Dependent Choice in Explicit Mathematics

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0 Introduction

In this thesis we'll present an axiom (dc) for dependent choice in explicit mathematics, and we'll give a proof-theoretical analysis of the resulting theory. The axiom we treat here was proposed by Jäger and enables us to embed the subsystem of analysis $(\Pi^1_0\text{-}\mathsf{CA})_{<\omega^\omega}\!\!\restriction$ into our theory in much the same way as it is embedded into the system $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$ in Cantini [3]. Hence we get $\varphi\omega 0$ as a lower bound. An upper bound for our theory with dependent choice is established by formalizing models within the theory $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$, introduced in Jäger [10] . This yields that the bound $\varphi\omega 0$ is sharp.

The fragment of explicit mathematics we start with is the theory $\mathsf{EETJ} + (\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$, i.e. we have just the usual axioms for elementary comprehension, an axiom for the constant j that allows us to build the disjoint union of an infinite family of types, and an induction principle for types. This theory corresponds to the subsystem of analysis $(\Sigma^1_1\text{-}\mathsf{AC})\!\restriction$. As $\mathsf{EETJ}+(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$, it lacks the possibility of forming hierarchies, i.e we can't form infinite sequence of sets, where each set is defined referring explicitly to its predecessors. In the theory $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$ this possibility is given by the axiom DC, stating, that given an arithmetic formula F such that $(\forall X)(\exists Y)F(X,Y)$ holds, there is a set (a hierarchy) Z, satisfying $(Z)_0 = X$ and $(\forall x)F([(Z)_x,(Z)_{x+1}]$. In explicit mathematics this is realized by introducing a new constant (dc) by the following axioms:

Dependent choice (dc).

$$
\begin{array}{ll}
(\mathsf{dc.1}) & \Re(a) \wedge (f : \Re \to \Re) \to (\mathsf{dc}(a,f) : \mathsf{N} \to \Re),\\
(\mathsf{dc.2}) & \Re(a) \wedge (f : \Re \to \Re) \to\\
& \mathsf{dc}(a,f)(\mathsf{0}) = a \wedge (\forall n \in \mathsf{N})[\mathsf{dc}(a,f)(\mathsf{s}_N n) = f(\mathsf{dc}(a,f)(n))].
\end{array}
$$

The strength of (dc) is due to the totality assertion in $(\mathsf{dc.1})$. Together with join this enables us to build hierarchies: If F is an elementary formula, let $Hier_F(\alpha,z)$ formalize the statement 'a type X named z is a hierarchy w.r.t. F up to α'. Then elementary comprehension allows us to define a term f_0 such that $Hier_F(\alpha,z)$ implies $Hier_F(\alpha+1,\mathsf{f}_0 z)$. Now (dc) yields $Hier_F(\alpha+n,\mathsf{dc}(\mathsf{f}_0,z)(n))$. Applying join, we can define a term f_1 such that $\mathsf{f}_1 z = \cup_{n\in\mathsf{N}}\{\mathsf{dc}(\mathsf{f}_0,z)(n)\}$, so that we get $Hier_F(a+\omega,\mathsf{f}_1 z)$. Iterating this process, we get for each $k \in \mathbb{N}$ a term f_k with $Hier_F(a+\omega^{\overline{k}},\mathsf{f}_k z)$. This suffices to perform the aforementioned embedding.

To formalize models within the theory $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ we are using methods described in Studer [15]. That is, we model the naming and elementhood relation by an inductively generated relation $P_{\mathcal{A}}(m,n,k)$ such that $\Re(a)$ translates to $P_{\mathcal{A}}(a^*,0,0)$ and $t \mathrel{\dot\in} a$ to $P_{\mathcal{A}}(a^*,t^*,1)$. We have just to take care that our operator form stays Π^0_1.

In our thesis we discuss two standard models, a recursion theoretic one, and a term model. In the recursion theoretic model, the universe is the set of the natural numbers, and $r \cdot s \simeq t$ is interpreted as $\{r^*\}(s^*) \simeq t^*$. Note that this abbreviates an Σ^0_1-formula. Now the constants are interpreted by appropriate codes for recursive functions. That the translation of $(\mathsf{dc.1})$ becomes provable in $\mathsf{FID^r}(\Pi^0_1)$ we add the following clause to our operator form $\mathcal{A}$:

$$(1) \qquad P(a,0,0) \wedge (\forall x)[P(x,0,0) \rightarrow P(\{f\}(x),0,0)] \rightarrow P(\mathsf{cl}^*(a,f),0,2).$$

This clause is to ensure, that if the translation of the premise of $(\mathsf{dc.1})$ holds, then there exists already a stage α such that

$$P^\alpha_{\mathcal{A}}(a,0,0) \wedge (\forall x)[P^\alpha_{\mathcal{A}}(x,0,0) \rightarrow P^\alpha_{\mathcal{A}}(\{f\}(x),0,0)]$$

holds. That allows us to prove the translation of $(\mathsf{dc.1})$ by $\Delta^{\mathbb{N}}_0$-induction on the natural numbers. In order to reformulate (1) such that it becomes Π^0_1 we introduce an auxiliary type noval^* (no value) with $(f,s) \in \mathsf{noval}^* \iff \neg\{f\}(s)\downarrow$. Hence we can check in $\mathcal{A}$ if $\{f^*\}(s^*)\downarrow$ holds by asking if $(f,s) \notin \mathsf{noval}^*$. Then the translation of $(f : \Re \rightarrow \Re)$ becomes equivalent to the Π^0_1-formula

$$(\forall x)[P(x,0,0) \rightarrow \{f\}(x)\downarrow] \wedge (\forall x)(\forall y)[P(x,0,0) \wedge \{f\}(x) = y \rightarrow P(y,0,0)].$$

In the term model, the universe consists of all codes for closed $\mathcal{L}_p$-terms. To model equality, we define a relation $\mathrm{Red1}_\rho$ on the codes for closed terms that models the behaviour of the constants, e.g. if t^* stands for the code of an $\mathcal{L}_p$-term t, then $\mathrm{Red1}_\rho((\mathsf{k}ab)^*, a^*)$ holds. Equality is then interpreted by the Σ^0_1-relation $\approx_\rho$, the reflexive, symmetric and transitive closure of $\mathrm{Red1}_\rho$. A problem is that $P_{\mathcal{A}}$ has to be closed w.r.t. $\approx_\rho$. This can't be achieved directly by a Π^0_1-operator form. But we find a primitive recursive function $bd(x)$ satisfying $\mathrm{Red1}_\rho(s,t) \Rightarrow t < bd(s)$, so we can close $P_{\mathcal{A}}$ under $\mathrm{Red1}_\rho$, that is, if we have $P_{\mathcal{A}}(m,n,1)$ and $\mathrm{Red1}_\rho(m,m')$ or $\mathrm{Red1}_\rho(n,n')$, then also $P_{\mathcal{A}}(m',n,1)$ or $P_{\mathcal{A}}(m,n',1)$. Fortunately, it turns out that closure w.r.t $\mathrm{Red1}_\rho$ already means closure w.r.t. $\approx_\rho$.

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1 A Lower Bound for $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$

In this chapter we introduce a new axiom (dc) in order to handle dependent choice in explicit mathematics, and we establish a lower bound for the theory $\mathsf{EETJ}+(\mathsf{dc})+(\mathsf{T}\text{-}\mathsf{I_N})$ by embedding the subsystem of analysis $(\Pi^1_0\text{-}\mathsf{CA})_{<\omega^\omega}$ into it.

1.1 The Theory EETJ

We formulate the theory EETJ in the language $\mathcal{L}_p$, a two-sorted language with individual variables $a, b, c, f, g, h, w, x, y, z, \ldots$ (possibly with subscript), and type variables $A, B, C, X, Y, Z, \ldots$. $\mathcal{L}_p$ includes individual constants k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projections), $\mathsf{0}$ (zero), s_N (successor on natural numbers), p_N (predecessor on natural numbers), d_N (definition by cases on natural numbers), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image) and j (join). Further $\mathcal{L}_p$ has a binary function symbol $\cdot$ for (partial) term application, unary relation symbols $\downarrow$ (defined) and N (natural numbers) as well as binary relation symbols $=$ (equality for individuals), $\in$ (elementhood between individuals and types) and $\Re$ (naming).
The individual terms $(r, s, t, \ldots)$ of $\mathcal{L}_p$ are inductively defined as follows:

1. Every individual variable and constant is an individual term.
2. If s, t are individual terms, then $(s \cdot t)$ is an individual term.

In the sequel we write (st), or just st instead of $(s \cdot t)$ and we adopt the convention of association to the left, i.e. $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2)\ldots s_n)$. Further, (t_1, t_2) stands for $\mathsf{p}t_0 t_1$ and $(t_1, \ldots t_n)$ for $(t_1, (t_2, \ldots, t_n))$. We also use vector notation to denote finite sequences of terms, e.g. $\vec{a}$ or $\vec{X}$ for $a_0, \ldots, a_{n-1}$ or $X_0, \ldots, X_{m-1}$, respectively. The length of these sequences is given by the context.
The $\mathcal{L}_p$-formulas $(F, G, H, \ldots)$ are inductively defined as follows:

1. $\mathsf{N}(t)$, $t\downarrow$, $(s = t)$, $(s \in X)$, $(X = Y)$ and $\Re(t, X)$ are (atomic) formulas.
2. If F and G are formulas, then $\neg F$, $(F \vee G$, $(F \wedge G)$ are formulas, too.
3. If F is a formula, then $(\forall x)F$, $(\exists x)F$, $(\forall X)F$, and $(\exists X)F$ are formulas, too.

If F is a formula, $F(\vec{x}, \vec{X})$ indicates that the variables $\vec{x}$, $\vec{X}$ may occur free in F. $F[\vec{t}/\vec{x}, \vec{Y}/\vec{X}]$ or short $F[\vec{t}, \vec{Y}]$ denotes the result of the simultaneous substitution of all free occurrences of the variables x_i and X_j in F by the terms t_i and the type variables Y_j. As usual we write $(F \to G)$ for $(\neg F \vee G)$ and $(F \leftrightarrow G)$ for $((F \to G) \wedge (F \to G))$. An $\mathcal{L}_p$-formula F is called elementary, if the relation symbol

$\Re$ does not occur in F, and F does not contain bounded type variables. We use the following abbreviations:

$$
\begin{array}{rcl}
t \in \mathsf{N} & :\equiv & \mathsf{N}(t), \\
X \subseteq Y & :\equiv & (\forall x)[x \in X \rightarrow x \in Y], \\
(\exists x \in \mathsf{N})F & :\equiv & (\exists x)(x \in \mathsf{N} \wedge F), \\
(\forall x \in \mathsf{N})F & :\equiv & (\forall x)(x \in \mathsf{N} \rightarrow F), \\
(\exists X \subseteq \mathsf{N})F & :\equiv & (\exists X)(X \subseteq \mathsf{N} \wedge F), \\
(\forall X \subseteq \mathsf{N})F & :\equiv & (\forall X)(X \subseteq \mathsf{N} \rightarrow F), \\
(t : \mathsf{N} \rightarrow \mathsf{N}) & :\equiv & (\forall x \in \mathsf{N})(tx \in \mathsf{N}), \\
(t : \mathsf{N}^{m+1} \rightarrow \mathsf{N}) & :\equiv & (\forall x \in \mathsf{N})(tx : \mathsf{N}^m \rightarrow \mathsf{N}).
\end{array}
$$

Our theory is based on partial term application. Hence it is not guaranteed that terms have a value, and t $\downarrow$ is read as 't is defined' or 't has a value'. So we introduce the relation of partial equality $\simeq$ by:

$$s \simeq t :\equiv (s\!\downarrow \vee \; t\!\downarrow) \rightarrow (s = t).$$

Now we are ready to state the axioms of the theory EETJ. The underlying logic of EETJ is the classical logic of partial terms with equality axioms for individuals, due to Beeson [2]. The first order part of the non-logical axioms consists of the following five groups of axioms that define the first-order theory BON of Feferman and Jäger [5].

I. Partial combinatory algebra.

(1) $\mathsf{k}xy = x$,

(2) $\mathsf{s}xy\!\downarrow \wedge \; \mathsf{s}xyz \simeq xz(yz)$.

II. Pairing and projections.

(3) $\mathsf{p}_0(x, y) = x \wedge \; \mathsf{p}_1(x, y) = y$.

III. Natural numbers.

(4) $0 \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(\mathsf{s}_N x \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(\mathsf{s}_N x \neq 0 \wedge \mathsf{p}_N(\mathsf{s}_N x) = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \rightarrow \mathsf{p}_N x \in \mathsf{N} \wedge \mathsf{s}_N(\mathsf{p}_N x) = x)$.

IV. Definition by cases on natural numbers.

(7) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \rightarrow \mathsf{d}_N xyab = x$,

(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \rightarrow \mathsf{d}_N xyab = y$.

V. Primitive recursion on N.

(9) $(f : \mathsf{N} \rightarrow \mathsf{N}) \wedge (g : \mathsf{N}^3 \rightarrow \mathsf{N}) \rightarrow (\mathsf{r}_N fg : \mathsf{N}^2 \rightarrow \mathsf{N})$,

(10) $(f : \mathsf{N} \rightarrow \mathsf{N}) \wedge (g : \mathsf{N}^3 \rightarrow \mathsf{N}) \wedge x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge h = \mathsf{r}_N fg$
$\rightarrow hx\mathsf{0} = fx \wedge hx(\mathsf{s}_N y) = gxy(hxy)$.

The axioms about primitive recursion on N are only required in the absence of strong enough induction principles. If we have e.g. type induction, ($\mathsf{T}\text{-}\mathsf{I}_\mathsf{N}$) primitive recursion on N can be proven (see below). Sometimes we want the application to be total. Therefore we add the axiom (Tot) that states $(\forall x)(\forall y)(xy\downarrow)$, i.e. every term has a value. (Tot) won't be an axiom of our theory, unless it is explicitly mentioned.

The second order part of EETJ deals with addressing and building types. The relation $\Re$ acts as a naming relation between individuals and types, i.e. $\Re(s, A)$ means that s is a name of the type A. Before we state the axioms, we define equality for types. We call two types A and B equal ($A = B$), if they have the same elements, i.e.

$$A = B :\equiv (\forall x)[x \in A \leftrightarrow x \in B].$$

The next axiom assures that types with the same elements also have the same names.

VI. Equality for types.

(EQ) $\Re(a, A) \wedge A = B \rightarrow \Re(a, B)$.

The axioms about explicit representation state that every type has a name (E.1) and that there are no homonyms (E.2).

VII. Explicit representation.

(E.1) $(\exists x)\Re(x, A)$,

(E.2) $\Re(a, B) \wedge \Re(a, C) \rightarrow B = C$.

To build types one has the following six principles which are equivalent to elementary comprehension (see below):

VIII. Natural numbers.

(N.1) $(\exists X)(\forall x)(x \in X \leftrightarrow \mathsf{N}(x))$,

(N.2) $(\forall x)(x \in A \leftrightarrow \mathsf{N}(x)) \rightarrow \Re(\mathsf{nat}, A)$.

IX. Identity.

(I.1) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)(x = (y, y)))$,

(I.2) $(\forall x)(x \in A \leftrightarrow (\exists y)(x = (y, y))) \rightarrow \Re(\mathsf{id}, A)$.

X. Complements.

(CO.1) $(\exists X)(\forall x)(x \in X \leftrightarrow x \notin B)$,

(CO.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow x \notin B) \rightarrow \Re(\mathsf{co}\ b, A)$.

XI. Intersections.

(INT.1) $(\exists X)(\forall x)(x \in X \leftrightarrow x \in B \wedge x \in C)$,

(INT.2) $\Re(b, B) \wedge \Re(c, C) \wedge (\forall x)(x \in A \leftrightarrow x \in B \wedge x \in C) \rightarrow \Re(\mathsf{int}(b, c), A)$.

XII. Domains.

(DOM.1) $(\exists X)(\forall x)(x \in X \leftrightarrow (\exists y)((x, y) \in B))$,

(DOM.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow (\exists y)((x, y) \in B)) \rightarrow \Re(\mathsf{dom}\ b, A)$.

XIII. Inverse images.

(INV.1) $(\exists X)(\forall x)(x \in X \leftrightarrow fx \in B)$,

(INV.2) $\Re(b, B) \wedge (\forall x)(x \in A \leftrightarrow fx \in B) \rightarrow \Re(\mathsf{inv}(b, f), A)$.

The axioms stated so far define the theory EET. As the following theorem due to Feferman and Jäger [6] shows, elementary comprehension is also available to define types. This proves to be useful in the sequel. In order to formulate the theorem, we introduce the notation

$$s \mathrel{\dot\in} t :\equiv (\exists X)[\Re(t, X) \wedge s \in X].$$

Theorem 1.1.1 *For every elementary $\mathcal{L}_p$-formula $F(x, \vec{y}, \vec{A})$ containing at most the indicated variables free, there exists a closed term t such that* EET *proves:*

(i) $\Re(\vec{a}, \vec{A}) \rightarrow \Re(t(\vec{y}, \vec{a}))$,

(ii) $\Re(\vec{a}, \vec{A}) \rightarrow (\forall x)[x \mathrel{\dot\in} t(\vec{y}, \vec{a}) \leftrightarrow F(x, \vec{y}, \vec{A})]$.

