Bar induction and ω model reflection

Gerhard Jäger* Thomas Strahm*

Abstract

We show that the principle of ω model reflection for Π^1_{n+1} formulas is equivalent over ACA_0 to the scheme of Π^1_n bar induction. This extends and refines previous results of Friedman and Simpson.

1 Introduction

The scheme of ω model reflection in second order arithmetic has been introduced in Friedman [1] and basically states that for every true formula A of second order arithmetic, possibly with parameters, there exists a countable ω model of the theory ACA_0 containing these parameters so that A is true in this model. In [1] it is also announced that the principle of ω model reflection is equivalent to the principle (BI) of bar induction. A detailed proof of this important result is given in Simpson [4].

Friedman's result provides the equivalence of the full scheme of ω model reflection and the full scheme of bar induction. It is of course a very natural question to ask how much ω model reflection corresponds to how much of bar induction. Building upon work of Friedman and Howard, Simpson [3] provides, among other results, the equivalence of ω model reflection restricted to Π^1_2 formulas and bar induction for Π^1_1 formulas.

In this paper we will establish the fine structure of the equivalence between ω model reflection and bar induction and show that for all $n \geq 1$, ω model reflection for Π^1_{n+1} formulas is equivalent (over ACA_0) to bar induction for Π^1_n formulas.

It is more or less obvious that $(\Pi^1_{n+1}\text{-}\mathsf{RFN})$ entails $(\Pi^1_n\text{-}\mathsf{BI})$. For establishing the converse direction we make use of Schütte's notion of deduction chain adapted for ω logic and a careful analysis of the logical complexities involved.

2 The theories $\Pi^1_n\text{-}\mathsf{BI}_0$ and $\Pi^1_{n+1}\text{-}\mathsf{RFN}_0$

It is the purpose of this section to introduce the theories $\Pi^1_n\text{-}\mathsf{BI}_0$ and $\Pi^1_{n+1}\text{-}\mathsf{RFN}_0$ for each natural number $n \geq 1$. We will immediately notice that $\Pi^1_n\text{-}\mathsf{BI}_0$ is contained in $\Pi^1_{n+1}\text{-}\mathsf{RFN}_0$ in a straightforward manner.

*Research supported by the Swiss National Science Foundation.

For technical reasons which will become clear in the next section of this paper, we choose a Tait-style formulation of the language $\mathcal{L}_2$ of second order arithmetic. In addition, it is convenient for our purpose to distinguish free and bound variables. More precisely, $\mathcal{L}_2$ includes *free number variables* $a, b, c, u, v, w, \ldots$ and *bound number variables* $x, y, z, \ldots$ as well as *free set variables* $U, V, V, \ldots$ and *bound set variables* $X, Y, Z, \ldots$. Further, $\mathcal{L}_2$ comprises the usual function and relation symbols $+, \cdot, 0, 1, =, <$ as well as the symbol $\in$ for elementhood between numbers and sets. In addition, there is a symbol $\sim$ for forming negative literals.

The *number terms* $r, s, t, \ldots$ of $\mathcal{L}_2$ are defined as usual; the *set terms* are just the set variables. Positive literals of $\mathcal{L}_2$ are all expressions $R(s_1, \ldots, s_n)$ and $(s \in U)$ for R a symbol for an n-ary relation symbol. The negative literals of $\mathcal{L}_2$ have the form $\sim E$ so that E a positive literal. We often write $(s \neq t)$ and $(s \notin U)$ instead of $\sim(s = t)$ and $\sim(s \in U)$, respectively. The formulas $A, B, C, \ldots$ of $\mathcal{L}_2$ are now generated from the positive and negative literals of $\mathcal{L}_2$ by closing under disjunction, conjunction as well as existential and universal number and set quantification. The negation $\neg A$ of an $\mathcal{L}_2$ formula A is defined by making use of De Morgan's laws and the law of double negation. Moreover, the remaining logical connectives are abbreviated as usual.

An $\mathcal{L}_2$ formula is called *arithmetic* if it does not contain bound set variables (but possibly free set variables); we write Π^1_0 for the collection of these formulas. Moreover, an $\mathcal{L}_2$ formula is called *proper* Π^1_n if it has the form

$$(\forall X_1)(\exists X_2) \ldots (Q_n X_n) A$$

with A arithmetic. The class of Π^1_n formulas consists of the proper Π^1_n formulas, and for all k less than n the proper Π^1_k formulas and their negations. A formula is called Σ^1_n if its negation is a Π^1_n formula. Obviously, every Π^1_n formula is logically equivalent a proper Π^1_n formula and similarly for Σ^1_n formulas.

