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RELATIONS BETWEEN PROPOSITIONAL NORMAL MODAL LOGICS: AN OVERVIEW

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Abstract The modal logic literature is notorious for multiple axiomatisations of the same logic and for conflicting overloading of axiom names. Many of the interesting interderivability results are still scattered over the often hard to obtain classics. We catalogue the most interesting axioms, their numerous variants, and explore the relationships between them in terms of interderivability as both axiom (schema) and as simple formulae. In doing so we introduce the Logics Workbench (LWB, see `http://lwbwww.unibe.ch:8080/LWBinfo.html`), a versatile tool for proving theorems in numerous propositional (nonclassical) logics. As a side effect we fulfill a call from the modal theorem proving community for a database of known theorems.

1 Introduction

The modal logic literature is quite diverse but most beginners turn to the introductory works by Hughes and Cresswell [HC68, HC84], Bull and Segerberg [BS86], Lemmon and Scott [LS77], or Chellas [Che80]. The classic by Segerberg [Seg71] is indispensable but is probably the hardest to get hold of (legally). Unfortunately the nomenclature used in these works is not uniform, and the proof techniques range from semantic, to proof-theoretic, to algebraic. Consequently, it is difficult to form a clear picture of the many interesting results contained in these works, and even harder to keep track of the many different versions of the basic axioms, their different names, and even the overloading of names. In what follows we try to give a general picture of this nomenclature, the interesting interconnections between some of these logics, and the interderivability of different axiomatisations of the same logic.

Our basic tool is the Logics Work Bench (LWB) [HJSS95], a program developed at the University of Bern that contains automated proof procedures based on modal Gentzen systems for numerous propositional (nonclassical) logics. Although the class of *modal* logics which have been automated in the LWB is relatively small, namely **K**, **KT**, **KT4**, **KT45** and **KW**, we can work with extensions of these logics by using a finite number of appropriate formula instances of the desired axioms, as explained below.

2 Definitions and Notational Conventions

We assume that **formulae** are built as usual from the classical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\top$, $\bot$ and the modal connectives $\Box$ and $\Diamond$. The binding strengths of the connectives from strongest to weakest is: $\neg$, $\Box$, $\Diamond$, $\wedge$, $\vee$, $\rightarrow$. Hence $\neg A \vee B \rightarrow C$ is read as $((\neg A) \vee B) \rightarrow C$. We also assume that $\rightarrow$ associates so that $A \rightarrow B \rightarrow C$ is to be read as $A \rightarrow (B \rightarrow C)$. We use $\Rightarrow$ to separate the two halves of Gentzen's sequents, and use $\vdash$ to denote derivability in a Hilbert system for a particular modal logic. This notation has been chosen carefully to avoid the confusion that may arise from the fact that the "deduction theorem" for modal logics is not the same as in classical logic. That is, $\Rightarrow$ is the metalevel analogue of $\rightarrow$ so that the usual deduction theorem holds for our modal Gentzen systems. But $\vdash$ is *not* the direct analogue of $\rightarrow$, although they are related as we explain below.

We assume that our modal logics are formulated by taking all classical propositional tautologies as axioms, and then adding a finite set of formulae (called *the axioms*) together with the inference rules of modus ponens, universal substitution, and necessitation. The formula $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is always one of the axioms (hence we deal only with normal modal logics). The name of the logic is obtained by concatenating the names of the axioms to the prefix **K** (note bolding and absence of commas in names of logics). A formula A is **derivable** in a modal logic **L** axiomatised in this manner iff there is a finite sequence of formulae $A_1, A_2, \ldots, A_n$ such that $A_n = A$ and each member of the sequence is either a classical tautology, an axiom, or is obtained from previous members of the sequence by one of the inference rules. If the formula A is derivable in **L** we write $\mathbf{L} \vdash A$ and call A a **theorem** of **L**. If A is an arbitrary formula and σ a substitution of formulae for propositional variables appearing in A, then the formula $A\sigma$ obtained by applying the substitution σ to A is called an **instance** of A. For example, if A is $\Box p \rightarrow p$ and σ is

$\{p := \Box(p \wedge q)\}$, then $A\sigma$ is $\Box\,\Box\,(p \wedge q) \rightarrow \Box(p \wedge q)$. In general, just using *instances* of axioms is not sufficient for our purposes, so let $\Box^0 A = A$ and $\Box^{n+1} A = \Box(\Box^n A)$ for $n \geq 0$. For any given $m \geq 0$, the formula $\Box^m(A\sigma)$ is a **modalised instance** of A. We use A^1, A^2, A^3, ...to denote modalised instances of A and use $\bigwedge_{i=1}^{k} A^i$ to denote the formula $A^1 \wedge A^2 \wedge \ldots \wedge A^k$ obtained by conjoining k modalised instances of A, for some finite $k \geq 1$.

Lemma 1 *If L is a normal modal logic, A, B are formulae, and there exist a finite number (say k) of modalised instances of A such that $\boldsymbol{L} \vdash \bigwedge_{i=1}^{k} A^i \rightarrow B$, then $\boldsymbol{LA} \vdash B$.*

Proof: Suppose that we can find k modalised instances A^i, $1 \leq i \leq k$ of A so that the *formula*

$$A^1 \wedge A^2 \wedge \ldots \wedge A^k \rightarrow B \tag{1}$$

is derivable in (is a theorem of) **L**. Since A is an *axiom* of **LA**, any instance $A\sigma$ is a *theorem* of **LA** by the rule of universal substitution; that is $\mathbf{LA} \vdash A\sigma$. Repeated uses of the rule of necessitation now allows us to deduce that any modalised instance A^j of A is also a theorem of **LA**; that is $\mathbf{LA} \vdash A^j$. And if (1) is derivable in **L**, then it will be derivable in **LA**. Since $((p \wedge q) \rightarrow r) \rightarrow (p \rightarrow (q \rightarrow r))$ is a classical tautology, we can now derive that

$$A^1 \rightarrow A^2 \rightarrow \ldots \rightarrow A^k \rightarrow B \tag{2}$$

is also derivable in **LA**. Since each antecedent A^i of (2) is a theorem of **LA**, k applications of modus ponens give us that B itself is a theorem of **LA**; that is $\mathbf{LA} \vdash B$. •

Now suppose that we wish to use the LWB to prove that a particular formula P is a theorem of the logic **KT4A** obtained by adding A as an axiom to **KT4**. Since the LWB does not have a proof procedure for **KT4A** we must use the one for **KT4** by finding enough modalised instances of A such that $\mathbf{KT4} \vdash \bigwedge_{i=1}^{k} A^i \rightarrow P$. Lemma 1 then allows us to conclude that $\mathbf{KT4A} \vdash P$.

