Metapredicative Subsystems of Analysis

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Sie als Leserin und Sie als Leser haben 145 Seiten vor sich – ich habe sie hinter mir. Ärgern Sie sich beim Durchlesen über unverständliche Argumente, mühsame Beweise oder zu knappe Formulierungen, dann verwünschen Sie *mich*. Stossen Sie beim Durchlesen auf elegante Konstruktionen, gute Ideen oder anschauliche Gedanken, so loben Sie all die Personen, die mir geholfen haben. Speziell erwähnen möchte ich Prof. Gerhard Jäger und Dr. Thomas Strahm. Beide hatten stets ein offenes Ohr für meine Fragen und Probleme. Beide haben ein erstes Manuskript dieser Dissertation durchgearbeitet, diskutiert und kritisiert. Beide waren oft zur Stelle mit wertvollen Hinweisen und Anhaltspunkten. Kurz: Beide unterstützten und motivierten mich während der ganzen Zeit und waren für eine entspannte, menschliche Atmosphäre verantwortlich. Dafür danke ich ihnen sehr.

Manchmal war es nur ein kurzes Gespräch, das Übermitteln eines LaTeX-makro oder die Englischkorrektur dieser Arbeit; manchmal das Übergeben von Unterlagen oder ein Literaturhinweis. Dabei konnte ich immer auf die Hilfe von jemandem zählen. All diesen Heinzelfrauen und -männern sei auch gedankt.

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Introduction

Context

The notion of *predicativity* in mathematics goes back to Poincaré. He had recognized that many antinomies which led to the – so-called – foundational crisis in mathematics at the beginning of the twentieth century use the same principle: the principle of *impredicative* definition [24]. A definition of a set is called impredicative if it contains a reference to a totality to which the set itself belongs. For instance the antinomies of Russell are of this type. The construction of the real numbers by Dedekind cuts uses impredicative definitions, too (cf. Weyl [33]). We refer also to Fraenkel [7] where several other such impredicative notions are presented and discussed.

The predicative standpoint of Poincaré was, to regard as given only the natural numbers with the unlimited principle of complete induction. Sets do not exist a priori but have to be introduced by definitions of the form

$$(\forall x)(x \in X \leftrightarrow \varphi(x)).$$

In order to avoid a vicious circle, we have to require that the meaning of the formula φ does not refer to a totality where X might belong to.

Later on, in the early sixties, Feferman and Schütte independently characterized predicativity in the framework of second order arithmetic. It was shown by Feferman and Schütte that Γ_0 is the proof-theoretic ordinal of predicative analysis (cf. Feferman [3, 4] and Schütte [26]). Since that time numerous theories have been found which are not predicative, but nevertheless have predicative strength in the sense that Γ_0 is an upper bound to their proof-theoretic ordinal. Typical examples are ATR_0 (cf. e.g. [29]), KPl_0 (cf. [12]) and KPi_0 (cf. [10]) etc..

The formal system of classical analysis is second order arithmetic with the full comprehension principle. It was baptized classical analysis, since classical mathematical analysis can be formalized in it. Often subsystems of classical analysis suffice as formal framework for particular parts of mathematical analysis. During the last decades a lot of such subsystems have been isolated and proof-theoretically investigated. The subsystems of analysis introduced in this thesis belong to *metapredicative* proof-theory. Metapredicative systems have proof-theoretic ordinals beyond Γ_0 but can still be treated by methods of predicative proof-theory only. Recently, numerous interesting metapredicative systems have been characterized. For previous work in metapredicativity the reader is referred to Jäger [11], Jäger, Kahle, Setzer and Strahm [13], Jäger and Strahm [16, 18], Kahle [19], Rathjen [25] and Strahm [30, 31, 32].

Up to the present the world of metapredicative subsystems of analysis was not so rich. There were ATR (proof-theoretic ordinal Γ_{ε_0}, e.g.[16]), $\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}$ (proof-theoretic ordinal $\varphi 1 \varepsilon_0 0$, [16]) and FTR (proof-theoretic ordinal $\varphi 2 0 \varepsilon_0$, [31]). We introduce in this thesis a lot of subsystems of second order arithmetic with proof-theoretic ordinals between Γ_0 and $\varphi \varepsilon_0 00$. (We use in the sequel the terms "second order arithmetic" and "analysis" as synonym.)

Three concepts are of central importance in this thesis: universes, reflections and hierarchies. Each subsystem, which we will introduce, deals with one of these concepts. We will prove equivalences of some subsystems and give a proof-theoretic analysis of all introduced subsystems.

Summary

First, a remark. Above we have used – and later on we will use – a notation system. Our notation system in this thesis is based on the n-ary φ or Veblen functions which are a straightforward generalization of the well-known binary φ function; in particular, no collapsing is used in this notation system. For instance, the ternary φ function is generated inductively as follows:

(i) $\varphi 0 \beta \gamma$ is just $\varphi \beta \gamma$.

(ii) If $\alpha > 0$, then $\varphi \alpha 0 \gamma$ denotes the γth ordinal which is strongly critical with respect to all functions $\lambda \xi, \eta . \varphi \alpha' \xi \eta$ for $\alpha' < \alpha$.

(iii) If $\alpha > 0$ and $\beta > 0$, then $\varphi \alpha \beta \gamma$ denotes the γth common fixed point of the functions $\lambda \xi . \varphi \alpha \beta' \xi$ for $\beta' < \beta$.

For example, $\varphi 10\alpha$ is Γ_α, and more generally, $\varphi 1 \alpha \beta$ denotes a Veblen hierarchy over $\lambda \alpha . \Gamma_\alpha$.

Universes

Universes play an important role in many systems of set theory and constructive mathematics. There is always the same idea behind universes: If there is given a formal theory T comprising certain existence axioms, then one may argue that there should also exist a set, a so-called universe, which satisfies these closure properties. Often one iterates this process. This basic idea can be found in theories of (iterated) admissibles [10], in Martin-Löf type theory [20, 22] and in explicit mathematics [21, 30] and [14]. It is the aim of the first part of this thesis to introduce universes in metapredicative analysis.

Of course, it is a question of point of view what a universe should be in second order arithmetic. There are two "degrees of freedom": First, what are the closure properties of a universe and secondly, how to define (or formalize) the notion "the set X is an element of the set M" in second order arithmetic.

Since we want to introduce universes which correspond in some sense e.g. to universes in explicit mathematics or to (non wellfounded) admissibles in admissible set theory we take as closure properties for universes in second order arithmetic only arithmetical comprehension and Σ^1_1 choice. Since there is no syntactical possibility to express "the set X is an element of the set M", we have to encode this. In this thesis we have chosen

$$\begin{aligned} X \text{ is in } M &:= X \mathrel{\dot{\in}} M \\ &:= (\exists k)(\forall x)(x \in X \leftrightarrow x \in (M)_k) \\ &:= (\exists k)(\forall x)(x \in X \leftrightarrow \langle x, k\rangle \in M). \end{aligned}$$

Hence, our universes will be countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$ (cf. [29]). We will introduce a predicate $\mathsf{U}(X)$ which says "X is a universe". It is a delicate question, whether we define $\mathsf{U}(X)$

$$\mathsf{U}(X) :\leftrightarrow X \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}, \tag{1}$$

or whether we take as given only the implication

$$\mathsf{U}(X) \rightarrow X \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}. \tag{2}$$

In case (1) each countable coded ω-model of $\Sigma^1_1\text{-}\mathsf{AC}$ is a universe, whereas in case (2) only particular countable coded ω-models X of $\Sigma^1_1\text{-}\mathsf{AC}$ have to be universes, exactly those with $\mathsf{U}(X)$. The formulation (2) gives us more liberty; we do not have to define $\mathsf{U}(X)$ by the equation (1), but it is also possible to define – if desired – $\mathsf{U}(X)$ by

$$\mathsf{U}(X) :\leftrightarrow X \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC} \wedge \varphi(X). \tag{3}$$

Here $\varphi(X)$ can be any formula. The definition (3) will restrict the whole world of universes to the world of countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$ with $\varphi(X)$. Since we want to have the liberty of definition (3) too, we take the implication (2) and not the definition (1).

We ensure the existence of universes by limit axioms. We discuss three types of limit axioms. A non-uniform variant

$$(\exists D)(X \mathrel{\dot{\in}} D \wedge \mathsf{U}(D)), \tag{4}$$

a uniform variant with a universe operator $\mathcal{U}$

$$X \mathrel{\dot{\in}} \mathcal{U}(X) \wedge \mathsf{U}(\mathcal{U}(X)) \tag{5}$$

and a minimal universe variant where we have the limit axiom (4) and for each rel-$\Delta^1_1(\mathsf{U})$ formula φ ($\varphi \in rel$-$\Delta^1_1(\mathsf{U})$ provable in the corresponding theory)

$$(\exists X)[\varphi(X) \wedge \mathsf{U}(X) \to (\exists D)(\varphi(D) \wedge \mathsf{U}(D) \wedge (\forall Z \mathrel{\dot\in} D)(\mathsf{U}(Z) \to \neg\varphi(Z)))]. \tag{6}$$

The class rel-$\Delta^1_1(\mathsf{U})$ of formulas corresponds essentially to the class of Δ^1_1 formulas extended by the predicate $\mathsf{U}(X)$. Axiom (4) leads to the theory NUT, (5) to the theory UUT and (6) to the theory MUT. All these theories also contain arithmetical comprehension, Σ^1_1 choice and formula induction. The formulas $\mathsf{U}(X)$ and $x \in \mathcal{U}(X)$ are arithmetic. For completeness, we mention that MUT contains also a linearity axiom

$$\mathsf{U}(D) \wedge \mathsf{U}(E) \to D \mathrel{\dot\in} E \vee D \doteq E \vee E \mathrel{\dot\in} D. \tag{7}$$

There are several points worth mentioning.

1. Our universes are countable coded ω-models of Σ^1_1-AC. If D is a universe, then there is an index k with e.g. $(D)_k = \{1, 2, 3\}$, since D is closed under arithmetical comprehension. We note that we know absolutely nothing about k. We know only the existence of k. If φ is an arithmetic formula and k with $(D)_k = \{x : \varphi(x)\}$, there is no chance to prove more about k than its existence. Of course, this is a consequence of our notion of universe – there are possible notions of universes such that there is information in the index.

2. In MUT the universes are ordered by axiom (7). If we had defined $\mathsf{U}(X)$ by (1), then the corresponding theory would be inconsistent. Here we see another advantage of our definition of the theories.

3. MUT has proof-theoretic ordinal $\varphi 1 \varepsilon_0 0$. If we had defined $\mathsf{U}(X)$ by (1), then for instance the theory "MUT without linearity (7)" would be much stronger. For instance, we can take for $\varphi(X)$ the arithmetic formula "X is a countable coded ω-model of Σ^1_1-AC". Then (6) implies the existence of a least countable coded ω-model of Σ^1_1-AC. From theorem VIII.4.23 [29] it follows that M is exactly the set of all hyperarithmetic sets. Hence this theory would be impredicative. (Perhaps, who knows, in some time it will be possible to give for such systems also a proof-theoretic analysis which uses methods of predicative proof-theory only.)

We write NUT_0, UUT_0, MUT_0 for the corresponding theories with set-induction instead of formula induction. We will show that all these systems with restricted induction have proof-theoretic ordinal Γ_0. In this sense NUT_0 corresponds to KPi_0 [10] and to UTN [21], and UUT_0 corresponds to EMU [30]. There is a linear ordering on admissibles in KPi_0. With respect to this ordering there are also least admissibles in KPi_0 (cf. theorem 6 in [31]). Hence also MUT_0 corresponds in some sense to KPi_0. Finally, we give a proof-theoretic analysis of the theories of universes. We will prove that $|\mathsf{NUT}| = \Gamma_{\varepsilon_0}$, $|\mathsf{UUT}| = \varphi 1 \varepsilon_0 0$ and

$\mathsf{MUT} = \varphi 1\varepsilon_0 0$. These results are analogous to

$$\begin{aligned} |\mathsf{EMU}| &= \varphi 1\varepsilon_0 0, \\ |\mathsf{KPi}_0 + \text{formula induction on the natural numbers}| &= \varphi 1\varepsilon_0 0. \end{aligned}$$

The proof-theoretic analysis of our theories of universes is similar to the proof-theoretic analysis of the mentioned other systems. Hierarchies of universes lead to the lower bounds. For the determination of the upper bound we introduce semi-formal systems T^0_α, E^0_α with set constants D^0_β, $\mathsf{D}^0_{<\gamma}$ for $\beta < \alpha$, $\gamma \leq \alpha$. Each D^0_β satisfies the closure properties of universes and for $\beta < \delta < \alpha$ we have $\mathsf{D}^0_\beta \dot{\in} \mathsf{D}^0_\delta$, $\mathsf{D}^0_{<\delta} \dot{\in} \mathsf{D}^0_\delta$. We can interpret e.g. MUT into $(\mathsf{T}^0_\alpha)_{\alpha<\varepsilon_0}$ and MUT_0 into $(\mathsf{T}^0_n)_{n\in\mathbb{N}}$. And with similar arguments which lead to $|\widehat{ID}_{<\varepsilon_0}| = \varphi 1\varepsilon_0 0$ (cf. [13]) we can prove $(\mathsf{T}^0_\alpha)_{\alpha<\varepsilon_0} = \varphi 1\varepsilon_0 0$. So we obtain $|\mathsf{MUT}| = \varphi 1\varepsilon_0 0$ and $|\mathsf{MUT}_0| = \Gamma_0$. By embedding UUT into a strengthening of MUT (with the same proof-theoretic ordinal as MUT) we conclude that $|\mathsf{UUT}| = |\mathsf{MUT}|$, and by embedding NUT into ATR we conclude that $|\mathsf{NUT}| = \mathsf{ATR} = \Gamma_{\varepsilon_0}$. We collect these proof-theoretic ordinals in the following scheme.

	universes	uniform universes	minimal universes
with set induction	$\varphi 100$	$\varphi 100$	$\varphi 100$
with formula induction	$\varphi 10\varepsilon_0$	$\varphi 1\varepsilon_0 0$	$\varphi 1\varepsilon_0 0$

Hierarchies and reflection principles of proof-theoretic strength $\varphi 200$ ($\varphi 20\varepsilon_0$)

After having motivated the concepts of universes we describe now principles which model hierarchies and reflections. In mathematics the concept of hierarchy is very useful. For instance in set theory there is the von Neumann hierarchy, the hierarchy of hereditarily finite sets, the constructible hierarchy, ... in recursion theory there is the hyperarithmetical hierarchy, the hierarchy of analytical sets and so on. There are also widely known subsystems of second order arithmetic which claim the existence of certain hierarchies. Especially we mention ATR_0. (ATR) is the following axioms scheme ($(Y)_{Za}$ is the disjoint union of all $(Y)_b$ for b Z-less than a)

(ATR) For all arithmetic $\mathcal{L}_2$ formulas $\varphi(x, X)$:
$WO(Z) \to (\exists Y)(\forall a \in \mathit{field}(Z))(\forall x)(x \in (Y)_a \leftrightarrow \varphi(x, (Y)_{Za}))$.

and ATR_0 is the theory (ATR) plus arithmetical comprehension plus set-induction. (ATR) is a very powerful predicative axioms scheme, since a lot of mathematics can be formalized in ATR_0 (cf. e.g. [29]). The proof-theoretic ordinal of ATR_0 is Γ_0, hence ATR_0 is a predicative subsystem of analysis. In this thesis we introduce several kinds of hierarchies. The corresponding theories will be metapredicative.

Secondly, we discuss certain kinds of reflections. Reflection schemes are important principles in set theory. There are also subsystems of analysis which model reflections e.g. Π^1_{n+1}-RFN where the central principle is ω-model reflection for Π^1_{n+1} formulas (cf. [29, 17]). In some sense, the limit axiom of theories of universes is also a reflection axiom. It claims the existence of a set which reflects a particular sentence, namely a finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{AC}) + (\mathsf{ACA})$. In the mentioned systems and in our theories reflection is always with respect to the relation $\dot\in$, in particular we adopt the notation φ^D for the formula φ where we replace all quantifiers $\forall X$ by $\forall X \dot\in D$ and $\exists X$ by $\exists X \dot\in D$.

A natural question arises: "How much reflection is necessary for proving the existence of certain hierarchies and vice versa; i.e. which reflection is equivalent to which hierarchy?". We prove in this thesis numerous such equivalences and show that the corresponding theories have proof-theoretic ordinal between $\varphi 100$ and $\varphi\varepsilon_0 00$.

The starting point is the equivalence – over ACA_0 – of (ATR) and the axiom

$$(\exists X)(Z \dot\in X \wedge X \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}).$$

The hard direction of this equivalence is proved in [29], theorem VIII.4.20. We prove in this thesis similar equivalences. The difference to the above equivalence is that our hierarchies and reflections are more powerful and hence proof-theoretically stronger. Furthermore, our hierarchies are in general not unique. It is instructive to try to construct within ATR_0 for a given wellordering Z a hierarchy Y with $(Y)_{Za} \dot\in (Y)_a$ and such that $(Y)_a$ is a countable coded ω-model of Σ^1_1-AC for each $a \in field(Z)$. We try to prove this by induction along Z. Of course we do not succeed in proving within ATR_0 the existence of such a hierarchy, but let us mention two problems which arise.

1. In order to prove

   $$(\exists Y)(\forall a \in field(Z))((Y)_{Za} \dot\in (Y)_a \wedge (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})$$

   by induction along Z, we have to show within ATR_0 that

   $$\begin{aligned}\{a \in field(Z) : (\forall b)(bZa \rightarrow (\exists Y)((Y)_{Zb} \dot\in (Y)_b \wedge \\ (Y)_b \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}))\}\end{aligned}$$

   is a set. In order to prove this we would need Σ^1_1 comprehension and this is not available in ATR_0.

2. We assume that we can use induction along Z in spite of problem 1. Then we distinguish three cases: $a = 0_Z$, a is a successor, a is a limit number. Only the third case gives rise to problems. Let us discuss these problems. Assume that a is a limit number. In theorem V.8.3 [29] it is shown that ATR_0 proves $(\Sigma^1_1\text{-}\mathsf{AC})$. Applying this to the induction hypothesis leads to a set Y such that for all b Z-less than a we have

$$((Y)_b)_{Zb} \mathrel{\dot\in} ((Y)_b)_b \wedge ((Y)_b)_b \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}.$$

In other words, each $(Y)_b$ is a desired hierarchy up to b. The problem is that we cannot prove the uniqueness of these hierarchies. More precisely: given d and b Z-less than a and given c Z-less than d and b we cannot prove

$$((Y)_b)_c = ((Y)_b)_d.$$

Hence, we cannot paste together the initial segments $(Y)_b$ in order to obtain a hierarchy up to a.

Now we ask for principles needed to prove the existence of hierarchies of countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$ by induction along the wellordering Z. We overcome the mentioned difficulties as follows:

1. Instead of

$$(\exists Y)(\forall a \in field(Z))((Y)_{Za} \mathrel{\dot\in} (Y)_a \wedge (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})$$

we prove

$$(\exists Y \mathrel{\dot\in} M)(\forall a \in field(Z))((Y)_{Za} \mathrel{\dot\in} (Y)_a \wedge \\ (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})$$

The crucial point here is that this formula is now equivalent to an arithmetic formula. Of course the set M is not an arbitrary set but a set with additional properties. Next, we will give some of these properties. We remember that this trick was already used, e.g., for the embedding of (ATR) into KPi_0 [10].

2. Again we distinguish the cases $a = 0_Z$, a is a successor, a is a limit number. If we can build in M countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$, then the cases $a = 0_Z$ and a is a successor go through. For instance the condition

$$M \text{ is a countable coded } \omega\text{-model of } \mathsf{ATR}$$

ensures this. There remains the limit case. Here we notice that we can extend hierarchies by using the axiom of dependent choice. Essentially we generalize the proof of the existence of fixed point hierarchies up to $\alpha < \varepsilon_0$ in $\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC}$ [16] in order to obtain hierarchies of countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$ along arbitrary wellorderings. Summing up, it is enough to have

$$M \text{ is a countable coded } \omega\text{-model of } \mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC}.$$

Carrying through all this in detail will lead to the implication (over ACA_0)

$$\begin{aligned} & (\forall X)(\exists Y)(X \dot{\in} Y \wedge Y \text{ is a countable coded } \omega\text{-model of } \mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC}) \\ & \quad \to \text{ there exist hierarchies of countable coded } \omega\text{-models of } \Sigma^1_1\text{-}\mathsf{AC} \\ & \qquad \text{along arbitrary wellorderings.} \end{aligned}$$

A crucial point is that in the premise of the above implication we ensure that Y is a model of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$. Were Y only a model of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{AC}$, then we would have no chance to prove the existence of the hierarchy of countable coded ω-models of Σ^1_1-AC. Notice that $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{AC}$ is equivalent to ATR. But $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ is not equivalent to ATR.

We now sketch how we can obtain the opposite direction. We use the method of pseudohierarchies. Pseudohierarchies are used, e.g in [29], in order to construct in ATR_0 countable coded ω-models of Σ^1_1-DC. It is very remarkable that this proof technique leads to such models. Of course, these models are no least models – we will always construct Σ^1_1 models and not Π^1_1 models. The existence of pseudohierarchies is assured by the following fact which can be proved in ACA_0.

$$\begin{aligned} & (\forall Z)(WO(Z) \to (\exists Y)(\forall a \in field(Z))((Y)_{Za} \dot{\in} (Y)_a \wedge \\ & \qquad\qquad (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})) \\ & \quad \to (\exists Z, Y)(LO(Z) \wedge \neg WO(Z) \wedge \\ & \qquad (\forall a \in field(Z))((Y)_{Za} \dot{\in} (Y)_a \wedge (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})). \end{aligned}$$

The existence of pseudohierarchies is claimed in the conclusion of the above implication. Pseudohierarchies of countable coded ω-models of Σ^1_1-AC look like hierarchies of countable coded ω-models of Σ^1_1-AC with the difference that the underlying linear ordering is not wellfounded. If we argue in ATR_0 we can prove more than stated in the above implication.

$$\begin{aligned} & (\forall Z)(WO(Z) \to (\exists Y)(\forall a \in field(Z))((Y)_{Za} \dot{\in} (Y)_a \wedge \\ & \qquad\qquad (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC})) \\ & \quad \to (\exists Z, Y, M^*)(LO(Z) \wedge \neg WO(Z) \wedge \\ & \qquad (\forall a \in field(Z))((Y)_{Za} \dot{\in} (Y)_a \wedge (Y)_a \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC} \\ & \qquad \wedge (WO(Z))^{M^*} \wedge Z, Y \dot{\in} M^* \wedge M^* \text{ is a countable coded } \omega\text{-model of } \mathsf{ACA})). \end{aligned} \tag{8}$$

The introduction of the set M^* is a technical trick. M^* is a countable coded ω-model of ACA and contains the linear ordering Z and the pseudohierarchy Y. Furthermore, all sets arithmetic in M^* are wellfounded with respect to Z. (In ATR_0 we would even be able to prove a stronger result: We can choose M^* such that all sets Δ^1_1 in M^* are wellfounded with respect to Z.) We give now a short sketch of the construction of a model M of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ with the aid of pseudohierarchies.

Let us choose Z, Y, M^* as described in the conclusion of the implication (8). Since Z is not wellfounded, there exists a function $\mathcal{F}$ such that $(\mathcal{F}(n))_{n \in \mathbb{N}}$ is an infinite descending sequence with respect to Z. We set

$$I := \{c : (\forall n)\langle c, \mathcal{F}(n)\rangle \in Z\}$$

and prove that I is closed downwards, not empty, not in M^* and unbounded with respect to Z. Our model M will be the set of all sets recursive in $(Y)_b$ for some $b \in I$. Since I is unbounded and each $(Y)_b$ is a countable coded ω-model of Σ^1_1-AC we immediately conclude that M is a model of ACA and ATR – since (ATR) is equivalent to

$$(\forall X)(\exists Y)(X \dot{\in} Y \wedge Y \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}).$$

The proof that M is a model of Σ^1_1-DC too, is more tricky. Here we need $(WO(Z))^{M^*}$ and $I \dot{\notin} M^*$. Choose an arithmetic formula φ and assume

$$(\forall X \dot{\in} M)(\exists Y \dot{\in} M)\varphi(X, Y).$$

We have to find a sequence $(U)_n$ with $U \dot{\in} M$, $(U)_0 = P$ and $\varphi((U)_n, (U)_{n+1})$ for all n and a given P in M. We construct the sequence $(U)_n$ together with a sequence $(b)_n$ as follows: Of course, we set $(U)_0 := P$ and $(b)_0$ is the Z-least c such that P is recursive in $(Y)_c$. Given $(U)_n$, let $(b)_{n+1}$ be the Z-least c Z-greater than $(b)_n$ and such that there exists a V, recursive in $(Y)_c$, with $\varphi((U)_n, V)$. We then set $(U)_n := \{x : \{e\}^{(Y)_{(b)_{n+1}}}(x) = 0\}$ where e is the least index with $\varphi((U)_n, \{x : \{e\}^{(Y)_{(b)_{n+1}}}(x) = 0\})$. Notice that here we need $(WO(Z))^{M^*}$ in order to choose a Z-least $(b)_{n+1}$. This unique choice of $(b)_{n+1}$ is the crucial point, since now the sequence $((b)_n)_{n\in\mathbb{N}}$ is welldefined. Hence we can choose a unique sequence $(U)_n$. Finally it can be shown that U is in fact in M, the proof uses $I \dot{\notin} M^*$. For a detailed account of all these arguments the reader is referred to lemma 30.

The crucial points in adapting the proof of lemma VIII.4.19 in [29] to our situation are:

1. In order to show that M is a model of ATR it suffices to show in M

$$(\forall X)(\exists Y)(X \dot{\in} Y \wedge Y \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-}\mathsf{AC}). \tag{9}$$

2. Instead of a pseudohyperarithmetical hierarchy which is used in lemma VIII.4.19 [29] we have a pseudohierarchy of countable coded ω-models of Σ^1_1-AC. And since I is unbounded with respect to Z the property (9) is easily proved: Take an X in M. Then we know that X is recursive in an $(Y)_b$ for a $b \in I$. But $(Y)_b \dot{\in} (Y)_{b+_Z 1}$ and $(Y)_{b+_Z 1}$ is recursive in $(Y)_{b+_Z 1}$, in particular $(Y)_{b+_Z 1} \dot{\in} M$. Hence the claim.

3. Pseudohierarchies of countable coded ω-models of Σ^1_1-AC have similar properties as pseudohyperarithmetical hierarchies. In fact, the properties of pseudohierarchies of countable coded ω-models of Σ^1_1-AC are in some sense better, stronger than those of pseudohyperarithmetical hierarchies. So the part of the proof of lemma VIII.4.19 [29], where it is shown that M satisfies Σ^1_1-DC, can be adapted to our situation.

Summing up, we have the equivalence of the following axioms over ACA_0.

a) $(\forall X)(\exists Y)(X \dot{\in} Y \wedge Y$ is a countable coded ω-model of $\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})$.

b) There exist hierarchies of countable coded ω-models of Σ^1_1-AC along arbitrary wellorderings.

Iterating the argument we can show in fact that a) and b) are equivalent to

c) There exist hierarchies of "countable coded ω-models of models of Σ^1_1-AC" along arbitrary wellorderings. That is, there exist hierarchies of countable coded ω-models of ATR along arbitrary wellorderings,

and so on.

The corresponding theories to the axioms a), b) and c) have proof-theoretic ordinal $\varphi 200$, respectively $\varphi 20\varepsilon_0$ if we have formula induction instead of set-induction. We will not give a proof-theoretic analysis but show that all these theories are equivalent to FTR_0, respectively FTR [31]. The axioms scheme (FTR) claims the existence of fixed point hierarchies along arbitrary wellorderings. Using Aczel's trick, mentioned in [6], we can construct fixed point hierarchies if we have hierarchies of countable coded ω-models of Σ^1_1-AC. The converse direction uses again – but now implicitly, via lemma VIII.4.19 [29] – the method of pseudohierarchies. We give the idea how to build a countable coded ω-model of Σ^1_1-AC with the aid of fixed points. In the following we mean by fixed point always a fixed point X of an appropriate X-positive formula $\mathcal{A}(x, X)$ with possibly further set and number variables.

1. We build a fixed point X such that for each wellordering recursive in Q with index a, $(X)_a$ is the π^0_1 jump hierarchy along the wellordering a starting with Q.
2. We build a fixed point Y such that $X \dot{\in} Y$ and Y is a countable coded ω-model of ACA.

It immediately follows from lemma VIII.4.19 [29] that there exists a countable coded ω-model of Σ^1_1-AC in Y. An iteration of this argument leads to a hierarchy of countable coded ω-models of Σ^1_1-AC. And we have shown the equivalence of hierarchies of fixed points and of countable coded ω-models of Σ^1_1-AC. In a certain sense this generalize the equivalence of (FP) and (ATR), a result of Avigad [1]. (FP) claims the existence of fixed points X for X-positive formulas. Moreover, we can use the presented method in order to embed Σ^1_1-AC into $\widehat{ID}_2$ (a new result too). Unfortunately the method does not yield an embedding of Σ^1_1-AC into $\widehat{ID}_1$.

Notice that the mentioned equivalences are of the intended type. Axiom b) is a kind of reflection axiom, a), c) and (FTR) claim the existence of certain hierarchies. The main results are listed in the following scheme.

	predicative	metapredicative
hierarchy	ATR_0	hierarchies of countable coded ω-models of Σ^1_1-AC fixed point hierarchies
reflection	there are countable coded ω-models of Σ^1_1-AC	there are countable coded ω-models of $\mathsf{ATR} + \Sigma^1_1$-DC
proof-theory (with set induction)	$\varphi 100$	$\varphi 200$

Hierarchies and reflection principles of proof-theoretic strength $\varphi\omega 00$ ($\varphi\varepsilon_0 00$)

In a next step we define predicates I_n:

$$\begin{aligned} \mathsf{I}_0(M) &:= M \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-DC}, \\ \mathsf{I}_{n+1}(M) &:= M \text{ is a countable coded } \omega\text{-model of } \Sigma^1_1\text{-DC} \\ &\quad \wedge (\forall X \mathrel{\dot\in} M)(\exists Y \mathrel{\dot\in} M)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_n(Y)). \end{aligned}$$

For instance, if we have $\mathsf{I}_1(M)$, then M is a model of $\mathsf{ATR} + \Sigma^1_1$-DC. The predicate I_n corresponds to n-inaccessibility, cf. [18]. We have motivated the equivalence of hierarchies Y, such that each stage $(Y)_b$ satisfies I_0, and of sets M with $\mathsf{I}_1(M)$. This is the equivalence of the above statements a) and b). Exactly the same proof technique leads to the equivalence of hierarchies Y, such that each stage $(Y)_b$ satisfies I_n, and of sets M with $\mathsf{I}_{n+1}(M)$. So we have the equivalence

d) $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_{n+1}(Y))$.

e) There exist hierarchies along arbitrary wellorderings such that each step satisfies I_{n+1}.

Notice that each M with $\mathsf{I}_{n+1}(M)$ is a countable coded ω-model of Σ^1_1-DC and reflects the Π^1_2 sentence

$$(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_n(Y)).$$

In other words, the Π^1_2 sentence $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_n(Y))$ is reflected on a countable coded ω-model of Σ^1_1-DC, namely M. A further reflection, a reflection of the Π^1_2 sentence $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_{n+1}(Y))$ on a countable coded ω-model of Σ^1_1-DC would lead to a set N with $\mathsf{I}_{n+2}(N)$. We ask: "Is there a theory which claims for each natural number $n \in \mathbb{N}$ the existence of a set M with $\mathsf{I}_n(M)$?". It is the aim of the last part of this introduction to present two such theories. We have mentioned that reflection of a special Π^1_2 formula on countable coded ω-models of Σ^1_1-DC gives successively sets which satisfy I_0, I_1, I_2, ... Hence we define

($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$) For all Π^1_2 formulas $\varphi[\vec z, \vec Z]$:
$\varphi[\vec z, \vec Z] \rightarrow (\exists M)(\vec Z \mathrel{\dot\in} M \wedge \varphi^M \wedge M$ is a countable coded ω-model of Σ^1_1-$\mathsf{DC})$.

The theory $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ is the theory ACA_0 plus ($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$). This principle is a reflection principle. And again we ask: "Are there hierarchies equivalent to this reflection principle?" The answer is: "Yes, there are." We argue within $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. Given $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_n(Y))$, an application of ($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$) leads to $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_{n+1}(Y))$. Remember that this is equivalent to hierarchies Y such that each stage $(Y)_b$ satisfies I_n, cf. the equivalence of d) and e). In other words, $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_n(Y))$ implies – in $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ – the existence of hierarchies Y with $\mathsf{I}_n((Y)_b)$ along arbitrary wellorderings. Hence we define

(Σ^1_1-TDC) For all Σ^1_1 formulas φ:
$(\forall a)(\forall X)(\exists Y)\varphi(X, Y, a) \wedge WO(Z)$
$\rightarrow (\exists Y)(\forall a \in field(Z))\varphi((Y)_{Za}, (Y)_a, a)$.

In terms of hierarchies the axioms scheme (Σ^1_1-TDC_0) reads as follows: If we know that for all X there is a set Y containing X and fulfilling φ, then there is along each wellordering Z a hierarchy such that each stage of the hierarchy satisfies s φ. The name Transfinite Dependent Choice is due to Gerhard Jäger.

The theory Σ^1_1-TDC_0 is the theory ACA_0 plus (Σ^1_1-TDC). The same line of argument which led to the equivalence of a) and b) above can be used in order to show the equivalence of (Σ^1_1-TDC) and ($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$) over ACA_0. Let us mention a few points.

1. With respect to Π^1_2 reflection there is a significant difference between reflection on models of Σ^1_1-DC and on models of Σ^1_1-AC. First, we observe that the premise of an

instance of (Σ^1_1-DC) is a Π^1_2 formula. Therefore, if we reflect a Π^1_2 formula on a model of Σ^1_1-DC, we can again apply (Σ^1_1-DC) within this model to that formula. We can do this again and again.

2. We have seen above that with the help of (Σ^1_1-DC) we can extend hierarchies, even non unique ones. This (together with point 1) leads to the fact that in models of (Σ^1_1-DC) we can build appropriate hierarchies. Thus there is a significant difference between Π^1_2 reflection on models of (Σ^1_1-DC) and on models of (Σ^1_1-AC) (cf. the collection of some results at the end of this introduction).

3. It is instructive to have a look at the following axioms scheme.

($weak$ Σ^1_1-TDC) For all Σ^1_1 formulas φ:

$$(\forall a)(\forall X)(\exists! Y)\varphi(X,Y,a) \wedge WO(Z) \\ \quad \rightarrow (\exists Y)(\forall a \in field(Z))\varphi((Y)_{Za},(Y)_a,a).$$

This scheme is less powerful than (Σ^1_1-TDC). We can prove ($weak$ Σ^1_1-TDC) in $\mathsf{ATR}_0 + \Sigma^1_1$-$\mathsf{IND}$, since in this theory ($\Sigma^1_1$-$\mathsf{TI}$) is available (cf. [29] lemma VIII.6.15). Therefore we can prove the existence of an Y with $(\forall a \in field(Z))\varphi((Y)_{Za},(Y)_a,a)$ by induction along the well-ordering Z. Since we have uniqueness of every stage, the limit case does not give rise to problems. That is, ($weak$ Σ^1_1-TDC) is much weaker than (Σ^1_1-TDC) (cf. the collection of some results at the end of this introduction). This is in contrast to (Σ^1_1-AC) and ($weak$ Σ^1_1-AC) (cf. [29] definition VIII.4.12) or (Σ^1_1-DC) and ($weak$ Σ^1_1-DC). Here Σ^1_1-AC and $weak$ Σ^1_1-AC have proof-theoretic strength $\varphi\varepsilon_0 0$, and so have Σ^1_1-DC and $weak$ Σ^1_1-DC.

Finally, we will give the proof theoretic ordinals for Σ^1_1-TDC_0 and Σ^1_1-TDC. The semi-formal systems needed for the proof-theoretic analysis of Σ^1_1-TDC_0 are introduced and discussed in chapter 3. Many technical tricks in that chapter are due to Thomas Strahm. That Σ^1_1-TDC, i.e. ACA plus (Σ^1_1-TDC), has proof-theoretic ordinal $\varphi\varepsilon_0 00$ is sketched, among other things, in chapter 4.

In order to emphasize the difference between models of Σ^1_1-DC and of Σ^1_1-AC we prove also the equivalence of (ATR) + (Σ^1_1-DC) and $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$. This leads to the proof-theoretic ordinal $\varphi 1\omega 0$ for $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{AC}}$, in contrast to $\varphi\omega 00$, the proof-theoretic ordinal of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$.

(Σ^1_1-TDC) is a natural extension of (Σ^1_1-DC). It solves the problem of building hierarchies which are not unique. The theory Σ^1_1-TDC_0 is metapredicative with proof-theoretic ordinal $\varphi\omega 00$. Hence, corresponding systems in set theory or in explicit mathematics are e.g.

systems of metapredicative Mahlo [18]. We collect these results in the following scheme.

	predicative	metapredicative	metapredicative
reflection hierarchy	$(\Pi^1_2\text{-}\mathsf{RFN})$ $(\Sigma^1_1\text{-}\mathsf{DC})$	$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$ $(\mathsf{ATR}) + (\Sigma^1_1\text{-}\mathsf{DC})$	$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ $(\Sigma^1_1\text{-}\mathsf{TDC})$
proof-theory (with set-induction)	$\varphi\omega 0$	$\varphi 1\omega 0$	$\varphi\omega 00$

Chapter 1

Universes

1.1 The theories NUT, UUT and MUT

In this section we set the languages, notations and abbreviations. Moreover, we introduce the theories of universes NUT, UUT and MUT. In NUT we have a non-uniform limit axiom, in UUT we have a uniform limit axiom. Also MUT is a theory with a non-uniform limit axiom, but beyond it, the universes are linearly ordered and we can choose minimal universes with respect to this ordering.

1.1.1 Languages, classes of formulas, abbreviations and definition of the proof-theoretic ordinal of a theory

$\mathcal{L}_1$: the language of first order arithmetic

As a rule we discuss second order arithmetic in this thesis. But sometimes we refer in definitions to first order formulas. Hence we introduce $\mathcal{L}_1$, the language of first order arithmetic. $\mathcal{L}_1$ includes *number variables* denoted by small letters, except r, s, t. Furthermore, there are symbols for all primitive recursive functions and relations, equality, a symbol $\sim$ for forming negative literals, as well as a unary relation symbol Q which we will use in the definition of the proof-theoretic ordinal below.

The *number terms* r, s, t of $\mathcal{L}_1$ are defined as usual. The *positive literals* of $\mathcal{L}_1$ are all expressions $(s = t)$, $K(s_1, \ldots, s_n)$, $\mathsf{Q}(s)$ for K a symbol for an n-ary primitive recursive relation. The *negative literals* of $\mathcal{L}_1$ have the form $({\sim}E)$ so that E is a positive literal. The *true literals* of $\mathcal{L}_1$ are all literals $(s = t)$, $K\vec{s}$ such that $(s = t)$, $K\vec{s}$ is true respectively.

$\mathcal{L}_2$: the language of second order arithmetic

We let $\mathcal{L}_2$ denote the language of second order arithmetic. $\mathcal{L}_2$ includes *number variables* (are denoted by small letters, except r, s, t), *set variables* (are denoted by capital letters, except R, S, T), symbols for all primitive recursive functions and relations, the symbol $\in$ for elementhood between numbers and sets, as well as equality in the first sort. Furthermore, there is a symbol $\sim$ for forming negative literals, as well as the unary relation symbol Q.

The *number terms* r, s, t of $\mathcal{L}_2$ are defined as usual; the *set terms* are just the set variables. *Positive literals* of $\mathcal{L}_2$ are all expressions $(s = t)$, $K(s_1, \ldots, s_n)$, $s \in X$, $\mathsf{Q}(s)$ for K a symbol for an n-ary primitive recursive relation. The *negative literals* of $\mathcal{L}_2$ have the form $(\sim E)$ so that E is a positive literal. We often write $(s \neq t)$ and $(s \notin X)$ instead of $\sim(s = t)$ and $\sim(s \in X)$. The *formulas* $\varphi, \psi, \theta, \ldots$ of $\mathcal{L}_2$ are generated from the positive and negative literals of $\mathcal{L}_2$ by closing against disjunction, conjunction, existential and universal number and set quantification. The *negation* $\neg\varphi$ of an $\mathcal{L}_2$ formula φ is defined by making use of De Morgan's laws and the law of double negation.

An $\mathcal{L}_2$ formula is called *arithmetic*, if it does not contain bound set variables (but possibly free set variables and the relation symbol Q); for the collection of these formulas we write Π^1_0 . Σ^1_1 is the collection of all arithmetic formulas and of all $\mathcal{L}_2$ formulas $\exists X \varphi(X)$ with $\varphi(X)$ from Π^1_0. Analogously Σ^1_k and Π^1_k are defined.

Extensions of $\mathcal{L}_2$: $\mathcal{L}_2(\mathsf{U})$, $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$

Language:
$\mathcal{L}_2(\mathsf{U})$ denotes the extension of $\mathcal{L}_2$ by the unary relation symbol U for being a universe and $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ denotes the extension of $\mathcal{L}_2(\mathsf{U})$ by the unary universe operator $\mathcal{U}$.

Terms:
The *number terms* r, s, t of $\mathcal{L}_2(\mathsf{U})$, $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ are the number terms of $\mathcal{L}_2$. The *set terms* R, S, T of $\mathcal{L}_2(\mathsf{U})$ are simply the set variables. The *set terms* R, S, T of $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ are the set variables and all expressions of the form $\mathcal{U}(S)$ so that S is a set term.

Formulas:
Positive literals of $\mathcal{L}_2(\mathsf{U})$ are the positive literals of $\mathcal{L}_2$ and all expressions $\mathsf{U}(X)$. *Positive literals* of $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ are all expressions $(s = t)$, $K(s_1, \ldots, s_n)$, $s \in S$, $\mathsf{Q}(s)$, $\mathsf{U}(S)$ for K a symbol for an n-ary primitive recursive relation. (Note that $s \in \mathcal{U}(S)$ is a positive literal of $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$.)

The *negative literals* and *formulas* of $\mathcal{L}_2(\mathsf{U})$, $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ are built in the same way like the negative literals and formulas of $\mathcal{L}_2$.

Special classes of formulas:

An $\mathcal{L}_2(\mathsf{U})$ resp. $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula is called *arithmetic*, if it does not contain bound set variables (but possibly set terms and the relation symbol Q); for the collection of these formulas we write $\Pi^1_0(\mathsf{U})$ resp. $\Pi^1_0(\mathsf{U},\mathcal{U})$. $\Sigma^1_1(\mathsf{U})$ resp. $\Sigma^1_1(\mathsf{U},\mathcal{U})$ is the collection of all arithmetic $\mathcal{L}_2(\mathsf{U})$ resp. $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formulas and of all formulas $\exists X\varphi(X)$ with $\varphi(X)$ from $\Pi^1_0(\mathsf{U})$ resp. $\Pi^1_0(\mathsf{U},\mathcal{U})$.

Abbreviations

In the sequel $\langle\ldots\rangle$ denotes a primitive recursive coding function for n-tuples $\langle t_1,\ldots,t_n\rangle$ with associated projections $(\cdot)_1,\ldots,(\cdot)_n$. Seq_n is the primitive recursive set of sequence numbers of length n. Seq denotes the primitive recursive set of sequence numbers. We write $s \in (S)_t$ for $\langle s,t\rangle \in S$ and $\vec{S}$ for $S_1,...,S_n$. Occasionally we use the abbreviations:

$$\begin{array}{rcl}
x \in S \oplus T &:=& Seq_2 x \wedge \\
&& [((x)_1 = 1 \wedge (x)_0 \in S) \vee ((x)_1 = 2 \wedge (x)_0 \in T)], \\
S = T &:=& (\forall x)(x \in S \leftrightarrow x \in T), \\
S \neq T &:=& \neg S = T, \\
S \dot{\in} T &:=& (\exists k)(\forall x)(x \in S \leftrightarrow \langle x,k\rangle \in T), \\
(\exists Y \dot{\in} S)\varphi(Y) &:=& (\exists Y)(Y \dot{\in} S \wedge \varphi(Y)), \\
(\forall Y \dot{\in} S)\varphi(Y) &:=& (\forall Y)(Y \dot{\in} S \rightarrow \varphi(Y)), \\
\vec{S} \dot{\in} T &:=& S_1 \dot{\in} T \wedge \ldots \wedge S_n \dot{\in} T, \\
S \doteq T &:=& (\forall X)(X \dot{\in} S \leftrightarrow X \dot{\in} T).
\end{array}$$

Furthermore, we write $\varphi[\vec{x},\vec{X}]$ if all free number variables of φ are among $\vec{x}$ and all free set variables of φ are among $\vec{X}$. We write $\varphi[\vec{x}\backslash\vec{t},\vec{X}\backslash\vec{S}]$ for the formula φ where all occurrences of x_i are substituted by t_i and all occurrences of X_i are substituted by S_i. Often we write directly $\varphi[\vec{t},\vec{S}]$ for $(\varphi[\vec{x},\vec{X}])[\vec{x}\backslash\vec{t},\vec{X},\vec{S}]$.

Proof-theoretic ordinal

In the sequel we will measure the proof-theoretic strength of formal theories in terms of their proof-theoretic ordinals. As usual we set for all primitive recursive relations $\prec$ and all formulas φ

$$\begin{array}{rcl}
Prog(\prec,\varphi) &:=& (\forall x)[(\forall y)(y \prec x \rightarrow \varphi(y)) \rightarrow \varphi(x)], \\
TI(\prec,\varphi) &:=& Prog(\prec,\varphi) \rightarrow (\forall x)\varphi(x).
\end{array}$$

The proof-theoretic ordinal of a theory T is defined by referring to transfinite induction for the anonymous relation Q. We say that an ordinal α is provable in T, if there is a primitive recursive wellordering $\prec$ of order type α so that $\mathsf{T} \vdash TI(\prec,\mathsf{Q})$. And the least ordinal which is not provable in T we call the proof-theoretic ordinal of T and is denoted by $|\mathsf{T}|$.