We conclude the description of our theory EETJ by stating a stronger type building axiom (J) for join, that enables us to form the disjoint union of an infinite family of types. If we write $A = \Sigma(B, f)$ for the statement

$$(\forall x)(x \in A \leftrightarrow x = (\mathsf{p}_0 x, \mathsf{p}_1 x) \wedge \mathsf{p}_0 x \in B \wedge (\exists X)[\Re(f(\mathsf{p}_0 x), X) \wedge \mathsf{p}_1 x \in X)],$$

join takes the form

XI. Join.

$$\text{(J)}\ \Re(a, A) \wedge (\forall x \in A)(\exists Y)\Re(fx, Y) \to (\exists Z)[\Re(\mathsf{j}(a, f), Z) \wedge Z = \Sigma(A, f)].$$

In order to get some proof-theoretical strength we need to endow our theory with certain forms of induction on N. The two induction principles we are interested in are:

Type induction on N.

$$(\mathsf{T\text{-}I_N})\quad \mathsf{0} \in A \wedge (\forall x \in \mathsf{N})(x \in A \to \mathsf{s}_N x \in A) \to (\forall x \in \mathsf{N})(x \in A),$$

Formula induction on N.

$$(\mathsf{F\text{-}I_N})\quad F(\mathsf{0}) \wedge [\forall x \in \mathsf{N})(F(x) \to F(\mathsf{s}_N x)] \to (\forall x \in \mathsf{N})(F(x)).$$

Of course, type induction is a special case of formula induction and therefore much weaker. Other forms of induction won't be treated here.

In the remaining of this section, we present some standard results concerning the combinators k and s. We define λ-abstraction and state a recursion theorem:

Definition 1.1.2 *Let t be a term of $\mathcal{L}_p$. Then $(\lambda x.t)$ is the term given by the following inductive definition:*

(i) $(\lambda x.t) :\equiv \mathsf{skk}$, *if* $t \equiv x$,

(ii) $(\lambda x.t) :\equiv \mathsf{k}t$, *if t is a variable different from x or a constant,*

(iii) $(\lambda x.t) :\equiv \mathsf{s}(\lambda x.s_1)(\lambda x.s_2)$, *if* $t \equiv (s_1 s_2)$.

Theorem 1.1.3 *(λ-abstraction) Let t, s be terms of $\mathcal{L}_p$. Then*

(i) $\mathsf{BON} \vdash (\lambda x.t)\downarrow$,

(ii) $\mathsf{BON} \vdash (\lambda x.t)x \simeq t$,

(iii) $\mathsf{BON} \vdash s\downarrow\ \to (\lambda x.t)s \simeq t[s/x]$.

Theorem 1.1.4 *(Recursion theorem) There is a closed term* rec *of $\mathcal{L}_p$ such that:*

$$\mathsf{BON} \vdash \mathsf{rec}f\downarrow \wedge\ \mathsf{rec}fx \simeq f(\mathsf{rec}f)x.$$

Proofs of these theorems can be found in Feferman [4] or Beeson [2].

In the presence of type induction $(\mathsf{T\text{-}I_N})$ the term rec helps us to prove the following theorem. A detailed proof can be found in [9].

Theorem 1.1.5 *(Primitive recursion on* N*) There is a closed term* $\tilde{\mathsf{r}}_N$ *that does not contain the constant* r_N*, such that* $\mathsf{EET} + (\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$ *proves:*

1. $(f : \mathsf{N} \to \mathsf{N}) \wedge (g : \mathsf{N}^3 \to \mathsf{N}) \to (\tilde{\mathsf{r}}_N f g : \mathsf{N}^2 \to \mathsf{N})$,
2. $(f : \mathsf{N} \to \mathsf{N}) \wedge (g : \mathsf{N}^3 \to \mathsf{N}) \wedge x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge h = \tilde{\mathsf{r}}_N f g$
   $\to hx\mathsf{0} = fx \wedge hx(\mathsf{s}_N y) = gxy(hxy)$.

□

1.2 The Axiom (dc)

Before we introduce the axiom (dc) for dependent choice in explicit mathematics, we like to have a look at the theory $(\Sigma^1_1\text{-}\mathsf{DC})\restriction$ of second order arithmetic to observe how dependent choice is formulated there.

Let $\mathbb{L}_2$ be a language of second order arithmetic, with number variables $(x, y, z, \ldots)$, set variables $(X, Y, Z, \ldots)$, the constant 0 (zero), symbols for all primitive recursive functions and relations, in particular a symbol S (successor), the symbol $\in$ for elementhood between numbers and sets as well as a symbol $=$ for equality in both sorts of variables. Terms $(r, s, t, \ldots)$ and formulas $(F, G, H, \ldots)$ of $\mathbb{L}_2$ are defined as usual. An $\mathbb{L}_2$-formula is called arithmetic, if it does not contain bounded set variables (but possibly free set variables); we write Π^1_0 for the collection of these formulas. The formulas of the form $(\exists X)F(X)$ $[(\forall X)F(X)]$, where F is Π^1_0 are called Σ^1_1-formulas [Π^1_1-formulas]. In the sequel $\langle \cdot, \cdot \rangle$ denotes a standard primitive recursive pairing function with associated primitive recursive projections $(\cdot)_0$ and $(\cdot)_1$. Further we'll write $s \in (X)_t$ for $\langle t, s \rangle \in X$, and if $\prec$ stands for a primitive recursive well-ordering, $s \in (X)_{\prec t}$ abbreviates $s = \langle (s)_0, (s)_1 \rangle \wedge (s)_0 \prec t \wedge s \in X$. Expressions of the form $(X)_s = (Y)_{\prec t}$ are read as $(\forall x)[x \in (X)_s \leftrightarrow x \in (Y)_{\prec t}]$. For every $k \in \mathbb{N}$, $\overline{k}$ denotes the k^{th} numeral, where $\overline{0} :\equiv 0$ and $\overline{k+1} :\equiv S\overline{k}$. Furthermore $x + 1$ stands for Sx.
Now we introduce the theory $(\Sigma^1_1\text{-}\mathsf{DC})\restriction$ and some related theories of second order arithmetic. They are all formulated in the language $\mathbb{L}_2$. The theory $\Pi^1_0\text{-}\mathsf{CA}$ comprises the usual axioms for the two-sorted predicate calculus with equality in both sorts and extensionality for sets, the axioms of Peano arithmetic PA, defining axioms for all primitive recursive functions and relations, the ordinary schema for arithmetic comprehension, i.e.

$$(\Pi^1_0\text{-}\mathsf{CA}) \qquad (\exists X)(\forall x)[x \in X \leftrightarrow F(x)]$$

where F is Π^1_0, and the induction schema

$$(\mathsf{IND\text{-}S}) \qquad F(0) \wedge (\forall x)[F(x) \to F(x+1)] \to (\forall x)F(x)$$

for all formulas F.
If the above schema is replaced by the axiom

$$(\mathsf{IND\text{-}A}) \qquad 0 \in X \wedge (\forall x)[x \in X \rightarrow (x+1) \in X] \rightarrow (\forall x)(x \in X)$$

we denote the resulting theory by $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$.
The theory $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$ can be seen as classical analogue of the theory $\mathsf{EET} + (\mathsf{T\text{-}I_N})$, whereas the conservative extension $\mathsf{EETJ} + (\mathsf{T\text{-}I_N})$ corresponds to the conservative extension $(\Sigma^1_1\text{-}\mathsf{AC})\!\restriction$ of $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$, i.e. the theory $(\Pi^1_0\text{-}\mathsf{CA})\!\restriction$ with the additional axiom schema (axiom of choice)

$$(\Sigma^1_1\text{-}\mathsf{AC}) \qquad (\forall x)(\exists Z)F(x,Z) \rightarrow (\exists Y)(\forall x)F[x,(Y)_x]$$

where F is Σ^1_1.
Like (J) in explicit mathematics, $(\Sigma^1_1\text{-}\mathsf{AC})$ enables us to form the disjoint union of a family of sets. If we replace the schema $(\Sigma^1_1\text{-}\mathsf{AC})$ in the theory $(\Sigma^1_1\text{-}\mathsf{AC})\!\restriction$ by the stronger schema (dependent choice)

$$(\Sigma^1_1\text{-}\mathsf{DC}) \qquad (\forall X)(\exists Y)F(X,Y) \rightarrow (\forall X)(\exists Z)[(Z)_0 = X \wedge (\forall x)F[(Z)_x,(Z)_{x+1}]]$$

where F is Σ^1_1, we get the theory $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$. It is important to note, that by means of the axiom $(\Sigma^1_1\text{-}\mathsf{DC})$ we can build the iterated jump-hierarchy along a well-ordering $\prec$ of order-type less than ω^ω, so that we can embed $(\Pi^1_0\text{-}\mathsf{CA})_{<\omega^\omega}\!\restriction$ into $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$, which shows, that $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$ is indeed stronger than $(\Sigma^1_1\text{-}\mathsf{AC})\!\restriction$. For details we refer to Cantini [3].
Our axiom (dc) is tailored such that $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})$ becomes an analogue of $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$, i.e. we want to be able to embed $(\Pi^1_0\text{-}\mathsf{CA})_{<\omega^\omega}\!\restriction$ into $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})$ in much the same way as it is embedded into $(\Sigma^1_1\text{-}\mathsf{DC})\!\restriction$ in the aforementioned paper [3]. It turns out that we get an adequate form of (dc) by extending the language $\mathcal{L}_p$ by the new constant dc, and by adding the axioms $(\mathsf{dc.1})$, $(\mathsf{dc.2})$ to the theory EETJ.

Dependent choice (dc).

$$\begin{array}{ll} (\mathsf{dc.1}) & \Re(a) \wedge (f : \Re \rightarrow \Re) \rightarrow (\mathsf{dc}(a,f) : \mathsf{N} \rightarrow \Re), \\ (\mathsf{dc.2}) & \Re(a) \wedge (f : \Re \rightarrow \Re) \rightarrow \\ & \mathsf{dc}(a,f)(0) = a \wedge (\forall n \in \mathsf{N})[\mathsf{dc}(a,f)(\mathsf{s}_N n) = f(\mathsf{dc}(a,f)(n))]. \end{array}$$

Here $(f : \Re \rightarrow \Re)$ stands for $(\forall x)(\Re(x) \rightarrow \Re(fx))$, $\Re(x)$ stands for $(\exists X)\Re(x,X)$ and $(t : \mathsf{N} \rightarrow \Re)$ abbreviates $(\forall x \in \mathsf{N})\Re(tx)$. Whereas the existence of a term $\tilde{\mathsf{dc}}$ satisfying $(\mathsf{dc.2})$ can be proven in $\mathsf{EETJ} + (\mathsf{T\text{-}I_N})$, type induction is not strong enough

to prove the totality of the function $(\tilde{\mathsf{dc}}(a, f) : \mathsf{N} \to \Re)$. We can't apply the premise $(f : \Re \to \Re)$ to show the induction step, because we can't express $\Re(\mathsf{dc}(a, f)(n))$ as an elementary formula. Of course (dc.1) becomes a theorem if we admit formula induction.
In the next section we present the theory $(\Pi^1_0\text{-CA})_{<\omega^\omega}$. Then we'll demonstrate how (J) and (dc) serve to build the iterated jump-hierarchy.

1.3 The Theories $(\Pi^1_0\text{-CA})_\alpha$ for $\alpha < \varepsilon_0$

For the description of these theories we fix a primitive recursive standard well-ordering $\prec$ of order-type ε_0. Without loss of generality we may assume that the field of $\prec$ is the set of all natural numbers and that 0 is the least element of $\prec$. Hence each natural number a codes an ordinal, say $ord(a)$, less than ε_0, and each ordinal $\alpha < \varepsilon_0$ is represented by an unique number, say $nr(\alpha)$. Moreover, there exist binary primitive recursive functions $\oplus$, and $\dot{\omega}$, that model the usual ordinal operations plus, times and exponentiation on these codes, that is:

- $\oplus(m, n) := nr[ord(m) + ord(n)]$,
- $\dot{\omega}(m, n) := nr[\omega^m \cdot n]$.

In order to keep notation as simple as possible, we'll write $(m+n)$ instead of $\oplus(m, n)$, if the context makes clear that m and n are codes for ordinals, $\omega^m \cdot n$ for $\dot{\omega}(m, n)$, ω^m for $\omega^m \cdot 1$ and ω for ω^1. If α denotes a fixed ordinal, then we identify α with $nr(\alpha)$ or $\overline{nr(\alpha)}$, respectively.

If $F(x, X, Y, a)$ is an arithmetic $\mathbb{L}_2$-formula with x, X, Y and a free, we can define *the F jump hierarchy along $\prec$ with parameter X* by the following transfinite recursion:

$$Y_a := \{x : F[x, X, (Y)_{\prec a}, a]\}.$$

We can formalize this definition by the arithmetic formula

$$Hier_F(a, X, Y) :\equiv (\forall b \prec a)(\forall x)[x \in (Y)_b \leftrightarrow F[x, X, (Y)_{\prec b}, b]],$$

that says *'Y is a jump-hierarchy along $\prec$ with parameter X up to a'*. If it is clear or unimportant which parameter X we refer to, it will be omitted.

If α is an ordinal less than ε_0 we denote by $(\Pi^1_0\text{-CA})_\alpha$ the theory that extends $\Pi^1_0\text{-CA}$ by the axiom schema

$$TI(\prec, \alpha, F) \qquad (\forall x \prec \alpha)[(\forall y \prec x)F(y) \to F(x)] \to (\forall x \prec \alpha)F(x)$$

for all $\mathbb{L}_2$-formulas F, and the axiom

$(\mathcal{H},\alpha)$ $\qquad\qquad (\forall X)(\exists Y)Hier_F[\alpha,X,Y]$

for all arithmetic formulas F. The theory $(\Pi^1_0\text{-CA})_\alpha\!\restriction$ is the theory $(\Pi^1_0\text{-CA})\!\restriction$ plus the axiom $(\mathcal{H},\alpha)$. The union of all the theories $(\Pi^1_0\text{-CA})_\beta$ with $\beta<\alpha$ is called $(\Pi^1_0\text{-CA})_{<\alpha}$; $(\Pi^1_0\text{-CA})_{<\alpha}\!\restriction$ is defined analogously.

Following Schütte's well-ordering proofs [13] for subsystems of predicative analysis we see that already $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$ has the proof-theoretical ordinal $\varphi\omega 0$.[1] Therefore it suffices to embed $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$ into $\mathsf{EETJ}+(\mathsf{dc})+(\mathsf{T}\text{-}\mathsf{I_N})$ in order to get $\varphi\omega 0$ as a lower bound.

1.4 The Embedding of $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$ into $\mathsf{EETJ}+(\mathsf{dc})+(\mathsf{T}\text{-}\mathsf{I_N})$

First we give an interpretation of the language $\mathbb{L}_2$ of second order arithmetic into $\mathcal{L}_p$. The number variables are interpreted as ranging over N and the set variables are ranging over the subtypes of N. Due to the recursion operator r_N each primitive recursive function can be represented by an $\mathcal{L}_p$-term, hence each primitive recursive function symbol f can be interpreted by a term f_p, and for every primitive recursive relation symbol R we find a term R_p that represents its characteristic function.
Now we assign to every $\mathbb{L}_2$-term t a $\mathcal{L}_p$-term t^N according to

- if t is a variable (symbol), then t^N is the same variable (symbol),
- if t is the constant 0, then t^N is the constant $\mathsf{0}$,
- if $t\equiv f(t_0,\dots t_{n-1})$, then $t^N :\equiv f_p t_0^N\cdots t_{n-1}^N$,

and to every $\mathbb{L}_2$-formula F we assign a $\mathcal{L}_p$-formula F^N according to the following inductive definition:

- if $F\equiv R(t_0,\dots t_{n-1})$, then $F^N :\equiv R_p t_0^N\cdots t_{n-1}^N=\mathsf{0}$,
- if $F\equiv t_1=t_2$, then $F^N :\equiv t_1^N=t_2^N$,
- if $F\equiv t\in X$, then $F^N :\equiv t^N\in X$,
- if $F\equiv(Qx)G(x)$, then $F^N :\equiv (Qx\in\mathsf{N})G^N(x)$, (where Q denotes $\forall$ or $\exists$),
- if $F\equiv(QX)G(X)$, then $F^N :\equiv (QX\subseteq\mathsf{N})G^N(X)$, (where Q denotes $\forall$ or $\exists$),

[1] Schütte's formula $\mathcal{R}(P,Q,t)$ is equivalent to the formula $Hier_F(t,P,Q)$ for a suitable $F\in\Pi^1_0$. So Schütte's lemma 12 becomes an instance of an axiom $(\mathcal{H},\alpha)$ and the claim follows by Schütte's lemma 10.

- if $F \equiv G$ j H, then $F^N :\equiv G^N$ j H^N, (where j stands for $\vee$ or $\wedge$),
- if $F \equiv \neg G$, then $F^N :\equiv \neg G^N$.