We presuppose standard notation for coding sequences of natural numbers: $\langle \ldots \rangle$ is a primitive recursive function for forming n tuples $\langle t_0, \ldots, t_{n-1} \rangle$; *Seq* denotes the primitive recursive set of sequence numbers; $lh(t)$ gives the length of the sequence coded by t, i.e. if $t = \langle t_0, \ldots, t_{n-1} \rangle$ then $lh(t) = n$; $(t)_i$ denotes the ith component of the sequence coded by t if $i < lh(t)$.

In the following we make use of the usual way of coding an infinite sequence of sets of natural numbers into a single one by writing $s \in (U)_t$ instead of $\langle s, t \rangle \in U$. Accordingly, we have for each $\mathcal{L}_2$ formula A its relativization to the set U, denoted by A^U, which is obtained from A by replacing all quantifiers $(\forall X)(\ldots X \ldots)$ and $(\exists X)(\ldots X \ldots)$ in A by $(\forall x)(\ldots (U)_x \ldots)$ and $(\exists x)(\ldots (U)_x \ldots)$, respectively. Note that A^U is always arithmetic. Finally, elementhood $U \dot{\in} V$ between sets has to be read in the obvious way as $(\exists x)(U = (V)_x)$.

A set U of natural numbers can be regarded as a binary relation by stipulating $s\ U\ t$ for $\langle s, t \rangle \in U$. In the sequel we let $\mathsf{LO}(U)$ denote the usual arithmetic formula which expresses that the binary relation U is a linear ordering. Furthermore, we use the following standard notions:

$$\begin{aligned}
\mathsf{PROG}(U, A) &:= (\forall x)[(\forall y)(y\ U\ x \to A(y)) \;\to\; A(x)], \\
\mathsf{TI}(U, A) &:= \mathsf{PROG}(U, A) \to (\forall x)A(x), \\
\mathsf{WF}(U) &:= (\forall X)\mathsf{TI}(U, X), \\
\mathsf{WO}(U) &:= \mathsf{LO}(U) \wedge \mathsf{WF}(U).
\end{aligned}$$

The stage is now set in order to introduce the theories Π^1_n-BI_0 and Π^1_{n+1}-RFN_0 for each natural number $n \geq 1$. All these theories comprise the standard system ACA_0 of second order arithmetic which includes comprehension for arithmetic formulas and induction on the natural numbers for sets. It is well-known that ACA_0 is finitely axiomatizable by a Π^1_2 sentence, say $\mathsf{F}_{\mathsf{ACA}}$, cf. [4].

The schema of Π^1_n bar induction comprises for each Π^1_n formula A the statement

$$(\Pi^1_n\text{-}\mathsf{BI}) \qquad\qquad (\forall X)[\mathsf{WO}(X) \;\to\; \mathsf{TI}(X, A)].$$

The theory Π^1_n-BI_0 extends ACA_0 by each instance of (Π^1_n-BI). Observe that induction on the natural numbers is available in ACA_0 + (Π^1_n-BI) (at least) for Π^1_n (and hence also for Σ^1_n) formulas.

Let us now turn to the schema of Π^1_{n+1} ω model reflection. Given a Π^1_{n+1} formula $A(U)$ which contains at most the set variable U free and no free number variables, this reflection principle guarantees the existence of a countable coded ω model of ACA_0 which contains U and models A. More formally, (Π^1_{n+1}-RFN) denotes the schema

$$(\Pi^1_{n+1}\text{-}\mathsf{RFN}) \qquad\qquad A(U) \;\to\; (\exists X)[U \,\dot{\in}\, X \;\wedge\; \mathsf{F}^X_{\mathsf{ACA}} \;\wedge\; A^X(U)]$$

for each Π^1_{n+1} formula $A(U)$ with at most U free. Accordingly, Π^1_{n+1}-RFN_0 denotes ACA_0 augmented with each instance of (Π^1_{n+1}-RFN). It is obvious that Π^1_{n+1} ω model reflection for Π^1_{n+1} formulas containing arbitrary set and number parameters follows from (Π^1_{n+1}-RFN) as stated above. This fact is stated in the following lemma.