1.1.2 Some subsystems of second order arithmetic

We introduce some (widely known) subsystems of second order arithmetic which we need in the sequel. We use the Tait-style formulation of $\mathcal{L}_2$ as presented and the following abbreviations:

$$\begin{array}{rcl} WO(X) & := & \text{formalization of "X codes a reflexive well-ordering"}, \\ x \in field(X) & := & (\exists y)(\langle x, y\rangle \in X \vee \langle y, x\rangle \in X), \\ x \in (Y)_{Za} & := & Seq_2 x \wedge x \in Y \wedge \langle (x)_1, a\rangle \in Z \wedge (x)_1 \neq a. \end{array}$$

$(Y)_{Za}$ is the disjoint union of all projections $(Y)_b$ with $\langle b, a\rangle \in Z$. For a wellordering Z we let 0_Z denote the least element in $field(Z)$ and for $a \in field(Z)$ we let $a +_Z 1$ denote the Z-least element in $field(Z)$ greater than a. Sometimes we write aZb for $\langle a, b\rangle \in Z$.

All subsystems are based on the usual axioms and rules for the two-sorted predicate calculus.

ACA:

The theory ACA includes defining axioms for all primitive recursive functions and relations, the induction scheme for arbitrary formulas of $\mathcal{L}_2$ and the axioms scheme

(ACA) For all arithmetic $\mathcal{L}_2$ formulas $\varphi(x)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x))$.

Σ^1_1-AC:

The theory Σ^1_1-AC extends ACA by the axioms scheme

(Σ^1_1-AC) For all $\mathcal{L}_2$ formulas $\varphi(x, X)$ in Σ^1_1:
$(\forall x)(\exists X)\varphi(x, X) \rightarrow (\exists X)(\forall x)\varphi(x, (X)_x)$.

ATR:

The theory ATR extends ACA by the axioms scheme

(ATR) For all arithmetic $\mathcal{L}_2$ formulas $\varphi(x, X)$:
$WO(Z) \rightarrow (\exists Y)(\forall a \in field(Z))(\forall x)(x \in (Y)_a \leftrightarrow \varphi(x, (Y)_{Za}))$.

Σ^1_1-DC:

The theory Σ^1_1-DC extends ACA by the axioms scheme

(Σ^1_1-DC) For all $\mathcal{L}_2$ formulas $\varphi(X,Y)$ in Σ^1_1:
$(\forall X)(\exists Y)\varphi(X,Y) \rightarrow (\exists Z)[(Z)_0 = X \wedge \forall u \varphi((Z)_u,(Z)_{u+1})]$.

Ax_{ACA} denotes a finite axiomatization of (ACA). We adopt the standard notation φ^X for the relativization of the $\mathcal{L}_2$ formula φ to X (for example $(\forall Y \varphi(Y))^X := (\forall Y \dot{\in} X)\varphi^X(Y)$). Then we can formulate the theory Π^1_{n+1}-RFN:

Π^1_{n+1}-RFN:

The theory Π^1_{n+1}-RFN extends ACA by the axioms scheme

(Π^1_{n+1}-RFN) For all $\mathcal{L}_2$ formulas $\varphi[x,Z]$ in Π^1_{n+1}:
$\varphi[x,Z] \rightarrow (\exists X)(Z \dot{\in} X \wedge (Ax_{\mathsf{ACA}})^X \wedge \varphi^X[x,Z])$.

T_0 denotes the theory T with set-induction instead of the induction scheme for arbitrary formulas.

1.1.3 The classes of formulas rel-$\Sigma^1_k(\mathsf{U})$, rel-$\Sigma^1_k(\mathsf{U},\mathcal{U})$, rel-$\Pi^1_k(\mathsf{U})$ and rel-$\Pi^1_k(\mathsf{U},\mathcal{U})$

Until now expressions such as $\mathsf{U}(\{x : \varphi(x)\})$ are not defined. So we have to explain, for instance, the meaning of $(\forall x)\mathsf{U}((X)_x)$. Moreover, if we want to be able to apply a choice axiom to a formula $(\forall x)(\exists X)\varphi(\mathsf{U}(X),x)$ the expression $\varphi(\mathsf{U}((X)_x),x)$ must be defined. There are several possibilities to do that. For instance we can extend our language to allow set terms such as $\{x : \varphi(x)\}$ for arithmetic formulas φ. This is a neat approach but for proof-theoretical investigations a little complicated. Therefore, we are going to take another route. We introduce new classes of formulas. First, we define the class of formulas rel-$\Pi^1_0(\mathsf{U})$ (*rel*ativ arithmetic $\mathcal{L}_2(\mathsf{U})$-formulas).

1. Every arithmetic $\mathcal{L}_2(\mathsf{U})$ formula is a rel-$\Pi^1_0(\mathsf{U})$ formula.
2. If φ and ψ are rel-$\Pi^1_0(\mathsf{U})$ formulas, so also are $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$.
3. If φ is a rel-$\Pi^1_0(\mathsf{U})$ formula, so also are $\exists x\varphi$ and $\forall x\varphi$.
4. If φ is a rel-$\Pi^1_0(\mathsf{U})$ formula, so also are $(\exists X \dot{\in} S)\varphi$ and $(\forall X \dot{\in} S)\varphi$.

rel-$\Sigma^1_1(\mathsf{U})$ is the collection of all rel-$\Pi^1_0(\mathsf{U})$ formulas and of all formulas $\exists X\varphi(X)$ with $\varphi(X)$ a rel-$\Pi^1_0(\mathsf{U})$ formula. Now rel-$\Pi^1_k(\mathsf{U})$ and rel-$\Sigma^1_k(\mathsf{U})$ are defined as usual. rel-$\Pi^1_k(\mathsf{U},\mathcal{U})$ and rel-$\Sigma^1_k(\mathsf{U},\mathcal{U})$ are analogously defined.

The stage is now set in order to explain what we mean by $\mathsf{U}(\{x : \varphi(x)\})$. Let φ be a $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula. Then we mean by $\mathsf{U}(\{x : \varphi(x)\})$ the expression

$$(\exists X)[(\forall x)(x \in X \leftrightarrow \varphi(x)) \wedge \mathsf{U}(X)].$$

And by $t \in \mathcal{U}(\{x : \varphi(x)\})$ we mean the expression

$$(\exists X)[(\forall x)(x \in X \leftrightarrow \varphi(x)) \wedge t \in \mathcal{U}(X)].$$

In the sequel we use $\mathsf{U}(\{x : \varphi(x)\})$ and $t \in \mathcal{U}(\{x : \varphi(x)\})$ as abbreviations for the expressions stated above. Later on, we will show in NUT_0 (resp. UUT_0) corresponding statements to the usual closure conditions of Δ^1_1 formulas. In particular we show that for $\varphi_1, \psi_1 \in rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$ and $\varphi_2, \psi_2 \in rel\text{-}\Pi^1_1(\mathsf{U},\mathcal{U})$ there are $\theta_1 \in rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$ and $\theta_2 \in rel\text{-}\Pi^1_1(\mathsf{U},\mathcal{U})$ such that we can prove in UUT_0 (this is one of our theories introduced below): $(i, j \in \{1, 2\})$

$$\begin{aligned}&\varphi_1 \leftrightarrow \varphi_2 \wedge \psi_1 \leftrightarrow \psi_2\\ &\quad \rightarrow (\varphi_i(\vec{S})[\vec{S}\backslash\{x : \vec{\psi}_j(x)\}] \leftrightarrow \theta_1) \wedge (\varphi_i(\vec{S})[\vec{S}\backslash\{x : \vec{\psi}_j(x)\}] \leftrightarrow \theta_2).\end{aligned}$$

1.1.4 Definition of the theories

First we define the theory of universes NUT (Non-uniform Universes Theory). It is formulated in $\mathcal{L}_2(\mathsf{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. The non-logical axioms comprise:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) *equality axioms*

$$\mathsf{U}(X) \wedge X = Y \rightarrow \mathsf{U}(Y).$$

(3) *set operations*

($rel\text{-}\Pi^1_0(\mathsf{U})\text{-}\mathsf{CA}$): For all $rel\text{-}\Pi^1_0(\mathsf{U})$ formulas $\varphi(x)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x))$.

($rel\text{-}\Sigma^1_1(\mathsf{U})\text{-}\mathsf{AC}$): For all $rel\text{-}\Sigma^1_1(\mathsf{U})$ formulas $\varphi(x, X)$:
$(\forall x)(\exists X)\varphi(x, X) \rightarrow (\exists X)(\forall x)\varphi(x, (X)_x)$.

(4) *closure conditions for universes*

(4.1) For all $rel\text{-}\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, \vec{Z}]$:
$\mathsf{U}(D) \wedge \vec{Z} \,\dot{\in}\, D \rightarrow (\exists Y \,\dot{\in}\, D)(\forall x)(x \in Y \leftrightarrow \varphi[x, \vec{z}, \vec{Z}])$.

(4.2) For all $rel\text{-}\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, X, Y, \vec{Z}]$:
$\mathsf{U}(D) \wedge \vec{Z} \mathrel{\dot{\in}} D \to (\forall x)(\exists Y \mathrel{\dot{\in}} D)(\exists X \mathrel{\dot{\in}} D)\varphi[x, \vec{z}, X, Y, \vec{Z}]$
$\to (\exists Y \mathrel{\dot{\in}} D)(\forall x)(\exists X \mathrel{\dot{\in}} D)\varphi[x, \vec{z}, X, (Y)_x, \vec{Z}]$.

(5) *non-uniform limit axioms*

$(\exists D)(X \mathrel{\dot{\in}} D \wedge \mathsf{U}(D))$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U})$.

The theory MUT (Minimal Universes Theory) is also formulated in $\mathcal{L}_2(\mathsf{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. It is a strengthening of NUT. The non-logical axioms comprise of:

(1)-(4) same as for NUT.

(5) (5.1) *non-uniform limit axioms*

$(\exists D)(X \mathrel{\dot{\in}} D \wedge \mathsf{U}(D))$.

(5.2) *linearity*

$\mathsf{U}(D) \wedge \mathsf{U}(E) \to D \mathrel{\dot{\in}} E \vee D \doteq E \vee E \mathrel{\dot{\in}} D$.

(5.3) *minimal universe axioms*

For all $\varphi(X) \in rel\text{-}\Sigma^1_1(\mathsf{U})$ and for all $\psi(X) \in rel\text{-}\Pi^1_1(\mathsf{U})$:
$(\psi(X) \leftrightarrow \varphi(X)) \wedge (\exists D)(\varphi(D) \wedge \mathsf{U}(D))$
$\to (\exists D)[\varphi(D) \wedge \mathsf{U}(D) \wedge (\forall X \mathrel{\dot{\in}} D)(\mathsf{U}(X) \to \neg\varphi(X))]$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U})$.

Finally, we introduce a uniform variant of NUT, the theory UUT (Uniform Universes Theory). It is formulated in $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. The non-logical axioms comprise of:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) *equality axioms*

(2.1) $\mathsf{U}(S) \wedge S = R \to \mathsf{U}(R)$.

(2.2) $S = R \to (\mathcal{U}(S) = \mathcal{U}(R))$.

(3) *set operations*

(rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$): For all rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ formulas $\varphi(x)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x))$.

(rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})$-$\mathsf{AC}$): For all rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})$ formulas $\varphi(x, X)$:
$(\forall x)(\exists X)\varphi(x, X) \to (\exists X)(\forall x)\varphi(x, (X)_x)$.

(4) *closure conditions for universes*

(4.1) For all rel-$\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, \vec{P}]$ and all $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ set terms $\vec{S}, R$:
$\mathsf{U}(R) \wedge \vec{S} \,\dot{\in}\, R \to (\exists Z \,\dot{\in}\, R)(\forall x)(x \in Z \leftrightarrow \varphi[x, \vec{z}, \vec{S}])$.

(4.2) For all rel-$\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, X, Y, \vec{P}]$ and all $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ set terms $\vec{S}, R$:
$\mathsf{U}(R) \wedge \vec{S} \,\dot{\in}\, R \to (\forall x)(\exists Y \,\dot{\in}\, R)(\exists X \,\dot{\in}\, R)\varphi[x, \vec{z}, X, Y, \vec{S}]$
$\to (\exists Y \,\dot{\in}\, R)(\forall x)(\exists X \,\dot{\in}\, R)\varphi[x, \vec{z}, X, (Y)_x, \vec{S}]$.

(5) *uniform limit axioms*

$S \,\dot{\in}\, \mathcal{U}(S) \wedge \mathsf{U}(\mathcal{U}(S))$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U},\mathcal{U})$.

NUT_0, MUT_0 and UUT_0 are taken to be the theories NUT, MUT, UUT with set-induction

$$(0 \in S \wedge (\forall x)(x \in S \to x + 1 \in S)) \to (\forall x)(x \in S),$$

instead of full induction (6). We end this section with some remarks.

1. NUT_0 is included in UUT_0 and MUT_0:

   A trivial induction on the length of the derivation $\mathsf{NUT}_0 \vdash A$ shows

   $$\mathsf{NUT}_0 \vdash A \Longrightarrow \mathsf{UUT}_0 \vdash A \text{ and } \mathsf{MUT}_0 \vdash A.$$

   Therefore, NUT_0 is included in UUT_0 and MUT_0.

2. Closure conditions in UUT_0:

   Note that the closure conditions for universes in UUT_0 are formulated for rel-$\Pi^1_0(\mathsf{U})$ formulas and not for rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ formulas. If we took, for instance,

   For all rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ formulas $\varphi[x, \vec{z}, \vec{S}]$:
   $\mathsf{U}(R) \wedge \vec{S} \,\dot{\in}\, R \to (\exists Z \,\dot{\in}\, R)(\forall x)(x \in Z \to \varphi[x, \vec{z}, \vec{S}])$,

then the corresponding theory would be inconsistent. Since in this case we can set $\varphi := x \in \mathcal{U}(X)$ and then the axiom yields

$$\mathsf{U}(\mathcal{U}(X)) \wedge X \dot{\in} \mathcal{U}(X) \to (\exists Z \dot{\in} \mathcal{U}(X))(Z = \mathcal{U}(X)).$$

We conclude that $\mathcal{U}(X) \dot{\in} \mathcal{U}(X)$ holds. This contradicts lemma 1b)

3. Motivation of the axioms:

   Using the axioms scheme (1) "the working" in the theories is more convenient. (2) assures the compatibility of the introduced symbols $\mathsf{U}, \mathcal{U}$ with the extensional equality of the sets.

   With our theories we intend to describe countable coded ω-models of Σ^1_1-AC. It is natural to demand at least the same set principles for dealing with these models. Therefore we have requested the axioms scheme (3). The closure conditions of these models are listed in (4). We have closure under arithmetical comprehension (4.1) and closure under Σ^1_1-choice (4.2).

   In (5) the existence of universes is assured with a limit axiom. In MUT we can choose these universes minimal with respect to rel-$\Delta^1_1(\mathsf{U})$ formulas and the given notion of linearity. In UUT we can uniformly choose universes.

   It is very important to remark that we can introduce in our theories universes only by the limit axioms (and the minimal universe axioms). All these axioms are only existence axioms. In a certain sense the universes are given implicitly. We have not *defined* the universes, in this sense the universes are not given explicitly. It is typical for this situation, that we can demand some further conditions for the universes such as linearity, minimality, such that the theory is still predicative, metapredicative respectively.

4. Universes and countable coded ω-models of Σ^1_1-AC:

   We have to mention the following: If there is a set X such that $\mathsf{U}(X)$ holds, we can define for example

   $$Y := \{\langle x, 2k+1\rangle : \langle x, k\rangle \in X\}.$$

   We see immediately that Y is also a countable coded ω-model of Σ^1_1-AC, but we can not prove that Y is a universe. In this sense we use the notation "universe" only for sets X with $\mathsf{U}(X)$. On the other hand we use the notation "countable coded ω-model of Σ^1_1-AC" for sets which satisfy the closure conditions (4.1) and (4.2) of universes. Each universe is a countable coded ω-model of Σ^1_1-AC but not vice versa.

   In our theories there are much more countable coded ω-models of Σ^1_1-AC than universes. Since we can embed ATR_0 into these theories (cf. lemma 5) we can even construct in our theories countable coded ω-models of Σ^1_1-AC (cf. theorem 7), because these models are defined explicitly (and of course because ATR_0 is strong enough). But we can not prove that these so constructed models are universes. That is, we can choose for example in MUT_0 a minimal universe but not a minimal countable coded ω-model of Σ^1_1-AC.

5. What about a uniform variant of MUT?

   We can create a lot of further theories by mixing the stated axioms (and further non-stated axioms). For instance we can replace the non-uniform limit axiom in MUT by a uniform limit axiom for minimal universes and adapt the other axioms of MUT. Later on we show that the proof-theoretic strength does not change. But it is an open question whether the stated linearity axiom of MUT is strong enough to define in MUT a universe operator. In a later section we prove that by the (in a certain sense stronger) linearity axiom

$$\mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X = Y \vee Y \mathrel{\dot\in} X.$$

   we can define in MUT a universe operator. This universe operator will be a minimal universe operator.

6. Our theories of universes in comparison with theories of universes in other contexts:

   Our theories are built in a similar way as the theories of universes in explicit mathematics, or theories about admissibles without foundation in the framework of set theory (cf. for example KPi^0 [10]). There is always the same structure: some ontological axioms and ground structures (here (1) and (2)), some set operations (here (3)), axioms about the properties of universes (here (4)), then the introducing of universes with the aid of limit axioms (here (5)) and finally some kind of induction (here (6)). The purpose of our theories of universes is not to give another possibility to deal with universes, rather to show that we can build similar theories (as for example KPi^0) in second order arithmetic and that these theories have the same proof-theoretic strength.

   Notice that our universes correspond to admissibles *without* foundation. The reason is that the properties of our universes are not strong. We have only closure under arithmetical comprehension and under the Σ^1_1-choice axiom. But for example we can not prove that our universes are equivalent (with respect to $\doteq$) to sets of the form $\{X \subseteq \omega : X \text{ is hyperarithmetical in } Z\}$. (That is, we can not prove that our universes are least (with respect to $\dot\in$) countable coded ω-models of Σ^1_1-AC.)

7. Universes as countable coded ω-models of Σ^1_1-DC:

   Our universes satisfy the axiom of Σ^1_1-choice. Later on, e.g. in chapter 2, the axiom of dependent choice will be central. Assume that we have "$\mathsf{U}(X)$ implies that X is a countable coded ω-model of Σ^1_1-DC" instead of "$\mathsf{U}(X)$ implies that X is a countable coded ω-model of Σ^1_1-AC". Is the corresponding theory of such universes proof-theoretically stronger than the theory NUT (or UUT, MUT)? We do not give a proof but only mention that the proof-theoretic strength does not change. There is the following reason for this fact: In the sequel we use that in ATR_0 we can prove the existence of countable coded ω-models of Σ^1_1-AC (theorem VIII.4.20 [29]). But the same theorem states also that ATR_0 proves the existence of countable coded ω-models of Σ^1_1-DC. This fact leads to the proof-theoretic equivalence of the mentioned theories.

But notice that the situation is different if we add rel-$\Sigma^1_1(\mathsf{U})$-DC to these theories. Then e.g. the adapted theory NUT will be proof-theoretic stronger than the original NUT.

1.2 Properties of $\mathsf{NUT}_0, \mathsf{UUT}_0, \mathsf{MUT}_0$

The purpose of this section is to present ontological properties of our theories, especially the closure properties of our classes of formulas. We often use these properties in the following tacitly. First we collect two properties of universes in lemma 1. Assertion a) is a kind of transitivity and assertion b) says that "a universe can not speak about itself".

Lemma 1 *In* NUT_0, UUT_0 *and* MUT_0 *we have:*

a) $\mathsf{U}(T) \wedge R \dot{\in} S \wedge S \dot{\in} T \rightarrow R \dot{\in} T$.

b) $\mathsf{U}(T) \rightarrow T \dot{\notin} T$.

Proof. Here and in the sequel we work informally in the theories.

a) Choose T with $\mathsf{U}(T)$, sets R, S in T with $R \dot{\in} S$ and $S \dot{\in} T$ and k with $(\forall x)(x \in R \leftrightarrow \langle x, k\rangle \in S)$. Now let $\varphi(x, k, S)$ be the formula

$$\varphi(x, k, S) := \langle x, k\rangle \in S.$$

An application of arithmetical comprehension in the universe T gets a Z in T with

$$(\forall x)(x \in Z \leftrightarrow \varphi(x, k, S)).$$

The definition of φ yields $Z = R$. Hence $R \dot{\in} T$.

b) Let us assume $\mathsf{U}(T)$ and $T \dot{\in} T$. We show by a diagonalization argument that this leads to a contradiction. By $T \dot{\in} T$ and closure of the universe T under arithmetical comprehension there exists a set Z in T with

$$(\forall x)[x \in Z \leftrightarrow (Seq_2 x \wedge (T)_{(x)_1} \dot{\notin} (T)_{(x)_1} \wedge (x)_0 \in (T)_{(x)_1})].$$

First, we prove

$$(\forall X \dot{\in} T)[X \neq \emptyset \to (X \dot{\in} Z \leftrightarrow X \dot{\notin} X)]. \tag{1.1}$$

Choose X in T such that $X \neq \emptyset$ holds. We have to show $X \dot{\in} Z \leftrightarrow X \dot{\notin} X$.
$\to$: Since X is in Z there is an index l with $X = (Z)_l$. The definition of Z yields

$$(\forall x)[x \in X \leftrightarrow ((T)_l \dot{\notin} (T)_l \wedge x \in (T)_l)].$$

Since X is not empty we can choose an x in X and conclude $(T)_l \dot{\notin} (T)_l$. Then we have $(\forall x)(x \in X \leftrightarrow x \in (T)_l)$. This is just $X = (T)_l$ and therefore $X \dot{\notin} X$.
$\leftarrow$: We have $X \dot{\notin} X$. Furthermore we know $X \dot{\in} T$. Therefore we can choose an index l with $X = (T)_l$. Since we have $X \dot{\notin} X$ we conclude

$$(\forall x)[x \in X \leftrightarrow ((T)_l \dot{\notin} (T)_l \wedge x \in (T)_l)].$$

By definition of Z we immediately get $X = (Z)_l$ and therefore $X \dot{\in} Z$. Thus we have proved (1.1).

In a next step we show

$$Z \neq \emptyset. \tag{1.2}$$

We use the injectivity of the coding function $\langle \cdot, \cdot \rangle$:

$$\langle x, y \rangle = \langle u, v \rangle \to x = u \wedge y = v. \tag{1.3}$$

In order to prove $Z \neq \emptyset$ we first prove

$$\neg(\forall x)(\exists l)\langle x, l \rangle = x. \tag{1.4}$$

By contradiction we assume $(\forall x)(\exists l)\langle x, l \rangle = x$. Since $\langle \cdot, \cdot \rangle$ is injective we get

$$(\forall x)(\exists! l)\langle x, l \rangle = x. \tag{1.5}$$

Now let us choose x and a unique l with $\langle x, l \rangle = x$. Again an application of (1.5) yields a k with $\langle \langle x, l{+}1 \rangle, k \rangle = \langle x, l{+}1 \rangle$. But this contradicts (1.3), since $\langle x, l{+}1 \rangle \neq x$ (we have $l + 1 \neq l$ and by assumption l is the unique number with $\langle x, l \rangle = x$). This proves (1.4). Hence we have

$$(\exists x)(\forall l)\langle x, l \rangle \neq x.$$

Now we choose z such that $(\forall l)\langle z, l \rangle \neq z$. Then $\{z\} \dot{\notin} \{z\}$. Finally, we know $\{z\} \dot{\in} T$ and we conclude $\{z\} \dot{\in} Z$. This proves (1.2). Now (1.1) and (1.2) yield the desired contradiction $Z \dot{\in} Z \leftrightarrow Z \dot{\notin} Z$. □

We notice that the proof of lemma 1b) does not use the closure property (4.2). This means: For each countable coded ω-model T of ACA we have $T \dot{\notin} T$. In a next step we prove that in NUT_0 (resp. UUT_0) we have (rel-$\Delta^1_1(\mathsf{U})$-CA) (resp. (rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$)).

Lemma 2 *For all* $\varphi_1 \in$ *rel*-$\Sigma^1_1(\mathsf{U})$, $\varphi_2 \in$ *rel*-$\Pi^1_1(\mathsf{U})$, $\psi_1 \in$ *rel*-$\Sigma^1_1(\mathsf{U},\mathcal{U})$, $\psi_2 \in$ *rel*-$\Pi^1_1(\mathsf{U},\mathcal{U})$ *we have:*

a) NUT_0 *proves (rel-*$\Delta^1_1(\mathsf{U})$*-*CA*), i.e., for* $i \in \{1,2\}$ *we have*

$$\mathsf{NUT}_0 \vdash (\varphi_1(x) \leftrightarrow \varphi_2(x)) \rightarrow (\exists X)(\forall x)(x \in X \leftrightarrow \varphi_i(x)).$$

b) UUT_0 *proves (rel-*$\Delta^1_1(\mathsf{U},\mathcal{U})$*-*$\mathsf{CA}$*), i.e., for* $i \in \{1,2\}$ *we have*

$$\mathsf{UUT}_0 \vdash (\psi_1(x) \leftrightarrow \psi_2(x)) \rightarrow (\exists X)(\forall x)(x \in X \leftrightarrow \psi_i(x)).$$

Proof. The proof is an imitation of the proof of the statement "Π^1_0-CA and Σ^1_1-AC imply Δ^1_1-CA". We only show a). b) can be proved analogously. Let us choose formulas $\exists X\varphi(x,X) \in$ *rel*-$\Sigma^1_1(\mathsf{U})$ and $\forall X\psi(x,X) \in$ *rel*-$\Pi^1_1(\mathsf{U})$ with

$$(\forall x)(\exists X\varphi(x,X) \leftrightarrow \forall X\psi(x,X))$$

and define the formula $\theta(x,X)$ by

$$\theta(x,Z) := [(\exists X\varphi(x,X)) \wedge 1 \in Z] \vee [\neg(\forall X\psi(x,X)) \wedge 1 \notin Z].$$

We conclude

$$\begin{aligned}\theta(x,Z) &\leftrightarrow (\exists X)(\varphi(x,X) \wedge 1 \in Z) \vee (\exists Y)(\neg\psi(x,Y) \wedge 1 \notin Z)\\ &\leftrightarrow (\exists H)(\exists X \dot{\in} H)(\exists Y \dot{\in} H)[H = X \oplus Y \wedge\\ &\qquad\qquad ((\varphi(x,X) \wedge 1 \in Z) \vee (\neg\psi(x,Y) \wedge 1 \notin Z))].\end{aligned}$$

Therefore, θ is equivalent to a *rel*-$\Sigma^1_1(\mathsf{U})$ formula. Furthermore, we have $\forall x\exists Z\theta(x,Z)$. Now we apply (*rel*-$\Sigma^1_1(\mathsf{U})$-$\mathsf{AC}$) to the formula $\forall x\exists Z\theta(x,Z)$ and conclude $\exists Z\forall x\theta(x,(Z)_x)$. We fix such a Z. The set $G := \{x : 1 \in (Z)_x\}$ satisfies

$$(\forall x)(x \in G \leftrightarrow \exists X\varphi(x,X)),$$

which yields the claim. □

The next lemma assures that we have in NUT_0 and UUT_0 properties which correspond to the usual closure conditions of the class of Σ^1_1-formulas (resp. Π^1_1-formulas).

Lemma 3 *We have*

a) *For all* $\varphi,\psi \in$ *rel*-$\Sigma^1_1(\mathsf{U})$ *(resp. rel-*$\Pi^1_1(\mathsf{U})$*) there are rel-*$\Sigma^1_1(\mathsf{U})$ *(resp. rel-*$\Pi^1_1(\mathsf{U})$*) formulas* $\theta_1,\ldots,\theta_7$ *such that* NUT_0 *proves*

1.	$(\varphi \wedge \psi) \leftrightarrow \theta_1$	*(resp.* $(\varphi \wedge \psi) \leftrightarrow \theta_1$*)*,
2.	$(\varphi \vee \psi) \leftrightarrow \theta_2$	*(resp.* $(\varphi \vee \psi) \leftrightarrow \theta_2$*)*,
3.	$\exists x\varphi \leftrightarrow \theta_3$	*(resp.* $\exists x\varphi \leftrightarrow \theta_3$*)*,
4.	$\forall x\varphi \leftrightarrow \theta_4$	*(resp.* $\forall x\varphi \leftrightarrow \theta_4$*)*,
5.	$(\exists X \dot{\in} Y)\varphi \leftrightarrow \theta_5$	*(resp.* $(\exists X \dot{\in} Y)\varphi \leftrightarrow \theta_5$*)*,
6.	$(\forall X \dot{\in} Y)\varphi \leftrightarrow \theta_6$	*(resp.* $(\forall X \dot{\in} Y)\varphi \leftrightarrow \theta_6$*)*,
7.	$\exists X\varphi \leftrightarrow \theta_7$	*(resp.* $\forall X\varphi \leftrightarrow \theta_7$*)*.

b) *For all $\varphi, \psi \in rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$ (resp. $rel\text{-}\Pi^1_1(\mathsf{U},\mathcal{U})$) there are $rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$ (resp. $rel\text{-}\Pi^1_1(\mathsf{U},\mathcal{U})$) formulas $\theta_1, \ldots, \theta_7$ such that UUT_0 proves*

1.	$(\varphi \wedge \psi) \leftrightarrow \theta_1$	(*resp.* $(\varphi \wedge \psi) \leftrightarrow \theta_1$),
2.	$(\varphi \vee \psi) \leftrightarrow \theta_2$	(*resp.* $(\varphi \vee \psi) \leftrightarrow \theta_2$),
3.	$\exists x\varphi \leftrightarrow \theta_3$	(*resp.* $\exists x\varphi \leftrightarrow \theta_3$),
4.	$\forall x\varphi \leftrightarrow \theta_4$	(*resp.* $\forall x\varphi \leftrightarrow \theta_4$),
5.	$(\exists X \dot{\in} Y)\varphi \leftrightarrow \theta_5$	(*resp.* $(\exists X \dot{\in} Y)\varphi \leftrightarrow \theta_5$),
6.	$(\forall X \dot{\in} Y)\varphi \leftrightarrow \theta_6$	(*resp.* $(\forall X \dot{\in} Y)\varphi \leftrightarrow \theta_6$),
7.	$\exists X\varphi \leftrightarrow \theta_7$	(*resp.* $\forall X\varphi \leftrightarrow \theta_7$).

Proof. Here we prove a), the proof of b) is similar. First we discuss the closure conditions of $rel\text{-}\Sigma^1_1(\mathsf{U})$. We assume $\exists Z\varphi_1(Z), \exists Z\varphi_2(Z) \in rel\text{-}\Sigma^1_1(\mathsf{U})$ (each formula in $rel\text{-}\Sigma^1_1(\mathsf{U})$ is equivalent to a formula $\exists X\psi(X)$ with $\psi \in rel\text{-}\Pi^1_0(\mathsf{U})$) and distinguish the following cases:

$\varphi \equiv \exists Z\varphi_1(Z) \wedge \exists Z\varphi_2(Z)$: The equivalence

$$\varphi \leftrightarrow (\exists Z)[(\exists X \dot{\in} Z)\varphi_1(X) \wedge (\exists Y \dot{\in} Z)\varphi_2(Y)]$$

yields the claim.

$\varphi \equiv \exists Z\varphi_1(Z) \vee \exists Z\varphi_2(Z)$: The equivalence $\varphi \leftrightarrow (\exists Z)[\varphi_1(Z) \vee \varphi_2(Z)]$ yields the claim.

$\varphi \equiv (\forall x)(\exists Z)\varphi_1(x, Z)$: First we prove

$$(\forall x)(\exists X)(\exists Y \dot{\in} X)\varphi_1(x, Y) \leftrightarrow (\exists X)(\forall x)(\exists Y \dot{\in} X)\varphi_1(x, Y). \tag{1.6}$$

The implication "$\leftarrow$" is trivial. Therefore, assume that

$$(\forall x)(\exists X)(\exists Y \dot{\in} X)\varphi_1(x, Y).$$

An application of ($rel\text{-}\Sigma^1_1(\mathsf{U})\text{-}\mathsf{AC}$) yields $(\exists X)(\forall x)(\exists Y \dot{\in} (X)_x)\varphi_1(x, Y)$. Now we choose a set X with $(\forall x)(\exists Y \dot{\in} (X)_x)\varphi_1(x, Y)$. There is the possibility that several projections $((X)_x)_l$ of $(X)_x$ satisfy $\varphi_1(x, ((X)_x)_l)$. We choose that projection $((X)_x)_k$ which satisfies $\varphi(x, ((X)_x)_k)$ and which has the least index k of all the projections $((X)_x)_l$ which satisfy $\varphi(x, ((X)_x)_l)$. We put all these projections into the set H:

$$\begin{array}{rcl} H & := & \{\langle n, x\rangle : (\exists k)[(\exists Y \dot{\in} X)[Y = (X)_x \wedge \\ & & \quad (\exists Z \dot{\in} Y)(Z = ((X)_x)_k \wedge \varphi_1(x, Z) \wedge n \in Z)] \wedge \\ & & \quad (\forall l < k)\neg(\exists Y \dot{\in} X)[Y = (X)_x \wedge \\ & & \qquad (\exists Z \dot{\in} Y)(Z = ((X)_x)_l \wedge \varphi_1(x, Z))]]\}. \end{array}$$

Since we have $\varphi_1 \in rel\text{-}\Pi^1_0(\mathsf{U})$, also $H \in rel\text{-}\Pi^1_0(\mathsf{U})$. So H is in fact a set and we have $(\forall x)(\exists Z \dot{\in} H)\varphi_1(x, Z)$. This implies (1.6). Now we conclude

$$\begin{array}{rcl} \varphi & \leftrightarrow & (\forall x)(\exists Z)\varphi_1(x, Z) \\ & \leftrightarrow & (\forall x)(\exists X)(\exists Z \dot{\in} X)\varphi_1(x, Z) \\ & \leftrightarrow & (\exists X)(\forall x)(\exists Z \dot{\in} X)\varphi_1(x, Z). \end{array}$$

$\varphi \equiv (\exists x)(\exists Z)\varphi_1(x, Z)$: immediate.

$\varphi \equiv (\forall X \dot\in Y)(\exists Z)\varphi_1(X, Z)$:

$$\begin{array}{rcl} \varphi & \leftrightarrow & (\forall X \dot\in Y)(\exists Z)(\exists H)(H = X \land \varphi_1(H, Z)) \\ & \leftrightarrow & (\forall X \dot\in Y)(\exists G)[(\exists Z \dot\in G)(\exists H \dot\in G)(H = X \land \varphi_1(H, Z))] \\ & \leftrightarrow & (\forall x)(\exists G)[(\exists Z \dot\in G)(\exists H \dot\in G)(H = (Y)_x \land \varphi_1(H, Z))]. \end{array}$$

Notice that the formula in the bracket $[\ldots]$ is a rel-$\Pi^1_0(\mathsf{U})$ formula. Hence $(\exists G)[\ldots]$ is a rel-$\Sigma^1_1(\mathsf{U})$ formula. But we have already shown that the formula $\forall x \psi$ is equivalent to a rel-$\Sigma^1_1(\mathsf{U})$ formula for $\psi \in rel$-$\Sigma^1_1(\mathsf{U})$. Thus $(\forall x)(\exists G)[\ldots]$ is equivalent to a rel-$\Sigma^1_1(\mathsf{U})$ formula.

$\varphi \equiv (\exists X \dot\in Y)(\exists Z)\varphi_1(X, Z)$: immediate.

$\varphi \equiv (\exists X)(\exists Z)\varphi_1(X, Z)$: The equivalence

$$\varphi \leftrightarrow (\exists Y)(\exists X \dot\in Y)(\exists Z \dot\in Y)\varphi_1(X, Z)$$

yields the claim.

The closure properties of the class of formulas rel-$\Pi^1_1(\mathsf{U})$ can now be proved in a similar way. $\square$

In the analysis we know that for all Δ^1_1 formulas ψ and φ the formula $\psi(X)[X\backslash\{x : \varphi(x)\}]$ is again a Δ^1_1 formula. We show now the corresponding property in NUT_0 and UUT_0.

Lemma 4 *We have*

a) *For all $\varphi_1, \psi_1 \in rel$-$\Sigma^1_1(\mathsf{U})$, $\varphi_2, \psi_2 \in rel$-$\Pi^1_1(\mathsf{U})$ there are $\theta_1 \in rel$-$\Sigma^1_1(\mathsf{U})$ and $\theta_2 \in rel$-$\Pi^1_1(\mathsf{U})$ such that NUT_0 proves for $i, j \in \{1, 2\}$*

$$\begin{array}{l} (\varphi_1 \leftrightarrow \varphi_2) \land (\psi_1 \leftrightarrow \psi_2) \\ \quad \to [(\varphi_i(X)[X\backslash\{u : \psi_j(u)\}] \leftrightarrow \theta_1) \land (\varphi_i(X)[X\backslash\{u : \psi_j(u)\}] \leftrightarrow \theta_2)]. \end{array}$$

b) *For all $\varphi_1, \psi_1 \in rel$-$\Sigma^1_1(\mathsf{U},\mathcal{U})$, $\varphi_2, \psi_2 \in rel$-$\Pi^1_1(\mathsf{U},\mathcal{U})$ there are $\theta_1 \in rel$-$\Sigma^1_1(\mathsf{U},\mathcal{U})$ and $\theta_2 \in rel$-$\Pi^1_1(\mathsf{U},\mathcal{U})$ such that UUT_0 proves for $i, j \in \{1, 2\}$*

$$\begin{array}{l} (\varphi_1 \leftrightarrow \varphi_2) \land (\psi_1 \leftrightarrow \psi_2) \\ \quad \to [(\varphi_i(S)[S\backslash\{u : \psi_j(u)\}] \leftrightarrow \theta_1) \land (\varphi_i(S)[S\backslash\{u : \psi_j(u)\}] \leftrightarrow \theta_2)]. \end{array}$$

Proof. Here we prove b). The proof of a) is almost the same. Choose formulas $\varphi_1, \psi_1 \in rel$-$\Sigma^1_1(\mathsf{U})$ and $\varphi_2, \psi_2 \in rel$-$\Pi^1_1(\mathsf{U})$ such that $\varphi_1 \leftrightarrow \varphi_2$ and $\psi_1 \leftrightarrow \psi_2$ holds. Furthermore, we set $i = j = 1$, since the other cases can be proved analogously. The proof is by induction on the build-up of φ_1. We discuss only the following two cases, because the remaining cases follows from lemma 3b) and the induction hypothesis.

$\varphi_1 \equiv \mathsf{U}(S)$: The definition gives

$$\mathsf{U}(S)[S\backslash\{u : \psi_1(u)\}] \leftrightarrow (\exists X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \wedge \mathsf{U}(X)].$$

We have $\psi_1 \leftrightarrow \psi_2$ and an application of lemma 2b) yields a unique set Z such that $x \in Z \leftrightarrow \psi_1(x)$ holds. Therefore, we conclude

$$\begin{aligned}&(\exists X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \wedge \mathsf{U}(X)]\\ &\quad \leftrightarrow (\forall X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \rightarrow \mathsf{U}(X)],\end{aligned}$$

as claimed.

$\varphi_1 \equiv t \in \mathcal{U}(S)$: The definition gives

$$t \in \mathcal{U}(S)[S\backslash\{u : \psi_1(u)\}] \leftrightarrow (\exists X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \wedge t \in \mathcal{U}(X)].$$

Again, we have $\psi_1 \leftrightarrow \psi_2$, and an application of lemma 2b) yields a unique set Z such that $x \in Z \leftrightarrow \psi_1(x)$ holds. Therefore, we conclude

$$\begin{aligned}&(\exists X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \wedge t \in \mathcal{U}(X)]\\ &\quad \leftrightarrow (\forall X)[(\forall x)(x \in X \leftrightarrow \psi_1(x)) \rightarrow t \in \mathcal{U}(X)],\end{aligned}$$

as claimed. □

In the sequel we often use the notion of a rel-$\Delta^1_1(\mathsf{U})$ (resp. rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$) formula. The class of rel-$\Sigma^1_1(\mathsf{U})$ (resp. rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})$) formulas and the class of rel-$\Pi^1_1(\mathsf{U})$ (resp. rel-$\Pi^1_1(\mathsf{U},\mathcal{U})$) formulas are defined purely syntactically. The notion of a rel-$\Delta^1_1(\mathsf{U})$ (resp. rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$) formula, however, is defined with respect to a theory: A formula φ is a rel-$\Delta^1_1(\mathsf{U})$ (resp. rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$) formula if there is a formula $\psi \in rel$-$\Sigma^1_1(\mathsf{U})$ (resp. rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})$) and a formula $\theta \in rel$-$\Pi^1_1(\mathsf{U})$ (resp. rel-$\Pi^1_1(\mathsf{U},\mathcal{U})$) such that the equivalences $\varphi \leftrightarrow \psi$ and $\varphi \leftrightarrow \theta$ can be proved in the theory. Strictly spoken, we have for each of ours theories a notion of rel-$\Delta^1_1(\mathsf{U})$ (resp. rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$). In the sequel it will be clear from the context which theory we mean.

The lemmas 2, 3 and 4 show that for theories which contain NUT_0 (resp. UUT_0) we have formula comprehension, (usual) closure conditions and (usual) replacement properties for rel-$\Delta^1_1(\mathsf{U})$ (resp. rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$).

1.3 ATR and NUT

We show that there is an embedding of ATR into NUT and of NUT into ATR.

The embedding of ATR_0 into NUT_0 corresponds exactly to the embedding of ATR_0 into KPi^0 (cf. [10]). Therefore we omit the proof of the following lemma.

Lemma 5 *For each $\mathcal{L}_2$ formula φ we have*

a) $\mathsf{ATR}_0 \vdash \varphi \implies \mathsf{NUT}_0 \vdash \varphi$,

b) $\mathsf{ATR} \vdash \varphi \implies \mathsf{NUT} \vdash \varphi$.

Now we use results of Simpson [29] to embed NUT_0 into ATR_0. In [29] it is shown that ATR_0 proves the existence of countable coded ω-models of Σ^1_1-AC. Simpsons definition of countable coded ω-models makes use of the notion of valuation functions (cf. definition VII.2.1 in [29]). Whereas our countable coded ω-models are sets which reflect (and not satisfy) appropriate properties. In order to apply the results of Simpson we proceed as follows. First we give a finite axiomatization $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ of $(\Sigma^1_1$-$\mathsf{AC}) + (\mathsf{ACA})$. Then we investigate Simpsons proof which leads to lemma VIII.4.19 in [29]. This investigation shows that more or less the same proof leads to the proposition: "ATR_0 proves the existence of a set D with $X \dot{\in} D$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D$". Then we can translate the predicate $\mathsf{U}(D)$ as "D is a countable coded ω-model of Σ^1_1-AC" and the embedding goes through. We give now the exact formulation. For this we need universal relations. For each n and m let $\pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$ be a universal Π^0_1 formula (of $\mathcal{L}_2$). This means that for each Π^0_1 formula φ (of $\mathcal{L}_2$) there is an integer e such that

$$(\forall x_1, \ldots, x_n)(\forall X_1, \ldots, X_m)(\varphi[x_1, \ldots, x_n, X_1, \ldots, X_m] \\ \leftrightarrow \pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]).$$

We mention especially that there is also an index e with

$$(\forall x)(\mathsf{Q}(x) \leftrightarrow \pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]).$$

Now the finite axiomatization is given by the formula $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$:

$$\begin{aligned} Ax_{\Sigma^1_1\text{-}\mathsf{AC}} \;:=\; & (\forall X, Y)(\exists Z)(Z = X \oplus Y) \wedge \\ & (\forall e, z)(\forall Z)(\exists Y)(\forall x)(x \in Y \leftrightarrow \pi^0_{1,2,1}(e, x, z, Z)) \wedge \\ & [(\forall e, z)(\forall Z)[(\forall x)(\exists Y)\pi^0_{1,2,2}(e, x, z, Y, Z) \\ & \qquad \rightarrow (\exists Y)(\forall x)\pi^0_{1,2,2}(e, x, z, (Y)_x, Z)]]. \end{aligned}$$

Again we adopt the standard notation φ^D for the relativization of the $\mathcal{L}_2$ formula φ to D (for example $(\forall X \varphi(X))^D := (\forall X \dot{\in} D)\varphi^D(X)$). The following lemma assures that the formula $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ serves the right role. Its proof is standard and therefore omitted.

Lemma 6 *Let φ be an instance of $(\Sigma^1_1$-$\mathsf{AC}) + (\mathsf{ACA})$. Then ACA_0 proves:*

$$(\forall \vec{z})(\forall \vec{Z})((Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D \wedge \vec{Z} \dot{\in} D \rightarrow \varphi^D[\vec{z}, \vec{Z}]).$$

Now, Simpsons theorem VIII.3.15 [29] and more or less the same proof which leads to lemma VIII.4.19 [29] yields the following theorem.

Theorem 7 $\mathsf{ATR}_0 \vdash (\exists D)(X \dot{\in} D \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D)$.

This theorem is the crucial point to ensure that our embedding of NUT_0 into ATR_0 goes through. We introduce a translation. For every $\mathcal{L}_2(\mathsf{U})$ formula we write φ^{Ax} for the $\mathcal{L}_2$ formula φ which is obtained by replacing each instance $\mathsf{U}(X)$ in φ by $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$. Then we have the following embedding theorem.

Theorem 8 *For all $\mathcal{L}_2(\mathsf{U})$ formulas φ we have*

- **a)** $\mathsf{NUT}_0 \vdash \varphi \implies \mathsf{ATR}_0 \vdash \varphi^{Ax}$.
- **b)** $\mathsf{NUT} \vdash \varphi \implies \mathsf{ATR} \vdash \varphi^{Ax}$.

Proof. We show b) by induction on the length of derivation $\mathsf{NUT} \vdash \varphi$ (the proof of the assertion a) is identical). We consider only the mathematical axioms (1)-(6) of NUT, the logical rules and logical axioms are easily verified.

(1) These are also axioms of ATR.

(2) Trivial.

