_F(R(x), x). \tag{4}$$

Further, notice that for $x \prec m \prec n$ the induction hypothesis of (1) implies because of $\gamma(x) + 1 \leq \gamma(m)$

$$\begin{aligned} \mathsf{E}\Omega^* \vdash \exists X^{\gamma(m)+1}\mathcal{H}_F(X, x) &\leftrightarrow \exists X^{\gamma(x)+1}\mathcal{H}_F(X, x) \\ &\leftrightarrow \exists X^{\gamma(m)}\mathcal{H}_F(X, x). \end{aligned}$$

Moreover, by Simpson [32], V.2.3, we have

$$\widehat{\mathsf{E}\Omega} \vdash \mathcal{H}_F(X, m) \wedge \mathcal{H}_F(Y, m) \to X = Y. \tag{5}$$

This yields, using Proposition 8,

$$\begin{aligned} \mathsf{E}\Omega^* \vdash R(n)^m &= \{\langle x, y\rangle : x \prec m \wedge \exists X^{\gamma(n)}(\mathcal{H}_F(X^{\gamma(n)}, x) \wedge F(X^{\gamma(n)}, y))\} \\ &= \{\langle x, y\rangle : x \prec m \wedge \exists X^{\gamma(m)}(\mathcal{H}_F(X^{\gamma(m)}, x) \wedge F(X^{\gamma(m)}, y))\} \\ &= R(m). \end{aligned}$$

Combining this with (4) we obtain

$$\mathsf{E}\Omega^* \vdash (\forall x \prec n)\mathcal{H}_F(R(n)^x, x),$$

from which we can conclude by (5), Proposition 8 and (2):

$$\mathsf{E}\Omega^* \vdash R(n) = \{\langle x, y\rangle : x \prec n \wedge F(R(n)^x, y)\}.$$

This is in fact (3); and (1) is a consequence of (3). $\square$

As an immediate consequence we obtain:

Proposition 10 *If* $\widehat{\mathsf{E\Omega}}_{<\varepsilon_0} \vdash F$*, then there is an* $\gamma < \varepsilon_0$ *and a* γ*-instance* F' *of* F *such that* $\mathsf{E\Omega}^* \mid\!\frac{\alpha}{\alpha}\, F'$ *with* $\alpha < \varepsilon_0$.

Theorem 11 $\mid \widehat{\mathsf{E\Omega}}_{<\varepsilon_0} \mid \leq \varphi(\varphi\varepsilon_0 0)0$.

PROOF Take an arithmetical ($\mathcal{L}_2$) sentence F with $\widehat{\mathsf{E\Omega}}_{<\varepsilon_0} \vdash F$. Then by Proposition 10 there is $\alpha < \varepsilon_0$ such that $\mathsf{E\Omega}^* \mid\!\frac{\alpha}{\alpha}\, F$. By Proposition 7 we obtain $\beta \leq \varphi\alpha\alpha$ such that $\mathsf{E\Omega}^* \mid\!\frac{\beta}{1}\, F$. Using the infinitary system $\mathsf{Z\Omega}$ of Jäger and Strahm [24], Theorem 39, we obtain an ordinal $\gamma < \omega^{\omega^{\beta}+1}$ such that $\mathsf{Z\Omega} \mid\!\frac{\gamma}{\gamma}\, F$. Using predicative cut elimination for $\mathsf{Z\Omega}$ ([24], Theorem 25), we obtain an ordinal $\delta \leq \varphi\gamma\gamma < \varphi(\varphi\varepsilon_0 0)0$ such that $\mathsf{Z\Omega} \mid\!\frac{\delta}{0}\, F$. But this implies $\mid \widehat{\mathsf{E\Omega}}_{<\varepsilon_0} \mid \leq \varphi(\varphi\varepsilon_0 0)0$, e.g. by [24], Theorem 29. $\Box$

In Section 7.3 we will see that Theorem 11 determines in fact the least upper bound for the proof-strength of $\widehat{\mathsf{E\Omega}}_{<\varepsilon_0}$.

7 Upper bounds

In this section we establish the exact proof-theoretic upper bounds of the three systems $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S\text{-}I_N})$, $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T\text{-}I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$, whereas the strength of the first system follows from known results (cf. Section 7.3). Hence, our main concern is about the latter two systems. In Section 7.1 we introduce Tait style reformulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T\text{-}I_N})$ and $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F\text{-}I_N})$, respectively, which make it possible to prove a partial cut elimination theorem. Quasi normal derivations of $\mathcal{T}_1$ and $\mathcal{T}_2$ are used in Section 7.2 in order to provide asymmetrical interpretations of $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T\text{-}I_N})$ and $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F\text{-}I_N})$ into $\mathfrak{S}_{\mathfrak{M}}(\omega)$ and $\mathfrak{S}_{\mathfrak{M}}(\varepsilon_0)$, respectively. In Section 7.3, finally, we sketch how the results of Section 7.1 and 7.2 can be formalized in second order theories with ordinals in order to yield the desired proof-theoretic upper bounds.

7.1 The Tait calculi $\mathcal{T}_1$ and $\mathcal{T}_2$

In the sequel we define two Tait calculi $\mathcal{T}_1$ and $\mathcal{T}_2$, which will be used for interpreting $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T\text{-}I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$, respectively. They enjoy partial cut elimination as it is needed for the asymmetrical interpretation described in the next paragraph. Tait calculi for systems of explicit mathematics with join have previously been studied in the literature in Marzetta [26, 27] and Glaß [17].