Lemma 1 *Let $A(U_1, \ldots, U_m)$ be a Π^1_{n+1} formula with at most the indicated set parameters and possibly additional number parameters. Then Π^1_{n+1}-RFN_0 proves*

$$A(U_1, \ldots, U_m) \;\to\; (\exists X)[U_1 \,\dot{\in}\, X \;\wedge \cdots \wedge\; U_m \,\dot{\in}\, X \;\wedge\; \mathsf{F}^X_{\mathsf{ACA}} \;\wedge\; A^X(U_1, \ldots, U_m)].$$

In this paper we aim at showing that (Π^1_n-BI) and (Π^1_{n+1}-RFN) are equivalent over ACA_0. One direction of this equivalence is immediate, namely that (Π^1_n-BI) follows from (Π^1_{n+1}-RFN).

Theorem 2 *We have that Π^1_{n+1}-RFN_0 proves each instance of (Π^1_n-BI).*

Proof. We work informally in Π^1_{n+1}-RFN_0 and assume that U is a linear ordering such that we have a failure of $\mathsf{TI}(U, A)$ for a specific Π^1_n formula A, i.e. there is an u so that

$$(\forall x)[(\forall y)(y \; U \; x \to A(y)) \;\to\; A(x)] \;\land\; \neg A(u). \tag{1}$$

Obviously (1) is equivalent to a Π^1_{n+1} statement and, hence, by (Π^1_{n+1}-RFN) there exists a countable coded ω model V of ACA_0 such that

$$(\forall x)[(\forall y)(y \; U \; x \to A^V(y)) \;\to\; A^V(x)] \;\land\; \neg A^V(u). \tag{2}$$

By arithmetic comprehension let $W = \{x : A^V(x)\}$. Then we have by (2) that $\mathsf{PROG}(U, W)$ and $u \notin W$. Hence, U is not a wellordering. This completes our argument. $\Box$

The rest of this paper is concerned with the proof of the reverse direction of the above theorem, i.e. that Π^1_n-BI_0 proves each instance of (Π^1_{n+1}-RFN).

3 Deduction chains in ω logic

In this section we deal with the construction of an ω model which reflects a specific Π^1_{n+1} formula. More precisely, given a Π^1_{n+1} formula $A(U)$ with set parameter U and a set $I \subset \mathbb{N}$ we first introduce an adaptation of Schütte deduction chains (cf. [2]) for ω logic depending on $A(U)$ and I. Then we show that there exists an ω model of ACA_0 which contains I and reflects $A(U)$ provided that the tree of all deduction chains for I and $A(U)$ is not wellfounded.

We have to introduce some notation. Terms and formulas of $\mathcal{L}_2$ are called *constant* if they do not contain free number variables. Then each constant $\mathcal{L}_2$ term t has a uniquely determined value $t^{\mathbb{N}}$ in the standard model $\mathbb{N}$. Moreover, each constant literal which is not of the form $t \in U$ or $t \notin U$ is either true or false in $\mathbb{N}$.

For introducing deduction chains we work with finite sequences $\Gamma, \Delta, \Phi, \Psi, \dots$ of constant $\mathcal{L}_2$ formulas rather than individual $\mathcal{L}_2$ formulas. Such a finite sequence Γ is said to be *axiomatic* if it contains a true literal or two formulas $s \notin U$ and $t \in U$ so that $s^{\mathbb{N}}$ is equal to $t^{\mathbb{N}}$. The *reducible* sequences are those sequences which are not axiomatic and contain at least one formula which is not a literal. The *reducible part* of a reducible sequence Γ is the rightmost formula in Γ which is not a literal.

The next step is to choose a subset I of the natural numbers and a constant Π^1_{n+1} formula $(\forall X)B(X, U)$ which contains U as it only free set variable. Furthermore, let

$$U_0, U_1, U_2, \dots$$

be an arbitrary but fixed enumeration of the free set variables of $\mathcal{L}_2$. In addition, we write the constant Π^1_2 formula $\mathsf{F}_{\mathsf{ACA}}$ as $(\forall X)C(X)$. Finally, $D_k[I]$ is defined to be the formula $(k \in U_0)$ provided that k belongs to I and $(k \notin U_0)$ otherwise.

Depending on a set of natural numbers I, a Π^1_{n+1} formula $(\forall X)B(X,U)$ and all the choices made above we now introduce the notion of deduction chain. Our approach is very similar to the one in Schütte [2]. However, it takes into account that we interpret the number quantifiers in the sense of ω logic as ranging over $\mathbb{N}$.

