Explicit mathematics: W-type, models

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1 Introduction

In this thesis we are going to define an analogue of Martin-Löf's W-type for explicit mathematics and show that in the framework of explicit mathematics with elementary comprehension and join the W-type is equivalent to the principle of inductive generation. Furthermore we describe a new model construction for $\mathsf{T_0}$ by a non-monotone inductive definition which we think is very natural. The construction will be formalized in Jäger's theory $\mathsf{FID}(\mathbb{I})$. We will show how in this model restricted inductions can be treated in an adequate way. Then we extend our model construction to systems of explicit mathematics with universes.

Systems of explicit mathematics were first presented by Feferman in the seventies. In [3] and [4] he introduced the theory $\mathsf{T_0}$. His aim was to give a formal framework to adequately treat constructive mathematics in a Bishop style way. These systems deal with functions and classes. But in the context of explicit mathematics one uses the notions of operations and types or classifications. This is to express the fact that a function is given by rules for mechanical computing and that classifications are successively defined or generated from preceding ones. From the beginning, systems of explicit mathematics turned out to be of a high proof-theoretic interest. Mainly for the analysis of subsystems of first and second order arithmetic and for systems of Kripke-Platek set theory. We will use here a axiomatization of $\mathsf{T_0}$ presented by Feferman and Jäger in [7], where the scheme for elementary comprehension is replaced by finitely many axioms. This version of $\mathsf{T_0}$ has some advantages in carrying out our model construction.

The W-type is well known in the context of Martin-Löf's type theory (c.f. [11]). He introduced the W-Type to have an intuitionistic analogue to ordinals. So elements of a W-type can be thought as well-founded trees. Feferman shows in [3] how to treat tree ordinals in $\mathsf{T_0}$ by means of inductive generation. But the relationship between the W-type and the principle of inductive generation has not been treated. To bring more light in the situation we are going to define the theory $\mathsf{T_W}$ as the theory $\mathsf{T_0}$ without inductive generation but with axioms for the W-type. We will prove that these two theories can be interpreted in each other. Hence in the presence of join the two principles are equivalent. The embedding of $\mathsf{T_0}$ in $\mathsf{T_W}$ can be done without join. To prove of the converse direction we need join, but we do not know if this is necessarily the case. If it is, then the W-type would be stronger than inductive generation.

One of the problems in the context of explicit mathematics is, to generate perspicuous and canonical models in a sufficiently weak theory in order to establish upper proof-theoretic bounds. In [3] and [4] Feferman gives a model construction for $\mathsf{T_0}$ using transfinite recursion. So the types indeed are successively generated from below. But he does not carry out his construction in a formal system; to model inductive generation he just takes the least fixed point. The first formal model construction is done in [2] and [4]. This model uses recursion theoretic methods. Instead of defining the classes in transfinite stages, one defines it simply to be the set of all

Δ^1_2 indices to model $\mathsf{T_0}$, or the set of all Δ^1_1 indices to get a model of $\mathsf{T_0}$ without inductive generation. One then gets the following proof-theoretic reductions: $\mathsf{T_0} \leq (\Sigma^1_2\text{-}\mathsf{AC}) + (\mathsf{BI})$ and $\mathsf{EET} + (\mathsf{F\text{-}I_N}) + (\mathsf{J}) \leq (\Sigma^1_1\text{-}\mathsf{AC})$. In both cases the use of the Addison-Kleene-Shoenfield substitution theorem is very important. One might say that these models don't adequately represent the idea of generating the types from previous ones.

Our aim is to give a formal model construction in terms of inductive definitions in order to represent the principle of successively building the types. For $\mathsf{EET} + (\mathsf{F\text{-}I_N}) + (\mathsf{J})$ such a construction can be performed in the theory $\widehat{\mathsf{ID}}_1$ of positive inductive definitions, where only the fixed point property is required (instead of *least* fixed point). For example this is done in Marzetta [12]. $\mathsf{T_0}$ is too strong to be modeled by a monotone inductive definition. Hence we have to make use of a theory of non-monotone inductive definition.

The first examples of non-monotone inductive definitions were those that appeared in the construction of systems of notations for large segments of the countable ordinals. We will consider only a special form of non-monotone operators invented by Richter [14]. Let Φ, Ψ be inclusive operators on ω - then we are interested in the operator $[\Phi, \Psi]$ defined by:

$$[\Phi,\Psi](X) := \begin{cases} \Phi(X) & \text{if } \Phi(X) \not\subseteq X \\ \Psi(X) & \text{if } \Phi(X) \subseteq X \end{cases} \quad \text{for } X \subseteq \omega.$$

There are several well known theories treating positive inductive definitions. For example the famous ID_1 (c.f. Buchholz et al. [2]) and the theory PA_Ω introduced by Jäger in [9]. For non-monotone inductive definitions there are no widely known theories. To handle non-monotone operators of the form $[\Phi, \Psi]$, where $[\Phi, \Psi]$ is an operator of a class $\mathbb{K}$, we can use Jägers theories $\mathsf{FID}(\mathbb{K})$, where FID stands for *first order inductive definitions* and $\mathbb{K}$ denotes a class of operator forms. For every such operator form $\mathcal{A} \in \mathbb{K}$ we have a relation symbol $P_\mathcal{A}$ for which $\mathsf{FID}(\mathbb{K})$ proves $\forall \vec{x}(\mathcal{A}(P_\mathcal{A}, \vec{x}) \to P_\mathcal{A}(\vec{x}))$. This is for each operator form $\mathcal{A}$ of $\mathbb{K}$ there exists a relation $P_\mathcal{A}$ that is closed under $\mathcal{A}$. Hence the proof-theoretic strength of $\mathsf{FID}(\mathbb{K})$ mainly depends on the complexity of the operator forms included in $\mathbb{K}$. If we add more complex operator forms to $\mathbb{K}$, then the strength of $\mathsf{FID}(\mathbb{K})$ increases.

The theories $\mathsf{FID}(\mathbb{K})$ are constructed along the lines of PA_Ω. $\mathsf{FID}(\mathbb{K})$ is formulated over a two-sorted language with number variables and variables for ordinals. The axioms include the axioms for Peano arithmetic and axioms stating that there exists a fixed point which is generated from below. We have induction on natural numbers and on ordinals. This is very important since in our model construction the inductions in $\mathsf{T_0}$ correspond directly to the induction principles in $\mathsf{FID}(\mathbb{K})$. Hence to obtain a model of $\mathsf{T_0}$ with restricted induction, the only thing to do is to restrict the corresponding induction-principle in $\mathsf{FID}(\mathbb{K})$. To carry out our model construction we use a $[\mathsf{POS}, \mathsf{QF}]$-operator form. The corresponding fixed point theory will be

$\mathsf{FID}(\mathbb{I})$. As mentioned above we will have two more theories with restricted inductions $\mathsf{FID}^{\mathsf{w}}(\mathbb{I})$ and $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ to model $\mathsf{EET} + (\mathsf{F}\text{-}\mathsf{I}_{\mathsf{N}}) + (\mathsf{J}) + (\mathsf{IG}\upharpoonright)$ and $\mathsf{T_0}\upharpoonright$, respectively.

Universes are a natural concept in mathematics. Martin-Löf introduced them to his theory of types (c.f. [11]) and in explicit mathematics they have been studied by Feferman [5] and Marzetta [12]. In constructive mathematics as well as in explicit mathematics types were built by type forming operations (e.g. elementary comprehension and join). Then one can consider the universe U of all these "small" types and regard this collection itself as a "large" type. With this large type we can construct by the type forming operations new types and collect them in a second universe which has to be regarded as an even "larger" type. This procedure can be iterated.

In the context of explicit mathematics universes have been proof-theoretically analyzed by Feferman [5] and Marzetta [12]. In both papers the analysis is done for systems of explicit mathematics with universes but without inductive generation. In this thesis we show how to model in our fixed point theories systems of explicit mathematics with universes and inductive generation. We extend our language by a new unary predicate $U(B)$ on types, which is true if B is a universe. Universes are types made up of names of types, closed under elementary comprehension and join. They are linearly ordered by inclusion and have to exist by a uniform limes axiom which states that every type is contained in a universe. We call this theory $\mathsf{T_0}$+U. We will carry out a model construction for this theory in $\mathsf{FID}(\mathbb{I})$.

I am grateful to Prof. Gerhard Jäger for introducing me to explicit mathematics and for guiding me during my work. I also would like to thank Dr. Thomas Strahm for the inspiring discussions on the thesis and all members of the research group for the friendly atmosphere.

2 The theory $\mathsf{T_0}$

The language $\mathcal{L}_p$ is two-sorted with *individual variables* $a, b, c, f, g, h, x, y, z, \ldots$ and *type variables* $A, B, C, X, Y, Z, \ldots$. Additionally $\mathcal{L}_p$ includes the following *individual constants*: k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing), 0 (zero), s_N (successor), p_N (predecessor), d_N (definition by cases on natural numbers), r_N (primitive recursion), **nat** (natural numbers), **id** (identity), **co** (complements), **int** (intersections), **dom** (domains), **inv** (inverse images), **j** (join), **i** (inductive generation). $\mathcal{L}_p$ has the binary function symbol $\cdot$ for (partial) term application. In $\mathcal{L}_p$ we have unary relation symbols $\downarrow$ (defined) and N (natural number) as well as binary relation symbols $=, \in$ and $\Re$ (naming).
The *individual terms* $(r, s, t, \ldots)$ of $\mathcal{L}_p$ are inductively defined as follows:

1. Every individual variable and constant is a individual term.

2. If s, t are individual terms, then $(s \cdot t)$ is an individual term.

We will drop $\cdot$ and only write (st) or st instead of $(s \cdot t)$ and we usually omit parenthesis with the implicit assumption that $\cdot$ associates to the left, i.e. rst stands for $((r \cdot s) \cdot t)$. We use (t_0, t_1) for $\mathsf{p} t_0 t_1$ and $(t_0, \ldots, t_n)$ for $(t_0, (t_1, \ldots, t_n)$.
The *formulas* $(F, G, H, \ldots)$ of $\mathcal{L}_p$ are inductively defined as follows:

1. $N(t)$, $t\downarrow$, $(s = t)$, $(s \in A)$, $(A = B)$ and $\Re(a, A)$ are (atomic) formulas.
2. If F and G are formulas, then $\neg F$, $(F \vee G)$, $(F \wedge G)$ are formulas, too.
3. If F is a formula, then $\forall x F$, $\exists x F$, $\forall X F$ and $\exists X F$ are formulas, too.

As usual we write $(F \to G)$ for $(\neg F \vee G)$ and $(F \leftrightarrow G)$ for $((F \to G) \wedge (G \to F))$. A formula F of $\mathcal{L}_p$ is called *elementary*, if the relation symbol $\Re$ does not occur in F and F does not contain bound type variables. We will use the following abbreviations:

$$\begin{aligned}
t \in N &:\equiv N(t), \\
(\exists x \in A)F &:\equiv \exists x(x \in A \wedge F), \\
(\forall x \in A)F &:\equiv \forall x(x \in A \to F), \\
(t : A \to A) &:\equiv (\forall x \in A)(tx \in A), \\
(t : A^{m+1} \to A) &:\equiv (\forall x \in N)(tx : A^m \to A).
\end{aligned}$$

The relation of *partial equality* is defined by:

$$s \simeq t :\equiv (s\downarrow \vee t\downarrow) \to (s = t).$$

Now we are ready to state the axioms of the theory $\mathsf{T_0}$. The logic of $\mathsf{T_0}$ is the logic of partial terms, cf [1]. The non-logical axioms consist of a first-order part, which corresponds to the theory BON of Feferman and Jäger [6]:

Partial combinatory algebra

(1) $\mathsf{k}xy = x$

(2) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq xz(yz)$

Pairing and projection

(3) $\mathsf{p}_0(x, y) = x \wedge \mathsf{p}_1(x, y) = y$

Natural Numbers

(4) $0 \in N \wedge (\forall x \in N)(\mathsf{s}_N x \in N)$

(5) $(\forall x \in N)(\mathsf{s}_N x \neq 0 \wedge \mathsf{p}_N(\mathsf{s}_N x) = x)$

(6) $(\forall x \in N)(x \neq 0 \to \mathsf{p}_N(\mathsf{s}_N x) \in N \wedge \mathsf{s}_N(\mathsf{p}_N x) = x)$

Definition by cases on natural numbers

(7) $u \in N \wedge v \in N \wedge u = v \to \mathsf{d}_N xyuv = x$

(8) $u \in N \wedge v \in N \wedge u \neq v \to \mathsf{d}_N xyuv = y$

Primitive recursion on N

(9) $(f : N \to N) \wedge (g : N^3 \to N) \to (\mathsf{r}_N f g : N^2 \to N)$

(10) $(f : N \to N) \wedge (g : N^3 \to N) \wedge x \in N \wedge y \in N \wedge h = \mathsf{r}_N f g \to$
$hx0 = fx \wedge hx(\mathsf{s}_N y) = gxy(hxy)$

With the axioms (1) and (2) we can define lambda abstraction, and we can prove a recursion theorem.

Definition 1 *Let t be a term of $\mathcal{L}_p$. Then $(\lambda x.t[x/a])$ is also a term of $\mathcal{L}_p$ defined inductively as follows (where $t[s/a]$ denotes the result of the simultaneous substitution of all the occurrences of the free variable a in t by the expression s):*

- $(\lambda x.t[x/a]) :\equiv \mathsf{skk}$, *if* $t \equiv a$,
- $(\lambda x.t[x/a]) :\equiv \mathsf{k}t$, *if t is a variable different from a or a constant,*
- $(\lambda x.t[x/a]) :\equiv \mathsf{s}(\lambda x.s_1[x/a])(\lambda x.s_2[x/a])$, *if* $t \equiv (s_1 s_2)$.