The language $\mathbb{L}_p^T$ of $\mathcal{T}_1$ and $\mathcal{T}_2$ is the extension of $\mathbb{L}_p$ by complementary relation symbols $\neq, \uparrow, \notin, \overline{N}$ and $\overline{\Re}$ for $=, \downarrow, \in, N$ and $\Re$, respectively. The atomic formulas built from the latter group of relation symbols are called *positive literals* and those obtained from relation symbols in the former group *negative literals*. The negation $\neg F$ of an $\mathbb{L}_p^T$ formula F is defined as usual by applying the law of double negation and de Morgan's laws. In the following we often identify $\mathbb{L}_p$ formulas with their translations into $\mathbb{L}_p^T$.

The Σ^+ [Π^-] formulas of $\mathbb{L}_p^T$ are inductively defined as follows:

1. The atomic formulas except those of the form $\overline{\Re}(s, A)$ [$\Re(s, A)$] are Σ^+ [Π^-] formulas.

2. The Σ^+ [Π^-] formulas are closed against disjunction, conjunction, object quantification and existential [universal] type quantification.

Note that F is a Σ^+ formula if and only if $\neg F$ is a Π^- formula. Moreover, Σ^+ formulas are upward persistent and Π^- formulas are downward persistent in $\mathfrak{S}_{\mathfrak{M}}(\alpha)$, respectively.

The *rank* $rn(F)$ of an $\mathbb{L}_p^T$ formula F is inductively defined as follows:

1. If F is Σ^+ or Π^-, then $rn(F) := 0$.

2. The rank of $\mathbb{L}_p^T$ formulas which are neither Σ^+ nor Π^- is defined as follows ($j \in \{\vee, \wedge\}, Q \in \{\exists, \forall\}$):

$$
\begin{aligned}
rn(F\ j\ G) &:= max(rn(F), rn(G)) + 1, \\
rn(QxF) &:= rn(F) + 2, \\
rn(QXF) &:= rn(F) + 1.
\end{aligned}
$$

Derivations in $\mathcal{T}_1$ and $\mathcal{T}_2$ are presented in a Tait-style manner. Accordingly, their axioms and rules of inference are formulated for finite sets of formulas, which have to be interpreted disjunctively. The capital Greek letters $\Gamma, \Lambda, \ldots$ denote finite sets of $\mathbb{L}_p^T$ formula, and we write, e.g., Γ, Λ, F, G for the union of Γ, Λ and $\{F, G\}$.

If $\Gamma(x_1, \ldots, x_n)$ is a finite set of $\mathbb{L}_p^T$ formulas, and if $s_1, \ldots, s_n$ are $\mathbb{L}_p^T$ terms, then the set

$$s_1\!\uparrow, \ldots, s_n\!\uparrow, \Gamma(s_1, \ldots, s_n)$$

is called a *faithful instance of* Γ.

In the following we need a reformulation of the join axiom (J), which will be adequate for the Tait calculi described below. Therefore, let us define two formulas $\mathcal{J}_1(a, f, A, Z)$ and $\mathcal{J}_2(a, f, z, A, Z)$ as follows:

$$
\begin{aligned}
\mathcal{J}_1(a, f, A, Z) \quad &:= \Re(\mathsf{j}(a, f), Z) \wedge Z = \Sigma(A, f), \\
\mathcal{J}_2(a, f, z, A, Z) &:= \exists X \exists Y [\Re(\mathsf{j}(a, f), Z) \wedge \\
&\qquad (z \in Z \rightarrow z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in A \wedge \Re(f(\mathsf{p}_0 z), X) \wedge \mathsf{p}_1 z \in X) \wedge \\
&\qquad (z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in A \wedge (\Re(f(\mathsf{p}_0 z), Y) \rightarrow \mathsf{p}_1 z \in Y) \rightarrow z \in Z)].
\end{aligned}
$$

Now the verification of the next claim is easy.

Lemma 12 *The following holds by pure logic and the representation axiom* ($\mathsf{E}.2$)*:*

$$(\forall x \in A)\exists Y \Re(fx, Y) \;\rightarrow\; [\,\exists Z \mathcal{J}_1(a, f, A, Z) \leftrightarrow \forall z \exists Z \mathcal{J}_2(a, f, z, A, Z)].$$

In particular, the above is true in $\mathfrak{S}_{\mathfrak{M}}(\alpha)$ *for each ordinal* α*.*

We are ready to introduce the Tait calculus $\mathcal{T}_1$. It comprises the following axioms and rules of inference.

I. Identity. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ formulas F with $rn(F) = 0$:

$$\Gamma,\, \neg F,\, F$$

II. Equality and Strictness for Objects. For all finite sets Γ of $\mathbb{L}_p^T$ formulas, all $\mathbb{L}_p^T$ formulas F with $rn(F) = 0$, all positive literals G of $\mathbb{L}_p^T$, all $\mathbb{L}_p^T$ terms s and t, all constants c of $\mathbb{L}_p^T$, and all variables x:

(1) $\Gamma,\, t\uparrow,\, t = t \qquad \Gamma,\, s \neq t,\, \neg F(s),\, F(t)$

(2) $\Gamma,\, \neg G(t),\, t\downarrow \qquad \Gamma,\, st\uparrow,\, s\downarrow \qquad \Gamma,\, st\uparrow,\, t\downarrow$

(3) $\Gamma,\, c\downarrow \qquad \Gamma,\, x\downarrow$

III. Equality for types. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ formulas F with $rn(F) = 0$:

$$\Gamma,\, A = A \qquad \Gamma,\, A \neq B,\, \neg F(A),\, F(B)$$

IV. Applicative and Ontological Axioms. These include all weakenings of faithful instances of the axioms (1)–(10), (μ.1), (μ.2), (EXT), ($\mathsf{E}.1$) and ($\mathsf{E}.2$).

V. Elementary Comprehension. These include all weakenings of faithful instances of ($\mathsf{ECA}.1$) and of the set

$$\neg\Re(\vec{b}, \vec{B}),\, \neg\forall x(x \in A \leftrightarrow F[x, \vec{a}, \vec{B}]),\, \Re(\mathsf{c}_e(\vec{a}, \vec{b}), A),$$

where $F[x, \vec{y}, \vec{Z}]$ is an elementary $\mathbb{L}_p^T$ formula with Gödelnumber e.