Definition 3 A *deduction chain* for a set of natural numbers I and a constant Π^1_{n+1} formula $(\forall X)B(X,U)$ is a finite sequence

$$\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$$

of finite sequences of constant $\mathcal{L}_2$ formulas which is constructed as follows.

1. Γ_0 is the sequence $\neg B(U_0, U_0),\ \neg C(U_0),\ \neg D_0[I]$.

2. For all natural numbers i less than k: Γ_i is not axiomatic.

3. For all natural numbers i less than k: If Γ_i is not reducible, then Γ_{i+1} is the sequence
$$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i.$$

4. For all natural numbers i less than k: If Γ_i is reducible with distinguished part E, then Γ_i can be written in the form
$$\Gamma_i',\ E,\ \Gamma_i''$$
so that Γ_i'' contains only literals. Then Γ_{i+1} is determined as follows:

   (a) If E is the formula $E_0 \vee E_1$, then Γ_{i+1} is the sequence
   $$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ E_0,\ E_1,\ \Gamma_i''.$$

   (b) If E is the formula $E_0 \wedge E_1$, then Γ_{i+1} is either the sequence
   $$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ E_0,\ \Gamma_i''$$
   or the sequence
   $$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ E_1,\ \Gamma_i''.$$

   (c) If E is the formula $(\exists x)F(x)$, then Γ_{i+1} is the sequence
   $$(\exists x)F(x),\ \neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ F(m),\ \Gamma_i''$$
   for m being is the least natural number so that $F(m)$ does not occur in the sequences $\Gamma_0, \ldots, \Gamma_i$.

   (d) If E is the formula $(\forall x)F(x)$, then Γ_{i+1} is the sequence
   $$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ F(m),\ \Gamma_i''$$
   for some natural number m.

(e) If E is the formula $(\exists X)F(X)$, then Γ_{i+1} is the sequence

$$(\exists X)F(X),\ \neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ F(U_m),\ \Gamma_i''$$

for m being the least natural number so that $F(U_m)$ does not occur in the sequences $\Gamma_0, \ldots, \Gamma_i$.

(f) If E is the formula $(\forall X)F(X)$, then Γ_{i+1} is the sequence

$$\neg B(U_{i+1}, U_0),\ \neg C(U_{i+1}),\ \neg D_{i+1}[I],\ \Gamma_i',\ F(U_m),\ \Gamma_i''$$

for m being the least natural number so that U_m does not occur in Γ_i.

It is obvious that the set of all deduction chains for I and $(\forall X)B(X, U)$ forms an ω branching tree. We call it the deduction tree for I and $(\forall X)B(X, U)$ and denote it by $\mathbb{DT}[I, (\forall X)B(X, U)]$. The following lemma (the proof of which is obvious) will be crucial for the next section.

Lemma 4 *Let I be a set of natural numbers and $(\forall X)B(X, U)$ a constant Π^1_{n+1} formula. Then all formulas in the deduction tree $\mathbb{DT}[I, (\forall X)B(X, U)]$ are constant Π^1_n formulas.*

In our further considerations we will distinguish between wellfounded deduction trees and non-wellfounded ones. The case of wellfounded deduction trees will be postponed to the next section.

Until the end of this section we assume that I is a set of natural numbers and $(\forall X)B(X, U)$ is a constant Π^1_{n+1} formula which contains at most the set variable U free so that $\mathbb{DT}[I, (\forall X)B(X, U)]$ is *not* wellfounded. Then there exists an infinite sequence

$$\Gamma_0, \Gamma_1, \Gamma_2, \ldots$$

so that each initial segment $\Gamma_0, \ldots, \Gamma_k$ of this infinite sequence belongs to the deduction tree $\mathbb{DT}[I, (\forall X)B(X, U)]$. Take Γ^* to be the set of all formulas which occur in some Γ_k. The following observations are easily verified:

(1) Γ^* does not contain literals which are true in $\mathbb{N}$.

(2) Γ^* does not contain formulas $s \notin U_i$ and $t \in U_i$ for constant terms s and t so that $s^{\mathbb{N}}$ is equal to $t^{\mathbb{N}}$.

(3) If Γ^* contains $E_0 \vee E_1$ then Γ^* contains E_0 and E_1.

(4) If Γ^* contains $E_0 \wedge E_1$ then Γ^* contains E_0 or E_1.