Theorem 2 *Let t, s be terms of $\mathcal{L}_p$. Then we can prove with (1) and (2):*

- $(\lambda x.t[x/a]) \downarrow$,
- $(\lambda x.t[x/a])a \simeq t$,
- $s \downarrow \to (\lambda x.t[x/a])s \simeq t[s/a]$.

Theorem 3 *(Recursion theorem) There is a closed term rec of $\mathcal{L}_p$ such that:*

$$\mathsf{rec}f \downarrow \wedge \mathsf{rec}fx \simeq f(\mathsf{rec}f)x.$$

Proofs for these theorems can be found for example in Feferman [3] or Beeson [1].

The relation $\Re$ acts as a naming relation between objects and types, i.e. $\Re(s, A)$ means that s is a name of the type A. While the representation of types by their names is intensional, the types themselves are extensional in the usual set-theoretical sense.

Extensionality

(EXT) $(\forall x)(x \in A \leftrightarrow x \in B) \to A = B$

The axioms about explicit representation state that every type has a name (E.1) and that there are no homonyms (E.2).

Explicit representation

(E.1) $(\exists x)\Re(x, A)$

(E.2) $\Re(a, B) \wedge \Re(a, C) \to B = C$

Definition 4 *Now we can introduce an element relation $\tilde{\in}$ between individuals and one between types $\dot{\in}$, a relation $\doteq$ for extensional equality between names and the unary relation $\Re$ on individuals:*

- $A \mathrel{\dot{\in}} B :\equiv \exists x(\Re(x, A) \wedge x \in B)$
- $a \mathrel{\tilde{\in}} b :\equiv \exists Y(\Re(b, Y) \wedge a \in Y)$
- $a \doteq b :\equiv \exists Y(\Re(a, Y) \wedge \Re(b, Y))$
- $\Re(a) :\equiv \exists A \Re(a, A)$

We could define $A \mathrel{\dot{\in}} B$ as $\forall x(\Re(x, A) \wedge x \in B)$, but this would lead us in serious troubles, since Minari [13] has shown that no universe can contain all the names of a type. Hence the axioms for universes would become inconsistent with this definition of $A \mathrel{\dot{\in}} B$.

To build types one has the principle of elementary comprehension. As mentioned above we use here a finite axiomatization of Feferman and Jäger [7] which is equivalent to the scheme usually used to state elementary comprehension.

Natural numbers

(N.1) $(\exists X)(\forall x)(x \in X \leftrightarrow N(x))$

(N.2) $(\forall x)(x \in A \leftrightarrow N(x)) \rightarrow \Re(\mathsf{nat}, A)$

Identity

(I.1) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)(x = (y, y)))$

(I.2) $(\forall x)(x \in A \leftrightarrow (\exists y)(x = (y, y))) \rightarrow \Re(\mathsf{id}, A)$

Complements

(CO.1) $(\exists X)(\forall x)(x \in X \leftrightarrow x \notin B)$

(CO.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow x \notin B) \rightarrow \Re(\mathsf{co}\ b, A)$

Intersections

(INT.1) $(\exists X)(\forall x)(x \in X \leftrightarrow x \in B \wedge x \in C)$

(INT.2) $\Re(b, B) \wedge \Re(c, C) \wedge (\forall x)(x \in A \leftrightarrow x \in B \wedge x \in C) \rightarrow \Re(\mathsf{int}(b, c), A)$

Domains

(DOM.1) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)((x, y) \in B))$

(DOM.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow (\exists y)((x, y) \in B)) \rightarrow \Re(\mathsf{dom}\ b, A)$

Inverse images

(INV.1) $(\exists X)(\forall x)(x \in X \leftrightarrow fx \in B)$

(INV.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow fx \in B) \rightarrow \Re(\mathsf{inv}(f, b), A)$

The axioms stated so far define the theory EET. To state elementary comprehension one normally extends $\mathcal{L}_p$ by new constants $(\mathsf{c}_e)_{e<\omega}$. Let $F[x,\vec{y},\vec{Z}]$ be an elementary formula of $\mathcal{L}_p$ with Gödelnumber e, then we have

- $\exists X \forall x(x \in X \leftrightarrow F[x,\vec{a},\vec{B}])$,
- $\Re(\vec{b},\vec{B}) \wedge \forall x(x \in A \leftrightarrow F[x,\vec{a},\vec{B}]) \rightarrow \Re(\mathsf{c}_e(\vec{a},\vec{b}), A)$.

This formulation of elementary comprehension is equivalent to our axiomatization. Besides elementary comprehension we have the stronger type building axioms for join and inductive generation.

Join. Let us write $A = \Sigma(B,f)$ for the statement

$\forall x(x \in A \leftrightarrow x = (\mathsf{p}_0 x, \mathsf{p}_1 x) \wedge \mathsf{p}_0 x \in B \wedge \exists X(\Re(f(\mathsf{p}_0 x), X) \wedge \mathsf{p}_1 x \in X))$,

i.e. A is the disjoint sum over all $x \in B$ of the types named by fx. Now the (uniform) axiom of join has the form

(J) $\quad \Re(a,A) \wedge (\forall x \in A)\exists Y \Re(fx, Y) \rightarrow \exists Z(\Re(\mathsf{j}(a,f), Z) \wedge Z = \Sigma(A,f))$.

Inductive generation. We can construct inductively generated types. This is given a type A and a relation R, we can build the R-accessible part of A. By $(\mathsf{IG.1})$ its name will be given uniformly in the names of A and R. $(\mathsf{IG.2})$ is a closure axiom and by $(\mathsf{IG\text{-}IND})$ we know that the inductively generated type is minimal.

$(\mathsf{IG.1})$ $\quad \Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{i}(a,b))$

$(\mathsf{IG.2})$ $\quad \Re(a) \wedge \Re(b) \wedge x \mathrel{\tilde{\in}} a \wedge (\forall y)((y,x) \mathrel{\tilde{\in}} b \rightarrow y \mathrel{\tilde{\in}} \mathsf{i}(a,b)) \rightarrow x \mathrel{\tilde{\in}} \mathsf{i}(a,b)$

We have the following principle of induction on inductively generated types:

$$(\mathsf{IG\text{-}IND}) \quad \Re(a) \wedge \Re(b) \wedge (\forall x \mathrel{\tilde{\in}} a)[\forall y((y,x) \mathrel{\tilde{\in}} b \rightarrow F(y)) \rightarrow F(x)] \\ \rightarrow (\forall x \mathrel{\tilde{\in}} \mathsf{i}(a,b))F(x)$$

where F is an arbitrary formula. By (IG) we denote $(\mathsf{IG.1})$+$(\mathsf{IG.2})$+$(\mathsf{IG\text{-}IND})$. As for natural numbers we also have the weaker form of type-induction on inductively generated types. By $(\mathsf{IG}\upharpoonright)$ we denote $(\mathsf{IG.1})$+$(\mathsf{IG.2})$ and the following axiom of type-induction:

$$\Re(a,A) \wedge \Re(b,B) \wedge (\forall x \in A)[\forall y((y,x) \in B \rightarrow y \in C) \rightarrow x \in C] \\ \rightarrow (\forall x \mathrel{\tilde{\in}} \mathsf{i}(a,b))(x \in C).$$

We are interested in two forms of induction on the natural numbers.

Type induction on N

$(\mathsf{T\text{-}I_N})$ $\quad 0 \in A \wedge (\forall x \in N)(x \in A \rightarrow \mathsf{s}_N x \in A) \rightarrow (\forall x \in N)(x \in A)$

Formula induction on N

$(\mathsf{F\text{-}I_N})$ $\quad F(0) \wedge (\forall x \in N)(F(x) \rightarrow F(\mathsf{s}_N x)) \rightarrow (\forall x \in N)(F(x))$

The theory $\mathsf{T_0}$ consists of EET+(J)+(IG)+$(\mathsf{F\text{-}I_N})$. We also are interested in the theory EET+(J)+$(\mathsf{IG}\upharpoonright)$+$(\mathsf{F\text{-}I_N})$, and by $\mathsf{T_0}\upharpoonright$ we denote the theory EET+(J)+$(\mathsf{IG}\upharpoonright)$+$(\mathsf{T\text{-}I_N})$.

3 The W-type

In [11] Martin-Löf introduces in his theory of types well-founded trees by the W-type. His aim is to have an constructive analogue to the concept of well-ordering and the principle of transfinite induction. In this section we will formulate axioms for well-founded trees in explicit mathematics. We define the new theory $\mathsf{T_W}$ as the theory $\mathsf{T_0}$ without the axioms of inductive generation but with our axioms for well-founded trees. Then it will be shown that $\mathsf{T_0}$ and $\mathsf{T_W}$ are equivalent.

The language $\mathcal{L}_w$ is the language $\mathcal{L}_p$ except that it does not contain the constant i and that it is augmented by new constants w and sup. Then $\mathsf{T_W}$ is formulated in $\mathcal{L}_w$ and consists of the axioms of $\mathsf{T_0}$ without (IG.1), (IG.2) and (IG-IND) but includes the axioms (W.1), (W.2) and (W-IND).

(W.1) $\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \to \Re(\mathsf{w}(a, f))$

(W.2) $\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \wedge b \mathrel{\tilde{\in}} a \wedge (\forall y \mathrel{\tilde{\in}} fb)(gy \mathrel{\tilde{\in}} \mathsf{w}(a, f)) \to \mathsf{sup}(b, g) \mathrel{\tilde{\in}} \mathsf{w}(a, f)$

(W-IND) $\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \wedge (\forall b, g)[b \mathrel{\tilde{\in}} a \wedge (\forall y \mathrel{\tilde{\in}} fb)F(gy) \to F(\mathsf{sup}(b, g))]$
$\to (\forall x \mathrel{\tilde{\in}} \mathsf{w}(a, f))F(x)$

The axiom corresponding to the W-elimination rule is (W-IND). This means we have full induction instead of a recursion operator on the W-type. Of course we can build by the recursion theorem a simple recursion operator on a W-type. But it will not be equivalent to the one of the elimination rule.

To get a intuitive picture of the meaning of the W-axioms one may interpret the occurring terms in the following way:

- The type A with name a: the collection of indices of "branching types".
- $b \in A$: a branching-index.
- fb: denotes the index-set of the immediate predecessors of $\mathsf{sup}(b, g)$.
- gq with $q \mathrel{\tilde{\in}} fb$ denotes a predecessor of $\mathsf{sup}(b, g)$.
- $\mathsf{sup}(b, g)$: a well-founded tree, which can be thought as the disjoint union of the subtrees gq with $q \mathrel{\tilde{\in}} fb$.
- $\mathsf{w}(a, f)$: the collection of all the well-founded trees $\mathsf{sup}(b, g)$.

To motivate that our formulation of the W-type in explicit mathematics is an adequate translation of Martin-Löf's W-type, we will treat here two examples of [11] in $\mathsf{T_W}$.

Example 5 *The first number class: Choose* $A := \{0, 1\}, \Re(a, A), B := \{\}, C{:=}\{t\}$, f *such that* $\Re(f0, B), \Re(f1, C)$. *Then start with* $\mathsf{w}(a, f)$ *being empty. Hence for all* g*:* $\mathsf{sup}(0, g) \mathrel{\tilde{\in}} \mathsf{w}(a, f)$, *since* $\Re(f0, B)$ *and* B *is empty. Then* $\mathsf{sup}(1, g) \mathrel{\tilde{\in}} \mathsf{w}(a, f)$

for those g with $(\exists h)(gt = \mathsf{sup}(0,h))$. In the next step we include those $\mathsf{sup}(1,g)$ to $\mathsf{w}(a,f)$ satisfying $(\exists h)(gt = \mathsf{sup}(1,h))$ and $\mathsf{sup}(1,h))$ was included to $\mathsf{w}(a,f)$ in the step before. This can be iterated until closure is reached. Then the procedure yields the first number class as in the theory of types ([11], p.82).

Example 6 *The second number class: We choose: $A := \{0,1,2\}, \Re(a,A), B{:=}\{\}$, $C{:=}\{t\}$, f such that $\Re(f0,B), \Re(f1,C), f2 \doteq \mathsf{nat}$. Then $\mathsf{w}(a,f)$ will name the second number class O_1. We define the successor operator to be $\lambda x.\mathsf{sup}(1,\lambda y.x)$ and the N-supremum sup_N as $\lambda x.\mathsf{sup}(2,x)$.*

Now we are ready to show that $\mathsf{T_0}$ and $\mathsf{T_W}$ can be interpreted in each other.

$\mathsf{T_W}$ in $\mathsf{T_0}$

To model $\mathsf{T_W}$ in $\mathsf{T_0}$ we have to build the types for well-founded trees using inductively generated types. Starting with a,f we are able to define a type P that will serve as the set of possible nodes of the tree, and we can define a type R of ordered pairs that will model the predecessor relation of the tree. With these two types we can define by inductive generation a new type that will satisfy the axioms for well-founded trees. The constant sup is interpreted as p.

Suppose we are given a,f with $\Re(a,A)$ and $(\forall x \mathrel{\tilde{\in}} a)\Re(fx)$ to model $\mathsf{w}(a,f)$. Then there is a type J with $\Re(\mathsf{j}(a,f),J)$. We obtain

$$x \in J \leftrightarrow (\exists u,v)(x = (u,v) \wedge u \in A \wedge v \mathrel{\tilde{\in}} fu).$$

We define the following two types:

- $P := \{\mathsf{sup}(b,g) | b \in A \wedge (\forall y \mathrel{\tilde{\in}} fb) gy\downarrow\}$
- $R := \{(y,x) | (\exists u,v)(x = \mathsf{sup}(u,v) \wedge \exists q((u,q) \in J \wedge vq = y)\}$

Let $\Re(p,P)$ and $\Re(r,R)$. We note that P and J depend uniformly on a,f, and R depends uniformly on J. Hence there is a term w such that $\mathsf{w}(a,f) = \mathsf{i}(p,r)$. We show that this term satisfies the axioms for well-founded trees.