VI. Type Induction. These include all weakenings of type induction (T-I_{N}).

VII. Logical Rules. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ formulas F and G:

$$\frac{\Gamma,\, F}{\Gamma,\, F \vee G} \qquad \frac{\Gamma,\, G}{\Gamma,\, F \vee G} \qquad \frac{\Gamma,\, F \qquad \Gamma,\, G}{\Gamma,\, F \wedge G}$$

VIII. Quantifier Rules. For all finite sets Γ of $\mathbb{L}_p^T$ formulas, all $\mathbb{L}_p^T$ formulas $F(u)$ and $G(A)$, and all $\mathbb{L}_p^T$ terms t, so that the usual variable conditions are satisfied:

$$\frac{\Gamma,\, t\downarrow \wedge F(t)}{\Gamma,\, \exists x F(x)} \qquad \frac{\Gamma,\, F(u)}{\Gamma,\, \forall x F(x)}$$

$$\frac{\Gamma,\, G(A)}{\Gamma,\, \exists X G(X)} \qquad \frac{\Gamma,\, G(A)}{\Gamma,\, \forall X G(X)}$$

IX. JOIN. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ terms r, s, t:

$$\frac{\Gamma,\ \Re(r, A)\ \wedge\ s\!\downarrow\ \wedge\ (\forall x \in A)\exists Y \Re(sx, Y)}{\Gamma,\ t\!\uparrow,\ \exists Z \mathcal{J}_2(r, s, t, A, Z)}$$

X. CUT RULES. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ formulas F:

$$\frac{\Gamma,\ F \qquad \Gamma,\ \neg F}{\Gamma}$$

This finishes the description of the Tait calculus $\mathcal{T}_1$. The calculus $\mathcal{T}_2$ is now obtained from $\mathcal{T}_1$ by substituting the axioms VI. for type induction by some kind of ω rule, namely:

XI. THE ω RULE. For all finite sets Γ of $\mathbb{L}_p^T$ formulas and all $\mathbb{L}_p^T$ terms t:

$$\frac{\ldots \Gamma,\ \overline{n} \neq t \ldots \quad \text{for all } n < \omega}{\Gamma, \neg N(t)}$$

In contrast to the usual ω rule, the above rule still allows one to prove a partial cut elimination theorem in a standard way; it is also used in Cantini [2].

In the following we write $\mathcal{T}$ for either $\mathcal{T}_1$ or $\mathcal{T}_2$. The derivability relation $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma$ is defined by induction on α as follows:

1. If Γ is an axiom of $\mathcal{T}$, then we have $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma$ for all ordinals α and all $k < \omega$.
2. If $\mathcal{T} \mid\!\frac{\alpha_i}{k}\ \Gamma_i$ and $\alpha_i < \alpha$ for every premise Γ_i of an inference rule or a cut whose rank is less than k, then we have $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma$ for the conclusion Γ of this rule.

Now the verification of the following lemma is a matter of routine.

Lemma 13 *We have for all finite sets Γ, Λ of $\mathbb{L}_p^T$ formulas, all $\mathbb{L}_p^T$ terms t, all ordinals α, β and all $k, l < \omega$:*

1. $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma(u) \quad \Longrightarrow \quad \mathcal{T} \mid\!\frac{\alpha}{k}\ t\!\uparrow,\ \Gamma(t)$.
2. $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma(A) \quad \Longrightarrow \quad \mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma(B)$.
3. $\mathcal{T} \mid\!\frac{\alpha}{k}\ \Gamma$ *and* $\Gamma \subset \Lambda$ *and* $\alpha \leq \beta$ *and* $k \leq l \quad \Longrightarrow \quad \mathcal{T} \mid\!\frac{\beta}{l}\ \Lambda$.

The stage is set in order to prove partial cut elimination for $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Since the main formulas of all axioms and rules (including the ω rule) have rank 0, we can eliminate all cuts of rank greater than 0. The proof is more or less standard apart from the presence of the logic of partial terms. For similar arguments the reader is referred to [29, 30].

Proposition 14 (Partial cut elimination) *We have for all finite sets Γ, Λ of $\mathbb{L}_p^T$ formulas, all $\mathbb{L}_p^T$ formulas F, all ordinals α, β and all $k < \omega$:*

1. $\mathcal{T} \vdash_{k+1}^{\alpha} \Gamma, F$ *and* $\mathcal{T} \vdash_{k+1}^{\beta} \Lambda, \neg F$ *and* $rn(F) \leq k+1 \implies \mathcal{T} \vdash_{k+1}^{(\alpha \# \beta)\cdot 3} \Gamma, \Lambda$.

2. $\mathcal{T} \vdash_{k+2}^{\alpha} \Gamma \implies \mathcal{T} \vdash_{k+1}^{6^{\alpha}} \Gamma$.

In the following, $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ are embedded into $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The tautology and identity lemma reads as usual.

Lemma 15 *We have for all finite sets Γ of $\mathbb{L}_p^T$ formulas, all $\mathbb{L}_p^T$ formulas F, and all $\mathbb{L}_p^T$ terms t and s.*

1. $\mathcal{T} \vdash_{0}^{2\cdot rn(F)} \Gamma, \neg F, F$.

2. $\mathcal{T} \vdash_{0}^{2\cdot rn(F)} \Gamma, s \neq t, \neg F(s), F(t)$.

3. $\mathcal{T} \vdash_{0}^{2\cdot rn(F)} \Gamma, A \neq B, \neg F(A), F(B)$.