Finally, if the free variables of the $\mathbb{L}_2$-formula F are among $\{\vec{x}, \vec{X}\}$, then we define the interpreted $\mathcal{L}_p$-formula F^I to be

$$F^I :\equiv (\vec{x} \in \mathsf{N} \wedge \vec{X} \subseteq \mathsf{N}) \rightarrow F^N,$$

where $\vec{x} \in \mathsf{N}$ $[\vec{X} \subseteq \mathsf{N}]$ stands for $x_0 \in \mathsf{N} \wedge \ldots \wedge x_{n-1} \in \mathsf{N}$ $[X_0 \subseteq \mathsf{N} \wedge \ldots \wedge X_{m-1} \subseteq \mathsf{N}]$. In order to show that the above introduced operation $\cdot^I$ defines indeed an embedding of $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$ into $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$ we have to prove that

Proposition 1.4.1 *For every axiom F of $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$ we have:*

$$\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I}_\mathsf{N}) \vdash F^I.$$

This is clear for the logical axioms and the axioms concerning the first order part of $(\Pi^1_0\text{-CA})_{<\omega^\omega}\!\restriction$. Also the interpretation of the induction axiom (IND-A) follows directly from $(\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$. However, to prove the axioms $(\mathcal{H}, \alpha)$ some extra work has to be done. Before we do so, we introduce some notational shorthands.
If R is a binary relation symbol of the language $\mathbb{L}_2$, we write for the $\mathcal{L}_p$-formula $(R(a, b))^N$ simply aRb instead of $R_p ab = \mathsf{0}$. Similarly, if f is a primitive recursive function symbol, we denote the $\mathcal{L}_p$-term $(f(\vec{x}))^N$ by $f(\vec{x})$ instead of $f_p\vec{x}$. Further, we want to substitute names for types in elementary $\mathcal{L}_p$-formulas. Therefore, if $F(\vec{X})$ is an elementary $\mathcal{L}_p$-formula, $\dot{F}[\vec{t}/\vec{X}]$ or short $\dot{F}[\vec{t}]$ denotes the formula that is the result of replacing each atomic subformula of F of the form $(s \in X_i)$ by $(s \,\dot{\in}\, t_i)$. Recall that $s \,\dot{\in}\, t :\equiv (\exists X)[\Re(t, X) \wedge s \in X]$.
Next we choose closed terms sec (section) and seg (initial segment) such that if s is a name of the type A, then $\mathsf{sec}(s, t)$ is a name of the type $(A)_t$ and $\mathsf{seg}(s, t)$ is a name of the type $(A)_{\prec t}$. That is, sec and seg are closed terms that satisfy

(sec.1) $\Re(a, A) \rightarrow \Re(\mathsf{sec}(a, y))$,
(sec.2) $\Re(a, A) \rightarrow (\forall x)[x \,\dot{\in}\, \mathsf{sec}(a, y) \leftrightarrow \langle y, x\rangle \in A]$.

(seg.1) $\Re(a, A) \rightarrow \Re(\mathsf{seg}(a, y))$,
(seg.2) $\Re(a, A) \rightarrow (\forall x)[x \,\dot{\in}\, \mathsf{seg}(a, y) \leftrightarrow x = \langle (x)_0, (x)_1\rangle \wedge (x)_0 \prec y \wedge x \in A]$.

Note that the existence of the terms sec and seg is ensured by theorem 1.1.1.
So far, the section $(A)_t$ and the initial segment $(A)_{\prec t}$ of a type A was defined w.r.t. the pairing function $\langle \cdot, \cdot \rangle$. We now define these notions also w.r.t. the pairing function $(\cdot, \cdot)$, what proves to be usefull if we are dealing with types obtained by

applying join. Further, we define equalitiy for names ($s \doteq t$) and 'equality of the natural part' of a name or a type:

$$\begin{array}{rcl} s \in [X]_t & :\equiv & (t,s) \in X, \\ s \in [X]_{\prec t} & :\equiv & s = (\mathsf{p}_0 s, \mathsf{p}_1 s) \wedge s \prec t \wedge s \in X, \\ s \doteq t & :\equiv & (\exists X)(\exists Y)[\Re(s,X) \wedge \Re(t,Y) \wedge X = Y], \\ X =_N Y & :\equiv & (\forall x \in \mathsf{N})[x \in X \leftrightarrow x \in Y], \\ s \doteq_N t & :\equiv & \Re(s) \wedge \Re(t) \wedge (\forall x \in \mathsf{N})[x \mathrel{\dot\in} s \leftrightarrow x \mathrel{\dot\in} t]. \end{array}$$

In the sequel we often deal with terms containing exactly one variable free. If t is such a term and y is a variable, we denote by t^y the result of replacing in t every free occurrence of a variable by y.
Let's turn to the proof of proposition 1.4.1. We have to show that EETJ+(dc)+$(\mathsf{T}\text{-}\mathsf{I_N})$ proves $(\mathcal{H}, \alpha)^I$ for $\alpha < \omega^\omega$. It suffices to show that $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ proves for all $k \in \mathbb{N}$

$$(*) \qquad (\forall Y)(\exists Z) Hier_F^N[\omega^{\overline{k}}, Y, Z],$$

for if we have an $\mathbb{L}_2$-formula $F(X)$, the $\mathcal{L}_p$-formula $F^N(X)$ just sees the 'natural part' of a set X. More precisely we have

Lemma 1.4.2 *If the free number variables of an $\mathbb{L}_2$-formula F are in $\{x_0, \ldots, x_{n-1}\}$, then* EET *proves:*

$$\vec{x} \in \mathsf{N} \rightarrow \Big(F^N(X) \leftrightarrow F^N[X \cap \mathsf{N}]\Big).$$

Proof: This is easily shown by induction on the definition of F. □

To prove $(*)$ we construct for each $k \in \mathbb{N}$ a term f_k that contains exactly one variable free, such that $\Re(y, Y)$, $\Re(z, Z)$ and $\dot{Hier}_F^N[a, y, z]$ implies $\dot{Hier}_F^N[a + \omega^{\overline{k}}, y, \mathsf{f}_k^y z]$.
In the following lemma we show how to get the term f_0, and in the next theorem we get terms f_k for each $k \in \mathbb{N}$.

Lemma 1.4.3 *There exists an $\mathcal{L}_p$-term f_0 that contains exactly one variable free such that* $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ *proves:*

(i) $\Re(y) \rightarrow (\mathsf{f}_0^y : \Re \rightarrow \Re)$,

(ii) $\Re(y) \wedge \Re(z) \wedge a \in \mathsf{N} \wedge \dot{Hier}_F^N[a, y, z] \rightarrow \dot{Hier}_F^N[a+1, y, \mathsf{f}_0^y z] \wedge$
$\mathsf{seg}(z, a) \doteq_N \mathsf{seg}(\mathsf{f}_0^y z, a)$.

PROOF: First observe that the formula $F^N[(x)_1, Y, (Z)_{\prec(x)_0}, (x)_0]$ is elementary. Therefore theorem 1.1.1 allows us to find a closed term t such that

(a) $\Re(y, Y) \wedge \Re(z, Z) \to \Re(t(y, z))$,

(b) $\Re(y, Y) \wedge \Re(z, Z) \to (\forall x)[x \mathrel{\dot\in} t(y, z) \leftrightarrow F^N[(x)_1, Y, (Z)_{\prec(x)_0}, (x)_0]]$.

Now we set $\mathsf{f}_0 :\equiv \lambda x.t(y, x)$. So f_0 contains exactly one variable free and satisfies (i). To show (ii) we work informally in $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and assume that (i) and the premise of (ii) hold. First we show that

$$\mathsf{seg}(z, a) \doteq_N \mathsf{seg}(\mathsf{f}_0^y z, a).$$

Let $x \in \mathsf{N}$. Then we have

$$\begin{aligned} x \mathrel{\dot\in} \mathsf{seg}(z, a) &\iff x = \langle (x)_0, (x)_1 \rangle \wedge (x)_0 \prec a \wedge (x)_1 \mathrel{\dot\in} \mathsf{sec}(z, (x)_0) \\ &\iff \dot{F}^N[(x)_1, y, \mathsf{seg}(z, (x)_0), (x)_0] \wedge (x)_0 \prec a \wedge x = \langle (x)_0, (x)_1 \rangle \\ &\iff (x)_1 \mathrel{\dot\in} \mathsf{sec}(\mathsf{f}_0^y z, (x)_0) \wedge (x)_0 \prec a \wedge x = \langle (x)_0, (x)_1 \rangle \\ &\iff x \mathrel{\dot\in} \mathsf{seg}(\mathsf{f}_0^y z, a) \end{aligned}$$

But this implies $\dot{Hier}_F^N[a, y, \mathsf{f}_0^y z]$. Moreover, for $w \in \mathsf{N}$

$$w \mathrel{\dot\in} \mathsf{sec}(\mathsf{f}_0^y z, a) \iff \dot{F}^N[w, y, \mathsf{seg}(z, a), a] \iff \dot{F}^N[w, y, \mathsf{seg}(\mathsf{f}_0^y z, a), a],$$

hence $\dot{Hier}_F^N[a + 1, y, \mathsf{f}_0^y z]$. □

With the help of (dc) we can now define terms f_k for every $k \in \mathbb{N}$ that take the hierarchy ω^k steps further:

Theorem 1.4.4 *For each $k \in \mathbb{N}$ there is an $\mathcal{L}_p$-term f_k that contains exactly one variable free such that $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ proves:*

(i) $\Re(y) \to (\mathsf{f}_k^y : \Re \to \Re)$,

(ii) $\Re(y) \wedge \Re(z) \wedge a \in \mathsf{N} \wedge \dot{Hier}_F^N[a, y, z] \to \dot{Hier}_F^N[a + \omega^{\overline{k}}, y, \mathsf{f}_k^y z] \wedge$
$\mathsf{seg}(z, a) \doteq_N \mathsf{seg}(\mathsf{f}_k^y z, a)$.

PROOF: We work informally in $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ and prove the claim by metainduction on k. We have already shown the case $k = 0$. For the induction step assume that we have already a term f_k containing exactly one variable free satisfying the assertions (i) and (ii).

Suppose now that $\Re(y, Y)$, $\Re(z, Z)$ and $a, n \in \mathsf{N}$ and $\dot{Hier}_F^N[a, y, z]$. Then the following holds:

(1) $\qquad \dot{Hier}_F^N[a + \omega^{\overline{k}} \cdot n, y, \mathsf{dc}(z, \mathsf{f}_k^y)(n)] \wedge \mathsf{seg}(z, a) \doteq_N \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(n), a)$.

We show (1) by type induction on n. Let C be the type with

$$\Re(\mathsf{j}(\mathsf{nat}, \mathsf{dc}(z, \mathsf{f}_k^y)), C),$$

so that we have

$$(\forall x \in \mathsf{N})[\Re(\mathsf{dc}(z, \mathsf{f}_k^y)(x), [C]_x)],$$

and (1) becomes equivalent to

$$(1') \qquad Hier_F^N[a + \omega^{\overline{k}} \cdot n, Y, [C]_n] \wedge \ (Z)_{\prec a} =_N ([C]_n)_{\prec a},$$

which is an elementary formula. For $n = 0$, there is nothing to show and the induction step follows if we apply the metainduction assertion (ii) to (1). That shows (1).

Further, if also $b, l \in \mathsf{N}$, (1) yields immediately that

$$(2) \qquad \dot{Hier}_F^N[b, y, \mathsf{dc}(z, \mathsf{f}_k^y)(n)] \rightarrow \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(n), b) \doteq_N \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(n + l), b),$$

because of $\mathsf{dc}\Big(\mathsf{dc}(z, \mathsf{f}_k^y)(n), \mathsf{f}_k^y\Big)(l) = \mathsf{dc}(z, \mathsf{f}_k^y)(n + l)$ (equal as terms !) as an easy induction on l yields.

Now we want to define a term f_{k+1} that contains exactly one variable free and satisfies

(a) $\Re(y) \rightarrow (\mathsf{f}_{k+1}^y : \Re \rightarrow \Re)$,

(b) $\Re(y) \wedge \Re(z) \rightarrow$
$(\forall x)[x \mathrel{\dot\in} \mathsf{f}_{k+1}^y z \leftrightarrow (\exists n_0 \in \mathsf{N})(\forall n \in \mathsf{N})[(n > n_0) \rightarrow (n, x) \mathrel{\dot\in} \mathsf{j}(\mathsf{nat}, \mathsf{dc}(z, \mathsf{f}_k^y))]$.

Observe that $(n, x) \mathrel{\dot\in} \mathsf{j}(\mathsf{nat}, \mathsf{dc}(z, \mathsf{f}_k^y))$ is equivalent to $x \mathrel{\dot\in} \mathsf{dc}(z, \mathsf{f}_k^y)(n)$. Note also that instead of (b), the requirement

$$(b') \qquad \Re(y) \wedge \Re(z) \rightarrow (\forall x)[x \mathrel{\dot\in} \mathsf{f}_{k+1}^y z \leftrightarrow (\exists n \in \mathsf{N})[(n, x) \mathrel{\dot\in} \mathsf{j}(\mathsf{nat}, \mathsf{dc}(z, \mathsf{f}_k^y))]]$$

wouldn't be adequate. Let e.g. $\dot{Hier}_F^N[a, y, z]$ and $x = \langle b, w \rangle$, $x \mathrel{\dot\in} \mathsf{dc}(z, \mathsf{f}_k^y)(n)$, with $a + \omega^{\overline{k}} \cdot n < b < a + \omega^{\overline{k+1}}$. It's possible that we have $\neg(x \mathrel{\dot\in} c)$, for all names c with $\dot{Hier}_F^N[b + 1, y, c]$. Of course we don't want that $x \mathrel{\dot\in} \mathsf{f}_{k+1}^y z$. However, if the requirement (b) holds, then we get an m such that $b < a + \omega^{\overline{k}} \cdot m$ and $x \mathrel{\dot\in} \mathsf{dc}(z, \mathsf{f}_k^y)(m)$.

Now we construct the term f_{k+1}. By theorem 1.1.1 we find a closed term t with

(c) $\Re(c, C) \rightarrow \Re(t(c))$,

(d) $\Re(c, C) \rightarrow (\forall x)[x \mathrel{\dot\in} t(c) \leftrightarrow (\exists n_0 \in \mathsf{N})(\forall n \in \mathsf{N})[(n > n_0) \rightarrow (n, x) \in C]]$.

We set $\mathsf{f}_{k+1} :\equiv \lambda x.t[\mathsf{j}(\mathsf{nat}, \mathsf{dc}(x, \mathsf{f}_k^y))]$. Clearly f_{k+1} contains exactly the variable y free, and if $\Re(y)$, $\Re(z)$ we have

$$\mathsf{f}_{k+1}^y z = t[\mathsf{j}(\mathsf{nat}, \mathsf{dc}(z, \mathsf{f}_k^y))].$$

By the choice of the term t and the metainduction assertion the term f_{k+1} has the properties (a) and (b).

For the following we assume that (i) and the premise of (ii) hold, and that n, n_0, m and l denote natural numbers. With this we show that

$$\text{(3)} \qquad \mathsf{seg}(\mathsf{f}_{k+1}^y z, a + \omega^{\overline{k}} \cdot m) \doteq_N \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(m), a + \omega^{\overline{k}} \cdot m).$$

Let $b, w \in \mathsf{N}$ and $\langle b, w \rangle \mathrel{\dot\in} \mathsf{seg}(\mathsf{f}_{k+1}^y z, a + \omega^{\overline{k}})$, so we have $b \prec a + \omega^{\overline{k}} \cdot m$ and $\langle b, w \rangle \mathrel{\dot\in} \mathsf{dc}(z, \mathsf{f}_k^y)(n)$ $(\forall n > n_0)$ for a certain n_0 by the definition of the term f_{k+1}. Now we choose n such that $n \geq m$, say $n = m + l$ for a certain l.

By (1) we have

$$\dot{Hier}_F^N[a + \omega^{\overline{k}} \cdot m, y, \mathsf{dc}(z, \mathsf{f}_k^y)(m)],$$

so (2) yields

$$\text{(4)} \qquad \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(m), a + \omega^{\overline{k}} \cdot m) \doteq_N \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(m + l), a + \omega^{\overline{k}} \cdot m).$$

That shows that $\langle b, w \rangle \mathrel{\dot\in} \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(m), a + \omega^{\overline{k}} \cdot m)$.

Now let $\langle b, w \rangle \mathrel{\dot\in} \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(m), a + \omega^{\overline{k}} \cdot m)$. Using (4) shows that we have also $\langle b, w \rangle \mathrel{\dot\in} \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(n), a + \omega^{\overline{k}} \cdot m)$ $(\forall n \geq m)$. Hence $\langle b, w \rangle \mathrel{\dot\in} \mathsf{seg}(\mathsf{f}_{k+1}^y z, a + \omega^{\overline{k}})$, and (3) is established. For $m = 0$, this yields immediately $\mathsf{seg}(z, a) \doteq_N \mathsf{seg}(\mathsf{f}_{k+1}^y z, a)$.

To conclude our proof we still have to show that $\dot{Hier}_F^N[a + \omega^{\overline{k+1}}, y, \mathsf{f}_{k+1}^y z]$ holds. However, if $b \in \mathsf{N}$, $b \prec a + \omega^{\overline{k+1}}$, there is an n_0 such that $b \prec a + \omega^{\overline{k}} \cdot n_0$. Using (3) we get

$$\begin{aligned} w \mathrel{\dot\in} \mathsf{sec}(\mathsf{f}_{k+1}^y z, b) &\iff w \mathrel{\dot\in} \mathsf{sec}(\mathsf{dc}(z, \mathsf{f}_k^y)(n_0), b) \\ &\iff \dot{F}^N[w, y, \mathsf{seg}(\mathsf{dc}(z, \mathsf{f}_k^y)(n_0), b), b] \\ &\iff \dot{F}^N[w, y, \mathsf{seg}(\mathsf{f}_{k+1}^y z, b), b]. \end{aligned}$$

Hence the theorem holds. □

Proposition 1.4.1 follows now directly form the above theorem, hence we could successfully embed $(\Pi_0^1\text{-CA})_{<\omega^\omega}\!\restriction$ into $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$. If T denotes one of our theories, we define the proof-theoretic ordinal $|\mathsf{T}|$ of the theory T in the usual way (cf. e.g. [11]). Now we can state our initially announced result.

Proposition 1.4.5 $\varphi\omega 0 \leq |(\Pi_0^1\text{-CA})_{<\omega^\omega}\!\restriction| \leq |\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})|$.

□

2 An Upper Bound for $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})$

In this chapter we'll establish an upper bound for $\mathsf{EETJ}+(\mathsf{dc})+(\mathsf{T\text{-}I_N})$ by formalizing the construction of models in the theory $\mathsf{FID}^{\mathsf{r}}(\mathsf{\Pi}^0_1)$, introduced in Jäger [10]. As a corollary we get that $\varphi\omega 0$ is a sharp upper bound for $\mathsf{FID}^{\mathsf{r}}(\mathsf{\Pi}^0_1)$.