(3) We prove only (rel-$\Sigma^1_1(\mathsf{U})$-AC), the proof of (rel-$\Pi^1_0(\mathsf{U})$-CA) is similar. Let us assume $((\forall x)(\exists X)\varphi(x,X))^{Ax}$ and $\varphi \in rel\text{-}\Sigma^1_1(\mathsf{U})$. We have to show (within ATR) $(\exists X)(\forall x)\varphi^{Ax}(x,(X)_x)$. First we notice

$$\begin{aligned} ((\forall x)(\exists X)\varphi(x,X))^{Ax} &\leftrightarrow (\forall x)(\exists X)\varphi^{Ax}(x,X), \\ ((\exists X)(\forall x)\varphi(x,(X)_x))^{Ax} &\leftrightarrow (\exists X)(\forall x)\varphi^{Ax}(x,(X)_x). \end{aligned}$$

Since $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$ is an arithmetic formula, the formula φ^{Ax} is equivalent to a Σ^1_1 formula θ, and we have $(\forall x)(\exists X)\theta(x,X)$. But we have the ($\Sigma^1_1$-$\mathsf{AC}$) axioms scheme in ATR. Hence,

$$(\exists X)(\forall x)\theta(x,(X)_x) \qquad \text{and} \qquad (\exists X)(\forall x)\varphi^{Ax}(x,(X)_x).$$

(4) Immediate from lemma 6.

(5) We have to show $(\exists D)(X \dot{\in} D \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D)$. But this is just theorem 7.

(6) Also in ATR we have the full induction scheme. □

The following corollary states the proof-theoretic strength of NUT_0 and NUT. Lemma 5 is used for the proof-theoretic lower bound of NUT_0 and NUT. Theorem 8 is used for the proof-theoretic upper bound of NUT_0 and NUT. This yields $|\mathsf{NUT}_0| = |\mathsf{ATR}_0|$ and $|\mathsf{NUT}| = |\mathsf{ATR}|$. The proof-theoretic ordinals of ATR_0 and ATR are known (cf. for example [1, 16]).

Corollary 9 *We have*

a) $|\mathsf{NUT}_0| = \Gamma_0$.

b) $|\mathsf{NUT}| = \Gamma_{\varepsilon_0}$.

1.4 An embedding of UUT_0 into $\mathsf{MUT}_0^=$

In this section we show that in a strengthening of MUT_0 we can define unique universes by using an appropriate rel-$\Delta^1_1(\mathsf{U})$ formula. This yields an embedding of UUT_0 into this strengthen theory. We do not know whether an embedding of UUT into MUT is possible, since we do not know how to define unique minimal universes with respect to the linear ordering of universes in MUT. Therefore, we strengthen the linearity axiom in such a way that we are able to show the existence of minimal (unique) universes. Then we can define a universe operator and the embedding goes through.

First we describe the strengthening of MUT_0. We add to the theory MUT_0 the linearity axioms

$$(\mathrm{Lin}^=) \quad \mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X = Y \vee Y \mathrel{\dot\in} X.$$

The difference between $(\mathrm{Lin}^=)$ and (Lin) is only a little point " $\cdot$ ". (Lin) are the axioms

$$(\mathrm{Lin}) \quad \mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X \doteq Y \vee Y \mathrel{\dot\in} X.$$

Notice that $X = Y$ means that X and Y are the same sets in fact. On the other hand, $X \doteq Y$ only implies that X and Y have the same projections (but not necessarily the same elements). $X = Y$ implies $X \doteq Y$ but not vice versa.

$\mathsf{MUT}_0^=$ denotes the theory $\mathsf{MUT}_0 + (\mathrm{Lin}^=)$ and $\mathsf{MUT}^=$ denotes the theory $\mathsf{MUT} + (\mathrm{Lin}^=)$. Later on, we will show that $\mathsf{MUT}^=$ and MUT have the same proof-theoretic strength.

In the theory MUT the universes are stratified in the following sense: All minimal universes over the empty set contain the same projections and all these universes build the first, lowest stratum. If for example the universes A, B are in the first stratum, then they have the same projections ($A \doteq B$), but they can have different indices for the same projections (i.e., we may have $(A)_k \neq (B)_k$). Now choose a universe D in this first stratum. Then the next stratum contains all minimal universes over D. That this second stratum does not depend on the choice of D is stated in lemma 16. That is, each universe C in the first stratum is contained in each universe of the second stratum; and so on. In the stratification of $\mathsf{MUT}^=$ each stratum contains only one universe. It is an open question whether $\mathsf{NUT} + (\mathrm{Lin}^=)$ is proof-theoretic stronger than NUT.

The uniqueness in $\mathsf{MUT}^=$ of the universes in a stratum entails that the following abbreviation is in fact a rel-$\Delta^1_1(\mathsf{U})$ formula.

$$minU(x, X) := (\exists Z)[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y) \wedge x \in Z].$$

In $\mathsf{MUT}^=_0$ the meaning of the formula $minU(x, X)$ is: x is in the minimal (unique !) universe which contains X. The following lemma is the formalization of this idea.

Lemma 10 *The following are theorems of* $\mathsf{MUT}^=_0$*:*

a) $[\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y)(\mathsf{U}(Y) \wedge X \dot{\in} Y \rightarrow Y = D \vee D \dot{\in} Y)] \leftrightarrow$
$[\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y \dot{\in} D)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)]$,

b) $(\exists! Z)[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)]$,

c) $minU(x, X) \leftrightarrow$
$(\forall Z)[[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)] \rightarrow x \in Z]$.

Proof.

a) $\rightarrow$: We assume $\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y)(\mathsf{U}(Y) \wedge X \dot{\in} Y \rightarrow Y = D \vee D \dot{\in} Y)$. Choose Y in D and assume $\mathsf{U}(Y)$. By contradiction we assume $X \dot{\in} Y$. We conclude $Y = D \vee D \dot{\in} Y$. If we have $Y = D$, we immediately conclude $D \dot{\in} D$, a contradiction. If $D \dot{\in} Y$, we conclude from $D \dot{\in} Y$ and $Y \dot{\in} D$ and lemma 1a) that $D \dot{\in} D$, again a contradiction.
$\leftarrow$: We assume $\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y \dot{\in} D)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)$ and choose Y with $\mathsf{U}(Y)$ and $X \dot{\in} Y$. If $Y \dot{\in} D$, then $X \dot{\notin} Y$, a contradiction. Hence $Y \dot{\notin} D$. Because of the linearity of universes this yields $Y = D \vee D \dot{\in} Y$.

b) Choose H and G such that $X \dot{\in} H \wedge \mathsf{U}(H) \wedge (\forall Y \dot{\in} H)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)$ and $X \dot{\in} G \wedge \mathsf{U}(G) \wedge (\forall Y \dot{\in} G)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)$ holds. The linearity of the universes yields $H = G \vee H \dot{\in} G \vee G \dot{\in} H$.
$H \dot{\in} G$: Then $X \dot{\notin} H$, a contradiction.
$G \dot{\in} H$: Then $X \dot{\notin} G$, again a contradiction.
Therefore $H = G$. The existence of G is assured by the limit axiom and the minimal universe axiom.

c) Follows from b). $\square$

We now give an embedding of UUT into $\mathsf{MUT}^=$. The idea is to interpret $x \in \mathcal{U}(S)$ as "x is in the minimal universe which contains S". $\mathsf{U}(S)$ will be interpreted essentially as $\mathsf{U}(S)$ (more precisely: $\mathsf{U}(\mathrm{S})$ will be interpreted as $\mathsf{U}(\{x : (x \in S)^{min}\})$). We define for each $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula φ an $\mathcal{L}_2(\mathsf{U})$ formula φ^{min}. It is inductively defined. If φ is an $\mathcal{L}_2$ literal, then $\varphi^{min} := \varphi$. Otherwise we set

1. $$\begin{aligned}(x \in \mathcal{U}(S))^{min} &:= \\ (minU(x,S))^{min} &= (\exists Z)[(\exists k)(\forall z)[(z \in S)^{min} \leftrightarrow \langle z,k\rangle \in Z] \wedge \mathsf{U}(Z) \wedge x \in Z \wedge \\ &\qquad (\forall Y \mathrel{\dot\in} Z)[\mathsf{U}(Y) \to \neg(\exists k)(\forall z)[(z \in S)^{min} \leftrightarrow \langle z,k\rangle \in Y]]],\end{aligned}$$
2. $(x \notin \mathcal{U}(S))^{min} := \neg(x \in \mathcal{U}(S))^{min}$,
3. $(\mathsf{U}(S))^{min} := (\exists Z)[(\forall x)(x \in Z \leftrightarrow (x \in S)^{min}) \wedge \mathsf{U}(Z)]$,
4. $(\neg\mathsf{U}(S))^{min} := \neg(\mathsf{U}(S))^{min}$,
5. $(\varphi \circ \psi)^{min} := \varphi^{min} \circ \psi^{min} \qquad \circ \in \{\wedge, \vee\}$,
6. $(\mathrm{Q}x\varphi)^{min} := \mathrm{Q}x\varphi^{min} \qquad \mathrm{Q} \in \{\exists, \forall\}$,
7. $(\mathrm{Q}X\varphi)^{min} := \mathrm{Q}X\varphi^{min} \qquad \mathrm{Q} \in \{\exists, \forall\}$.

Theorem 11 *For all $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formulas φ we have:*

a) $\mathsf{UUT}_0 \vdash \varphi \implies \mathsf{MUT}_0^= \vdash \varphi^{min}$.

b) $\mathsf{UUT} \vdash \varphi \implies \mathsf{MUT}^= \vdash \varphi^{min}$.

Proof. We show a) by induction on the length of the derivation $\mathsf{UUT}_0 \vdash \varphi$ (an analogous argument shows b)). The logical rules and logical axioms are easily dealt with. Let us consider the mathematical axioms (1)-(6) of UUT_0.

(1) We have these axioms also in $\mathsf{MUT}_0^=$.

(2) An easy induction on the build-up of set terms implies the claim.

(3) If φ is a rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ formula, then we know from lemma 10c) and the closure properties of rel-$\Delta^1_1(\mathsf{U})$ formulas (cf. lemmas 3, 4 and the remarks after lemma 4) that φ^{min} is a rel-$\Delta^1_1(\mathsf{U})$ formula. But in $\mathsf{MUT}_0^=$ we have (rel-$\Delta^1_1(\mathsf{U})$-CA) (lemma 2). This immediately proves the translation of (rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$). For the proof of the translation of (rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})$-$\mathsf{AC}$) we notice that in $\mathsf{MUT}_0^=$ we have (rel-$\Sigma^1_1(\mathsf{U})$-AC) and that for $\varphi \in rel$-$\Sigma^1_1(\mathsf{U},\mathcal{U})$ the formula φ^{min} is equivalent to a rel-$\Sigma^1_1(\mathsf{U})$ formula (again lemmas 3, 4).

(4) First we prove by induction on the build-up of the $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ set term S

$$(\exists! Z)(\forall x)(x \in Z \leftrightarrow (x \in S)^{min}). \tag{1.7}$$

If S is a set variable, the claim is trivial. Therefore we assume that S has the form $\mathcal{U}(R)$. We have to show

$$(\exists! Z)(\forall x)(x \in Z \leftrightarrow (minU(x,R))^{min}).$$

By the induction hypothesis we can choose a unique set H with $(\forall x)(x \in H \leftrightarrow (x \in R)^{min})$. Then we have to show

$$(\exists! Z)(\forall x)(x \in Z \leftrightarrow minU(x,H)).$$

$minU(x,H)$ is a rel-$\Delta^1_1(\mathsf{U})$ formula and we therefore immediately get the claim (1.7).

We now show the translation of axiom (4.1). We choose a rel-$\Pi^1_0(\mathsf{U})$ formula $\varphi[x,\vec{z},\vec{P}]$, $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ set terms $\vec{S},R$ and have to prove

$$(\mathsf{U}(R))^{min} \wedge (\vec{S} \mathrel{\dot\in} R)^{min} \rightarrow (\exists Z \mathrel{\dot\in} R)^{min}(\forall x)(x \in Z \leftrightarrow \varphi[x,\vec{z},\vec{S}]^{min}). \tag{1.8}$$

Because of (1.7) there are unique sets D and $Y_1,\ldots,Y_n$ such that

$$(\forall x)(x \in D \leftrightarrow (x \in R)^{min}) \qquad \text{and} \qquad (\forall x)((x \in S_i)^{min} \leftrightarrow x \in Y_i).$$

Notice that $(\mathsf{U}(R))^{min}$ is the same as $\mathsf{U}(\{x : (x \in R)^{min}\})$ and that $(\mathsf{U}(S_i))^{min}$ is the same as $\mathsf{U}(\{x : (x \in S_i)^{min}\})$. We use this to transform (1.8) into

$$\mathsf{U}(D) \wedge \vec{Y} \mathrel{\dot\in} D \rightarrow (\exists Z \mathrel{\dot\in} D)(\forall x)(x \in Z \leftrightarrow \varphi^{min}[x,\vec{z},\vec{Y}]).$$

$\varphi^{min}[x,\vec{z},\vec{Y}]$ is obtained from $\varphi[x,\vec{z},\vec{S}]^{min}$ by replacing all subformulas $(x \in S_i)^{min}$ by $x \in Y_i$ and by replacing all subformulas $(\mathsf{U}(S_i))^{min}$ by $\mathsf{U}(Y_i)$. Hence $\varphi^{min}[x,\vec{z},\vec{Y}]$ is a rel-$\Pi^1_0(\mathsf{U})$ formula and the claim immediately follows from the closure of D under rel-$\Pi^1_0(\mathsf{U})$-CA.

Finally, we show the translation of axiom (4.2). We choose a rel-$\Pi^1_0(\mathsf{U})$ formula $\varphi[x,\vec{z},X,Y,\vec{P}]$, $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ set terms $\vec{S},R$ and have to prove

$$\begin{aligned}(\mathsf{U}(R))^{min} \wedge (\vec{S} \mathrel{\dot\in} R)^{min} \rightarrow (\forall x)(\exists Y \mathrel{\dot\in} R)^{min}(\exists X \mathrel{\dot\in} R)^{min}\varphi[x,\vec{z},X,Y,\vec{S}]^{min}\\ \rightarrow (\exists Y \mathrel{\dot\in} R)^{min}(\forall x)(\exists X \mathrel{\dot\in} R)^{min}\varphi[x,\vec{z},X,(Y)_x,\vec{S}]^{min}.\end{aligned}$$

Again we choose sets D and $Y_1,\ldots,Y_n$ with

$$(\forall x)(x \in D \leftrightarrow (x \in R)^{min}) \qquad \text{and} \qquad (\forall x)((x \in S_i)^{min} \leftrightarrow x \in Y_i).$$

The same arguments as in the case (4.1) imply that it is enough to prove

$$\begin{aligned}\mathsf{U}(D) \wedge \vec{Y} \mathrel{\dot\in} D \rightarrow (\forall x)(\exists Y \mathrel{\dot\in} D)(\exists X \mathrel{\dot\in} D)\varphi[x,\vec{z},X,Y,\vec{Y}]^{min}\\ \rightarrow (\exists Y \mathrel{\dot\in} D)(\forall x)(\exists X \mathrel{\dot\in} D)\varphi[x,\vec{z},X,(Y)_x,\vec{Y}].\end{aligned}$$

Again we can show $\varphi^{min} \in rel$-$\Pi^1_0(\mathsf{U})$. Hence the closure of D under rel-Σ^1_1-AC yields the claim.

(5) We have to prove $(S \dot{\in} \mathcal{U}(S) \wedge \mathsf{U}(\mathcal{U}(S)))^{min}$. The same arguments as in (4) produce unique sets X and D with

$$(\forall x)(x \in X \leftrightarrow (x \in S)^{min}) \qquad \text{and} \qquad (\forall x)(x \in D \leftrightarrow (x \in \mathcal{U}(R))^{min}).$$

It hence remains to show $X \dot{\in} D \wedge \mathsf{U}(D)$. This immediately follows from the definition of X and D.

(6) We have to show

$$0 \in S \wedge (\forall x)(x \in S \to x+1 \in S) \to (\forall x)(x \in S)$$

In (4) we have proved that there is a set Z with $x \in Z \leftrightarrow (x \in S)^{min}$. Since set induction is an axiom of $\mathsf{MUT}_0^=$ we get the claim. □

For the proof-theoretic strength we get the following corollary.

Corollary 12 *We have the following proof-theoretic reductions:*

a) $|\mathsf{UUT}_0| \leq |\mathsf{MUT}_0^=|$.

b) $|\mathsf{UUT}| \leq |\mathsf{MUT}^=|$.

1.5 Lower bounds of UUT and MUT

In this section we show that UUT and MUT prove transfinite induction for each initial segment of the ordinal $\varphi 1 \varepsilon_0 0$. We follow the presentation in [30]. (Here we give wellordering proofs although it is also possible to embed other theories, as for instance $\widehat{\mathrm{ID}}_{<\varepsilon_0}$.)

In the sequel we presuppose the same ordinal-theoretic facts as given in section 2 of [13]. Namely, we let Φ_0 denote the least ordinal greater than 0 which is closed under all n-ary φ functions, and we assume that a standard notation system of order type Φ_0 is given in a straightforward manner. We write $\prec$ for the corresponding primitive recursive wellordering. We assume without loss of generality that the field of $\prec$ is the set of all natural numbers and that 0 is the least element with respect to $\prec$. Hence, each natural number codes an ordinal less than Φ_0. When working in UUT or MUT in this section, we let $a, b, c, \ldots$ range over the field of $\prec$, and ℓ denotes limit notations. There exist primitive recursive functions acting on the codes of this notation system which corresponds to the usual operations on ordinals. In the sequel it is often convenient in order to simplify notation to use ordinals and ordinal operations instead of their codes and primitive recursive analogues. Then (for example) ω and $\omega + \omega$ stand for the natural numbers whose order type with respect to $\prec$ are ω and $\omega + \omega$. Finally, let us put as usual

$$\begin{aligned} Prog(\varphi) &:= (\forall a)[(\forall b \prec a)\varphi(b) \to \varphi(a)], \\ TI(\varphi, a) &:= Prog(\varphi) \to (\forall b \prec a)\varphi(b). \end{aligned}$$

If we want to stress the relevant induction variable of a formula φ, we sometimes write $Prog(\lambda a.\varphi(a))$ instead of $Prog(\varphi)$. If S is a set term, then $Prog(S)$ and $TI(S,a)$ have their obvious meanings.

1.5.1 Hierarchies of universes

It is our aim to derive $(\forall X)TI(X,\alpha)$ in UUT and in MUT for each ordinal α less than $\varphi 1 \varepsilon_0 0$. A crucial step towards this aim is the following: Given a set term S, we can build a transfinite hierarchy H of universes above a universe containing S along $\prec$. In UUT we choose $\mathcal{U}(S)$ and in MUT we choose a minimal universe as universe containing S.

Hierarchies of universes in UUT

We let $Hier(S,H,a)$ denote the formula which formalizes the property "H is a hierarchy of universes above S along $\prec$ up to a".

$$\begin{aligned} Hier(S,H,a) \quad := \quad & (\forall x)[x \in (H)_0 \leftrightarrow x \in \mathcal{U}(S)] \wedge \\ & (\forall b)[0 \prec b \preceq a \rightarrow (\forall x)(x \in (H)_b \leftrightarrow \mathcal{U}((H)_{\prec b}))]. \end{aligned}$$

We remember that $(H)_{\prec b}$ is the disjoint union of all $(H)_c$ with $c \prec b$. The uniqueness of such hierarchies is proved by transfinite induction up to ordinals α less than ε_0, which is available in UUT.

Lemma 13 *For all ordinals α less than ε_0 we have:*

$$\mathsf{UUT} \vdash (\forall a \prec \alpha)[Hier(S,H,a) \wedge Hier(S,G,a) \rightarrow (\forall b \prec a)((H)_b = (G)_b)].$$

We mention two ontological properties of such hierarchies of universes:

Lemma 14 *In UUT we have:*

- **a)** $Hier(S,H,a) \rightarrow (\forall b \preceq a)\mathsf{U}((H)_b)$,
- **b)** $Hier(S,H,a) \rightarrow (\forall b,c)(c \prec b \preceq a \rightarrow (H)_c \dot{\in} (H)_b)$.

Proof. Each step $(H)_b$ of the hierarchy H is of the form $\mathcal{U}(S)$ or $\mathcal{U}((H)_{\prec b})$ for $b \preceq a$ and for $Hier(S,H,a)$. But we know $\mathsf{U}(\mathcal{U}(S))$ for all set terms S. This yields a). In order to prove assertion b) we assume $Hier(S,H,a)$ and $c \prec b \preceq a$. We know $(H)_c \dot{\in} (H)_{\prec b}$, $(H)_{\prec b} \dot{\in} \mathcal{U}((H)_{\prec b})$ and $(H)_b = \mathcal{U}((H)_{\prec b})$. $(H)_b$ is a universe and lemma 1a) yields $(H)_c \dot{\in} (H)_b$. □

The next lemma states the existence of such hierarchies up to ordinals less than ε_0.

Lemma 15 *For all ordinals α less than ε_0 we have:*

$$\mathsf{UUT} \vdash (\forall a \prec \alpha)(\exists Y)Hier(S, Y, a).$$

Proof. Choose α less than ε_0 . We prove the assertion by induction up to α. For a with $a \prec \alpha$ we distinguish three cases:

$a = 0$: trivial.

$a + 1$: Using the induction hypothesis we can choose H with $Hier(S, H, a)$. We set

$$(Z)_c := \begin{cases} (H)_c & \text{if } c \prec a+1, \\ \mathcal{U}((H)_{\prec c}) & \text{if } c = a+1, \\ \emptyset & \text{if } c \succ a+1 \end{cases}$$

and conclude $Hier(S, Z, a + 1)$.

$a = \ell$: We know $(\forall c \prec a)(\exists Y)Hier(S, Y, c)$. An application of $(rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})\text{-}\mathsf{AC})$ yields a set Y with

$$(\forall c \prec a)Hier(S, (Y)_c, c).$$

Set $Z := \{\langle x, d\rangle : (\exists c)[d \prec c \prec a \wedge \langle x, d\rangle \in (Y)_c]\}$. Since we have uniqueness of such hierarchies, it can be proved

$$d \prec c \prec a \to ((Y)_c)_d = (Z)_d,$$

and therefore we conclude $(\forall c \prec a)Hier(S, Z, c)$. We set

$$(H)_c := \begin{cases} (Z)_c & \text{if } c \prec a, \\ \mathcal{U}((Z)_{\prec c}) & \text{if } c = a, \\ \emptyset & \text{if } c \succ a \end{cases}$$

and get $Hier(S, H, a)$. □

Hierarchies of universes in MUT

In MUT the proof of the existence of hierarchies of universes is more complex than in UUT. There are two reasons worth mentioning.

1. In UUT we can build unique hierarchies. This means that having $Hier(R, H, a)$ and $Hier(R, G, a)$ we have also $(H)_b = (G)_b$ for $b \prec a$ and $a \prec \alpha < \varepsilon_0$. It is an open question whether this is also possible in MUT. In MUT we use hierarchies which are almost unique, i.e., we can show for these hierarchies $(H)_b \doteq (G)_b$.

2. The first reason implies the second. Typically, one proves the existence of hierarchies with induction on the length a of hierarchies. In the case $a = \ell$ one needs a choice axiom in order to get an $(Y)_{\prec\ell}$ so that $(Y)_b$ is a hierarchy with length b for $b \prec \ell$. Here problems arise in MUT. Assume that we have $(Y)_{\prec\ell}$ and $(Z)_{\prec\ell}$ with $(Y)_b \doteq (Z)_b$ and $\mathsf{U}((Y)_b)$, $\mathsf{U}((Z)_b)$ for all b with $b \prec \ell$. Then choose minimal universes D, E with $(Y)_{\prec\ell} \dot\in D$ and $(Z)_{\prec\ell} \dot\in E$. The difficulty is now to show $D \doteq E$. We do not know how to prove this in MUT. Therefore we choose a minimal universe F with

$$(\forall b \prec \ell)(Y)_b \dot\in F$$

(instead of $(Y)_{\prec\ell} \dot\in D$). This is done with the minimal universe axiom (5.3) of MUT. Then we know that this F is almost unique. We also have $(\forall b \prec \ell)(Z)_b \dot\in F$.

We therefore define the hierarchies as follows:

$$\begin{aligned} H(X,Y,a) \quad := \quad & X \dot\in (Y)_0 \wedge \mathsf{U}((Y)_0) \wedge (\forall Z \dot\in (Y)_0)(\mathsf{U}(Z) \to X \dot\notin Z) \wedge \\ & (\forall b)[0 \prec b \preceq a \to [(\forall c \prec b)(Y)_c \dot\in (Y)_b \wedge \mathsf{U}((Y)_b) \wedge \\ & \qquad (\forall Z \dot\in (Y)_b)(\mathsf{U}(Z) \to (\exists c \prec b)(Y)_c \dot\notin Z)]]. \end{aligned}$$

The next lemma is crucial for the following proofs.

Lemma 16 MUT_0 *proves*

$$\mathsf{U}(X) \wedge \mathsf{U}(Y) \wedge \mathsf{U}(Z) \wedge X \doteq Y \wedge Y \dot\in Z \to X \dot\in Z.$$

Proof. Choose X, Y, Z with $X \doteq Y$, $Y \dot\in Z$, $\mathsf{U}(X)$, $\mathsf{U}(Y)$ and $\mathsf{U}(Z)$. By contradiction we assume $X \doteq Z$ or $Z \dot\in X$:
$X \doteq Z$: $Y \dot\in Z$ yields $Y \dot\in X$, and with $X \doteq Y$ we have $Y \dot\in Y$ (lemma 1a)), a contradiction.
$Z \dot\in X$: $X \doteq Y$ yields $Z \dot\in Y$, and with $Y \dot\in Z$ we have $Y \dot\in Y$, a contradiction.
The claim follows now from the linearity axioms. □

Lemma 17 MUT *proves for all ordinals α less than ε_0:*

a) $H(X,Y,a) \wedge a \prec \alpha \to (\forall b \prec a)(Y)_b \dot\in (Y)_a$,

b) $H(X,Y,a) \wedge a \prec \alpha \to (\forall b \preceq a)\mathsf{U}((Y)_b)$,

c) $H(X,Y,a) \wedge H(X,Z,a) \wedge a \prec \alpha \to (\forall b \preceq a)(Y)_b \doteq (Z)_b$.

Proof. We prove c). a) and b) are easily shown. We show c) by transfinite induction up to α. Let $a \prec \alpha$. We distinguish three cases:

$a = 0$: Assume $H(X,Y,a)$, $H(X,Z,a)$ and $a \prec \alpha$. We have to show that $(Y)_0 \doteq (Z)_0$. We know that $(Y)_0$ and $(Z)_0$ are minimal universes which contain X. By contradiction we assume $(Y)_0 \dot{\in} (Z)_0$. But then $H(X,Z,0)$ yields $X \dot{\notin} (Y)_0$ a contradiction. Analogously we conclude $(Z)_0 \dot{\notin} (Y)_0$, thus $(Y)_0 \doteq (Z)_0$.

$a+1$: Assume $H(X,Y,a+1)$, $H(X,Z,a+1)$ and $a+1 \prec \alpha$. The induction hypothesis yields $(\forall b \preceq a)(Y)_b \doteq (Z)_b$. We have to prove $(Y)_{a+1} \doteq (Z)_{a+1}$. We assume by contradiction $(Y)_{a+1} \dot{\in} (Z)_{a+1}$. Then $H(X,Z,a+1)$ gives a c with $c \prec a+1$ and $(Z)_c \dot{\notin} (Y)_{a+1}$. But this is a contradiction, since by lemma 16 and $(Z)_c \doteq (Y)_c$ we have $(Z)_c \dot{\in} (Y)_{a+1}$.

Analogously we conclude $(Z)_{a+1} \dot{\notin} (Y)_{a+1}$, thus $(Z)_{a+1} \doteq (Y)_{a+1}$.

$a = \ell$: Again we assume $H(X,Y,a)$, $H(X,Z,a)$ and $a \prec \alpha$. The induction hypothesis yields $(\forall b \prec a)(Z)_b \doteq (Y)_b$. From lemma 16 we conclude

$$(\forall b \prec a)(Z)_b \dot{\in} (Y)_a. \tag{1.9}$$

We assume by contradiction $(Y)_a \dot{\in} (Z)_a$. Then $H(X,Z,a)$ gives a c with $c \prec a$ and $(Z)_c \dot{\notin} (Y)_a$. A contradiction to (1.9). Analogously we conclude $(Z)_a \dot{\notin} (Y)_a$ and therefore $(Z)_a \doteq (Y)_a$. □

Now we can prove the existence of hierarchies of universes.

Lemma 18 *For all ordinals α less than ε_0 we have:*

$$\mathsf{MUT} \vdash (\forall a \prec \alpha)(\exists Y)H(X,Y,a).$$

Proof. We prove the claim by transfinite induction up to α. Let $a \prec \alpha$. We distinguish three cases:

$a = 0$: Using the limit axiom (5.1) we get a universe over X, and using the minimal universe axiom (5.3) we can choose this universe minimal. We take this minimal universe as $(Y)_0$.

$a+1$: The induction hypothesis yields a Z with $H(X,Z,a)$. Again by the limit axiom and the minimal universe axiom we can choose a minimal universe D with $(Z)_a \dot{\in} D$. We set

$$(Y)_c := \begin{cases} (Z)_c & \text{if } c \prec a+1, \\ D & \text{if } c = a+1, \\ \emptyset & \text{if } c \succ a+1. \end{cases}$$

We conclude $H(X,Y,a+1)$.

$a = \ell:$ The induction hypothesis yields $(\forall b \prec a)(\exists Y)H(X, Y, b)$. An application of (*rel*-$\Sigma^1_1(\mathsf{U})$-AC) yields a set Y with

$$(\forall b \prec a)H(X, (Y)_b, b).$$

We set $(Z)_b := ((Y)_{b+1})_b$ and show

$$(\forall b \prec a)H(X, Z, b) \tag{1.10}$$

by transfinite induction up to a:

$b = 0:$ The claim follows immediately.

$b \succ 0:$ We know $(\forall c \prec b)H(X, Z, c)$ and have to prove $H(X, Z, b)$. It suffices to show

$$(\forall c \prec b)(Z)_c \mathrel{\dot\in} (Z)_b \wedge \mathsf{U}((Z)_b) \wedge$$
$$(\forall H \mathrel{\dot\in} (Z)_b)(\mathsf{U}(H) \rightarrow (\exists c \prec b)(Z)_c \mathrel{\dot\notin} H).$$

$\mathsf{U}((Z)_b)$ is trivial. We next show $(\forall c \prec b)(Z)_c \mathrel{\dot\in} (Z)_b$. Choose c with $c \prec b$ and have to prove $(Z)_c \mathrel{\dot\in} (Z)_b$. We know $c \prec c{+}1 \prec b{+}1 \prec a$ and $H(X, (Y)_{c+1}, c{+}1)$ and $H(X, (Y)_{b+1}, b{+}1)$. Thus we also have $H(X, (Y)_{b+1}, c{+}1)$ and from lemma 17c) we conclude

$$((Y)_{c+1})_c \doteq ((Y)_{b+1})_c. \tag{1.11}$$

Therefore by lemma 16

$$((Y)_{c+1})_c \mathrel{\dot\in} ((Y)_{b+1})_b.$$

This is just $(Z)_c \mathrel{\dot\in} (Z)_b$. We next prove the minimality condition. Choose H in $(Z)_b$ with $\mathsf{U}(H)$. We have $b{+}1 \prec a$. This yields $H(X, (Y)_{b+1}, b{+}1)$, in particular $H(X, (Y)_{b+1}, b)$. Hence (we know $(Z)_b = ((Y)_{b+1})_b$ and $H \mathrel{\dot\in} ((Y)_{b+1})_b$)

$$(\exists c \prec b)((Y)_{b+1})_c \mathrel{\dot\notin} H.$$

Because of $((Y)_{b+1})_c \doteq (Z)_c$ (cf. (1.11)) we have $(Z)_c \mathrel{\dot\notin} H$ and therefore

$$(\exists c \prec b)(Z)_c \mathrel{\dot\notin} H,$$

as claimed. This finishes the proof of (1.10).

We set $\varphi(D) := (\forall c \prec a)(Z)_c \mathrel{\dot\in} D$ and apply the minimal universe axiom. This yields a minimal universe E with

$$\varphi(E) \wedge (\forall H \mathrel{\dot\in} E)(\mathsf{U}(H) \rightarrow \neg\varphi(H)).$$

We set

$$(G)_c := \begin{cases} (Z)_c & \text{if } c \prec a, \\ E & \text{if } c = a, \\ \emptyset & \text{if } c \succ a, \end{cases}$$

and show $H(X,G,a)$. Again it suffices to show

$$(\forall c \prec a)((G)_c \dot{\in} (G)_a) \wedge \mathsf{U}((G)_a) \wedge$$
$$(\forall H \dot{\in} (G)_a)(\mathsf{U}(H) \rightarrow (\exists c \prec a)(G)_c \dot{\notin} H).$$

$\mathsf{U}((G)_a)$ is trivial. We show $(\forall c \prec a)((G)_c \dot{\in} (G)_a)$. Choose c with $c \prec a$. We have to show $(G)_c \dot{\in} (G)_a$, that is $(Z)_c \dot{\in} E$. But this follows immediately from $\varphi(E)$. It remains to show the minimality condition. To this end, choose H in $(G)_a$ with $\mathsf{U}(H)$. i.e., $H \dot{\in} E$. Since E is a minimal universe with $\varphi(E)$ we know $(\exists c \prec a)(Z)_c \dot{\notin} H$. But this is the claim. □

1.5.2 Wellordering proofs for UUT and MUT

We follow the presentation in [30]. We sketch the wellordering proof only for UUT. It is evident that exactly the same arguments lead to a wellordering proof for MUT.

Crucial for carrying out the wellordering proof in UUT is the very natural notion $I^c_{R,H}(a)$ of *transfinite induction up to a for all sets belonging to a universe $(H)_b$ with $b \prec c$ (and $Hier(R,H,c)$)* which is given as follows:

$$I^c_{R,H}(a) := (\forall b \prec c)(\forall Y \dot{\in} (H)_b)TI(Y,a).$$

The next lemma tells us that $I^c_{R,H}$ can be represented by a set in $(H)_c$:

Lemma 19 *For each ordinal α less than ε_0 the following is a theorem of* UUT*:*

$$(\forall c \prec \alpha)[Hier(R,H,\alpha) \rightarrow (\exists Z \dot{\in} (H)_c)(\forall x)(x \in Z \leftrightarrow I^c_{R,H}(x))].$$

Proof. Assuming $c \prec \alpha$ and $Hier(R,H,\alpha)$ we know by definition

$$b \prec c \rightarrow ((H)_{\prec c})_b = (H)_b \qquad \text{and} \qquad (H)_{\prec c} \dot{\in} (H)_c.$$

Therefore $(\forall b \prec c)(\forall Y \dot{\in} (H)_b)TI(Y,a)$ is equivalent to a rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ formula with set parameter $H_{\prec c}$ in $(H)_c$. By closure of $(H)_c$ under rel-$\Pi^1_0(\mathsf{U},\mathcal{U})$ comprehension we conclude the existence of a set Z in $(H)_c$ with $Z = I^c_{R,H}$. □

In the next theorem we use the binary relation $\uparrow$

$$a \uparrow b := (\exists c,\ell)(b = c + a \cdot \ell)$$

and the abbreviation

$$\begin{aligned}&Main_\alpha(a) :=\\ &\quad (\forall X,Y)(\forall b,c)[c \preceq \alpha \wedge \omega^{1+a} \uparrow c \wedge Hier(X,Y,c) \wedge I^c_{X,Y}(b) \rightarrow I^c_{X,Y}(\varphi 1ab)].\end{aligned}$$

We omit the proof of the following theorem, because the statements correspond to analogous results in [30] and [13].

Theorem 20 *For each ordinal α less than ε_0 we can prove in* UUT*:*

a) $(\forall X, Y)(\forall \ell, \alpha)[\ell \prec a \wedge Hier(X, Y, \alpha) \wedge I^{\ell}_{X,Y}(a) \rightarrow I^{\ell}_{X,Y}(\varphi a0)]$,

b) $(\forall X, Y)(\forall \ell)[\ell \prec \alpha \wedge Hier(X, Y, \alpha) \rightarrow Prog(\lambda a.I^{\ell}_{X,Y}(\Gamma_a))]$,

c) $Prog(\lambda a.Main_{\alpha}(a))$.

And for each ordinal α less than $\varphi 1\varepsilon_0 0$ the following is a theorem of UUT*:*

$$(\forall X)TI(X, \alpha).$$

We collect the lower bounds in a corollary.

Corollary 21 *We have*

a) $\varphi 1\varepsilon_0 0 \leq |\mathsf{MUT}|$.

b) $\varphi 1\varepsilon_0 0 \leq |\mathsf{UUT}|$.

In chapter 4 we will show that these bounds are sharp.

1.6 A uniform fixed point theory

Jäger and Strahm [15] have proposed a uniform variant UFP of Avigads fixed point theory FP [1]. In this section we present an embedding of this uniform fixed point theory into UUT. First, we show that in UUT we can define a uniform fixed point operator and then give the embedding by interpreting $\mathcal{F}^{\mathcal{A}}$ by this fixed point operator.

1.6.1 The theory UFP

The theory UFP is formulated in the extension $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ of $\mathcal{L}_2$ in which we have a 2-ary operation symbol $\mathcal{F}^{\mathcal{A}}$ for each X positive $\mathcal{L}_2$ operator form $\mathcal{A}(X, Y, x, y)$. The *set terms* $R, S, T, \ldots$ of $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ are the set variables plus all expressions of the form $\mathcal{F}^{\mathcal{A}}(R, s)$ so that R is a set term and s a number term. The *formulas* of $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ are defined as usual. In the sequel we write $\Pi^1_0(\mathcal{F}^{\mathcal{A}})$ for the collection of the $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ formulas without bound set variables. The theory UFP is based on the usual axioms and rules for the two sorted predicate calculus. The non-logical axioms of UFP comprise:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) $(\Sigma^1_1(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{AC})$.

(3) *fixed point axioms*

(3.1) $S = R \to \mathcal{F}^{\mathcal{A}}(R, x) = \mathcal{F}^{\mathcal{A}}(S, x)$.

(3.2) For all X positive operator forms $\mathcal{A}[X, Y, x, y]$:
$(\forall Y)(\forall x, y)(x \in \mathcal{F}^{\mathcal{A}}(Y, y) \leftrightarrow \mathcal{A}[\mathcal{F}^{\mathcal{A}}(Y, y), Y, x, y])$.

(4) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$.

UFP_0 denotes the theory UFP with set induction instead of full formula induction. The uniform fixed point theory that Jäger and Strahm have proposed does not contain $(\Sigma^1_1(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{AC})$. We have added $(\Sigma^1_1(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{AC})$ for the following reason: In a uniform fixed point theory without $(\Sigma^1_1(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{AC})$ we have $(\Pi^1_0(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{CA})$ (from the fixed point axioms) and $(\Sigma^1_1\text{-}\mathsf{AC})$ (since we have ATR_0, cf. [1]). In this case it seems natural to us to allow the choice axioms also for $\Sigma^1_1(\mathcal{F}^{\mathcal{A}})$ formulas.

1.6.2 Embedding of UFP into UUT

We argue in UUT_0 and imitate Aczel's trick, cf. [6]. Let us choose a Π^0_1 universal formula $\pi^0_{1,3,2}[e, z, x, y, Z, X]$. Then En_S denotes the formula

$$En_S[e, z, x, y, Z] := (\exists X \,\dot{\in}\, S)\pi^0_{1,3,2}[e, z, x, y, Z, X].$$

Standard arguments prove the following lemma.

Lemma 22 *We have*

a) *For all arithmetic $\mathcal{L}_2$ formulas $\varphi[z, x, y, Z, X]$ there is a natural number e with*

$$\mathsf{UUT}_0 \vdash En_{\mathcal{U}(Z)}[e, z, x, y, Z] \leftrightarrow (\exists X \,\dot{\in}\, \mathcal{U}(Z))\varphi[z, x, y, Z, X].$$

b) *For all X positive operator forms $\mathcal{A}[X, Y, x, y]$ there is an e such that*

$$\mathsf{UUT}_0 \vdash En_{\mathcal{U}(Z)}[e, e, x, y, Z] \leftrightarrow \mathcal{A}[\hat{u}En_{\mathcal{U}(Z)}[e, e, u, y, Z], Z, x, y].$$

There are two crucial aspects. Firstly, $\{x : En_{\mathcal{U}(Z)}[e, e, x, y, Z]\}$ defines a set, and secondly, the construction of such a fixed point is uniform in the parameters. Now let us formulate the embedding of UFP_0 into UUT_0. We define for each $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ formula φ an $\mathcal{L}_2(\mathsf{U}, \mathcal{U})$ formula φ^U. We obtain φ^U by replacing in φ each $t \in \mathcal{F}^{\mathcal{A}}(R, s)$ by

$$(\exists Z)[(\forall x)(x \in Z \leftrightarrow (x \in R)^U) \wedge En_{\mathcal{U}(Z)}[e, e, t, s, Z]]$$

where we choose e such that it is the least number such that UUT_0 proves

$$En_{\mathcal{U}(Z)}[e, e, x, y, Z] \leftrightarrow \mathcal{A}[\hat{u}En_{\mathcal{U}(Z)}[e, e, u, y, Z], Z, x, y].$$

Note that $(t \in \mathcal{F}^{\mathcal{A}}(R, y,))^U$ is in fact a rel-$\Delta^1_1(\mathsf{U}, \mathcal{U})$ formula provable in UUT_0. In other words, in UUT_0 there is a set Z with

$$(\forall x)(x \in Z \leftrightarrow (x \in \mathcal{F}^{\mathcal{A}}(R, y))^U).$$

Now, the formulation of the embedding is no surprise.

Theorem 23 *For all $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ formulas φ we have*

$$\mathsf{UFP}_0 \vdash \varphi \Longrightarrow \mathsf{UUT}_0 \vdash \varphi^U.$$

Proof. By induction on the deduction length. Here we discuss only the fixed point axiom. The other axioms and rules are (also) easily dealt with. We have to show

$$(\forall Y)(\forall y, x)((x \in \mathcal{F}^{\mathcal{A}}(Y, y))^U \leftrightarrow \mathcal{A}[\mathcal{F}^{\mathcal{A}}(Y, y), Y, x, y]^U).$$

We know

$$(t \in \mathcal{F}^{\mathcal{A}}(Y, y))^U \leftrightarrow En_{\mathcal{U}(Y)}(e, e, t, y, Y).$$

The claim follows now from lemma 22b). □

Of course we have a corresponding embedding for the theories with full formula induction. Furthermore, since we have in UFP_0 also $(\Sigma^1_1(\mathcal{F}^{\mathcal{A}})\text{-}\mathsf{AC})$, we can show with similar arguments as for UUT and MUT that we have $\varphi 1\varepsilon_0 0 \leq |\mathsf{UFP}|$. Therefore we conclude from the results of chapter 4:

Corollary 24 *We have*

a) $|\mathsf{UFP}_0| = |\mathsf{UUT}_0| = \Gamma_0$.

b) $|\mathsf{UFP}| = |\mathsf{UUT}| = \varphi 1\varepsilon_0 0$.

1.7 Inconsistencies

In this section we prove that some kinds of theories of countable coded ω-models of Σ^1_1-AC and of theories of universes are inconsistent. All our inconsistency proofs are based on the fact that our theories state nothing about the indexes of the sets in a countable coded ω-model of Σ^1_1-AC or in a universe. We may thus encode "too much" information in these indexes, which leads to the inconsistency of these theories.

We have mentioned that ATR_0 proves the existence of countable coded ω-models of Σ^1_1-AC. In the next lemma we show that contrary to the universes of MUT_0 we can not linearly order these models. Again we let $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ denote our finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{AC}) + (\mathsf{ACA})$.

Lemma 25

$$\mathsf{ATR}_0 + (\forall X, Y)[(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y \to X \dot{\in} Y \vee X \doteq Y \vee Y \dot{\in} X]$$

is inconsistent.

Proof. Choose sets G and H with $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^G$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^H$ and $G \dot{\in} H$. We construct a set D, a further countable coded ω-model of Σ^1_1-AC, in such a way that $D \doteq G$ and the indexes of the set $\{0\}$ in D are of the form $2k+2$ with $k \in H$. We define

$$\begin{array}{lcl} B & := & \{\langle x, 3k+3\rangle : x \in (G)_k \wedge (\forall l)((G)_k = (G)_l \to k \le l)\}, \\ C & := & \{\langle x, y\rangle : \langle x, y\rangle \in B \wedge ((B)_y = \{0\} \to x = l)\}, \\ D & := & \{\langle x, 3k+3\rangle : \langle x, 3k+3\rangle \in C\} \cup \{\langle 0, 2k+2\rangle : k \in H\}, \\ E & := & \{x : \langle 0, 2x+2\rangle \in D\}. \end{array}$$

In B we take all non-empty sets in G and the least index of these sets. This leads to

$$G \doteq B \wedge (\forall X \dot{\in} B)(X \neq \emptyset \to (\exists! r) X = (B)_r).$$

C is B without the set $\{0\}$:

$$\{0\} \dot{\notin} C \wedge (\forall X)(X \neq \emptyset \to (X \dot{\in} C \leftrightarrow X \dot{\in} B)).$$

B contains the same sets as D. i.e., $D \doteq G$. And all indexes of the set $\{0\}$ in D are of the form $2k+2$ with $k \in H$. Therefore $E = H$. But $G \dot{\in} H$, $D \doteq G$ yield $D \dot{\in} H$ (cf. lemma 16). With arithmetical comprehension in H we conclude therefore $E \dot{\in} H$, and with $E = H$ we get $H \dot{\in} H$, a contradiction. $\Box$

We know (chapter 4) that MUT_0 is consistent. Therefore the theory $\mathsf{NUT}_0 + (\textit{linearity})$ is also consistent. But it is not consistent if we demand in addition that we have with $\mathsf{U}(X)$ and $X \doteq Y$ also $\mathsf{U}(Y)$. The proof of this inconsistency is similar to the proof of the lemma above and is therefore omitted.

Lemma 26

$$\begin{array}{lcl} \mathsf{NUT}_0 & + & (\forall X, Y)(\mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \dot{\in} Y \vee X \doteq Y \vee Y \doteq X) \\ & + & (\forall X, Y)(\mathsf{U}(X) \wedge X \doteq Y \to \mathsf{U}(Y)) \end{array}$$

is inconsistent.