In this way, we can "prove" theorems in the logic **KT4A** even though the LWB itself is capable of proving theorems only in the sublogic **KT4** (say). It is easy to generalise this process to use only the logic **K** by looking for modalised instances of the axioms T and 4 as well, and attempting to prove that:

$$\mathbf{K} \vdash A^1 \wedge A^2 \wedge \ldots \wedge A^i \wedge T^1 \wedge T^2 \wedge \ldots \wedge T^j \wedge 4^1 \wedge 4^2 \wedge \ldots \wedge 4^k \rightarrow P$$

For many logics, the literature contains more than one axiomatisation, say $\mathbf{KA_1A_2A}$ and $\mathbf{KA_1A_2B}$ where the axioms A_1 and A_2 are common but the axioms A and B differ. We can show the equivalence between these axiomatisations using the technique described above by proving that each logic contains the axioms of the other as theorems. Clearly, this is not an efficient method since *we* have to provide the instances to the LWB. However, it is possible to extend the LWB with a programming language that allows us to write procedures to systematically search for the right instances. But as this is work in progress we do not pursue this point here; see [Ach95] or see the LWB home page `http://lwbwww.unibe.ch:8080/LWBinfo.html`.

In the rest of this paper we do the following:

- in Section 3 we collect together in one table the many different axioms that are used in the literature, their different names, their many different shapes, and the sources for these variations;
- in Section 4.1 we explore some of the relationships between the fifteen basic logics obtained from combinations of the axioms T, D, 4, 5, and B;
- in Section 4.2 we explore the relationships between the axioms G, H, L, and some of their variants;
- in Section 4.3 we explore the relationships between the axioms M, Pt, and some of their variants;
- in Section 4.4 we explore the relationships between the axioms W, W_0, and Z;
- in Section 4.5 we explore relations between various versions of the axioms named after Dummett, Grzegorczyk, and McKinsey particularly as extensions of **KT4**;
- we provide figures that summarize the tables in a more readable form;
- and finally, in Section 5 we present a well-known sequent calculus for the logic **K**, which allows the reader to hand-check the results obtained by the LWB;
- in the appendix we show the LWB sessions that were used to obtain our data.

As a side-effect we provide a database of known theorems and their relationships for many propositional modal logics, thus answering a call for such a database from the modal theorem proving community (where testing an implementation calls for such a database).

3 Formulas

The tables below list all the formulae we consider and the names we use for them. The literature contains many variants of the formulae known as *Dum* and *Grz* and so we list these as well. In the next few sections we consider the relationships between many of these formulae.

name	formula
D	$\Box p \to \Diamond p$
D_2	$\Diamond \top$
T	$\Box p \to p$
4	$\Box p \to \Box \Box p$
4_M	$\Box p \wedge \Diamond q \to \Diamond(\Box p \wedge q)$
5	$\Diamond p \to \Box \Diamond p$
5_M	$\Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge q)$
B	$p \to \Box \Diamond p$
B_M	$p \wedge \Diamond q \to \Diamond(\Diamond p \wedge q)$
G	$\Diamond \Box p \to \Box \Diamond p$
G_0	$\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
H	$\Box(p \vee q) \wedge \Box(\Box p \vee q) \wedge \Box(p \vee \Box q) \to \Box p \vee \Box q$
H^+	$\Box(\Box p \vee q) \wedge \Box(p \vee \Box q) \to \Box p \vee \Box q$
L	$\Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p)$
L^+	$\Box(\Box p \to q) \vee \Box(\Box q \to p)$
L^{++}	$\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$
M	$\Box \Diamond p \to \Diamond \Box p$
Pt	$\Box(p \vee \Diamond p) \to \Diamond(p \wedge \Box p)$
W	$\Box(\Box p \to p) \to \Box p$
W_0	$\Box \Diamond \top \to \Box \bot$
Z	$\Box(\Box p \to p) \to (\Diamond \Box p \to \Box p)$
Dum	$\Box(\Box(p \to \Box p) \to p) \to (\Diamond \Box p \to p)$
Dum_1	$\Box(\Box(p \to \Box p) \to p) \to (\Diamond \Box p \to \Box p)$
Grz	$\Box(\Box(p \to \Box p) \to p) \to p$
Grz_1	$\Box(\Box(p \to \Box p) \to p) \to \Box p$
F	$(\Diamond \Box p \to q) \vee \Box(\Box q \to p)$
H_s	$p \to \Box(\Diamond p \to p)$
P	$\Diamond \Box \Diamond p \to (p \to \Box p)$
R	$\Diamond \Box p \to (p \to \Box p)$
X	$\Box \Box p \to \Box p$
Zem	$\Box \Diamond \Box p \to (p \to \Box p)$

variants of M, Dum, Grz	
name	formula
M	$\Box \Diamond p \rightarrow \Diamond \Box p$
M_2	$\Diamond \Box (p \rightarrow \Box p)$
M_3	$\Box \Diamond p \wedge \Box \Diamond q \rightarrow \Diamond (p \wedge q)$
Dum	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond \Box p \rightarrow p)$
Dum_1	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond \Box p \rightarrow \Box p)$
Dum_2	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond \Box p \rightarrow p)$
Dum_3	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond \Box p \rightarrow \Box p)$
Dum_4	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond \Box p \rightarrow p \vee \Box p)$
Grz	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$
Grz_1	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p$
Grz_2	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow p$
Grz_3	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow \Box p$
Grz_4	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p \vee \Box p$
Grz_5	$\Box(\Box(p \rightarrow \Box q) \rightarrow \Box q) \wedge \Box(\Box(\neg p \rightarrow \Box q) \rightarrow \Box q) \rightarrow \Box q$

The following table shows the names used in the literature. Note that the formulas are only equal modulo substitutions of the form $\{p := \neg p\}$.