(5) If Γ^* contains $(\exists x)F(x)$ then Γ^* contains $F(n)$ for all natural numbers n.

(6) If Γ^* contains $(\forall x)F(x)$ then Γ^* contains $F(n)$ for some natural number n.

(7) If Γ^* contains $(\exists X)F(X)$ then Γ^* contains $F(U_m)$ for all natural numbers m.

(8) If Γ^* contains $(\forall X)F(X)$ then Γ^* contains $F(U_m)$ for some natural number m.

(9) Γ^* contains the formulas $\neg B(U_m, U_0)$ for all natural numbers m.

(10) Γ^* contains the formulas $\neg C(U_m)$ for all natural numbers m.

(11) Γ^* contains the formulas $\neg D_m[I]$ for all natural numbers m.

Based on Γ^* we now assign a subset $\mathsf{val}(U_m)$ of the natural numbers to all free set variables U_m:

$$\mathsf{val}(U_m) \ := \ \{ t^{\mathbb{N}} : t \text{ is a constant } \mathcal{L}_2 \text{ term and } (t \notin U_m) \text{ belongs to } \Gamma^* \}$$

From property (11) and definition of the formulas $D_k[I]$ we can immediately conclude that $\mathsf{val}(U_0)$ is our given set I. Furthermore, let $\mathbb{M}$ be the disjoint union of the values $\mathsf{val}(U_m)$ of all free set variables, i.e.

$$\mathbb{M} \ := \ \{\langle i, m\rangle : m \in \mathbb{N} \text{ and } i \in \mathsf{val}(U_m)\}.$$

In view of the properties (1)–(8) a simple induction on the length of the constant formula F yields that

$$(\star) \qquad\qquad F \in \Gamma^* \qquad \Longrightarrow \qquad \mathbb{N} \not\models F^{\mathbb{M}}$$

This implies, in particular, because of properties (9) and (10) that the set $\mathbb{M}$ validates arithmetic comprehension and our given Π^1_{n+1} formula in the sense that

$$\mathbb{N} \models (\forall X)B(X, I)^{\mathbb{M}} \qquad \text{and} \qquad \mathbb{N} \models (\forall X)C(X)^{\mathbb{M}}.$$

In other words, starting off from the assumption that $\mathbb{DT}[I, (\forall X)B(X, U)]$ is not wellfounded we could produce an ω model of ACA_0 which contains our initial set I and reflects our given Π^1_{n+1} formula.

Careful analysis of the previous arguments shows that they can be formalized in comparatively weak theories, for example in Π^1_1-BI_0. After standard coding of the syntax of $\mathcal{L}_2$, the notion of deduction tree is obviously arithmetic in the given set I. Hence, depending on the Gödelnumber g of the formula $(\forall X)B(X, U)$ we can find an arithmetic $\mathcal{L}_2$ formula $\mathcal{DT}_g(V, w)$ so that $\mathcal{DT}_g(I, s)$ expresses that s is a sequence number coding a finite sequence of constant $\mathcal{L}_2$ formulas which belongs to the deduction tree $\mathbb{DT}[I, (\forall X)B(X, U)]$.

The only critical part of the formalization of the preceding argument is the proof of the formalized version of $(\star)$. However, by making use of a Δ^1_1 truth definition of the arithmetic formulas one can show by Π^1_1 induction on the natural numbers that the arithmetic formulas $F^{\mathbb{M}}$ are false for all F in Γ^*. Because of the property of such truth definitions as stated in Lemma 8 below we obtain $(\forall X)B(X, I)^{\mathbb{M}}$ and $(\forall X)C(X)^{\mathbb{M}}$ from the truth of these two formulas. Therefore, we have the following theorem.

Theorem 5 *Let $A(U)$ be a Π^1_{n+1} formula which contains at most the set variable U free and assume that g is its Gödelnumber. Then Π^1_1-BI_0 proves*

$$(\exists f)(\forall x)\mathcal{DT}_g(V, \bar{f}(x)) \;\rightarrow\; (\exists X)[V \,\dot{\in}\, X \;\wedge\; \mathsf{F}^X_{\mathsf{ACA}} \;\wedge\; A^X(V)].$$[1]

4 Wellfounded deduction trees

The previous theorem corresponds more or less to Schütte's principal semantic lemma. In this section we are mainly concerned with an analogue of the so-called principal syntactic lemma, which deals in our present situation with wellfounded deduction trees.