In lemma 16 we have proved

$$\mathsf{U}(X) \wedge \mathsf{U}(Y) \wedge \mathsf{U}(Z) \wedge X \doteq Y \wedge Y \dot{\in} Z \to X \dot{\in} Z.$$

We do not have this property for arbitrary X, Y. It is crucial that X and Y are universes. Otherwise, there would be an inconsistency.

Lemma 27

$$\mathsf{NUT}_0 + (\forall X, Y, Z)(X \doteq Y \wedge Y \mathrel{\dot\in} Z \wedge \mathsf{U}(Z) \to X \mathrel{\dot\in} Z)$$

is inconsistent.

Proof. Choose a universe Z and set

$$\begin{array}{rcl} X & := & \{\langle 1, x\rangle : x \in Z\} \cup \{\langle 2, x\rangle : x \notin Z\}, \\ Y & := & \{\langle 1, 2k\rangle : k = k\} \cup \{\langle 2, 2k+1\rangle : k = k\}. \end{array}$$

We conclude $Y \mathrel{\dot\in} Z$, $X \doteq Y$ and $X \mathrel{\dot\notin} Z$ (since $Z = \{x : \langle 1, x\rangle \in X\}$). □

These results show that in our theory of universes it is crucial to control the importation of universes by the universe existence axioms. Only with these axioms we can "build" universes, since a set X is a universe only if we have $\mathsf{U}(X)$. If we consider theories without a universe predicate U but with the possibility of constructing sets X with $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$ (for instance ATR_0), we have to add further properties of these sets cautiously.

Chapter 2

Reflections and hierarchies

2.1 The theories $\mathsf{K}_n\mathsf{TR}$, FTR and $(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$

In this section we introduce theories equivalent to FTR. The theory FTR is introduced in [31]. In FTR there is a so-called Fixed point Transfinite Recursion principle, which demands the existence of fixed point hierarchies along arbitrary given wellorderings. We discuss two kinds of such theories. The theories $\mathsf{K}_n\mathsf{TR}$ claim the existence of hierarchies with certain properties along wellorderings. On the other side we define a theory $(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$ which assures the existence of countable coded ω-models of $\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}$. We have not found a better name for this theory. As a motivation for this name we remember that in the theory $\Pi^1_2\text{-}\mathsf{RFN}$ there is for each Π^1_2 sentence a model of ACA which reflects this Π^1_2 sentence. Analogously there is in $(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$ for each finite axiomatization of $(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})$ a model of ACA which reflects this axiomatization. Moreover, we collect in this section some elementary properties of these theories. First, we define the predicate K_n. Again we write $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ for our finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{AC})+(\mathsf{ACA})$.

$$\begin{array}{rcl}\mathsf{K}_0(M) & := & (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M,\\ \mathsf{K}_{n+1}(M) & := & (\forall X \mathrel{\dot\in} M)(\exists Y \mathrel{\dot\in} M)(X \mathrel{\dot\in} Y \wedge \mathsf{K}_n(Y)) \wedge (Ax_{\mathsf{ACA}})^M.\end{array}$$

We let Ax_{ACA} denote a finite axiomatization of (ACA). For a better understanding of the predicate K_n we mention the following lemma.

Lemma 28 *The following holds*

a) $\mathsf{ATR}_0 \vdash (\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$.

b) *For each instance φ of* (ATR)*:*

$$\mathsf{ACA}_0 \vdash (\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y) \rightarrow \varphi.$$

Proof.

a) That ATR_0 proves $(\forall X)(\exists Y)(X \dot{\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$ is stated in theorem 7.

b) Similar to the embedding of ATR_0 into NUT_0 (cf. lemma 5) it can be proved in ACA_0 for each arithmetic formula ψ

$$\begin{aligned}&(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D \wedge Z, \vec{X} \dot{\in} D \wedge WO(Z)\\ &\quad \rightarrow (\exists Y \dot{\in} D)(\forall a \in field(Z))(\forall x)(x \in (Y)_a \leftrightarrow \psi[x, a, \vec{z}, (Y)_{Za}, \vec{X}])\end{aligned}$$

by induction on the wellordering Z. □

We conclude from lemma 28 that each set M with $\mathsf{K}_1(M)$ is a countable coded ω-model of ATR. We define now two hierarchy predicates. The formula $FHier_{\mathcal{A}}(Z, Y)$ expresses that Y is a hierarchy of fixed points along Z with respect to the arithmetic, X positive formula $\mathcal{A}(X, Y, x, y)$. $\mathcal{A}$ may contain further free set and number variables.

$$\begin{aligned}&FHier_{\mathcal{A}}(Z, Y) :=\\ &\quad (\forall a \in field(Z))(\forall x)[x \in (Y)_a \leftrightarrow \mathcal{A}((Y)_a, (Y)_{Za}, x, a)].\end{aligned}$$

Corresponding to $FHier_{\mathcal{A}}(Z, Y)$ the formula $\mathsf{K}_n Hier(Z, X, Y)$ expresses that Y is a hierarchy along Z in such a way that all projections $(Y)_b$ with $b \prec a$ are also projections of $(Y)_a$ and that we have $\mathsf{K}_n((Y)_a)$ and $X \dot{\in} (Y)_a$ for all a in $field(Z)$.

$$\begin{aligned}&\mathsf{K}_n Hier(Z, X, Y) :=\\ &\quad (\forall a \in field(Z))[\mathsf{K_n}((Y)_a) \wedge (Y)_{Za} \dot{\in} (Y)_a \wedge X \dot{\in} (Y)_a].\end{aligned}$$

Finally let $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}$ denote a finite axiomatization of the axioms $(\mathsf{ATR}) + (\Sigma^1_1\text{-}\mathsf{DC}) + (\mathsf{ACA})$.

$$\begin{aligned}Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}} \;:=\;& (\forall X)(\exists Y)(X \dot{\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y) \wedge\\ & (\forall e, z)(\forall E, Z)(\exists X)(\forall x)(x \in X \leftrightarrow \pi^0_{1,2,2}(e, x, z, E, Z)) \wedge\\ & (\forall e, z)(\forall E, Z)[(\forall X)(\exists Y)\pi^0_{1,1,3}(e, z, X, Y, Z) \rightarrow\\ &\quad (\exists D)(\forall u)[(D)_0 = E \wedge \pi^0_{1,1,3}(e, z, (D)_u, (D)_{u+1}, Z)]].\end{aligned}$$

The formula $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}$ is not a direct axiomatization of $\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC}$. But it can be proved with lemma 28 and with similar arguments as in lemma 6 that ACA_0 proves for each instance φ of $(\mathsf{ATR}) + (\Sigma^1_1\text{-}\mathsf{DC}) + (\mathsf{ACA})$

$$(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M \rightarrow \varphi^M.$$

Hence, the notation $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}$ is justified.

The stage is now set in order to define the theories FTR, $\mathsf{K}_n\mathsf{TR}$ and $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$. These theories are formulated in $\mathcal{L}_2$. FTR extends ACA by the axioms scheme

(FTR) For all arithmetic, X positive formulas $\mathcal{A}$:
$WO(Z) \rightarrow (\exists Y) FHier_{\mathcal{A}}(Z, Y)$.

The theory $\mathsf{K}_n\mathsf{TR}$ extends ACA by the axioms

($\mathsf{K}_n\mathsf{TR}$) $WO(Z) \rightarrow (\exists Y)\mathsf{K}_n Hier(Z, X, Y)$.

The theory $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$ extends ACA by the axioms

($(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$) $(\exists M)[X \,\dot{\in}\, M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M]$.

FTR_0, $\mathsf{K}_n\mathsf{TR}_0$, $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}_0$ denote the corresponding theories with set-induction instead of formula induction. Note that $\mathsf{K}_0\mathsf{TR}$ claims the existence of hierarchies of countable coded ω-models of $\Sigma^1_1\text{-}\mathsf{AC}$ and that $\mathsf{K}_1\mathsf{TR}$ proves the existence of hierarchies of countable coded ω-models of ATR (cf. lemma 28). We collect some results, which we use in the sequel often tacitly, in a lemma.

Lemma 29 *We have*

- **a)** FTR_0 *proves each instance of* (ATR).
- **b)** $\mathsf{K}_n\mathsf{TR}_0$ *proves each instance of* (ATR).

Proof.

a) FTR_0 proves for each arithmetic, X positive formula $\mathcal{A}$

$$(\exists X)(\forall x)(x \in X \leftrightarrow \mathcal{A}(x, X)).$$

Now with the result of Avigad ([1]), that FP_0 and ATR_0 are equivalent, we can conclude (ATR).

b) $\mathsf{K}_n\mathsf{TR}_0$ proves $(\forall X)(\exists Y)(X \,\dot{\in}\, Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$. The claim follows now from lemma 28. □

2.2 Equivalence of $\mathsf{K}_n\mathsf{TR}$, FTR and $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$

In this section we prove the equivalence of $\mathsf{K}_n\mathsf{TR}$, FTR and $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}$. Then we can conclude that the proof-theoretic strength of all these theories is $\varphi 20\varepsilon_0$.

2.2.1 $(\mathsf{K_0TR})$ implies $((\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN})$ over ACA_0

The proof of this implication is essentially a generalization of the proof of lemma VIII.4.19 in [29]. It uses the method of pseudohierarchies. For unexplained notations we refer to [29].

Lemma 30 ACA_0 *proves*

$$
\begin{aligned}
&(\forall Z,G)[WO(Z) \to (\exists Y)\mathsf{K}_0 Hier(Z,G,Y)]\\
&\qquad \to (\exists M)[X \mathrel{\dot\in} M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M].
\end{aligned}
$$

Proof. Since later on we use the same proof idea, we prove the claim in detail. Choose a set X and assume $(\mathsf{K_0TR})$. First we show the existence of sets $Z, Y, M^\star$ with

$$
\begin{aligned}
&\neg WO(Z) \wedge Z, Y \mathrel{\dot\in} M^\star \wedge (WO(Z))^{M^\star} \wedge (Ax_{\mathsf{ACA}})^{M^\star} \wedge\\
&(\forall a \in field(Z))[(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_a} \wedge (Y)_{Za} \mathrel{\dot\in} (Y)_a \wedge X, Z \mathrel{\dot\in} (Y)_a]
\end{aligned}
\tag{2.1}
$$

This is the corresponding statement to lemma VIII.4.18 in [29]. Therefore, the proof is nearly the same: We know that hierarchies of countable coded ω-models of Σ^1_1-AC exist:

$$
WO(Z) \to (\exists Y)(\forall a \in field(Z))[(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_a} \wedge (Y)_{Za} \mathrel{\dot\in} (Y)_a \wedge X, Z \mathrel{\dot\in} (Y)_a].
$$

In particular we have $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$. Hence, we conclude from lemma 28 that we also have each instance of (ATR). And with (ATR) we can construct countable coded ω-models of ACA (cf. for example theorem VIII.1.13 in [29]). Therefore, we conclude that

$$
\begin{aligned}
WO(Z) \to (\exists Y, M)[&Z, Y \mathrel{\dot\in} M \wedge (WO(Z))^M \wedge (Ax_{\mathsf{ACA}})^M \wedge\\
&(\forall a \in field(Z))[(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_a} \wedge (Y)_{Za} \mathrel{\dot\in} (Y)_a \wedge X, Z \mathrel{\dot\in} (Y)_a]]
\end{aligned}
\tag{2.2}
$$

This equation has the form $WO(Z) \to \varphi(Z)$ with φ in Σ^1_1. But WO is a complete Π^1_1 predicate and we have (theorem V.1.9 in [29])

$$
\neg(\forall Z)(\varphi(Z) \leftrightarrow WO(Z)).
$$

This and (2.2) yields the sets $Z, Y, M^\star$ which satisfy (2.1). In the sequel we let $Z, Y, M^\star$ be sets with the properties (2.1). Since Z is not a wellordering, there exists a function $\mathcal{F}$ with

$$
(\forall n)(\langle \mathcal{F}(n+1), \mathcal{F}(n)\rangle \in Z \wedge \mathcal{F}(n+1) \neq \mathcal{F}(n)).
$$

Let I be a set which contains the elements beneath $\{\mathcal{F}(n) : n \in \omega\}$:

$$
I := \{c : (\forall n)(\langle c, \mathcal{F}(n)\rangle \in Z)\}.
$$

I has the following properties:

$$I \dot{\notin} M^\star, \tag{2.3}$$
$$I \neq \emptyset, \tag{2.4}$$
$$(\forall b \in I)(cZb \rightarrow c \in I), \tag{2.5}$$
$$(\forall b \in I)(\exists c \in I)(bZc \wedge b \neq c), \tag{2.6}$$

where we write cZb for $\langle c, b\rangle \in Z$. We prove these properties: Assume $I \dot{\in} M^\star$. Then $(field(Z)\backslash I) \dot{\in} M^\star$ too. Therefore, $(field(Z)\backslash I)$ is wellfounded, a contradiction, and we have $I \dot{\notin} M^\star$. It immediately follows $I \neq \emptyset$, then otherwise we would have $I \dot{\in} M^\star$. Property (2.5) follows from the definition of I. It remains to prove (2.6). We assume that there is a b in $field(Z)$ with $I = \{c : cZb\}$. Then we know $b +_Z 1 \notin I$ (we write again $b +_Z 1$ for the Z-successor of b) and there is an n with $\mathcal{F}(n)Z(b +_Z 1)$. But this gives $\mathcal{F}(n+1) \in I$, a contradiction. Thus

$$(\forall b \in field(Z))(I \neq \{c : cZb\}).$$

Now we assume again by contradiction that there is a b in I with

$$(\forall c \in I)cZb.$$

Then we conclude $I = \{c : cZb\}$. This is the desired contradiction.

The stage is set in order to define our model M of ATR and Σ^1_1-DC. It consists of all sets which are recursive in $(Y)_b$ for a b in I:

$$M := \{\langle m, \langle e, b\rangle\rangle : b \in I \wedge (\forall x)(\exists y)(\{e\}^{(Y)_b}(x) = y) \wedge \{e\}^{(Y)_b}(m) = 0\}.$$

We have to verify that M is the desired model. We notice that X is in M. First, we prove $(Ax_{\mathsf{ACA}})^M$. We choose $\vec{Z} \dot{\in} M$ and an arithmetic formula $\varphi[x, \vec{z}, \vec{Z}]$ and have to show

$$(\exists X \dot{\in} M)(\forall x)(x \in X \leftrightarrow \varphi[x, \vec{z}, \vec{Z}]).$$

There exists a $b \in I$ such that for all Z_i there exists an e_i with $Z_i = (M)_{\langle e_i, b\rangle}$. Furthermore, we know $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_{b+_Z 1}}$. Since $(Y)_b$ is in $(Y)_{b+_Z 1}$, all sets Z_i are in $(Y)_{b+_Z 1}$ too. Hence, there is a set X in $(Y)_{b+_Z 1}$ with

$$(\forall x)(x \in X \leftrightarrow \varphi[x, \vec{z}, \vec{Z}]).$$

Since X is in $(Y)_{b+_Z 1}$, there exists a g with

$$(\forall x)(x \in X \leftrightarrow \{g\}^{(Y)_{b+_Z 1}}(x) = 0) \wedge (\forall x)(\exists y)\{g\}^{(Y)_{b+_Z 1}}(x) = y.$$

I has no upper bound in I (property 2.6), hence $b +_Z 1 \in I$ and $X \dot{\in} M$. Now we prove $(Ax_{\mathsf{ATR}})^M$. We choose a C in M and e, b with $C = (M)_{\langle e, b\rangle}$. We have to show that there is a D in M with $C \dot{\in} D$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D$. Since we know

$$m \in C \leftrightarrow \{e\}^{(Y)_b}(m) = 0,$$

C is arithmetic in $(Y)_b$. Furthermore, we know $(Y)_b \mathrel{\dot\in} (Y)_{b+_Z 1}$ and $(Ax_{\Sigma^1_1\text{-AC}})^{(Y)_{b+_Z 1}}$. Hence $C \mathrel{\dot\in} (Y)_{b+_Z 1}$. We set $D := (Y)_{b+_Z 1}$ and conclude that $C \mathrel{\dot\in} D$ and $(Ax_{\Sigma^1_1\text{-AC}})^D$. Since I has no upper bound in I (property (2.6)) we know $b +_Z 1 \in I$ and we can find a g with

$$(\forall n)(n \in D \leftrightarrow \{g\}^{(Y)_{b+_Z 1}}(n) = 0) \wedge (\forall x)(\exists y)\{g\}^{(Y)_{b+_Z 1}}(x) = y.$$

This is $D = (M)_{\langle g, b+_Z 1\rangle}$. It remains finally to show $(Ax_{\Sigma^1_1\text{-DC}})^M$. We follow the proof in [29]. Let H denote the set of all sets recursive in $(Y)_b$ for some b in Z: $(H \mathrel{\dot\in} M^\star)$

$$H := \{\langle m, \langle e, b\rangle\rangle : b \in field(Z) \wedge (\forall x)(\exists y)(\{e\}^{(Y)_b}(x) = y) \wedge \{e\}^{(Y)_b}(m) = 0\}.$$

We assume

$$(\forall U \mathrel{\dot\in} M)(\exists V \mathrel{\dot\in} M)\varphi[\vec{z}, U, V, \vec{P}] \wedge \vec{P} \mathrel{\dot\in} M.$$

Furthermore we can assume $\varphi \in \Pi^1_0$. Choose an U in M. We have to show that there is a V in M with

$$(V)_0 = U \wedge (\forall u)\varphi[\vec{z}, (V)_u, (V)_{u+1}, \vec{P}]. \tag{2.7}$$

We can choose e_0, b_0 with b_0 in I and $U = (M)_{\langle e_0, b_0\rangle}$ and such that for all parameters P_i there is a g with $P_i = (M)_{\langle g, b_0\rangle}$. Reasoning within $M^\star$, we choose a sequence p as follows. We give first an informal description. We set $(p)_0 = \langle e_0, b_0\rangle$. Given $(p)_{n-1}$ let b be the Z-least c such that $\langle((p)_{n-1})_1, c\rangle \in Z$ and such that there is an e with $\varphi[x, z, (H)_{(p)_{n-1}}, (H)_{\langle e,c\rangle}, \vec{P}]$. Here we need $(WO(Z))^{M^*}$ in order to choose c as the Z-least c. And since we have

$$(\forall U \mathrel{\dot\in} M)(\exists V \mathrel{\dot\in} M)\varphi[\vec{z}, U, V, P]$$

and property (2.5) we know $c \in I$ if $((p)_{n-1})_1 \in I$. Then pick the $<$-least e with this property and set $(p)_n := \langle e, b\rangle$. The formula $\psi(n, p)$ formalizes this process.

$$\begin{aligned}
&\psi(n, p) := \\
&\quad Seq_{n+1}(p) \wedge ([n = 0 \wedge (p)_0 = \langle e_0, b_0\rangle] \vee \\
&\quad [n > 0 \wedge \\
&\quad (\exists e, b)[(p)_n = \langle e, b\rangle \wedge b \in field(Z) \wedge \\
&\qquad (\forall k < n)(\langle((p)_k)_1, b\rangle \in Z) \wedge \\
&\qquad \varphi[\vec{z}, (H)_{(p)_{n-1}}, (H)_{\langle e,b\rangle}, \vec{P}] \wedge \\
&\qquad (\forall f)(\varphi[\vec{z}, (H)_{(p)_{n-1}}, (H)_{\langle f,b\rangle}, \vec{P}] \rightarrow e \le f) \wedge \\
&\qquad (\forall c \in field(Z))([(\exists f)\varphi[\vec{z}, (H)_{(p)_{n-1}}, (H)_{\langle f,c\rangle}, \vec{P}] \wedge \\
&\qquad\quad (\forall k < n)(\langle((p)_k)_1, c\rangle \in Z)] \rightarrow bZc)]]).
\end{aligned}$$

By induction on n we can prove

$$(\forall n)(\exists! p)\psi(n, p) \wedge [\psi(n, t) \rightarrow (\forall k \le n)((t)_k)_1 \in I].$$

We set

$$J := \{c : c \in field(Z) \wedge (\exists n,k)(\psi(n,k) \wedge \langle c, ((k)_n)_1\rangle \in Z)\}.$$

J is arithmetic with parameters $Z, H, \vec{P}$. Hence J is in $M^\star$ and $field(Z)\backslash J$ is in $M^\star$ too. Thus we can choose a Z-least $b^\star$ with $b^\star$ in $field(Z)\backslash J$. Since $J \subseteq I$ and $J \neq I$ ($I \dot{\notin} M^\star$ and $J \dot{\in} M^\star$) we have $b^\star \in I$. J, I are closed downwards, thus $J = \{c : cZb^\star \wedge c \neq b^\star\}$. We set

$$H_{b^\star} := \{\langle m, \langle e, b\rangle\rangle : bZb^\star \wedge (\forall x)(\exists y)(\{e\}^{(Y)_b}(x) = y) \wedge \{e\}^{(Y)_b}(m) = 0\}.$$

and can express $\psi(n,p)$ as a formula arithmetic in $Z, H_{b^\star}, \vec{P}$. Since all these parameters are in M, also the set $\{\langle n,k\rangle : \psi(n,k)\}$ is in M and we see that

$$V := \{\langle m,n\rangle : (\exists k)(\psi(n,k) \wedge \{((k)_n)_0\}^{((Y)_{Zb^\star})_{((k)_n)_1}}(m) = 0)\}$$

is in M. This V satisfies (2.7). □

2.2.2 $((\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN})$ implies (FTR) over ACA_0

We first sketch the proof idea. We have to construct a fixed point hierarchy for a given arithmetic, X positive formula $\mathcal{A}$ and a given wellordering Z. We choose a model M with Z and all set parameters $\vec{Q}$ of $\mathcal{A}$ in M and with $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M$. Now we would like to prove by induction on the wellordering Z

$$(\forall b \in field(Z))(\exists Y \dot{\in} M)(\forall c,x)[\langle c,b\rangle \in Z \rightarrow$$
$$(x \in (Y)_c \leftrightarrow \mathcal{A}((Y)_c, (Y)_{Zc}, x, c))].$$

There is no problem for $b = 0_Z$ and $b = a +_Z 1$. But when b is a limit number there are difficulties, since the hierarchies do not have to be unique. Therefore, we choose another way. We will build fixed point hierarchies similar to the construction of fixed point hierarchies along the wellordering $\prec\restriction \alpha$ ($\alpha < \varepsilon_0$) in $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ in [16]. i.e., we would like to prove by induction on the wellordering Z a statement like: "If we have built in a model M of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ a fixed point hierarchy along Z up to a we can extend in M this hierarchy up to $a +_Z b$" (cf. lemma 2 and 3 in [16]). Our proof of this statement is essentially an adaption of lemma 2 and lemma 3 in [16] to our situation. All what we have to do is to show that we can define in M a "+-operation" on the wellordering Z and that we can define in M "fundamental sequences" for limit numbers of Z. Then we can imitate the proof of lemma 2 and lemma 3 in our model M of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$.

We first show the existence of "fundamental sequences". Sometimes we work extremely informally in the sequel. For example, we often simply write "$\mathcal{F}$" or "$\mathcal{G}$" for "F is a function" or "G is a function" and use the notation "$\mathcal{F}(n)$" for the unique m with $\langle n,m\rangle \in F$. We prove the existence of fundamental sequences in the theory ATR_0. Of course we do not need the full strength of ATR_0 for this proof. We take ATR_0, since we have ATR_0 in the sequel.

Lemma 31 *There is an arithmetic formula* $\varphi(c,n,X,Z)$ *such that* ATR_0 *proves*

$$\begin{aligned}&WO(Z) \wedge (\forall b \in field(Z))(\exists c)(bZc \wedge b \neq c)\\ &\to (\exists! \mathcal{F})[(\forall c \in field(Z))(\forall n)(\varphi(c,n,\mathcal{F}_{<n},Z) \leftrightarrow \mathcal{F}(n) = c) \wedge\\ &\qquad\qquad (\forall n)(\mathcal{F}(n) \in field(Z) \wedge \langle \mathcal{F}(n), \mathcal{F}(n+1)\rangle \in Z) \wedge \mathcal{F}(n) \neq \mathcal{F}(n+1) \wedge\\ &\qquad\qquad (\forall b \in field(Z))(\exists n)\langle b, \mathcal{F}(n)\rangle \in Z].\end{aligned}$$

Proof. Choose a wellordering Z. We construct - reasoning in ATR_0 - the function $\mathcal{F}$ as follows: We distinguish the two cases:

1. $n \in field(Z)$: $\mathcal{F}(n)$ is the Z-least c, which is Z-greater than n and Z-greater than all $\mathcal{F}(k)$ with $k < n$.
2. $n \notin field(Z)$: $\mathcal{F}(n)$ is the Z-least c, which is Z-greater than all $\mathcal{F}(k)$ with $k < n$.

The following formula $\varphi(c,n,X,Z)$ is the desired formula and formalizes this construction:

$$\begin{aligned}&\varphi(c,n,X,Z) :=\\ &\quad [n \in field(Z) \wedge nZc \wedge n \neq c \wedge\\ &\quad (\forall k < n)(\forall y)(y \in (X)_k \to (yZc \wedge y \neq c)) \wedge\\ &\quad (\forall b)[(nZb \wedge n \neq b \wedge (\forall k < n)(\forall y)(y \in (X)_k \to (yZb \wedge y \neq b))) \to cZb]] \vee\\ &\quad [n \notin field(Z) \wedge\\ &\quad (\forall k < n)(\forall y)(y \in (X)_k \to (yZc \wedge y \neq c)) \wedge\\ &\quad (\forall b)[(\forall k < n)(\forall y)(y \in (X)_k \to (yZb \wedge y \neq b)) \to cZb]]\end{aligned}$$

From (ATR) [1] we conclude $(\exists Y)(\forall n,c)(\langle c,n\rangle \in Y \leftrightarrow \varphi(c,n,(Y)_{<n}))$. Choose such an Y and define $\mathcal{F}$ as

$$\mathcal{F}(n) = c :\leftrightarrow c \in (Y)_n.$$

Then we conclude

$$(\forall n)(\langle \mathcal{F}(n), \mathcal{F}(n+1)\rangle \in Z \wedge \mathcal{F}(n) \neq \mathcal{F}(n+1))$$

and

$$c \in field(Z) \to cZ\mathcal{F}(c).$$

This is exactly the claim, since it can be proved by induction on c that $\mathcal{F}$ is unique. □

We are going now to define a $+_Z$-operation on a wellordering Z. In general it is not possible to get a total operation "$+_Z$" on an arbitrary wellordering Z. Therefore, we

[1] As mentioned, we do not need here the full strength of (ATR); a weaker axioms scheme would serve the right role too.

first define a wellordering ω^Z which is closed under a canonical addition. Furthermore, we can show that Z is isomorphic to an initial section of ω^Z (notice that ATR_0 implies comparability of countable wellorderings, cf. [29] lemma V.2.9).

We define $field(\omega^Z)$ as the set of all sequence numbers p with $(p)_n$ in $field(Z)$ and with $\langle (p)_n, (p)_{n-1}\rangle \in Z$. This definition is in analogy to the codes of ordinals in cartan normalform (cf. for example [23] section 7 and 8). The wellordering $\prec_{\omega^Z}$ on $field(\omega^Z)$ corresponds to the wellordering on these codes:

$$\begin{aligned}
&p \in field(\omega^Z) := \\
&\qquad Seq(p) \wedge (\forall n < lh(p))((p)_n \in field(Z) \wedge (n > 0 \to \langle (p)_n, (p)_{n-1}\rangle \in Z), \\
&p \prec_{\omega^Z} q := \\
&\qquad [lh(p) < lh(q) \wedge (\forall n < lh(p))(p)_n = (q)_n] \vee \\
&\qquad (\exists n < min[lh(p), lh(q)])[\langle (p)_n, (q)_n\rangle \in Z \wedge (\forall k < n)(p)_k = (q)_k].
\end{aligned}$$

Again we refer to [29] for unexplained notations. The following lemma corresponds to a part of theorem 5.4.1 in [8]. The proof uses standard arguments and we omit it.

Lemma 32 ACA_0 *proves*

$$WO(Z) \to WO(\omega^Z).$$

In the next lemma we establish the existence of a total $+_{\omega^Z}$-operation on the wellordering ω^Z.

Lemma 33 ACA_0 *proves:*

There is a unique operation $+_{\omega^Z}$ on $field(\omega^Z) \times field(\omega^Z)$ with the following properties: For all a, b, c in $field(\omega^Z)$ we have

a) $0_{\omega^Z} +_{\omega^Z} b = b$,

b) $(a +_{\omega^Z} b) +_{\omega^Z} c = a +_{\omega^Z} (b +_{\omega^Z} c)$,

c) $b \prec_{\omega^Z} c \to a +_{\omega^Z} b \prec_{\omega^Z} a +_{\omega^Z} c$,

d) $a \preceq_{\omega^Z} b \to a +_{\omega^Z} c \preceq_{\omega^Z} b +_{\omega^Z} c$,

e) $(\exists! b)(a \prec_{\omega^Z} c \to a +_{\omega^Z} b = c)$,

f) $Lim(b) \to a +_{\omega^Z} b = \sup\{a +_{\omega^Z} d : \langle d, b\rangle \in \omega^Z \wedge d \neq b\}$.

Proof. Choose a wellordering Z. We define the addition on $field(\omega^Z)$ analogous to the addition of two codes of ordinals in cartan normalform. Let $\varphi(x,p,q)$ be the following formula (the intention is $x = p +_{\omega^Z} q$):

$$\begin{aligned}
&\varphi(x,p,q) := \\
&\quad (\forall k < lh(p))(\langle (p)_k, (q)_0\rangle \in Z \wedge (p)_k \neq (q)_0 \wedge x = q) \vee \\
&\quad (\exists k)[(q)_0 Z (p)_k \wedge lh(x) = k+1+lh(q) \wedge Seq(x) \wedge \\
&\qquad (p)_{k+1} Z (q)_0 \wedge (p)_{k+1} \neq (q)_0] \wedge \\
&\qquad (\forall n \leq k)(x)_k = (p)_k \wedge (\forall n < lh(q))(x)_{k+1+n} = (q)_n.
\end{aligned}$$

Since we have

$$(\forall p,q \in field(\omega^Z))(\exists! x \in field(\omega^Z))\varphi(x,p,q),$$

we can define $+_{\omega^Z}$ as

$$p +_{\omega^Z} q = x :\leftrightarrow \varphi(x,p,q).$$

Now the properties a) – f) can be verified with standard arguments. □

In ATR_0 we can compare the wellorderings Z and ω^Z. This means, there is a comparison map $\mathcal{F}$ (cf. [29] section V.2). We claim that $\mathcal{F}$ is an isomorphism from Z onto some initial section of ω^Z. By contradiction we assume that $\mathcal{F}$ is an isomorphism from ω^Z onto $\{c : cZa \wedge c \neq a\}$ for $a \in field(Z)$. We have also an injective, order preserving map $\mathcal{G}$ from Z into ω^Z, namely $\mathcal{G}(a) := \langle a\rangle$. Thus $\mathcal{F} \circ \mathcal{G}$ is an order preserving, injective map from Z into $\{c : cZa \wedge c \neq a\}$, a contradiction. Therefore, $\mathcal{F}$ has to be an isomorphism from Z onto some initial section of ω^Z (provable in ATR_0).

For a wellordering Z we introduce some notations. For each limit number ℓ in $field(\omega^Z)$ we get a unique $\mathcal{F}$ from lemma 31 by inserting

$$\{\langle a,b\rangle : \langle a,b\rangle \in \omega^Z \wedge \langle a,\ell\rangle \in \omega^Z \wedge \langle b,\ell\rangle \in \omega^Z \wedge a \neq \ell \wedge b \neq \ell\}$$

for the wellordering Z in lemma 31. We let $\ell[n]$ denote this fundamental sequence $\mathcal{F}(n)$. We have $\ell[n] \prec_{\omega^Z} \ell[n+1]$ and $\ell[n] \overset{n\to\infty}{\to} \ell$. With the aid of the unique $+_{\omega^Z}$-operation (lemma 33) we can choose a unique element c in $field(\omega^Z)$ with $\ell[n] +_{\omega^Z} c = \ell[n+1]$. We write $\ell^-[n]$ for this unique c. Moreover we write $H_{\mathcal{A}}(a,b,\omega^Z,X,Y)$ for the formalization of: "If X is a fixed point hierarchy along ω^Z up to a, then Y is such a hierarchy along ω^Z up to $a +_{\omega^Z} b$ with the same projections beneath a". Let $\mathcal{A}(X,Y,x,y)$ be an arithmetic, X positive formula. Then we set

$$\begin{aligned}
&Hier_{\mathcal{A}}(a,\omega^Z,Y) := \\
&\quad (\forall c \prec_{\omega^Z} a)(\forall x)[x \in (Y)_c \leftrightarrow \mathcal{A}((Y)_c, (Y)_{\omega^Z c}, x, c)],
\end{aligned}$$

$$\begin{aligned}
&H_{\mathcal{A}}(a,b,\omega^Z,X,Y) := \\
&\quad Hier_{\mathcal{A}}(a,\omega^Z,X) \\
&\quad \to Hier_{\mathcal{A}}(a +_{\omega^Z} b, \omega^Z, Y) \wedge (\forall c \prec_{\omega^Z} a)(X)_c = (Y)_c.
\end{aligned}$$

We can now begin to adapt lemma 2 and lemma 3 in [16] to our context.

Lemma 34 *For each arithmetic, X positive formula $\mathcal{A}(X,Y,x,y)$ Σ^1_1-DC proves*

$$\begin{aligned}&WO(Z) \wedge (\forall n)(\forall X)(\exists Y) H_{\mathcal{A}}(a +_{\omega^Z} \ell[n], \ell^-[n], \omega^Z, X, Y)\\ &\rightarrow (\forall X)(\exists Y)[Hier_{\mathcal{A}}(a +_{\omega^Z} \ell[0], \omega^Z, X)\\ &\qquad\qquad \rightarrow (Hier_{\mathcal{A}}(a +_{\omega^Z} \ell, \omega^Z, Y) \wedge (\forall b \prec_{\omega^Z} a +_{\omega^Z} \ell[0])(X)_b = (Y)_b)].\end{aligned}$$

Proof. Exactly the same proof as for lemma 2 in [16]. □

Lemma 35 *For each arithmetic, X positive formula $\mathcal{A}(X,Y,x,y)$ ACA_0 proves*

$$\begin{aligned}&(\forall X)(\exists M)[X \dot\in M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M]\\ &\qquad \rightarrow (\forall Z)[WO(Z) \rightarrow (\exists Y) FHier_{\mathcal{A}}(Z,Y)].\end{aligned}$$

Proof. It is essentially the same proof as for lemma 3 in [16]. Choose $\mathcal{A}$ and a wellordering E. Choose M with E and all set parameters $\vec{Q}$ of $\mathcal{A}$ in M and with $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M$. We set

$$Z := \{\langle 2n, 2m\rangle : \langle n,m\rangle \in E\} \cup \{\langle 2n, 1\rangle : n \in field(E)\}.$$

We immediately conclude that $WO(Z)$ and $Z \dot\in M$. Z is roughly spoken "E plus 1". We have introduced Z because we need a bit more than E, exactly: one step more. We know that

$$\{b : b \in field(\omega^Z) \wedge (\forall a)(\forall X \dot\in M)(\exists Y \dot\in M) H_{\mathcal{A}}(a, b, \omega^Z, X, Y)\}$$

is a set. Thus we can show by transfinite induction along ω^Z

$$(\forall a, b \in field(\omega^Z))(\forall X \dot\in M)(\exists Y \dot\in M) H_{\mathcal{A}}(a, b, \omega^Z, X, Y),$$

where we use lemma 34 in the case that b is a limit number. This immediately yields a fixed point hierarchy for $\mathcal{A}$ along ω^Z up to each a in $field(\omega^Z)$ and finally a fixed point hierarchy for $\mathcal{A}$ along E. □

2.2.3 (FTR) implies ($\mathsf{K_0TR}$) over ACA_0

A crucial point in the proof of this implication is the construction of countable coded ω-models of Σ^1_1-AC with the aid of fixed points. If this is done, we have only to iterate the procedure. In order to build these countable coded ω-models of Σ^1_1-AC we first construct a countable coded ω-model M of ACA in which we can prove the existence of the π^0_1 jump hierarchy along each wellordering recursive in a given set Q. Then we can apply lemma VIII.4.19 in [29] to get a countable coded ω-model of Σ^1_1-AC in M, which contains Q.

Let $\pi^0_1[x, X]$ denote a complete Π^0_1 predicate. We write $\overset{+}{\pi}[x, X, Y]$ and $\overset{-}{\pi}[x, X, Y]$ for the X and Y positive formulas with the following properties:

$$\begin{aligned}\pi^0_1[x, X] &\leftrightarrow \overset{+}{\pi}[x, X, \neg X],\\ \neg\pi^0_1[x, X] &\leftrightarrow \overset{-}{\pi}[x, X, \neg X].\end{aligned}$$

Now we define three formulas which finally lead to the desired countable coded ω-model of ACA equipped with the mentioned properties. Again we do not formalize everything.

1. $\mathcal{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle)$ is a formula positive in X. The intended interpretation is that for each fixed point X of $\mathcal{A}$ and for each wellordering recursive in Q with index a, $(X)_a$ is the π^0_1 jump hierarchy along a starting with Q.

$$\begin{aligned}
&\mathcal{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle) :=\\
&\quad \text{“}a\text{ codes a }Q\text{-recursive linear ordering }\prec^Q_a\text{ and}\\
&\quad \text{there is a least element }0_a\text{”} \wedge\\
&\quad [(x = 0_a \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 = 0_a \wedge (z)_0 \in Q) \vee\\
&\quad\ (x = 0_a \wedge y = 1 \wedge [\neg Seq_2 z \vee (z)_1 \neq 0_a \vee (z)_0 \notin Q]) \vee\\
&\quad\ (0_a \prec^Q_a x \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 \prec^Q_a x\\
&\quad\ \ \wedge \overset{+}{\pi}[(z)_0, (((X)_a)_{(z)_1})_0, (((X)_a)_{(z)_1})_1]) \vee\\
&\quad\ (0_a \prec^Q_a x \wedge y = 1 \wedge [\neg Seq_2 z \vee \neg((z)_1 \prec^Q_a x)\\
&\quad\ \ \vee \overset{-}{\pi}[(z)_0, (((X)_a)_{(z)_1})_0, (((X)_a)_{(z)_1})_1]])].
\end{aligned}$$

2. $\mathcal{B}(Y, X, \langle\langle z, y\rangle, x\rangle)$ is a formula positive in Y. The intended interpretation is that for each fixed point Y of $\mathcal{B}$, Y is the π^0_1 jump hierarchy along $<$ starting with X.

$$\begin{aligned}
&\mathcal{B}(Y, X, \langle\langle z, y\rangle, x\rangle) :=\\
&\quad (x = 0 \wedge y = 0 \wedge z \in X) \vee\\
&\quad (x = 0 \wedge y = 1 \wedge z \notin X) \vee\\
&\quad (0 < x \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 < x\\
&\quad\ \wedge \overset{+}{\pi}[(z)_0, ((Y)_{(z)_1})_0, ((Y)_{(z)_1})_1]) \vee\\
&\quad (0 < x \wedge y = 1 \wedge [\neg Seq_2 z \vee \neg((z)_1 < x)\\
&\quad\ \vee \overset{-}{\pi}[(z)_0, ((Y)_{(z)_1})_0, ((Y)_{(z)_1})_1]]).
\end{aligned}$$

3. $\mathcal{C}(Z, Y, \langle m, \langle e, x\rangle\rangle)$ is a formula positive in Z. The intended interpretation is that for each fixed point Z of $\mathcal{C}$, $(Z)_{\langle e,x\rangle}$ is a set recursive in $((Y)_x)_0$, namely $\{m : \{e\}^{((Y)_x)_0}(m) = 0\}$.

Notice that the set variable Z do not occur in $\mathcal{C}(Z, Y, \langle m, \langle e, x \rangle \rangle)$.

$$\begin{aligned}&\mathcal{C}(Z, Y, \langle m, \langle e, x, \rangle \rangle) := \\ &\qquad (\forall z)(\exists n)(\{e\}^{((Y)_x)_0}(z) = n) \land \{e\}^{((Y)_x)_0}(m) = 0.\end{aligned}$$

The next lemma establishes that the listed formulas serve the right role.

Lemma 36 *We can prove in* ACA_0*: If we have sets* X, Y, Z, Q *with*

$$\begin{aligned}&(\forall z, y, x, a)(\langle\langle\langle z, y\rangle, x\rangle, a\rangle \in X \leftrightarrow \mathcal{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle)) \land \\ &(\forall z, y, x)(\langle\langle z, y\rangle, x\rangle \in Y \leftrightarrow \mathcal{B}(Y, X, \langle\langle z, y\rangle, x\rangle)) \land \\ &(\forall m, e, x)(\langle m, \langle e, x\rangle\rangle \in Z \leftrightarrow \mathcal{C}(Z, Y, \langle m, \langle e, x\rangle\rangle)),\end{aligned}$$

then we can conclude

- **a)** $(\forall x, z)[\langle\langle z, 0\rangle, x\rangle \in Y \leftrightarrow \langle\langle z, 1\rangle, x\rangle \notin Y]$,
- **b)** $(Ax_{\mathsf{ACA}})^Z \land X \dot{\in} Z \land Q \dot{\in} Z$,
- **c)** *(for each* Q*-recursive wellordering* a *there exists the* π^0_1 *jump hierarchy along* a*, starting with* $Q)^Z$*,*
- **d)** $(\exists M \dot{\in} Z)(Q \dot{\in} M \land (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M)$.

Proof.

a) $\{x : (\forall z)(\langle\langle z, 0\rangle, x\rangle \in Y \leftrightarrow \langle\langle z, 1\rangle, x\rangle \notin Y)\}$ is a set. Therefore, the claim can be proved by induction along $<$.

b) $(((Y)_x)_0)_{x\in\omega}$ is by definition the π^0_1 jump hierarchy along $<$ (cf. a)), and Z is the union of all sets recursive in a $((Y)_x)_0$. It can be proved by standard arguments that in this situation we have $(Ax_{\mathsf{ACA}})^Z$ (cf. for instance [29] Theorem VIII.1.13). Furthermore, we know $((Y)_0)_0 = X$ and $((((X)_a)_{0_a})_0)_{0_a} = Q$ for an appropriate a. Thus $Q \dot{\in} Z$ holds.

c) Choose a Q-recursive wellordering a with $(WO(a))^Z$. We know $Q \dot{\in} Z$ and $(Ax_{\mathsf{ACA}})^Z$. That is, all subsets of $field(a)$ in Z are wellfounded with respect to a. Now we know that the set

$$N := \{b : b \prec^Q_a a \land (\forall z)(\langle\langle\langle z, 0\rangle, b\rangle, a\rangle \in X \leftrightarrow \langle\langle\langle z, 1\rangle, b\rangle, a\rangle \notin X)\}$$

is arithmetic in Q, X. Therefore, $N \dot{\in} Z$. Again transfinite induction along a shows

$$(\forall b \prec^Q_a a)(\forall z)(\langle\langle\langle z, 0\rangle, b\rangle, a\rangle \in X \leftrightarrow \langle\langle\langle z, 1\rangle, b\rangle, a\rangle \notin X).$$

Thus, $E := \{\langle z, b\rangle : b \prec^Q_a a \land \langle\langle\langle z, 0\rangle, b\rangle, a\rangle \in X\}$ is the hyperarithmetical hierarchy along a and $E \dot{\in} Z$.

d) From lemma VIII.4.19 in [29] and a), b), c) we immediately conclude the claim. $\Box$

Now we can begin to iterate the whole thing. For technical reasons we define for two wellorderings Z, Y the wellordering $Z \otimes Y$, where we have between two neighboring elements of $field(Z)$ the wellordering Y once more.

$$\begin{aligned} Z \otimes Y \ &:= \ \{\langle\langle x, m\rangle, \langle y, n\rangle\rangle : x, y \in field(Z) \wedge m, n \in field(Y) \wedge \\ &\qquad [(x = y \wedge mYn) \vee (xZy \wedge x \neq y)]\}. \end{aligned}$$

Since the proof of the following lemma uses only standard arguments, we omit it.

Lemma 37 ACA_0 *proves*

$$WO(Z) \wedge WO(Y) \to WO(Z \otimes Y).$$

We let $4_<$ denote the restriction of the wellordering $<$ on $\{0, 1, 2, 3\}$. i.e., for a wellordering Z $Z \otimes 4_<$ is a wellordering with exactly three further elements between two neighboring elements of $field(Z)$. Again we write $0_{Z\otimes 4_<}$ for the $Z \otimes 4_<$-least element of $field(Z \otimes 4_<)$. Furthermore, we write $Y_{\langle Zk,3\rangle}$ for the disjoint union of all $(Y)_{\langle b,3\rangle}$ with $\langle b, k\rangle \in Z$ and $b \neq k$:

$$Y_{\langle Zk,3\rangle} := \{\langle x, \langle y, 3\rangle\rangle : \langle y, k\rangle \in Z \wedge y \neq k \wedge \langle x, \langle y, 3\rangle\rangle \in (Y)_{(Z\otimes 4_<)\langle k,0\rangle}\}.$$

We give first an informal description of the iteration and afterwards we give a formalization. So, choose a wellordering Z and a set Q. We construct a hierarchy Y along $Z \otimes 4_<$ as follows:

0. $(Y)_{\langle 0_Z,0\rangle}$ satisfies

$$\langle\langle\langle z, y\rangle, x\rangle, b\rangle \in (Y)_{\langle 0_Z,0\rangle} \leftrightarrow \mathcal{A}((Y)_{\langle 0_Z,0\rangle}, Q \oplus Z, \langle\langle\langle z, y\rangle, x\rangle, b\rangle).$$

1. $(Y)_{\langle 0_Z,1\rangle}$ satisfies

$$\langle\langle z, y\rangle, x\rangle \in (Y)_{\langle 0_Z,1\rangle} \leftrightarrow \mathcal{B}((Y)_{\langle 0_Z,1\rangle}, (Y)_{\langle 0_Z,0\rangle}, \langle\langle z, y\rangle, x\rangle).$$

2. $(Y)_{\langle 0_Z,2\rangle}$ satisfies

$$\langle m, \langle e, x\rangle\rangle \in (Y)_{\langle 0_Z,2\rangle} \leftrightarrow \mathcal{C}((Y)_{\langle 0_Z,2\rangle}, (Y)_{\langle 0_Z,1\rangle}, \langle m, \langle e, x\rangle\rangle).$$

3. $(Y)_{\langle 0_Z,3\rangle} := ((Y)_{\langle 0_Z,2\rangle})_{\langle e,x\rangle}$ such that $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_{\langle 0_Z,3\rangle}}$ and $Q \oplus Z \,\dot{\in}\, (Y)_{\langle 0_Z,3\rangle}$ and $\langle e, x\rangle$ is the $<$-least index with this property.