here	[BS86]	[Seg71]	[LS77]	[HC68]	[HC84]	[Che80]	[Gol87]
D	D		D (p. 50)		D	D	D
D_2		D (p. 47)					
T	T	T (p. 47)	T (p. 50)		T	T	T
4	4	4 (p. 47)	4 (p. 50)		4	4	4
5	E	E (p. 47)	E (p. 50)		5	5	5
B	B	B (p. 47)	B (p. 50)	B	B	B	B
G	G	G (p. 47)	G (p. 50)	$G1$	$G1$	G	
G_0		G_0 (p. 47)					
H	H		H (p. 69)			H	
H^+			H^+ (p. 69)			H^+	
L		Lem_0 (p. 47)	H_0^+ (p. 80)		$D1_0$		L
L^+		Lem (p. 47)	H_0 (p. 80)		$D1$	L^+	L_1
L^{++}				$D2$		L^{++}	
M	M	M (p. 107)	M (p. 47)	Kb	M	G_c	
M_3				Ka			
W	W	W (p. 84)			W	Gr	W
W_0		W_0 (p. 93)			W_0		
Z		Z (p. 84)					Z
Dum	Dum			$N1$	$N1$		
Dum_1		Dum_1 (p. 107)					Dum
Grz	Grz		Grz (p. 81)	$J1$	$J1$		
Grz_1		Grz_1 (p. 107)					
Grz_5		(p. 108)					
F		F (p. 161)					
H_s		H (p. 148)		$H1$			
P		P (p. 152)					
R		R (p. 160)		$R1$			
X							X
Zem		Zem (p. 152)					

We give the page numbers in the case of [Seg71] and [LS77] because there is no index in these books.

Some variants of Dum and Grz are mentioned in [Seg71] (p. 107).

Variants of M are discussed in [Sob64].

If you want to find out whether a certain formula is considered in this paper, the following table may help you. It contains a negation normal form of the formula, ordered according to their modal depth (md) and the number of occurring variables (vars). However note that the negation normal form is not unique, and that renaming as well as substitutions of the form $\{p := \neg p\}$ do not matter in this context.

formula	a negation normal form	md	vars
D_2	$\Diamond\top$	1	0
W_0	$\Diamond\Box\bot \vee \Box\bot$	2	0
D	$\Diamond p \vee \Diamond\neg p$	1	1
T	$p \vee \Diamond\neg p$	1	1
4	$\Diamond\neg p \vee \Box\Box p$	2	1
5	$\Box\neg p \vee \Box\Diamond p$	2	1
B	$\neg p \vee \Box\Diamond p$	2	1
G	$\Box\Diamond p \vee \Box\Diamond\neg p$	2	1
M	$\Diamond\Box p \vee \Diamond\Box\neg p$	2	1
Pt	$\Diamond(p \wedge \Box p) \vee \Diamond(\neg p \wedge \Box\neg p)$	2	1
W	$\Diamond(\neg p \wedge \Box p) \vee \Box p$	2	1
H_s	$\neg p \vee \Box(p \vee \Box\neg p)$	2	1
R	$\neg p \vee \Box p \vee \Box\Diamond\neg p$	2	1
X	$\Box p \vee \Diamond\Diamond\neg p$	2	1
Z	$\Box p \vee \Box\Diamond\neg p \vee \Diamond(\neg p \wedge \Box p)$	2	1
Dum	$p \vee \Box\Diamond\neg p \vee \Diamond(\neg p \wedge \Box(\neg p \vee \Box p))$	3	1
Grz	$p \vee \Diamond(\neg p \wedge \Box(\neg p \vee \Box p))$	3	1
M_2	$\Diamond\Box(\neg p \vee \Box p)$	3	1
P	$\neg p \vee \Box p \vee \Box\Diamond\Box\neg p$	3	1
Zem	$\neg p \vee \Box p \vee \Diamond\Box\Diamond\neg p$	3	1
4_M	$\Diamond\neg p \vee \Box\neg q \vee \Diamond(\Box p \wedge q)$	2	2
5_M	$\Box\neg p \vee \Box\neg q \vee \Diamond(\Diamond p \wedge q)$	2	2
B_M	$\neg p \vee \Box\neg q \vee \Diamond(\Diamond p \wedge q)$	2	2
G_0	$\Box(p \vee \Diamond q) \vee \Box(\neg p \vee \Diamond\neg q)$	2	2
H	$\Box p \vee \Box q \vee \Diamond(\neg p \wedge \neg q) \vee \Diamond(\neg p \wedge \Diamond\neg q) \vee \Diamond(\Diamond\neg p \wedge \neg q)$	2	2
H^+	$\Box p \vee \Box q \vee \Diamond(\neg p \wedge \Diamond\neg q) \vee \Diamond(\Diamond\neg p \wedge \neg q)$	2	2
L	$\Box(p \vee \neg q \vee \Diamond\neg q) \vee \Box(q \vee \neg p \vee \Diamond\neg p)$	2	2
L^+	$\Box(p \vee \Diamond\neg q) \vee \Box(\Diamond\neg p \vee q)$	2	2
L^{++}	$\Box(\Box p \vee \Diamond\neg q) \vee \Box(\Diamond\neg p \vee \Box q)$	2	2
M_3	$\Diamond\Box\neg p \vee \Diamond\Box\neg q \vee \Diamond(p \wedge q)$	2	2
F	$q \vee \Box\Diamond\neg p \vee \Box(\Diamond\neg q \vee p)$	2	2
Grz_5	$\Box q \vee \Diamond(\Box(p \vee \Box q) \wedge \Diamond\neg q) \vee \Diamond(\Box(\neg p \vee \Box q) \wedge \Diamond\neg q)$	3	2

Note that some variants of Dum and Grz are not listed here.

4 Tables and diagrams

We now consider the relationships between some of the formulae found in the literature.

4.1 Relations concerning D, T, 4, 5, B

The most basic normal modal logics are obtained by adding combinations of the formulae T, D, 4, 5 and B as axioms to the basic normal modal logic **K**. There are fifteen distinct logics obtained in this way and the relationships between them are quite well known (see e.g. [LS77] or [Che80] for a complete diagram). In the table, we simply confirm some of these relationships, and show the equivalence between different versions of these axioms.

The axioms 4_M, 5_M and B_M form the basis for very versatile resolution-style proof systems for their respective logics [AM86].