Theorem 7 $\mathsf{T_0}$ *proves* (W.1), (W.2) *and* (W-IND).

PROOF First we show (W.1). That is $\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \rightarrow \Re(\mathsf{i}(p,r)$. This holds by the definitions of P and R.
Now we show that (W.2) holds. We have to check

$$\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \wedge b \mathrel{\tilde{\in}} a \wedge (\forall y \mathrel{\tilde{\in}} fb)(gy \mathrel{\tilde{\in}} \mathsf{i}(p,r)) \rightarrow \mathsf{sup}(b,g) \mathrel{\tilde{\in}} \mathsf{i}(p,r)$$

We have $b \mathrel{\tilde{\in}} a$ and $(\forall y \mathrel{\tilde{\in}} fb)(gy \mathrel{\tilde{\in}} \mathsf{i}(p,r))$, hence $b \in A$ and $(\forall y \mathrel{\tilde{\in}} fb)gy\downarrow$ and therefore we find that $\mathsf{sup}(b,g) \in P$. Furthermore $(\forall y \mathrel{\tilde{\in}} fb)(gy \mathrel{\tilde{\in}} \mathsf{i}(p,r))$ yields

$$(b,y) \in J \rightarrow gy \mathrel{\tilde{\in}} \mathsf{i}(p,r). \tag{1}$$

It remains to show

$$(\forall y)((y, \mathsf{sup}(b,g)) \in R \to y \mathrel{\tilde{\in}} \mathsf{i}(p,r)).$$

Let $(y, \mathsf{sup}(b,g)) \in R$. Then we find that $(\exists q)((b,q) \in J \wedge gq = y)$. Hence by (1) $y \mathrel{\tilde{\in}} \mathsf{i}(b,r)$. Therefore we have $\mathsf{sup}(b,g) \mathrel{\tilde{\in}} \mathsf{i}(b,r)$.

We have to show (W-IND). Assume

$$\Re(a) \wedge (\forall x \mathrel{\tilde{\in}} a)\Re(fx) \wedge (\forall b,g)[b \mathrel{\tilde{\in}} a \wedge (\forall u \mathrel{\tilde{\in}} fb)F(gu) \to F(\mathsf{sup}(b,g))]. \qquad (2)$$

In order to show $(\forall x \mathrel{\tilde{\in}} \mathsf{w}(a,f))F(x)$ we use (IG-IND). If we can show

$$(\forall x \in P)[(\forall y)((y,x) \in R \to F(y)) \to F(x)],$$

then we know by (IG.2) that $(\forall x \in \mathsf{i}(p,r))F(x)$ holds and this implies the theorem. Suppose

$$x \in P \wedge (\forall y)((y,x) \in R \to F(y)). \qquad (3)$$

As a consequence of $x \in P$ we obtain $(\exists b,g)(x = \mathsf{sup}(b,g) \wedge b \mathrel{\tilde{\in}} a \wedge (\forall u \mathrel{\tilde{\in}} fb)(gu\downarrow))$. Now we show $(\forall u \mathrel{\tilde{\in}} fb)F(gu)$. Let $u \mathrel{\tilde{\in}} fb$ hence $(b,u) \in J$ and $(gu, \mathsf{sup}(b,g)) \in R$. By (3) we have $F(gu)$. Finally we find that $F(x)$ is a consequence of (2). Therefore (W-IND) holds. $\square$

T_0 in T_W

Given p, r we will construct the type named by $\mathsf{i}(p,r)$ that satisfies the axioms for inductive generation. Assuming $\Re(p,P)$ this is done in the following way. Every $b \in P$ will be interpreted as a branching index. Hence we can set the type A with name a of all "branching types" to be P. There exists a function f such that for each $b \in P$ fb denotes the type consisting exactly of the r-predecessors of b. Using A and f we can define a well-founded tree $\mathsf{w}(a,f)$. In the description of the W-type we see that fb is only an index of a predecessor and not the predecessor itself. So we will make use of a term s chosen such that $sx = \mathsf{sup}(x,s)$. Hence to define our inductively generated type we have to pick up those q such that $\mathsf{sup}(q,s)$ is an element of $\mathsf{w}(a,f)$.

Suppose we have p, r with $\Re(p,P)$ and $\Re(r,R)$. Our aim is to construct $\mathsf{i}(p,r)$ using the W-type. First we set $a := p$. Then for every $t \mathrel{\tilde{\in}} a$ there exists z with $\Re(z, \{y | (y,t) \in R\})$. We choose f such that $\Re(ft, \{y|(y,t) \in R\})$ for every $t \mathrel{\tilde{\in}} a$. Such f exists because elementary comprehension is uniform and $\{y|(y,t) \in R\}$ is an elementary type. Then we choose s such that $sx = \mathsf{sup}(x,s)$. Now we set $I := \{q \mid \mathsf{sup}(q,s) \mathrel{\tilde{\in}} \mathsf{w}(a,f)\}$ and $\mathsf{i}(p,r)$ names I. Such i exists since elementary comprehension is uniform and f depends uniform on r.

Lemma 8 *There exists a term s in T_W such that $sx = \mathsf{sup}(x,s)$.*

Proof Set $f := \lambda y.\lambda x.\mathsf{sup}(x,y)$, then define $s := \mathsf{rec} f$. By the recursion theorem we have $sx = \mathsf{rec} fx = f(\mathsf{rec} f)x = fsx = \mathsf{sup}(x,s)$ for all x.
Equality holds because $\mathsf{rec} f \downarrow$ by the recursion theorem and $(\forall x, y) fxy \downarrow$ by the definition of f. $\square$

Theorem 9 $\mathsf{T_W}$ *proves* (IG.1)

Proof This theorem follows from the construction of I. $\square$

Theorem 10 $\mathsf{T_W}$ *proves* (IG.2)

Proof We have to show $(\forall x)[x \in P \wedge (\forall y)((y,x) \in R \to y \in I) \to x \in I]$.
Assume $x \in P \wedge (\forall y)((y,x) \in R \to y \in I)$. We have to show $\mathsf{sup}(x,s) \tilde{\in} \mathsf{w}(a,f)$. So we prove $x \tilde{\in} a \wedge (\forall y \tilde{\in} fx)(sy \tilde{\in} \mathsf{w}(a,f))$. $x \tilde{\in} a$ is a consequence of $a = p$. Let $y \tilde{\in} fx$, then $(y,x) \in R$ by the definition of f, and hence $y \in I$. Therefore we have $sy = \mathsf{sup}(y,s) \tilde{\in} \mathsf{w}(a,f)$. $\square$

Theorem 11 $\mathsf{T_W}$ *proves* (IG-IND).

Proof Assume

$$(\forall x \in P)[(\forall y)((y,x) \in R \to F(y)) \to F(x)], \tag{4}$$

then we have to show $(\forall x \in I)F(x)$.
We show that

$$(\forall b, g)[b \tilde{\in} a \wedge (\forall y \tilde{\in} fb)G(gy) \to G(\mathsf{sup}(b,g))] \tag{5}$$

holds, where

$$G(q) :\equiv (\forall x)((q = \mathsf{sup}(x,s) \wedge x \in P) \to F(x)).$$

Then by (W-IND) we conclude $(\forall x \tilde{\in} \mathsf{w}(a,f))G(x)$.
Assume $x \in I$, hence $\mathsf{sup}(x,s) \in W$ and $G(\mathsf{sup}(x,s))$ holds. Consequently we can conclude F(x).
It remains to show (5). Fix b, g and suppose that $b \tilde{\in} a \wedge (\forall y \tilde{\in} fb)G(gy)$ holds. This is

$$b \tilde{\in} a \wedge (\forall y \tilde{\in} fb)(\forall x)((gy = \mathsf{sup}(x,s) \wedge x \in P) \to F(x)).$$

Now we show $G(\mathsf{sup}(b,g))$. We distinguish the following two cases:

1. $\mathsf{sup}(b,g) = \mathsf{sup}(x,s) \wedge x \in P$: Here we have $b = x$ and $g = s$. Then for any y with $(y,x) \in R$ we have $y \tilde{\in} fb$ and hence $G(gy)$. This is $G(sy)$ and we conclude $F(y)$. But now by (4) we know $F(x)$. Consequently we obtain $G(\mathsf{sup}(b,g))$.

2. $\mathsf{sup}(b,g) \neq \mathsf{sup}(x,s) \vee x \notin P$: In this case $G(\mathsf{sup}(b,g))$ trivially holds.

$\square$

4 The theories $\mathsf{FID}(\mathbb{K})$

Let $\mathcal{L}_1$ be one of the usual first order languages with number variables $x, y, z, \ldots$, possibly with subscripts, a constant 0, function and relation symbols for all primitive recursive functions and relations. In particular we will use the following function symbols: $\boldsymbol{S}_N$ for the successor- and $\boldsymbol{Pd}_N$ for the predecessor-function. $\langle ., . \rangle$ denotes the usual pairing function and $(.)_0, (.)_1$ will be used for the projections. $\lambda x.\langle a, x\rangle$ denotes the function that maps a numeral b to the ordered pair $\langle a, b\rangle$.
$\mathcal{L}_1(P)$ is the extension of $\mathcal{L}_1$ by an n-ary relation symbol P. An $\mathcal{L}_1(P)$ formula $F(P, x_1, \ldots, x_n)$ is called *n-ary operator form*, if it contains at most $x_1, \ldots, x_n, P$ free.
Now let $\mathbb{K}$ be a set of operator forms. Then we extend $\mathcal{L}_1$ to a new language $\mathcal{L}_\mathbb{K}$ by adding *ordinal variables* α, β, γ,..., a new binary relation $<$ on the ordinals and a new (n+1)-ary relation $P_\mathcal{A}$ for each n-ary operator form $\mathcal{A} \in \mathbb{K}$.
The *number terms s,t,...* of $\mathcal{L}_\mathbb{K}$ are the terms of $\mathcal{L}_1$; the *ordinal terms* of $\mathcal{L}_\mathbb{K}$ are the ordinal variables. The *formulas F,G,H,...* are inductively defined as follows:

1. If R is an n-ary relation symbol of $\mathcal{L}_1$, then $R(s_1, ..., s_n)$ is an (atomic) formula of $\mathcal{L}_\mathbb{K}$.

2. $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_\mathcal{A}(\alpha, s_1, ..., s_n)$ are (atomic) formulas of $\mathcal{L}_\mathbb{K}$. We write $P^\alpha_\mathcal{A}(s_1, ..., s_n)$ for $P_\mathcal{A}(\alpha, s_1, ..., s_n)$.

3. If F and G are formulas of $\mathcal{L}_\mathbb{K}$, then $\neg F$, $(F \wedge G)$ and $(F \vee G)$ are formulas of $\mathcal{L}_\mathbb{K}$.

4. If F is a formula of $\mathcal{L}_\mathbb{K}$, then $(\forall x)F$ and $(\exists x)F$ are formulas of $\mathcal{L}_\mathbb{K}$.

5. If F is a formula of $\mathcal{L}_\mathbb{K}$, then $(\forall \alpha)F$ and $(\exists \alpha)F$ are formulas of $\mathcal{L}_\mathbb{K}$.

6. If F is a formula of $\mathcal{L}_\mathbb{K}$, then $(\forall \alpha < \beta)F$ and $(\exists \alpha < \beta)F$ are formulas of $\mathcal{L}_\mathbb{K}$. In this case we say the ordinal quantifier is *bounded*.

Again we write $(F \to G)$ for $(\neg F \vee G)$ and $(F \leftrightarrow G)$ for $((F \to G) \wedge (G \to F))$. Often we omit parenthesis with the assumption that $\neg$ binds stringer than $\wedge$ and $\vee$. We will use vector notation for finite sequences of terms and write for example $\vec{x}$ for $x_1, \ldots, x_n$. The length of such a sequence is given by the context. By $\Delta^\mathbb{O}_0$ we denote the class of $\mathcal{L}_\mathbb{K}$ formulas where all ordinal quantifiers are bounded.
We use the following additional abbreviations:

- $P^{<\alpha}_\mathcal{A} := (\exists \beta < \alpha) P^\beta_\mathcal{A}$
- $P_\mathcal{A}(\vec{s}) := (\exists \alpha) P^\alpha_\mathcal{A}(\vec{s})$

In the sequel we will need the following classes of formulas:

- An $\mathcal{L}_1(P)$ formula F belongs to POS, if every occurrence of P in F is positive.

- QF denotes the class of quantifier-free $\mathcal{L}_1(P)$ formulas.
- Π^0_0 and Σ^0_0 denote the class of $\mathcal{L}_1(P)$ formulas F, where $F \in \mathsf{QF}$ or all quantifiers occurring in F are bounded. This is F is of the form $\forall x((x < a) \to G(x))$ or $\exists x((x < a) \wedge G(x))$ and $G \in \Pi^0_0$.
- A $\mathcal{L}_1(P)$ formula F is a member of Π^0_1, if F is of the form $\forall x G(x)$ and $G \in \Sigma^0_0$.
- $[\mathsf{M}, \mathsf{N}]$ is the class of formulas of the form

$$\mathcal{A}_0(P, \vec{s}) \vee [(\forall \vec{x})(\mathcal{A}_0(P, \vec{x}) \to P(\vec{x})) \wedge \mathcal{A}_1(P, \vec{s})]$$

  where $\mathcal{A}_0 \in \mathsf{M}$ and $\mathcal{A}_1 \in \mathsf{N}$.
- $\mathbb{I} = [\mathsf{POS}, \mathsf{QF}]$

With operator forms of a class $[\mathsf{M}, \mathsf{N}]$ we can treat in our theories operators of the form

$$[\Phi, \Psi](X) := \begin{cases} \Phi(X) & \text{if } \Phi(X) \not\subseteq X \\ \Psi(X) & \text{if } \Phi(X) \subseteq X \end{cases} \quad \text{for } X \subseteq \omega.$$

This construction of inductive operators was invented by Richter in [14]. Such an operator works as follows. First Φ is applied until closure under Φ is achieved. Then Ψ is applied once and again Φ will be applied till closure is reached and Ψ will again be applied once. This procedure will go on until closure under Φ and Ψ is achieved. It is important to remark that if X is closed under Φ, then $[\Phi, \Psi](X) = \Psi(X)$. Operator forms of the class $[\mathsf{POS}, \mathsf{QF}]$ have the advantage that the non-monotone part is only in the second operator form and thus very controllable.