4. $(Y)_{\langle 1_Z,0\rangle}$ satisfies

$$\langle\langle\langle z, y\rangle, x\rangle, b\rangle \in (Y)_{\langle 1_Z,0\rangle} \leftrightarrow \mathcal{A}((Y)_{\langle 1_Z,0\rangle}, (Y)_{\langle 0_Z,3\rangle}, \langle\langle\langle z, y\rangle, x\rangle, b\rangle).$$

$\cdots$ and so on.

And now the formalization: $(a \in field(Z \otimes 4_<))$

$$
\begin{aligned}
&\mathcal{A}^\star((Y)_a, (Y)_{(Z\otimes 4_<)a}, n, a) := \\
&\quad (a = 0_{Z\otimes 4_<} \wedge (\exists z, y, x, b)[n = \langle\langle\langle z, y\rangle, x\rangle, b\rangle \wedge \mathcal{A}((Y)_a, Q \oplus Z, n)]) \vee \\
&\quad (\langle 0_{Z\otimes 4_<}, a\rangle \in Z \otimes 4_< \wedge 0_{Z\otimes 4_<} \neq a \wedge (\exists k \in field(Z))[\\
&\quad (a = \langle k, 0\rangle \wedge (\exists z, y, x, b)(n = \langle\langle\langle z, y\rangle, x\rangle, b\rangle \wedge \mathcal{A}((Y)_a, Y_{\langle Zk,3\rangle}, n))) \vee \\
&\quad (a = \langle k, 1\rangle \wedge (\exists z, y, x)(n = \langle\langle z, y\rangle, x\rangle \wedge \mathcal{B}((Y)_a, ((Y)_{(Z\otimes 4_<)a})_{\langle k,0\rangle}, n))) \vee \\
&\quad (a = \langle k, 2\rangle \wedge (\exists m, e, x)(n = \langle m, \langle e, x\rangle\rangle \wedge \mathcal{C}((Y)_a, ((Y)_{(Z\otimes 4_<)a})_{\langle k,1\rangle}, n))) \vee \\
&\quad (a = \langle k, 3\rangle \wedge (\exists e, x)(n \in (((Y)_{(Z\otimes 4_<)a})_{\langle k,2\rangle})_{\langle e,x\rangle} \\
&\qquad\qquad \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(((Y)_{(Z\otimes 4_<)a})_{\langle k,2\rangle})_{\langle e,x\rangle}} \\
&\qquad\qquad \wedge (Q \oplus Z \oplus Y_{\langle Zk,3\rangle}) \mathrel{\dot\in} (((Y)_{(Z\otimes 4_<)a})_{\langle k,2\rangle})_{\langle e,x\rangle} \wedge \\
&\qquad\qquad (\forall l)(l < \langle e, x\rangle \to \\
&\qquad\qquad\qquad Q \oplus Z \oplus Y_{\langle Zk,3\rangle} \mathrel{\dot\notin} (((Y)_{(Z\otimes 4_<)a})_{\langle k,2\rangle})_l \\
&\qquad\qquad\qquad \vee\neg(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(((Y)_{(Z\otimes 4_<)a})_{\langle k,2\rangle})_l})))]).
\end{aligned}
$$

Since $Z \otimes 4_<$ is a wellordering, FTR_0 yields a set Y with

$$(\forall a \in field(Z \otimes 4_<))(\forall n)(n \in (Y)_a \leftrightarrow \mathcal{A}^\star((Y)_a, (Y)_{(Z\otimes 4_<)a}, n, a)).$$

We set $P := \{\langle x, a\rangle : \langle x, \langle a, 3\rangle\rangle \in Y \wedge a \in field(Z)\}$ and collect the properties of P in a lemma.

Lemma 38 *With the above definitions, notations and assumptions we have*

$$(\forall a \in field(Z))((Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(P)_a} \wedge (P)_{Za} \mathrel{\dot\in} (P)_a \wedge Q \mathrel{\dot\in} (P)_a).$$

Proof. We prove the claim by transfinite induction along the wellordering Z. Choose $a \in field(Z)$. We distinguish the two cases:

$0_Z = a$: The definition of P yields $(P)_{0_Z} = (Y)_{\langle 0_Z,3\rangle}$. We know there is a countable coded ω-model of Σ^1_1-AC in $(Y)_{\langle 0_Z,2\rangle}$ containing $Q \oplus Z$ (lemma 36d)). We choose that model with the least index. This is just $(Y)_{\langle 0_Z,3\rangle}$. Therefore $Q \mathrel{\dot\in} (P)_{0_Z}$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(P)_{0_Z}}$. Moreover, $(P)_{Z0_Z} = \emptyset$ and $\emptyset \mathrel{\dot\in} (P)_{0_Z}$.

$0_Z Za$: Again we have $(P)_a = (Y)_{\langle a,3\rangle}$ and therefore $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(P)_a}$ and $Q \mathrel{\dot\in} (P)_a$. It remains to show $(P)_{Za} \mathrel{\dot\in} (P)_a$. We know $Q \oplus Z \oplus Y_{\langle Za,3\rangle} \mathrel{\dot\in} (P)_a$ and

$$\langle x, b\rangle \in (P)_{Za} \leftrightarrow \langle x, \langle b, 3\rangle\rangle \in Y_{\langle Za,3\rangle}.$$

Therefore, we can conclude $(P)_{Za} \dot{\in} (P)_a$. $\square$

The next lemma establishes the result.

Lemma 39 *There is an arithmetic, in X positive formula $\mathcal{A}^\star$ with set parameters X, Y, Z, Q such that* ACA_0 *proves*

$$[WO(Z) \wedge (\exists Y) FHier_{\mathcal{A}^\star}(Z, Y)] \to (\exists Y)\mathsf{K}_0 Hier(Z, Q, Y).$$

2.2.4 $(\mathsf{K}_0\mathsf{TR})$ implies $(\mathsf{K}_n\mathsf{TR})$ over ACA_0

We show this implication with metamathematical induction on n. We assume $\mathsf{K}_n\mathsf{TR}_0$ and prove $(\mathsf{K}_{n+1}\mathsf{TR})$. i.e., given a wellordering Z and a set X, we have to show the existence of an Y with

$$(\forall a \in field(Z))[[(\forall G)(\exists H)(G \dot{\in} H \wedge \mathsf{K}_n(H))]^{(Y)_a} \wedge (Ax_{\mathsf{ACA}})^{(Y)_a} \wedge (Y)_{Za} \dot{\in} (Y)_a \wedge X \dot{\in} (Y)_a].$$

Since we have $(\mathsf{K}_n\mathsf{TR})$, there is a hierarchy E along the wellordering $Z\otimes <$ ($<$ is the canonical linear ordering on the natural numbers) with $\mathsf{K}_n((E)_a)$ and $(E)_{(Z\otimes<)a} \dot{\in} (E)_a$ for all a in $field(Z\otimes <)$ and $X \oplus Z \dot{\in} (E)_{0_{Z\otimes<}}$. Now we build for each a in $field(Z)$ a set $(G)_a$ which consists of all projections of $(E)_{\langle b,k\rangle}$ with $\langle b, a\rangle \in Z$. G will be the desired hierarchy.

Lemma 40 ACA_0 *proves*

$$\begin{aligned}&(\forall X, Z)(\exists Y)[WO(Z) \to \mathsf{K}_0 Hier(Z, X, Y)]\\ &\quad \to (\forall D, E)(\exists F)[WO(D) \to \mathsf{K}_n Hier(D, E, F)].\end{aligned}$$

Proof. By metamathematical induction on n. The claim is trivial for $n = 0$. Therefore, we assume $(\mathsf{K}_n\mathsf{TR})$ and show $(\mathsf{K}_{n+1}\mathsf{TR})$. We have to prove $(\exists G)\mathsf{K}_{n+1}Hier(Z, X, G)$. Choose a wellordering Z and a set X. We can assume that for all b in $field(Z)$ there is a c in $field(Z)$ with $\langle b, c\rangle \in Z \wedge b \neq c$. With the aid of $(\mathsf{K}_n\mathsf{TR})$ we get an E with

$$(\forall a \in field(Z\otimes <))[\mathsf{K}_n((E)_a) \wedge (E)_{(Z\otimes<)a} \dot{\in} (E)_a \wedge X \oplus Z \dot{\in} (E)_a].$$

Now we set for all a in $field(Z)$:

$$(G)_a := \{\langle x, \langle e, \langle b, k\rangle\rangle\rangle : x \in ((E)_{\langle b,k\rangle})_e \wedge \langle b, a\rangle \in Z\}.$$

and prove that G is a hierarchy along Z with $\mathsf{K}_{n+1}((G)_a)$, $(G)_{Za} \dot{\in} (G)_a$ and $X \dot{\in} (G)_a$. We use transfinite induction on the wellordering Z.

$a = 0_Z:$ $(G)_{0_Z} = \{\langle x, \langle e, \langle 0_Z, k\rangle\rangle\rangle : x \in ((E)_{\langle 0_Z,k\rangle})_e\}$. Since we have $X \dot\in (E)_{\langle 0_Z,0\rangle}$ we get $X \dot\in (G)_{0_Z}$. Now we will show $\mathsf{K}_{n+1}((G)_{0_Z})$. (Then we can conclude $(G)_{Z0_Z} \dot\in (G)_{0_Z}$ too, since $(G)_{Z0_Z}$ is the empty set.) We have to show

$$[(\forall D)(\exists H)(D \dot\in H \wedge \mathsf{K}_n(H))]^{(G)_{0_Z}} \wedge (Ax_{\mathsf{ACA}})^{(G)_{0_Z}}.$$

Choose a D in $(G)_{0_Z}$. That is $D = ((E)_{\langle 0_Z,k\rangle})_e$ for appropriate e, k. We know $(E)_{\langle 0_Z,k\rangle} \dot\in (E)_{\langle 0_Z,k+1\rangle}$. Hence with $H := (E)_{\langle 0_Z,k\rangle}$ we conclude $D \dot\in H \dot\in (G)_{0_Z}$ and $\mathsf{K}_n(H)$. It remains to show $(Ax_{\mathsf{ACA}})^{(G)_{0_Z}}$. We choose an arithmetic formula φ and prove

$$\vec{A} \dot\in (G)_{0_Z} \to (\exists B \dot\in (G)_{0_Z})(\forall x)(x \in B \leftrightarrow \varphi[x, \vec{z}, \vec{A}]).$$

For each parameter A_i there exist e_i, k_i with $A_i = ((E)_{\langle 0_Z,k_i\rangle})_{e_i}$. Choose n with $k_i < n$ for all i. We know $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(E)_{\langle 0_Z,k_i\rangle}}$ and $(E)_{(Z\otimes<)\langle 0_Z,k_i\rangle} \dot\in (E)_{\langle 0_Z,k_i\rangle}$. Hence $\vec{A} \dot\in (E)_{\langle 0_Z,n\rangle}$. $(E)_{\langle 0_Z,n\rangle}$ is a model of (ACA) too. So there exists f with

$$(\forall x)(x \in ((E)_{\langle 0_Z,n\rangle})_f \leftrightarrow \varphi[x, \vec{z}, \vec{A}]).$$

Moreover, $((E)_{\langle 0_Z,n\rangle})_f \dot\in (G)_{0_Z}$. This is the claim.

$a +_Z 1:$ $(G)_{a+_Z1} = \{\langle x, \langle e, \langle b, k\rangle\rangle\rangle : x \in ((E)_{\langle b,k\rangle})_e \wedge \langle b, a +_Z 1\rangle \in Z\}$. Since we have for all b with $\langle b, a\rangle \in Z$ $(E)_{\langle b,k\rangle} = ((E)_{(Z\otimes<)\langle a+_Z1,0\rangle})_{\langle b,k\rangle}$ and $(E)_{(Z\otimes<)\langle a+_Z1,0\rangle} \dot\in (E)_{\langle a+_Z1,0\rangle}$ we can conclude $(G)_a \dot\in (G)_{a+_Z1}$ and $(G)_{Za} \dot\in (G)_a$. With $X \dot\in (G)_a$ we conclude $X \dot\in (G)_{a+_Z1}$ too. Finally similar arguments as in the case $a = 0_Z$ lead to $\mathsf{K}_{n+1}((G)_{a+_Z1})$ and $(Ax_{\mathsf{ACA}})^{(G)_{a+_Z1}}$.

$a = \ell:$ With analogous arguments as in the cases above we can prove the claim. □

2.2.5 Summary

We collect the results of the preceding paragraphs in the following theorem

Theorem 41 *Over* ACA_0 *the following axioms schemes are equivalent:*

a) $(\mathsf{K}_n\mathsf{TR})$,

b) $((\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN})$,

c) (FTR).

Proof. lemma 30 shows the implication a)→b), lemma 35 shows the implication b)→c) and lemma 39 shows the implication $(\mathsf{FTR}) \to (\mathsf{K}_0\mathsf{TR})$. Finally we have also $(\mathsf{K}_0\mathsf{TR}) \to (\mathsf{K}_n\mathsf{TR})$ (lemma 40). □

The proof-theoretic strength of FTR and FTR_0 is stated in [31]. Therefore, we have the following corollary:

Corollary 42 *We have the following proof-theoretic strengths:*

a) $|\mathsf{K_0TR}| = |(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN}| = |\mathsf{FTR}| = \varphi 20\varepsilon_0$.

b) $|\mathsf{K_0TR_0}| = |(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN_0}| = |\mathsf{FTR_0}| = \varphi 200$.

2.3 The axioms schemes $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ and $\Sigma^1_1\text{-}\mathsf{TDC}$

Similar to the predicates K_n we can define predicates I_n:

$$\begin{array}{rcl} \mathsf{I}_0(M) & := & (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M, \\ \mathsf{I}_{n+1}(M) & := & (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge (\forall X \dot\in M)(\exists Y \dot\in M)(X \dot\in Y \wedge \mathsf{I}_n(Y)), \end{array}$$

where $Ax_{\Sigma^1_1\text{-}\mathsf{DC}}$ denotes a finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{DC}) + (\mathsf{ACA})$. I_{n+1} corresponds to a "n-fold reflection" of the theory $(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN_0}$ on models of Σ^1_1-DC. We remind the reader to the discussion of universes in chapter 1. We have introduced there our universes as models of Σ^1_1-AC; and we have noted that it would also be possible to introduce the universes as models of Σ^1_1-DC without changing the proof-theoretic strength. In contrast it makes here a significant difference if we replace $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$ by $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M$ in the definition of $\mathsf{I}_{n+1}(M)$. But if we replace $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M$ by $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$ in the definition of $\mathsf{I}_0(M)$, the proof-theoretic strength of the theories introduced below would be the same. We have set here $\mathsf{I}_0(M) := (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M$ instead of $\mathsf{I}_0(M) := (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$, since our universes in chapter 1 are models of Σ^1_1-AC and not of Σ^1_1-DC.

Each M with $\mathsf{I}_n(M)$ is among other things a countable coded ω-model of Σ^1_1-DC for $n > 0$. Each M with $\mathsf{I}_1(M)$ is a countable coded ω-model of $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ (cf. lemma 28). For $n > 0$ it is important that we have $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$. $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M$ would change the situation drastically. Below, we will motivate this requirement.

There is a sequence $(\mathsf{I}_n\text{-}\mathsf{RFN})_{n\geq 0}$ of theories I_n-RFN:

$$\begin{array}{rcl} \mathsf{I}_0\text{-}\mathsf{RFN} & := & \mathsf{ACA} + (\forall X)(\exists Y)(X \dot\in Y \wedge \mathsf{I}_0(Y)), \\ \mathsf{I}_1\text{-}\mathsf{RFN} & := & \mathsf{ACA} + (\forall X)(\exists Y)(X \dot\in Y \wedge \mathsf{I}_1(Y)), \\ & & \vdots \\ \mathsf{I}_{n+2}\text{-}\mathsf{RFN} & := & \mathsf{ACA} + (\forall X)(\exists Y)(X \dot\in Y \wedge \mathsf{I}_{n+2}(Y)), \\ & & \vdots \end{array}$$

The question is, whether there is a theory which is the limit of the above sequence of theories, the closure of the above theories respectively. i.e., is there a theory which corresponds to the union $\bigcup_{n\geq 0} \mathsf{I}_n$-$\mathsf{RFN}$? The answer is "yes" and it is the purpose of this section to introduce such theories. First, we remark the following fact, which is an immediate conclusion of the definition.

$$\mathsf{ACA}_0 \vdash \mathsf{I}_n(M) \to (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M. \tag{2.8}$$

We have mentioned above that it is very important to have in the definition of $\mathsf{I}_{n+1}(M)$ $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$. If we replace in the definition of $\mathsf{I}_{n+1}(M)$ $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$ by $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M$, then the resulting predicates are equivalent – over ACA_0 – to the predicate $\mathsf{K}_m(M)$ introduced in this chapter. Another possibility is to introduce a predicate $\tilde{\mathsf{I}}_n$:

$$\begin{array}{rcl} \tilde{\mathsf{I}}_0(M) & := & \mathsf{I}_0(M), \\ \tilde{\mathsf{I}}_1(M) & := & \mathsf{I}_1(M), \\ \tilde{\mathsf{I}}_{n+2}(M) & := & (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M \wedge (\forall X \mathrel{\dot{\in}} M)(\exists Y \mathrel{\dot{\in}} M)(X \mathrel{\dot{\in}} Y \wedge \tilde{\mathsf{I}}_{n+1}(Y)). \end{array}$$

It can be shown that the predicate $\tilde{\mathsf{I}}_{n+1}$ is, in fact, much weaker than I_{n+1}. i.e. we have $\mathsf{I}_n(M) \to \tilde{\mathsf{I}}_n(M)$ but not vice versa. Here we discuss the predicate I_n instead of $\tilde{\mathsf{I}}_n$, because we have seen in theorem 41 that $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$ is a natural additional condition for models M of ATR. There are further reasons for the additional property $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$. First we are interested in theories substantially stronger than $\varphi 200$. With the predicates $\tilde{\mathsf{I}}_n$ the corresponding theories are not stronger than $\varphi 2\varepsilon_0\varepsilon_0$. Secondly we would like to build hierarchies in our models. And in lemma 34 and 35 we have seen that for this the axioms scheme (Σ^1_1-DC) is very important (since we have no uniformation, no foundation as, e.g., a bar-rule or something like that). In models of Σ^1_1-AC it is not possible to prove the existence of these desired hierarchies. We refer also to lemma 48 for an illustration of this fact.

There are hierarchies equivalent to models M with $\mathsf{I}_n(M)$. We write $\mathsf{I}_n Hier(Z, X, Y)$ for the statement "Y is a hierarchy along Z with $\mathsf{I}_n((Y)_a)$, $X \mathrel{\dot{\in}} (Y)_a$ and $(Y)_{Za} \mathrel{\dot{\in}} (Y)_a$ for all a in $field(Z)$":

$$\begin{array}{l} \mathsf{I}_n Hier(Z, X, Y) := \\ \qquad (\forall a \in field(Z))[\mathsf{I}_n((Y)_a) \wedge (Y)_{Za} \mathrel{\dot{\in}} (Y)_a \wedge X \mathrel{\dot{\in}} (Y)_a] \end{array}$$

and $\mathsf{I}_n\mathsf{TR}$ denotes the theory ACA extended by the axioms

$$(\mathsf{I}_n\mathsf{TR}) \qquad\qquad WO(Z) \to (\exists Y)\mathsf{I}_n Hier(Z, X, Y).$$

$\mathsf{I}_n\mathsf{TR}_0$ denotes $\mathsf{I}_n\mathsf{TR}$ with set-induction instead of full induction. Now with similar arguments that lead to lemma 30 and 35 we can prove the following equivalence (lemma 43). The direction b) $\to$ a) is proved by constructing the hierarchy in an appropriate model. The converse direction uses again the method of pseudohierarchies. Since we do not use lemma 43 in the sequel we omit the proof.

Lemma 43 ACA_0 *proves the equivalence of*

a) $(\mathsf{I}_n\mathsf{TR})$,

b) $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \land \mathsf{I}_{n+1}(Y))$.

Let us return to the definition of the predicates I_n. From $\mathsf{I}_n(M)$ $(n > 0)$ we take first that M is a countable coded ω-model of Σ^1_1-DC (cf. 2.8) and secondly that M reflects a Π^1_2 sentence. This leads to the conjecture that the theory ACA extended by the axioms scheme

$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ For all Π^1_2 formulas $\varphi[\vec{z},\vec{Z}]$:
$\varphi[\vec{z},\vec{Z}] \to (\exists M)[\vec{Z} \mathrel{\dot\in} M \land (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \land \varphi^M]$

is (at least proof-theoretically) equivalent to the union $\bigcup_{n\geq 0} \mathsf{I}_n$-$\mathsf{RFN}$. $Ax_{\Sigma^1_1\text{-}\mathsf{DC}}$ denotes a finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{DC})+(\mathsf{ACA})$ and we write $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ (resp. $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$) for the theory $\mathsf{ACA} + ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ (resp. $\mathsf{ACA}_0 + ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$. Since theorem VIII.5.12 [29] states the equivalence of $(\Pi^1_2\text{-}\mathsf{RFN})$ and $(\Sigma^1_1\text{-}\mathsf{DC})$ over ACA_0, the above axioms scheme expresses a certain kind of self-reference: For all Π^1_2 formulas φ there is a model M_1, which reflects φ and in which there is a model M_2 which reflects φ and in which there is a model M_3 .... Here we have used that $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ includes the axioms scheme

$(\Sigma^1_3\text{-}\mathsf{RFN})$ For all Σ^1_3 formulas $\varphi[\vec{x},\vec{X}]$:
$\varphi \to (\exists M)(\vec{X} \mathrel{\dot\in} M \land (Ax_{\Pi^1_2\text{-}\mathsf{RFN}})^M \land \varphi^M)$,

and that the formula

$$\psi \to (\exists M)(\vec{Z} \mathrel{\dot\in} M \land (Ax_{\Pi^1_2\text{-}\mathsf{RFN}})^M \land \psi^M)$$

is equivalent to a Σ^1_3 formula φ if ψ is a Π^1_2 formula.

Lemma 43 establishes the equivalence of $\mathsf{I}_n\mathsf{TR}$ and I_{n+1}-RFN. Therefore, we expect an analogous equivalence for the theory $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. Indeed there are hierarchies equivalent to the models of Σ^1_1-DC which reflect Π^1_2 formulas. We define the theory Σ^1_1-TDC as the theory ACA extended by

$(\Sigma^1_1\text{-}\mathsf{TDC})$ For all Σ^1_1-formulas φ:
$(\forall a)(\forall X)(\exists Y)\varphi(X,Y,Z,a) \land WO(Z)$
$\quad \to (\exists Y)[(Y)_{0_Z} = Q \land (\forall a)(0_Z Z a \land a \neq 0_Z$
$\qquad \to \varphi((Y)_{Za},(Y)_a,Z,a))]$.

Again we write Σ^1_1-TDC_0 for the corresponding theory with set-induction instead of full induction. Σ^1_1-TDC (Σ^1_1 Transfinite Dependent Choice) is a "transfinite" strengthening of Σ^1_1-DC. If we replace Z by $<$ in (Σ^1_1-TDC) we get immediately (Σ^1_1-DC). The possibility of building transfinite hierarchies in the sense of (Σ^1_1-TDC) is very useful when there is no kind of uniformization principles. Assume that for all X there is an Y with $\varphi(X,Y)$ (φ arithmetic). In general there is no possibility to build (for example) in $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ hierarchies Y along an arbitrary wellordering Z with $\varphi((Y)_{Za},(Y)_a)$ for a in $field(Z)$. The reason is that in general we do not have $(\exists! Y)\varphi(X,Y)$. We also refer to lemma 49. And in $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ there is no uniformization principle for this situation. But with the aid of (Σ^1_1-TDC) we have such hierarchies (axiomatically). In the next section we prove the equivalence of Σ^1_1-TDC and $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. And later on, we give the proof-theoretic ordinal of Σ^1_1-TDC_0. It will be $\varphi\omega 00$.

2.4 Equivalence of (Σ^1_1-TDC) and ($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$) over ACA_0

In this section we use the same proof ideas which led to lemma 30 and 35 in order to show the equivalence of $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ and Σ^1_1-TDC. But first we state in the next lemma that Σ^1_1-TDC_0 contains all relevant subsystems of analysis with proof-theoretical strength less than $\varphi\omega 00$. In this sense, the theory Σ^1_1-TDC_0 is built with care. We omit the proof, because the claims are trivial or stated before.

Lemma 44 Σ^1_1-TDC_0 *proves*

- **a)** (ACA),
- **b)** (Σ^1_1-AC),
- **c)** (Σ^1_1-DC),
- **d)** (ATR),
- **e)** ($\mathsf{I}_n\mathsf{TR}$).

Lemma 45 ACA_0 *proves*

$$(\Sigma^1_1\text{-}\mathsf{TDC}) \rightarrow ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}).$$

Proof. As mentioned, the proof is a generalization of lemma VIII.4.19 in [29]. Further we use similar arguments as in the proof of theorem VIII.5.12 (again in [29]) for the "Π^1_2 reflecting" property. We argue in ACA_0. We choose a set U and assume φ ($\varphi \in \Pi^1_2$); it is

$$\varphi = (\forall X)(\exists Y)\psi(X,Y)$$

with ψ an arithmetic formula. We have to show the existence of a set M with

$$U \mathrel{\dot\in} M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M.$$

Again $\pi^0_{1,k,n}$ is our universal Π^0_1 predicate. We define first a formula θ and give then the intended interpretation.

$$\begin{array}{lll}
\multicolumn{3}{l}{\theta(X,Y,Z,a) :=} \\
\multicolumn{3}{l}{\;a \in field(Z) \wedge} \\
\qquad a = 0_Z & \to & (Y)_2 = \{\langle m,e\rangle : \pi^0_{1,1,1}[e,m,U]\} \wedge \\
\qquad a \neq 0_Z & \to & [(Y)_0 = \{\langle x,\langle\langle e,f\rangle,b\rangle\rangle : bZa \wedge b \neq a \wedge \{f\}^{(((X)_b)_2)_e}\text{total} \\
 & & \qquad\qquad\qquad \wedge\{f\}^{(((X)_b)_2)_e}(x) = 0\} \wedge \\
 & & (\forall e)\psi(((Y)_0)_e,((Y)_1)_e) \wedge \\
 & & (Y)_2 = \{\langle m,e\rangle : \pi^0_{1,1,1}[e,m,(Y)_1 \oplus (Y)_0 \oplus X]\}].
\end{array}$$

For a linear ordering Z with Z-least element 0_Z the interpretation of the formula θ is as follows: If $a = 0_Z$ then $(Y)_2$ is the Π^0_1-jump of U. If $a \neq 0_Z$ then $(Y)_0$ is the set of all sets recursive in $(((X)_b)_2)_e$ for a Z-smaller b than a. Later on, the sets in $(Y)_0$ will build the desired model. In order to ensure that this model reflects the formula φ, we require $(\forall e)\psi(((Y)_0)_e,((Y)_1)_e)$. i.e. for each $(((X)_b)_2)_e$-recursive set D there is a set $((Y)_1)_e$ in $(Y)_1$ with $\psi(D,((Y)_1)_e)$. Finally, $(Y)_2$ is the Π^0_1-jump of $(Y)_1 \oplus (Y)_0 \oplus X$. We take the Π^0_1-jump in order to have that the desired model reflects (ACA) and $(\Sigma^1_1\text{-}\mathsf{DC})$. First we prove

$$(\forall P)(\exists Q)\theta(P,Q,Z,a). \tag{2.9}$$

In the case $a = 0_Z$ (2.9) follows immediately. Therefore, we assume $0_Z Za$ and $a \neq 0_Z$ and we choose a set P. There is no problem to build a set E with

$$E := \{\langle x,\langle\langle e,f\rangle,b\rangle\rangle : bZa \wedge b \neq a \wedge \{f\}^{(((P)_b)_2)_e}\text{total} \wedge \{f\}^{(((P)_b)_2)_e}(x) = 0\}.$$

We know $(\forall X)(\exists Y)\psi(X,Y)$ and therefore

$$(\forall e)(\exists Y)\psi((E)_e,Y).$$

An application of $(\Sigma^1_1\text{-}\mathsf{AC})$ yields a set Y with

$$(\forall e)\psi((E)_e,(Y)_e).$$

We set

$$\begin{array}{lll}
(Q)_0 & := & E, \\
(Q)_1 & := & Y, \\
(Q)_2 & := & \{\langle m,e\rangle : \pi^0_{1,1,1}[e,m,Y \oplus E \oplus P]\}.
\end{array}$$

This proves (2.9). We apply (Σ^1_1-TDC) to the formula θ and conclude (with (2.9))

$$\begin{aligned} WO(Z) \to (\exists Y)[((Y)_{0_Z})_2 = \{\langle m, e\rangle : \pi^0_{1,1,1}[e, m, U]\}\wedge \\ (\forall a)(0_Z Z a \wedge a \neq 0_Z \to \theta((Y)_{Za}, (Y)_a, Z, a))]. \end{aligned}$$

The same arguments as in the proof of lemma 30 yield sets $Y, Z, M^\star$ with

$$\begin{aligned} &\neg WO(Z) \wedge Z, Y, U \,\dot{\in}\, M^\star \wedge (Ax_{\mathsf{ACA}})^{M^\star} \wedge (WO(Z))^{M^\star} \wedge \\ &((Y)_{0_Z})_2 = \{\langle m, e\rangle : \pi^0_{1,1,1}[e, m, U]\} \wedge \\ &(\forall a)(0_Z Z a \wedge a \neq 0_Z \to \theta((Y)_{Za}, (Y)_a, Z, a)). \end{aligned}$$

i.e., Y is a pseudohierarchy starting with $((Y)_{0_Z})_2 = \{\langle m, e\rangle : \pi^0_{1,1,1}[e, m, U]\}$. For $a \neq 0_Z$ $(Y)_a$ consists of three parts: $((Y)_a)_0$ is the collection of all sets recursive in $(((Y)_b)_2)_e$ for b Z-smaller than a; then we search for each set Q in $((Y)_a)_0$ a set D with $\psi(Q, D)$ and put these D into $((Y)_a)_1$; finally we take as $((Y)_a)_2$ the Π^0_1-jump of $((Y)_a)_1 \oplus ((Y)_a)_0 \oplus (Y)_{Za}$. Again we can choose a set $I \subset field(Z)$ with

$$\begin{aligned} &I \,\dot{\notin}\, M^\star, \\ &I \neq \emptyset, \\ &(\forall b \in I)(cZb \to c \in I), \\ &(\forall b \in I)(\exists c \in I)(bZc \wedge b \neq c). \end{aligned}$$

The stage is set up in order to define our model M:

$$M := \{\langle x, \langle\langle e, f\rangle, b\rangle\rangle : b \in I \wedge (\exists a \in I)(bZa \wedge b \neq a \wedge \langle x, \langle\langle e, f\rangle, b\rangle\rangle \in ((Y)_a)_0)\}.$$

We only show φ^M. The remaining properties, $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$ and $U \,\dot{\in}\, M$, are proved with analogous arguments as in Lemma VIII.4.19 [29] (or lemma 30).

Choose an $X = (M)_{\langle\langle e,f\rangle,b\rangle} = (((Y)_{b+_Z 1})_0)_{\langle\langle e,f\rangle,b\rangle}$. Notice that we can choose $b +_Z 1$ instead of an $a \in I$ with bZa and $b \neq a$, since we have for all $a \in I$ with bZa and $b \neq a$ the equality $(Y_{Za})_b = (Y_{Z(b+_Z 1)})_b$. We know

$$(\forall e, f)(\forall b \in I)\psi((((Y)_{b+_Z 1})_0)_{\langle\langle e,f\rangle,b\rangle}, (((Y)_{b+_Z 1})_1)_{\langle\langle e,f\rangle,b\rangle}).$$

For appropriate g and h we can now conclude

$$\begin{aligned} x \in (((Y)_{b+_Z 1})_1)_{\langle\langle e,f\rangle,b\rangle} \;&\leftrightarrow\; \langle x, \langle\langle e, f\rangle, b\rangle\rangle \in ((Y)_{b+_Z 1})_1 \\ &\leftrightarrow\; \langle\langle x, \langle\langle e, f\rangle, b\rangle\rangle, g\rangle \in ((Y)_{b+_Z 1})_2 \\ &\leftrightarrow\; \langle x, \langle\langle e, f\rangle, b\rangle\rangle \in ((((Y)_{Z(b+_Z 2)})_{b+_Z 1})_2)_g \\ &\leftrightarrow\; \langle x, \langle\langle g, h\rangle, b +_Z 1\rangle\rangle \in ((Y)_{b+_Z 2})_0 \\ &\leftrightarrow\; x \in (M)_{\langle\langle g,h\rangle,b+_Z 1\rangle}. \end{aligned}$$

Therefore, we have $\psi(X, (M)_{\langle\langle g,h\rangle,b+_Z 1\rangle})$ and this proves the claim. □

We establish the converse direction in the following lemma.

Lemma 46 ACA_0 *proves*

$$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}) \to (\Sigma^1_1\text{-}\mathsf{TDC}).$$

Proof. We work in ACA_0 and proceed as in the proof of the implication

$$((Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})\text{-}\mathsf{RFN}) \to (\mathsf{FTR})$$

(cf. lemma 31 – 35). Assume $\varphi := (\forall X)(\exists Y)\psi(X, Y, E, a)$ with ψ arithmetic. Moreover, we assume $WO(E)$ and choose a set Q. We apply the axiom $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ to the formula

$$\varphi \wedge (\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$$

and get a model M with

$$\varphi^M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge Q, E \mathrel{\dot\in} M.$$

Again we use a step more than E. Therefore, we set (cf. proof of lemma 35)

$$Z := \{\langle 2n, 2m\rangle : \langle n, m\rangle \in E\} \cup \{\langle 2n, 1\rangle : n \in field(E)\}$$

and define the predicates ($a \in field(\omega^Z)$)

$$\begin{aligned} Hier(a, X) &:= (\forall c \prec_{\omega^Z} a)\psi((X)_{\omega^Z c}, (X)_c), \\ H(a, b, X, Y) &:= Hier(a, X) \to [Hier(a +_{\omega^Z} b, Y) \wedge \\ &\qquad\qquad\qquad\qquad (\forall c \prec_{\omega^Z} a)(X)_c = (Y)_c], \end{aligned}$$

where we use the same notations for the wellordering $\prec_{\omega^Z}$ which is closed under the ordinal addition $+_Z$ and which has ω^Z as field. We use also again the notation $\ell[u]$ for a fundamental sequence of a limit number ℓ in $field(\omega^Z)$. We prove now by transfinite induction on the wellordering $\prec_{\omega^Z}$

$$(\forall b \in field(\omega^Z))(\forall a \in field(\omega^Z))(\forall X \mathrel{\dot\in} M)(\exists Y \mathrel{\dot\in} M)H(a, b, X, Y).$$

We distinguish three cases. Since we have φ^M and $Q, E \mathrel{\dot\in} M$ the cases $b = 0_{\omega^Z}$ and $b +_{\omega^Z} 1$ are proved with standard arguments. Therefore, we assume $b = \ell$. But in this case we can argue as in lemma 34 and 35 and we can prove - by the same arguments - the corresponding statements, which yield the claim. □

We collect the main result of this section in a theorem.

Theorem 47 ACA_0 *proves the equivalence of the following axioms schemes*

a) $(\Sigma^1_1\text{-}\mathsf{TDC})$,

b) $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$.

That there is a difference between the reflection on models of Σ^1_1-DC and on models of Σ^1_1-AC is stated in lemma 48. For its formulation we need the axioms scheme $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$

$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$ For all Π^1_2 formulas $\varphi[\vec{z},\vec{Z}]$:
$\varphi[\vec{z},\vec{Z}] \to (\exists M)[\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M \wedge \varphi^M]$.

Then we can prove the following equivalence:

Lemma 48 ACA_0 *proves the equivalence of the following axioms schemes*

a) $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$,

b) $(\mathsf{ATR}) + (\Sigma^1_1\text{-}\mathsf{DC})$.

Proof. First, we assume b) and prove a). In theorem VIII.5.12 [29] the equivalence of $(\Sigma^1_1$-$\mathsf{DC})$ and $(\Pi^1_2$-$\mathsf{RFN})$ is proved. Since we have $(\Sigma^1_1$-$\mathsf{DC})$ we also have $(\Pi^1_2$-$\mathsf{RFN})$. Let us write ψ for the following Π^1_2 formula

$$\psi := (\forall X)(\exists Y)(X \,\dot{\in}\, Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y).$$

Now choose a Π^1_2 formula $\varphi[\vec{z},\vec{Z}]$. We have to prove the existence of a model M of Σ^1_1-AC which contains $\vec{Z}$ and reflects φ. Since we have (ATR) we have ψ. Hence $\varphi \wedge \psi$. Since $\varphi \wedge \psi$ is equivalent to a Π^1_2 formula we obtain by an application of $(\Pi^1_2$-$\mathsf{RFN})$ a set M with

$$\vec{Z} \,\dot{\in}\, M \wedge \varphi^M \wedge \psi^M.$$

We can prove in ATR $(\Sigma^1_1$-$\mathsf{AC})$. Hence we conclude from ψ^M

$$(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M.$$

This is the claim. Now, we assume a) and prove b). It is trivial that $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$ implies $(\Pi^1_2$-$\mathsf{RFN})$, hence $(\Sigma^1_1$-$\mathsf{DC})$. Moreover, we conclude from $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$ immediately

$$(\forall X)(\exists Y)(X \,\dot{\in}\, Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y).$$

And this includes (ATR). $\square$

In [16] there is proved $|\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC}| = \varphi 1 \varepsilon_0 0$. Hence, the proof-theoretic strength of $\mathsf{ACA} + (\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}}$ is $\varphi 1 \varepsilon_0 0$ too. In contrast to $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ which has proof-theoretic ordinal $\varphi \varepsilon_0 0 0$. Finally, we establish the different strength of *weak* Σ^1_1-TDC and Σ^1_1-TDC.

($weak\ \Sigma^1_1$-TDC) For all Σ^1_1-formulas φ:

$$(\forall a)(\forall X)(\exists! Y)\varphi(X,Y,Z,a) \wedge WO(Z)$$
$$\rightarrow (\exists Y)[(Y)_{0_Z} = Q \wedge (\forall a)(0_Z Z a \wedge a \neq 0_Z$$
$$\rightarrow \varphi((Y)_{Za},(Y)_a,Z,a))].$$

Lemma 49 $\mathsf{ATR} + (\Sigma^1_1\text{-}\mathsf{IND})$ *proves* ($weak\ \Sigma^1_1$-TDC).

Proof. From lemma VIII.6.15 in [29] we conclude that we have (Σ^1_1-TI) too. Now we can construct directly the desired hierarchy using this transfinite induction scheme. □

2.5 Lower bound of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$

We are also interested in the proof-theoretic strength of Σ^1_1-TDC_0. Therefore, we give in this section a wellordering proof for the theory $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$. This yields the lower bound

$$\varphi\omega 00 \leq |(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}| = |\Sigma^1_1\text{-}\mathsf{TDC}_0|.$$

In [13] wellordering proofs for the theories $\widehat{ID}_\alpha$ are given. For our theory $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ we have only to iterate that procedure. For each n and each ordinal $\alpha < \varphi(n+1)00$ we will prove

$$\mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash (\forall X)TI(X,\alpha).$$

We use the same ordinal-theoretic facts as given in chapter 1. (For instance we write $\prec$ for the primitive recursive wellordering corresponding to the notation system of order type Φ_0.) Again we assign fundamental sequences $(\ell[n])_{n\geq 0}$ to each limit ordinal ℓ. We can assume $\ell[u] \prec \ell[u+1]$ and $0 \prec \ell[u]$ for all u. We choose $\ell^-[u]$ to denote the unique ordinal such that $\ell[u] + \ell^-[u] = \ell[u+1]$. We use in this section the following abbreviations:

$$Hier_{n,Q}(a,Y) :=$$
$$(\forall c \prec a)((Y)_{\prec c} \,\dot{\in}\, (Y)_c \wedge Q \,\dot{\in}\, (Y)_c \wedge \mathsf{I}_n((Y)_c)),$$

$$H_{n,Q}(a,b,X,Y) :=$$
$$Hier_{n,Q}(a,X) \rightarrow [Hier_{n,Q}(a+b,Y) \wedge (\forall c \prec a)(X)_c = (Y)_c],$$

$$I_{e,Y}(a) :=$$
$$(\forall d \prec e)(\forall X \,\dot{\in}\, (Y)_d)TI(X,a),$$

$$Main^n(a) :=$$
$$(\forall Q,Y)(\forall b,c)[\omega^{1+a} \uparrow c \wedge Hier_{n,Q}(c,Y) \wedge I_{c,Y}(b)$$
$$\rightarrow I_{c,Y}(\varphi(n+1)ab)].$$

We specify the steps of the wellordering proof in the following lemmas.

Lemma 50 ACA_0 *proves*

$$\begin{aligned}
&(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge Q \mathrel{\dot\in} M \rightarrow \\
&[(\forall u)(\forall X)(\exists Y)H_{n,Q}(a+\ell[u],\ell^-[u],X,Y) \\
&\rightarrow (\forall X)(\exists Y)[Hier_{n,Q}(a+\ell[0],X) \\
&\qquad\qquad \rightarrow Hier_{n,Q}(a+\ell,Y) \wedge (\forall c \prec a+\ell[0])(X)_c = (Y)_c]]^M.
\end{aligned}$$

Proof. This lemma corresponds to lemma 2 in [16]. The proof is therefore the same. □

Lemma 51 ACA_0 *proves*

$$\begin{aligned}
&(\forall X)TI(X,e) \wedge \mathsf{I}_{n+1}(M) \\
&\rightarrow ([(\forall b \prec e)(\forall a)(\forall Q)(\exists Y)H_{n,Q}(a,b,X,Y)]^M \\
&\qquad \wedge[(\forall b \prec e)(\forall Q)(\exists Y)Hier_{n,Q}(b,Y)]^M).
\end{aligned}$$

Proof. This lemma corresponds to lemma 3 in [16]. Therefore, the proof is again identical. □

We have mentioned that our wellordering proof is only an iteration of the wellordering proof for $\widehat{ID}_\alpha$. In the next lemma we collect the main results for the theory $\mathsf{I}_1\text{-}\mathsf{RFN}_0$. Roughly spoken, they correspond to the beginning of the iteration.

Lemma 52 ACA_0 *proves*

- **a)** $Hier_{0,Q}(\ell,Y) \wedge I_{\ell,Y}(b) \rightarrow I_{\ell,Y}(\varphi 0 b 0)$,
- **b)** $(Ax_{\mathsf{ACA}})^M \rightarrow (Prog(\lambda a.Main^0(a)))^M$.

Proof. The proof for a) can be extracted and adapted to the present context from [5]. b) corresponds to Main Lemma I in [13], the proof is essentially the same. □

The induction steps are given in the next lemmas.

Lemma 53 ACA_0 *proves*

$$\begin{aligned}
&(\forall Q,Y)[Hier_{n,Q}(\ell,Y) \rightarrow Prog(\lambda a.I_{\ell,Y}(\varphi(n+1)0a))] \wedge (Ax_{\mathsf{ACA}})^M \\
&\qquad\qquad \rightarrow (Prog(\lambda a.Main^n(a)))^M.
\end{aligned}$$

Proof. We give only a rough sketch, because the proof is again nearly the same as for Main Lemma I in [13]. Assume

$$(\forall Q, Y)[Hier_{n,Q}(\ell, Y) \to Prog(\lambda a.I_{\ell,Y}(\varphi(n+1)0a))] \tag{2.10}$$

and choose a set M with $(Ax_{\mathsf{ACA}})^M$. In order to prove

$$[(\forall b \prec a)Main^n(b) \to Main^n(a)]^M$$

we distinguish the three cases $a = 0$, $a + 1$, $a = \ell$. i.e., we have to prove

i) $(Main^n(0))^M$,

ii) $(Main^n(a) \to Main^n(a+1))^M$,

iii) $[(\forall u)(Main^n(\ell[u])) \to Main^n(\ell)]^M$.

ii) and iii) are proved in the same way as the corresponding cases in the proof of Main Lemma I in [13]. Therefore, we show only i). Assume $a = 0$. Choose b, c, Q, Y, M with

$$\omega \uparrow c \wedge Hier_{n,Q}(c, Y) \wedge I_{c,Y}(b) \wedge Q, Y \mathrel{\dot\in} M \wedge (Ax_{\mathsf{ACA}})^M.$$

There are c_0 and ℓ with $c = c_0 + \omega \cdot \ell$ and there is a fundamental sequence $\ell[u]$ with $\ell[u] > 0$ for all u. We set $c[u] := c_0 + \omega \cdot \ell[u]$ and have to show

$$(\forall X \mathrel{\dot\in} (Y)_{c[u]})TI(X, \varphi(n+1)0b).$$

With $c[u+1] \prec c$, $Lim(c[u+1])$ and (2.10) we conclude that

$$Prog(\lambda a.I_{c[u+1],Y}(\varphi(n+1)0a))$$

and with $I_{c,Y}(b)$ and $\{a : I_{c[u+1],Y}(\varphi(n+1)0a)\} \mathrel{\dot\in} (Y)_{c[u+2]}$ we get finally

$$I_{c[u+1],Y}(\varphi(n+1)0b).$$

In particular $(\forall X \mathrel{\dot\in} (Y)_{c[u]})TI(X, \varphi(n+1)0b)$. □

Lemma 54 ACA_0 *proves*

$$(\forall M)[(Ax_{\mathsf{ACA}})^M \to (Prog(\lambda a.Main^n(a)))^M] \wedge Hier_{n+1,Q}(\ell, Y) \wedge I_{\ell,Y}(a) \\ \to I_{\ell,Y}(\varphi(n+1)a0).$$

Proof. Assume

$$(\forall M)[(Ax_{\mathsf{ACA}})^M \to (Prog(\lambda a.Main^n(a)))^M] \wedge Hier_{n+1,Q}(\ell, Y) \wedge I_{\ell,Y}(a)$$

and choose d, X with $d \prec \ell$ and $X \dot{\in} (Y)_d$. We have to show

$$TI(X, \varphi(n+1)a0).$$

With $I_{\ell,Y}(a)$ we also have $I_{\ell,Y}(\omega^{1+a+1} \cdot \omega)$. Now we apply lemma 51. Lemma 51 is provable in $(Y)_{d+2}$ since $(Y)_{d+2}$ is a model of (ACA). We have also $[(\forall X)TI(X, \omega^{1+a+1} \cdot \omega)]^{(Y)_{d+2}}$ and $(Y)_{d+1} \dot{\in} (Y)_{d+2}$. We set in that lemma $M := (Y)_{d+1}$, $e := \omega^{1+a+1} \cdot \omega$ and get a hierarchy P in $(Y)_{d+1}$ with

$$Hier_{n,X}(\omega^{1+a} \cdot \omega, P).$$

We know $(Ax_{\mathsf{ACA}})^{(Y)_{d+1}}$ and conclude therefore

$$(Prog(\lambda c.Main^n(c)))^{(Y)_{d+1}}.$$

We use $I_{\ell,Y}(a)$ and $\{c : (Main^n(c))^{(Y)_{d+1}}\} \dot{\in} (Y)_{d+2}$ in order to conclude that

$$(Main^n(a))^{(Y)_{d+1}}.$$

In particular we have $I_{\omega^{1+a}\cdot\omega,P}(\varphi(n+1)a0)$ and therefore

$$TI(X, \varphi(n+1)a0).$$

This is the claim. $\square$

The iteration of the preceding lemmas leads to the following lemma:

Lemma 55 ACA_0 *proves*

- **a)** $Hier_{n+1,Q}(\ell, Y) \wedge I_{\ell,Y}(a) \rightarrow I_{\ell,Y}(\varphi(n+1)a0)$,
- **b)** $(Ax_{\mathsf{ACA}})^M \rightarrow (Prog(\lambda a.Main^n(a)))^M$.