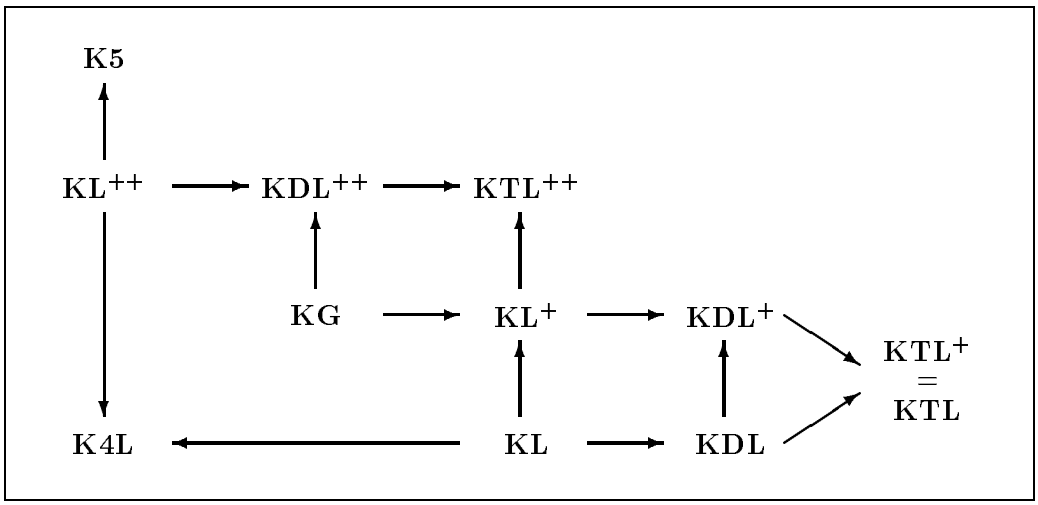
$\mathbf{K4} = \mathbf{K4_M}$	$\mathbf{K5} = \mathbf{K5_M}$	$\mathbf{KB} = \mathbf{KB_M}$

$\mathbf{KA_1} \ldots \mathbf{A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
D	D_2	D
D_2	D	D_2
4	4_M	4
4_M	4	$4_M\{q := \neg \Box p\}$
5	5_M	5
5_M	5	$5_M\{q := \neg \Diamond p\}$
B	B_M	B
B_M	B	$B_M\{q := \neg \Diamond p\}$
T	D	$T\{p := \bot\}$
T, 5	4	$T\{p := \Diamond\neg p\}$, $5\{p := \Box p\}$, $\Box 5\{p := \neg p\}$
T, 5	B	$T\{p := \neg p\}$, 5
4, B	5	$\Box 4\{p := \neg p\}$, $B\{p := \Diamond p\}$
5, B	4	$\Box 5\{p := \neg p\}$, $B\{p := \Box p\}$
D, 4, B	T	D, 4, $B\{p := \neg p\}$

4.2 Relations concerning $G, G_0, H, H^+, L, L^+, L^{++}$

The formulae G, H, and L also have numerous versions. The table below clears up some of these relationships and the figures show these more explicitly. For example, we have $\mathbf{KH} = \mathbf{KL}$ since both $\mathbf{K} \vdash H^1 \to L$ and $\mathbf{K} \vdash L \to H$, where $H^1 = H\{p := p \wedge \Box p \to q, q := q \wedge \Box q \to p\}$.

$\mathbf{KG} = \mathbf{KDG_0}$ $\qquad$ $\mathbf{KL} = \mathbf{KH}$ $\qquad$ $\mathbf{KL^+} = \mathbf{KH^+}$



$\mathbf{KA_1 \ldots A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
G	G_0	$G\{p := q\}$
D, G_0	G	$\Box D$, $G_0\{p := \Box p, q := p\}$
L	H	L
H	L	$H\{p := p \wedge \Box p \to q, q := q \wedge \Box q \to p\}$
L^+	H^+	L^+
H^+	L^+	$H^+\{p := \Box p \to q, q := \Box q \to p\}$
L^+	L	L^+
T, L	L^+	$\Box T, \Box T\{p := q\}, L$
T, L^{++}	L^+	$\Box T, \Box T\{p := q\}, L^{++}$
$4, L$	L^{++}	$4\{p := p \wedge \Box p \to q\}, 4\{p := q \wedge \Box q \to p\}, \Box 4, \Box 4\{p := q\}, L$
5	L^{++}	$5\{p := \Box p\}, \Box 5\{p := \neg p\}$
D, L^{++}	G	$\Box D\{p := \neg p\}, L^{++}\{q := \neg p\}$
5	G	$5, 5\{p := \neg p\}$

4.3 Relations concerning M, Pt

The axioms discussed in this section are loosely grouped around the famed McKinsey-axiom M, which attracted attention mainly because of its second order nature. The formula M appears to be one of the simplest axioms defining a pure second order property (M is not canonical), thus falling out of the class of Sahlqvist axioms [Sah73].

In the presence of 4, however, M defines a first order property: every point in the frame has a successor which is a (reflexive) dead end point. This property is also defined by Pt except that Pt *is* a Sahlqvist-axiom. Thus, in the last two rows here we show that $\mathbf{K4M} \vdash Pt$; $\mathbf{KPt} \vdash M$; and hence that $\mathbf{K4Pt} = \mathbf{K4M}$. The other axioms mentioned are variations of M.

$$\mathbf{KT4M} = \mathbf{KT4M_2} = \mathbf{KT4M_3}$$

$$\mathbf{K4M} = \mathbf{K4Pt}$$

$$\mathbf{KD} \longrightarrow \mathbf{KM} \longrightarrow \mathbf{KPt} \longrightarrow \mathbf{K4M}$$

$\mathbf{KA_1} \ldots \mathbf{A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
$T, 4, M$	M_2	$T\{p := M\}$, $\Box 4$, $\Box M$
$T, 4, M_2$	M	$T\{p := M_2\}$, $4\{p := \Diamond\neg p\}$, M_2
$4, M$	M_3	$4\{p := \Diamond q \wedge (\neg p \vee \neg q)\}$, M
M_3	M	$M_3\{q := \neg p\}$
M	D	M
$4, M$	Pt	$4\{p := (p \vee \Diamond p) \wedge (\neg p \vee \Diamond\neg p)\}$, $\Box 4\{p := p \wedge (p \vee \Diamond p) \wedge (\neg p \vee \Diamond\neg p)\}$, $\Box 4\{p := \neg p \wedge (p \vee \Diamond p) \wedge (\neg p \vee \Diamond\neg p)\}$, M
Pt	M	Pt

4.4 Relations concerning W, W_0, Z

The formula we call W (and sometimes called G) has only one variant but it is related to the formula Z, used by Goldblatt [Gol87] to describe the (temporal) logic of the frame $\langle \omega, \leq \rangle$ where ω is the set of natural numbers and $\leq$ is the usual ordering on the natural numbers.

We first show that W enforces transitivity by showing that 4 is a theorem of **KW**. We then show the relationships between W, its cousin W_0 and Z.