2.1 The Theories $\mathsf{FID}(\mathbb{K})$

Let $\mathbb{L}_1$ be a standard first order language with number variables $x, y, z, \ldots$, possibly with subscripts, a constant 0, symbols for all primitive recursive functions and relations, in particular symbols S_N for the successor and Pd_N for the predecessor function. Then $\mathbb{L}_1(P)$ is the extension of $\mathbb{L}_1$ by a fresh n-ary relation symbol P. An $\mathbb{L}_1(P)$-formula $F(P, \vec{x})$ is called *n-ary operator form* if it contains at most $x_0, \ldots, x_{n-1}, P$ free.

Now let $\mathbb{K}$ be a collection of operator forms. Then we extend $\mathbb{L}_1$ to the language $\mathbb{L}_{\mathbb{K}}$ by adding ordinal variables $\alpha, \beta, \gamma, \ldots$, a binary relation symbol $\prec$ for the less relation on the ordinals and an $(n+1)$-ary relation symbol $P_{\mathcal{A}}$ for each n-ary operator form $\mathcal{A} \in \mathbb{K}$. The number terms $s, t, \ldots$ of $\mathbb{L}_{\mathbb{K}}$ are the terms of $\mathbb{L}_1$, the ordinal terms of $\mathbb{L}_{\mathbb{K}}$ are the ordinal variables. The formulas $F, G, H, \ldots$ are inductively defined as follows:

1. If R is an n-ary relation symbol of $\mathbb{L}_1$, then $R(\vec{s})$ is an (atomic) formula of $\mathbb{L}_{\mathbb{K}}$.

2. $(\alpha \prec \beta)$, $(\alpha = \beta)$ and $P_{\mathcal{A}}(\alpha, \vec{s})$ are (atomic) formulas of $\mathbb{L}_{\mathbb{K}}$.

3. If F and G are formulas, then $\neg F$, $(F \wedge G)$ and $(F \vee G)$ are formulas of $\mathbb{L}_{\mathbb{K}}$.

4. If F is a formula, then $(\forall x)F$ and $(\exists x)F$ are formulas of $\mathbb{L}_{\mathbb{K}}$.

5. If F is a formula, then $(\forall \alpha)F$ and $(\exists \alpha)F$ are formulas of $\mathbb{L}_{\mathbb{K}}$.

We are mainly interested in two classes of operator forms, $\mathsf{\Pi}^0_1$ and POS. The class $\mathsf{\Pi}^0_1$ comprises all $\mathbb{L}_1(P)$-formulas F of the form $(\forall x)G$ where G contains solely bounded number quantifiers, where bounded ordinal [number] quantifiers are quantifiers appearing in the context $(Q\alpha \prec \beta)$, $[(Qx < t)]$. F belongs to POS if each occurrence of P in F is positive. $\Delta^{\mathbb{O}}_0$ denotes the $\mathbb{L}_{\mathbb{K}}$-formulas which do not contain unbounded ordinal quantifiers. Further $\langle \cdot, \ldots, \cdot \rangle$ denotes the usual primitive recursive function for forming n-tuples, Seq is the primitive recursive set of sequence numbers, $lh(t)$ gives the length of the sequence coded by t, i.e. if $t = \langle t_0, \ldots, t_{n-1} \rangle$ then $lh(t) = n$, and $(t)_i$ denotes the i^{th} component of the sequence coded by t. We'll write $(m)_{i,j}$ for $((m)_i)_j$, $1, 2, 3, \ldots$ for $\overline{1}, \overline{2}, \overline{3}, \ldots$, $x + 1$ for $\mathsf{S}_N x$ and $x \dot{-} 1$ for $\mathsf{Pd}_N x$. Additional abbreviations are:

- $P^{\alpha}_{\mathcal{A}}(\vec{s}) :\equiv P_{\mathcal{A}}(\alpha, \vec{s})$,
- $P^{\prec\alpha}_{\mathcal{A}}(\vec{s}) :\equiv (\exists\beta \prec \alpha) P^{\beta}_{\mathcal{A}}(\vec{s})$,
- $P_{\mathcal{A}}(\vec{s}) :\equiv (\exists\alpha) P^{\alpha}_{\mathcal{A}}(\vec{s})$.

Now we present three $\mathbb{L}_{\mathbb{K}}$-theories which differ in the strength of their induction principles. The weakest of these theories is $\mathsf{FID}^{\mathsf{r}}(\mathbb{K})$ and consists of the following axioms:

I. Number-theoretic axioms. The axioms of PA, except complete induction on the natural numbers.

II. Linearity of $\prec$ on the ordinals.

$$(\mathsf{L}, \prec) \quad \alpha \not\prec \alpha \wedge (\alpha \prec \beta \wedge \beta \prec \gamma \to \alpha \prec \gamma) \wedge (\alpha \prec \beta \vee \alpha = \beta \vee \beta \prec \alpha).$$

III. Operator axioms. For every operator form $\mathcal{A} \in \mathbb{K}$:

$$(\mathsf{OP.1}) \quad (\forall \vec{x})[P^{\alpha}_{\mathcal{A}}(\vec{x}) \leftrightarrow P^{\prec\alpha}_{\mathcal{A}}(\vec{x}) \vee \mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, \vec{x})],$$
$$(\mathsf{OP.2}) \quad (\forall \vec{x})[\mathcal{A}(P_{\mathcal{A}}, \vec{x}) \to P_{\mathcal{A}}(\vec{x})].$$

IV. $\Delta^{\mathbb{O}}_0$-induction on the natural numbers.

$$(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_N) \quad F(0) \wedge (\forall x)[F(x) \to F(x+1)] \to (\forall x) F(x), \text{ for all } F \in \Delta^{\mathbb{O}}_0.$$

V. $\Delta^{\mathbb{O}}_0$-induction on the ordinals.

$$(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}}) \quad (\forall\alpha)[(\forall\beta \prec \alpha) F(\beta) \to F(\alpha)] \to (\forall\alpha) F(\alpha), \text{ for all } F \in \Delta^{\mathbb{O}}_0.$$

$\mathsf{FID}^{\mathsf{w}}(\mathbb{K})$ is the extension of $\mathsf{FID}^{\mathsf{r}}(\mathbb{K})$ by the following schema of complete induction on the natural numbers:

$$(\mathsf{F}\text{-}\mathsf{IND}_N) \qquad F(0) \wedge (\forall x)[F(x) \to F(x+1)] \to (\forall x) F(x),$$

for all $\mathbb{L}_{\mathbb{K}}$-formulas F. $\mathsf{FID}(\mathbb{K})$ is the extension of $\mathsf{FID}^{\mathsf{w}}(\mathbb{K})$ by the following schema of induction on the ordinals:

$$(\mathsf{F}\text{-}\mathsf{IND}_{\mathbb{O}}) \qquad (\forall\alpha)[(\forall\beta \prec \alpha) F(\beta) \to F(\alpha)] \to (\forall\alpha) F(\alpha),$$

for all $\mathbb{L}_{\mathbb{K}}$-formulas F.

The operator axioms stated above are tailored according to the usual treatment of monotone or nonmonotone inductive definitions as described for example in Richter [12]. The first ones (OP.1) formalize that the sets $P^{\alpha}_{\mathcal{A}}$ are the stages of the inductive definition generated by the operator form $\mathcal{A}(P,\vec{x})$; then one says that $P_{\mathcal{A}}$ is the set inductively defined by $\mathcal{A}(P,\vec{x})$. The axioms $(\mathsf{OP.2})$ are closure properties which implicitly require that there are sufficiently many ordinals in $\mathsf{FID}(\mathbb{K})$ and its subsystems, so that the process of forming the stages of the inductive definition with clauses from $\mathbb{K}$ comes to an end. The least ordinal $|\mathbb{K}|$ such that for all operator-forms $\mathcal{A} \in \mathbb{K}$ $(\forall \vec{x})(P^{\alpha}_{\mathcal{A}}(\vec{x}) \leftrightarrow P^{\prec\alpha}_{\mathcal{A}}(\vec{x}))$ holds is called *the closure ordinal of the class* $\mathbb{K}$. One has $|\Pi^0_1| = \omega^{ck}_1$ (Gandy, unpublished) and $|\mathsf{POS}| = \omega^{ck}_1$ (Spector [14]).

2.2 A Recursion Theoretic Model

In this section we'll formalize a recursion theoretic model of $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I_N})$ in $\mathsf{FID^r}(\Pi^0_1)$. Thereby we adapt the construction given in Studer [15]. The main ideas are sketched below:

The individual variables of $\mathcal{L}_p$ are ranging over the natural numbers of $\mathsf{FID^r}(\Pi^0_1)$, and application is modeled in the usual recursion theoretic way: $x \cdot y \simeq z$ is interpreted as $\{x^*\}(y^*) \simeq z^*$. To the constants we assign appropriate codes for functions, e.g. to k we assign a numeral k^* such that $\{\{\mathsf{k}^*\}(x)\}(y) = x$ holds. Types are identified with their names, so that the type variables of $\mathcal{L}_p$ are ranging over the natural numbers coding names. The type structure is modeled by the inductively defined relation $P_{\mathcal{A}}(m,n,k)$. $P_{\mathcal{A}}(m,0,0)$ is to express that m codes a type, and $P_{\mathcal{A}}(m,n,1)$ states that n is an element of the type coded by m. The terms nat^* and id^* are meant to code the types named nat and id. noval^* (no value) codes the auxiliary type $\{(f,x) : \neg(fx)\downarrow\}$, so we can check the totality of a function f by the Π^0_1-formula $(\forall x)[P_{\mathcal{A}}(x,0,0) \rightarrow \neg P_{\mathcal{A}}(\mathsf{noval}^*, \langle f,x\rangle, 1)]$. The interpretations co^*, int^*, ...of the $\mathcal{L}_p$-terms co , int, ...are given by codes of total functions: If a codes the type A, $\{\mathsf{co}^*\}(a)$ codes A's complement, and if a and b code the types A and B, $\{\mathsf{int}^*\}(\langle a,b\rangle)$ codes the type $A \cap B$. Because of existential quantifiers are not admitted in the operator form $\mathcal{A}$, we sometimes have to model the complement of a type first, e.g. $\{\mathsf{codom}^*\}(a)$ codes the type named co $(\mathsf{dom}\ a)$, and the type named $\mathsf{dom}\ a$ is then coded by the term $\{\mathsf{co}^*\}(\{\mathsf{codom}^*\}(a))$. So far the relation $P_{\mathcal{A}}$ could be defined by a monotone inductive definition, i.e. $P_{\mathcal{A}}$ could be defined within $\mathsf{FID^r}(\mathsf{POS})$ (cf. Studer [15]). However, the limited strength of $\mathsf{FID^r}(\mathsf{POS})$ doesn't allow to model the axiom (dc). In particular, we can't verify the condition $(f : \Re \rightarrow \Re)$ by a positive operator form. That is where we need the full strength of $\mathsf{FID^r}(\Pi^0_1)$. What we want to ensure is the following: Provided we have a function $(f : \Re \rightarrow \Re)$ and a name a, then there should be a stage α such that $P^{\prec\alpha}_{\mathcal{A}}(m,0,0) \rightarrow P^{\prec\alpha}_{\mathcal{A}}(\{f\}(m),0,0)$ and $P^{\prec\alpha}_{\mathcal{A}}(a,0,0)$ holds for all m. We mark this stage by adding the triple $(\{\mathsf{cl}^*\}(\langle a,f\rangle),0,2)$ to the relation $P_{\mathcal{A}}$. That makes the statement $(\mathsf{dc}(a,f) : \mathsf{N} \rightarrow \Re)$ correspond to the claim $(\forall n)P^{\prec\alpha}_{\mathcal{A}}(\{\{\mathsf{dc}^*\}(\langle a,f\rangle)\}(n),0,0)$, which is $\Delta^{\mathbb{O}}_0$ and therefore provable by $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_N)$.

Let's outline the above given sketch. With Kleene brackets in mind, we extend the language $\mathbb{L}_1$ by the 2-place function symbol '$\{\}$' to the language $\mathbb{L}_1^p$. The terms and formulas of $\mathbb{L}_1^p$ are defined as usual, and we write $\{s\}(t)$ for the $\mathbb{L}_1^p$-term $\{\}(s,t)$. Now we further extend the language $\mathbb{L}_1^p$ to $\mathbb{L}_1^p(P)$ and then to $\mathbb{L}_{\Pi_1^0}^p$ in the same way we extended $\mathbb{L}_1$ to $\mathbb{L}_1(P)$ and $\mathbb{L}_{\Pi_1^0}$, but with the difference that we only add a relation symbol to $\mathbb{L}_{\Pi_1^0}^p$ if it belongs also to $\mathbb{L}_{\Pi_1^0}$. The purpose of introducing the languages $\mathbb{L}_1^p(P)$ and $\mathbb{L}_{\Pi_1^0}^p$ is to have Kleene brackets at disposal.

An $\mathbb{L}_{\Pi_1^0}^p$-formula F is meant to abbreviate the $\mathbb{L}_{\Pi_1^0}$-formula $F^\star$, where the restriction of $\cdot^\star$ to the atoms of $\mathbb{L}_{\Pi_1^0}^p$ is given by the following inductive definition. For $\mathbb{L}_1^p(P)$-formulas $\cdot^\star$ is defined as below, but with the first and last clause properly adjusted.

- If F is the formula $(y = z)$, $(\mathsf{c} = z)$, $(\alpha = \beta)$ or $(\alpha \prec \beta)$, then $F^\star$ is the formula $(y = z)$, $(\mathsf{c} = z)$,$(\alpha = \beta)$ or $(\alpha \prec \beta)$, where c denotes the constants of $\mathbb{L}_{\Pi_1^0}^p$.
- If $F \equiv (\{x\}(y) = z)$, then $F^\star :\equiv (\exists u)[\mathsf{T}(x, \langle y\rangle, u) \wedge (u)_0 = z]$.
- If $F \equiv (f(\vec{s}) = z)$, then $F^\star :\equiv (\exists \vec{x})[(\vec{s} = \vec{x})^\star \wedge f(\vec{x}) = z]$, for every function symbol f of $\mathbb{L}_{\Pi_1^0}$.
- If $F \equiv (\{r\}(s) = z)$ then $F^\star :\equiv (\exists x)(\exists y)[(r = x)^\star \wedge (s = y)^\star \wedge (\{x\}(y) = z)^\star]$.
- If $F \equiv (s = t)$, then $F^\star :\equiv (\exists x)[(s = x)^\star \wedge (t = x)^\star]$.
- If $F \equiv (R(\alpha, \vec{s}))$, then $F^\star :\equiv (\exists \vec{x})[(\vec{s} = \vec{x})^\star \wedge R(\alpha, \vec{x})]$, for every relation symbol R of $\mathbb{L}_{\Pi_1^0}^p$.

The map $\cdot^\star$ extends canonically to all $\mathbb{L}_{\Pi_1^0}^p$-formulas. The definition of $(\{x\}(y) = z)^\star$ is the usual way of introducing Kleene brackets. Observe that the relation $\mathsf{T}(x, \langle y\rangle, u)$ (Kleene's T-predicate) is primitive recursive. $\mathsf{T}(x, \langle y\rangle, u)$ states that $u = \langle z, x, y, v_0, \dots, v_{n-1}\rangle$ codes the computation of the value z by the function with code x, given the input y. For details confer Hinman [8]. Further, if s,t are $\mathbb{L}_1^p$-terms, then $t\downarrow$ abbreviates the formula $(\exists x)(t = x)^\star$, and $s \simeq t$ stands for the formula $(s\downarrow \vee\, t\downarrow) \to (s = t)^\star$.

The ordinary recursion theorem allows us to define the $\mathbb{L}_{\Pi_1^0}$-numerals $\mathsf{k}^*, \mathsf{s}^*, \mathsf{p}^*, \mathsf{p}_0^*, \mathsf{p}_1^*$, s_N^*, p_N^*, d_N^*, and dc^* such that the following formulas hold. These numerals are adequate interpretations of the corresponding $\mathcal{L}_p$-constants $\mathsf{k}, \mathsf{s}, \mathsf{p}, \mathsf{p}_0, \mathsf{p}_1, \mathsf{s}_N$, p_N, d_N and dc.

- $\{\{\mathsf{k}^*\}(x)\}(y) = y$,

- $\{\{\mathsf{s}^*\}(x)\}(y)\downarrow$,
- $\{\{\{\mathsf{s}^*\}(x)\}(y)\}(z) \simeq \{\{x\}(z)\}(\{y\}(z))$,
- $\{\{\mathsf{p}^*\}(x)\}(y) = \langle x, y\rangle$,
- $\{\mathsf{p}_0^*\}(\langle x, y\rangle) = x$,
- $\{\mathsf{p}_1^*\}(\langle x, y\rangle) = y$,
- $\{\mathsf{s}_\mathsf{N}^*\}(x) = x + 1$,
- $\{\mathsf{p}_\mathsf{N}^*\}(x) = x \dot{-} 1$,
- $z_0 = z_1 \to \{\{\{\{\mathsf{d}_\mathsf{N}^*\}(x)\}(y)\}(z_0)\}(z_1) = x$,
- $z_0 \neq z_1 \to \{\{\{\{\mathsf{d}_\mathsf{N}^*\}(x)\}(y)\}(z_0)\}(z_1) = y$,
- $\{\{\mathsf{dc}^*\}(\langle x, y\rangle)\}(0) = x$,
- $\{\{\mathsf{dc}^*\}(\langle x, y\rangle)\}(z + 1) \simeq \{y\}(\{\{\mathsf{dc}^*\}(\langle x, y\rangle)\}(z))$.