Proof. We prove a) and b) simultaneously by metainduction on n. Assume $n = 0$. Then b) follows from lemma 52b). Then we can apply lemma 54 for $n = 0$ and get a). Now assume $n > 0$. The induction hypothesis is

$$\begin{array}{ll} Hier_{n,Q}(\ell, Y) \wedge I_{\ell,Y}(a) \rightarrow I_{\ell,Y}(\varphi na0) & \text{and} \\ (Ax_{\mathsf{ACA}})^M \rightarrow (Prog(\lambda a.Main^{n-1}(a)))^M & \end{array}$$

First we show

$$Hier_{n,Q}(\ell, Y) \rightarrow Prog\lambda a.I_{\ell,Y}(\varphi(n+1)0a).$$

We assume $Hier_{n,Q}(\ell, Y)$ and have to show

$$(\forall b \prec a)I_{\ell,Y}(\varphi(n+1)0b) \rightarrow I_{\ell,Y}(\varphi(n+1)0a).$$

We distinguish three cases: $a = 0$, $a + 1$ and $Lim(a)$.

Case $a = 0$: We have to show $I_{\ell,Y}(\varphi(n+1)00)$. For this we set $z_0 := 0$ and $z_{k+1} := \varphi n z_k 0$. An easy induction on k yields $(\forall k) I_{\ell,Y}(z_k)$, the claim.
Case $a + 1$: We can assume $I_{\ell,Y}(\varphi(n+1)0a)$ and have to show $I_{\ell,Y}(\varphi(n+1)0(a+1))$. For this we set $z_0 = \varphi(n+1)0a + 1$ and $z_{k+1} := \varphi n z_k 0$. Again we can prove $(\forall k) I_{\ell,Y}(z_k)$, the claim.
Case $Lim(a)$: We can assume $(\forall b \prec a) I_{\ell,Y}(\varphi(n+1)0b)$ and have to show $I_{\ell,Y}(\varphi(n+1)0a)$. But a is a limit number and therefore the claim follows immediately.

Now we can apply lemma 53 and get b). And with lemma 54 we get a) too. □

We formulate the main theorem of this section.

Theorem 56 *Define the sequence $(\gamma_k^n)_{k\geq 0}$ as follows: $\gamma_0^n := 0$ and $\gamma_{k+1}^n := \varphi n \gamma_k^n 0$. Then we have for all k*

$$\mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash (\forall X) TI(X, \gamma_k^n).$$

Proof. We distinguish the two cases $n = 0$ and $n > 0$. If $n = 0$, then we can prove the claim with standard arguments of predicative proof theory (cf. for instance [5]). If $n > 0$, we prove the claim by metainduction on k. The case $k = 0$ is trivial. Therefore, assume $k > 0$ and $(\forall X) TI(X, \gamma_k^n)$. Choose a set X. Then we have to show $TI(X, \varphi n \gamma_k^n 0)$.

We have not only $(\forall X) TI(X, \gamma_k^n)$ but also $(\forall X) TI(X, \omega^{1+\gamma_k^n+1} \cdot \omega)$. Choose M with $X \dot{\in} M$ and $\mathsf{I}_n(M)$. Then we get with lemma 51 a hierarchy P in M with

$$Hier_{n-1,X}(\omega^{1+\gamma_k^n} \cdot \omega, P).$$

Then lemma 55b) yields

$$(Prog(\lambda a.Main^{n-1}(a)))^M.$$

We conclude from the definition of $Main^{n-1}(a)$

$$I_{\gamma_k^n,P}(\varphi n \gamma_k^n 0).$$

Thus the claim. □

In the following corollary we collect the proof-theoretic lower bounds.

Corollary 57 *We have*

a) $\varphi(n+1)00 \leq |\mathsf{I}_n\text{-}\mathsf{RFN}_0|$.

b) $\varphi\omega 00 \leq |(\Pi_2^1\text{-}\mathsf{RFN})_0^{\Sigma_1^1\text{-}\mathsf{DC}}|$.

Proof. Together with theorem 56 and $\varphi(n+1)00 = \sup_{k\geq 0} \gamma_k^n$ we conclude a). For b) we notice that an easy metainduction on n shows

$$(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash (\forall X)(\exists Y)(X \mathrel{\dot{\in}} Y \wedge \mathsf{I}_n(Y)).$$

$\square$

In chapter 4 we will show that these lower bounds are sharp.

Chapter 3

Semi-formal systems

3.1 The semi-formal systems T^n_α and E^n_α

In this section we introduce for each $n \in \mathbb{N}$ and each ordinal $\alpha \in \Phi_0$ semi-formal systems T^n_α and E^n_α. Essentially T^n_α and E^n_α axiomatisize the statement "There is a hierarchy D with $\mathsf{I}_n Hier(\prec\restriction\alpha, \emptyset, D)$". Later on we will use T^n_α and E^n_α in order to determine upper bounds for $\mathsf{MUT}^=$ and $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$.

3.1.1 Definition of the theories $\bar{\mathsf{T}}^n_\alpha$

We first define theories $\bar{\mathsf{T}}^n_\alpha$. The semi-formal systems T^n_α and E^n_α will be Tait-style versions of $\bar{\mathsf{T}}^n_\alpha$. $\bar{\mathsf{T}}^n_\alpha$ is formulated in the language $\mathcal{L}_2(\mathsf{D}^n)$. $\mathcal{L}_2(\mathsf{D}^n)$ is the extension of $\mathcal{L}_2$ by the new unary relation symbol D^n ($n \in \mathbb{N}$). The *set terms* of $\mathcal{L}_2(\mathsf{D}^n)$ are the set variables. *Formulas* are defined as usual.

As in the wellordering proofs we write $\prec$ for the primitive recursive wellordering corresponding to the notation system of order type Φ_0. And often we again simplify notation by using directly ordinals $\alpha, \beta, \gamma, \dots$ instead of their codes. Moreover we write $t \in \mathsf{D}^n$ for $\mathsf{D}^n(t)$. Corresponding to this noatation we write also $t \in \mathsf{D}^n_b$ for $\mathsf{D}^n(\langle t, b\rangle)$, $t \in \mathsf{D}^n_{\prec b}$ for $t = \langle (t)_0, (t)_1\rangle \wedge (t)_1 \prec b \wedge \mathsf{D}^n(t)$. Analogously we use $X \dot{\in} \mathsf{D}^n_b$, $\mathsf{D}^n_b \dot{\in} X$,.... Furthermore we will write simply $\pi^0_1[e, x_1, \dots, x_n, X_1, \dots, X_m]$ for the universal Π^0_1-predicate $\pi^0_{1,n,m}[e, x_1, \dots, x_n, X_1, \dots, X_m]$. We mention once more that there is an index e such that $\mathsf{Q}(x) \leftrightarrow \pi^0_1[e, x, \vec{z}, \vec{X}]$.

We now define the theories $\bar{\mathsf{T}}^n_\alpha$. Choose $\alpha \in \Phi_0$ and $n \in \mathbb{N}\backslash\{0\}$. $\bar{\mathsf{T}}^n_\alpha$ is formulated in the language $\mathcal{L}_2(\mathsf{D}^n)$ and is based on the usual axioms and rules for the two-sorted predicate calculus. The non logical axioms comprise:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) *closure conditions for* D^n_b *($b \prec \alpha$).*

(2.1) $$b \prec \alpha \wedge Y, Z \mathrel{\dot\in} \mathsf{D}^n_b \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^n_b) X = Y \oplus Z.$$

(2.2) $$b \prec \alpha \wedge Z \mathrel{\dot\in} \mathsf{D}^n_b \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^n_b)(\forall x)(x \in X \leftrightarrow \pi^0_1[e, x, z, Z]).$$

(2.3) $$\begin{aligned} & b \prec \alpha \wedge U, Z \mathrel{\dot\in} \mathsf{D}^n_b \wedge (\forall X \mathrel{\dot\in} \mathsf{D}^n_b)(\exists Y \mathrel{\dot\in} \mathsf{D}^n_b)\pi^0_1[e, z, X, Y, Z] \\ & \quad \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^n_b)[(X)_0 = U \wedge (\forall u)\pi^0_1[e, z, (X)_u, (X)_{u+1}, Z]]. \end{aligned}$$

(3) *hierarchy and reflection properties.*

(3.1) $$b \prec \alpha \rightarrow \mathsf{D}^n_{\prec b} \mathrel{\dot\in} \mathsf{D}^n_b.$$

(3.2) $$b \prec \alpha \rightarrow (\forall X \mathrel{\dot\in} \mathsf{D}^n_b)(\exists Y \mathrel{\dot\in} \mathsf{D}^n_b)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_{n-1}(Y)).$$

(4) *set-induction.*

$$(0 \in X \wedge (\forall x)(x \in X \rightarrow x + 1 \in X)) \rightarrow (\forall x)(x \in X).$$

For $n = 0$ we define $\bar{\mathsf{T}}^0_\alpha$ as the theory formulated in the language $\mathcal{L}_2(\mathsf{D}^0)$ and based on the usual axioms and rules for the two-sorted predicate calculus. The non logical axioms comprise:

(1) as for $\bar{\mathsf{T}}^1_\alpha$.

(2) *closure conditions for* D^0_b *($b \prec \alpha$).*

(2.1) $$b \prec a \wedge Y, Z \mathrel{\dot\in} \mathsf{D}^0_b \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_b) X = Y \oplus Z.$$

(2.2) $$b \prec \alpha \wedge Z \mathrel{\dot\in} \mathsf{D}^0_b \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_b)(\forall x)(x \in X \leftrightarrow \pi^0_1[e, x, z, Z]).$$

(2.3) $$\begin{aligned} & b \prec \alpha \wedge Z \mathrel{\dot\in} \mathsf{D}^0_b \wedge (\forall x)(\exists X \mathrel{\dot\in} \mathsf{D}^0_b)\pi^0_1[e, x, z, X, Z] \\ & \quad \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_b)(\forall x)\pi^0_1[e, x, z, (X)_x, Z]. \end{aligned}$$

(3) *hierarchy properties.*

$$b \prec \alpha \rightarrow \mathsf{D}^0_{\prec b} \mathrel{\dot\in} \mathsf{D}^0_b.$$

(4) *set-induction.*

$$(0 \in X \wedge (\forall x)(x \in X \rightarrow x + 1 \in X)) \rightarrow (\forall x)(x \in X).$$

3.1.2 The semi-formal systems T_α^n

The semi-formal system T_α^n corresponds to the theory $\bar{\mathsf{T}}_\alpha^n$. T_α^n is formulated with bounded second order quantifiers $\exists X \mathrel{\dot\in} \mathsf{D}_\beta^n$ and $\forall X \mathrel{\dot\in} \mathsf{D}_\beta^n$ $\beta < \alpha$. We formulate $(\exists X)(X \mathrel{\dot\in} \mathsf{D}_\beta^n \wedge \varphi(X))$ as $(\exists X \mathrel{\dot\in} \mathsf{D}_\beta^n)\varphi(X)$ where $(\exists X \mathrel{\dot\in} \mathsf{D}_\beta^n)$ is a bounded quantifier. Of course we have the same for the dual formula $(\forall X)(X \mathrel{\dot\in} \mathsf{D}_\beta^n \rightarrow \varphi(X))$. Notice that in $\bar{\mathsf{T}}_\alpha^n$ we have used $(\exists X \mathrel{\dot\in} \mathsf{D}_\beta^n)\varphi(X)$ as an abbreviation for $(\exists X)(X \mathrel{\dot\in} \mathsf{D}_\beta^n \wedge \varphi(X))$. In T_α^n, $(\exists X \mathrel{\dot\in} \mathsf{D}_\beta^n)\varphi(X)$ is in fact a formula and not an abbreviation.

T_α^n is based on the language $\mathcal{L}_\alpha^n$. $\mathcal{L}_\alpha^n$ is the extension of $\mathcal{L}_2$ by new unary relation symbols D_β^n for each $\beta < \alpha$ and new unary relation symbols $\mathsf{D}_{<\gamma}^n$ for each $\gamma \leq \alpha$. The *set terms* of $\mathcal{L}_\alpha^n$ are the set variables. The $\mathcal{L}_\alpha^n$ *formulas* are the $\mathcal{L}_1$ literals and all formulas $t \in X$, $t \notin X$, $\mathsf{D}_\beta^n(t)$, $\neg\mathsf{D}_\beta^n(t)$, $\mathsf{D}_{<\gamma}^n(t)$, $\neg\mathsf{D}_{<\gamma}^n(t)$ for each set variable X, all number terms t and all ordinals $\beta < \alpha$, $\gamma \leq \alpha$. Furthermore, the class of $\mathcal{L}_\alpha^n$ formulas is closed under $\wedge, \vee, \forall x, \exists x, \exists X \mathrel{\dot\in} \mathsf{D}_\beta^n, \forall X \mathrel{\dot\in} \mathsf{D}_\beta^n, \exists X, \forall X$ for each $\beta < \alpha$. The exact meaning of the bounded second order quantifiers will be given in the definition of T_α^n. Again we write for instance $t \in \mathsf{D}_\beta^n$ for $\mathsf{D}_\beta^n(t)$, $t \in \mathsf{D}_{<\beta}^n$ for $\mathsf{D}_{<\beta}^n(t)$ etc. We take as $\mathcal{L}_\alpha^n$ *formulas of* T_α^n the $\mathcal{L}_\alpha^n$ formulas without free number variables.

We let $\Gamma, \Lambda, \ldots$ range over finite sets of $\mathcal{L}_\alpha^n$ formulas; we often write (for instance) Γ, φ for the union of Γ and $\{\varphi\}$. We first introduce the Tait-calculus T_α^n. It is an extension of the classical Tait-calculus [28] by the non logical axioms of $\bar{\mathsf{T}}_\alpha^n$. It contains the following axioms and rules of inference:

1. Ontological axioms I. For all finite sets Γ of $\mathcal{L}_\alpha^n$ formulas of T_α^n, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$, all set variables X and all $\beta < \alpha, \gamma \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X, s \notin X \qquad \text{and} \qquad \Gamma, \mathsf{Q}(t), \neg\mathsf{Q}(s)$$

$$\text{and} \qquad \Gamma, t \in \mathsf{D}_\beta^n, s \notin \mathsf{D}_\beta^n \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}_{<\gamma}^n, s \notin \mathsf{D}_{<\gamma}^n.$$

2. Propositional rules. For all finite sets Γ of $\mathcal{L}_\alpha^n$ formulas of T_α^n and all $\mathcal{L}_\alpha^n$ formulas φ and ψ of T_α^n:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

3. Quantifier rules. For all finite sets Γ of $\mathcal{L}_\alpha^n$ formulas of T_α^n, all $\beta < \alpha$, all $\mathcal{L}_\alpha^n$ formulas

φ and ψ of T^n_α, all closed number terms s, all set variables Y:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)},$$

$$\frac{\Gamma, \psi(Y)}{\Gamma, (\exists X)\psi(X)}, \qquad \frac{\Gamma, \psi(Y)}{\Gamma, (\forall X)\psi(X)} \quad (vc),$$

$$\frac{\Gamma, Y \mathrel{\dot\in} \mathsf{D}^n_\beta \land \psi(Y)}{\Gamma, (\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)}, \qquad \frac{\Gamma, Y \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi(Y)}{\Gamma, (\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)} \quad (vc),$$

We remarked rules with (vc) if they have to respect the usual variable conditions. That is, Y does not occur in Γ and does not occur in $\psi(X)$.

4. Ontological axioms II. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta \leq \alpha$, all closed terms s so that $Seq_2 s$ is false, all closed terms t such that $Seq_2 t$, $Seq_2 (t)_0$ and $\beta \preceq (t)_1$ is true:

$$\Gamma, s \notin \mathsf{D}^n_{<\beta} \qquad \text{and} \qquad \Gamma, t \notin \mathsf{D}^n_{<\beta}.$$

5. Ontological rules III. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta \leq \alpha$, $\gamma < \beta$, all closed terms t so that $Seq_2 t$ and $(t)_1 = \gamma$ is true:

$$\frac{\Gamma, (t)_0 \in \mathsf{D}^n_\gamma}{\Gamma, t \in \mathsf{D}^n_{<\beta}}, \qquad \frac{\Gamma, (t)_0 \notin \mathsf{D}^n_\gamma}{\Gamma, t \notin \mathsf{D}^n_{<\beta}}.$$

6. Closure axioms. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all closed number terms e, r, all set variables U, V and all $\beta < \alpha$:

$$\Gamma, (U, V \mathrel{\dot\notin} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)(X = U \oplus V),$$
$$\Gamma, (U \mathrel{\dot\notin} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)(\forall x)(x \in X \leftrightarrow \pi^0_1[e, x, r, U, \mathsf{D}^n_{<\beta}]).$$

7. Closure rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all closed number terms e, r, all $\beta < \alpha$, all set variables U, V and if $n = 0$:

$$\frac{\Gamma, (U \mathrel{\dot\notin} \mathsf{D}^0_\beta), (\forall x)(\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)\pi^0_1[e, x, r, X, U, \mathsf{D}^0_{<\beta}]}{\Gamma, (U \mathrel{\dot\notin} \mathsf{D}^0_\beta), (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(\forall x)\pi^0_1[e, x, r, (X)_x, U, \mathsf{D}^0_{<\beta}]},$$

and if $n > 0$:

$$\frac{\Gamma, (U, V \mathrel{\dot\notin} \mathsf{D}^n_\beta), (\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)(\exists Y \mathrel{\dot\in} \mathsf{D}^n_\beta)\pi^0_1[e, r, X, Y, V, \mathsf{D}^n_{<\beta}]}{\Gamma, (U, V \mathrel{\dot\notin} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)[(X)_0 = U \land (\forall u)\pi^0_1[e, r, (X)_u, (X)_{u+1}, V, \mathsf{D}^n_{<\beta}]]}.$$

8. Reflection axioms. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta < \alpha$, all set variables U and if $n > 0$:

$$\Gamma, U \mathrel{\dot\notin} \mathsf{D}^n_\beta, (\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)(U \mathrel{\dot\in} X \land \mathsf{I}_{n-1}(X)).$$

9. Cut rules. For all finite sets Γ of $\mathcal{L}_\alpha^n$ formulas of T_α^n and for all $\mathcal{L}_\alpha^n$ formulas φ of T_α^n:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

Here, in the reflection axioms, we have used the predicate I_n. We have defined this predicate in chapter 2. There we have used second order quantifiers, e.g.

$$\mathsf{I}_{n+2}(M) := (Ax_{\Sigma_1^1\text{-}\mathsf{DC}})^M \wedge (\forall X \mathrel{\dot\in} M)(\exists X \mathrel{\dot\in} M)(X \mathrel{\dot\in} Y \wedge \mathsf{I}_{n+1}(Y)),$$

and we have introduced $\forall X \mathrel{\dot\in} M$ as an abbreviation for $\forall X(X \mathrel{\dot\in} M \to \ldots)$. Hence, there are second order quantifiers in this definition. In order to avoid these second order quantifiers, we take a first order reformulation of I_n. We replace all $(\forall X \mathrel{\dot\in} M)\varphi(X)$ (resp. $(\exists X \mathrel{\dot\in} M)\varphi(X)$) by $(\forall k)\varphi((M)_k)$ (resp. $(\exists k)\varphi((M)_k)$) and let again I_n denote this first order reformulation (of the old predicate I_n).

In order to prove a partial cut elimination, we have to introduce a cut rank. Choose an $\mathcal{L}_\alpha^n$ formula φ of T_α^n. We set $rk(\varphi) = 0$ iff in φ there are no unbounded second order quantifiers $\exists X, \forall X$. Otherwise we set

1. If φ is a formula $\psi \wedge \theta$ or $\psi \vee \theta$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.
2. If φ is a formula $\exists x\psi$, $\forall x\psi$, $\exists X\psi$ or $\forall X\psi$, then $rk(\varphi) := rk(\psi) + 1$.
3. If φ is a formula $(\exists X \mathrel{\dot\in} \mathsf{D}_\gamma^n)\psi$ or $(\forall X \mathrel{\dot\in} \mathsf{D}_\gamma^n)\psi$, then $rk(\varphi) := rk(\psi) + 2$ $(\gamma < \alpha)$.

The notion $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\beta}{m}} \Gamma$ is used to express that Γ is provable in T_α^n by a proof of depth less than or equal to β and so that all its cut formulas have ranks less than m. We write $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{<\beta}{<m}} \Gamma$ if there exists a $\gamma < \beta$ and a $k < m$ with $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\gamma}{k}} \Gamma$. We write $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{<\beta}{<\omega}} \Gamma$ if there exists a $\gamma < \beta$ and a k with $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\gamma}{k}} \Gamma$. Finally we write $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\beta}{<\omega}} \Gamma$ if there exists a k with $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\beta}{k}} \Gamma$. All these definitions lead to the following partial cut elimination. The proof is standard and hence omitted. We set $\omega_0(\gamma) := \gamma$ and $\omega_{k+1}(\gamma) := \omega^{\omega_k(\gamma)}$.

Lemma 58 $\mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\gamma}{k+1}} \Gamma \implies \mathsf{T}_\alpha^n \mathrel{\big|\!\frac{\omega_k(\gamma)}{1}} \Gamma$.

In the following we need some embeddings. We discuss here the embedding of $\bar{\mathsf{T}}_\alpha^n$ into T_α^n. This embedding is obtained by interpreting D^n by $\mathsf{D}_{<\alpha}^n$, thus establishing a translation $(\ldots)^\alpha$ of $\mathcal{L}_2(\mathsf{D}^n)$ into $\mathcal{L}_\alpha^n$. The embedding of $\bar{\mathsf{T}}_\alpha^n$ into T_α^n can now be given straightforwardly. Since the proof is standard, we omit it.

Lemma 59 *We have for all $\mathcal{L}_2(\mathsf{D}^n)$ sentences φ*

$$\bar{\mathsf{T}}_\alpha^n \vdash \varphi \implies \mathsf{T}_\alpha^n \mathrel{\big|\!\frac{<\omega+\omega}{<\omega}} \varphi^\alpha.$$

3.1.3 The semi-formal systems E^n_α

We introduce the systems E^n_α; they are first order reformulations of T^n_α. We formulate E^n_α in the first order part of $\mathcal{L}^n_\alpha$. The *formulas of* E^n_α are the formulas of T^n_α in which no set variables occur. We now give the definition of the Tait-calculus E^n_α.

1. Ontological axioms I. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all $\beta < \alpha, \gamma \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, \mathsf{Q}(t), \neg\mathsf{Q}(t)$$

$$\text{and} \qquad \Gamma, t \in \mathsf{D}^n_\beta, s \notin \mathsf{D}^n_\beta \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^n_{<\gamma}, s \notin \mathsf{D}^n_{<\gamma}.$$

2. Propositional rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α and all $\mathcal{L}^n_\alpha$ formulas φ and ψ of E^n_α:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

3. Quantifier rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta < \alpha$, all closed number terms s and all $\mathcal{L}^n_\alpha$ formulas φ and ψ of E^n_α:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)}.$$

4. Ontological axioms II. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta \leq \alpha$, all closed terms s so that $Seq_2 s$ is false, all closed terms t such that $Seq_2 t$, $Seq_2(t)_0$ and $\beta \preceq (t)_1$ is true:

$$\Gamma, s \notin \mathsf{D}^n_{<\beta} \qquad \text{and} \qquad \Gamma, t \notin \mathsf{D}^n_{<\beta}.$$

5. Ontological rules III. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta \leq \alpha$, $\gamma < \beta$, all closed terms t so that $Seq_2 t$ and $(t)_1 = \gamma$ is true:

$$\frac{\Gamma, (t)_0 \in \mathsf{D}^n_\gamma}{\Gamma, t \in \mathsf{D}^n_{<\beta}}, \qquad \frac{\Gamma, (t)_0 \notin \mathsf{D}^n_\gamma}{\Gamma, t \notin \mathsf{D}^n_{<\beta}}.$$

6. Closure axioms. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms e, r, s, t and all $\beta < \alpha$:

$$\Gamma, (\exists k)(\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_t \oplus (\mathsf{D}^n_\beta)_s,$$
$$\Gamma, (\exists k)(\forall x)(x \in (\mathsf{D}^n_\beta)_k \leftrightarrow \pi^0_1[e, x, r, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]).$$

7. Closure rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms e, r, s, t, all $\beta < \alpha$ and if $n = 0$:

$$\frac{\Gamma, (\forall x)(\exists k)\pi^0_1[e, x, r, (\mathsf{D}^0_\beta)_k, (\mathsf{D}^0_\beta)_t, \mathsf{D}^0_{<\beta}]}{\Gamma, (\exists k)(\forall x)\pi^0_1[e, x, r, ((\mathsf{D}^0_\beta)_k)_x, (\mathsf{D}^0_\beta)_t, \mathsf{D}^0_{<\beta}]},$$

and if $n > 0$:

$$\frac{\Gamma, (\forall k)(\exists l)\pi_1^0[e, r, (\mathsf{D}^n_\beta)_k, (\mathsf{D}^n_\beta)_l, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]}{\Gamma, (\exists k)[((\mathsf{D}^n_\beta)_k)_0 = (\mathsf{D}^n_\beta)_s \wedge (\forall u)\pi_1^0[e, r, ((\mathsf{D}^n_\beta)_k)_u, ((\mathsf{D}^n_\beta)_k)_{u+1}, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]]}.$$

8. Reflection axioms. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms t, all $\beta < \alpha$ and if $n > 0$:

$$\Gamma, (\exists k)((\mathsf{D}^n_\beta)_t \mathrel{\dot\in} (\mathsf{D}^n_\beta)_k \wedge \mathsf{I}_{n-1}((\mathsf{D}^n_\beta)_k)).$$

9. Cut rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α and for all $\mathcal{L}^n_\alpha$ formulas φ of E^n_α:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In a next step we give a partial cut elimination for E^n_α. The situation here is more complicated than for T^n_α. We have in E^n_α, for instance, that the formula $(\exists k)\varphi((\mathsf{D}^n_\beta)_k)$ corresponds to $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)\varphi(X)$. The problem is that we want to characterize formulas $(\exists k)\varphi(k)$ with subformulas of type $\langle s, k\rangle \in \mathsf{D}^n_\beta$ ($k \notin FV(s)$) but not with, e.g, a subformula of type $k \in \mathsf{D}^n_\beta$ or $k = 0$. In order to define such an appropriate class of formulas we introduce (nominal) symbols $*_i$ ($i \in \mathbb{N}$) which are different from all symbols in $\mathcal{L}^n_\alpha$. We now define the classes ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$.

Definition 60 We fix an $\alpha \in \Phi_0$, a $\beta < \alpha$ and an $n \in \mathbb{N}$. The classes ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ are inductively defined as follows:

1. For all number terms $\vec{s}, t$ of $\mathcal{L}_1$, all $\gamma < \beta$, all primitive recursive relation symbols K of $\mathcal{L}_1$ and all $*_i$ the following expressions are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$: $K\vec{s}$, $\neg K\vec{s}$, $\mathsf{Q}(t)$, $\neg\mathsf{Q}(t)$, $t \in \mathsf{D}^n_\gamma$, $t \notin \mathsf{D}^n_\gamma$, $t \in \mathsf{D}^n_{<\gamma}$, $t \notin \mathsf{D}^n_{<\gamma}$, $\langle t, *_i\rangle \in \mathsf{D}^n_\beta$, $\langle t, *_i\rangle \notin \mathsf{D}^n_\beta$, $t \in \mathsf{D}^n_{<\beta}$, $t \notin \mathsf{D}^n_{<\beta}$. (We write $\langle t, *_i\rangle \in \mathsf{D}^n_\beta$ for $\mathsf{D}^n_\beta(\langle t, *_i\rangle)$.)

2. If φ, ψ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$), then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$).

3. If φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$), then $\exists x\varphi$ and $\forall x\varphi$ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$).

4. If $\varphi(*_i)$ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$), then $\exists x\varphi[*_i\backslash x]$ (resp. $\forall x\varphi[*_i\backslash x]$) is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$). Here we write $\varphi[*_i\backslash x]$ for the expression φ where all occurrences of $*_i$ are substituted by x.

There is one point worth mentioning. If φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ or in ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ and of the form $t \in \mathsf{D}^n_\gamma$, then γ is strict less than β. And if φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ or in ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ and of the form $\langle t, *_i\rangle \in \mathsf{D}^n_\gamma$, then γ is (syntactically) equal to β.

Further we define that the class $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)^c$ (resp. $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)^c$) is the subset of $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)$) which contain all expressions in which there are no free number variables. For a given φ in $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)^c$ or in $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)^c$ and for $\vec{*} = *_1, \ldots, *_k$ we write $\varphi[\vec{*}]$ if all $*_i$ occurring in φ are among $*_1, \ldots, *_k$. Often we write only $\varphi[\vec{t}]$ for $\varphi[\vec{*}][\vec{*}\backslash\vec{t}]$. Notice that $\varphi[\vec{t}]$ is an $\mathcal{L}^n_\alpha$ formula of E^n_α. Analogously we write $\Gamma[\vec{*}]$ if all $*_i$ occurring in a φ in Γ are listed in $\vec{*}$ and if Γ is a finite subset of $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)^c$. And again we write $\Gamma[\vec{t}]$ for $\Gamma[\vec{*}][\vec{*}\backslash\vec{t}]$.

We can now define the rank $rk(\varphi)$ of a $\mathcal{L}^n_\alpha$ formula φ of E^n_α. We set $rk(\varphi) = 0$ iff there is a $\vec{t}$ and a $\psi[\vec{*}]$ in $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)^c$ or $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)^c$ for a $\beta < \alpha$ such that $\varphi \equiv \psi[\vec{t}]$. Otherwise we set

1. If φ is a formula $t \in \mathsf{D}^n_{<\alpha}$, $t \notin \mathsf{D}^n_{<\alpha}$, $t \in \mathsf{D}^n_\beta$ or a formula $t \notin \mathsf{D}^n_\beta$ ($\beta < \alpha$), then $rk(\varphi) := 1$.

2. If φ is a formula $\psi \wedge \theta$ or $\psi \vee \theta$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.

3. If φ is a formula $\exists x\psi$ or $\forall x\psi$, then $rk(\varphi) := rk(\psi) + 1$.

Concerning clause 1 of this rank definition of E^n_α, we give some explanations. First, assume that α is a limit number. Then each $t \in \mathsf{D}^n_\beta$ with $\beta < \alpha$ has rank 0, since $t \in \mathsf{D}^n_\beta$ is an element of $ess\text{-}\Sigma^1_1(\mathsf{D}^n_{\beta+1})^c$ and $\beta + 1 < \alpha$. $t \in \mathsf{D}^n_{<\alpha}$ has rank 1 for each term t. Secondly, we assume that α is a successor ordinal. We write $\alpha - 1$ for the predecessor of α. Then each $t \in \mathsf{D}^n_\beta$ with $\beta < \alpha - 1$ has rank 0 and $\langle r, s\rangle \in \mathsf{D}^n_{\alpha-1}$ has rank 0 too. If t is a term different of all terms $\langle r, s\rangle$, then $t \in \mathsf{D}^n_{\alpha-1}$ has rank 1. Again the rank of $t \in \mathsf{D}^n_{<\alpha}$ is 1 and the rank of $t \in \mathsf{D}^n_{<\beta}$ is 0 for $\beta < \alpha$.

The notion $\mathsf{E}^n_\alpha \overset{\beta}{\underset{m}{\vdash}} \Gamma$ is defined as for T^n_α but now with the above cut ranks. Again we omit the proof of the following lemma.

Lemma 61 $\mathsf{E}^n_\alpha \overset{\gamma}{\underset{k+1}{\vdash}} \Gamma \implies \mathsf{E}^n_\alpha \overset{\omega_k(\gamma)}{\underset{1}{\vdash}} \Gamma$.

In a next step we embed $\mathsf{T}^n_{\alpha+1}$ into $\mathsf{E}^n_{\alpha+1}$. In order to achieve this, we inductively define for each $\mathcal{L}^n_{\alpha+1}$ formula φ of $\mathsf{T}^n_{\alpha+1}$ a $\mathcal{L}^n_{\alpha+1}$ formula φ^* of $\mathsf{E}^n_{\alpha+1}$. If in φ there is no occurence of $\forall X \dot\in \mathsf{D}^n_\beta$ and of $\exists X \dot\in \mathsf{D}^n_\beta$ for all $\beta < \alpha + 1$, then we set $\varphi^* := \varphi$. Otherwise we set

1. If φ is of the form $\theta \vee \psi$ ($\theta \wedge \psi$ respectively), then we set $\varphi^* := \theta^* \vee \psi^*$ ($\theta^* \wedge \psi^*$ respectively).

2. If φ is of the form $\exists x\psi$ ($\forall x\psi$, $\exists X\psi$, $\forall X\psi$ respectively), then we set $\varphi^* := \exists x\psi^*$ ($\forall x\psi^*$, $\exists X\psi^*$, $\forall X\psi^*$ respectively).

3. If φ is of the form $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)$ $((\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)$ respectively), then we set $\varphi^* := (\exists k)\psi^*((\mathsf{D}^n_\beta)_k)$ $((\forall k)\psi^*((\mathsf{D}^n_\beta)_k)$ respectively) for $\beta < \alpha$.

This translation leads to the following embedding. For $\vec{t} = t_1, \dots, t_n$ we write $(\mathsf{D}^n_\alpha)_{\vec{t}}$ for $(\mathsf{D}^n_\alpha)_{t_1}, \dots, (\mathsf{D}^n_\alpha)_{t_n}$. $\Gamma[(\mathsf{D}^n_\alpha)_{\vec{t}}]$ is a shorthand for $\Gamma[\vec{X}][\vec{X}\backslash(\mathsf{D}^n_\alpha)_{\vec{t}}]$.

Lemma 62 *Assume that Γ is a set of $\mathsf{T}^n_{\alpha+1}$ formulas without occurences of unbounded set quantifiers $\exists X, \forall X$. Then we have for all closed number terms $\vec{t}$*

$$\mathsf{T}^n_{\alpha+1} \vdash^{\gamma}_{1} \Gamma[\vec{X}] \quad \Longrightarrow \quad \mathsf{E}^n_{\alpha+1} \vdash^{\omega^{\omega^\gamma}}_{<\omega} \Gamma^*[(\mathsf{D}^n_\alpha)_{\vec{t}}].$$

Proof. The proof is by induction on γ. If Γ is an ontological axiom I or II, the claim is immediate. If Γ is the conclusion of a propositional rule, of an ontological rule III or of a cut rule, the claim follows immediately from the induction hypothesis. We now discuss the quantifier rules. By assumption we do not have to deal with the $(\exists X)$- and $(\forall X)$-rule. The $(\exists x)$- and $(\forall x)$-rule follows immediately from the induction hypothesis. There remain the cases of the bounded second order quantifiers. First we discuss the $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule. We assume that $\Gamma[\vec{X}]$ is the conclusion of the $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule $(\beta \leq \alpha)$. Then there exists a $\gamma_0 < \gamma$ and a set variable Z with

$$\mathsf{T}^n_{\alpha+1} \vdash^{\gamma_0}_{1} \Gamma[\vec{X}], Z \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi(Z) \tag{3.1}$$

The induction hypothesis yields

$$\mathsf{E}^n_{\alpha+1} \vdash^{\omega^{\omega^{\gamma_0}}}_{<\omega} \Gamma^*[(\mathsf{D}^n_\alpha)_{\vec{t}}], (\mathsf{D}^n_\alpha)_r \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_\alpha)_r)$$

for all closed number terms $r, \vec{t}$ such that $X_i \equiv Z$ implies $t_i = r$. An application of the $(\exists x)$-rule leads to

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega^{\omega^\gamma}}_{<\omega} \Gamma^*[(\mathsf{D}^n_\alpha)_{\vec{t}}], (\exists k)((\mathsf{D}^n_\alpha)_k \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_\alpha)_k)).$$

We now prove

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} \neg(\exists k)((\mathsf{D}^n_\alpha)_k \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_\alpha)_k)), (\exists k)\psi^*((\mathsf{D}^n_\beta)_k). \tag{3.2}$$

Then a cut implies the claim. Notice that we have

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} (\mathsf{D}^n_\alpha)_t \neq (\mathsf{D}^n_\beta)_r, \neg\psi^*((\mathsf{D}^n_\alpha)_t), \psi^*((\mathsf{D}^n_\beta)_r)$$

for all closed terms t, r. Then we conclude for all closed terms t that

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} (\mathsf{D}^n_\alpha)_t \mathrel{\dot\in} \mathsf{D}^n_\beta \rightarrow \neg\psi^*((\mathsf{D}^n_\alpha)_t), (\exists k)\psi^*((\mathsf{D}^n_\beta)_k).$$

We can show this uniformly in t. Hence the $(\forall x)$-rule implies (3.2). Now, we discuss the $(\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule. Hence we assume that $\Gamma[\vec{X}]$ is the conclusion of the $(\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule $(\beta \leq \alpha)$. Then there exists a $\gamma_0 < \gamma$ and a set variable Y which does not occur in $\Gamma[\vec{X}]$ with

$$\mathsf{T}^n_{\alpha+1} \vdash^{\gamma_0}_{1} \Gamma[\vec{X}], Y \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi(Y)$$

The induction hypothesis yields

$$\mathsf{E}^n_{\alpha+1} \vdash^{\omega^{\omega^{\gamma_0}}}_{<\omega} \Gamma^*[(\mathsf{D}^n_\alpha)_{\vec{t}}], (\mathsf{D}^n_\alpha)_r \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi^*((\mathsf{D}^n_\alpha)_r) \tag{3.3}$$

for all closed terms $\vec{t}, r$. Since we can prove with finite deduction length ($\beta \leq \alpha$)

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} \neg(\forall k)((\mathsf{D}^n_\alpha)_k \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi^*((\mathsf{D}^n_\alpha)_k)), (\forall k)\psi^*((\mathsf{D}^n_\beta)_k),$$

a cut together with the $(\forall x)$-rule implies the claim. There remain the closure and reflection properties. As an illustration we prove closure under disjoint union. We have to prove (for instance)

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} (\mathsf{D}^n_\alpha)_t \mathrel{\dot\notin} \mathsf{D}^n_\beta, (\mathsf{D}^n_\alpha)_s \mathrel{\dot\notin} \mathsf{D}^n_\beta, (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\alpha)_t \oplus (\mathsf{D}^n_\alpha)_s)$$

for closed terms s, t. We have

$$\mathsf{E}^n_{\alpha+1} \vdash^{0}_{0} (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_{r_1} \oplus (\mathsf{D}^n_\beta)_{r_2})$$

for all closed terms r_1 and r_2 and hence

$$\mathsf{E}^n_{\alpha+1} \vdash^{<\omega}_{<\omega} (\mathsf{D}^n_\alpha)_t \neq (\mathsf{D}^n_\beta)_{r_1}, (\mathsf{D}^n_\alpha)_s \neq (\mathsf{D}^n_\beta)_{r_2}, (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\alpha)_t \oplus (\mathsf{D}^n_\alpha)_s).$$

Since we have this for all closed terms r_1, r_2, the $(\forall x)$-rule implies the claim. $\square$

The following lemma will be used in the asymmetric interpretation. It states that in $\mathsf{E}^0_{\alpha+1}$ the projections $(\mathsf{D}^0_\alpha)_t$ are first order analogues of the second order variables X. Usual second order systems have a substitution property: If they prove $\Gamma[\vec{X}]$, then they prove $\Gamma[\vec{Y}]$ too. We prove in lemma 63 the corresponding property for the system $\mathsf{E}^0_{\alpha+1}$: If we can prove $\Gamma[\vec{t}]$ (as mentioned we write $\Gamma[\vec{t}]$ for $\Gamma[\vec{*}][\vec{*}\backslash\vec{t}]$) we can also prove $\Gamma[\vec{s}]$ (for $t_i = t_j \Rightarrow s_i = s_j$). Of course we can not prove this for arbitrary sets Γ of formulas; but only for formulas which have a second order analogue. That is, we prove this substitution property for formulas in $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$. In fact, it would be possible to prove the substitution property for a larger class of such second order analogue but we do not want to introduce further classes of formulas. We refer also to lemma 62. There it is proved that free set variables (in $\mathsf{T}^n_{\alpha+1}$) are represented by projections (in $\mathsf{E}^n_{\alpha+1}$).

Notice that this – perhaps surprising – substitution property reflects a typical quality of countable coded ω-models. Assume that M is such a countable coded ω-model, e.g. of ACA. Then the projections $(M)_k$ are the sets in M. The number variable k is the index of the set $(M)_k$ in M. We know absolutely nothing about this index. If there is given an index k we have no more information than the fact "k is an index". Perhaps, this can serve as motivation for the following, mentioned lemma. We write only $s = t$ for "$s = t$ is true" (s, t closed number terms).

Lemma 63 *Assume that $\Gamma[\vec{*}]$ is a finite subset of ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup$ ess-$\Pi^1_1(\mathsf{D}^0_\alpha)^c$. We assume that*

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma}_{1} \Gamma[\vec{t}].$$

Then we have for all n-tuples $\vec{s}$ of closed terms s_i $(1 \leq i \leq n)$ such that for all i, j $(1 \leq i, j \leq n)$ $t_i = t_j$ implies $s_i = s_j$ that

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma}_{1} \Gamma[\vec{s}].$$

Proof. The proof is by induction on γ. We have put the desired property directly into the closure conditions. Therefore, the case of the closure axioms and the rules follows immediately from the induction hypothesis. If Γ is the conclusion of a propositional rule, of an ontological rule III or of a cut rule, the claim is immediate from the induction hypothesis. The case of the ontological axioms II is also trivial. There remain the cases of the ontological axioms I and of the quantifier rules. Let us discuss the ontological axioms I. Here we have only to discuss the case of the following axioms, since the other cases are trivial:

$$\Lambda[\vec{t}], r_1 \in (\mathsf{D}^0_\alpha)_{t_{n+1}}, r_2 \notin (\mathsf{D}^0_\alpha)_{t_{n+2}}$$

such that $\vec{t} = (t_1, \ldots, t_n)$ and $t_{n+1} = t_{n+2}$. Choose an n-tuple $\vec{s}$ and s_{n+1}, s_{n+2} such that $t_i = t_j$ implies $s_i = s_j$ $(1 \leq i, j \leq n+2)$. We have to prove

$$\Lambda[\vec{s}], r_1 \in (\mathsf{D}^0_\alpha)_{s_{n+1}}, r_2 \notin (\mathsf{D}^0_\alpha)_{s_{n+2}}$$

But this is again an axiom, since $s_{n+1} = s_{n+2}$. Now we discuss the quantifier rules. Γ is a subset of ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup$ ess-$\Pi^1_1(\mathsf{D}^0_\alpha)^c$. First, we assume that Γ is the conclusion of the $(\exists x)$-rule. Then the main formula of the conclusion is of type $\exists k\varphi(k)$. If there occur no $(\mathsf{D}^0_\alpha)_k$ in φ the claim follows immediately from the induction hypothesis. If there occur a $(\mathsf{D}^0_\alpha)_k$ in φ, then k occurs in φ only in $(\mathsf{D}^0_\alpha)_k$. Hence there are a $\gamma_0 < \gamma$ and a closed number term r such that

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_0}_{1} \Gamma[\vec{t}], \varphi[\vec{t}, r].$$

We fix $\vec{s}$ such that $t_i = t_j$ implies $s_i = s_j$. Then the induction hypothesis yields

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_0}_{1} \Gamma[\vec{s}], \varphi[\vec{s}, r'].$$

We have written r' instead of r, since it is possible that the application of the induction hypothesis changes r too. Now the $(\exists x)$-rule implies the claim. Finally we discuss the $(\forall x)$-rule. Here the main formula of the conclusion is of type $\forall k\varphi(k)$. Again we discuss only the case where $(\mathsf{D}^0_\alpha)_k$ occurs in φ. Then there are $\gamma_r < \gamma$ such that

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_r}_{1} \Gamma[\vec{t}], \varphi[\vec{t}, r]$$

for all closed number terms r. We fix an $\vec{s}$ such that $t_i = t_j$ implies $s_i = s_j$ $(1 \leq i, j \leq n)$. Choose an r such that $r \neq t_i$ for all i $(1 \leq i \leq n)$. Then an application of the induction hypothesis leads to

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_{r_2}}_{1} \Gamma[\vec{s}], \varphi[\vec{s}, r].$$

for all closed terms r. Then the $(\forall x)$-rule gives the claim. $\square$

3.2 Finite reduction

3.2.1 Reduction of $\mathsf{E}^0_{\alpha+1}$ to E^0_α

As mentioned, our reductions are adaptions of the reductions presented in [2]. We thus have to introduce further semi-formal systems $H_\nu\mathsf{E}^0_\alpha$ in which we have in addition iterated arithmetical comprehension up to $\nu \in \Phi_0$. Then we prove an asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $H_\nu\mathsf{E}^0_\alpha$. The next step will be the elimination of "H_ν" in $H_\nu\mathsf{E}^0_\alpha$. To achieve this we introduce a system RA_α of ramified analysis. The first order part of RA_α essentially corresponds to E^0_α. We can embed $H_\nu\mathsf{E}^0_\alpha$ into RA_α. There is also a partial (second) cut elimination in RA_α. Finally, we will embed the first order fragment of RA_α into E^0_α. This will yield the desired reduction.