By $|\mathcal{A}|$ we denote the *closure ordinal* of an operator form $\mathcal{A}$. That is the least ordinal α, such that $(\forall \vec{x})(P^\alpha_\mathcal{A}(\vec{x}) \leftrightarrow P^{<\alpha}_\mathcal{A}(\vec{x}))$ holds. Let $\mathcal{C}$ be a class of operator forms. Then then the closure ordinal of $\mathcal{C}$ is defined in the following way: $|\mathcal{C}| := \mathbf{sup}\{|\mathcal{A}| : \mathcal{A} \in \mathcal{C}\}$. There are the following well-known results about closure ordinals:

- $|\Pi^0_0| = \omega$.
- (Spector [15]) $|\mathsf{POS}| = \omega^{ck}_1$, the first non-recursive ordinal.
- (Gandy, unpublished) $|\Pi^0_1| = \omega^{ck}_1$.
- (Richter [14]) $|[\Pi^0_1, \Pi^0_0]|$ = the first recursive inaccessible ordinal.
- (Richter [14]) $|[\Pi^0_1, \Pi^0_1]|$ = the first recursively Mahlo ordinal.

Now we introduce three $\mathcal{L}_\mathbb{K}$ theories which differ in the strength of their induction principles. The weakest of these theories is $\mathsf{FID}^\mathsf{r}(\mathbb{K})$ and consists of the following axioms.

I. Number-theoretic axioms. These include the usual axioms of PA except for complete induction on the natural numbers.

II. Linearity of $<$ on the ordinals.

$$\sigma \nless \sigma \wedge (\sigma < \tau \wedge \tau < \eta \rightarrow \sigma < \eta) \wedge (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

III. Operator axioms. For every operator form $\mathcal{A}(P, \vec{x})$ in $\mathbb{K}$:

$(\mathsf{OP}.1)$ $\quad \forall \vec{x}[P^{\alpha}_{\mathcal{A}}(\vec{x}) \leftrightarrow P^{<\alpha}_{\mathcal{A}}(\vec{x}) \vee \mathcal{A}(P^{<\alpha}_{\mathcal{A}}, \vec{x})]$.

$(\mathsf{OP}.2)$ $\quad \forall \vec{x}[\mathcal{A}(P_{\mathcal{A}}, \vec{x}) \rightarrow P_{\mathcal{A}}(\vec{x})]$.

IV. $\Delta^{\mathbb{O}}_0$-induction on the natural numbers. For all $\Delta^{\mathbb{O}}_0$-formulas $F(x)$:

$(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathsf{N}})$ $\quad F(0) \wedge \forall x(F(x) \rightarrow F(x+1)) \rightarrow \forall x F(x)$.

V. $\Delta^{\mathbb{O}}_0$-induction on the ordinals. For all $\Delta^{\mathbb{O}}_0$-formulas $F(\xi)$:

$(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ $\quad \forall \xi((\forall \eta < \xi)F(\eta) \rightarrow F(\xi)) \rightarrow \forall \xi F(\xi)$.

$\mathsf{FID}^{\mathsf{w}}(\mathbb{K})$ is the extension of $\mathsf{FID}^{\mathsf{r}}(\mathbb{K})$ by the following scheme of complete induction on the natural numbers:

$(\mathsf{F}\text{-}\mathsf{IND}_{\mathsf{N}})$ $\quad F(0) \wedge \forall x(F(x) \rightarrow F(x+1)) \rightarrow \forall x F(x)$

for all $\mathcal{L}_{\mathbb{K}}$ formulas $F(x)$. $\mathsf{FID}(\mathbb{K})$ is the extension of $\mathsf{FID}^{\mathsf{w}}(\mathbb{K})$ by the following scheme of induction on the ordinals:

$(\mathsf{F}\text{-}\mathsf{IND}_{\mathbb{O}})$ $\quad \forall \xi((\forall \eta < \xi)F(\eta) \rightarrow F(\xi)) \rightarrow \forall \xi F(\xi)$.

for all $\mathcal{L}_{\mathbb{K}}$ formulas $F(x)$.

Now we will turn our attention to the first operator axiom $(\mathsf{OP}.1)$. In general an operator Γ need not have a fixed point P, that is there need not exist P with $\Gamma(P) = P$. But we know that every inclusive operator has fixed point. An operator Γ is called *inclusive*, if and only if $P \subseteq \Gamma(P)$ holds for every P. Since we do not claim $P(\vec{x}) \rightarrow \mathcal{A}(P, \vec{x})$ for our operator forms, they need not define an inclusive operator and there need not exists a fixed point of $\mathcal{A}$, this is a relation P with $\forall \vec{x}(P(\vec{x}) \leftrightarrow \mathcal{A}(P, \vec{x}))$. What we actually do in the first operator axiom is to define an operator $\Gamma_{\mathcal{A}}$ by $\Gamma_{\mathcal{A}}(P) := \{\vec{x} | P(\vec{x}) \vee \mathcal{A}(P, \vec{x})\}$ which in fact is inclusive.

In general $P_{\mathcal{A}}$ will not be a fixed point of $\mathcal{A}$ but by $(\mathsf{OP}.2)$ it will be closed under $\mathcal{A}$. What we have is that $P_{\mathcal{A}}$ is a fixed point of $\Gamma_{\mathcal{A}}$, but not necessarily the least one. At this point the interaction of the operator axiom $(\mathsf{OP}.1)$ and the axiom of induction on the ordinals $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ is very important. Those two axioms together tell us that the fixed point is built up from below. This is how we can prove that a certain number is not an element of the fixed point. If we would drop the principle of induction on the ordinals the whole theory would become trivial. $P_{\mathcal{A}}$ could then always be the whole universe. The operator axiom $(\mathsf{OP}.2)$ ensures the existence of so many ordinals that the fixed point really can be reached. In a theory without induction on ordinals we could say nothing about the form of the fixed point. It could always be $\mathbb{N}^n$. For an example of this fact have look at the proof of theorem 27, where the use of $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ is very important.

Theories of first order inductive definitions can be embedded in systems of Kripke-Platek set-theory in a natural way. So we have the following proof-theoretical results by Jäger: $\mathsf{FID}^{\mathsf{r}}(\mathbb{I}) \leq \mathsf{KPi}^{\mathsf{r}}$, $\mathsf{FID}^{\mathsf{w}}(\mathbb{I}) \leq \mathsf{KPi}^{\mathsf{w}}$ and $\mathsf{FID}(\mathbb{I}) \leq \mathsf{KPi}$. Furthermore we have $\mathsf{FID}^{\mathsf{w}}([\mathsf{POS}, \Pi^0_0]) \leq \mathsf{KPi}^{\mathsf{w}}$ and $\mathsf{FID}([\mathsf{POS}, \Pi^0_0]) \leq \mathsf{KPi}$.

5 The model construction

In this section we are going to formalize the construction of a model of $\mathsf{T_0}$ in $\mathsf{FID}(\mathbb{I})$. First we want informally describe the main idea of the construction. In our model the range of the variables is the set of the natural numbers and $fx \simeq y$ is interpreted as $\{f\}(x) \simeq y$. Thus the applicative part is modeled in a recursion theoretic way. The idea for modeling the type structure is to code types as numbers and to generate by a non-monotone inductive definition simultaneously the collection of all codes for types and their extensions by a membership relation. That is we define the relation $P_{\mathcal{A}}(m, n, k)$ with the following meaning: $P_{\mathcal{A}}(m, 0, 0)$ says m codes the name of a type and $P_{\mathcal{A}}(m, n, 1)$ means m models the name of a type and n is one of its elements.

Now we would like to show why we need a non-monotone inductive definition. First we examine the case of modeling $\mathsf{T_0}$ without inductive generation. There we could use a positive inductive definition to model the type structure. We had to define an additional relation $P_{\mathcal{A}}(m, n, 2)$ to express that n is not a member of the type named by m (co-extension). Using this relation we could define the extension of the complement type by a $P_{\mathcal{A}}$-positive formula. Here it is very important to see that all three, the naming-, the extension and the co-extension-relation are built at one and the same stage. Hence if at a certain stage a number is known to represent a type, then at the next stage we can model the complement of that type, since we know that the extension is also already modeled. Marzetta's model construction for $\mathsf{EET} + (\mathsf{J}) + (\mathsf{F\text{-}I_N})$ in $\widehat{\mathsf{ID}}_1$ is based exactly on this idea (cf. [12]).

Now we examine the case with inductive generation. The extension of an inductively generated type can not be defined in one step and the co-extension of such a type can not be built till the extension is completely defined. Otherwise we would have numbers that belong to a type and to its complement which is contradictory. Hence we have the problem of deciding whether the extension of an inductively generated type is completely modeled or not. This can not be done by a positive formula, and this is the place where the $[\mathsf{POS}, \mathsf{QF}]$ operator comes into the picture. We can define our operator form in such a way that, if we have closure under the first operator, which is positive, then the extensions of all types already defined are completely modeled, especially the extensions of inductively generated types are complete. There will no more elements be added to them. Hence we can define by the second operator the complement types. Since this operator must not be defined by a positive formula, we can dispense with the modeling of the co-extension. The construction can be carried out only be using the relations for naming and extensions. We want to summarize what happens when we generate the codes from below using our $[\mathsf{POS}, \mathsf{QF}]$ operator. We apply the first operator over and over until closure is reached. At this stage we know that for all the types we already have the extensions are completely built up. After that we apply the second operator once in order to define complements. This yields us all the complements of those types. Then we use again the first operator to build further types, then the second one for the complements once, and so on until closure under both operators is achieved.

In the sequel we will use Kleene brackets in a liberal way. One could define them using primitive recursive functions and relations.

Definition 12 *We define the numerals $\boldsymbol{k}, \boldsymbol{s}, \boldsymbol{p}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{s}_N, \boldsymbol{p}_N, \boldsymbol{d}_N, \boldsymbol{r}_N$ of $\mathcal{L}_{\mathbb{I}}$ such that $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ proves the following:*

- $\{\{\boldsymbol{k}\}(x)\}(y) = y$
- $\{\{\boldsymbol{s}\}(x)\}(y) \downarrow$
- $\{\{\{\boldsymbol{s}\}(x)\}(y)\}(z) \simeq \{\{x\}(z)\}(\{y\}(z))$
- $\{\{\boldsymbol{p}\}(x)\}(y) = \langle x, y\rangle$
- $\{\boldsymbol{p}_0\}(\langle x, y\rangle) = x$
- $\{\boldsymbol{p}_1\}(\langle x, y\rangle) = y$
- $\{\boldsymbol{s}_N\}(x) = \boldsymbol{S}_N(x)$
- $\{\boldsymbol{p}_N\}(x) = \boldsymbol{Pd}_N(x)$
- $z_1 = z_2 \rightarrow \{\{\{\{\boldsymbol{d}_N\}(x)\}(y)\}(z_1)\}(z_2) = x$
- $z_1 \neq z_2 \rightarrow \{\{\{\{\boldsymbol{d}_N\}(x)\}(y)\}(z_1)\}(z_2) = y$
- $\{\{\{\{\boldsymbol{r}_N\}(f)\}(g)\}(x)\}(0) = \{f\}(z)$
- $\{\{\{\{\boldsymbol{r}_N\}(f)\}(g)\}(x)\}(\boldsymbol{S}_N(y)) = \{\{\{g\}(x)\}(y)\}(\{\{\{\{\boldsymbol{r}_N\}(f)\}(g)\}(x)\}(y))$

These numerals exist by ordinary recursion theory (c.f. Hinman [8]).

Definition 13 *We define $\boldsymbol{nat}, \boldsymbol{id}, \boldsymbol{co}, \boldsymbol{int}, \boldsymbol{dom}, \boldsymbol{inv}, \boldsymbol{j}, \boldsymbol{i}$ of $\mathcal{L}_{\mathbb{I}}$ such that the following holds:*

- $\boldsymbol{nat} = \langle 1, 0\rangle$
- $\boldsymbol{id} = \langle 2, 0\rangle$
- $\{\boldsymbol{co}\} = \lambda x.\langle 3, x\rangle$
- $\{\boldsymbol{int}\} = \lambda x.\langle 4, x\rangle$
- $\{\boldsymbol{dom}\} = \lambda x.\langle 5, x\rangle$

- $\{\boldsymbol{inv}\} = \lambda x.\langle 6, x\rangle$
- $\{\boldsymbol{j}\} = \lambda x.\langle 7, x\rangle$
- $\{\boldsymbol{i}\} = \lambda x.\langle 8, x\rangle$

These numerals will be used to model the names of the types. For example the term $\mathsf{co}\ \mathsf{id}$ will be modeled by $\{\boldsymbol{co}\}(\boldsymbol{id}) = \langle 3, \langle 2, 0\rangle\rangle$.