Further we need the $\mathbb{L}^p_{\Pi^0_1}$-numerals $\mathsf{nat}^*, \mathsf{id}^*, \mathsf{co}^*, \mathsf{int}^*, \mathsf{codom}^*, \mathsf{dom}^*, \mathsf{coinv}^*, \mathsf{inv}^*, \mathsf{coj}^*$, j^* and cl^* to code the type structure. We choose them such that the following holds:

- $\mathsf{nat}^* :\equiv \langle 1, 0\rangle$,
- $\mathsf{id}^* :\equiv \langle 2, 0\rangle$,
- $\mathsf{noval}^* :\equiv \langle 3, 0\rangle$,
- $\{\mathsf{int}^*\}(x) = \langle 4, x\rangle$,
- $\{\mathsf{co}^*\}(x) = \langle 5, x\rangle$,
- $\{\mathsf{codom}^*\}(x) = \langle 6, x\rangle$,
- $\{\mathsf{coinv}^*\}(x) = \langle 7, x\rangle$,
- $\{\mathsf{coj}^*\} = \langle 8, x\rangle$,
- $\{\mathsf{cl}^*\}(x) = \langle 9, x\rangle$,
- $\{\mathsf{dom}^*\}(x) = \{\mathsf{co}^*\}(\{\mathsf{codom}^*\}(x))$,
- $\{\mathsf{inv}^*\}(x) = \{\mathsf{co}^*\}(\{\mathsf{coinv}^*\}(x))$,
- $\{\mathsf{j}^*\}(x) = \{\mathsf{co}^*\}(\{\mathsf{coj}^*\}(x))$.

Now the stage is set to present the operator form $\mathcal{A}$. Observe that $\mathcal{A}$ is indeed (equivalent to) a Π^0_1-formula: Kleene brackets appear only in the context $\neg P(\{f(x)\}(y), n, k)$ which translates to the $\mathbb{L}_1(P)$-formula

$$\neg[(\exists e)(\exists z)(f(x) = e \wedge (\exists u)[\mathsf{T}(e, \langle y\rangle, u) \wedge (u)_0 = z] \wedge P(z, n, k)],$$

and $\neg[\{(n)_0\}((n)_1)\downarrow]$ which translates to an $\mathbb{L}_1(P)$-formula equivalent to

$$(\forall u)\neg[\mathsf{T}((n)_0, \langle (n)_1\rangle, u)].$$

To keep our definition of $\mathcal{A}$ readable, we write e.g. $m = \{\mathsf{int}^*\}(\langle a, b\rangle) \wedge P(a, 0, 0)$ for the statement

$$m = \langle (m)_0, (m)_1\rangle \wedge (m)_0 = 4 \wedge (m)_1 = (\langle (m)_{1,0}, (m)_{1,1}\rangle) \wedge P((m)_{1,0}, 0, 0).$$

Definition 2.2.1 *$\mathcal{A}(P, m, n, k)$ is the disjunction of the following formulas:*

1a) $m = \mathsf{noval}^* \wedge n = 0 \wedge k = 0,$
1b) $m = \mathsf{noval}^* \wedge n = \langle (n)_0, (n)_1\rangle \wedge \neg\{(n)_0\}((n)_1)\downarrow \wedge\ k = 1,$

2a) $m = \mathsf{nat}^* \wedge n = 0 \wedge k = 0,$
2b) $m = \mathsf{nat}^* \wedge k = 1,$

3a) $m = \mathsf{id}^* \wedge n = 0 \wedge k = 0,$
3b) $m = \mathsf{id}^* \wedge n = \langle (n)_0, (n)_0\rangle \wedge k = 1,$

4a) $m = \{\mathsf{int}^*\}(\langle a, b\rangle) \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge n = 0 \wedge k = 0,$
4b) $m = \{\mathsf{int}^*\}(\langle a, b\rangle) \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge P(a, n, 1) \wedge P(b, n, 1) \wedge k = 1,$

5a) $m = \{\mathsf{co}^*\}(a) \wedge P(a, 0, 0) \wedge n = 0 \wedge k = 0,$
5b) $m = \{\mathsf{co}^*\}(a) \wedge P(a, 0, 0) \wedge \neg P(a, n, 1) \wedge k = 1,$

6a) $m = \{\mathsf{codom}^*\}(a) \wedge P(a, 0, 0) \wedge n = 0 \wedge k = 0,$
6b) $m = \{\mathsf{codom}^*\}(a) \wedge P(a, 0, 0) \wedge (\forall q)\neg P(a, \langle n, q\rangle, 1) \wedge k = 1,$

7a) $m = \{\mathsf{coinv}^*\}(\langle a, f\rangle) \wedge P(a, 0, 0) \wedge n = 0 \wedge k = 0,$
7b) $m = \{\mathsf{coinv}^*\}(\langle a, f\rangle) \wedge P(a, 0, 0) \wedge \neg P(a, \{f\}(n), 1) \wedge k = 1,$

8a) $m = \{\mathsf{coj}^*\}(\langle a, f\rangle) \wedge P(a, 0, 0) \wedge (\forall x)[P(a, x, 1) \rightarrow \neg P(\mathsf{noval}^*, \langle f, x\rangle, 1)] \wedge$
$(\forall x)(\forall y)[P(a, x, 1) \wedge \{f\}(x) = y \rightarrow P(y, 0, 0)] \wedge n = 0 \wedge k = 0,$
8b) $m = \{\mathsf{coj}^*\}(\langle a, f\rangle) \wedge P(a, 0, 0) \wedge (\forall x)[P(a, x, 1) \rightarrow \neg P(\mathsf{noval}^*, \langle f, x\rangle, 1)] \wedge$
$(\forall x)(\forall y)[P(a, x, 1) \wedge \{f\}(x) = y \rightarrow P(y, 0, 0)] \wedge$
$\neg[n = \langle (n)_0, (n)_1\rangle \wedge P(a, (n)_0, 1) \wedge P(\{f\}((n)_0), (n)_1, 1)] \wedge k = 1,$

9) $m = \{\mathsf{cl}^*\}(\langle a, f\rangle) \wedge P(a, 0, 0) \wedge (\forall x)[P(x, 0, 0) \rightarrow \neg P(\mathsf{noval}^*, \langle f, x\rangle, 1)] \wedge$
$(\forall x)(\forall y)[P(x, 0, 0) \wedge \{f\}(x) = y \rightarrow P(y, 0, 0)] \wedge n = 0 \wedge k = 2.$

Observe that the type noval^* is generated at the very first stage, i.e. ${P_{\mathcal{A}}}^0(\mathsf{noval}^*, 0, 0)$ holds. Hence when we refer to the type noval^* in 8) and 9), we know that it's already built. Moreover, we have the following lemmas.

Lemma 2.2.2 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash P^\beta_{\mathcal{A}}(m,n,k) \to (\exists \alpha \preceq \beta)[P^\alpha_{\mathcal{A}}(m,n,k) \wedge \neg P^{\prec\alpha}_{\mathcal{A}}(m,n,k)]$.

PROOF: We show the claim by $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on β. $(\mathsf{OP}.1)$ yields $P^{\prec\beta}_{\mathcal{A}}(m,n,k)$ or $\mathcal{A}(P^{\prec\beta}_{\mathcal{A}}, m, n, k)$. If $P^{\prec\beta}_{\mathcal{A}}(m,n,k)$ holds the claim follows by the IH (induction hypothesis), if $\neg P^{\prec\beta}_{\mathcal{A}}(m,n,k)$, then β is a witness for α. □

Lemma 2.2.3 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash \mathcal{A}(P_{\mathcal{A}}, m, n, k) \to (\exists \alpha)\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, n, k)$.

PROOF: Let $\mathcal{A}(P_{\mathcal{A}}, m, n, k)$. By $(\mathsf{OP}.2)$ we have $P_{\mathcal{A}}(m,n,k)$. Now lemma 2.2.2 gives us an α such that $P^\alpha_{\mathcal{A}}(m,n,k)$ and $\neg P^{\prec\alpha}_{\mathcal{A}}(m,n,k)$, hence by $(\mathsf{OP}.1)$ we have $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, n, k)$. □

Lemma 2.2.4 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash P^\alpha_{\mathcal{A}}(m,n,1) \to P^\alpha_{\mathcal{A}}(m,0,0)$.

PROOF: By $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. If $P^{\prec\alpha}_{\mathcal{A}}(m,n,1)$, the IH applies, otherwise we have $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, n, 1)$. Now the definition of $\mathcal{A}$ yields immediately $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, 0, 0)$, that is $P^\alpha_{\mathcal{A}}(m,0,0)$. □

Lemma 2.2.5 *(Persistence Lemma)* $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$P^\alpha_{\mathcal{A}}(m,0,0) \wedge P^\beta_{\mathcal{A}}(m,n,1) \to P^\alpha_{\mathcal{A}}(m,n,1).$$

PROOF: By $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. If $P^{\prec\alpha}_{\mathcal{A}}(m,0,0)$, the IH applies, otherwise we have $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, 0, 0)$. Then there is a $\beta' \preceq \beta$ with $P^{\beta'}_{\mathcal{A}}(m,n,1)$ and $\neg P^{\prec\beta'}_{\mathcal{A}}(m,n,1)$, hence $\mathcal{A}(P^{\prec\beta'}_{\mathcal{A}}, m, n, 1)$ and $\alpha \preceq \beta'$ by lemma 2.2.4. If $m = \mathsf{nat}^*$ or $m = \mathsf{id}^*$, the claim is obvious. If $m = \{\mathsf{int}^*\}(\langle a, b\rangle)$, then $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, 0, 0)$ and $\mathcal{A}(P^{\prec\beta'}_{\mathcal{A}}, m, n, 1)$, therefore $P^{\prec\alpha}_{\mathcal{A}}(a,0,0)$ and $P^{\prec\alpha}_{\mathcal{A}}(b,0,0)$ as well as $P^{\prec\beta}_{\mathcal{A}}(a,n,1)$ and $P^{\prec\beta}_{\mathcal{A}}(b,n,1)$. Now the IH yields $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, n, 1)$. If $m = \{\mathsf{codom}^*\}(a)$, then $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, 0, 0)$ and $\mathcal{A}(P^{\prec\beta'}_{\mathcal{A}}, m, n, 1)$, therefore $P^{\prec\alpha}_{\mathcal{A}}(a,0,0)$ as well as $(\forall q)[\neg P^{\prec\beta'}_{\mathcal{A}}(a, \langle n, q\rangle, 1)$. We may assume that $\alpha \preceq \beta'$, so $(\forall q)[\neg P^{\prec\beta'}_{\mathcal{A}}(a, \langle n, q\rangle, 1)]$ implies $(\forall q)[\neg P^{\prec\alpha}_{\mathcal{A}}(a, \langle n, q\rangle, 1)]$, hence $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}}, m, n, 1)$. The remaining cases are shown similarly. □

In order to give a translation $\cdot^I$ from $\mathcal{L}_p$ into $\mathbb{L}_{\Pi^0_1}$ it suffices to give a translation $\cdot^*$ from $\mathcal{L}_p$ into $\mathbb{L}^p_{\Pi^0_1}$. The translation $\cdot^I$ is then given by the composition $(\cdot^*)^\star$.

First we define $*$ for $\mathcal{L}_p$-terms:

- If t is a individual variable x or a type variable X, then t^* is the number variable x^*, X^* respectively; where $\cdot^*$ maps individual and type variables one-one to number variables of $\mathbb{L}_{\Pi^0_1}$, i.e. syntactically different variables of $\mathcal{L}_p$ are mapped on syntactically different variables of $\mathbb{L}_{\Pi^0_1}$.

- If t is the constant $\mathsf{0}, \mathsf{k}, \mathsf{s}, \ldots$, then t^* is the constant $0, \mathsf{k}^*, \mathsf{s}^*, \ldots$.

- If $t \equiv r \cdot s$, then $t^* :\equiv \{r^*\}(s^*)$.

Now we extend $*$ to $\mathcal{L}_p$-formulas:

$$\begin{array}{rcl}
(r = s)^* & :\equiv & (r^* = t^*), \\
[\mathsf{N}(t)]^* & :\equiv & (\exists x)(t^* = x), \\
(t\downarrow)^* & :\equiv & (\exists x)(t^* = x), \\
(t \in X)^* & :\equiv & P_{\mathcal{A}}(X^*, t^*, 1), \\
[\Re(t, X)]^* & :\equiv & P_{\mathcal{A}}(t^*, 0, 0) \wedge (\forall x)(P_{\mathcal{A}}(t^*, x, 1) \leftrightarrow P_{\mathcal{A}}(X^*, x, 1)), \\
(X = Y)^* & :\equiv & (\forall x)(P_{\mathcal{A}}(X^*, x, 1) \leftrightarrow P_{\mathcal{A}}(Y^*, x, 1)), \\
(\neg F)^* & :\equiv & \neg F^*, \\
(F \mathrel{\mathrm{j}} G)^* & :\equiv & F^* \mathrel{\mathrm{j}} G^*, \\
[(Qx)F]^* & :\equiv & (Qx^*)F^*, \\
[(\exists X)F]^* & :\equiv & (\exists X^*)(P_{\mathcal{A}}(X^*, 0, 0) \wedge F^*), \\
[(\forall X)F]^* & :\equiv & (\forall X^*)(P_{\mathcal{A}}(X^*, 0, 0) \rightarrow F^*).
\end{array}$$

As usual j denotes the connectives $\wedge$ and $\vee$, and Q stands for a quantifier.
Now we have the following proposition:

Proposition 2.2.6 *For every axiom F of $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$ with its free type variables among $X_0, \ldots, X_{n-1}$ we have*

$$\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash P_{\mathcal{A}}(X^*_0, 0, 0) \wedge \ldots \wedge P_{\mathcal{A}}(X^*_{n-1}, 0, 0) \rightarrow F^I.$$

PROOF: The logical axioms and the axioms concerning the constants and the natural numbers are shown as in Studer [15]. From the axioms concerning elementary comprehension, we show exemplary the axiom (INV). For $(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$ and (dc), the proof is given below, too.

The axiom (INV.1) translates to

$$P_{\mathcal{A}}(B^*, 0, 0) \rightarrow (\exists X^*)[P_{\mathcal{A}}(X^*, 0, 0) \wedge (\forall x^*)(P_{\mathcal{A}}(X^*, x^*, 1) \leftrightarrow \neg P_{\mathcal{A}}(B^*, \{f^*\}(x^*), 1))].$$

$P_{\mathcal{A}}(B^*, 0, 0)$ implies $P_{\mathcal{A}}(\{\mathsf{coinv}^*\}(\langle B^*, f^*\rangle), 0, 0)$. Now we show that

$$(\forall x^*)[P_{\mathcal{A}}(\{\mathsf{coinv}^*\}(\langle B^*, f^*\rangle), x^*, 1) \leftrightarrow \neg P_{\mathcal{A}}(B^*, \{f^*\}(x^*), 1)].$$

Assume that $P_{\mathcal{A}}(\{\mathsf{coinv}^*\}(\langle B^*, f^*\rangle), x^*, 1)$. As before we can find an α such that $\mathcal{A}(P_{\mathcal{A}}^{\prec\alpha}, \{\mathsf{coinv}^*\}(\langle B^*, f^*\rangle), x^*, 1)$ holds. So $P_{\mathcal{A}}^{\prec\alpha}(B^*, 0, 0)$ and $\neg P_{\mathcal{A}}^{\prec\alpha}(B^*, \{f^*\}(x^*), 1)$ by the definition of $\mathcal{A}$. Now persistence yields $\neg P_{\mathcal{A}}(B^*, \{f^*\}(x^*), 1)$. Because of $\{\mathsf{inv}^*\}(x^*) = \{\mathsf{co}^*\}(\{\mathsf{coinv}^*\}(x^*))$, X^* can be witnessed by $\{\mathsf{inv}^*\}(\langle B^*, f^*\rangle)$. The translation of ($\mathsf{INV.2}$) follows from the translation of ($\mathsf{INV.1}$) and lemma 2.2.4.
The translation of ($\mathsf{T\text{-}I_N}$) is equivalent to

$$
\begin{aligned}
&P_{\mathcal{A}}(A^*, 0, 0) \rightarrow \\
&[P_{\mathcal{A}}(A^*, 0, 1) \wedge (\forall x^*)[P_{\mathcal{A}}(A^*, x^*, 1) \rightarrow P_{\mathcal{A}}(A^*, x^* + 1, 1)] \rightarrow (\forall x^*) P_{\mathcal{A}}(A^*, x^*, 1)].
\end{aligned}
$$

Assume $P_{\mathcal{A}}(A^*, 0, 0)$. By ($\mathsf{OP.2}$) we find an α such that $P_{\mathcal{A}}^{\alpha}(A^*, 0, 0)$ holds, hence $P_{\mathcal{A}}(A^*, n, 1)$ implies $P_{\mathcal{A}}^{\alpha}(A^*, n, 1)$ by persistence. Therefore the conclusion becomes equivalent to

$$
P_{\mathcal{A}}^{\alpha}(A^*, 0, 1) \wedge (\forall x^*)[P_{\mathcal{A}}^{\alpha}(A^*, x^*, 1) \rightarrow P_{\mathcal{A}}^{\alpha}(A^*, x^* + 1, 1)] \rightarrow (\forall x^*) P_{\mathcal{A}}^{\alpha}(A^*, x^*, 1),
$$

and follows immediately with ($\Delta_0^{\mathbb{O}}$-$\mathsf{IND_N}$).
The translation of ($\mathsf{dc.2}$) holds due to the choice of the numeral dc^*. It remains to show the translation of ($\mathsf{dc.1}$), which is equivalent to

$$
\begin{aligned}
&P_{\mathcal{A}}(a, 0, 0) \wedge (\forall x)[P_{\mathcal{A}}(x, 0, 0) \rightarrow P_{\mathcal{A}}(\{f\}(x), 0, 0)] \rightarrow \\
&(\forall n) P_{\mathcal{A}}(\{\{\mathsf{dc}^*\}(\langle a, f\rangle)\}(n), 0, 0).
\end{aligned}
$$

Under the assumption that the premise holds, we have $\mathcal{A}(P_{\mathcal{A}}, \{\mathsf{cl}^*\}(\langle a, f\rangle), 0, 2)$. Hence lemma 2.2.3 yields $\mathcal{A}(P_{\mathcal{A}}^{\prec\alpha}, \{\mathsf{cl}^*\}, (\langle a, f\rangle), 0, 2)$ for some α, therefore we have $P_{\mathcal{A}}^{\prec\alpha}(a, 0, 0) \wedge (\forall x) P_{\mathcal{A}}^{\prec\alpha}(x, 0, 0) \rightarrow P_{\mathcal{A}}^{\prec\alpha}(\{f\}(x), 0, 0)$. Now ($\Delta_0^{\mathbb{O}}$-$\mathsf{IND}_N$) enables us to prove $(\forall n) P_{\mathcal{A}}^{\prec\alpha}(\{\{\mathsf{dc}^*\}(\langle a, f\rangle)\}(n), 0, 0)$, and we are done.