The class of *arithmetic $\mathcal{L}^0_\alpha$ formulas of* E^0_α contains all $\mathcal{L}^0_\alpha$ formulas φ such that no quantifier $\exists X \mathrel{\dot\in} \mathsf{D}^0_\gamma$, $\forall X \mathrel{\dot\in} \mathsf{D}^0_\gamma$, $\exists X$, $\forall X$ occurs in φ ($\gamma < \alpha$). We introduce now the Tait-calculus $H_\nu\mathsf{E}^0_\alpha$. It is formulated in $\mathcal{L}^0_\alpha$. The *formulas of* $H_\nu\mathsf{E}^0_\alpha$ are those of T^0_α which do not contain bounded second order quantifiers. In particular we allow unbounded second order quantifiers. $H_\nu\mathsf{E}^0_\alpha$ includes all axioms and rules of E^0_α extended to formulas of $H_\nu\mathsf{E}^0_\alpha$. In addition there are quantifier rules for unbounded second order quantification, as well as the following scheme

Iterated arithmetical comprehension. For all finite sets Γ of $\mathcal{L}^0_\alpha$ formulas of $H_\nu\mathsf{E}^0_\alpha$, all arithmetic $\mathcal{L}^0_\alpha$ formulas $\varphi[x,y,Z,Y]$ of E^0_α and all set variables Y:

$$\Gamma, (\exists X)(\forall x)(\forall c \prec \nu)(x \in (X)_c \leftrightarrow \varphi[x, c, (X)_{\prec c}, Y]).$$

In $H_\nu\mathsf{E}^0_\alpha$ we need a rank definition for the definition of the notion of deduction $\vdash^{\delta}_{k}$, which is defined as before. For simplicity we set $rk(\varphi) := 0$ iff there are either no unbounded second order universal quantifiers $\forall X$ or no unbounded second order existence quantifiers $\exists X$ in φ.

We can now define in $H_\nu\mathsf{E}^0_\alpha$ the hyperarithmetical hierarchy H (up to ν) and predicates I^S_c, E^S_c. We do not give the exact definitions of all these things (cf. [2]), but introduce them only informally:

1. $H^S_0 := \{x : x \in S\}$,
   $H^S_a := \{x : j(H^S_{\prec a}, x)\}$, where j is a Π^0_1-complete predicate.
2. $\mathcal{H}^S_a := \{Y : Y$ is recursive in a H^S_b with $b \preceq a\}$.
3. $I^S_a := \{e : e$ is an index of an element of $\mathcal{H}^S_a\}$
   $= \{\langle\langle f, b\rangle, \langle g, b\rangle\rangle : b \preceq a \wedge (\forall u)(\Sigma^S_b(f,u) \leftrightarrow \neg\Sigma^S_b(g,u))\}$, where the predicate Σ^S_b enumerates all sets Σ^0_1 in H^S_b.
   $E^S_a := \{\langle x, e\rangle : e \in I^S_a \wedge \Sigma^S_a(((e)_0)_0, x)\}$.

In the following we will prove an asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $H_\nu\mathsf{E}^0_\alpha$. It corresponds essentially to the asymmetric interpretation of Σ^1_1-AC into $(\Pi^1_0\text{-}\mathsf{CA})_{<\varepsilon_0}$ in [2]. The only difference is that our situation is more complicated. We first give a translation.

Definition 64 For each expression φ in ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)$ or in ess-$\Pi^1_1(\mathsf{D}^0_\alpha)$ we inductively define $\varphi^{\beta,\gamma,\nu}$ as follows:

1. If there is no occurrence of D^0_α in φ, then $\varphi^{\beta,\gamma,\nu} := \varphi$.
2. $(\langle t, *_i\rangle \in \mathsf{D}^0_\alpha)^{\beta,\gamma,\nu} := \langle t, *_i\rangle \in E_\nu^{\mathsf{D}^0_{<\alpha}}$ and $(\langle t, *_i\rangle \notin \mathsf{D}^0_\alpha)^{\beta,\gamma,\nu} := \langle t, *_i\rangle \notin E_\nu^{\mathsf{D}^0_{<\alpha}}$.
3. If φ ist of the form $\theta\wedge\psi$ (resp. $\theta\vee\psi$), then $\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu}\wedge\psi^{\beta,\gamma,\nu}$ (resp. $\theta^{\beta,\gamma,\nu}\vee\psi^{\beta,\gamma,\nu}$).
4. If φ ist of the form $\exists k\psi(k)$ (resp. $\forall k\psi(k)$) such that there is no $(\mathsf{D}^0_\alpha)_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := \exists k\psi^{\beta,\gamma,\nu}(k)$ (resp. $\forall k\psi^{\beta,\gamma,\nu}(k)$).
5. If φ is of the form $\exists k\psi((\mathsf{D}^0_\alpha)_k)$ (resp. $(\forall k\psi((\mathsf{D}^0_\alpha)_k))$ such that there is a $(\mathsf{D}^0_\alpha)_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := (\exists k \in I_\gamma^{\mathsf{D}^0_{<\alpha}})\psi^{\beta,\gamma,\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_k)$ (resp. $(\forall k \in I_\beta^{\mathsf{D}^0_{<\alpha}})\psi^{\beta,\gamma,\nu}((E_\beta^{\mathsf{D}^0_{<\alpha}})_k)$).

In clause 2 we have given a translation of $\langle t, *_i\rangle \in \mathsf{D}^0_\alpha$. In the following we need a translation of $\langle t, s\rangle \in \mathsf{D}^0_\alpha$ for s a closed number term. We set

$$(\langle t, s\rangle \in \mathsf{D}^0_\alpha)^{\beta,\gamma,\nu} := (\langle t, *_i\rangle \in \mathsf{D}^0_\alpha)^{\beta,\gamma,\nu}[*_i\backslash s].$$

We extend this translation to all expressions $\varphi[\vec{*}]$ in ess-$(\Sigma^1_1\mathsf{D}^0_\alpha)^c \cup ess$-$\Pi^1_1(\mathsf{D}^0_\alpha)^c$ by setting $\varphi[\vec{t}]^{\beta,\gamma,\nu} := (\varphi[\vec{*}]^{\beta,\gamma,\nu})[\vec{*},\backslash\vec{t}]$. Notice that for s a closed number term the formulas $t \in (\mathsf{D}^0_\alpha)_s$ and $t \notin (\mathsf{D}^0_\alpha)_s$ are interpreted symmetrically, whereas the quantifiers $\exists k\psi((\mathsf{D}^0_\alpha)_k)$, $\forall k\psi((\mathsf{D}^0_\alpha)_k)$ are interpreted asymmetrically.

We will give an asymmetric interpretation. It is typical for such situations that there is a persistency property. Here we deal with infinite deduction lengths and we have not put the whole persistency in the axioms and rules of our systems. Therefore, our persistency is a little bit more complicated as in similar cases.

Lemma 65 *For all finite sets $\Gamma[\vec{*}] \cup \{\varphi[\vec{*}]\}$ of expressions in ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess$-$\Pi^1_1(\mathsf{D}^0_\alpha)^c$ and for all ordinals $\nu, \rho, \rho', \gamma, \gamma', \delta$ with $\nu > \rho > \rho'$, $\gamma < \gamma' < \nu$ we have for all closed number terms $\vec{t}$*

$$H_{\nu+1}\mathsf{E}^0_\alpha \,\frac{\ \delta\ }{<\omega}\, \Gamma[\vec{t}], \varphi[\vec{t}]^{\rho,\gamma,\nu} \quad\Longrightarrow\quad H_{\nu+1}\mathsf{E}^0_\alpha \,\frac{<\delta+\omega}{<\omega}\, \Gamma[\vec{t}], \varphi[\vec{t}]^{\rho',\gamma',\nu}.$$

Proof. By induction on δ. As an illustration we discuss the case of the $(\exists x)$-rule. Assume that we have

$$H_{\nu+1}\mathsf{E}^0_\alpha \,\frac{\ \delta\ }{<\omega}\, \Gamma, (\exists k\psi(k))^{\delta,\gamma,\nu}.$$

If ψ is not of the form $k \in I_\gamma^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta,\gamma,\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_k)$, then the claim follows immediately from the induction hypothesis. Therefore, we assume that ψ is of the form $k \in I_\gamma^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta,\gamma,\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_k)$. That is, there is a $\delta_0 < \delta$ and a closed term t with

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{\delta_0}{\underset{<\omega}{\vdash}} \Gamma, (\exists k\psi(k))^{\delta,\gamma,\nu}, t \in I_\gamma^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta,\gamma,\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_t).$$

The induction hypothesis implies

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\delta_0+\omega}{\underset{<\omega}{\vdash}} \Gamma, (\exists k\psi(k))^{\delta',\gamma',\nu}, t \in I_\gamma^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta',\gamma',\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_t).$$

It follows immediately from the definition of the index set I_a^S that

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega}{\underset{<\omega}{\vdash}} t \notin I_\gamma^{\mathsf{D}^0_{<\alpha}}, t \in I_{\gamma'}^{\mathsf{D}^0_{<\alpha}},$$

and by induction on the build-up of φ we can prove

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega}{\underset{<\omega}{\vdash}} t \notin I_\gamma^{\mathsf{D}^0_{<\alpha}}, \neg\varphi^{\delta',\gamma',\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_t), \varphi^{\delta',\gamma',\nu}((E_{\gamma'}^{\mathsf{D}^0_{<\alpha}})_t).$$

Hence

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega}{\underset{<\omega}{\vdash}} \neg(t \in I_\gamma^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta',\gamma',\nu}((E_\gamma^{\mathsf{D}^0_{<\alpha}})_t)), t \in I_{\gamma'}^{\mathsf{D}^0_{<\alpha}} \wedge \varphi^{\delta',\gamma',\nu}((E_{\gamma'}^{\mathsf{D}^0_{<\alpha}})_t).$$

A cut and the $(\exists x)$-rule imply the claim. $\square$

Theorem 66 *For all finite subsets $\Gamma[\vec{*}]$ of ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup$ ess-$\Pi^1_1(\mathsf{D}^0_\alpha)^c$ and for all ordinals $\beta, \gamma, \nu \in \Phi_0$ with $\beta + \omega^\gamma < \nu$ we have for all closed number terms $\vec{t}$*

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma}{\underset{1}{\vdash}} \Gamma[\vec{t}] \quad\Longrightarrow\quad H_{\nu+1}\mathsf{E}^0_\alpha \overset{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{\underset{<\omega}{\vdash}} \vec{t} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma,\nu}.$$

Proof. The proof is by induction on γ. We have to discuss the cases 1-9 of $\mathsf{E}^0_{\alpha+1}$. If Γ is an axiom of case 1, the claim follows immediately, since we can prove in $H_{\nu+1}\mathsf{E}^0_\alpha$ $\neg\varphi, \varphi$ with finite deduction length. Also the cases 2,4,5 follow immediately. And since $\mathsf{E}^0_{\alpha+1}$ does not contain the case 8 there remain the cases 3,6,7,9. We write in this proof $\varphi^{\delta,\varepsilon}$ for $\varphi^{\delta,\varepsilon,\nu}$.

Case 3. We have only to deal with the $(\forall x)$-rule and the $(\exists x)$-rule. We discuss first the $(\exists x)$-rule. Hence, assume that $\Gamma[\vec{t}]$ is the conclusion of the $(\exists x)$-rule. There is a $\gamma_0 < \gamma$ and a closed term t_{n+1} such that

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_0}{\underset{1}{\vdash}} \Gamma[\vec{t}], \varphi(t_{n+1})[\vec{t}].$$

If no $(\mathsf{D}^0_\alpha)_{t_{n+1}}$ occurs in φ, the claim follows easily from the induction hypothesis. Therefore, we assume that $(\mathsf{D}^0_\alpha)_{t_{n+1}}$ occurs in φ. Thus we have

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_0}{\underset{1}{\vdash}} \Gamma[\vec{t}], \varphi[\vec{t}]((\mathsf{D}^0_\alpha)_{t_{n+1}}).$$

We prefer here – and sometimes also later on – to write $\varphi[\vec{t}]((\mathsf{D}^0_\alpha)_{t_{n+1}})$ instead of $\varphi[t_{n+1},\vec{t}]$, since later on we have also to control the ordinal ε in $E_\varepsilon^{\mathsf{D}^0_{<\alpha}}$. Using lemma 63 and the induction hypothesis, we obtain for all closed terms $\vec{s}=(s_1,\dots,s_n)$ and s_{n+1} such that $t_i=t_j$ implies $s_i=s_j$ $(1\le i,j\le n+1)$

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}}{\underset{<\omega}{\vdash}} \vec{s}, s_{n+1}\notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{s}]^{\beta,\beta+\omega^{\gamma_0}}, \varphi[\vec{s}]^{\beta,\beta+\omega^{\gamma_0}}((\mathsf{E}_\nu^{\mathsf{D}^0_{<\alpha}})_{s_{n+1}}).$$

We can prove with finite deduction length $s_{n+1}\notin I_\beta^{\mathsf{D}^0_{<\alpha}}$, $s_{n+1}\in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}}$. Then we use the $\wedge$-rule, the $(\exists k)$-rule and persistency. Hence

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} \vec{s}, s_{n+1}\notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{s}]^{\beta,\beta+\omega^{\gamma}}, (\exists k\in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})\varphi[\vec{s}]^{\beta,\beta+\omega^{\gamma}}((E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k).$$

We have this for all $\vec{s}, s_{n+1}$ which satisfy the condition above. If there is a t_i $(1\le i\le n)$ with $t_i=t_{n+1}$ we can set $\vec{s}:=\vec{t}, s_{n+1}:=t_i$ and we are done. If there is no t_i with $t_{n+1}=t_i$ we distinguish two cases: If $n\ge 1$, we set $\vec{s}:=\vec{t}$ and $s_{n+1}:=t_1$. If $n=0$ we use the $(\forall x)$-rule and obtain

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} (\forall k)(k\notin I_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma^{\beta,\beta+\omega^{\gamma}}, (\exists k\in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})\varphi^{\beta,\beta+\omega^{\gamma}}((E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k).$$

We can show with finite deduction length $\neg(\forall k)(k\notin I_\beta^{\mathsf{D}^0_{<\alpha}})$. Hence, a cut implies the claim. Now, we discuss the $(\forall x)$-rule. We assume that $\Gamma[\vec{t}]$ is the conclusion of the $(\forall x)$-rule. Hence there is for each closed term r a $\gamma_r<\gamma$ such that

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_r}{\underset{1}{\vdash}} \Gamma[\vec{t}], \varphi(r)[\vec{t}].$$

If no $(\mathsf{D}^0_\alpha)_r$ occurs in φ, the claim follows easily from the induction hypothesis. Therefore, we assume that $(\mathsf{D}^0_\alpha)_r$ occurs in φ. Thus we have

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_r}{\underset{1}{\vdash}} \Gamma[\vec{t}], \varphi[\vec{t}]((\mathsf{D}^0_\alpha)_r)$$

for all closed terms r. We apply the induction hypothesis and obtain with the aid of persistency for all closed terms r

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} \vec{t}, r\notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}}, \varphi[\vec{t}]^{\beta,\beta+\omega^{\gamma}}((E_\nu^{\mathsf{D}^0_{<\alpha}})_r).$$

The $\vee$-rule and $(\forall x)$-rule imply

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} \vec{t}\notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}}, (\forall k\in I_\beta^{\mathsf{D}^0_{<\alpha}})\varphi[\vec{t}]^{\beta,\beta+\omega^{\gamma}}((E_\nu^{\mathsf{D}^0_{<\alpha}})_k).$$

Since we can prove with finite deduction length

$$\neg(\forall k\in I_\beta^{\mathsf{D}^0_{<\alpha}})\varphi[\vec{t}]^{\beta,\beta+\omega^{\gamma}}((E_\nu^{\mathsf{D}^0_{<\alpha}})_k), (\forall k\in I_\beta^{\mathsf{D}^0_{<\alpha}})\varphi[\vec{t}]^{\beta,\beta+\omega^{\gamma}}((E_\beta^{\mathsf{D}^0_{<\alpha}})_k),$$

a cut implies the claim.

Case 6. We discuss the second axioms, the first are proved with similar arguments. We have to prove

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{\underset{<\omega}{\vdash}} t \notin I_\beta^{\mathsf{D}^0_{<\alpha}}, (\exists k \in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)(x \in (E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k \leftrightarrow \pi^0_1[e,x,r,(E_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]).$$

We introduce a theory M. M is $\bar{\mathsf{T}}^0_\alpha$ plus iterated arithmetical comprehension up to $\nu+1$ and set induction up to $\nu+1$. Then we can prove in M

$$t \notin I_\beta^{\mathsf{D}^0_{<\alpha}} \vee (\exists k \in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)(x \in (E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k \leftrightarrow \pi^0_1[e,x,r,(E_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]).$$

There is a translation $(\ldots)^\alpha$ too (cf. lemma 59), i.e., we have

$$M \vdash \varphi \Longrightarrow H_{\nu+1}\mathsf{T}^0_\alpha \overset{<\omega^{\nu+1}+\omega}{\underset{<\omega}{\vdash}} \varphi^\alpha$$

where $H_{\nu+1}\mathsf{T}^0_\alpha$ is T^0_α plus iterated arithmetical comprehension over T^0_α up to $\nu+1$. Since we have in $H_{\nu+1}\mathsf{T}^0_\alpha$ and in $H_{\nu+1}\mathsf{E}^0_\alpha$ arithmetical comprehension these two calculus are equivalent. That is

$$H_{\nu+1}\mathsf{T}^0_\alpha \overset{\delta}{\underset{<\omega}{\vdash}} \Gamma \Longrightarrow H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\delta+\omega}{\underset{<\omega}{\vdash}} \Gamma.$$

In particular we can conclude

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega}{\underset{<\omega}{\vdash}} \varphi^\alpha.$$

Hence

$$\begin{aligned} H_{\nu+1}\mathsf{E}^0_\alpha \ \overset{<\omega^{\nu+1}+\omega}{\underset{<\omega}{\vdash}} \ & (t \notin I_\beta^{\mathsf{D}^0_{<\alpha}})^\alpha, \\ & ((\exists k \in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)(x \in (E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k \leftrightarrow \pi^0_1[e,x,r,(E_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]))^\alpha. \end{aligned}$$

Moreover, we can prove in $H_{\nu+1}\mathsf{E}^0_\alpha$ with finite deduction length

$$\neg(\psi(\mathsf{D}^0))^\alpha, \psi(\mathsf{D}^0_{<\alpha}) \qquad \text{and} \qquad (\psi(\mathsf{D}^0))^\alpha, \neg\psi(\mathsf{D}^0_{<\alpha})$$

for each $\mathcal{L}_2(\mathsf{D}^0)$ formula ψ. Now cuts imply the claim.

Case 7. We know

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma}{\underset{1}{\vdash}} \Gamma[\vec{t}], (\exists k)(\forall x)\pi^0_1[e,x,r,((\mathsf{D}^0_\alpha)_k)_x, (\mathsf{D}^0_\alpha)_r, \mathsf{D}^0_{<\alpha}]$$

and have to prove

$$\begin{aligned} H_{\nu+1}\mathsf{E}^0_\alpha \ \overset{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{\underset{<\omega}{\vdash}} \ & (\vec{t}, r \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma}, \\ & (\exists k \in I_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)\pi^0_1[e,x,r,((E_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k)_x, (E_\nu^{\mathsf{D}^0_{<\alpha}})_r, \mathsf{D}^0_{<\alpha}]. \end{aligned}$$

We know that there exists a $\gamma_0 < \gamma$ with

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_0}{\underset{1}{\vdash}} \Gamma[\vec{t}], (\forall x)(\exists k)\pi^0_1[e,x,r,(\mathsf{D}^0_\alpha)_k, (\mathsf{D}^0_\alpha)_r, \mathsf{D}^0_{<\alpha}].$$

An application of the induction hypothesis leads to

$$\begin{aligned} H_{\nu+1}\mathsf{E}^0_\alpha \;\; \overset{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}}{\underset{<\omega}{\vdash}} \;\; & (\vec{t}, r \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma_0}}, \\ & (\forall x)(\exists k \in I_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}})\pi^0_1[e, x, r, (E_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}})_k, (E_\nu^{\mathsf{D}^0_{<\alpha}})_r, \mathsf{D}^0_{<\alpha}]. \end{aligned}$$

Again we let M denote the theory introduced in case 6. Arguing as in case 6 it is enough to show

$$\begin{aligned} M \;\;\vdash\;\; & (\vec{t}, r \in I_\beta^{\mathsf{D}^0_{<\alpha}} \\ & \to (\Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma_0}} \vee (\forall x)(\exists k \in I_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}})\pi^0_1[e, x, r, (E_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}})_k, (E_\nu^{\mathsf{D}^0_{<\alpha}})_r, \mathsf{D}^0_{<\alpha}])) \\ & \quad \to (\vec{t}, r \in I_\beta^{\mathsf{D}^0_{<\alpha}} \\ & \quad\quad \to (\Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}} \vee (\exists k \in I_{\beta+\omega^{\gamma}}^{\mathsf{D}^0_{<\alpha}})(\forall x)\pi^0_1[e, x, r, ((E_{\beta+\omega^{\gamma}}^{\mathsf{D}^0_{<\alpha}})_k)_x, (E_\nu^{\mathsf{D}^0_{<\alpha}})_r, \mathsf{D}^0_{<\alpha}])). \end{aligned}$$

But this can be shown as, e.g., in [2].

Case 9. Choose $\gamma_0, \gamma_1 < \gamma$ with φ in ess-$\Sigma^1_1(\mathsf{D}^0_\alpha)$ and with

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_0}{\underset{1}{\vdash}} \Gamma[\vec{t}], \varphi[\vec{t}, \vec{r}], \tag{3.4}$$

$$\mathsf{E}^0_{\alpha+1} \overset{\gamma_1}{\underset{1}{\vdash}} \Gamma[\vec{t}], \neg\varphi[\vec{t}, \vec{r}]. \tag{3.5}$$

We have to prove

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} \vec{t} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}}.$$

Now we apply the induction hypothesis to (3.4) and (3.5). In the application to (3.5) we choose $\beta + \omega^{\gamma_0}$ instead of β.

$$\begin{aligned} H_{\nu+1}\mathsf{E}^0_\alpha \;\; \overset{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}}{\underset{<\omega}{\vdash}} \;\; & (\vec{r}, \vec{t} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma_0}}, \varphi[\vec{t}, \vec{r}]^{\beta,\beta+\omega^{\gamma_0}}, \\ H_{\nu+1}\mathsf{E}^0_\alpha \;\; \overset{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}}{\underset{<\omega}{\vdash}} \;\; & (\vec{r}, \vec{t} \notin I_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}, \\ & (\neg\varphi)[\vec{t}, \vec{r}]^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}. \end{aligned}$$

Since we have $\varphi \in ess$-$\Sigma^1_1(\mathsf{D}^0_\alpha)$, we have

$$(\neg\varphi)^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}} \equiv \neg\varphi^{\beta+\omega^{\gamma_0}+\omega^{\gamma_1},\beta+\omega^{\gamma_0}} \equiv \neg\varphi^{\beta,\beta+\omega^{\gamma_0}}.$$

Hence a cut and persistency yield

$$H_{\nu+1}\mathsf{E}^0_\alpha \overset{<\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma}}}{\underset{<\omega}{\vdash}} (\vec{t}, \vec{r} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), (\vec{t}, \vec{r} \notin I_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}}.$$

Now we use that we can prove in $H_{\nu+1}\mathsf{E}^0_\alpha$ with finite deduction length

$$(\vec{t}, \vec{r} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), (\vec{t}, \vec{r} \in I_{\beta+\omega^{\gamma_0}}^{\mathsf{D}^0_{<\alpha}}).$$

Again, by a cut we obtain

$$H_{\nu+1}\mathsf{E}^0_\alpha \mid\!\frac{<\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{<\omega}\; (\vec{t}, \vec{r} \notin I_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma}.$$

By using lemma 63 we can do this for all appropriate closed terms $\vec{r}$ and arguing as in the case 3 we obtain the claim by choosing $r_j := t_i$ or by eliminating $r_j \notin I_\beta^{\mathsf{D}^0_{<\alpha}}$. □

In a next step we reduce $H_{\nu+1}\mathsf{E}^0_\alpha$ to E^0_α. This reduction together with the asymmetric interpretation of theorem 66 will lead to an interpretation of $\mathsf{E}^0_{\alpha+1}$ into E^0_α. As mentioned we introduce a semi-formal system RA_α. RA_α is essentially an extension of $\mathsf{RA}^\star$ of Schütte (cf. [27]) by E^0_α. The language $\mathcal{L}_{RA_\alpha}$ of RA_α is similar to $\mathcal{L}^0_\alpha$. We have set variables $X^\beta, Y^\beta, Z^\beta, \ldots$ for all $\beta \in \Phi_0$, and we have all predicates of $\mathcal{L}^0_\alpha$. The *number terms* of $\mathcal{L}_{RA_\alpha}$ are those of $\mathcal{L}_2$. The *set terms* $R, S, T, \ldots$ of $\mathcal{L}_{RA_\alpha}$ are defined simultaneously with the *formulas* of $\mathcal{L}_{RA_\alpha}$:

1. Each X^β is a set term.
2. If φ is a $\mathcal{L}_{RA_\alpha}$ formula, then $\{x : \varphi\}$ is a set term.
3. $K\vec{t}$, $\neg K\vec{t}$, $\mathsf{Q}(t)$, $\neg\mathsf{Q}(t)$, $t \in \mathsf{D}^0_\beta$, $t \notin \mathsf{D}^0_\beta$, $t \in \mathsf{D}^0_{<\gamma}$, $t \notin \mathsf{D}^0_{<\gamma}$ are $\mathcal{L}_{RA_\alpha}$ formulas for K a primitive recursive relation symbol and $\beta < \alpha$, $\gamma \leq \alpha$.
4. $(t \in T), (t \notin T)$ are $\mathcal{L}_{RA_\alpha}$ formulas for number terms t and set terms T.
5. $\mathcal{L}_{RA_\alpha}$ formulas are closed under $\wedge, \vee, \exists x, \forall x, \exists X^\beta, \forall X^\beta$ for $\beta > 0$.

The *level* of a set term and the level of a formula φ is defined by

$$\begin{aligned} lev(T) &:= \max(\{0\} \cup \{\alpha : X^\alpha \text{ occurs in } T\}), \\ lev(\varphi) &:= \max(\{0\} \cup \{\alpha : X^\alpha \text{ occurs in } \varphi\}). \end{aligned}$$

Definition 67 The rank $rk(\varphi)$ of an $\mathcal{L}_{RA_\alpha}$ formula φ and of RA_α is inductively defined as follows: If in φ there is no occurrence of an X^β or a $\{x : \psi\}$, then $rk(\varphi) := 0$. Otherwise:

1. If φ is a formula $(t \in X^\beta)$ or $(t \notin X^\beta)$, then $rk(\varphi) := \max\{1, \omega \cdot \beta\}$.
2. If φ is a formula $(t \in \{x : \psi\})$ or $(t \notin \{x : \psi\})$, then $rk(\varphi) := rk(\psi) + 1$.
3. If φ is a formula $(\psi \vee \theta)$ or $(\psi \wedge \theta)$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.
4. If φ is a formula $(\exists x \psi)$ or $(\forall x \psi)$, then $rk(\varphi) := rk(\psi) + 1$.
5. If φ is a formula $(\exists X^\beta)\psi(X^\beta)$ or $(\forall X^\beta)\psi(X^\beta)$, then $rk(\varphi) := \max(\omega \cdot lev(\varphi), rk(\psi(X^0)) + 1)$.

Notice that $rk(\varphi) = rk(\neg\varphi)$. We make the following observations:

1. If $lev(\varphi) = \gamma$, then $\omega\gamma \leq rk(\varphi) < \omega(\gamma + 1)$.
2. If $lev(T) < \gamma$, then $rk(\varphi(T)) < rk(\exists X^\gamma \varphi(X^\gamma))$.

RA_α is defined as a Tait-calculus ($\alpha \in \Phi_0$). The axioms and rules are given below. Notice that the properties just remarked lead to a partial cut elimination lemma.

1. Logical axioms. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all set variables X^β, all true $\mathcal{L}_1$ literals φ, all closed number terms s, t with identical value and all ordinals γ, δ with $\gamma < \alpha, \delta \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X^\beta, s \notin X^\beta \qquad \text{and} \qquad \Gamma, \mathsf{Q}(t), \neg\mathsf{Q}(s)$$

$$\text{and} \qquad \Gamma, t \in \mathsf{D}^0_\gamma, s \notin \mathsf{D}^0_\gamma \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^0_{<\delta}, s \notin \mathsf{D}^0_{<\delta}.$$

2. Propositional rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas and all $\mathcal{L}_{RA_\alpha}$ formulas φ and ψ:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

3. Set term rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all $\mathcal{L}_{RA_\alpha}$ formulas φ and all closed number terms t:

$$\frac{\Gamma, \varphi(t)}{\Gamma, t \in \{x : \varphi(x)\}}, \qquad \frac{\Gamma, \neg\varphi(t)}{\Gamma, t \notin \{x : \varphi(x)\}}.$$

4. Quantifier rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all set terms T, all closed number terms s and all $\mathcal{L}_{RA_\alpha}$ formulas $\varphi(s), \psi(T)$:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)},$$

$$\frac{\Gamma, \psi(T)}{\Gamma, (\exists X^\beta)\psi(X^\beta)} \; lev(T) < \beta, \qquad \frac{\Gamma, \psi(T) \quad \text{for all set terms } T \text{ with } lev(T) < \beta}{\Gamma, (\forall X^\beta)\psi(X^\beta)}.$$

5. E^0_α axioms and rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, for all axioms Λ_1 and all rules $\frac{\Lambda_2}{\Lambda_3}$ of the ontological axioms II and rules III and closure axioms Λ_1 and rules $\frac{\Lambda_2}{\Lambda_3}$ of E^0_α:

$$\Gamma, \Lambda_1 \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

6. Cut rules. For all finite sets Γ of closed $\mathcal{L}_{RA_\alpha}$ formulas and for all $\mathcal{L}_{RA_\alpha}$ formulas φ:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In the following theorem we collect the main results about RA_α. For the formulation we need the notion of a γ-instance.

Definition 68 Take an $\mathcal{L}^0_\alpha$ formula φ of $H_\nu\mathsf{E}^0_\alpha$ (notice that then there are no bounded second order quantifiers in φ). The $\mathcal{L}_{RA_\alpha}$ formula φ^γ is a γ-instance of φ if φ^γ is obtained from φ by

- free set variables are replaced by set terms of $\mathcal{L}_{RA_\alpha}$ with $lev < \gamma$.
- bound set variables get the superscript γ.

Theorem 69 *The following holds:*

a) *For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas we have:*

$$\mathsf{RA}_\alpha \vdash^{\gamma}_{1+\beta+\omega^\delta} \Gamma \quad\Longrightarrow\quad \mathsf{RA}_\alpha \vdash^{\varphi\rho\gamma}_{1+\beta} \Gamma.$$

b) *For all finite sets Γ of $\mathcal{L}^0_\alpha$ formulas of $H_\nu\mathsf{E}^0_\alpha$, we have for all $\omega^{\nu+1}$-instances $\Gamma^{\omega^{\nu+1}}$ of Γ:*

$$H_\nu\mathsf{E}^0_\alpha \vdash^{\gamma}_{1} \Gamma \quad\Longrightarrow\quad \mathsf{RA}_\alpha \vdash^{\omega^{\omega^{\nu+3}+\omega^\gamma}}_{\omega^{\omega^{\nu+3}+\omega^\gamma}} \Gamma^{\omega^{\nu+1}}.$$

c) *For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas without set terms X^β, $\{x : \varphi(x)\}$ we have*

$$\mathsf{RA}_\alpha \vdash^{\gamma}_{1} \Gamma \quad\Longrightarrow\quad \mathsf{E}^0_\alpha \vdash^{\gamma}_{<\omega} \Gamma.$$

Proof. The proof of the partial (second) cut elimination a) is standard and is hence omitted (cf. for instance [23] theorem 18.4). The proof of b) is by induction on γ. All cases beside the iterated arithmetical comprehension can be shown by standard arguments. The relevant arguments for the embedding of iterated arithmetical comprehension in RA_α can be extracted from [9] Proposition 9. Finally, an easy induction on γ shows c). □

Corollary 70 *For all finite sets $\Gamma \subset (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha))^c \cup (ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha))^c$ without an occurrence of D^0_α we have*

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma}_{1} \Gamma \quad\Longrightarrow\quad \mathsf{E}^0_\alpha \vdash^{<\varphi\varepsilon(\gamma)0}_{1} \Gamma.$$

Proof. We assume that $\mathsf{E}^0_{\alpha+1} \vdash^{\gamma}_{1} \Gamma$. By theorem 66 there exist ordinals ν, ξ less than $\varepsilon(\gamma)$ with

$$H_\nu\mathsf{E}^0_\alpha \vdash^{\xi}_{<\omega} \Gamma.$$

We conclude from theorem 69a) and 69b)

$$\mathsf{RA}_\alpha \vdash^{<\varphi\varepsilon(\gamma)0}_{1} \Gamma.$$

And from theorem 69c) and lemma 61

$$\mathsf{E}^0_\alpha \vdash^{<\varphi\varepsilon(\gamma)0}_{1} \Gamma.$$

□

3.2.2 The semi-formal systems $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$

In theorem 66 we have interpreted $\mathsf{E}^0_{\alpha+1}$ into "Iterated arithmetical comprehension over E^0_α". In the following we give an asymmetric interpretation of $\mathsf{E}^{n+1}_{\alpha+1}$ into "E^n_μ over E^{n+1}_α". We will introduce in this subsection e.g. a semi-formal system $\mathsf{E}^n_\nu[\mathsf{E}^{n+1}_\alpha]$, which corresponds to "$\mathsf{E}^n_\nu$ over E^{n+1}_α".

For natural numbers $n, n+1, \ldots, n+k$ and ordinals $\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k} \in \Phi_0$ we define a language $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$. $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ is an extension of $\mathcal{L}_1$ by the predicates $\mathsf{D}^i_{\beta_i}$, $\mathsf{D}^i_{<\gamma_i}$ for each i with $n \leq i \leq n+k$ and all ordinals β_i, γ_i with $\beta_i < \alpha_i$, $\gamma_i \leq \alpha_i$. The *formulas* of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ are built in analogy to E^m_α: All $\mathcal{L}_1$ literals and $t \in \mathsf{D}^i_{\beta_i}$, $t \notin \mathsf{D}^i_{\beta_i}$, $t \in \mathsf{D}^i_{<\gamma_i}$, $t \notin \mathsf{D}^i_{<\gamma_i}$ are formulas of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ for $n \leq i \leq n+k$, $\beta_i < \alpha_i$, $\gamma_i \leq \alpha_i$. Moreover, the formulas of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ are closed under $\wedge$, $\vee$, $\exists x$, $\forall x$. We take as $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ *formulas of* $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$ the $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ formulas without free number variables.

The Tait-calculus $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$ contains the following axioms and rules of inference:

1. Ontological axioms I. For all finite sets Γ of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ formulas of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all $\beta_i < \alpha_i$, $\gamma_i \leq \alpha_i$, $1 \leq i \leq n$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, \mathsf{Q}(t), \neg\mathsf{Q}(s)$$
$$\text{and} \qquad \Gamma, t \in \mathsf{D}^i_{\beta_i}, s \notin \mathsf{D}^i_{\beta_i} \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^i_{<\gamma_i}, s \notin \mathsf{D}^i_{<\gamma_i}.$$

2. Propositional and quantifier rules. Rules for $\wedge$, $\vee$, $\exists x$, $\forall x$ (ω-rule).

3. Ontological axioms II and rules III. For all finite sets Γ of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ formulas of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$ and for all ontological axioms II Λ_1 and ontological rules III $\frac{\Lambda_2}{\Lambda_3}$ of the systems $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$:

$$\Gamma, \Lambda_1, \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

4. $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$ axioms and rules. For all finite sets Γ of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ formulas of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$, for all closure and reflection axioms Λ_1 and for all closure rules $\frac{\Lambda_2}{\Lambda_3}$ of the systems $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$:

$$\Gamma, \Lambda_1, \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

5. Inclusion axioms. For all finite sets Γ of $\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$ formulas of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$, all i with $n \leq i < n+k$ and all ordinals $\beta_i < \alpha_i$:

$$\Gamma, (\exists k)((\mathsf{D}^i_{\beta_i})_k = \mathsf{D}^{i+1}_{<\alpha_{i+1}}).$$

6. Cut rules. The usual cut rules.

For $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$ we can introduce classes corresponding to $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)$ and $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)$ $(\beta < \alpha_n)$. We do not give the explicit definition here. Since we do not want to introduce more terminology, we write again $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)$ and $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)$ for these classes. Furthermore we can prove a reduction of $\mathsf{E}^{n+1}_{\alpha+1}$ to $\mathsf{E}^n_\nu[\mathsf{E}^{n+1}_\alpha]$. In order to achieve this, we extend the methods of the preceding subsections, which led to theorem 66, to the case $\mathsf{E}^{n+1}_{\alpha+1}$. Again we define a translation for formulas in $ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1}_\alpha)$ and in $ess\text{-}\Pi^1_1(\mathsf{D}^{n+1}_\alpha)$.

Definition 71 For each expression φ in $ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1}_\alpha)$ or in $ess\text{-}\Pi^1_1(\mathsf{D}^{n+1}_\alpha)$ we inductively define $\varphi^{\beta,\gamma,\nu}$ as follows:

1. If there is no occurrence of D^{n+1}_α in φ, then $\varphi^{\beta,\gamma,\nu} := \varphi$.
2. $(\langle t, *_i\rangle \in \mathsf{D}^{n+1}_\alpha)^{\beta,\gamma,\nu} := \langle t, *_i\rangle \in \mathsf{D}^n_\nu$ and $(\langle t, *_i\rangle \notin \mathsf{D}^{n+1}_\alpha)^{\beta,\gamma,\nu} := \langle t, *_i\rangle \notin \mathsf{D}^n_\nu$.
3. If φ is of the form $\theta \wedge \psi$ (resp. $\theta \vee \psi$), then $\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu} \wedge \psi^{\beta,\gamma,\nu}$ (resp. $\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu} \vee \psi^{\beta,\gamma,\nu}$).
4. If φ is of the form $\exists x\psi$ (resp. $\forall x\psi$) such that there is no $(\mathsf{D}^{n+1}_\alpha)_x$ in φ, then $\varphi^{\beta,\gamma,\nu} := \exists x\psi^{\beta,\gamma,\nu}$ (resp. $\varphi^{\beta,\gamma,\nu} := \forall x\psi^{\beta,\gamma,\nu}$).
5. If φ is of the form $(\exists k)\psi((\mathsf{D}^{n+1}_\alpha)_k)$ (resp. $(\forall k)\psi((\mathsf{D}^{n+1}_\alpha)_k)$) such that there is a $(\mathsf{D}^{n+1}_\alpha)_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := (\exists k)\psi^{\beta,\gamma,\nu}((\mathsf{D}^n_\gamma)_k)$ (resp. $\varphi^{\beta,\gamma,\nu} := (\forall k)\psi^{\beta,\gamma,\nu}((\mathsf{D}^n_\beta)_k)$).

We now formulate the asymmetric interpretation. It corresponds to the asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $H_\nu\mathsf{E}^0_\alpha$. Also the proof is very similar, hence we omit it. For the proof of the closure under Σ^1_1-DC we refer to [2].

Theorem 72 *For all finite subsets $\Gamma[\vec{*}]$ of $ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1}_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n+1}_\alpha)^c$ and for all ordinals $\beta,\gamma,\nu \in \Phi_0$ with $\beta + \omega^\gamma < \nu$ we have for all closed number terms $\vec{t}$*

$$\mathsf{E}^{n+1}_{\alpha+1} \vdash^{\gamma}_{1} \Gamma[\vec{t}] \quad\Longrightarrow\quad \mathsf{E}^n_{\nu+1}[\mathsf{E}^{n+1}_\alpha] \vdash^{\omega^{\beta+\omega^\gamma}}_{<\omega} \vec{t} \notin I^{\mathsf{D}^{n+1}_{<\alpha}}_\beta, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma,\nu}.$$

Corollary 73 *For all finite sets $\Gamma \subset ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1}_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n+1}_\alpha)^c$ without occurrences of D^{n+1}_α we have*

$$\mathsf{E}^{n+1}_{\alpha+1} \vdash^{\gamma}_{<\omega} \Gamma \quad\Longrightarrow\quad \textit{there is an ordinal } \nu \textit{ less than } \varepsilon(\gamma) \textit{ with}$$
$$\mathsf{E}^n_\nu[\mathsf{E}^{n+1}_\alpha] \vdash^{<\varepsilon(\gamma)}_{<\omega} \Gamma.$$

Proof. We assume that $\mathsf{E}^{n+1}_{\alpha+1} \vdash^{\gamma}_{<\omega} \Gamma$. Lemma 61 yields

$$\mathsf{E}^{n+1}_{\alpha+1} \vdash^{<\varepsilon(\gamma)}_{1} \Gamma.$$

By theorem 72 there exist ordinals ν, ξ less than $\varepsilon(\gamma)$ with

$$\mathsf{E}^{n}_{\nu}[\mathsf{E}^{n+1}_{\alpha}] \vdash^{\xi}_{<\omega} \Gamma,$$

the claim. $\square$

3.3 Transfinite reduction

The transfinite reductions in our context are very similar to the reduction of transfinitely many fixed points (cf. [13] Main Lemma II) or to the reduction of transfinitely many n-inaccessibles (cf. [18] Theorem 10). Roughly spoken, the hard part is the finite reduction, since usually for that we need asymmetric interpretations and embeddings and "back-embeddings". On the other hand, when we inspect the proofs of the transfinite reductions we see that nearly nothing happens: The initial step of the induction follows from the finite reduction, and the induction step essentially follows from the induction hypothesis. Again we distinguish two cases: E^0_α and E^{n+1}_α. We start with the first case.

3.3.1 Transfinite reduction of E^0_α

The following theorem corresponds to Main Lemma II in [13]. Also the proof is very similar.

Theorem 74 *Assume $\mathsf{E}^0_{\beta+\omega^{1+\rho}} \vdash^{\alpha}_{1} \Gamma$ for a finite set*

$$\Gamma \subset \bigcup_{\delta<\beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c).$$

Then we have for all ordinals ξ less than $\omega^{1+\rho}$:

$$\Gamma \subset \bigcup_{\delta<\beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c) \quad \Longrightarrow \quad \mathsf{E}^0_{\beta+\xi} \vdash^{\varphi 1\rho\alpha}_{1} \Gamma.$$

Proof. We follow the proof of Main Lemma II in [13]. We prove the claim by main induction on ρ and side induction on α. We distinguish the cases $\rho = 0$, ρ is a successor or ρ is a limit ordinal. Here we discuss only the case $\rho = 0$, since the other cases are nearly identical with the corresponding cases in the proof of Main Lemma II in [13].

Let us assume that $\rho = 0$ and that Γ is a finite set of $\mathcal{L}^0_{\beta+n}$ formulas of $\mathsf{E}^0_{\beta+n}$ for some natural number n so that $\mathsf{E}^0_{\beta+\omega} \vdash^{\alpha}_{1} \Gamma$. If Γ is an axiom of $\mathsf{E}^0_{\beta+n}$, then the claim is trivial. Furthermore, if Γ is the conclusion of a rule different from the cut rule, the claim is immediate from the induction hypothesis. Hence, the only critical case comes up if Γ is the conclusion of a cut-rule. Then there exist a natural number $m \geq n$, ordinals $\alpha_0, \alpha_1 < \alpha$ and an $\mathcal{L}^0_{\beta+m}$ formula φ such that all $\mathsf{D}^0_\theta, \mathsf{D}^0_{<\lambda}$ in φ fulfills $\lambda, \theta < \beta + m$ and so that

$$\mathsf{E}^0_{\beta+\omega} \vdash^{\alpha_0}_{1} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^0_{\beta+\omega} \vdash^{\alpha_1}_{1} \Gamma, \neg\varphi.$$

By the induction hypothesis we can conclude that

$$\mathsf{E}^0_{\beta+m} \vdash^{\varphi 10\alpha_0}_{1} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^0_{\beta+m} \vdash^{\varphi 10\alpha_1}_{1} \Gamma, \neg\varphi$$

and an application of the cut-rule yields

$$\mathsf{E}^0_{\beta+m} \vdash^{\gamma}_{<\omega} \Gamma$$

for $\gamma := \max(\varphi 10\alpha_0, \varphi 10\alpha_1) + 1$. And partial cut elimination gives $\mathsf{E}^0_{\beta+m} \vdash^{<\varepsilon(\gamma)}_{1} \Gamma$. If $m = n$, we are done. Otherwise, successive application of corollary 70 (finite reduction) and partial cut elimination gives $\mathsf{E}^0_{\beta+n} \vdash^{\varphi 10\alpha}_{1} \Gamma$. □

Notice that we have proved in corollary 70 a reduction of $\mathsf{E}^0_{\alpha+1}$ to E^0_α. There is no difficulties to generalize this to a reduction of $\mathsf{E}^0_{\alpha+1}[\mathsf{E}^1_{\alpha_1}[\ldots \mathsf{E}^k_{\alpha_k}]\ldots]]$ to $\mathsf{E}^0_{\alpha}[\mathsf{E}^1_{\alpha_1}[\ldots \mathsf{E}^k_{\alpha_k}]\ldots]]$. Hence, we can use this in order to prove a generalised version of theorem 74. Since the proof of this generalisation is more or less the same, we omit it.