Definition 14 *Here we introduce the notion of a partial term in $\mathcal{L}_{\mathbb{I}}$.*

- *The free variables of $\mathcal{L}_{\mathbb{I}}$ are partial terms of $\mathcal{L}_{\mathbb{I}}$.*
- *If f is an n-ary function symbol of $\mathcal{L}_{\mathbb{I}}$ and $s_1, \ldots, s_n$ are partial terms, then $f(s_1, \ldots, s_n)$ is a partial term of $\mathcal{L}_{\mathbb{I}}$.*
- *If $s, t_1, \ldots, t_n$ are partial terms, then also $\{s\}(t_1, \ldots, t_n)$.*

Definition 15 *For every type variable X and every term r of $\mathcal{L}_p$ we define a variable $\hat{X}$, respectively a partial term $\hat{r}$ of $\mathcal{L}_{\mathbb{I}}$ as follows:*

- *Individual and type variables are translated one to one into number variables of $\mathcal{L}_{\mathbb{I}}$. Hence both kinds of variables of $\mathcal{L}_p$ are translated into number variables, but the translations of the type variables $\hat{X}$ have to be different from all the translations of individual variables $\hat{x}$.*
- *Constants:* $\hat{\mathsf{k}} :\equiv \boldsymbol{k}, \hat{\mathsf{s}} :\equiv \boldsymbol{s}, \hat{\mathsf{s}}_N :\equiv \boldsymbol{s}_N, \hat{\mathsf{p}}_N :\equiv \boldsymbol{p}_N, \hat{\mathsf{p}} :\equiv \boldsymbol{p}, \ldots,$
  $\widehat{\mathsf{nat}} :\equiv \boldsymbol{nat}, \widehat{\mathsf{id}} :\equiv \boldsymbol{id}, \widehat{\mathsf{co}} :\equiv \boldsymbol{co}, \ldots$
- *Compound terms:* $\widehat{rs} :\equiv \{\hat{r}\}(\hat{s})$.

With this definition we obtain $\{\{\hat{\mathsf{p}}\}(a)\}(b) = \langle a, b\rangle$. We are now at the point where we can define an operator form $\mathcal{A}$ that will describe the inductive definition of the naming and extension relations of our model construction.

Definition 16 *By $\mathcal{A}$ we denote the formula*

$$\mathcal{A}_0(P, \vec{s}) \vee [(\forall \vec{x})(\mathcal{A}_0(P, \vec{x}) \to P(\vec{x})) \wedge \mathcal{A}_1(P, \vec{s})],$$

where $\mathcal{A}_0(P, m, n, k)$ is the following formula:

$$(m = \widehat{\mathsf{nat}} \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \widehat{\mathsf{nat}} \wedge k = 1) \vee$$

$$(m = \widehat{\mathsf{id}} \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \widehat{\mathsf{id}} \wedge \exists q(n = \langle q, q\rangle) \wedge k = 1) \vee$$

$$\exists b, c[m = \{\widehat{\mathsf{int}}\}(b,c) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge P(\{\widehat{\mathsf{co}}\}(c),0,0) \wedge n = 0 \wedge k = 0] \vee$$

$$\exists b, c[m = \{\widehat{\mathsf{int}}\}(b,c) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge P(\{\widehat{\mathsf{co}}\}(c),0,0) \wedge P(b,n,1) \wedge P(c,n,1) \wedge k = 1] \vee$$

$$\exists b[m = \{\widehat{\mathsf{dom}}\}(b) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge n = 0 \wedge k = 0] \vee$$

$$\exists b[m = \{\widehat{\mathsf{dom}}\}(b) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge \exists q P(b,\langle n,q\rangle,1) \wedge k = 1] \vee$$

$$\exists f, b[m = \{\widehat{\mathsf{inv}}\}(f,b) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge n = 0 \wedge k = 0] \vee$$

$$\exists f, b[m = \{\widehat{\mathsf{inv}}\}(f,b) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge P(b,\{f\}(n),1) \wedge k = 1] \vee$$

$$\exists a, f[m = \{\hat{\boldsymbol{\jmath}}\}(a,f) \wedge \forall q(P(\{\widehat{\mathsf{co}}\}(a),q,1) \vee P(\{f\}(q),0,0)) \wedge P(\{\widehat{\mathsf{co}}\}(a),0,0) \wedge n = 0 \wedge k = 0] \vee$$

$$\exists a, f[m = \{\hat{\boldsymbol{\jmath}}\}(a,f) \wedge \forall q(P(\{\widehat{\mathsf{co}}\}(a),q,1) \vee P(\{f\}(q),0,0)) \wedge P(\{\widehat{\mathsf{co}}\}(a),0,0) \wedge n = \langle \{\hat{\mathsf{p}}_0\}(n), \{\hat{\mathsf{p}}_1\}(n)\rangle \wedge P(a,\{\hat{\mathsf{p}}_0\}(n),1) \wedge P(\{f\}(\{\hat{\mathsf{p}}_0\}(n)),\{\hat{\mathsf{p}}_1\}(n),1) \wedge k = 1] \vee$$

$$\exists a, b[m = \{\hat{\boldsymbol{\imath}}\}(a,b) \wedge P(\{\widehat{\mathsf{co}}\}(a),0,0) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge n = 0 \wedge k = 0] \vee$$

$$\exists a, b[m = \{\hat{\boldsymbol{\imath}}\}(a,b) \wedge P(\{\widehat{\mathsf{co}}\}(a),0,0) \wedge P(\{\widehat{\mathsf{co}}\}(b),0,0) \wedge P(a,n,1) \wedge \forall q(P(\{\widehat{\mathsf{co}}\}(b),\langle q,n\rangle,1) \vee P(m,q,1)) \wedge k = 1]$$

and $\mathcal{A}_1(P,m,n,k)$ *is defined as:*

$$(m = \langle 3,(m)_1\rangle \wedge P((m)_1,0,0) \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \langle 3,(m)_1\rangle \wedge P((m)_1,0,0) \wedge \neg P((m)_1,n,1) \wedge k = 1)$$

Here we have to comment on the occurrences of all the terms $\widehat{\mathsf{co}}$ in the operator form. As mentioned earlier, if we have closure under $\mathcal{A}_0$, then the extensions of the types already defined are completely built up and their complement will be included to P in the next stage by $\mathcal{A}_1$. That is to check whether a type is completely modeled, we have to test whether its complement is yet included to P.

Now we emphasize on the special form of $\mathcal{A}_0$ and $\mathcal{A}_1$. $\mathcal{A}_0$ is positive in P and there is no quantification occurring in $\mathcal{A}_1$. Therefore our operator form $\mathcal{A}$ belongs to $[\mathsf{POS},\mathsf{QF}]$ and hence to $\mathbb{I}$.

Lemma 17 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:*

$$P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1) \to (\exists \beta < \alpha)(P^{\beta}_{\mathcal{A}}(m,0,0) \wedge \neg P^{\beta}_{\mathcal{A}}(m,n,1)).$$

Proof By $\Delta^{\mathbb{O}}_0$-induction on α. We distinguish the following two cases:

1. $P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$: The proposition follows immediately from the induction hypothesis.

2. $\neg P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m), n, 1)$: Then we have $\mathcal{A}_1(P^{<\alpha}_{\mathcal{A}}, \{\widehat{\mathsf{co}}\}(m), n, 1)$. By the definition of $\mathcal{A}_1$ we obtain $P^{<\alpha}_{\mathcal{A}}(m, 0, 0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(m, n, 1)$.

$\square$

Lemma 18 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:*
$(P^{\alpha}_{\mathcal{A}}(m, 0, 0) \vee P^{\alpha}_{\mathcal{A}}(m, n, 1)) \rightarrow$
$(m = \widehat{\mathsf{nat}} \vee$
$m = \widehat{\mathsf{id}} \vee$
$(\exists b)(m = \{\widehat{\mathsf{co}}\}(b) \wedge P^{<\alpha}_{\mathcal{A}}(b, 0, 0)) \vee$
$(\exists b, c)(m = \{\widehat{\mathsf{int}}\}(b, c) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), 0, 0) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(c), 0, 0)) \vee$
$(\exists b)(m = \{\widehat{\mathsf{dom}}\}(b) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), 0, 0)) \vee$
$(\exists b, f)(m = \{\widehat{\mathsf{inv}}\}(f, b) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), 0, 0)) \vee$
$(\exists b, f)(m = \{\hat{\jmath}\}(b, f) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), 0, 0) \wedge$
$\quad (\forall q)(P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), q, 1) \vee P^{<\alpha}_{\mathcal{A}}(\{f\}(q), 0, 0))) \vee$
$(\exists b, c)(m = \{\hat{\imath}\}(b, c) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(b), 0, 0) \wedge P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(c), 0, 0)))$.

Proof By $\Delta^{\mathbb{O}}_0$-induction on α. We distinguish the following two cases:

1. $P^{<\alpha}_{\mathcal{A}}(m, 0, 0)$: The proposition follows by the induction hypothesis.

2. $\neg P^{<\alpha}_{\mathcal{A}}(m, 0, 0)$: Here we have $\mathcal{A}(P^{<\alpha}_{\mathcal{A}}, m, 0, 0)$. Then the proposition follows directly from the definition of $\mathcal{A}$.

$\square$

Lemma 19 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:* $P^{\alpha}_{\mathcal{A}}(m, n, 1) \rightarrow P^{\alpha}_{\mathcal{A}}(m, 0, 0)$.

Proof By $\Delta^{\mathbb{O}}_0$-induction on α. We distinguish the following two cases:

1. $P^{<\alpha}_{\mathcal{A}}(m, n, 1)$: The proposition follows by the induction hypothesis and ($\mathsf{OP.1}$).

2. $\neg P^{<\alpha}_{\mathcal{A}}(m, n, 1)$: In this case $\mathcal{A}(P^{<\alpha}_{\mathcal{A}}, m, n, 1)$ holds. Then we apply lemma 18 in order to distinguish the different possible forms of m. The fact that each case holds is a consequence of the definition of $\mathcal{A}$.

$\square$

Lemma 20 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:*
$\forall u, v, w[\mathcal{A}_0(P^{<\alpha}_{\mathcal{A}}, u, v, w) \rightarrow P^{<\alpha}_{\mathcal{A}}(u, v, w)] \wedge P^{<\alpha}_{\mathcal{A}}(m, 0, 0) \wedge \beta \geq \alpha \wedge P^{\beta}_{\mathcal{A}}(m, x, 1)$
$\longrightarrow P^{<\alpha}_{\mathcal{A}}(m, x, 1)$

Proof Proof by $\Delta_0^{\mathbb{O}}$-induction on β. We distinguish the different cases for m:

$m = \widehat{\mathsf{nat}}$: By the assumption we have $P_{\mathcal{A}}^{\beta}(m, x, 1)$. Hence with the definition of $\mathcal{A}_0$ we find $P_{\mathcal{A}}^{<\alpha}(m, x, 1)$.

$m = \widehat{\mathsf{id}}$: By the assumption we have $P_{\mathcal{A}}^{\beta}(m, x, 1)$ and with the definition of $\mathcal{A}_0$ we find $\exists q(x = \langle q, q\rangle)$. Hence $P_{\mathcal{A}}^{<\alpha}(m, x, 1)$.

$m = \{\widehat{\mathsf{co}}\}(b)$: There exist $\gamma < \alpha$ with $P_{\mathcal{A}}^{\gamma}(m, 0, 0) \wedge \neg P_{\mathcal{A}}^{<\gamma}(m, 0, 0)$ and $\delta \leq \beta$ with $P_{\mathcal{A}}^{\delta}(m, x, 1) \wedge \neg P_{\mathcal{A}}^{<\delta}(m, x, 1)$. Hence we obtain $P_{\mathcal{A}}^{<\delta}(b, 0, 0) \wedge \neg P_{\mathcal{A}}^{<\delta}(b, x, 1)$. Assume $\delta > \gamma$. Then we know $\neg P_{\mathcal{A}}^{<\gamma}(b, x, 1)$. With $P_{\mathcal{A}}^{<\gamma}(b, 0, 0)$ and

$$P_{\mathcal{A}}^{\gamma}(u, v, w) \leftrightarrow \mathcal{A}_1(P_{\mathcal{A}}^{<\gamma}, u, v, w) \vee P_{\mathcal{A}}^{<\gamma}(u, v, w)$$

we find $P_{\mathcal{A}}^{\gamma}(m, x, 1)$ which contradicts our assumption. Hence $\delta \leq \gamma$. But in this case we have $P_{\mathcal{A}}^{<\alpha}(m, x, 1)$.

$m = \{\widehat{\mathsf{int}}\}(b, c)$: We have $P_{\mathcal{A}}^{<\beta}(b, x, 1) \wedge P_{\mathcal{A}}^{<\beta}(c, x, 1)$. Because of $P_{\mathcal{A}}^{<\alpha}(m, 0, 0)$ we know $P_{\mathcal{A}}^{<\alpha}(b, 0, 0) \wedge P_{\mathcal{A}}^{<\alpha}(c, 0, 0)$. With the induction hypothesis we obtain $P_{\mathcal{A}}^{<\alpha}(b, x, 1) \wedge P_{\mathcal{A}}^{<\alpha}(c, x, 1)$, and because $P_{\mathcal{A}}^{<\alpha}$ is closed under $\mathcal{A}_0$ we can conclude $P_{\mathcal{A}}^{<\alpha}(m, x, 1)$.

$m = \{\widehat{\mathsf{dom}}\}(b)$: similarly.

$m = \{\widehat{\mathsf{inv}}\}(f, b)$: similarly.

$m = \{\hat{\jmath}\}(b, f)$: similarly.

$m = \{\hat{\imath}\}(b, c)$: We have $P_{\mathcal{A}}^{<\beta}(b, x, 1) \wedge \forall q[P_{\mathcal{A}}^{<\beta}(\{\widehat{\mathsf{co}}\}(c), \langle q, x\rangle, 1) \vee P_{\mathcal{A}}^{<\beta}(m, q, 1)]$. We know $P_{\mathcal{A}}^{<\alpha}(b, 0, 0) \wedge P_{\mathcal{A}}^{<\alpha}(\{\widehat{\mathsf{co}}\}(c), 0, 0)$. With the induction hypothesis we obtain $P_{\mathcal{A}}^{<\alpha}(b, x, 1) \wedge \forall q[P_{\mathcal{A}}^{<\alpha}(\{\widehat{\mathsf{co}}\}(c), \langle q, x\rangle, 1) \vee P_{\mathcal{A}}^{<\alpha}(m, q, 1)]$. Therefore we conclude $P_{\mathcal{A}}^{<\alpha}(m, x, 1)$.