□

Hence we have established that the theory $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})$ isn't stronger than the theory $\mathsf{FID^r}(\Pi_1^0)$. By techniques presented in Jäger [10] and standard methods from proof-theory it can be shown that the proof-theoretic strength of the theory $\mathsf{FID^r}(\Pi_1^0)$ is at most $\varphi\omega 0$. That allows us to state the following proposition.

Proposition 2.2.7 $|\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})| \leq |\mathsf{FID^r}(\Pi_1^0)| \leq \varphi\omega 0$.

□

Together with proposition 1.4.5 from chapter 1, this yields:

Corollary 2.2.8 $|\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})| = |\mathsf{FID^r}(\Pi_1^0)| = \varphi\omega 0$.

□

In the final section of our thesis we show, that also $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N}) + (\mathsf{Tot})$ can be embedded into $\mathsf{FID^r}(\Pi_1^0)$.

2.3 A Term Model

In this section we'll formalize a total term model of $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N}) + (\mathsf{Tot})$ in $\mathsf{FID^r}(\Pi^0_1)$. Thereby we proceed basically as in the previous section. However, this times things are a bit more complicate. We give a rough sketch:
The individual variables of $\mathcal{L}_p$ are ranging over the $\mathbb{L}_{\Pi^0_1}$-terms coding the closed $\mathcal{L}_p$-terms, and the application is modeled in the usual term model way: If s^* codes the $\mathcal{L}_p$-term s and t^* codes the $\mathcal{L}_p$-term t, then $\mathsf{s}^* \circ t^*$ is given by $(s \cdot t)^*$. The constants are modeled together with equality. Equality is interpreted by the Σ^0_1-relation $\approx_\rho$. That way we force the constants to behave appropriate, e.g. we have $((\mathsf{k}^* \circ x) \circ y) \approx_\rho x$. The natural numbers are represented by the $\mathbb{L}_{\Pi^0_1}$-terms n with $n \approx_\rho \mathrm{Num}(x)$ for some x, where $\mathrm{Num}(x)$ denotes the Gödelnumber of the x^{th} natural number. As before, types are identified with their names, so that the type variables of $\mathcal{L}_p$ are ranging over the $\mathbb{L}_{\Pi^0_1}$-terms coding names. The type structure is modeled by the inductively defined relation $P_\mathcal{A}(m,n,k)$. Again, $P_\mathcal{A}(m,0,0)$ is to express that m is a name, and $P_\mathcal{A}(m,n,k)$ states that n is an element of the type named m. The type structure and the axiom (dc) are modeled as before. However, when we generate the relation $P_\mathcal{A}$, we have to take into account that equality is interpreted by the relation $\approx_\rho$. We can't add just one representative t of the equivalence class $[t]_{\approx_\rho}$ to $P_\mathcal{A}$ and model the naming relation by interpreting $\Re(a)$ by $(\exists x)[P_\mathcal{A}(x,0,0) \wedge a \approx_\rho x]$: To check if we can include e.g. the triple $(\mathsf{j}^* \circ (a,f)^\star, 0, 0)$ into $P_\mathcal{A}$ we have to check if $(\forall x)[P_\mathcal{A}(a,x,1) \to P_\mathcal{A}(f \circ x, 0, 0)]$. But in general, the terms $f \circ x$ aren't canonical representatives of their equivalence classes, and the test $(\exists x)(f \circ x \approx_\rho y \wedge P(y,0,0)$ can't be performed for the operator form $\mathcal{A}$ is Π^0_1. So if we want to model the type structure by the relation $P_\mathcal{A}$ as in the previous section, $P_\mathcal{A}$ has to be closed under $\approx_\rho$, i.e. if $P_\mathcal{A}(m,n,1)$ and $m' \approx_\rho m$ and $n' \approx_\rho n$ then $P_\mathcal{A}(m',n',1)$.

Let's make thing precise. First we assign to each constant c of $\mathcal{L}_p$ and to the function symbol $\cdot$ Gödelnumbers $\ulcorner c \urcorner$ and $\ulcorner \cdot \urcorner$ such that $\ulcorner c \urcorner$ and $\ulcorner \cdot \urcorner$ aren't elements of Seq. Then the Gödelnumber of a compound term (st) can be defined by

$$\ulcorner st \urcorner = \langle \ulcorner \cdot \urcorner, \ulcorner s \urcorner, \ulcorner t \urcorner \rangle.$$

Due to this definition we have a primitive recursive relation $\mathrm{CTer}(x)$, indicating that x is the Gödelnumber of a closed term, and a primitive recursive function $\mathrm{Num}(x)$ satisfying $\mathrm{Num}(x) = \ulcorner \overline{x} \urcorner$, i.e. $\mathrm{Num}(x)$ is the Gödelnumber of the x^{th} numeral of $\mathcal{L}_p$. Further let $\ulcorner \mathrm{codom} \urcorner$ be a natural number that isn't in Seq and different from all the Gödelnumbers $\ulcorner \cdot \urcorner$ and $\ulcorner c \urcorner$.
Next, we define a translation $\cdot^*$ from the $\mathcal{L}_p$-terms into the $\mathbb{L}_{\Pi^0_1}$-terms in the following way:

- If t is a individual variable x or a type variable X, then t^* is the number variable x^*, X^* respectively; where $\cdot^*$ maps individual and type variables one-one to number variables of $\mathbb{L}_{\Pi^0_1}$, i.e. syntactically different variables of $\mathcal{L}_p$ are mapped on different variables of $\mathbb{L}_{\Pi^0_1}$.

- If t is an individual constant, then

$$t^* :\equiv \begin{cases} \langle \ulcorner \cdot \urcorner, \mathsf{co}^*, \ulcorner \mathrm{codom} \urcorner \rangle & \text{if } t \equiv \mathsf{dom}\ , \\ \ulcorner t \urcorner & \text{otherwise.} \end{cases}$$

- If $t \equiv r \cdot s$, then $t^* :\equiv \langle \ulcorner \cdot \urcorner, r^*, s^* \rangle$.

Further we define the primitive recursive functions $\circ$, $(\cdot,\cdot)^\star$, $[\cdot]_0$ and $[\cdot]_1$ by:

- $x \circ y := \langle \ulcorner \cdot \urcorner, x, y \rangle$
- $(x,y)^\star := \langle \ulcorner \cdot \urcorner, \langle \ulcorner \cdot \urcorner, \mathsf{p}^*, x \rangle, y \rangle$
- $[n]_0 := (n)_{1,2}$
- $[n]_1 := (n)_2$

If s, t are $\mathcal{L}_p$-terms then $(s \cdot t)^* = s^* \circ t^*$, $(s,t)^* = (s^*, t^*)^\star$, and $[([n]_0, [n]_1)^\star]_0 = [n]_0$ and $[([n]_0, [n]_1)^\star]_1 = [n]_1$.

Now we focus on the relation $\approx_\rho$ that is to interpret equality. It is based on a binary relation $\boldsymbol{\rho}$ (the notion of reduction) on the $\mathcal{L}_p$-terms, that is tailored to model the behaviour of the $\mathcal{L}_p$-constants. The relation $\boldsymbol{\rho}$ is given by the following redex-contractum pairs, where t_0, t_1, t_2 are $\mathcal{L}_p$ terms, m, n are natural numbers with $m \neq n$ and $\overline{m}, \overline{n}$ are the corresponding numerals of $\mathcal{L}_p$.

$$\begin{array}{rcl}
\mathsf{k}t_0t_1 & \boldsymbol{\rho} & t_0, \\
\mathsf{s}t_0t_1t_2 & \boldsymbol{\rho} & t_0t_2(t_0t_1), \\
\mathsf{p}_0(\mathsf{p}t_0t_1) & \boldsymbol{\rho} & t_0, \\
\mathsf{p}_1(\mathsf{p}t_0t_1) & \boldsymbol{\rho} & t_1, \\
\mathsf{p}_N(\mathsf{s}_N\overline{m}) & \boldsymbol{\rho} & \overline{m}, \\
\mathsf{d}_Nt_0t_1\overline{m}\,\overline{m} & \boldsymbol{\rho} & t_0, \\
\mathsf{d}_Nt_0t_1\overline{m}\,\overline{n} & \boldsymbol{\rho} & t_1, \\
\mathsf{dc}(\mathsf{p}t_0t_1)0 & \boldsymbol{\rho} & t_0, \\
\mathsf{dc}(\mathsf{p}t_0t_1)(\mathsf{s}_N\overline{n}) & \boldsymbol{\rho} & t_1(\mathsf{dc}(\mathsf{p}t_0t_1)\overline{n}).
\end{array}$$

This notion of reduction induces the binary relation $\rightarrow_\rho$ of one step $\boldsymbol{\rho}$ reduction (the compatible closure of $\boldsymbol{\rho}$) and the binary relation $\twoheadrightarrow_\rho$ of $\boldsymbol{\rho}$ reduction (the reflexive

transitive closure of $\rightarrow_\rho$). We remark that $\twoheadrightarrow_\rho$ satisfies the Church Rosser property (cf. e.g. Barendregt [1]).
Further we need a formalized version Red_ρ of the relation $\twoheadrightarrow_\rho$ on the Gödelnumbers of the closed terms of $\mathcal{L}_p$. For that purpose, let $\mathrm{RedCon}_\rho(x,y)$ be a primitive recursive relation formalizing the notion of reduction $\boldsymbol{\rho}$. Then a formalized version $\mathrm{Red1}_\rho(x,y)$ of $\rightarrow_\rho$ can be described by the following primitive recursive definition:

$$\mathrm{Red1}_\rho(x,y) := \mathrm{CTer}(x) \wedge \mathrm{CTer}(y) \wedge \mathrm{Red1}^*_\rho(x,y),$$

where $\mathrm{Red1}^*_\rho(x,y)$ is the disjunction of the following formulas:

(1) $\mathrm{RedCon}_\rho(x,y)$,

(2) $x = \langle \ulcorner \cdot \urcorner, (x)_1, (x)_2 \rangle \wedge y = \langle \ulcorner \cdot \urcorner, (x)_1, (y)_2 \rangle \wedge \mathrm{Red1}_\rho((x)_2, (y)_2)$,

(3) $x = \langle \ulcorner \cdot \urcorner, (x)_1, (x)_2 \rangle \wedge y = \langle \ulcorner \cdot \urcorner, (y)_1, (x)_2 \rangle \wedge \mathrm{Red1}_\rho((x)_1, (y)_1)$.

In order to formalize the reflexive, transitive closure $\twoheadrightarrow_\rho$ of $\rightarrow_\rho$ we define an intermediate predicate $\mathrm{RedSeq}_\rho(x,y,z)$ with the intended meaning that x codes a reduction sequence from the closed term with Gödelnumber y to the closed term with Gödelnumber z with respect to $\rightarrow_\rho$:

$$\mathrm{RedSeq}_\rho(x,y,z) := \mathrm{Seq}(x) \wedge \mathrm{CTer}(y) \wedge \mathrm{CTer}(z) \wedge \mathrm{Red}^*_\rho(x,y,z),$$

where $\mathrm{Red}^*_\rho(x,y,z)$ is the disjunction of the following formulas:

(1) $lh(x) = 1 \wedge x = \langle y \rangle \wedge y = z$,

(2) $lh(x) > 1 \wedge y = (x)_0 \wedge z = (x)_{lh(x) \dot{-} 1} \wedge (\forall i < lh(x) \dot{-} 1)\mathrm{Red1}_\rho((x)_i, (x)_{i+1})$.

The formalization Red_ρ of $\twoheadrightarrow_\rho$ is then given by the $\mathbf{\Sigma}^0_1$-formula

$$\mathrm{Red}_\rho(y,z) :\equiv (\exists x)(\mathrm{RedSeq}_\rho(x,y,z).$$

It is well known (cf. e.g. Girard [7]) that the Church Rosser property can already be proven in PRA, therefore we have the theorem

Theorem 2.3.1 $\mathsf{FID}^{\mathsf{r}}(\mathsf{\Pi}^0_1)$ *proves:*

$$(\forall x)(\forall y)(\forall z)[\mathrm{Red}_\rho(x,y) \wedge \mathrm{Red}_\rho(x,z) \rightarrow (\exists w)(\mathrm{Red}_\rho(y,w) \wedge \mathrm{Red}_\rho(z,w))].$$

□

By the above theorem we can define the equivalence relation $\approx_\rho$ on $\mathbb{L}_2$-terms by the $\mathbf{\Sigma}^0_1$-formula

$$s \approx_\rho t :\iff (\exists x)[\mathrm{Red}_\rho(s,x) \wedge \mathrm{Red}_\rho(t,x)],$$

that is s and t have a common Red_ρ-reduct. In the definition of the operator form $\mathcal{A}$ the reflexive closure $\mathrm{Red1}^{\mathrm{r}}_\rho$ of the relation $\mathrm{Red1}_\rho$ plays a major role. An important property of the relation $\mathrm{Red1}^{\mathrm{r}}_\rho$ is stated in the following lemma.

Lemma 2.3.2 *There is a primitive recursive function $bd(x)$ such that*

$$\mathrm{Red1}^{\mathrm{r}}_{\rho}(m, m') \rightarrow m' < bd(m).$$

PROOF: We define the function $bd(x)$ by:

$$bd(x) := \begin{cases} bd[(t_0 t_2)^*] \circ bd[(t_1 t_2)^*], & \text{if } x = (\mathsf{s} t_0 t_1 t_2)^*, \\ bd[t_0^*] \circ bd[(\mathsf{dc}(\mathsf{p} t_0 t_1)\overline{n})^*], & \text{if } x = (\mathsf{dc}(\mathsf{p} t_0 t_1)(\mathsf{s}_N \overline{n}))^*, \\ bd(t_0) \circ bd(t_1), & \text{if } x = t_0 \circ t_1 \wedge x \neq (\mathsf{dc}(\mathsf{p} r_0 r_1)(\mathsf{s}_N \overline{n}))^* \wedge \\ & x \neq (\mathsf{s} l_0 l_1 l_2)^*, \\ x + 1, & \text{otherwise.} \end{cases}$$

It is immediate from that definition that $bd(x)$ is primitive recursive and satisfies the demanded property. □

Before we present the operator form $\mathcal{A}$, we describe informally how we manage to close $P_{\mathcal{A}}$ w.r.t. $\approx_\rho$: With an operator form that is Π^0_1 we can't test if $\mathrm{Red}_\rho(n, n')$ or $n' \approx_\rho n$. However, given a closed term n, we can check if there is a closed term n' such that $\mathrm{Red1}^{\mathrm{r}}_{\rho}(n, n')$ by the $\Delta^{\mathbb{O}}_0$-formula $(\exists n' < bd(n))\mathrm{Red1}^{\mathrm{r}}_{\rho}(n, n')$. But note that the converse, i.e. given a term n, is there a term n' such that $\mathrm{Red1}^{\mathrm{r}}_{\rho}(n', n)$, is not decidable by a Π^0_1-predicate. To model $P_{\mathcal{A}}$ we use a combined operator form. The first operator $\mathcal{A}_0$ is POS and $\Delta^{\mathbb{O}}_0$, and the second operator $\mathcal{A}_1$ is Π^0_1, so that the whole operator form $\mathcal{A}$ is still Π^0_1. What the first operator does is this: If m' is a term coding a type and $\mathrm{Red1}^{\mathrm{r}}_{\rho}(m, m')$, then m is to code the same type, and if n' is in the extension of a type and $\mathrm{Red1}^{\mathrm{r}}_{\rho}(n, n')$, then so is n. We apply the first operator over and over again until closure is reached, and it turns out that at this stage $P_{\mathcal{A}}$ is also closed under $\approx_\rho$. Then we apply the second operator once in order to get representatives of the types $\mathsf{co}\ a$, $\mathsf{int}(a, b)$ etc., then we apply again the first operator over and over again until closure is reached (w.r.t. the first operator and w.r.t. $\approx_\rho$), then the second operator once, and so on until closure under both operators is achieved.
Now we are in the position to define the operator form $\mathcal{A}$. To keep our notation intuitive, we write e.g. $m = \mathsf{int}^* \circ (a, b)^\star \wedge P_{\mathcal{A}}(a, 0, 0)$ for the statement $m = (m)_1 \circ (m)_2 \wedge (m)_1 = \mathsf{int}^* \wedge (\exists x)[x = (m)_2 \rightarrow x = ([x]_0, [x]_1)^\star \wedge P([x]_0, 0, 0)]$.