Now the embeddings are more or less standard. In particular, the join axiom (J) is proved by making use of the corresponding inference rule IX. The only unusual point is the derivation of formula induction $(\mathsf{F}\text{-}\mathsf{I_N})$ in $\mathcal{T}_2$ by making use of our special ω rule.

Lemma 16 *We have for all $\mathbb{L}_p^T$ formulas F and all $\mathbb{L}_p^T$ terms t:*

$$\mathcal{T}_2 \vdash_{rn(F)+1}^{\omega} \neg F(0), \neg(\forall x \in N)(F(x) \to F(x')), \neg N(t), F(t).$$

Proof One first proves by an easy induction on n that

$$\mathcal{T}_2 \vdash_{1}^{4\cdot(rn(F)+n+1)} \neg F(0), \neg(\forall x \in N)(F(x) \to F(x')), F(\overline{n}), \tag{6}$$

where essential use is made of Lemma 15.1. Furthermore, we have by Lemma 15.2 that

$$\mathcal{T}_2 \vdash_{0}^{2\cdot rn(F)} \overline{n} \neq t, \neg F(\overline{n}), F(t). \tag{7}$$

If we apply (weakening and) cut to (6) and (7) we get

$$\mathcal{T}_2 \vdash_{rn(F)+1}^{4\cdot(rn(F)+n+1)+1} \neg F(0), \neg(\forall x \in N)(F(x) \to F(x')), \overline{n} \neq t, F(t) \tag{8}$$

for all $n < \omega$. Now the claim is immediate by a use of the ω rule. $\square$

We are ready to state the embedding lemma. For similar results cf. e.g. [29, 30].

Lemma 17 *We have for all $\mathbb{L}_p$ formulas F:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N}) \vdash F \implies (\exists n, k < \omega)\ \mathcal{T}_1 \vdash_{k}^{n} F$.

2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N}) \vdash F \implies (\exists \alpha < \omega \cdot 2)(\exists k < \omega)\ \mathcal{T}_2 \vdash_{k}^{\alpha} F$.

By making use of partial cut elimination (Proposition 14), the embedding lemma entails the following reduction of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$, respectively.

Theorem 18 *We have for all $\mathbb{L}_p$ formulas F:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N}) \vdash F \implies (\exists n < \omega)\ \mathcal{T}_1 \vdash_{1}^{n} F$.

2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N}) \vdash F \implies (\exists \alpha < \varepsilon_0)\ \mathcal{T}_2 \vdash_{1}^{\alpha} F$.

7.2 Asymmetrical interpretations

In the following we provide asymmetrical interpretations of $\mathcal{T}_1$ and $\mathcal{T}_2$ into initial segments of the standard structures $\mathfrak{S}_{\mathfrak{M}}$. In particular, if $\mathfrak{M}$ is a model of $\mathsf{BON}(\mu)$ with a standard interpretation of N, then the $\forall\Sigma^+$ fragment of $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu)+(\mathsf{J})+(\mathsf{F}\text{-}\mathsf{I_N})$ can be modeled in $\mathfrak{S}_{\mathfrak{M}}(\omega)$ and $\mathfrak{S}_{\mathfrak{M}}(\varepsilon_0)$, respectively.

Asymmetrical interpretations are a well-known technique in proof theory, cf. e.g. [3, 19, 30]. They have previously been applied in the context of explicit mathematics in Marzetta [26, 27] and Glaß [16, 17].

Before we turn to the interpretation itself, let us state essential persistency properties of Σ^+ and Π^- formulas w.r.t. the standard structures $\mathfrak{S}_{\mathfrak{M}}(\alpha)$.

Lemma 19 *Let $\mathfrak{M}=(M,\dots)$ be a model of $\mathsf{BON}(\mu)$. Further, let $\alpha\le\beta$, $\vec{U}\in T_\alpha$ and $\vec{m}\in M$. Then we have for all Σ^+ formulas $F[\vec{A},\vec{a}]$ and all Π^- formulas $G[\vec{A},\vec{a}]$:*

1. $\mathfrak{S}_{\mathfrak{M}}(\alpha)\models F[\vec{U},\vec{m}] \quad\Longrightarrow\quad \mathfrak{S}_{\mathfrak{M}}(\beta)\models F[\vec{U},\vec{m}]$.
2. $\mathfrak{S}_{\mathfrak{M}}(\beta)\models G[\vec{U},\vec{m}] \quad\Longrightarrow\quad \mathfrak{S}_{\mathfrak{M}}(\alpha)\models G[\vec{U},\vec{m}]$.

In the sequel let us assume that $\Gamma[\vec{A},\vec{a}]$ is a set of Σ^+ and Π^- formulas. Further, if $\mathfrak{M}=(M,\dots)$ is a model of $\mathsf{BON}(\mu)$, then we write

$$\mathfrak{S}_{\mathfrak{M}}(\alpha,\beta)\models\Gamma[\vec{U},\vec{m}] \qquad (\vec{U}\in T_\alpha, \vec{m}\in M),$$

provided that one of the following conditions is satisfied:

(1) there is a Π^- formula $F[\vec{A},\vec{a}]$ in Γ so that $\mathfrak{S}_{\mathfrak{M}}(\alpha)\models F[\vec{U},\vec{m}]$;

(2) there is a Σ^+ formula $G[\vec{A},\vec{a}]$ in Γ so that $\mathfrak{S}_{\mathfrak{M}}(\beta)\models G[\vec{U},\vec{m}]$.