□

The following lemma says that, if at a certain stage α we have $P_{\mathcal{A}}^{\alpha}(\{\widehat{\mathsf{co}}\}(a), 0, 0)$, then the coding of the extension of the type represented by a is complete. This is in later stages no new elements will be added to the extension of the type named by a. With this lemma we get upper bounds for the stages where certain types are completely modeled. This is also the reason for all the $\{\widehat{\mathsf{co}}\}$ functions in the definition of $\mathcal{A}$. For example we can build a type $\{\widehat{\mathsf{dom}}\}(b)$ not till the extension of the type named by b is complete. By the following lemma this can be tested by $P_{\mathcal{A}}^{\alpha}(\{\widehat{\mathsf{co}}\}(b), 0, 0)$, this is by checking if its complement is yet built.

Lemma 21 *We can prove in* $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$*:*

$$P_{\mathcal{A}}^{\beta}(\{\widehat{\mathsf{co}}\}(a), 0, 0) \wedge \beta \leq \gamma \longrightarrow (\forall x)(P_{\mathcal{A}}^{<\beta}(a, x, 1) \leftrightarrow P_{\mathcal{A}}^{\gamma}(a, x, 1)).$$

PROOF The direction from left to right is satisfied because of (OP.1).
From right to left: There exists $\alpha \le \beta$ such that $P^{<\alpha}_{\mathcal{A}}$ is closed under $\mathcal{A}_0$ and $P^{<\alpha}_{\mathcal{A}}(a,0,0)$ holds. Hence lemma 20 applies. □

Lemma 22 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves the following statement:*

$$P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0) \longrightarrow (\forall n)(\neg P^{\alpha}_{\mathcal{A}}(m,n,1) \leftrightarrow P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)).$$

PROOF From left to right: From $P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0)$ it follows that there exists $\beta \le \alpha$ satisfying $P^{\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0)$ and $\neg P^{<\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0)$. From $\neg P^{\alpha}_{\mathcal{A}}(m,n,1)$ follows $\neg P^{<\beta}_{\mathcal{A}}(m,n,1)$ since we have (OP.1). By the definition of $\mathcal{A}_1$ we get $P^{\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$ and hence $P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$.
From right to left: We prove this direction by $\Delta^{\mathbb{O}}_0$-induction on the ordinals. Assume $P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$. Now we distinguish whether n was included to $\{\widehat{\mathsf{co}}\}(m)$ at stage α or before.

1. $P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$: Hence $P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0)$ by lemma 19. With the induction hypothesis we obtain $\neg P^{<\alpha}_{\mathcal{A}}(m,n,1)$ and with lemma 21 we conclude $\neg P^{\alpha}_{\mathcal{A}}(m,n,1)$.

2. $\neg P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1)$: Hence $\mathcal{A}_1(P^{<\alpha}_{\mathcal{A}},\{\widehat{\mathsf{co}}\}(m),n,1)$. This is $\neg P^{<\alpha}_{\mathcal{A}}(m,n,1)$. Again with lemma 21 we conclude $\neg P^{\alpha}_{\mathcal{A}}(m,n,1)$.

□

In order to define a translation from $\mathcal{L}_p$ to $\mathcal{L}_{\mathbb{I}}$ we need the notion of the value of a partial term of $\mathcal{L}_{\mathbb{I}}$. The formula $V_t(z)$ can be read as "the partial term t has the value z." We then can define a relation on partial terms to be the relation on the values of the partial terms.

Definition 23 *We assign to every partial term t of $\mathcal{L}_{\mathbb{I}}$ a formula $V_t(z)$ of $\mathcal{L}_{\mathbb{I}}$ in the following way:*

$$\begin{aligned}
V_a(z) &:\equiv a = z, \text{ for every individual variable } a \text{ of } \mathcal{L}_p.\\
V_{f(s_1,\dots,s_n)}(z) &:\equiv \exists x_1,\dots x_n[V_{s_1}(x_1)\wedge\cdots\wedge V_{s_n}(x_n)\wedge f(x_1,\dots,x_n)=z].\\
V_{\{r\}(s_1,\dots,s_n)}(z) &:\equiv \exists x,y_1,\dots,y_n[V_r(x)\wedge V_{s_1}(y_1)\wedge\cdots\wedge V_{s_n}(y_n)\wedge\\
&\qquad \{x\}(y_1,\dots,y_n)=z].
\end{aligned}$$

Definition 24 *Let $s,t,s_1,\dots,s_n$ be partial terms of $\mathcal{L}_{\mathbb{I}}$ and R an n-ary relation symbol of $\mathcal{L}_{\mathbb{I}}$, then*

$$\begin{aligned}
s = t &:\equiv \exists x(V_s(x)\wedge V_t(x))\\
R(s_1,\dots,s_n) &:\equiv \exists x_1,\dots x_n[V_{s_1}(x_1)\wedge\cdots\wedge V_{s_n}(x_n)\wedge R(x_1,\dots,x_2).
\end{aligned}$$

There is an ambiguity coming with these definitions since terms of $\mathcal{L}_{\mathbb{I}}$ are also partial terms, hence $s = t$ for example can be viewed as formula of $\mathcal{L}_{\mathbb{I}}$ or as abbreviation for $\exists x(V_s(x) \wedge V_t(x))$. But this fact will not disturb us, for we can prove in $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ that $\forall x(V_s(x) \leftrightarrow s = x)$ for every term s of $\mathcal{L}_{\mathbb{I}}$.

Definition 25 *We define following abbreviations which we will use to define the translation of formulas:*

$$\begin{array}{rcl} \mathsf{R}(a) & :\equiv & P_{\mathcal{A}}(a, 0, 0) \\ \mathsf{E}(x, a) & :\equiv & P_{\mathcal{A}}(a, x, 1) \\ \mathsf{ExtEq}(a, b) & :\equiv & (\forall x)(\mathsf{E}(x, a) \leftrightarrow \mathsf{E}(x, b)) \end{array}$$

Now we can complete the translation from $\mathcal{L}_p$ into $\mathcal{L}_{\mathbb{I}}$ by the following definition of the translation * for formulas.

Definition 26 *Let F be a formula of $\mathcal{L}_p$. Then we define the $\mathcal{L}_{\mathbb{I}}$ formula F^* inductively as follows:*

$$\begin{array}{rcl} (s\downarrow)^* & :\equiv & \exists x V_{\hat{s}}(x) \\ (s = t)^* & :\equiv & \hat{s} = \hat{t} \\ (N(s))^* & :\equiv & \exists x V_{\hat{s}}(x) \\ (s \in B)^* & :\equiv & \mathsf{E}(\hat{s}, \hat{B})) \\ (A = B)^* & :\equiv & \mathsf{ExtEq}(\hat{A}, \hat{B}) \\ (\Re(s, A))^* & :\equiv & \mathsf{R}(\hat{s}) \wedge \mathsf{ExtEq}(\hat{s}, \hat{A})) \\ (\neg F)^* & :\equiv & \neg F^* \\ (F \vee G)^* & :\equiv & F^* \vee G^* \\ (F \wedge G)^* & :\equiv & F^* \wedge G^* \\ (\exists x F(x))^* & :\equiv & \exists \hat{x} F(\hat{x})^* \\ (\forall x F(x))^* & :\equiv & \forall \hat{x} F(\hat{x})^* \\ (\exists X F(X))^* & :\equiv & \exists \hat{X}(\mathsf{R}(\hat{X}) \wedge F(\hat{X})^*) \\ (\forall X F(X))^* & :\equiv & \forall \hat{X}(\mathsf{R}(\hat{X}) \rightarrow F(\hat{X})^*) \end{array}$$

Theorem 27 $\mathsf{FID}(\mathbb{I})$ *proves* $\mathsf{R}(\hat{A}_1) \wedge \cdots \wedge \mathsf{R}(\hat{A}_n) \rightarrow F^*$ *for every axiom F of* $\mathsf{T_0}$ *having its free type variables among $A_1, \ldots, A_n$.*

We are not going to perform the whole proof because the first part of the proof can be done as in Marzetta [12] and all the proofs for the axioms simulating elementary comprehension follow the same pattern which will be exposed for the cases of natural numbers and intersections.

Logical axioms: The logical axioms are easily checked. The only remarkable point is that one needs the fact that $V_t(a) \wedge V_t(b) \to a = b$ for all terms t, which is proved by induction on the setup of t.

Mathematical axioms (1)-(10): The combinatorial axioms and the axioms for pairing, projection, natural numbers, definition by numerical cases and primitive recursion on N are checked in the same way as the logical axioms are.

Extensionality (EXT) is trivial, since equality of types is defined as having the same elements. The ontological axiom ($\mathsf{E}.1$) $(\exists x)\Re(x, A)$ is checked by taking the witness $\hat{A}$. The no homonyms axiom ($\mathsf{E}.2$) is satisfied because $\mathbf{ExtEq}$ is transitive and symmetrical.

Natural numbers: ($\mathsf{N}.1$): Choose $\widehat{\mathbf{nat}}$ for $\hat{X}$. We have to check that $\widehat{\mathbf{nat}}$ has the correct extension. This is $(\forall x)P_{\mathcal{A}}(\widehat{\mathbf{nat}}, x, 1)$ which is fulfilled by the definition of $\mathcal{A}_0$. ($\mathsf{N}.2$): Surely $\mathbf{R}(\widehat{\mathbf{nat}})$. Furthermore if $(\forall x)(x \in A \leftrightarrow N(x))^*$, then we can show $(\forall x)P_{\mathcal{A}}(\hat{A}, x, 1)$. Hence we have $\mathbf{ExtEq}(\widehat{\mathbf{nat}}, \hat{A})$ and thus the translation of $\Re(\mathbf{nat}, A)$ holds.

Intersections: ($\mathsf{INT}.1$): Choose $\{\widehat{\mathbf{int}}\}(\hat{B}, \hat{C})$ for $\hat{X}$. We have to show

$$(\forall x)(\mathbf{E}(x, \{\widehat{\mathbf{int}}\}(\hat{B}, \hat{C})) \leftrightarrow \mathbf{E}(x, \hat{B}) \wedge \mathbf{E}(x, \hat{C})).$$

From left to right is shown by ($\Delta_0^{\mathbb{O}}$-$\mathsf{IND}_{\mathbb{O}}$). From right to left: Assume $\mathbf{E}(x, \hat{B})$, this is $P_{\mathcal{A}}(\hat{B}, x, 1)$. By lemma 19 we obtain $P_{\mathcal{A}}(\hat{B}, 0, 0)$ and hence $P_{\mathcal{A}}(\{\widehat{\mathbf{co}}\}(\hat{B}), 0, 0)$. Analogously we get $P_{\mathcal{A}}(\{\widehat{\mathbf{co}}\}(\hat{C}), 0, 0)$. Finally by the definition of $\mathcal{A}_0$ we find that $\mathbf{E}(x, \{\widehat{\mathbf{int}}\}(\hat{B}, \hat{C}))$ holds.
($\mathsf{INT}.2$): If $\mathbf{R}(\hat{b})$ and $\mathbf{R}(\hat{c})$ hold, then also $\mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{b}))$ and $\mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{c}))$. This yields $\mathbf{R}(\{\widehat{\mathbf{int}}\}(\hat{b}, \hat{c}))$. If $(\forall x)(x \in A \leftrightarrow x \in B \wedge x \in C)^*$ holds, then we find that $\mathbf{ExtEq}(\{\widehat{\mathbf{int}}\}(\hat{b}, \hat{c}), \hat{A})$ and therefore $\Re(\mathbf{int}(b, c), A)^*$ holds.

The other axioms for elementary comprehension are checked in the same way as the axioms for natural numbers and intersections.

Join (J): Under the assumptions $\mathbf{R}(\hat{A})$, $\mathbf{R}(\hat{a})$, $\mathbf{ExtEq}(\hat{a}, \hat{A})$ and $\mathbf{R}(\{\hat{f}\}(x))$ for all x with $\mathbf{E}(x, \hat{A})$ we have to produce a $\hat{Z}$ with $\mathbf{R}(\hat{Z})$, and the elements of $\hat{Z}$ are exactly the elements of the join of $\hat{f}$ over $\hat{A}$. By the construction of $P_{\mathcal{A}}$ we have $\mathbf{R}(\{\boldsymbol{\jmath}\}(\hat{a}, \hat{f}))$ so we can set $\hat{Z} = \{\boldsymbol{\jmath}\}(\hat{a}, \hat{f})$ and verify that $\hat{Z}$ has the desired properties.