See also the figure in Section 4.5.

$$\mathbf{KW} = \mathbf{KW_0Z}$$

$\mathbf{KA_1} \ldots \mathbf{A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
W	4	$W\{p := p \wedge \Box p\}$
W	W_0	$W\{p := \bot\}$
W	Z	W
W_0, Z	W	W_0, Z
W	Grz_1	$W\{p := p \wedge \Box(\Box p \rightarrow p)\}, \Box\Box\ W$
$4, Z$	Dum_1	$\Box 4, Z$

4.5 Relations concerning Dum, Grz

The formulae known as Dum and Grz are two of the most bizarre formulae that occur in the literature. They each have numerous variants and the relationships between these variants has attracted considerable attention in the literature. For example, Segerberg [Seg71] proves (using semantic means) that these variants are "deductively equivalent"* [Seg71, page 108] in **KT4**. A detailed syntactical analysis of these axioms based on K4 (explicitly benefitting of the usage of modal rules), has been undertaken in [Hei95].

In the table below we explore the relationships between Dum and each of its variants in turn. For example, the first row indicates that

$$\mathbf{K} \vdash 4^1 \wedge 4^2 \wedge Dum \wedge Dum^1 \rightarrow Dum_1$$

where 4^1, 4^2 and Dum^1 are the required modalised instance shown in the table. Hence

$$\mathbf{K4} \vdash Dum \wedge Dum^1 \rightarrow Dum_1 \text{ and } \qquad \mathbf{K4Dum} \vdash Dum_1$$

*For two schemata S and S', Segerberg (page 8) uses the term "S implies S' in **L**" to mean that adding every substitutional instance of S as an axiom to **L** allows us to derive every substitutional instance of S'. He then defines two schemata to be "equivalent in **L**" if each implies the other in **L**. But he does not actually define what "deductively equivalent" means.

And the second row indicates that

$$\mathbf{K} \vdash T \wedge Dum_1 \rightarrow Dum$$

hence

$$\mathbf{KT} \vdash Dum_1 \rightarrow Dum \qquad \text{and} \qquad \mathbf{KTDum_1} \vdash Dum$$

Since **KT4** is an extension of both **KT** and **K4**, we have both

$$\mathbf{KT4} \vdash Dum \wedge Dum^1 \rightarrow Dum_1 \qquad \text{and} \qquad \mathbf{KT4} \vdash Dum_1 \rightarrow Dum$$

And from this we can conclude that

$$\mathbf{KT4Dum} \vdash Dum_1 \qquad \text{and} \qquad \mathbf{KT4Dum_1} \vdash Dum$$

Thus not only have we shown that $\mathbf{KT4Dum} = \mathbf{KT4Dum_1}$ but we have given the exact substitutions necessary to prove using LWB that "Dum_1 is deductively equivalent to Dum in **KT4**". As we shall show later, proofs from LWB can be turned into syntactic proofs in a modal Gentzen system for **K**, from which it is possible to extract a Hilbert-style derivation.

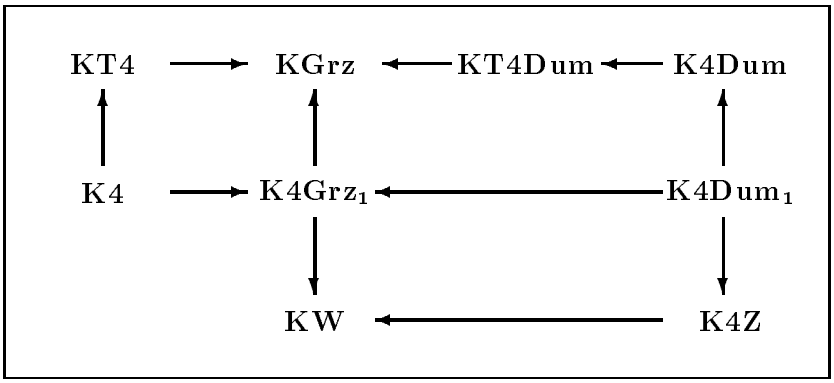
The variants of *Grz* are even more numerous. Again, we explore the relationship between *Grz* and each of its variants in turn. Once again, we can deduce that in the presence of *T* and *4*, these variants are all "deductively equivalent" (as axiom schema and as formulae) in the sense described above.

Note also that **KGrz** has each of *T*, *4*, *M* and *Dum* as theorems, and that **KT4MDum** has *Grz* as a theorem. Thus we have that $\mathbf{KT4MDum} = \mathbf{KGrz}$.

$$\mathbf{KT4Dum} = \mathbf{KTDum_1} = \mathbf{KTDum_2} = \mathbf{KTDum_3} = \mathbf{KTDum_4}$$

$$\mathbf{KT4Grz} = \mathbf{KTGrz_1} = \mathbf{KTGrz_2} = \mathbf{KTGrz_3} = \mathbf{KTGrz_4} = \mathbf{KTGrz_5}$$

$$\mathbf{KGrz} = \mathbf{KT4MDum}$$



$\mathbf{KA_1} \ldots \mathbf{A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
$4, Dum$	Dum_1	$4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, Dum, $Dum\{p := p \to \Box p\}$
T, Dum_1	Dum	T, Dum_1
T, Dum	Dum_2	$\Box T$, Dum
T, Dum_2	Dum	$T\{p := Dum_2\}$, $\Box T\{p := p \to \Box p\}$, $\Box Dum_2$
$T, 4, Dum$	Dum_3	$\Box T$, $4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, Dum, $Dum\{p := p \to \Box p\}$
T, Dum_3	Dum	T, $T\{p := Dum_3\}$, $\Box T\{p := p \to \Box p\}$, $\Box Dum_3$
Dum	Dum_4	Dum
T, Dum_4	Dum	T, Dum_4
Grz	Grz_1	$Grz\{p := (\Box(p \to \Box p) \to p) \wedge 4\{p := \Box(p \to \Box p) \to p\}\}$, $\Box Grz$
T, Grz_1	Grz	T, $T\{p := Grz_1\}$, $\Box Grz_1$
Grz	Grz_2	Grz, $\Box Grz$
T, Grz_2	Grz	$\Box T\{p := p \to \Box p\}$, $T\{p := \neg \Box (\Box(p \to \Box p) \to p)\}$, Grz_2
Grz	Grz_3	$Grz\{p := (\Box(p \to \Box p) \to \Box p) \wedge Grz$ $\quad \wedge 4\{p := (\Box(p \to \Box p) \to \Box p) \wedge Grz\}\}$, $\Box Grz$
T, Grz_3	Grz	T, $\Box T\{p := p \to \Box p\}$, Grz_3
Grz	Grz_4	Grz, $Grz\{p := (\Box(p \to \Box p) \to p) \wedge 4\{p := \Box(p \to \Box p) \to p\}\}$
T, Grz_4	Grz	T, $T\{p := Grz_4\}$, $\Box Grz_4$
Grz	Grz_5	$Grz\{p := (\Box(p \to \Box q) \to \Box q) \wedge 4\{p := \Box(p \to \Box q) \to \Box q\}\}$, $Grz\{p := (\Box((p \to \Box q) \to \Box(p \to \Box q)) \to (p \to \Box q))$ $\quad \wedge 4\{p := \Box((p \to \Box q) \to \Box(p \to \Box q)) \to (p \to \Box q)\}\}$, $\Box Grz\{p := p \to \Box q\}$
$T, 4, Grz_5$	Grz	$T\{p := \Box(p \to \Box p) \to p\}$, $\Box \Box$ $T\{p := p \to \Box p\}$, $4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, $Grz_5\{q := p \to \Box p\}$
Grz	T	Grz
Grz	4	$Grz\{p := p \wedge 4\}$
Grz	M	$Grz\{p := \Diamond \neg p\}$, $Grz\{p := \Diamond \neg p \wedge 4\{p := \Diamond \neg p\}\}$, $Grz\{p := \neg \Box (p \to \Box p) \wedge 4\{p := \neg \Box (p \to \Box p)\}\}$, $\Box Grz$
Grz	Dum	Grz
Grz_1	Dum_1	Grz_1
T, M, Dum	Grz	$\Box T\{p := \neg p\}$, M, Dum