Definition 2.3.3 *(The operator form $\mathcal{A}$):*

$\mathcal{A}(P, m, n, k) :\equiv \mathcal{A}_0(P, m, n, k) \vee [(\forall \vec{x})(\mathcal{A}_0(P, \vec{x}) \rightarrow P(\vec{x})) \wedge \mathcal{A}_1(P, m, n, k)]$*, where*

$$\begin{aligned} \mathcal{A}_0(P, m, n, k) :\equiv & (\exists m' < bd(m))[\mathrm{Red1}^{\mathrm{r}}_{\rho}(m, m') \wedge P(m', 0, 0) \wedge n = k = 0] \vee \\ & (\exists m' < bd(m))[\mathrm{Red1}^{\mathrm{r}}_{\rho}(m, m') \wedge P(m', n, 1) \wedge k = 1] \vee \\ & (\exists n' < bd(n))[\mathrm{Red1}^{\mathrm{r}}_{\rho}(n, n') \wedge P(m, n', 1) \wedge k = 1], \end{aligned}$$

and $\mathcal{A}_1(P, m, n, k)$ *is the disjunction of the following formulas:*

1a) $m = \mathsf{nat}^* \wedge n = 0 \wedge k = 0$,
1b) $m = \mathsf{nat}^* \wedge \mathrm{CTer}(n) \wedge (\exists x < n)(\mathrm{Num}(x) = n) \wedge k = 1$,

2a) $m = \mathsf{id}^* \wedge n = 0 \wedge k = 0$,
2b) $m = \mathsf{id}^* \wedge \mathrm{CTer}(n) \wedge n = ([n]_0, [n]_0)^\star \wedge k = 1$,

3a) $m = \mathsf{int}^* \circ (a, b)^\star \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge n = 0 \wedge k = 0$,
3b) $m = \mathsf{int}^* \circ (a, b)^\star \wedge P(a, 0, 0) \wedge P(b, 0, 0) \wedge P(a, n, 1) \wedge P(b, n, 1) \wedge k = 1$,

4a) $m = \mathsf{co}^* \circ a \wedge P(a, 0, 0) \wedge n = 0 \wedge k = 0$,
4b) $m = \mathsf{co}^* \circ a \wedge P(a, 0, 0) \wedge \mathrm{CTer}(n) \wedge \neg P(a, n, 1) \wedge k = 1$,

5a) $m = \mathsf{codom}^* \circ a \wedge P(a, 0, 0) \wedge n = 0 \wedge k = 0$,
5b) $m = \mathsf{codom}^* \circ a \wedge P(a, 0, 0) \wedge \mathrm{CTer}(n) \wedge (\forall q)\neg P(a, (n, q)^\star, 1) \wedge k = 1$,

6a) $m = \mathsf{inv}^* \circ (a, f)^\star \wedge P(a, 0, 0) \wedge \mathrm{CTer}(f) \wedge n = 0 \wedge k = 0$,
6b) $m = \mathsf{inv}^* \circ (a, f)^\star \wedge P(a, 0, 0) \wedge \mathrm{CTer}(f) \wedge P(a, f \circ n, 1) \wedge k = 1$,

7a) $m = \mathsf{j}^* \circ (a, f)^\star \wedge P(a, 0, 0) \wedge \mathrm{CTer}(f) \wedge (\forall x)[P(a, x, 1) \to P(f \circ x, 0, 0)] \wedge$
$n = 0 \wedge k = 0$,
7b) $m = \mathsf{j}^* \circ (a, f)^\star \wedge P(a, 0, 0) \wedge \mathrm{CTer}(f) \wedge (\forall x)[P(a, x, 1) \to P(f \circ x, 0, 0)] \wedge$
$n = ([n]_0, [n]_1)^\star \wedge P(a, [n]_0, 1) \wedge P(f \circ [n]_0, [n]_1, 1) \wedge k = 1$,

8) $m = \mathsf{cl}^* \circ (a, f)^\star \wedge P(a, 0, 0) \wedge \mathrm{CTer}(f) \wedge (\forall x)[P(x, 0, 0) \to P(f \circ x, 0, 0)] \wedge$
$n = 0 \wedge k = 2$.

In the sequel const denotes the set $\{\mathsf{int}, \mathsf{co}\ , \mathsf{codom}\ , \mathsf{inv}, \mathsf{j}, \mathsf{cl}\}$. Then the expression $(\exists \mathsf{c} \in \mathsf{const})A(\mathsf{c}^*)$ is meant to abbreviate the formula $A[\mathsf{int}^*] \vee \ldots \vee A[\mathsf{cl}^*]$. In addition we set $\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}}) :\equiv (\forall \vec{x})(\mathcal{A}_0(P^{\prec\alpha}_{\mathcal{A}}, \vec{x}) \to P^{\prec\alpha}_{\mathcal{A}}(\vec{x}))$.

Lemma 2.3.4 *For all* c *in* const $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$\mathrm{Red}_\rho(\mathsf{c}^* \circ t, s) \to (\exists t')(s = \mathsf{c}^* \circ t' \wedge \mathrm{Red}_\rho(t, t')).$$

Proof: For no c in const there is a redex-contractum pair of the form $\mathsf{c}\ldots\ \boldsymbol{\rho}\ \mathsf{c}\ldots$, therefore, for c in const $\mathrm{Red1}^{\mathrm{r}}_\rho(\mathsf{c}^* \circ t, m)$ holds if and only if $m = \mathsf{c}^* \circ t'$ and $\mathrm{Red1}^{\mathrm{r}}_\rho(t, t')$. Now $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathsf{N}})$ on the length of the reduction sequence s yields the claim. □

Lemma 2.3.5 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$m = (a, b)^\star \wedge \mathrm{Red}_\rho(m, m') \to (\exists a')(\exists b')[\mathrm{Red}_\rho(a, a') \wedge \mathrm{Red}_\rho(b, b') \wedge m' = (a', b')^\star$$

Proof: There is no redex-contractum pair of the form $\mathsf{p}\ldots\ \boldsymbol{\rho}\ \mathsf{p}\ldots$, therefore $\mathrm{Red1}^{\mathrm{r}}_{\rho}((a,b)^{\star},m')$ holds if and only if $m'=(a',b')^{\star}$ with $\mathrm{Red1}^{\mathrm{r}}_{\rho}(a,a')$ and $\mathrm{Red1}^{\mathrm{r}}_{\rho}(b,b')$. Again $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathsf{N}})$ yields the claim. □

Lemma 2.3.6 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)\vdash P^{\alpha}_{\mathcal{A}}(m,0,0)\to(\exists t)(\exists\mathsf{c}\in\mathsf{const})\mathrm{Red}_{\rho}(m,\mathsf{c}^{*}\circ t)$.

Proof: By $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. If $P^{\prec\alpha}_{\mathcal{A}}(m,0,0)$ the IH applies. If $\mathcal{A}_0(P^{\prec\alpha}_{\mathcal{A}},m,0,0)$ there is an m' with $\mathrm{Red1}^{\mathrm{r}}_{\rho}(m,m')$ and $P^{\prec\alpha}_{\mathcal{A}}(m',0,0)$, hence by IH there is a term t and a c in const with $\mathrm{Red}_{\rho}(m',\mathsf{c}^{*}\circ t)$. The definition of Red_{ρ} yields $\mathrm{Red}_{\rho}(m,\mathsf{c}^{*}\circ t)$. If $\neg\mathcal{A}_0(P^{\prec\alpha}_{\mathcal{A}},m,0,0)$ and $\mathcal{A}(P^{\prec\alpha}_{\mathcal{A}},m,0,0)$, then m is of the form $\mathsf{c}^{*}\circ t$ for a c in const and we are done. □

By the definition of $\mathcal{A}$ we have that $P_{\mathcal{A}}$ is 'closed upwards' under $\mathrm{Red1}^{\mathrm{r}}_{\rho}$, i.e. if $\mathrm{Red1}^{\mathrm{r}}_{\rho}(m,m')\wedge P_{\mathcal{A}}(m',n,k)$ then $P_{\mathcal{A}}(m,n,k)$ $(k\in\{0,1\})$, and if $\mathrm{Red1}^{\mathrm{r}}_{\rho}(n,n')\wedge P_{\mathcal{A}}(m,n',1)$ then $P_{\mathcal{A}}(m,n,1)$. We aim to show that $P_{\mathcal{A}}$ is closed under $\approx_{\rho}$.

Lemma 2.3.7 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:* $k=0\vee k=1$ *implies*

(i) $\mathrm{Red}_{\rho}(m,m')\wedge\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})\to[P^{\prec\alpha}_{\mathcal{A}}(m',n,k)\to P^{\prec\alpha}_{\mathcal{A}}(m,n,k)]$

(ii) $\mathrm{Red}_{\rho}(n,n')\wedge\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})\to[P^{\prec\alpha}_{\mathcal{A}}(m,n',1)\to P^{\prec\alpha}_{\mathcal{A}}(m,n,1)]$

Proof: Let $F(P^{\prec\alpha}_{\mathcal{A}},m,m',n,k):\equiv\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})\to[P^{\prec\alpha}_{\mathcal{A}}(m,n,k)\to P^{\prec\alpha}_{\mathcal{A}}(m',n,k)]$. Now $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathsf{N}})$ on l shows $(\forall l)(\forall s)(\forall m)(\forall m')[\mathrm{RedSeq}_{\rho}(s,m,m')\wedge lh(s)=l+1\to F(P^{\prec\alpha}_{\mathcal{A}},m,m',n,k)]$: $l=0$ implies $m=m'$, so there is nothing to show. For the induction step note, that if $lh(s)=l+2$ then there is a reduction sequence s' and a term m_0 with $lh(s')=l+1$, $\mathrm{RedSeq}_{\rho}(s',m_0,m')$ and $\mathrm{Red1}^{\mathrm{r}}_{\rho}(m,m_0)$. Now the IH yields $P^{\prec\alpha}_{\mathcal{A}}(m_0,n,k)$ and the definition of $\mathcal{A}_0$ implies $\mathcal{A}_0(P^{\prec\alpha}_{\mathcal{A}},m,n,k)$, so by $\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})$ we get $P^{\prec\alpha}_{\mathcal{A}}(m,n,k)$. This proves (i), (ii) is shown the same way. □

In order to formulate the next lemma, we set

$$G(\alpha,\beta,m,n,k):\equiv(\forall\gamma)[(\beta\prec\gamma\wedge(\gamma\prec\alpha\vee\gamma=\alpha)\wedge\mathcal{C}(P^{\prec\gamma}_{\mathcal{A}}))\to P^{\prec\gamma}_{\mathcal{A}}(m,n,k)]$$

$G(\alpha,\beta,m,n,k)$ expresses that $P^{\prec\gamma}_{\mathcal{A}}(m,n,k)$ holds for all γ with $\beta\prec\gamma\preceq\alpha$ and $\mathcal{C}(P^{\prec\gamma}_{\mathcal{A}})$.

Lemma 2.3.8 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:* $k=0\vee k=1$ *implies*

(i) $\mathrm{Red}_{\rho}(m_{\top},m_0)\wedge\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})\wedge\beta\prec\alpha\to[P^{\beta}_{\mathcal{A}}(m_{\top},n,k)\to G(\alpha,\beta,m_0,n,k)]$

(ii) $\mathrm{Red}_{\rho}(n_{\top},n_0)\wedge\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})\wedge\beta\prec\alpha\to[P^{\beta}_{\mathcal{A}}(m,n_{\top},1)\to G(\alpha,\beta,m,n_0,k)]$

Proof: We prove (i) and (ii) simultaneously by $\Delta_0^{\mathbb{O}}$-induction on β. We show the induction step for the case (i): If $P_{\mathcal{A}}^{\prec\beta}(m_\top, n, k)$ then the claim follows from the IH. If $\mathcal{A}(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k)$, we distinguish two cases:

(a) $\mathcal{A}_0(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k)$. By the definition of $\mathcal{A}_0$ there is

(a.1) an m_1 with $\mathrm{Red1}_\rho^{\mathrm{r}}(m_\top, m_1) \wedge P_{\mathcal{A}}^{\prec\beta}(m_1, n, k)$. Red_ρ has the Church-Rosser property, hence there is a $m_\bot$ such that $\mathrm{Red}_\rho(m_0, m_\bot) \wedge \mathrm{Red}_\rho(m_1, m_\bot)$. By IH we have $G(\alpha, \beta, m_\bot, n, k)$ and therefore $G(\alpha, \beta, m_0, n, k)$ by the previous lemma.

(a.2) an n' with $\mathrm{Red1}_\rho^{\mathrm{r}}(n, n') \wedge P_{\mathcal{A}}^{\prec\beta}(m_\top, n', k)$. By IH we have $G(\alpha, \beta, m_0, n', k)$ and therefore $G(\alpha, \beta, m_0, n, k)$ by the previous lemma.

$$\begin{tikzcd} & m_\top \arrow[dl] \arrow[dr] & \\ m_0 \arrow[dr] & & m_1 \arrow[dl] \\ & m_\bot & \end{tikzcd}$$

(b) $\mathcal{A}(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k) \wedge \neg\mathcal{A}_0(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k)$. The definition of $\mathcal{A}$ implies $\mathcal{C}(P_{\mathcal{A}}^{\prec\beta})$ and $\mathcal{A}_1(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k)$. So $m_\top$ is of the form $\mathsf{c}^* \circ t$ and m_0 is of the form $\mathsf{c}^* \circ t'$ where $\mathrm{Red}_\rho(t, t')$ for a c in const. We just consider the case where c is the term j. So $m_\top = \mathsf{j}^* \circ (a, f)^\star$ for suitable terms a and f, and $m_0 = \mathsf{j}^* \circ (a', f')^\star$ where $\mathrm{Red}_\rho(a, a')$ or $\mathrm{Red}_\rho(f, f')$. Assume $\mathcal{A}_1(P_{\mathcal{A}}^{\prec\beta}, m_\top, n, k)$ holds because of

$P_{\mathcal{A}}^{\prec\beta}(a, 0, 0) \wedge (\forall x)[P_{\mathcal{A}}^{\prec\beta}(a, x, 1) \rightarrow P_{\mathcal{A}}^{\prec\beta}(f \circ x, 0, 0)] \wedge$
$n = ([n]_0, [n]_1)^\star \wedge P_{\mathcal{A}}^{\prec\beta}(a, [n]_0, 1) \wedge P_{\mathcal{A}}^{\prec\beta}(f \circ [n]_0, [n]_1, 1) \wedge k = 1.$

(the case $k = 0$ is shown similarly). $\mathcal{C}(P_{\mathcal{A}}^{\prec\beta})$ expresses that $P_{\mathcal{A}}^{\prec\beta}$ is closed under $\mathcal{A}_0$, hence by IH and lemma 2.3.7 we have

$P_{\mathcal{A}}^{\prec\beta}(a', 0, 0) \wedge (\forall x)[P_{\mathcal{A}}^{\prec\beta}(a', x, 1) \rightarrow P_{\mathcal{A}}^{\prec\beta}(f' \circ x, 0, 0)] \wedge$
$n = ([n]_0, [n]_1)^\star \wedge P_{\mathcal{A}}^{\prec\beta}(a', [n]_0, 1) \wedge P_{\mathcal{A}}^{\prec\beta}(f' \circ [n]_0, [n]_1, 1) \wedge k = 1.$

This yields $\mathcal{A}_1(P_{\mathcal{A}}^{\prec\beta}, \mathsf{j}^* \circ (a', f')^\star, n, k)$, hence $P_{\mathcal{A}}^{\beta}(m_0, n, k)$, and $G(\alpha, \beta, m_0, n, k)$ trivially follows.