Inductive generation: Use $\{\boldsymbol{\imath}\}(\hat{a}, \hat{b})$ to verify the axioms for inductive generation.
To check ($\mathsf{IG}.1$) assume $\mathbf{R}(\hat{a})$ and $\mathbf{R}(\hat{b})$. Then we have $\mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{a}))$ and $\mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{b}))$ and this yields $\mathbf{R}(\{\boldsymbol{\imath}\}(\hat{a}, \hat{b}))$.
To verify ($\mathsf{IG}.2$) we assume

$$\mathbf{R}(\hat{a}) \wedge \mathbf{R}(\hat{b}) \wedge \mathbf{E}(x, \hat{a}) \wedge (\forall y)(\mathbf{E}(\langle y, x\rangle, \hat{b}) \to \mathbf{E}(y, \{\boldsymbol{\imath}\}(a, b))).$$

Thus we have

$$\mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{a})) \wedge \mathbf{R}(\{\widehat{\mathbf{co}}\}(\hat{b})) \wedge P_{\mathcal{A}}(\hat{a}, x, 1) \wedge (\forall y)(\neg P_{\mathcal{A}}(\hat{b}, \langle y, x\rangle, 1) \vee P_{\mathcal{A}}(\{\boldsymbol{\imath}\}(a, b), y, 1)).$$

With lemma 22 we obtain $(\forall y)(P_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{b}), \langle y, x\rangle, 1) \vee P_{\mathcal{A}}(\{\hat{\boldsymbol{\imath}}\}(a, b), y, 1))$. Since $P_{\mathcal{A}}$ is closed under $\mathcal{A}_0$ we get $\mathsf{E}(x, \{\hat{\boldsymbol{\imath}}\}(a, b)))$.
To verify the induction principle on inductively generated types $(\mathsf{IG\text{-}IND})$ we need induction on ordinals in $\mathsf{FID}(\mathbb{I})$. We define:

- $G^* :\equiv (\forall x)[\mathsf{E}(x, \hat{a}) \to [(\forall y)(\mathsf{E}(\langle y, x\rangle, \hat{b}) \to F^*(y)) \to F^*(x)]]$
- $H(\alpha) :\equiv \mathsf{R}(\hat{a}) \wedge \mathsf{R}(\hat{b}) \wedge G^* \to \forall x(P^{\alpha}_{\mathcal{A}}(\{\hat{\boldsymbol{\imath}}\}(\hat{a}, \hat{b}), x, 1) \to F^*(x))$

We show $\mathsf{FID}(\mathbb{I}) \vdash \forall\alpha H(\alpha)$ by induction on α. Under the assumption $(\forall\beta < \alpha)H(\beta)$ we show $H(\alpha)$. Let $\mathsf{R}(\hat{a}) \wedge \mathsf{R}(\hat{b}) \wedge G^*$ hold. Then assume $P^{\alpha}_{\mathcal{A}}(\{\hat{\boldsymbol{\imath}}\}(\hat{a}, \hat{b}), x, 1)$, hence we have $P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{a}), 0, 0)$ and $P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{b}), 0, 0)$. Then for any y with $P_{\mathcal{A}}(\hat{b}, \langle y, x\rangle, 1)$ lemma 21 implies $P^{<\alpha}_{\mathcal{A}}(\hat{b}, \langle y, x\rangle, 1)$. We conclude $P^{<\alpha}_{\mathcal{A}}(\{\hat{\boldsymbol{\imath}}\}(\hat{a}, \hat{b}), y, 1)$ by the definition of $\mathcal{A}$. With the induction hypothesis we get $F^*(y)$. By the premise G^* it follows $F^*(x)$.

Induction $(\mathsf{F\text{-}I_N})$: Every instance of the induction scheme in $\mathsf{T_0}$ translates into a formula which follows in few steps from an instance of the induction scheme on natural numbers in $\mathsf{FID}(\mathbb{I})$.

6 Restricted inductions

In the preceding section we built a model for $\mathsf{T_0}$ in $\mathsf{FID}(\mathbb{I})$. The only cases where we used the full induction principles of $\mathsf{FID}(\mathbb{I})$ was to prove the translations of the full induction principles of $\mathsf{T_0}$. So it is not surprising that we can show the correspondence of the restrictions of the inductions in $\mathsf{T_0}$ and in $\mathsf{FID}(\mathbb{I})$. This is we can prove type induction on natural numbers in $\mathsf{FID^r}(\mathbb{I})$ and type induction on inductively generated types in $\mathsf{FID^w}(\mathbb{I})$.

Remark 28 *We examine the proof of theorem 27: The induction principle* $(\mathsf{F\text{-}IND_N})$ *was only used to show that* $(\mathsf{F\text{-}I_N})$ *holds and* $(\mathsf{F\text{-}IND_{\mathbb{O}}})$ *was necessary to show* $(\mathsf{IG\text{-}IND})$. *All the lemmas and the rest of theorem 27 have been proven by the use of* $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND_N})$ *and* $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND_{\mathbb{O}}})$ *only.*

Theorem 29 $\mathsf{FID^w}(\mathbb{I})$ *proves* $\mathsf{R}(\hat{A}_1) \wedge \cdots \wedge \mathsf{R}(\hat{A}_n) \to F^*$ *for every axiom F of* $\mathsf{EET} + (\mathsf{J}) + (\mathsf{IG}\restriction) + (\mathsf{F\text{-}I_N})$ *having its free type variables among $A_1, \ldots, A_n$.*

Proof In the light of remark 28 it is sufficient to examine the case for the induction principle for IG-types in the proof of theorem 27. If we have only type induction then $F(x)$ equal to $x \in C$. Hence there exists β with

$$P^{\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{a}), 0, 0) \wedge P^{\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{b}), 0, 0) \wedge P^{\beta}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(\hat{C}), 0, 0).$$

Now we can define $G^{*\beta}$ as G^* where each occurrence of $P_{\mathcal{A}}$ is replaced by $P^{<\beta}_{\mathcal{A}}$. Then we denote by $H^{\beta}(\alpha)$ the formula

$$P^{<\beta}_{\mathcal{A}}(\hat{a}, 0, 0) \wedge P^{<\beta}_{\mathcal{A}}(\hat{b}, 0, 0) \wedge G^{*\beta} \to (\forall x)(P^{\alpha}_{\mathcal{A}}(\{\hat{\boldsymbol{\imath}}\}(a, b), x, 1) \to P^{<\beta}_{\mathcal{A}}(\hat{C}, x, 1)).$$

Hence $H^{\beta}(\alpha)$ is a $\Delta^{\mathbb{O}}_0$-formula and the theorem holds. $\square$

Theorem 30 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves* $\mathsf{R}(\hat{A}_1) \wedge \cdots \wedge \mathsf{R}(\hat{A}_n) \to F^*$ *for every axiom* F *of* $\mathsf{T_0} \restriction$ *having its free type variables among* $A_1, \ldots, A_n$.

PROOF It only remains to proof in $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$:

$$[0 \in A \wedge (\forall x \in N)(x \in A \to \mathsf{s}_N x \in A) \to (\forall x \in N)(x \in A)]^*.$$

Let $(0 \in A)^*$. Then we have A is a type, so $\exists \beta P^{\beta}_{\mathcal{A}}(\{\widehat{\mathbf{co}}\}(\hat{A}), 0, 0)$. Let

$$((\forall x \in N)(x \in A \to \mathsf{s}_N x \in A))^*$$

hold, then by lemma 21 we know

$$P^{<\beta}_{\mathcal{A}}(\hat{A}, 0, 1) \wedge (\forall x \in N)(P^{<\beta}_{\mathcal{A}}(\hat{A}, x, 1) \to P^{<\beta}_{\mathcal{A}}(\hat{A}, \hat{\mathsf{s}}_N x, 1).$$

With $(\Delta^{\mathbb{O}}_{\mathsf{0}}\text{-}\mathsf{IND_N})$ follows $(\forall x \in N) P^{<\beta}_{\mathcal{A}}(\hat{A}, x, 1)$. Hence $((\forall x \in N)(x \in A))^*$. □

7 Universes

Universes are a well known extension of Martin-Löf's type theories. In [10, 11] universes are introduced as sets closed under certain specified set forming operations. In the context of explicit mathematics universes are first used by Feferman in [5]. Marzetta defines in [12] the theory UTN in which he examines universes over the theory $\mathsf{EET} + (\mathsf{J})$. There the existence of the universes is ensured by a limes axiom as it is done for admissible sets in Kripke-Platek set theory. We will show that we can perform in $\mathsf{FID}(\mathbb{I})$ a model construction for the theory $\mathsf{T_0}$ with axioms for universes which we will call $\mathsf{T_0{+}U}$. Our universes will be closed under elementary comprehension and join but not under inductive generation. This is in contrast to Martin-Löf's formulation of universes in [11]. His universes are closed under the formation of well-founded trees. So if it would be a complete analogue, our universes should be closed under inductive generation. But we think the resulting theory would be stronger than $\mathsf{FID}(\mathbb{I})$. In $\mathsf{T_0{+}U}$ universes must exist by a uniform limes axiom.

The language $\mathcal{L}_u$ is the language $\mathcal{L}_p$ with a new unary predicate U on types, which is intended to be true on those types which are universes and false otherwise. Terms and formulas of $\mathcal{L}_u$ are defined as usual. The theory $\mathsf{T_0{+}U}$ consists of the axioms of $\mathsf{T_0}$ and includes the following additional axioms for universes:

- Ontological Axioms
  - $U(A) \to \mathbf{nat} \in A$
  - $U(A) \wedge x \in A \to \Re(x)$
  - $U(A) \wedge U(B) \to (A \dot{\in} B \vee A = B \vee B \dot{\in} A)$
  - $U(A) \wedge U(B) \wedge A \dot{\in} B \to \forall x(x \in A \to x \in B)$

- Closure properties
  - $U(A) \rightarrow \mathsf{id} \in A$
  - $U(A) \wedge b \in A \rightarrow \mathsf{co}\ b \in A$
  - $U(A) \wedge b \in A \wedge c \in A \rightarrow \mathsf{int}(b, c) \in A$
  - $U(A) \wedge b \in A \rightarrow \mathsf{dom}\ b \in A$
  - $U(A) \wedge b \in A \rightarrow \mathsf{inv}(f, b) \in A$
  - $U(A) \wedge b \in A \wedge \Re(b, B) \wedge (\forall q \in B)(fq \in A) \rightarrow \mathsf{j}(b, f) \in A$
- The limes axiom
  - $\Re(a) \rightarrow \exists Z[\Re(\mathsf{lim}\ a, Z) \wedge U(Z) \wedge a \in Z]$

In this section our aim is to construct a model for the theory $\mathsf{T_0}$+U. The model will be built along the same lines as our model for $\mathsf{T_0}$ was. This is we will define inductively the codes for the names of types and their extensions. Universes will be just another kind of types having names of types as elements. Since we can not check if the closure properties are satisfied, neither by only positive formulas nor only by quantifier free formulas, we have to modify $\mathcal{A}_0$ and $\mathcal{A}_1$ in the following way: Closure under $\mathcal{A}_0$ should model the closure properties of an universe. This is closure under elementary comprehension, or the equivalent operations, and join. In $\mathcal{A}_1$ we can collect all the types built so far. They will automatically satisfy the demanded closure properties. In the definition 16 we simultaneously defined a naming relation and a membership relation. Here we have to define simultaneously a third relation $P_{\mathcal{A}}(m, n, 2)$ which says that n is not a member of the type coded by m. Hence this models a kind of complement-extension relation. Due to this relation, we can now model the behavior of terms of the form $\mathsf{co}\ b$ by a positive formula. This is in $\mathcal{A}_0$. If this would not be the case, closure under $\mathcal{A}_0$ would not imply that the relation generated so far satisfies the closure conditions for universes. We still need $\mathcal{A}_1$ to model the complement extension of inductively generated types and to model the universes which will be coded by $\langle 9, x\rangle$. So we define $\boldsymbol{lim}$ such that $\{\boldsymbol{lim}\} :\equiv \lambda x.\langle 9, x\rangle$ and we extend the translation from $\mathcal{L}_p$ to $\mathcal{L}_{\mathbb{I}}$ by $\widehat{\mathsf{lim}} :\equiv \boldsymbol{lim}$. Now we can define the operator form $\mathcal{A}$ which will define the relation $P_{\mathcal{A}}$ of our model.

Definition 31 *By $\mathcal{A}$ we denote the formula*

$$\mathcal{A}_0(P, \vec{s}) \vee [(\forall \vec{x})(\mathcal{A}_0(P, \vec{x}) \rightarrow P(\vec{x})) \wedge \mathcal{A}_1(P, \vec{s})],$$

where $\mathcal{A}_0(P, m, n, k)$ is the following formula:

$(m = \widehat{\mathsf{nat}} \wedge n = 0 \wedge k = 0) \vee$
$(m = \widehat{\mathsf{nat}} \wedge k = 1) \vee$
$(m = \widehat{\mathsf{nat}} \wedge n \neq n \wedge k = 2) \vee$

$$(m = \widehat{\mathsf{id}} \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \widehat{\mathsf{id}} \wedge \exists q(n = \langle q, q\rangle) \wedge k = 1) \vee$$
$$(m = \widehat{\mathsf{id}} \wedge \neg\exists q(n = \langle q, q\rangle) \wedge k = 2) \vee$$

$$\exists b[m = \{\widehat{\mathsf{co}}\}(b) P(b, 0, 0) \wedge \wedge n = 0 \wedge k = 0] \vee$$
$$\exists b[m = \{\widehat{\mathsf{co}}\}(b) P(b, 0, 0) \wedge P(b, n, 2) \wedge k = 1] \vee$$
$$\exists b[m = \{\widehat{\mathsf{co}}\}(b) P(b, 0, 0) \wedge P(b, n, 1) \wedge k = 2] \vee$$

$$\exists b, c[m = \{\widehat{\mathsf{int}}\}(b, c) \wedge P(b, 0, 0) \wedge P(c, 0, 0) \wedge n = 0 \wedge k = 0] \vee$$
$$\exists b, c[m = \{\widehat{\mathsf{int}}\}(b, c) \wedge P(b, 0, 0) \wedge P(c, 0, 0) \wedge P(b, n, 1) \wedge P(c, n, 1) \wedge k = 1] \vee$$
$$\exists b, c[m = \{\widehat{\mathsf{int}}\}(b, c) \wedge P(b, 0, 0) \wedge P(c, 0, 0) \wedge (P(b, n, 2) \vee P(c, n, 2)) \wedge \\ k = 2] \vee$$

$$\exists b[m = \{\widehat{\mathsf{dom}}\}(b) P(b, 0, 0) \wedge \wedge n = 0 \wedge k = 0] \vee$$
$$\exists b[m = \{\widehat{\mathsf{dom}}\}(b) \wedge P(b, 0, 0) \wedge \exists q P(b, \langle n, q\rangle, 1) \wedge k = 1] \vee$$
$$\exists b[m = \{\widehat{\mathsf{dom}}\}(b) \wedge P(b, 0, 0) \wedge \forall q P(b, \langle n, q\rangle, 2) \wedge k = 2] \vee$$