4.6 Miscellaneous relations

The following table contains some miscellaneous relationships between some of the formulae that have been used predominantly as extensions of **KT4**.

$\mathbf{KT4L^+Zem = KT4R}$
$\mathbf{KT4MR = KT4P}$

$\mathbf{KA_1 \ldots A_k} \vdash B$		
$A_1, \ldots, A_k$	B	modalised instances of $A_1, \ldots, A_k$
R	L^+	$R\{p := \Box p \to q\}$
T, R	Zem	$T\{p := \Diamond \Box p\}$, R
$T, 4, L^+, Zem$	R	$\Box\Box\, T\{p := 4\}$, $\Box\Box\, T\{p := \neg(\Box \Diamond \Box p \wedge \Box p)\}$, $\Box 4$, $L^+\{p := \Diamond \Box p, q := \neg \Box p\}$, Zem
$T, 4, P$	M	$T\{p := \Diamond p\}$, $\Box T\{p := \neg \Box \Diamond p\}$, $4\{p := \Diamond p\}$, $\Box P$
T, P	R	$\Box\Box\, T\{p := \neg p\}$, P
$4, M, R$	P	$4\{p := \neg \Box p\}$, $\Box M$, R
T, P	H_s	$\Box\Box\Box T$, P, $\Box P\{p := \neg p\}$
$T, 4, R$	Dum	$T\{p := \Box(p \to \Box p) \to p\}$, $\Box 4\{p := p \wedge (\Box(p \to \Box p) \to p)\}$, $R\{p := p \to \Box p\}$

5 Checking the results and obtaining purely syntactic proofs

Since we give the instances required for the proofs, it is easy to check all the results using a theorem prover for **K**.

One possibility is the LWB. You can use it via the World Wide Web (WWW). Type the address `http://lwbwww.unibe.ch:8080/LWBinfo.html` in your WWW browser, choose the item `run a session` and type in your request.

It is also possible to check the results by hand (although it is a bit tiring in some cases). One can for example make a backward search in the following sequent calculus where Γ, Δ, Σ, and Π stand for (perhaps empty) multisets of formulas. (See e.g. [Fit83], [Gor92] or [Wal90] for more information on such calculi.)

axioms:

$$P, \Gamma \Rightarrow P, \Delta \qquad \Gamma \Rightarrow \top, \Delta \qquad \bot, \Gamma \Rightarrow \Delta$$

rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\mathrm{l}\wedge) \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ (\mathrm{r}\wedge)$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\mathrm{l}\vee) \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ (\mathrm{r}\vee)$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\mathrm{l}\to) \qquad \frac{A, \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ (\mathrm{r}\to)$$

$$\frac{\Gamma \Rightarrow A, \Delta}{\neg A, \Gamma \Rightarrow \Delta}\ (\mathrm{l}\neg) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (\mathrm{r}\neg)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Diamond A, \Box\Gamma, \Sigma \Rightarrow \Diamond\Delta, \Pi}\ (\mathrm{l}\Diamond) \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Box\Gamma, \Sigma \Rightarrow \Box A, \Diamond\Delta, \Pi}\ (\mathrm{r}\Box)$$

For example, the claim that $\mathbf{K} \vdash W\{p := p \wedge \Box p\} \to 4$ can be checked using the above calculus for **K** as shown below, where we use C as an abbreviation for $p \wedge \Box p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, \Box p, p \Rightarrow \Box p}\ (\mathrm{r}\Box)}{C, p \Rightarrow \Box p}\ (\mathrm{l}\wedge)}{\Box C, \Box p \Rightarrow \Box\Box p}\ (\mathrm{r}\Box) \qquad \dfrac{\dfrac{\dfrac{\dfrac{p, \Box p \Rightarrow p}{C \Rightarrow p}\ (\mathrm{l}\wedge)}{p, \Box C \Rightarrow \Box p}\ (\mathrm{r}\Box)}{p, \Box C \Rightarrow C}\ (\mathrm{r}\wedge)}{p \Rightarrow \Box C \to C}\ (\mathrm{r}\to)}{\Box p \Rightarrow \Box(\Box C \to C), \Box\Box p}\ (\mathrm{r}\Box)}{\Box(\Box C \to C) \to \Box C, \Box p \Rightarrow \Box\Box p}\ (\mathrm{l}\to)$$

$$\dfrac{\dfrac{\Box(\Box C \to C) \to \Box C \Rightarrow \Box p \to \Box\Box p}{\Rightarrow (\Box(\Box C \to C) \to \Box C) \to \Box p \to \Box\Box p}\ (\mathrm{r}\to)}{}\ (\mathrm{r}\to)$$

It is easy (but tedious) to convert such a Gentzen proof into a Hilbert-style derivation. First we replace each sequent $C_1, \ldots, C_m \Rightarrow D_1, \ldots, D_n$ that occurs in the Gentzen proof by the formula $\neg C_1 \vee \ldots \vee \neg C_m \vee D_1 \vee \ldots \vee D_n$. Every axiom of the Gentzen proof thus becomes an axiom of a Hilbert-style derivation. It remains to show that it is possible to convert each Gentzen rule application into a sequence of steps in a Hilbert-style derivation.