Hence, $\mathfrak{S}_{\mathfrak{M}}(\alpha,\beta)\models\Gamma[\vec{U},\vec{m}]$ means that the disjunction of the formulas in $\Gamma[\vec{U},\vec{m}]$ is true in $\mathfrak{S}_{\mathfrak{M}}$, where universal type quantifiers range over T_α and existential type quantifiers range over T_β, and similarly for the representation relation $\Re$. Finally, a model $\mathfrak{M}$ of $\mathsf{BON}(\mu)$ is called *standard*, if the interpretation of the predicate N consists of standard numbers only, i.e. if

$$N^{\mathfrak{M}}=\{\bar{n}^{\mathfrak{M}}:n\in\omega\}.$$

Theorem 20 (Asymmetrical interpretation) *Let $\mathfrak{M}=(M,\dots)$ be a standard model of $\mathsf{BON}(\mu)$. Assume further that $\mathcal{T}\;\frac{\alpha}{1}\;\Gamma[\vec{A},\vec{a}]$ for a set Γ of Σ^+ and Π^- formulas. Then we have for all ordinals β:*

$$\vec{U}\in T_\beta \text{ and } \vec{m}\in M \quad\Longrightarrow\quad \mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^\alpha)\models\Gamma[\vec{U},\vec{m}].$$

PROOF The assertion is proved by induction on α. In the following we only discuss two relevant cases, namely join and cut. In all other cases the claim follows from persistency (Lemma 19), the definition of $\mathfrak{S}_{\mathfrak{M}}$ and the induction hypothesis. In particular, type induction ($\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}}$) and the ω rule are trivially valid since $\mathfrak{M}$ is standard.

Let us first assume that $\Gamma[\vec{A},\vec{a}]$ is the conclusion of the inference rule IX. for the join. Then there are terms $r[\vec{a}], s[\vec{a}], t[\vec{a}]$ and an $A \in \vec{A}$ so that $t\uparrow,\ \exists Z \mathcal{J}_2(r,s,t,A,Z) \subset \Gamma$. Furthermore, we have an $\alpha_0 < \alpha$ so that

$$\mathcal{T} \mathrel{\vert\!\frac{\alpha_0}{1}} \Gamma[\vec{A},\vec{a}],\ \Re(r[\vec{a}],A) \wedge s[\vec{a}]\downarrow \wedge (\forall x \in A)\exists Y \Re(s[\vec{a}]x,Y). \tag{9}$$

Let us fix $\beta, \vec{U} \in T_\beta$ and $\vec{m} \in M$. Then the induction hypothesis yields

$$\mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^{\alpha_0}) \models \Gamma[\vec{U},\vec{m}],\ \Re(r[\vec{m}],U) \wedge s[\vec{m}]\downarrow \wedge (\forall x \in U)\exists Y \Re(s[\vec{m}]x,Y), \tag{10}$$

where $U \in \vec{U}$. If $\mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^{\alpha_0}) \models \Gamma[\vec{U},\vec{m}]$, then the claim is immediate by persistency. Otherwise, we have

$$\mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0}) \models \Re(r[\vec{m}],U) \wedge s[\vec{m}]\downarrow \wedge (\forall x \in U)\exists Y \Re(s[\vec{m}]x,Y). \tag{11}$$

By construction of $\mathfrak{S}_{\mathfrak{M}}$ we can conclude from (11) that

$$\mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0}+1) \models \exists Z(\Re(\mathsf{j}(r[\vec{m}],s[\vec{m}]),Z) \wedge Z = \Sigma(U,s[\vec{m}])). \tag{12}$$

Now the claim is immediate from Lemma 12.

As a second illustrative example let us consider the case where $\Gamma[\vec{A},\vec{a}]$ is the conclusion of a cut rule. Then the cut formula has rank 0, i.e. there is a Σ^+ formula $F[\vec{A},\vec{a}]$ and $\alpha_0,\alpha_1 < \alpha$ so that

$$\mathcal{T} \mathrel{\vert\!\frac{\alpha_0}{1}} \Gamma[\vec{A},\vec{a}],\ F[\vec{A},\vec{a}] \quad \text{and} \quad \mathcal{T} \mathrel{\vert\!\frac{\alpha_1}{1}} \Gamma[\vec{A},\vec{a}],\ \neg F[\vec{A},\vec{a}]. \tag{13}$$

Choose $\beta, \vec{U} \in T_\beta$ and $\vec{m} \in M$. We have to show $\mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^{\alpha}) \models \Gamma[\vec{U},\vec{m}]$. If we apply the induction hypothesis to (13) with β and $\beta+2^{\alpha_0}$, respectively, then we get

$$\begin{aligned} \mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^{\alpha_0}) &\models \Gamma[\vec{U},\vec{m}],\ F[\vec{U},\vec{m}], && (14)\\ \mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0},\beta+2^{\alpha_0}+2^{\alpha_1}) &\models \Gamma[\vec{U},\vec{m}],\ \neg F[\vec{U},\vec{m}]. && (15) \end{aligned}$$

Observe that $\beta+2^{\alpha_0}+2^{\alpha_1} \leq \beta+2^{\alpha}$. Hence, if it is

$$(i)\ \mathfrak{S}_{\mathfrak{M}}(\beta,\beta+2^{\alpha_0}) \models \Gamma[\vec{U},\vec{m}] \quad \text{or} \quad (ii)\ \mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0},\beta+2^{\alpha_0}+2^{\alpha_1}) \models \Gamma[\vec{U},\vec{m}],$$

then our assertion immediately follows by persistency. But one of (i) and (ii) applies, since otherwise (14) and (15) imply

$$\mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0}) \models F[\vec{U},\vec{m}] \quad \text{and} \quad \mathfrak{S}_{\mathfrak{M}}(\beta+2^{\alpha_0}) \models \neg F[\vec{U},\vec{m}]. \tag{16}$$

This, however, is not possible, and hence our claim is proved. □

Let us call the universal closure of a Σ^+ formula a $\forall\Sigma^+$ sentence. By combining the previous theorem with Theorem 18, we have established the following result.