In the sequel it will be convenient to have a more general class of Π^1_n formulas, the so-called *extended Π^1_n formulas* or $\mathsf{E}\Pi^1_n$ formulas.

Definition 6 The class of $\mathcal{L}_2$ formulas $\mathsf{E}\Pi^1_n$ is defined by induction on n as follows.

1. $\mathsf{E}\Pi^1_0$ coincides with the class of arithmetic formulas.

2. $\mathsf{E}\Pi^1_{n+1}$ is the least class of formulas which contains the $\mathsf{E}\Pi^1_n$ formulas and their negations and is closed under disjunction, conjunction, bounded numerical quantification as well as universal number and universal set quantification.

The following lemma will be needed below. It readily entails that Π^1_n-BI_0 proves bar induction also for $\mathsf{E}\Pi^1_n$ formulas.

Lemma 7 *For each $\mathsf{E}\Pi^1_n$ formula A there exists a Π^1_n formula B with the same free variables as A such that Π^1_n-BI_0 proves $A \leftrightarrow B$.*

Proof. This lemma is proved by induction on n. The proof is routine except for the fact that one uses Σ^1_n induction on the natural numbers in order to be able to exchange a bounded universal number quantifier with an existential set quantifier. But (Π^1_n-BI) entails Π^1_n induction on the natural numbers and, hence, also Σ^1_n induction is available in Π^1_n-BI_0. $\Box$

For the following arguments we make use of the fact that for all $n \geq 1$ there exists a standard truth definition $\mathsf{Tr}_n(u, V, W)$ for all constant Π^1_n formulas which itself is an $\mathsf{E}\Pi^1_n$ formula. Informally, $\mathsf{Tr}_n(u, V, W)$ is interpreted as:

u is the Gödelnumber of a constant Π^1_n formula which is true in the standard model provided that the set variable U_0 is interpreted by V and the set variable U_i for $i > 0$ is interpreted by $(W)_{i-1}$.

We omit mentioning all the standard properties of this truth definition and confine ourselves to the following lemma.

[1] As usual $f, g, \ldots$ range over functions and $\bar{f}$ denotes the course of value of the function f, i.e. $\bar{f}(u) = \langle f(0), \ldots, f(u-1)\rangle$.

Lemma 8 *1. Let $A(U_0, \ldots, U_k)$ be a constant Π^1_n formula which contains at most the set variables $U_0, \ldots, U_k$ free and assume that g is its Gödelnumber. Then Π^1_1-BI_0 proves that*

$$A(V, (W)_0, \ldots, (W)_{k-1}) \;\leftrightarrow\; \mathsf{Tr}_n(g, V, W).$$

2. Let $(\forall X)B(X, U)$ be a Π^1_{n+1} formula which contains at most the set variable U free and assume that β is the primitive recursive function which assigns the Gödelnumber of the formula $\neg B(U_k, U_0)$ to each natural number k. Then Π^1_1-BI_0 proves

$$(\forall X)B(X, V) \;\leftrightarrow\; (\forall X)\neg\mathsf{Tr}_n(\beta(u), V, X).$$

The second assertion of this lemma is an immediate consequence of the first. Its proof uses a standard argument and is left to the reader.

Theorem 9 *Let $(\forall X)B(X, U)$ be a constant Π^1_{n+1} formula which contains at most the set variable U free and assume that g is its Gödelnumber. Further, let $\mathsf{Ass}(V)$ denote the formula*

$$(\forall X)B(X, V) \;\wedge\; (\forall f)(\exists x)\neg\mathcal{DT}_g(V, \bar{f}(x)).$$

Then Π^1_n-BI_0 proves

$$\mathsf{Ass}(V) \;\wedge\; \mathcal{DT}_g(V, s) \;\rightarrow\; (\forall Z)(\exists w < lh(s))\mathsf{Tr}_n((s)_w, V, Z).$$

Proof. In the following we work informally in Π^1_n-BI_0 and assume

$$(\forall X)B(X, V) \;\wedge\; (\forall f)(\exists x)\neg\mathcal{DT}_g(V, \bar{f}(x)). \tag{1}$$