An application of a unary classical rule is converted using an appropriate classical tautology and then one application of modus ponens, while an application of a binary classical rule requires two tautologies and two applications of modus ponens. For example

$$\frac{C \Rightarrow A, D \quad C \Rightarrow B, D}{C \Rightarrow A \wedge B, D}\,(\mathrm{r}\wedge)$$

is converted into

$$\dfrac{\dfrac{\neg C \vee A \vee D \to \neg C \vee B \vee D \to \neg C \vee A \wedge B \vee D \quad \neg C \vee A \vee D}{\neg C \vee B \vee D \to \neg C \vee A \wedge B \vee D} \qquad \neg C \vee B \vee D}{\neg C \vee A \wedge B \vee D}$$

The conversion of a (r$\Box$) or (l$\Diamond$) application is slightly more difficult. We show it only for a special case; the general case follows immediately.

$$\frac{\Rightarrow A, D}{\Rightarrow \Box A, \Diamond D}\,(\mathrm{r}\Box)$$

is converted into

$$\dfrac{\dfrac{A \vee D \quad A \vee D \to \neg D \to A}{\neg D \to A}}{\dfrac{\Box(\neg D \to A) \qquad \Box(\neg D \to A) \to \Box\neg D \to \Box A}{\Box\neg D \to \Box A}}$$

With the classical tautology $\Box\neg D \to \Box A \to \Box A \vee \neg\,\Box\,\neg D$ and an application of modus ponens we can conclude $\Box A \vee \Diamond D$.

The following "proof" of $W\{p := p \wedge \Box p\} \to \not\downarrow$ was produced by the LWB. The symbol $\Rightarrow$ separates the right and left halves of a sequent, and each half of a sequent is divided into four lists (enclosed in `[ ]`) classifying the formulae as "literals", "main connective classical", "main connective a $\Box$" and "main connective a $\Diamond$":

```
>  load(k);

k> set("infolevel", 5);

k> c := p & box p;

k> provable( (box(box c -> c) -> box c) -> box p -> box box p );
```

```
K |- A for given expression A?
(r ->)
(r ->)
1.[],[box(box(p & box p) -> p & box p) -> box(p & box p)],[box p],[]
  =>[],[],[box box p],[]
  (l ->)
1.1.[],[],[box p],[]
    =>[],[],[box(box(p & box p) -> p & box p), box box p],[]
    (r box) with  box(box(p & box p) -> p & box p)
    (r ->)
1.1.1.[p],[],[box(p & box p)],[]=>[],[p & box p],[],[]
      (r &)
1.1.1.1.[p],[],[box(p & box p)],[]=>[p],[],[],[]
        (axiom).

1.1.1.2.[p],[],[box(p & box p)],[]=>[],[],[box p],[]
        (r box) with  box p
        (l &)
1.1.1.2.1.[p],[],[box p],[]=>,[p],[],[],[]
          (axiom).

1.2.[],[],[box(p & box p), box p],[]=>[],[],[box box p],[]
    (r box) with  box box p
    (l &)
1.2.1.[p],[],[box p],[]=>[],[],[box p],[]
      (axiom).

true
```

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Appendix

Formulas

```
> load(k);
```

```
> D := box p -> dia p;
  D2 := dia true;
  T := box p -> p;
  A4 := box p -> box box p;
  A4m := box p & dia q -> dia(box p & q);
  A5 := dia p -> box dia p;
  A5m := dia p & dia q -> dia(dia p & q);
  B := p -> box dia p;
  Bm := p & dia q -> dia(dia p & q);
```

```
> G := dia box p -> box dia p;
  G0 := dia(p & box q) -> box(p v dia q);
  H := box(p v q) & box(box p v q) & box(p v box q) -> box p v box q;
  Hp := box(box p v q) & box(p v box q) -> box p v box q;
  L := box(p & box p -> q) v box(q & box q -> p);
  Lp := box(box p -> q) v box(box q -> p);
  Lpp := box(box p -> box q) v box(box q -> box p);
```

```
> M := box dia p -> dia box p;
  Pt := box(p v dia p) -> dia(p & box p);
```

```
> W := box(box p -> p) -> box p;
  W0 := box dia true -> box false;
  Z := box(box p -> p) -> (dia box p -> box p);
```

```
> Dum := box(box(p -> box p) -> p) -> (dia box p -> p);
  Dum1 := box(box(p -> box p) -> p) -> (dia box p -> box p);
  Grz := box(box(p -> box p) -> p) -> p;
  Grz1 := box(box(p -> box p) -> p) -> box p;
```

```
> F := (dia box p -> q) v box(box q -> p);
  Hs := p -> box(dia p -> p);
  P := dia box dia p -> (p -> box p);
  R := dia box p -> (p -> box p);
  X := box box p -> box p;
```

```
  Zem := box dia box p -> (p -> box p);

> M2 := dia box(p -> box p);
  M3 := box dia p & box dia q -> dia(p & q);

> Dum2 := box(box(p -> box p) -> box p) -> (dia box p -> p);
  Dum3 := box(box(p -> box p) -> box p) -> (dia box p -> box p);
  Dum4 := box(box(p -> box p) -> p)     -> (dia box p -> p v box p);

> Grz2 := box(box(p -> box p) -> box p) -> p;
  Grz3 := box(box(p -> box p) -> box p) -> box p;
  Grz4 := box(box(p -> box p) -> p)     -> p v box p;
  Grz5 := box(box(p -> box q) -> box q) & box(box(~p -> box q) -> box q)
          -> box q;
```

Relations concerning D, T, 4, 5, B

```
> k::provable( D  ->  D2 );
  true
> k::provable( D2  ->  D );
  true
> k::provable( A4  ->  A4m );
  true
> k::provable( A4m{~ box p/q}  ->  A4 );
  true
> k::provable( A5  ->  A5m );
  true
> k::provable( A5m{~ dia p/q}  ->  A5 );
  true
> k::provable( B  ->  Bm );
  true
> k::provable( Bm{~ dia p/q}  ->  B );
  true

> k::provable( T{false/p}  ->  D );
  true

> k::provable( T{dia ~p/p} & A5{box p/p} & box A5{~p/p}  ->  A4 );
  true
> k::provable( T{~p/p} & A5  ->  B );
```

```
  true
> k::provable( box A4{~p/p} & B{dia p/p}  ->  A5 );
  true
> k::provable( box A5{~p/p} & B{box p/p}  ->  A4 );
  true
> k::provable( D & A4 & B{~p/p}  ->  T );
  true
```