Theorem 21 *Let $\mathfrak{M}$ be a standard model of* $\mathsf{BON}(\mu)$*. Then we have for all* $\forall\Sigma^+$ *sentences* F *of* $\mathbb{L}_p$*:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N}) \vdash F \implies \mathfrak{S}_{\mathfrak{M}}(\omega) \models F.$
2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N}) \vdash F \implies \mathfrak{S}_{\mathfrak{M}}(\varepsilon_0) \models F.$

In the next paragraph we indicate how the results of the previous two paragraphs can be formalized in theories of ordinals with (iterated) elementary comprehension in order to get the final proof-theoretic reductions.

7.3 Final proof-theoretic equivalences

In this paragraph we sketch the final proof-theoretic upper bounds of the systems $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N})$, $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$. Essential use will be made of the results of the previous two paragraphs as well as the second order theories with ordinals of Section 6.

Let us begin with the theory $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N})$, whose treatment is particularly simple by making use of Feferman and Jäger [13]. Recall that BON denotes the restriction of EET to the language L_p.

Theorem 22 $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N})$ *is a conservative extension of* $\mathsf{BON}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N})$.

Proof Let us establish this claim by an easy model-theoretic argument. Assume that $\mathfrak{M}$ is a model of $\mathsf{BON}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N})$ and consider the standard structure $\mathfrak{S}_{\mathfrak{M}}$ above $\mathfrak{M}$. By Proposition 2 we have that $\mathfrak{S}_{\mathfrak{M}} \models \mathsf{EET}(\mu) + (\mathsf{J})$ and, in addition,

$$\mathfrak{S}_{\mathfrak{M}} \models F \iff \mathfrak{M} \models F$$

holds for all L_p formulas F. In particular, $\mathfrak{S}_{\mathfrak{M}} \models (\mathsf{S}\text{-}\mathsf{I_N})$. Now the claim of the theorem is immediate by Gödel completeness. $\Box$

Let us mention that it is also possible to provide a proof-theoretic proof of this theorem; it makes use of arguments similar to the ones of the previous two paragraphs.

By [13] we know that $\mathsf{BON}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N})$ is proof-theoretically equivalent to (the first order part of) $\mathsf{R}\text{-}\widehat{\mathsf{E}\Omega}$, which in turn has the proof-theoretic strength of PA by Jäger [21] (cf. Proposition 6). Hence, we have established the following corollary.

Corollary 23 *We have the following proof-theoretic equivalences:*

$$\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{S}\text{-}\mathsf{I_N}) \equiv \mathsf{R}\text{-}\widehat{\mathsf{E}\Omega} \equiv \mathsf{PA}.$$

The upper bounds of the theories $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ are obtained by formalizing the results of the previous two paragraphs in $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, respectively, thus yielding the desired proof-theoretic equivalences. In the sequel we will only sketch these reductions, since the details of formalization are fairly standard.

Let us first mention that in all our reductions the applicative fragment L_p of $\mathbb{L}_p$ is treated in exactly the same way as in Feferman and Jäger [13], namely by making use of fixed point theories with ordinals. In particular, there exists a Σ^Ω formula $App(x, y, z)$ which interprets the L_p formula $(xy \simeq z)$, and this translation is lifted to a translation of all L_p formulas in an obvious way.

The *first step* in reducing $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ to $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, respectively, is provided by Theorem 18. In the case of the theory $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ a straightforward formalization of infinitary derivations and cut elimination procedures is needed within $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, cf. Schwichtenberg [31] for similar arguments.

The *second step* of our reductions consists in formalizing Theorem 20 in $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$, respectively. As far as this formalization is concerned, let us make the following general remarks:

(i) it is sufficient to consider structures $\mathfrak{S}_{\mathfrak{M}}(\alpha)$ for a *fixed* α less than ω or ε_0, respectively, since we are working with a fixed derivation in the systems $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$, respectively;

(ii) for the same reason we can restrict ourselves to only finitely many instances of the elementary comprehension axiom (ECA);

(iii) as a consequence of (ii), we get that a structure $\mathfrak{S}_{\mathfrak{M}}(\alpha + 1)$ can be described *elementary* from $\mathfrak{S}_{\mathfrak{M}}(\alpha)$ in our theories with ordinals and elementary comprehension.

Furthermore, let us mention that in the case of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ some straightforward formal truth definitions have to be described in $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$. Summing up, structures $\mathfrak{S}_{\mathfrak{M}}(\alpha)$ can be formalized in $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ and $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$ for a fixed $\alpha < \omega$ and $\alpha < \varepsilon_0$, respectively. Hence, Theorem 20 can be formalized in these systems, and we have established the following theorem.

Theorem 24 *We have the following embeddings:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ *can be embedded into* $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$.
2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ *can be embedded into* $\widehat{\mathsf{E}\Omega}_{<\varepsilon_0}$.

It is in fact already possible to embed $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ into the first order part of $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$; observe that $\mathsf{W}\text{-}\widehat{\mathsf{E}\Omega}$ is a conservative extension over its first order part.

Now the final proof-theoretic strength of the systems $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ is available by Feferman and Jäger [13] (Corollary 6), Theorem 5 and Theorem 11. Furthermore, together with the results of Feferman [10], the proof-theoretic strength of $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N})$ can be related to the well-known fixed-point theories $\widehat{\mathsf{ID}}_1$ and $\widehat{\mathsf{ID}}_2$, respectively.

Corollary 25 *We have the following proof-theoretic equivalences:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{T}\text{-}\mathsf{I_N}) \equiv \mathsf{W}\text{-}\widehat{\mathsf{E}\Omega} \equiv \widehat{\mathsf{ID}}_1 \equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0}$.

2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{F}\text{-}\mathsf{I_N}) \equiv \widehat{\mathsf{E}\Omega}_{<\varepsilon_0} \equiv \widehat{\mathsf{ID}}_2 \equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi\varepsilon_0 0}$.

Let us finish this paragraph by mentioning that these proof-theoretic equivalences always yield conservative extensions for (at least) arithmetic statements. This is immediate from the corresponding reductions sketched above.