Hence, we know that the deduction tree T for V and $(\forall X)B(X, U)$,

$$T = \{u : \mathcal{DT}_g(V, u)\} \tag{2}$$

is a *wellfounded* tree. Consequently, by reasoning in ACA_0 only (cf. Simpson [4]), the canonical *Kleene-Brouwer ordering* $\mathsf{KB}(T)$ *of* T is a wellordering. Now let $E(u)$ denote the $\mathcal{L}_2$ formula

$$E(u) \;:=\; \mathcal{DT}_g(V, u) \;\rightarrow\; (\forall Z)(\exists w < lh(u))\mathsf{Tr}_n((u)_w, V, Z). \tag{3}$$

Clearly, $E(u)$ is an $\mathsf{E}\Pi^1_n$ formula and, therefore, transfinite induction is available with respect to $E(u)$ along $\mathsf{KB}(T)$ in Π^1_n-BI_0 by Lemma 7. The verification of

$$\mathsf{PROG}(\mathsf{KB}(T), E) \tag{4}$$

is now a straightforward matter and follows essentially from the construction of T. One only has to recall that all formulas occurring in T are Π^1_n formulas (cf. Lemma 4) so that Tr_n is the adequate truth predicate for these. Moreover, in each step of the proof of (4) one makes use of our assumption $(\forall X)B(X, V)$ which by Lemma 8 entails that the formulas $\neg B(U_k, U_0)$ are false, and a similar argument is used for the formulas $\neg C(U_k)$ and $\neg D_k[V]$. Hence, (4) holds and an application of (Π^1_n-BI) yields $(\forall x)E(x)$ as claimed. This finishes our argument. $\Box$

Corollary 10 Let $(\forall X)B(X,U)$ be a constant Π^1_{n+1} formula which contains at most the set variable U free and assume that g is its Gödelnumber. Then Π^1_n-BI_0 proves

$$(\forall X)B(X,V) \;\rightarrow\; (\exists f)(\forall x)\mathcal{DT}_g(V,\bar{f}(x)).$$

Proof. Again work informally in Π^1_n-BI_0 and assume $(\forall X)B(X,V)$. If the deduction tree for V and $(\forall X)B(X,U)$ were wellfounded, then an application of our theorem to the (code of the) root

$$\Gamma_0 \;=\; \neg B(U_0,U_0),\; \neg C(U_0),\; \neg D_0[V]$$

of this tree yields an immediate contradiction since all the formulas in Γ_0 are false when U_0 is interpreted by V. $\square$

Combining this corollary with Theorem 5 gives us that indeed Π^1_n-BI_0 proves each instance of $(\Pi^1_{n+1}$-$\mathsf{RFN})$.

Corollary 11 *Let $A(U)$ be a Π^1_{n+1} formula which contains at most the set variable U free. Then Π^1_n-BI_0 proves*

$$A(V) \;\rightarrow\; (\exists X)[V \mathrel{\dot{\in}} X \;\wedge\; \mathsf{F}^X_{\mathsf{ACA}} \;\wedge\; A^X(V)].$$

Together with Theorem 2 this entails the main result of this article.

Main Theorem *We have that the theories Π^1_n-BI_0 and Π^1_{n+1}-RFN_0 are equivalent.*

We want to emphasize again that the theories Π^1_n-BI_0 and Π^1_{n+1}-RFN_0 have been defined for $n \geq 1$ only, and the above equivalence is only true for such n: the corresponding theory Π^1_0-BI_0 is the theory ACA_0 whereas the crucial principle entailed by Π^1_1-RFN_0 is the fact that each set is contained in an ω model of ACA_0; the proof-theoretic ordinal of Π^1_1-RFN_0 should be $\varphi 20$.

References

[1] Friedman, H. Some systems of second order arithmetic and their use. In *Proceedings of the International Congress of Mathematicians, Vancouver 1974* (1975), vol. 1, Canadian Mathematical Congress, pp. 235–242.

[2] Schütte, K. *Proof Theory.* Springer, Berlin, 1977.

[3] Simpson, S. G. Σ^1_1 and Π^1_1 transfinite induction. In *Logic Colloquium '80*, D. van Dalen, D. Lascar, and J. Smiley, Eds. North Holland, Amsterdam, 1982, pp. 239–253.

[4] Simpson, S. G. *Subsystems of Second Order Arithmetic.* Perspectives in Mathematical Logic. Springer-Verlag, 1998. in preparation.

Address

Institut für Informatik und angewandte Mathematik, Universität Bern, Neubrückstrasse 10, CH-3012 Bern, Switzerland, {jaeger,strahm}@iam.unibe.ch

October 19, 1998