Relations concerning G, G_0, H, H^+ L, L^+, L^{++}

```
> k::provable( G{q/p} ->  G0 );
  true
> k::provable( box D & G0{box p/p, p/q}  ->  G );
  true

> k::provable( L  ->  H );
  true
> k::provable( H{p & box p -> q/p, q & box q -> p/q}  ->  L );
  true
> k::provable( Lp  ->  Hp );
  true
> k::provable( Hp{box p -> q/p, box q -> p/q}  ->  Lp );
  true

> k::provable( Lp  ->  L );
  true
> k::provable( box T & box T{q/p}  & L  ->  Lp );
  true
> k::provable( box T & box T{q/p} & Lpp  ->  Lp );
  true
> k::provable(   A4{p & box p -> q/p} & A4{q & box q -> p/p}
              & box A4 & box A4{q/p}
              & L  ->  Lpp );
  true
> k::provable( A5{box p/p} & box A5{~p/p}  ->  Lpp );
  true

> k::provable( box D{~p/p} & Lpp{~p/q}   ->  G );
  true
```

```
> k::provable( A5 & A5{~p/p}  ->  G );
  true
```

Relations concerning M, Pt

```
> k::provable( T{M/p} & box A4 & box M  ->  M2 );
  true
> k::provable( T{M2/p} & A4{dia ~p/p} & M2  ->  M );
  true
> k::provable( A4{dia q & (~p v ~q)/p} & M  ->  M3 );
  true
> k::provable( M3{~p/q}  ->  M );
  true

> k::provable( M  ->  D );
  true

> k::provable(       A4{     (p v dia p) & (~p v dia ~p)/p}
              & box A4{ p & (p v dia p) & (~p v dia ~p)/p}
              & box A4{~p & (p v dia p) & (~p v dia ~p)/p}
              & M
              ->  Pt );
  true
> k::provable( Pt  ->  M );
  true
```

Relations concerning W, W_0, Z

```
> k::provable( W{p & box p/p}  ->  A4 );
  true
> k::provable( W{false/p}  ->  W0 );
  true
> k::provable( W  ->  Z );
  true
> k::provable( W0 & Z  ->  W );
  true
> k::provable( W{p & box(box p -> p)/p} & box box W  ->  Grz1 );
  true
> k::provable( box A4 & Z  ->  Dum1 );
```

```
  true
```

Relations concerning *Dum*, *Grz*

```
> k::provable(    A4{box(p -> box p) -> p/p} & box A4
                & Dum & Dum{p -> box p/p}
                ->  Dum1 );
  true
> k::provable( T & Dum1  ->  Dum );
  true
> k::provable( box T & Dum  ->  Dum2 );
  true
> k::provable(   T{Dum2/p} & box T{p -> box p/p}
               & box Dum2
               ->  Dum );
  true
> k::provable(    box T
                & A4{box(p -> box p) -> p/p} & box A4
                & Dum & Dum{p -> box p/p}
                ->  Dum3 );
  true
> k::provable(   T & T{Dum3/p} & box T{p -> box p/p}
               & box Dum3
               ->  Dum);
  true
> k::provable( Dum  ->  Dum4 );
  true
> k::provable( T & Dum4  ->  Dum );
  true

> k::provable( Grz{(box(p -> box p) -> p) & A4{box(p -> box p) -> p/p}/p}
               & box Grz
               ->  Grz1 );
  true
> k::provable( T & T{Grz1/p} & box Grz1  ->  Grz );
  true
> k::provable( Grz & box Grz  ->  Grz2 );
  true
```

```
> k::provable(   box T{p -> box p/p}
              & T{~ box(box(p -> box p) -> p)/p}
              & Grz2
              ->  Grz );
  true
> k::provable(   Grz{(box(p -> box p) -> box p) & Grz
                     & A4{(box(p -> box p) -> box p) & Grz/p}/p}
              & box Grz
              ->  Grz3 );
  true
> k::provable( T & box T{p -> box p/p} & Grz3  ->  Grz );
  true
> k::provable(
       Grz
     & Grz{(box(p -> box p) -> p) & A4{box(p -> box p) -> p/p}/p}
     ->  Grz4 );
  true
> k::provable( T & T{Grz4/p} & box Grz4  ->  Grz );
  true
> k::provable(
       Grz{(box(p -> box q) -> box q)
           & A4{box(p -> box q) -> box q/p}/p}
     & Grz{(box((p -> box q) -> box(p -> box q)) -> (p -> box q))
           & A4{box((p -> box q) -> box(p -> box q)) -> (p -> box q)/p}/p}
     & box Grz{ p -> box q/p}
     ->  Grz5 );
  true
> k::provable(   T{box(p -> box p) -> p/p}
              & box box T{p -> box p/p}
              & A4{box(p -> box p) -> p/p} & box A4
              & Grz5{p -> box p/q}  ->  Grz );
  true

> k::provable( Grz  ->  T );
  true
> k::provable( Grz{p & A4/p}  ->  A4 );
  true
> k::provable(   Grz{dia ~p/p}
```

```
              & Grz{dia ~p & A4{dia ~p/p}/p}
              & Grz{~ box(p -> box p) & A4{~ box(p -> box p)/p}/p}
              & box Grz
              ->  M );
  true
> k::provable( Grz  ->  Dum );
  true
> k::provable( Grz1  ->  Dum1 );
  true
> k::provable( box T{~p/p} & M & Dum  ->  Grz );
  true
```

Miscellaneous relations

```
> k::provable( R{box p -> q/p}  ->  Lp );
  true
> k::provable( T{dia box p/p} & R  ->  Zem );
  true
> k::provable(   box box T{A4/p}
              & box box T{~(box dia box p & box p)/p}
              & box A4
              & Lp{dia box p/p, ~ box p/q} & Zem
              ->  R );
  true

> k::provable(   T{dia p/p} & box T{~ box dia p/p}
              & A4{dia p/p}
              & box P  ->  M );
  true
> k::provable( box box T{~p/p} & P  ->  R );
  true
> k::provable( A4{~ box p/p} & box M & R  ->  P );
  true

> k::provable( box box box T & P & box P{~p/p}  ->  Hs );
  true
> k::provable(   T{box(p -> box p) -> p/p}
              & box A4{p & (box(p -> box p) -> p)/p}
              & R{p -> box p/p}
              ->  Dum );
  true
```