8 Extensions

Using support of standard structures it is also possible to determine the proof-theoretic strength of systems of explicit mathematics with μ-operator and inductive generation (IG), cf. Feferman [5, 9], which asserts the existence of the accessible part of binary relations w.r.t. arbitrary $\mathbb{L}_p$-definable classes. $(\mathsf{IG})\!\upharpoonright$ is a weakening of (IG) asserting the existence of the accessible part of a relation w.r.t. types only.

Theorem 26 *We have the following proof-theoretic equivalences:*

1. $\mathsf{EET}(\mu) + (\mathsf{IG})\!\upharpoonright + (\mathsf{T}\text{-}\mathsf{I_N}) \equiv (\Pi^1_1\text{-}\mathsf{CA})\!\upharpoonright$.

2. $\mathsf{EET}(\mu) + (\mathsf{IG})\!\upharpoonright + (\mathsf{F}\text{-}\mathsf{I_N}) \equiv (\Pi^1_1\text{-}\mathsf{CA})$.

3. $\mathsf{EET}(\mu) + (\mathsf{IG}) + (\mathsf{F}\text{-}\mathsf{I_N}) \equiv (\Pi^1_1\text{-}\mathsf{CA}) + (\mathsf{BI})$.

4. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{IG}) + (\mathsf{F}\text{-}\mathsf{I_N}) \equiv (\Delta^1_2\text{-}\mathsf{CA}) + (\mathsf{BI})$.

Here $\subseteq$ is obtained by formalizing standard structures, cf. Section 4, which respect inductive generation, cf. Feferman [5, 9], in the corresponding subsystem of analysis. We make use of the fact that, using Π^1_1-comprehension, the relation *App* is a set.
$\supseteq$ follows from the fact that the subsystems of analysis are proof-theoretically reducible to those systems of explicit mathematics without μ-operator, cf. [9, 14, 20]. Moreover, the presence of μ does not seem to facilitate these reductions.

Combining the method of standard structures with asymmetrical interpretations, cf. Section 7, one can also handle the presence of the join axiom in the following situations.

Theorem 27 *We have the following proof-theoretic equivalences:*

1. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{IG})\!\upharpoonright + (\mathsf{T}\text{-}\mathsf{I_N}) \equiv (\Pi^1_1\text{-}\mathsf{CA})\!\upharpoonright$.

2. $\mathsf{EET}(\mu) + (\mathsf{J}) + (\mathsf{IG})\!\upharpoonright + (\mathsf{F}\text{-}\mathsf{I_N}) \equiv (\Pi^1_1\text{-}\mathsf{CA})_{<\varepsilon_0}$.

Further, we mention that also the methods of Glaß [17] for non-uniform type existence principles are applicable to the context where the μ-operator is present. It is possible to obtain similar results to [17]. For example we obtain with the non-uniform version (J^-) of the join axiom:

$$(\mathsf{J}^-) \qquad\qquad (\forall x \in A)\exists Y \Re(fx, Y) \;\rightarrow\; \exists Z(Z = \Sigma(A, f)).$$

Theorem 28 $\mathsf{EET}(\mu) + (\mathsf{J}^-) + (\mathsf{F}\text{-}\mathsf{I}_\mathsf{N}) \equiv \widehat{\mathsf{E}\Omega}_{<\omega^2} \equiv (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varphi 2\varepsilon_0}$.

We notice that the proof uses a combination of methods of this paper and of [17]. It is similarly to the proof of Theorem 11 and Section 5 and makes use of a sharpening of Proposition 9 for the special case $\alpha = \omega$ and the fact that $\widehat{\mathsf{E}\Omega}_{<\omega^2} = \widehat{\mathsf{E}\Omega} + (\mathsf{ECA}_\omega)$.

We finish this section by mentioning two possible strengthenings of the applicative axioms which do not raise the proof-theoretic strength of the theories considered in this article. The *totality axiom* (Tot) expresses that application is always total, i.e.

$$(\mathsf{Tot}) \qquad\qquad (\forall x)(\forall y)(xy\downarrow),$$

and the *extensionality axiom* (Ext) claims that operations are extensional in the following sense:

$$(\mathsf{Ext}) \qquad\qquad (\forall x)(fx \simeq gx) \;\rightarrow\; (f = g).$$

It is established in Jäger and Strahm [25] that the presence of (Tot) and (Ext) does not raise the proof-theoretic strength of various theories including the non-constructive μ-operator, and one readily verifies that the methods used there carry over to theories of types and names.

We finish this paper by taking up Feferman's conjecture (ii) which we have mentioned in the introduction. Although the results of this article disprove (ii), the question arises whether there is a natural subsystem of T_1 of the *same* strength as predicative analysis. A partial answer to this question is provided in Marzetta and Strahm [28], where a system of explicit mathematics with *universes* plus μ operator is studied, whose ordinal strength is shown to be exactly Γ_0. An alternative approach consists in setting up systems of explicit mathematics with μ plus a form of the *bar rule*; details will be presented elsewhere.

References

[1] Beeson, M. J. *Foundations of Constructive Mathematics: Metamathematical Studies.* Springer, Berlin, 1984.

[2] Cantini, A. *Logical frameworks for truth and abstraction.* To appear.

[3] Cantini, A. On the relationship between choice and comprehension principles in second order arithmetic. *Journal of Symbolic Logic 51* (1986), 360–373.

[4] FEFERMAN, S. Formal theories for transfinite iteration of generalized inductive definitions and some subsystems of analysis. In *Intuitionism and Proof Theory, Proceedings of the Summer Conference at Buffalo, New York, 1968*, A. Kino, J. Myhill, and R. E. Vesley, Eds. North Holland, Amsterdam, 1970, pp. 303–326.