$$\exists f, b[m = \{\widehat{\mathsf{inv}}\}(f, b) \wedge P(b, 0, 0) \wedge n = 0 \wedge k = 0] \vee$$
$$\exists f, b[m = \{\widehat{\mathsf{inv}}\}(f, b) \wedge P(b, 0, 0) \wedge P(b, fn, 1) \wedge k = 1] \vee$$
$$\exists f, b[m = \{\widehat{\mathsf{inv}}\}(f, b) \wedge P(b, 0, 0) \wedge P(b, fn, 2) \wedge k = 2] \vee$$

$$\exists a, f[m = \{\hat{\boldsymbol{\jmath}}\}(a, f) \wedge P(a, 0, 0) \wedge \forall q(P(a, q, 2) \vee P(\{f\}(q), 0, 0)) \wedge \\ n = 0 \wedge k = 0] \vee$$
$$\exists a, f[m = \{\hat{\boldsymbol{\jmath}}\}(a, f) \wedge P(a, 0, 0) \wedge \forall q(P(a, q, 2) \vee P(\{f\}(q), 0, 0)) \wedge \\ n = \langle \{\hat{\mathsf{p}}_0\}(n), \{\hat{\mathsf{p}}_1\}(n)\rangle \wedge P(\{f\}(\{\hat{\mathsf{p}}_0\}(n)), \{\hat{\mathsf{p}}_1\}(n), 1) \wedge \\ P(a, \{\hat{\mathsf{p}}_0\}(n), 1) \wedge k = 1] \vee$$
$$\exists a, f[m = \{\hat{\boldsymbol{\jmath}}\}(a, f) \wedge P(a, 0, 0) \wedge \forall q(P(a, q, 2) \vee P(\{f\}(q), 0, 0)) \wedge \\ (n \neq \langle \{\hat{\mathsf{p}}_0\}(n), \{\hat{\mathsf{p}}_1\}(n)\rangle \vee P(\{f\}(\{\hat{\mathsf{p}}_0\}(n)), \{\hat{\mathsf{p}}_1\}(n), 2) \vee \\ P(a, \{\hat{\mathsf{p}}_0\}(n), 2)) \wedge k = 2] \vee$$

$$\exists a, b[m = \{\hat{\boldsymbol{\imath}}\}(a, b) \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge n = 0 \wedge k = 3] \vee$$
$$\exists a, b[m = \{\hat{\boldsymbol{\imath}}\}(a, b) \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge P(a, n, 1) \wedge k = 1 \wedge \\ \forall q(P(a, q, 2) \vee P(b, \langle q, n\rangle, 2) \vee P(\{\hat{\boldsymbol{\imath}}\}(a, b), q, 1))]$$

and $\mathcal{A}_1(P, m, n, k)$ is defined as:

$$(m = \langle 8, (m)_1\rangle \wedge P(m, 0, 3) \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \langle 8, (m)_1\rangle \wedge P(m, 0, 3) \wedge \neg P(m, n, 1) \wedge k = 2) \vee$$

$$(m = \langle 9, (m)_1\rangle \wedge \neg P(m, 0, 0) \wedge P((m)_1, 0, 0) \wedge n = 0 \wedge k = 0) \vee$$
$$(m = \langle 9, (m)_1\rangle \wedge \neg P(m, 0, 0) \wedge P((m)_1, 0, 0) \wedge P(n, 0, 0) \wedge k = 1) \vee$$
$$(m = \langle 9, (m)_1\rangle \wedge \neg P(m, 0, 0) \wedge P((m)_1, 0, 0) \wedge \neg P(n, 0, 0) \wedge k = 2)$$

Here we have to remark on the condition $\neg P(m,0,0)$ in $\mathcal{A}_1$. That is to build an universe we require that this universe has not yet been generated. Hence for each universe there is only one stage where it gets defined and where elements are put to its extension. Then the universe is completely built up and in later stages no more elements will be added to it.

At this point we want to focus on the definition of the inductively generated types in the model. All other types are generated in one step. This means we get the code of a type, its extension and its complement-extension at the same stage. This is completely different for inductive generation (as the name says). This is the reason why $k = 3$ occurs in the formula modeling inductive generation. We construct the extension of that type without having its code. But in $\mathcal{A}_1$ we can check whether the extension is built and then add the code of the type. Hence when we have the code of a type we know its extension (and its complement-extension). Therefore the following lemmas hold.

Lemma 32 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:*

$$P^{\alpha}_{\mathcal{A}}(m,0,0) \longrightarrow (\forall n)(\neg P^{\alpha}_{\mathcal{A}}(m,n,1) \leftrightarrow P^{\alpha}_{\mathcal{A}}(m,n,2))$$

This lemma is proved with induction on α as in the proof of lemma 22. But one can not proof the direction from left to right immediately. Both directions are shown in the induction step. One gets a new case for m:

$m = \{\widehat{\mathsf{lim}}\}(b)$: There is $\beta \le \alpha$ with $P^{\beta}_{\mathcal{A}}(m,0,0)$ and $\neg P^{<\beta}_{\mathcal{A}}(m,0,0)$.
From right to left: Let $P^{\alpha}_{\mathcal{A}}(m,x,1)$. By the definition of $\mathcal{A}_1$ we get $P^{\beta}_{\mathcal{A}}(m,x,1)$ and $P^{<\beta}_{\mathcal{A}}(x,0,0)$. This yields $\neg P^{\beta}_{\mathcal{A}}(m,x,2)$ and $\neg P^{\alpha}_{\mathcal{A}}(m,x,2)$.
From left to right: Let $\neg P^{\alpha}_{\mathcal{A}}(m,x,2)$, so $\neg P^{\beta}_{\mathcal{A}}(m,x,2)$. Consequently it is not $\neg P^{<\beta}_{\mathcal{A}}(x,0,0)$. We conclude $P^{<\beta}_{\mathcal{A}}(x,0,0)$ and $P^{\beta}_{\mathcal{A}}(m,x,1)$ and $P^{\alpha}_{\mathcal{A}}(m,x,1)$.

To complete the analogy we state the following corollary:

Corollary 33 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves:*

$$P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),0,0) \longrightarrow (\forall n)(\neg P^{\alpha}_{\mathcal{A}}(m,n,1) \leftrightarrow P^{\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{co}}\}(m),n,1))$$

The following lemma states, that if P is a closed under $\mathcal{A}_0$, then P is closed under elementary comprehension and join. This is exactly what we want the universes to be. So we can put with $\mathcal{A}_1$ all the types defined by the relation P closed under $\mathcal{A}_0$ as elements in a new type. This new type will code an universe of our construction.

Lemma 34 $\mathsf{FID}^{\mathsf{r}}(\mathbb{I})$ *proves the following formulas:*

- $\forall \vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \longrightarrow P(\widehat{\mathsf{nat}},0,0)$
- $\forall \vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \longrightarrow P(\widehat{\mathsf{id}},0,0)$

- $\forall a[\forall\vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \wedge P(a,0,0) \longrightarrow P(\{\widehat{\mathsf{co}}\}(a),0,0)]$
- $\forall a,b[\forall\vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \wedge P(a,0,0) \wedge P(b,0,0) \longrightarrow P(\{\widehat{\mathsf{int}}\}(a,b),0,0)]$
- $\forall a[\forall\vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \wedge P(a,0,0) \longrightarrow P(\{\widehat{\mathsf{dom}}\}(a),0,0)]$
- $\forall a,f[\forall\vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \wedge P(a,0,0) \longrightarrow P(\{\widehat{\mathsf{inv}}\}(f,a),0,0)]$
- $\forall a[\forall\vec{x}[\mathcal{A}_0(P,\vec{x}) \to P(\vec{x})] \wedge P(a,0,0) \wedge \forall q[P(a,q,2) \vee P(\{f\}(q),0,0)]$
  $\longrightarrow P(\{\hat{\jmath}\}(a,f),0,0)]$

Proof If we had in the conclusion $\mathcal{A}_0(P,\text{'type'},0,0)$, then every implication would hold by the definition of $\mathcal{A}_0$. Because P is supposed to be closed under $\mathcal{A}_0$ the lemma trivially holds. □

We extend the definition 26 for F^* by the following clause for the predicate U:

- $U(X)^* :\equiv \hat{X} = \langle 9, (\hat{X})_1\rangle \wedge \mathsf{R}(\hat{X})$

Now we can show that we have indeed a model for $\mathsf{T_0}$+U.

Theorem 35 $\mathsf{FID}(\mathbb{I})$ *proves* $\mathsf{R}(\hat{A}_1) \wedge \cdots \wedge \mathsf{R}(\hat{A}_n) \to F^*$ *for every axiom F of* $\mathsf{T_0}$+U *having its free type variables among* $A_1,\ldots,A_n$.

Proof If F is an axiom of $\mathsf{T_0}$, then the theorem holds by the same reasoning as in theorem 27. So we proof the theorem here only for the axioms for universes.

$U(A) \to \mathsf{nat} \in A$: Assume $U(A)^*$. Then we know $(\exists\alpha)(P^{\alpha}_{\mathcal{A}}(\hat{A},0,0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(\hat{A},0,0))$. Hence $P^{<\alpha}_{\mathcal{A}}$ is closed under $\mathcal{A}_0$. So by lemma 34 we get $P^{<\alpha}_{\mathcal{A}}(\widehat{\mathsf{nat}},0,0)$ and consequently $P^{\alpha}_{\mathcal{A}}(\hat{A},\widehat{\mathsf{nat}},1)$.

$U(A) \wedge x \in A \to \Re(x)$: This follows immediately from the definition of $\mathcal{A}_1$.

$U(A) \wedge U(B) \to [A \mathrel{\dot\in} B \vee A = B \vee B \mathrel{\dot\in} A]$: Assume $(U(A) \wedge U(B))^*$, then there exist α,β with $P^{\alpha}_{\mathcal{A}}(\hat{A},0,0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(\hat{A},0,0)$ and $P^{\beta}_{\mathcal{A}}(\hat{B},0,0) \wedge \neg P^{<\beta}_{\mathcal{A}}(\hat{B},0,0)$. Now we have three cases:

1. If $\alpha < \beta$ holds, then we have $P^{<\beta}_{\mathcal{A}}(\hat{A},0,0)$ and therefore $P^{\beta}_{\mathcal{A}}(\hat{B},\hat{A},1)$. Hence $(A \mathrel{\dot\in} B)^*$.
2. If $\beta < \alpha$ we get by a symmetrical argument $P^{\alpha}_{\mathcal{A}}(\hat{A},\hat{B},1)$.
3. Let $\alpha = \beta$. For all x we have $P^{\alpha}_{\mathcal{A}}(\hat{A},x,1) \Rightarrow P^{<\alpha}_{\mathcal{A}}(x,0,0) \Rightarrow P^{\alpha}_{\mathcal{A}}(\hat{B},x,1)$. $\neg P^{\alpha}_{\mathcal{A}}(\hat{A},x,1) \Rightarrow \neg P^{<\alpha}_{\mathcal{A}}(x,0,0) \Rightarrow \neg P^{\alpha}_{\mathcal{A}}(\hat{B},x,1)$. Hence by extensionality for types we get $(A = B)^*$.

$U(A) \wedge U(B) \wedge A \mathrel{\dot{\in}} B \to \forall x(x \in A \to x \in B)$: Let $(U(A) \wedge U(B) \wedge A \mathrel{\dot{\in}} B)^*$ hold. Then there exist α, β with $P^\alpha_{\mathcal{A}}(\hat{A}, 0, 0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(\hat{A}, 0, 0)$ and $P^\beta_{\mathcal{A}}(\hat{B}, 0, 0) \wedge \neg P^{<\beta}_{\mathcal{A}}(\hat{B}, 0, 0)$. Now we look at $\exists x(\Re(x, A) \wedge x \in B)^*$. As a consequence of $U(A)^*$ we obtain $\hat{x} = \langle 9, (\hat{x})_1 \rangle$ and because of $(x \in B)^*$ we have $\alpha < \beta$. Let $(x \in A)^*$, hence $P^{<\alpha}_{\mathcal{A}}(\hat{x}, 0, 0)$ and consequently $P^{<\beta}_{\mathcal{A}}(\hat{x}, 0, 0)$. This yields $(x \in B)^*$.

Closure axioms: Assume $U(A)^*$ and let α be such that $P^\alpha_{\mathcal{A}}(\hat{A}, 0, 0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(\hat{A}, 0, 0)$. Then $(x \in A)^*$ if and only if $P^{<\alpha}_{\mathcal{A}}(\hat{x}, 0, 0)$ and $P^{<\alpha}_{\mathcal{A}}$ is closed under $\mathcal{A}_0$ (by the definition of $\mathcal{A}$). With lemma 34 the closure axioms are satisfied.

Limes axiom: We have $\Re(a)^*$. Hence $\exists \alpha P^\alpha_{\mathcal{A}}(\hat{a}, 0, 0)$. Then there is a $\beta \geq \alpha$ such that $P^\beta_{\mathcal{A}}$ is closed under $\mathcal{A}_0$. Then we have $\mathcal{A}_1(P^\beta_{\mathcal{A}}, \{\widehat{\mathsf{lim}}\}(\hat{a}), 0, 0)$ and finally $P_{\mathcal{A}}(\{\widehat{\mathsf{lim}}\}(\hat{a}), 0, 0)$.
$\{\widehat{\mathsf{lim}}\}(\hat{a})$ codes a type that is an universe because of its definition in $\mathcal{A}_1$.
Again let α be such that $P^\alpha_{\mathcal{A}}(\{\widehat{\mathsf{lim}}\}(\hat{a}), 0, 0) \wedge \neg P^{<\alpha}_{\mathcal{A}}(\{\widehat{\mathsf{lim}}\}(\hat{a}), 0, 0)$ holds. From $P^\alpha_{\mathcal{A}}(\{\widehat{\mathsf{lim}}\}(\hat{a}), 0, 0)$ we infer $P^{<\alpha}_{\mathcal{A}}(\hat{a}, 0, 0)$ and thus $P^\alpha_{\mathcal{A}}(\{\widehat{\mathsf{lim}}\}(\hat{a}), \hat{a}, 1)$. $\square$

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