□

Lemma 2.3.9 $\mathsf{FID}^{\mathsf{r}}(\Pi_1^0)$ *proves:* $k = 0 \vee k = 1$ *implies*

(i) $\mathrm{Red}_\rho(m, m') \wedge \mathcal{C}(P_{\mathcal{A}}^{\prec\alpha}) \rightarrow [P_{\mathcal{A}}^{\alpha}(m, n, k) \leftrightarrow P_{\mathcal{A}}^{\alpha}(m', n, k)]$

(ii) $\mathrm{Red}_\rho(n,n') \wedge \mathcal{C}(P_{\mathcal{A}}^{\prec\alpha}) \to [P_{\mathcal{A}}^{\alpha}(m,n,1) \leftrightarrow P_{\mathcal{A}}^{\alpha}(m,n',1)]$

Proof: Let $P_{\mathcal{A}}^{\alpha}(m,n,k)$. If $P_{\mathcal{A}}^{\prec\alpha}(m,n,k)$, the previous lemma applies. If we have $\neg P_{\mathcal{A}}^{\prec\alpha}(m,n,k)$, $\mathcal{A}_0(P_{\mathcal{A}}^{\prec\alpha},m,n,k)$ is impossible because of $\mathcal{C}(P_{\mathcal{A}}^{\prec\alpha})$, therefore $\mathcal{A}_1(P_{\mathcal{A}}^{\prec\alpha},m,n,k)$. Let e.g. $m = \mathsf{int}^* \circ (a,b)^\star$, then $m' = \mathsf{int}^* \circ (a',b')^\star$ where $\mathrm{Red}_\rho(a,a')$ or $\mathrm{Red}_\rho(b,b')$. By the previous lemma $P_{\mathcal{A}}^{\prec\alpha}(a,n,1)$ and $P_{\mathcal{A}}^{\prec\alpha}(b,n,1)$ implies $P_{\mathcal{A}}^{\prec\alpha}(a',n,1)$ and $P_{\mathcal{A}}^{\prec\alpha}(b',n,1)$, hence $P_{\mathcal{A}}^{\alpha}(m',n,1)$. The other direction is shown similarly. □

Corollary 2.3.10 $\mathsf{FID}^{\mathsf{r}}(\Pi_1^0)$ *proves:* $k=0 \vee k=1$ *implies*

(i) $(m \approx_\rho m') \wedge \mathcal{C}(P_{\mathcal{A}}^{\prec\alpha}) \to [P_{\mathcal{A}}^{\alpha}(m,n,k) \leftrightarrow P_{\mathcal{A}}^{\alpha}(m',n,k)]$

(ii) $(n \approx_\rho n') \wedge \mathcal{C}(P_{\mathcal{A}}^{\prec\alpha}) \to [P_{\mathcal{A}}^{\alpha}(m,n,1) \leftrightarrow P_{\mathcal{A}}^{\alpha}(m,n',1)]$

Proof: This follows immediately form the definition of $\approx_\rho$ and the above lemma. □

Lemma 2.3.11 *For all* c *in* const $\mathsf{FID}^{\mathsf{r}}(\Pi_1^0)$ *proves:*

$$P_{\mathcal{A}}^{\alpha}(\mathsf{c}^* \circ m,n,k) \wedge \neg P_{\mathcal{A}}^{\prec\alpha}(\mathsf{c}^* \circ m,n,k) \to \mathcal{C}(P_{\mathcal{A}}^{\prec\alpha}) \wedge \mathcal{A}_1(P_{\mathcal{A}}^{\prec\alpha},\mathsf{c}^* \circ m,n,k).$$

Proof: By $(\Delta_0^{\mathbb{O}}\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. We show that for c in const $\mathcal{A}_0(P_{\mathcal{A}}^{\prec\alpha},\mathsf{c}^* \circ m,n,k)$ is impossible. In this case there are m',n' with $\mathrm{Red1}^{\mathrm{r}}_\rho(m,m')$ and $\mathrm{Red1}^{\mathrm{r}}_\rho(n,n')$ such that $P_{\mathcal{A}}^{\prec\alpha}(\mathsf{c}^* \circ m',n',k)$, and therefore there is a $\beta \prec \alpha$ with $P_{\mathcal{A}}^{\beta}(\mathsf{c}^* \circ m',n',k)$ and $\neg P_{\mathcal{A}}^{\prec\beta}(\mathsf{c}^* \circ m',n',k)$. The IH yields $\mathcal{C}(P_{\mathcal{A}}^{\prec\beta})$, and due to lemma 2.3.9 we have $P_{\mathcal{A}}^{\beta}(\mathsf{c}^* \circ m,n,k)$, what contradicts the premise. □

Lemma 2.3.12 $\mathsf{FID}^{\mathsf{r}}(\Pi_1^0) \vdash P_{\mathcal{A}}^{\alpha}(m,n,k) \to (\exists\beta)[\mathcal{C}(P_{\mathcal{A}}^{\prec\beta}) \wedge P_{\mathcal{A}}^{\prec\beta}(m,n,k)]$.

Proof: $(\mathsf{OP.2})$ asserts that $(\forall\vec{x})[\mathcal{A}_0(P_{\mathcal{A}},\vec{x}) \to P_{\mathcal{A}}(\vec{x})]$. With $P_{\mathcal{A}}^{\alpha}(m,n,k)$ we have also $P_{\mathcal{A}}(m,n,k)$. This yields $\mathcal{A}(P_{\mathcal{A}},\mathsf{int}^* \circ (m,m)^\star,n,k)$, so again by $(\mathsf{OP.2})$ and lemma 2.2.2 there is a stage β such that $P_{\mathcal{A}}^{\beta}(\mathsf{int}^* \circ (m,m)^\star,n,k)$ and $\neg P_{\mathcal{A}}^{\prec\beta}(\mathsf{int}^* \circ (m,m)^\star,n,k)$. Now lemma 2.3.11 yields $\mathcal{C}(P_{\mathcal{A}}^{\prec\beta})$ and $\mathcal{A}_1(P_{\mathcal{A}}^{\prec\beta},\mathsf{int}^* \circ (m,m)^\star,n,k)$, that is $P_{\mathcal{A}}^{\prec\beta}(m,n,k)$. □

Now we are able to show that $P_{\mathcal{A}}$ is closed under $\approx_\rho$:

Lemma 2.3.13 $\mathsf{FID}^{\mathsf{r}}(\Pi_1^0)$ *proves:* $k=0 \vee k=1$ *implies*

(i) $(m \approx_\rho m') \to [P_{\mathcal{A}}(m,n,k) \leftrightarrow P_{\mathcal{A}}(m',n,k)]$

(ii) $(n \approx_\rho n') \to [P_{\mathcal{A}}(m,n,1) \leftrightarrow P_{\mathcal{A}}(m,n',1)]$

PROOF: $P_{\mathcal{A}}(m,n,k)$ implies $\mathcal{A}(P_{\mathcal{A}}, \mathsf{int}^* \circ (m,m)^\star, n, k)$. Then there is an α with $P^{\alpha}_{\mathcal{A}}(\mathsf{int}^* \circ (m,m)^\star, n, k)$ and $\neg P^{\prec\alpha}_{\mathcal{A}}(\mathsf{int}^* \circ (m,m)^\star, n, k)$, so lemma 2.3.11 yields $\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})$ and $P^{\prec\alpha}_{\mathcal{A}}(m,n,k)$. Now corollary 2.3.10 yields the claim. □

Lemma 2.3.14 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash P^{\alpha}_{\mathcal{A}}(m,n,1) \to P^{\alpha}_{\mathcal{A}}(m,0,0) \wedge \mathrm{CTer}(n)$.

PROOF: By $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. □

Next we proof the Persistence Lemma, which states that the extension of a type doesn't change anymore after closure w.r.t. $\mathcal{A}_0$ is reached.

Lemma 2.3.15 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}}) \wedge P^{\prec\alpha}_{\mathcal{A}}(m,0,0) \wedge \mathcal{C}(P^{\prec\beta}_{\mathcal{A}}) \wedge P^{\prec\beta}_{\mathcal{A}}(m,n,1) \to P^{\prec\alpha}_{\mathcal{A}}(m,n,1)$$

PROOF: By $(\Delta^{\mathbb{O}}_0\text{-}\mathsf{IND}_{\mathbb{O}})$ on α. Because of $\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})$, $\mathcal{C}(P^{\prec\beta}_{\mathcal{A}})$, lemma 2.3.6 and lemma 2.3.7 we may assume that m is of the form $\mathsf{c}^* \circ t$ for a c in const. Let e.g. $m = \mathsf{int}^* \circ (a,b)^\star$. So there is an $\alpha' \prec \alpha$ such that $P^{\alpha'}_{\mathcal{A}}(m,0,0)$ and $\neg P^{\prec\alpha'}_{\mathcal{A}}(m,n,k)$. Now lemma 2.3.11 implies $\mathcal{C}(P^{\prec\alpha'}_{\mathcal{A}})$, $P^{\prec\alpha'}_{\mathcal{A}}(a,0,0)$ and $P^{\prec\alpha'}_{\mathcal{A}}(b,0,0)$. On the other hand there is a $\beta' \prec \beta$ with $P^{\beta'}_{\mathcal{A}}(m,n,1)$ and $\neg P^{\prec\beta'}_{\mathcal{A}}(m,n,1)$, hence lemma 2.3.11 yields $\mathcal{C}(P^{\prec\beta'}_{\mathcal{A}})$ and $\mathcal{A}_1(P^{\prec\beta'}_{\mathcal{A}}, m, n, 1)$, hence $P^{\prec\beta}_{\mathcal{A}}(a,n,1)$ and $P^{\prec\beta}_{\mathcal{A}}(b,n,1)$. Now IH yields $P^{\prec\alpha'}_{\mathcal{A}}(a,n,1)$ and $P^{\prec\alpha'}_{\mathcal{A}}(b,n,1)$, therefore $P^{\prec\alpha}_{\mathcal{A}}(m,n,1)$. □

Lemma 2.3.16 *(Persistence Lemma)* $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}}) \wedge P^{\prec\alpha}_{\mathcal{A}}(m,0,0) \wedge P^{\beta}_{\mathcal{A}}(m,n,1) \to P^{\prec\alpha}_{\mathcal{A}}(m,n,1)$$

PROOF: By lemma 2.3.12 there is a β' such that $\mathcal{C}(P^{\prec\beta'}_{\mathcal{A}})$ and $P^{\prec\beta'}_{\mathcal{A}}(m,n,1)$. Now the previous lemma yields the claim. □

Lemma 2.3.17 $\mathsf{FID}^{\mathsf{r}}(\Pi^0_1)$ *proves:*

$$P_{\mathcal{A}}(\mathsf{co}^* \circ m, n, 1) \leftrightarrow P_{\mathcal{A}}(m,0,0) \wedge \neg P_{\mathcal{A}}(m,n,1)$$

PROOF: If $P_{\mathcal{A}}(\mathsf{co}^* \circ m, n, 1)$ holds, then by $(\mathsf{OP.2})$ and lemma 2.3.14 we get an α with $\mathcal{C}(P^{\prec\alpha}_{\mathcal{A}})$ and $\mathcal{A}_1(P^{\prec\alpha}_{\mathcal{A}}, \mathsf{co}^* \circ m, n, 1)$, so $\neg P^{\prec\alpha}_{\mathcal{A}}(m,n,1)$. Now $\neg P_{\mathcal{A}}(m,n,1)$ follows by persistence.
If $P_{\mathcal{A}}(m,0,0)$ and $\neg P_{\mathcal{A}}(m,n,1)$ then $\mathcal{A}(P_{\mathcal{A}}, \mathsf{co}^* \circ m, n, 1)$, so the claim holds by $(\mathsf{OP.2})$.

□

Lemma 2.3.18 *If $t(x)$ is an $\mathcal{L}_p$-term and x is a free number or type variable of $\mathcal{L}_p$, then the following holds:*

$$\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash (u \approx_\rho v) \to (t^*[u/x^*] \approx_\rho t^*[v/x^*]).$$

Proof: The claim is shown by induction on the setup of the $\mathcal{L}_p$-term t. □

We extend the translation $\cdot^*$ such that for every formula F of $\mathcal{L}_p$ F^* is a formula of $\mathbb{L}_{\Pi^0_1}$.

$$\begin{array}{rcl}
(r = s)^* & :\equiv & (r^* \approx_\rho s^*), \\
(t \downarrow)^* & :\equiv & 0 = 0, \\
[\mathsf{N}(t)]^* & :\equiv & P_{\mathcal{A}}(\mathsf{nat}^*, t^*, 1) \\
(t \in X)^* & :\equiv & P_{\mathcal{A}}(X^*, t^*, 1), \\
[\Re(t, X)]^* & :\equiv & P_{\mathcal{A}}(t^*, 0, 0) \wedge (\forall x)(P_{\mathcal{A}}(t^*, x, 1) \leftrightarrow P_{\mathcal{A}}(X^*, x, 1)), \\
(X = Y)^* & :\equiv & (\forall x)(P_{\mathcal{A}}(X^*, x, 1) \leftrightarrow P_{\mathcal{A}}(Y^*, x, 1)), \\
(\neg F)^* & :\equiv & \neg F^*, \\
(F \text{ j } G)^* & :\equiv & F^* \text{ j } G^*, \\
[(Qx)F]^* & :\equiv & (Qx^*)F^*, \\
[(\exists X)F]^* & :\equiv & (\exists X^*)(P_{\mathcal{A}}(X^*, 0, 0) \wedge F^*, \\
[(\forall X)F]^* & :\equiv & (\forall X^*)(P_{\mathcal{A}}(X^*, 0, 0) \to F^*,
\end{array}$$

As usual j denotes the connectives $\wedge$ or $\vee$ and Q stands for a quantifier.

Lemma 2.3.19 *If $F(x)$ is an $\mathcal{L}_p$-formula and x is a free number or type variable of $\mathcal{L}_p$, then the following holds:*

$$\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash u \approx_\rho v \to (F^*[u/x^*] \leftrightarrow F^*[v/x^*]).$$

Proof: The claim is shown by induction on the setup of the $\mathcal{L}_p$-formula F. We just show an illustrative case:
Be $F(x) \equiv \mathsf{N}(t(x))$. Then $F^*(x^*) \equiv P_{\mathcal{A}}(\mathsf{nat}^*, t^*(x^*), 1)$. Now lemma 2.3.18 yields $(t(u/x))^* \approx_\rho (t(v/x))^*$, and lemma 2.3.13 yields the claim.

□

Proposition 2.3.20 *For every axiom F of $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$ with its free type variables among $X_0, \ldots, X_{n-1}$ we have:*

$$\mathsf{FID}^{\mathsf{r}}(\Pi^0_1) \vdash P_{\mathcal{A}}(X^*_0, 0, 0) \wedge \ldots \wedge P_{\mathcal{A}}(X^*_{n-1}, 0, 0) \to F^*.$$

Proof: The equality axioms hold due lemma 2.3.19, and the other logical axioms are easily checked. The axioms concerning the constants follow directly from the definition of the relation $\approx_\rho$.

The axioms about the natural numbers are shown as follows: The function Num is given by

$$\begin{array}{rcl} \mathrm{Num}(0) & = & 0^*, \\ \mathrm{Num}(x+1) & = & \mathsf{s}_{\mathsf{N}}^* \circ \mathrm{Num}(x) \end{array}$$

The axiom

$$\mathsf{0} \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(\mathsf{s}_N x \in \mathsf{N})$$

translates to

$$P_{\mathcal{A}}(\mathsf{nat}^*, 0^*, 1) \wedge (\forall x)[P_{\mathcal{A}}(\mathsf{nat}^*, x, 1) \rightarrow P_{\mathcal{A}}(\mathsf{nat}^*, \mathsf{s}_{\mathsf{N}}^* \circ x, 1)],$$

which holds, because we have $\mathrm{CTer}(0^*)$ and $\mathrm{Num}(0) = 0^*$, as well as $x \approx_\rho \mathrm{Num}(y)$ implies $\mathsf{s}_{\mathsf{N}}^* \circ x \approx_\rho \mathsf{s}_{\mathsf{N}}^* \circ \mathrm{Num}(y) = \mathrm{Num}(y+1)$. The other axioms are shown similarly. Extensionality (EXT) is built in the translation of $(X = Y)$. The axiom ($\mathsf{E.1}$) $(\exists x)(\Re(x, X)$ is checked by taking the witness X^*, and ($\mathsf{E.2}$) again holds by its translation.
Now we verify the axioms for elementary comprehension. To show ($\mathsf{N.1}$) we witness X^* by nat^*. Because $\mathrm{Num}(x) > x$ holds, nat^* has the correct extension. Also ($\mathsf{N.2}$), and the axioms concerning the identity type follow directly from the definitions of $\mathcal{A}$ and $\cdot^*$. ($\mathsf{CO.1}$) and ($\mathsf{CO.2}$) are due to lemma 2.3.17.
In ($\mathsf{INT.1}$) X^* is witnessed by $\mathsf{int}^* \circ (B^*, C^*)^\star$. We have to show:

$$(\forall x)[P_{\mathcal{A}}(\mathsf{int}^* \circ (B^*, C^*)^\star, x, 1) \leftrightarrow P_{\mathcal{A}}(B^*, x, 1) \wedge P_{\mathcal{A}}(C^*, x, 1)].$$

Let $P_{\mathcal{A}}(\mathsf{int}^* \circ (B^*, C^*)^\star, x, 1)$. Then we find a α with $P_{\mathcal{A}}^{\alpha}(\mathsf{int}^* \circ (B^*, C^*)^\star, x, 1)$ and $\neg P_{\mathcal{A}}^{\prec\alpha}(\mathsf{int}^* \circ (B^*, C^*)^\star, x, 1)$. Now lemma 2.3.11 yields $\mathcal{C}(P_{\mathcal{A}}^{\prec\alpha})$ and $P_{\mathcal{A}}^{\prec\alpha}(B^*, x, 1)$ and $P_{\mathcal{A}}^{\prec\alpha}(C^*, x, 1)$. By lemma 2.3.14 and persistence we get $P_{\mathcal{A}}(B^*, x, 1)$ and $P_{\mathcal{A}}(C^*, x, 1)$. For the other direction note the ($\mathsf{OP.2}$) yields $\mathcal{C}(P_{\mathcal{A}})$. Because of $\mathcal{A}_1(P_{\mathcal{A}}, \mathsf{int}^* \circ (B^*, C^*)^\star, x, 1)$ holds, ($\mathsf{OP.2}$) yields the claim. ($\mathsf{INT.2}$) follows by ($\mathsf{OP.2}$), too.
In ($\mathsf{DOM.1}$) X^* is witnessed by $\mathsf{co}^* \circ (\mathsf{codom}^* \circ B^*)$. As above we show

$$(\forall x)[P_{\mathcal{A}}(\mathsf{codom}^* \circ B^*, x, 1) \leftrightarrow (\forall q)\neg P_{\mathcal{A}}(B^*, (x, q)^\star, 1)].$$

By lemma 2.3.17 we then get

$$(\forall x)[P_{\mathcal{A}}(\mathsf{co}^* \circ (\mathsf{codom}^* \circ B^*), x, 1) \leftrightarrow (\exists q)P_{\mathcal{A}}(B^*, (x, q)^\star, 1)].$$

The remaining axioms concerning elementary comprehension are shown the same way, and (dc) is shown as in the previous section. □

This means that the result about the strength of the theory $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N})$ from the previous section hold also for the theory $\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N}) + (\mathsf{Tot})$.

Corollary 2.3.21 $|\mathsf{EETJ} + (\mathsf{dc}) + (\mathsf{T\text{-}I_N}) + (\mathsf{Tot})| = |\mathsf{FID^r}(\Pi^0_1)| = \varphi\omega 0$.

□

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