[5] FEFERMAN, S. A language and axioms for explicit mathematics. In *Algebra and Logic*, J. Crossley, Ed., vol. 450 of *Lecture Notes in Mathematics*. Springer, Berlin, 1975, pp. 87–139.

[6] FEFERMAN, S. A theory of variable types. *Revista Colombiana de Matemáticas* (1975), 95–105.

[7] FEFERMAN, S. Theories of finite type related to mathematical practice. In *Handbook of Mathematical Logic*, J. Barwise, Ed. North Holland, Amsterdam, 1977, pp. 913–971.

[8] FEFERMAN, S. Recursion theory and set theory: a marriage of convenience. In *Generalized recursion theory II, Oslo 1977*, J. E. Fenstad, R. O. Gandy, and G. E. Sacks, Eds., vol. 94 of *Stud. Logic Found. Math.* North Holland, Amsterdam, 1978, pp. 55–98.

[9] FEFERMAN, S. Constructive theories of functions and classes. In *Logic Colloquium '78*, M. Boffa, D. van Dalen, and K. McAloon, Eds. North Holland, Amsterdam, 1979, pp. 159–224.

[10] FEFERMAN, S. Iterated fixed-point theories: application to Hancock's conjecture. In *The Patras Symposium*, G. Metakides, Ed. North Holland, Amsterdam, 1982, pp. 171–196.

[11] FEFERMAN, S. Hilbert's program relativized: proof-theoretical and foundational studies. *Journal of Symbolic Logic*, 53 (1988), 364–384.

[12] FEFERMAN, S., AND JÄGER, G. Systems of explicit mathematics with non-constructive μ-operator. Part II. *Annals of Pure and Applied Logic*. To appear.

[13] FEFERMAN, S., AND JÄGER, G. Systems of explicit mathematics with non-constructive μ-operator. Part I. *Annals of Pure and Applied Logic 65*, 3 (1993), 243–263.

[14] FEFERMAN, S., AND SIEG, W. Proof-theoretic equivalences between classical and constructive theories for analysis. In *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, W. Buchholz, S. Feferman, W. Pohlers, and W. Sieg, Eds., vol. 897 of *Lecture Notes in Mathematics*. Springer, Berlin, 1981, pp. 78–142.

[15] FRIEDMAN, H. Iterated inductive definitions and Σ^1_2-AC. In *Intuitionism and Proof Theory, Proceedings of the Summer Conference at Buffalo, New York, 1968*, A. Kino, J. Myhill, and R. E. Vesley, Eds. North Holland, Amsterdam, 1970, pp. 435–442.

[16] GLASS, T. On power set in explicit mathematics. *Journal of Symbolic Logic*. To appear.

[17] GLASS, T. Understanding uniformity in Feferman's explicit mathematics. *Annals of Pure and Applied Logic 75*, 1–2 (1995), 89–106.

[18] HINMAN, P. G. *Recursion-Theoretic Hierarchies*. Springer, Berlin, 1978.

[19] JÄGER, G. Beweistheorie von *KPN*. *Archiv für mathematische Logik und Grundlagenforschung 20* (1980), 53–64.

[20] JÄGER, G. A well-ordering proof for Feferman's theory T_0. *Archiv für mathematische Logik und Grundlagenforschung 23* (1983), 65–77.

[21] JÄGER, G. Fixed points in Peano arithmetic with ordinals. *Annals of Pure and Applied Logic 60*, 2 (1993), 119–132.

[22] JÄGER, G., AND POHLERS, W. Eine beweistheoretische Untersuchung von $(\Delta^1_2\text{-CA}) + (\mathrm{BI})$ und verwandter Systeme. In *Sitzungsberichte der Bayerischen Akademie der Wissenschaften*. 1982, pp. 1–28.

[23] JÄGER, G., AND STRAHM, T. Some theories with positive induction of ordinal strength $\varphi\omega 0$. *Journal of Symbolic Logic*. To appear.

[24] JÄGER, G., AND STRAHM, T. Second order theories with ordinals and elementary comprehension. *Archive for Mathematical Logic 34* (1995), 345–375.

[25] JÄGER, G., AND STRAHM, T. Totality in applicative theories. *Annals of Pure and Applied Logic 74*, 2 (1995), 105–120.

[26] MARZETTA, M. Universes in the theory of types and names. In *Computer Science Logic CSL'92* (Berlin, 1993), E. Börger et al., Ed., vol. 702 of *Lecture Notes in Computer Science*, Springer, pp. 340–351.

[27] MARZETTA, M. *Predicative Theories of Types and Names*. PhD thesis, Institut für Informatik und angewandte Mathematik, Universität Bern, 1994.

[28] MARZETTA, M., AND STRAHM, T. The μ quantification operator in explicit mathematics with universes and iterated fixed point theories with ordinals. In preparation.

[29] POHLERS, W. *Proof Theory: An Introduction*, vol. 1407 of *Lecture Notes in Mathematics*. Springer, Berlin, 1988.

[30] Schütte, K. *Proof Theory*. Springer, Berlin, 1977.

[31] Schwichtenberg, H. Proof theory: Some applications of cut-elimination. In *Handbook of Mathematical Logic*, J. Barwise, Ed. North Holland, Amsterdam, 1977, pp. 867–895.

[32] Simpson, S. G. Subsystems of second order arithmetic. Tech. rep., Pennsylvania State University, 1986. Chapters II–V and VII.

[33] Takahashi, S. Monotone inductive definitions in a constructive theory of functions and classes. *Annals of Pure and Applied Logic 42* (1989), 255–297.

Addresses

Thomas Glaß, Siemens AG, Zentralabteilung Forschung und Entwicklung, D-81730 München, Germany, Thomas.Glass@zfe.siemens.de

Thomas Strahm, Institut für Informatik und angewandte Mathematik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland, strahm@iam.unibe.ch*

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*Corresponding address