Theorem 75 *Assume* $\mathsf{E}^0_{\beta+\omega^{1+\rho}}[\mathsf{E}^1_{\alpha_1}[\ldots \mathsf{E}^k_{\alpha_k}]\ldots]] \vdash^{\alpha}_{1} \Gamma$ *for a finite set*

$$\Gamma \subset \bigcup_{\delta < \beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c).$$

Then we have for all ordinals ξ less than $\omega^{1+\rho}$:

$$\Gamma \subset \bigcup_{\delta < \beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c) \quad \Longrightarrow \quad \mathsf{E}^0_{\beta+\xi}[\mathsf{E}^1_{\alpha_1}[\ldots \mathsf{E}^k_{\alpha_k}]\ldots]] \vdash^{\varphi 1\rho\alpha}_{1} \Gamma.$$

3.3.2 Transfinite reduction of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots \mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$

We give in this subsection the central theorem which leads to the proof-theoretic upper bound of Σ^1_1-TDC_0.

Theorem 76 *Assume* $\mathsf{E}^n_{\beta+\omega^{1+\rho}}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]] \overset{\alpha}{\underset{1}{\vdash}} \Gamma$ *for a finite subset*

$$\Gamma \subset \bigcup_{\delta<\beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^n_\delta)^c) \cup ess\text{-}\Pi^1_1(\mathsf{D}^n_\delta)^c).$$

Then we have for all ξ *less than* $\omega^{1+\rho}$*:*

$$\Gamma \subset \bigcup_{\delta<\beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^n_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^n_\delta)^c) \quad \Longrightarrow \quad \mathsf{E}^n_{\beta+\xi}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]] \overset{\varphi(n+1)\rho\alpha}{\underset{1}{\vdash}} \Gamma.$$

Proof. The proof is by metainduction on n. The case $n = 0$ is exactly theorem 75. It remains to prove the claim for $n > 0$. Therefore, we assume $n > 0$. Notice that we have the induction hypothesis for all natural numbers k. For simplicity we set $k = 0$, $k > 0$ can be proved similarly. We prove the claim by main induction on ρ and side induction on α. We distinguish the cases $\rho = 0$, ρ is a successor or ρ is a limit ordinal. Here we discuss again only the case $\rho = 0$, since the other two cases are nearly identical with the corresponding cases in the proof of Main Lemma II in [13].

Let us assume $\rho = 0$ and that Γ is a finite set of $\mathcal{L}^n_{\beta+l}$ formulas of $\mathsf{E}^n_{\beta+\omega}$ for some natural number l so that $\mathsf{E}^n_{\beta+\omega} \overset{\alpha}{\underset{1}{\vdash}} \Gamma$. Again, the only critical case comes up if Γ is the conclusion of a cut-rule. Then there exist a natural number $m \geq l$, ordinals $\alpha_0, \alpha_1 < \alpha$ and a $\mathcal{L}^n_{\beta+m}$ formula φ such that all $\mathsf{D}^n_\theta, \mathsf{D}^n_{<\lambda}$ in φ fulfills $\lambda, \theta < \beta + m$ and so that

$$\mathsf{E}^n_{\beta+\omega} \overset{\alpha_0}{\underset{1}{\vdash}} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^n_{\beta+\omega} \overset{\alpha_1}{\underset{1}{\vdash}} \Gamma, \neg\varphi.$$

By the induction hypothesis we can conclude that

$$\mathsf{E}^n_{\beta+m} \overset{\varphi(n+1)0\alpha_0}{\underset{1}{\vdash}} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^n_{\beta+m} \overset{\varphi(n+1)0\alpha_1}{\underset{1}{\vdash}} \Gamma, \neg\varphi.$$

An application of the cut rule yields

$$\mathsf{E}^n_{\beta+m} \overset{\gamma}{\underset{<\omega}{\vdash}} \Gamma$$

for $\gamma := \max(\varphi(n+1)0\alpha_0, \varphi(n+1)0\alpha_1) + 1$. And a partial cut elimination gives

$$\mathsf{E}^n_{\beta+m} \overset{<\varepsilon(\gamma)}{\underset{1}{\vdash}} \Gamma.$$

If $m = l$, we are done. Otherwise an application of corollary 73 yields

$$\mathsf{E}^{n-1}_\nu[\mathsf{E}^n_{\beta+m-1}] \overset{<\varepsilon(\gamma)}{\underset{<\omega}{\vdash}} \Gamma$$

for a ν less than $\varepsilon(\gamma)$. We have not proved partial cut elimination for $\mathsf{E}^{n-1}_\nu[\mathsf{E}^n_{\beta+m-1}]$. But it is clear that we can do this as for E^{n-1}_ν. Hence

$$\mathsf{E}^{n-1}_\nu[\mathsf{E}^n_{\beta+m-1}] \overset{<\varepsilon(\gamma)}{\underset{1}{\vdash}} \Gamma.$$

Now, we use the metainduction hypothesis and conclude that

$$\mathsf{E}_0^{n-1}[\mathsf{E}^n_{\beta+m-1}] \overset{<\varphi n\varepsilon(\gamma)0}{\underset{1}{\vdash}} \Gamma.$$

Since $\mathsf{E}_0^{n-1}[\mathsf{E}^n_{\beta+m-1}]$ is just $\mathsf{E}^n_{\beta+m-1}$ we have $\mathsf{E}^n_{\beta+m-1} \overset{<\varphi\varepsilon(\gamma)0}{\underset{1}{\vdash}} \Gamma$. We do this again and again until we have $m-1=l$. Therefore $\mathsf{E}^n_{\beta+l} \overset{\varphi(n+1)0\alpha}{\underset{1}{\vdash}} \Gamma$. □

3.3.3 Proof-theoretic upper bound of T^{n+1}_m and T^0_α

In this subsection we collect the results of the preceding subsections. Moreover we will present these results in such a form that we can directly apply them in the proof-theoretic analysis of our theories. We write $\mathsf{PA}^\star$ for a Tait-style reformulation (with ω-rule) of the Peano arithmetic PA (cf. for instance [23]).

Theorem 77 *Assume that α is an ordinal less than Φ_0 given in the form*

$$\alpha = \omega^{1+\alpha_n} + \omega^{1+\alpha_{n-1}} + \ldots + \omega^{1+\alpha_1} + m$$

for ordinals $\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1$ and $m < \omega$. We set

$$\begin{array}{ll} (\alpha|0) := \varepsilon(\alpha) \qquad \textit{and} & (\alpha|m+1) := \varphi(\alpha|m)0 \\ \textit{and} \qquad \delta := \varphi 1\alpha_n(\varphi 1\alpha_{n-1}(\ldots\varphi 1\alpha_1(\alpha|m)\ldots)). & \end{array}$$

Then we have for all sentences φ of $\mathcal{L}_1$ and for all ordinals $\nu < \varepsilon(\alpha)$:

$$\mathsf{T}^0_\alpha \overset{\nu}{\underset{<\omega}{\vdash}} \varphi \quad\Longrightarrow\quad \mathsf{PA}^\star \overset{<\delta}{\underset{0}{\vdash}} \varphi.$$

Proof. We assume $\mathsf{T}^0_\alpha \overset{\nu}{\underset{<\omega}{\vdash}} \varphi$. From lemma 58 and lemma 62 we conclude that $\mathsf{E}^0_\alpha \overset{<\varepsilon(\nu)}{\underset{1}{\vdash}} \varphi$. (If α is not a successor we can prove a similar embedding as it is given in lemma 62.) Applying m-times corollary 70 leads to

$$\mathsf{E}^0_{\omega^{1+\alpha_n}+\ldots+\omega^{1+\alpha_1}} \overset{<(\alpha|m)}{\underset{1}{\vdash}} \varphi.$$

We now use n-times theorem 74 and conclude

$$\mathsf{E}^0_0 \overset{<\delta}{\underset{1}{\vdash}} \varphi.$$

We can embed E^0_0 into PA^* and obtain the claim by predicative cut elimination (in PA^*). □

Theorem 78 *We set $\gamma_{\nu,0} := \varepsilon(\nu)$ and $\gamma_{\nu,k+1} := \varphi n\gamma_{\nu,k}0$ for $n > 0$. Then we have for all sentences of $\mathcal{L}_1$ and for $n > 0$:*

$$\mathsf{T}^n_m \overset{\nu}{\underset{<\omega}{\vdash}} \varphi \quad\Longrightarrow\quad \mathsf{PA}^\star \overset{\gamma_{\nu,m}}{\underset{0}{\vdash}} \varphi.$$

Proof. We assume $\mathsf{T}^n_m \vdash^{\nu}_{<\omega} \varphi$. Lemma 58 and lemma 62 lead to $\mathsf{E}^n_m \vdash^{<\varepsilon(\nu)}_{1} \varphi$. If $m = 0$, we embed E^n_0 into $\mathsf{PA}^\star$ and get the claim by predicative cut elimination. Now we assume $m > 0$. We conclude from corollary 73 that there is an ordinal α less than $\varepsilon(\nu)$ with

$$\mathsf{E}^{n-1}_{\alpha}[\mathsf{E}^n_m] \vdash^{\alpha}_{<\omega} \varphi.$$

An application of theorem 76 gives

$$\mathsf{E}^{n-1}_{0}[\mathsf{E}^n_{m-1}] \vdash^{\varphi n \alpha \alpha}_{1} \varphi.$$

This is $\mathsf{E}^n_{m-1} \vdash^{\varphi n \alpha \alpha}_{1} \varphi$. We distinguish two cases. First we assume $m - 1 = 0$. In this case we can embed E^n_{m-1} into $\mathsf{PA}^\star$ and a predicative cut elimination yields the claim. Next, assume $m - 1 > 0$. Applying corollary 73 and theorem 76 $(m-1)$-times and arguing as in the case $m - 1 = 0$ leads to the claim. □

Chapter 4

Proof-theoretic strengths

4.1 An interpretation of $\mathsf{MUT}^=$ into T^0_α

In this section we give an asymmetric interpretation of $\mathsf{MUT}^=$ into T^0_α. We sum up the proceeding: First we show that without loss of generality we can take the minimality condition (5.3) in $\mathsf{MUT}^=$ only for the rel-$\Pi^1_0(\mathsf{U})$ formulas instead for the whole class of rel-$\Delta^1_1(\mathsf{U})$ formulas. Considering this we will introduce the corresponding Tait-style reformulation $(\mathsf{MUT}^=)^T$ of MUT. Then we prove an asymmetric interpretation of $(\mathsf{MUT}^=)^T$ into T^0_α. This leads finally to the interpretation of $\mathsf{MUT}^=$ into T^0_α $(\alpha < \varepsilon_0)$.

As mentioned we start with the reduction of the minimality condition (5.3) for rel-$\Delta^1_1(\mathsf{U})$ to rel-$\Pi^1_0(\mathsf{U})$.

Lemma 79 *Let T denote the theory $\mathsf{MUT}^=$ where the minimal universe axiom (5.3) is formulated only for rel-$\Pi^1_0(\mathsf{U})$ formulas. Then T proves the (full) minimal universe axiom (5.3).*

Proof. We argue in T. Choose rel-$\Pi^1_0(\mathsf{U})$ formulas φ, ψ with

$$((\exists Z)\varphi(Z,E) \leftrightarrow (\forall Z)\psi(Z,E)) \wedge (\exists D)(\mathsf{U}(D) \wedge (\exists Z)\varphi(Z,D)).$$

We have to show that there is a minimal universe F with $(\exists Z)\varphi(Z,F)$. We can choose a universe E such that there is a universe D in E such that $(\exists Z)\varphi(Z,D)$ holds. Now set

$$H := \{\langle x,k\rangle : x \in (E)_k \wedge (\exists Z)\varphi(Z,(E)_k) \wedge \mathsf{U}((E)_k)\}.$$

H is a rel-$\Delta^1_1(\mathsf{U})$ set, since we have

$$\begin{aligned}\langle x,k\rangle \in H &\leftrightarrow (\forall Z)(x \in (E)_k \wedge \psi(Z,(E)_k) \wedge \mathsf{U}((E)_k)) \\ &\leftrightarrow (\exists Z)(x \in (E)_k \wedge \varphi(Z,(E)_k) \wedge \mathsf{U}((E)_k)).\end{aligned}$$

We also know
$$\mathsf{U}(X) \to (X \dot{\in} H \leftrightarrow (X \dot{\in} E \wedge (\exists Z)\varphi(Z, X))).$$
Hence the universe D is in H. An application of the minimal universe axiom (of T) to the formula $D \dot{\in} H$ yields a universe F such that
$$F \dot{\in} H \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \to X \dot{\notin} H).$$
Hence, we conclude
$$F \dot{\in} E \wedge (\exists Z)\varphi(Z, F) \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \to (X \dot{\notin} E \vee \neg(\exists Z)\varphi(Z, X))).$$
But we have for all universes X in F that X is in E. Therefore
$$(\exists Z)\varphi(Z, F) \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \to \neg(\exists Z)\varphi(Z, X)).$$
This is the claim. $\square$

Now we give a Tait-style version $(\mathsf{MUT}^{=})^T$ of $\mathsf{MUT}^{=}$. Again $\Gamma, \Lambda, \ldots$ are finite sets of $\mathcal{L}_2(\mathsf{U})$ formulas and Γ, φ is a shorthand for $\Gamma \cup \{\varphi\}$. The system $(\mathsf{MUT}^{=})^T$ contains the following axioms and rules of inference:

1. Ontological axioms I. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all set variables X:
$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X, s \notin X \qquad \text{and} \qquad \Gamma, \mathsf{U}(X), \neg\mathsf{U}(X).$$

2. Propositional and quantifier rules. These include the usual Tait-style inference rules for the propositional connectives and all sorts of quantifiers (especially the ω-rule).

3. Ontological axioms II. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas and all set variables X, Y:
$$\Gamma, \neg\mathsf{U}(X), X \neq Y, \mathsf{U}(Y).$$

4. Set axioms and rules. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas and all rel-$\Pi^1_0(\mathsf{U})$ formulas φ:
$$\Gamma, (\exists X)(x \in X \leftrightarrow \varphi(x)), \qquad \frac{\Gamma, (\forall x)(\exists X)\varphi(x, X)}{\Gamma, (\exists X)(\forall x)\varphi(x, (X)_x)}.$$

5. Closure axioms. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas, all set variables X, Z, D and all rel-$\Pi^1_0(\mathsf{U})$ formulas φ:
$$\Gamma, \neg\mathsf{U}(D), X \dot{\notin} D, Z \dot{\notin} D, X \oplus Z \dot{\in} D,$$
$$\Gamma, \neg\mathsf{U}(D), Z \dot{\notin} D, (\exists Y \dot{\in} D)(\forall x)(x \in Y \leftrightarrow \varphi[x, Z]),$$
$$\Gamma, \neg\mathsf{U}(D), Z \dot{\notin} D, \neg(\forall x)(\exists Y \dot{\in} D)\varphi[x, Y, Z], (\exists Y \dot{\in} D)(\forall x)\varphi[x, (Y)_x, Z].$$

6. Universe axioms. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas, all set variables X, D, E and all $\mathcal{L}_2(\mathsf{U})$ formulas φ in rel-$\Pi^1_0(\mathsf{U})$:

$$\Gamma, (\exists Z)(X \dot{\in} Z \land \mathsf{U}(Z)),$$
$$\Gamma, \neg\mathsf{U}(D), \neg\mathsf{U}(E), D \dot{\in} E, D = E, E \dot{\in} D,$$
$$\Gamma, (\forall Z)(\neg\mathsf{U}(Z) \lor \neg\varphi(Z)), (\exists Z)[\mathsf{U}(Z) \land \varphi(Z) \land (\forall F \dot{\in} Z)(\mathsf{U}(F) \to \neg\varphi(F))].$$

7. Cut rules. For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas and all $\mathcal{L}_2(\mathsf{U})$ formulas φ:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In a next step we define the classes of $\mathcal{L}_2(\mathsf{U})$ formulas $essrel$-$\Sigma^1_1(\mathsf{U})$ and $essrel$-$\Pi^1_1(\mathsf{U})$. They correspond to ess-Σ^1_1 and ess-Π^1_1 (cf. for example [2]).

Definition 80 The $essrel$-$\Sigma^1_1(\mathsf{U})$ ($essrel$-$\Pi^1_1(\mathsf{U})$) formulas are inductively defined as follows:

1. Each rel-$\Pi^1_0(\mathsf{U})$ formula is an $essrel$-$\Sigma^1_1(\mathsf{U})$ and an $essrel$-$\Pi^1_1(\mathsf{U})$ formula.

2. If φ, ψ are $essrel$-$\Sigma^1_1(\mathsf{U})$ (resp. $essrel$-$\Pi^1_1(\mathsf{U})$) formulas, then so also are $\varphi \lor \psi$, $\varphi \land \psi$, $\forall x\varphi$, $\exists x\varphi$, $(\forall X \dot{\in} Y)\varphi$, $(\exists X \dot{\in} Y)\varphi$, $\exists X\varphi$ (resp. $\forall X\varphi$).

Definition 81 The rank $rk(\varphi)$ of an $\mathcal{L}_2(\mathsf{U})$ formula φ and of $(\mathsf{MUT}^=)^T$ is inductively defined as follows:

If φ is an $essrel$-$\Sigma^1_1(\mathsf{U})$ or an $essrel$-$\Pi^1_1(\mathsf{U})$ formula, then $rk(\varphi) := 0$. Otherwise:

1. If φ is a formula $\psi \lor \theta$ or $\psi \land \theta$, then $rk(\varphi) := max(rk(\psi), rk(\theta)) + 1$.

2. If φ is a formula $\exists x\psi$, $\forall x\psi$, $\exists X\psi$, $\forall X\psi$, then $rk(\varphi) := rk(\psi) + 1$.

Corresponding to this rank we have partial cut elimination. Furthermore, we can embed $\mathsf{MUT}^=$ into $(\mathsf{MUT}^=)^T$. Again the proof is standard and we omit it.

Lemma 82 *For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas and all $\mathcal{L}_2(\mathsf{U})$ formulas φ we have:*

a) $(\mathsf{MUT}^=)^T \mathrel{\vert\!\frac{\alpha}{k+1}} \Gamma \implies (\mathsf{MUT}^=)^T \mathrel{\vert\!\frac{\omega_k(\alpha)}{1}} \Gamma,$

b) $\mathsf{MUT}^= \vdash \varphi[\vec{x}] \implies (\mathsf{MUT}^=)^T \mathrel{\vert\!\frac{<\omega+\omega}{<\omega}} \varphi[\vec{t}\,]$ *for all closed terms t.*

Now we define the translation which is used in the asymmetric interpretation.

Definition 83 For each $\mathcal{L}_2(\mathsf{U})$ formula φ we define the $\mathcal{L}^0_\gamma$ formula $\varphi^{\alpha,\beta,\gamma}$ of T^0_γ inductively as follows: $(\alpha, \beta < \gamma)$

1. If φ does not contain a subformula $\mathsf{U}(X)$ and is in rel-$\Pi^1_0(\mathsf{U})$, then $\varphi^{\alpha,\beta,\gamma} := \varphi$.

2. If φ is of the form $\mathsf{U}(X)$, then $\varphi^{\alpha,\beta,\gamma} := (\exists d \prec \gamma)(X = (\mathsf{D}^0_{<\gamma})_d)$.

3. If φ is of the form $\neg\mathsf{U}(X)$, then $\varphi^{\alpha,\beta,\gamma} := (\forall d \prec \gamma)(X \neq (\mathsf{D}^0_{<\gamma})_d)$.

4. If φ is of the form $\psi \wedge \theta$ (resp. $\psi \vee \theta$), then $\varphi^{\alpha,\beta,\gamma} := \psi^{\alpha,\beta,\gamma} \wedge \theta^{\alpha,\beta,\gamma}$ (resp. $\varphi^{\alpha,\beta,\gamma} := \psi^{\alpha,\beta,\gamma} \vee \theta^{\alpha,\beta,\gamma}$).

5. If φ is of the form $\exists x\psi$ (resp. $\forall x\psi$), then $\varphi^{\alpha,\beta,\gamma} := \exists x\psi^{\alpha,\beta,\gamma}$ (resp. $\varphi^{\alpha,\beta,\gamma} := \forall x\psi^{\alpha,\beta,\gamma}$).

6. If φ is of the form $(\exists X \dot\in Y)\psi(X)$ (resp. $(\forall X \dot\in Y)\psi(X)$), then $\varphi^{\alpha,\beta,\gamma} := (\exists X \dot\in Y)\psi^{\alpha,\beta,\gamma}(X)$ (resp. $\varphi^{\alpha,\beta,\gamma} := (\forall X \dot\in Y)\psi^{\alpha,\beta,\gamma}(X)$).

7. If φ is of the form $\exists X\psi$ and $\psi \not\equiv X \dot\in Y \wedge \theta(X)$ for each θ, Y, then $\varphi^{\alpha,\beta,\gamma} := (\exists X \dot\in \mathsf{D}^0_\beta)\psi^{\alpha,\beta,\gamma}(X)$.

8. If φ is of the form $\forall X\psi$ and $\psi \not\equiv X \dot\in Y \to \theta(X)$ for each θ, Y, then $\varphi^{\alpha,\beta,\gamma} := (\forall X \dot\in \mathsf{D}^0_\alpha)\psi^{\alpha,\beta,\gamma}(X)$.

We remember that $\exists X \dot\in Y$ (resp. $\forall X \dot\in Y$) is an abbreviation for $(\exists X)(X \dot\in Y \wedge \ldots)$ (resp. $(\forall X)(X \dot\in Y \to \ldots)$ in T^0_γ. Whereas the quantifiers $\exists X \dot\in \mathsf{D}^0_\beta$ (resp. $\forall X \dot\in \mathsf{D}^0_\beta$) are no abbreviations, they belong to the language of T^0_γ. Sometimes we use the terminology $\varphi^{\alpha,\beta,\gamma}$ also in the theory $\bar{\mathsf{T}}^0_\delta$. Then we mean with $\varphi^{\alpha,\beta,\gamma}$ the formula $\varphi^{\alpha,\beta,\gamma}$ where we write instead of the set constants $\mathsf{D}^0_\nu, \mathsf{D}^0_{<\xi}$ the expressions $\mathsf{D}^0_\nu, \mathsf{D}^0_{\prec\xi}$ and where we write instead of the bounded second order quantifiers $\exists X \dot\in \mathsf{D}^0_\beta$, $\forall X \dot\in \mathsf{D}^0_\beta$ the expressions $(\exists X)(X \dot\in \mathsf{D}^0_\beta \wedge \ldots)$, $(\forall X)(X \dot\in \mathsf{D}^0_\beta \to \ldots)$. In the next lemma we formulate the persistency of our translation.

Lemma 84 *For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas, all $\mathcal{L}_2(\mathsf{U})$ formulas φ and all ordinals $\alpha, \beta, \beta', \gamma, \gamma', \delta$ with $\beta' < \beta < \delta$, $\gamma < \gamma' < \delta$, $\delta \leq \alpha$, $\alpha \in \Phi_0$ we have:*

$$\mathsf{T}^0_\alpha \,\frac{\rho}{<\omega}\, \Gamma, \varphi^{\beta,\gamma,\delta} \quad \Longrightarrow \quad \mathsf{T}^0_\alpha \,\frac{<\rho+\omega}{<\omega}\, \Gamma, \varphi^{\beta',\gamma',\delta}.$$

Proof. The proof is by induction on ρ. As an example we discuss the cases where $\varphi^{\beta,\gamma,\delta}$ is mainformula of the $(\exists X)$-rule and where $\varphi^{\beta,\gamma,\delta}$ is mainformula of the $(\forall X \dot\in \mathsf{D}^0_\beta)$-rule.

1. We assume $\varphi^{\beta,\gamma,\delta}$ is mainformula of the $(\exists X)$-rule. In this case φ is of the form $(\exists X \dot\in Y)\theta(X)$. There is a $\rho_0 < \rho$ with:

$$\mathsf{T}^0_\alpha \,\frac{\rho_0}{<\omega}\, \Gamma, X \dot\in Y \wedge \theta^{\beta,\gamma,\delta}(X), \varphi^{\beta,\gamma,\delta}.$$

   The induction hypothesis and an application of the $(\exists X)$-rule yields the claim.

2. We assume $\varphi^{\beta,\gamma,\delta}$ is mainformula of the $(\forall X \mathrel{\dot\in} \mathsf{D}^0_\beta)$-rule. There is a a $\rho_0 < \rho$ and a formula ψ with:
$$\mathsf{T}^0_\alpha \mathrel{\frac{\;\rho_0\;}{<\omega}} \Gamma, X \mathrel{\dot\in} \mathsf{D}^0_\beta \to \psi^{\beta,\gamma,\delta}(X), \varphi^{\beta,\gamma,\delta}.$$
An application of the induction hypothesis and $\vee$-exportation yields
$$\mathsf{T}^0_\alpha \mathrel{\frac{<\rho+\omega}{<\omega}} \Gamma, X \mathrel{\dot\notin} \mathsf{D}^0_\beta, \psi^{\beta',\gamma',\delta}(X), \varphi^{\beta',\gamma',\delta}.$$
In T^0_α we can deduce for $\beta' < \beta < \alpha$ with finite deduction length $X \mathrel{\dot\in} \mathsf{D}^0_\beta, X \mathrel{\dot\notin} \mathsf{D}^0_{\beta'}$. Hence a cut implies
$$\mathsf{T}^0_\alpha \mathrel{\frac{<\rho+\omega}{<\omega}} \Gamma, X \mathrel{\dot\notin} \mathsf{D}^0_{\beta'}, \psi^{\beta',\gamma',\delta}(X), \varphi^{\beta',\gamma',\delta}.$$
Thus, the $\vee$-rule and the $(\forall X \mathrel{\dot\in} \mathsf{D}^0_{\beta'})$-rule yield the claim. □

The following technical lemma will be used in the asymmetric interpretation.

Lemma 85 *For all $\mathcal{L}_2(\mathsf{U})$ formulas φ in rel-$\Pi^1_0(\mathsf{U})$ there exists an $\mathcal{L}_2(\mathsf{D}^0)$ formula ψ in Π^1_0 such that $\bar{\mathsf{T}}^0_\alpha$ proves for all $\beta < \alpha, \gamma < \alpha, \delta \leq \alpha$ and without use of the set-induction axiom (5):*
$$\vec{Z} \mathrel{\dot\in} \mathsf{D}^0_c \wedge c \prec \delta \to (\varphi^{\beta,\gamma,\delta}[\vec{z}, \vec{Z}] \leftrightarrow \psi[\vec{z}, y, \vec{Z}, Y][y/c, Y/\mathsf{D}^0_{\prec c}]).$$

Proof. The proof is by induction on the build-up of φ. We only discuss two cases:

1. $\varphi \equiv \mathsf{U}(Z)$: We know $(\mathsf{U}(Z))^{\beta,\gamma,\delta} = (\exists d \prec \delta)(Z = \mathsf{D}^0_d)$. We set
$$\psi := (\exists d \prec y)(Z = (Y)_d).$$
Assume $Z \mathrel{\dot\in} \mathsf{D}^0_c$ and $c \prec \delta$. For $d \succeq c$ we know $\mathsf{D}^0_d \mathrel{\dot\notin} \mathsf{D}^0_c$. Hence we conclude
$$\begin{aligned}(\exists d \prec \delta)(Z = \mathsf{D}^0_d) &\leftrightarrow (\exists d \prec c)(Z = (\mathsf{D}^0_{\prec c})_d)\\ &\leftrightarrow (\exists d \prec y)(Z = (Y)_d)[y/c, Y/\mathsf{D}^0_{\prec c}].\end{aligned}$$

2. $\varphi \equiv (\forall U \mathrel{\dot\in} X)\psi[\vec{z}, \vec{Z}, X, U]$: We apply the induction hypothesis to the formula ψ. This yields an $\mathcal{L}_2(\mathsf{D}^0)$ formula θ in Π^1_0 with
$$\vec{Z}, X, U \mathrel{\dot\in} \mathsf{D}^0_c \wedge c \prec \delta \to (\psi^{\beta,\gamma,\delta}[\vec{z}, \vec{Z}, X, U] \leftrightarrow \theta[\vec{z}, y, \vec{Z}, Y, X, U][y/c, Y/\mathsf{D}^0_{\prec c}]).$$
Assume $X, \vec{Z} \mathrel{\dot\in} \mathsf{D}^0_c$ and $c \prec \delta$. Then we conclude (notice that for $U \mathrel{\dot\in} X$ we have $U \mathrel{\dot\in} \mathsf{D}^0_c$)
$$\begin{aligned}\varphi^{\beta,\gamma,\delta}[\vec{z}, \vec{Z}] &\leftrightarrow (\forall U \mathrel{\dot\in} X)\psi^{\beta,\gamma,\delta}[\vec{z}, \vec{Z}, X, U]\\ &\leftrightarrow (\forall U \mathrel{\dot\in} X)\theta[\vec{z}, y, \vec{Z}, Y, X, U][y/c, Y/\mathsf{D}^0_{\prec c}]\\ &\leftrightarrow (\forall k)\theta[\vec{z}, y, \vec{Z}, Y, X, U][U/(X)_k, y/c, Y/\mathsf{D}^0_{\prec c}].\end{aligned}$$

□

We are ready to state the asymmetric interpretation.

Theorem 86 *For all finite sets Γ of $\mathcal{L}_2(\mathsf{U})$ formulas and all ordinals α, β, γ with $\beta + \omega^\gamma < \alpha < \varepsilon_0$ we have:*

$$(\mathsf{MUT}^=)^T \mathrel{\big|\!\frac{\gamma}{1}} \Gamma[\vec{X}] \quad \Longrightarrow \quad \mathsf{T}^0_\alpha \mathrel{\big|\!\frac{\omega^{\beta+\omega^\gamma}}{<\omega}} \vec{X} \dot{\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^\gamma,\alpha}[\vec{X}].$$

Proof. This theorem is proved by induction on γ. We write in this proof only $\varphi^{\delta,\lambda}$ for $\varphi^{\delta,\lambda,\alpha}$. We discuss four exemplary cases: minimal universe axiom, rel-$\Pi^1_0(\mathsf{U})$-AC-rule, $(\forall X)$-rule and cut.

1. Assume that in Γ occurs an instance of the minimal universe axiom:

   Choose a rel-$\Pi^1_0(\mathsf{U})$ formula φ. We assume that all free set parameters of φ are among $\vec{X}$. Then we have to show

$$\begin{aligned} \mathsf{T}^0_\alpha \mathrel{\big|\!\frac{\omega^{\beta+\omega^\gamma}}{<\omega}} \; & \vec{X} \dot{\notin} \mathsf{D}^0_\beta, (\forall Z \dot{\in} \mathsf{D}^0_\beta)((\forall d \prec \alpha)(Z \neq (\mathsf{D}^0_{<\alpha})_d) \vee (\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z)), \\ & (\exists Z \dot{\in} \mathsf{D}^0_{\beta+\omega^\gamma})[(\exists d \prec \alpha)(Z = (\mathsf{D}^0_{<\alpha})_d) \wedge \varphi^{\beta,\beta+\omega^\gamma}(Z) \wedge \\ & \qquad (\forall F \dot{\in} Z)((\exists d \prec \alpha)(F = (\mathsf{D}^0_{<\alpha})_d) \rightarrow (\neg\varphi)^{\beta,\beta+\omega^\gamma}(F))]. \end{aligned} \tag{4.1}$$

   First we show within $\bar{\mathsf{T}}^0_\alpha$ that $TI(\beta, H)$ implies without use of set-induction

$$\begin{aligned} & \vec{X} \dot{\in} \mathsf{D}^0_\beta \wedge (\exists Z \dot{\in} \mathsf{D}^0_\beta)((\exists d \prec \alpha)(Z = \mathsf{D}^0_d) \wedge \neg(\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z)) \\ & \rightarrow (\exists Z \dot{\in} \mathsf{D}^0_{\beta+\omega^\gamma})[(\exists d \prec \alpha)(Z = \mathsf{D}^0_d) \wedge \varphi^{\beta,\beta+\omega^\gamma}(Z) \wedge \\ & \qquad (\forall F \dot{\in} Z)((\exists d \prec \alpha)(F = \mathsf{D}^0_d) \rightarrow (\neg\varphi)^{\beta,\beta+\omega^\gamma}(F))]. \end{aligned} \tag{4.2}$$

   Choose an $\mathcal{L}_2(\mathsf{D}^0)$ formula ψ in Π^1_0 such that $\bar{\mathsf{T}}^0_\alpha$ proves (lemma 85)

$$\vec{X} \dot{\in} \mathsf{D}^0_c \wedge c \prec \alpha \rightarrow (\varphi^{\beta,\beta+\omega^\gamma}(Z) \leftrightarrow \psi(y, Z, Y)[y \backslash c, Y \backslash \mathsf{D}^0_{\prec c}]).$$

   Since φ is in rel-$\Pi^1_0(\mathsf{U})$ we know $(\neg\varphi)^{\beta,\beta+\omega^\gamma} \equiv \neg\varphi^{\beta,\beta+\omega^\gamma}$ and $\neg(\neg\varphi)^{\beta,\beta+\omega^\gamma} \equiv \varphi^{\beta,\beta+\omega^\gamma}$. By assumption there is a Z such that $Z = \mathsf{D}^0_d$, $d \prec \alpha$ and $\neg(\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z)$ holds. Hence we can choose $Z, \vec{X} \dot{\in} \mathsf{D}^0_\beta$, $d \prec \alpha$ such that $Z = \mathsf{D}^0_d \wedge \psi(\beta, Z, \mathsf{D}^0_{\prec\beta})$ holds. We have to prove

$$\begin{aligned} & (\exists G \dot{\in} \mathsf{D}^0_{\beta+\omega^\gamma})[(\exists e \prec \alpha)(G = \mathsf{D}^0_e) \wedge \psi(\beta, G, \mathsf{D}^0_{\prec\beta}) \wedge \\ & \qquad (\forall F \dot{\in} G)((\exists e \prec \alpha)(F = \mathsf{D}^0_e) \rightarrow \neg\psi(\beta, F, \mathsf{D}^0_{\prec\beta}))]. \end{aligned} \tag{4.3}$$

   We define $H := \{c : c \prec \beta \wedge \psi(\beta, \mathsf{D}^0_c, \mathsf{D}^0_{\prec\beta})\}$. We have $\mathsf{D}^0_d \dot{\in} \mathsf{D}^0_\beta$, hence $d \prec \beta, d \in H, H \neq \emptyset$. Therefore, we can choose a least c with $c \in H$, since we have assumed $TI(\beta, H)$. This immediately proves (4.3).

Let θ denote the formula (4.2). Then we can prove in T^0_α with finite deduction length (notice that we have introduced a translation $(\ldots)^\alpha$ of formulas of $\mathcal{L}_2(\mathsf{D}^0)$ which led to an embedding of $\bar{\mathsf{T}}^0_\alpha$ into T^0_α)

$$\neg TI(\beta, Y), \theta^\alpha.$$

Furthermore, standard arguments show

$$\mathsf{T}^0_\alpha \overset{\omega\beta}{\underset{<\gamma}{\vdash}} TI(\beta, Y)$$

and a cut imply

$$\mathsf{T}^0_\alpha \overset{<\omega^{\beta+\omega^\gamma}}{\underset{<\omega}{\vdash}} \theta^\alpha.$$

We let Λ denote the set of formulas in equation (4.1). Since we can prove in T^0_α with finite deduction length $\neg\theta^\alpha, \Lambda$, we obtain the claim by an application of the cut-rule.

2. Assume that Γ is the conclusion of the rel-$\Pi^1_0(\mathsf{U})$-AC-rule:

There is a $\gamma_0 < \gamma$ and a rel-$\Pi^1_0(\mathsf{U})$ formula φ with

$$(\mathsf{MUT}^=)^T \overset{\gamma_0}{\underset{1}{\vdash}} \Gamma[\vec{Z}], (\forall x)(\exists X)\varphi[x, \vec{z}, X, \vec{Z}]. \tag{4.4}$$

And we have to prove

$$\mathsf{T}^0_\alpha \overset{\omega^{\beta+\omega^\gamma}}{\underset{<\omega}{\vdash}} \vec{Z} \mathrel{\dot{\notin}} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^\gamma}[\vec{Z}], (\exists X \mathrel{\dot{\in}} \mathsf{D}^0_{\beta+\omega^\gamma})(\forall x)\varphi^{\beta,\beta+\omega^\gamma}[x, \vec{z}, (X)_x, \vec{Z}].$$

Two cases have to be to distinguished: φ is of the form $X \mathrel{\dot{\in}} U \wedge \theta$ for a θ, U or φ is different of $X \mathrel{\dot{\in}} U \wedge \theta$ for all θ, U. We only discuss the first case, because the second is very similar – in fact a bit easier.

So, choose a rel-$\Pi^1_0(\mathsf{U})$ formula θ with $\varphi \equiv X \mathrel{\dot{\in}} U \wedge \theta$. We have to show in T^0_α with deduction length $\omega^{\beta+\omega^\gamma}$

$$\vec{Z}, U \mathrel{\dot{\notin}} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^\gamma}[\vec{Z}], (\exists X \mathrel{\dot{\in}} \mathsf{D}^0_{\beta+\omega^\gamma})(\forall x)((X)_x \mathrel{\dot{\in}} U \wedge \theta^{\beta,\beta+\omega^\gamma}[x, \vec{z}, (X)_x, \vec{Z}, U]).$$

We apply the induction hypothesis to (4.4) and get in T^0_α with deduction length $\omega^{\beta+\omega^{\gamma_0}}$

$$U, \vec{Z} \mathrel{\dot{\notin}} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma_0}}[\vec{Z}], (\forall x)(\exists X)(X \mathrel{\dot{\in}} U \wedge \theta^{\beta,\beta+\omega^{\gamma_0}}[x, \vec{z}, X, \vec{Z}, U]).$$

A similar argumentation as in case 1 yields that it is enough to prove in $\bar{\mathsf{T}}^0_\alpha$ – but now without using set-induction –

$$\begin{aligned} &U, \vec{Z} \mathrel{\dot{\in}} \mathsf{D}^0_\beta \wedge (\forall x)(\exists X)(X \mathrel{\dot{\in}} U \wedge \theta^{\beta,\beta+\omega^{\gamma_0}}[x, \vec{z}, X, \vec{Z}, U]) \\ &\quad \rightarrow (\exists X \mathrel{\dot{\in}} \mathsf{D}^0_{\beta+\omega^\gamma})(\forall x)((X)_x \mathrel{\dot{\in}} U \wedge \theta^{\beta,\beta+\omega^\gamma}[x, \vec{z}, (X)_x, \vec{Z}, U]). \end{aligned}$$

Again an application of lemma 85 yields a $\mathcal{L}_2(\mathsf{D}^0)$ formula ψ in Π^1_0 such that $\bar{\mathsf{T}}^0_\alpha$ proves

$$U, X, \vec{Z} \mathrel{\dot\in} \mathsf{D}^0_\beta \to (\theta^{\beta,\beta+\omega^{\gamma_0}}[x, \vec{z}, X, \vec{Z}, U] \leftrightarrow \psi[x, \vec{z}, y, X, \vec{Z}, Y, U][y/\beta, Y/\mathsf{D}^0_{\prec\beta}]).$$

We assume $U, \vec{Z} \mathrel{\dot\in} \mathsf{D}^0_\beta$ and conclude in $\bar{\mathsf{T}}^0_\alpha$

$$\begin{aligned}
&(\forall x)(\exists X)(X \mathrel{\dot\in} U \wedge \theta^{\beta,\beta+\omega^{\gamma_0}}[x, \vec{z}, X, \vec{Z}, U]) \\
&\quad\to\quad (\forall x)(\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(X \mathrel{\dot\in} U \wedge \psi[x, \vec{z}, \beta, X, \vec{Z}, \mathsf{D}^0_{\prec\beta}, U]) \\
&\quad\to\quad (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(\forall x)((X)_x \mathrel{\dot\in} U \wedge \psi[x, \vec{z}, \beta, (X)_x, \vec{Z}, \mathsf{D}^0_{\prec\beta}, U]) \\
&\quad\to\quad (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(\forall x)((X)_x \mathrel{\dot\in} U \wedge \theta^{\beta,\beta+\omega^{\gamma_0}}[x, \vec{z}, (X)_x, \vec{Z}, U]) \\
&\quad\to\quad (\exists X \mathrel{\dot\in} \mathsf{D}^0_{\beta+\omega^\gamma})(\forall x)((X)_x \mathrel{\dot\in} U \wedge \theta^{\beta,\beta+\omega^\gamma}[x, \vec{z}, (X)_x, \vec{Z}, U]).
\end{aligned}$$

3. Assume that Γ is the conclusion of the $(\forall Y)$-rule:

There is a $\gamma_0 < \gamma$ and a formula φ with

$$(\mathsf{MUT}^=)^T \vdash^{\gamma_0}_{1} \Gamma[\vec{X}], \varphi[\vec{X}, Y] \tag{4.5}$$

such that Y do not occur in $\Gamma[\vec{X}]$. Again we distinguish two cases: φ is of the form $Y \mathrel{\dot\in} Z \to \theta$ for a θ, Z or φ is different of $Y \mathrel{\dot\in} Z \to \theta$ for all θ, Z. Here we discuss only the first case, because the second is similar.

Let us choose θ with $\varphi \equiv Y \mathrel{\dot\in} Z \to \theta$. (There is an i with $X_i \equiv Z$.) We have to show

$$\mathsf{T}^0_\alpha \vdash^{\omega^{\beta+\omega^\gamma}}_{<\omega} \vec{X} \mathrel{\dot\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^\gamma}[\vec{X}], (\forall Y \mathrel{\dot\in} Z)\theta^{\beta,\beta+\omega^\gamma}.$$

We apply the induction hypothesis to (4.5) and get

$$\mathsf{T}^0_\alpha \vdash^{\omega^{\beta+\omega^{\gamma_0}}}_{<\omega} \vec{X}, Y, Z \mathrel{\dot\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma_0}}[\vec{X}], Y \mathrel{\dot\in} Z \to \theta^{\beta,\beta+\omega^{\gamma_0}}.$$

In T^0_α we can prove with finite deduction length

$$Y \mathrel{\dot\notin} Z, Z \mathrel{\dot\notin} \mathsf{D}^0_\beta, Y \mathrel{\dot\in} \mathsf{D}^0_\beta.$$

Therefore, with a cut (and $\vee$-exportation, $\vee$-importation) we get

$$\mathsf{T}^0_\alpha \vdash^{<\omega^{\beta+\omega^\gamma}}_{<\omega} \vec{X}, Z \mathrel{\dot\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma_0}}[\vec{X}], Y \mathrel{\dot\in} Z \to \theta^{\beta,\beta+\omega^{\gamma_0}}.$$

Notice that there is an i with $X_i \equiv Z$. Hence, persistency and the $(\forall Y)$-rule yield the claim.

4. Assume, that Γ is the conclusion of a cut:

   There are $\gamma_0 < \gamma, \gamma_1 < \gamma$ and (without loss of generality) a formula φ in $essrel$-$\Sigma^1_1(\mathsf{U})$ with
$$(\mathsf{MUT}^=)^T \vdash^{\gamma_0}_{1} \Gamma[\vec{X}], \varphi[\vec{Y}] \qquad \text{and} \qquad (\mathsf{MUT}^=)^T \vdash^{\gamma_1}_{1} \Gamma[\vec{X}], \neg\varphi[\vec{Y}].$$
   An application of the induction hypothesis yields (in (4.7) we have applied the induction hypothesis with "$\beta := \beta + \omega^{\gamma_0}$")
$$\mathsf{T}^0_\alpha \quad \vdash^{\omega^{\beta+\omega^{\gamma_0}}}_{<\omega} \quad \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma_0}}[\vec{X}], \varphi^{\beta,\beta+\omega^{\gamma_0}}[\vec{Y}], \tag{4.6}$$
$$\begin{aligned}\mathsf{T}^0_\alpha \quad \vdash^{\omega^{\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}}_{<\omega} \quad & \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_{\beta+\omega^{\gamma_0}}, \Gamma^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}[\vec{X}], \\ & (\neg\varphi)^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}[\vec{Y}].\end{aligned} \tag{4.7}$$
   With induction on the build-up of φ in $essrel$-$\Sigma^1_1(\mathsf{U})$ we can prove with finite deduction length
$$(\neg\varphi)^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}} \equiv \neg\varphi^{\beta,\beta+\omega^{\gamma_0}}.$$
   Hence, we conclude form (4.7)
$$\begin{aligned}\mathsf{T}^0_\alpha \quad \vdash^{<\omega^{\beta+\omega^{\gamma}}}_{<\omega} \quad & \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_{\beta+\omega^{\gamma_0}}, \Gamma^{\beta+\omega^{\gamma_0},\beta+\omega^{\gamma_0}+\omega^{\gamma_1}}[\vec{X}], \\ & \neg\varphi^{\beta,\beta+\omega^{\gamma_0}}[\vec{Y}]\end{aligned} \tag{4.8}$$
   Persistency and a cut applied to (4.6) and (4.8) yields
$$\mathsf{T}^0_\alpha \vdash^{<\omega^{\beta+\omega^{\gamma}}}_{<\omega} \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_{\beta+\omega^{\gamma_0}}, \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma}}[\vec{X}].$$
   Notice that in T^0_α we can prove with finite deduction length
$$\vec{Y} \dot{\notin} \mathsf{D}^0_\beta, \vec{Y} \dot{\in} \mathsf{D}^0_{\beta+\omega^{\gamma_0}}.$$
   Cuts imply
$$\mathsf{T}^0_\alpha \vdash^{<\omega^{\beta+\omega^{\gamma}}}_{<\omega} \vec{X}, \vec{Y} \dot{\notin} \mathsf{D}^0_\beta, \Gamma^{\beta,\beta+\omega^{\gamma}}[\vec{X}].$$
   By the same argumentation as in the proof of theorem 66 we obtain the claim. (If not all Y_i are among X, the elimination of these Y_i is as in the proof of theorem 66.)

□

Now we have done the work for $\mathsf{MUT}^=$. We can carry-out an analogous analysis of the theory $\mathsf{MUT}^=_0$, with the difference that then only finitely many D^0_n ($n \in \mathbb{N}$) are necessary. Instead of a rigorous proof, we give a short sketch of this procedure:

1. We fix a Tait-style reformulation $(\mathsf{MUT}^=_0)^T$ of $\mathsf{MUT}^=_0$. It looks like $(\mathsf{MUT}^=)^T$ but instead of the ω-rule we take the $(\forall x)$-rule; we have also to add set-induction.

2. As for $(\mathsf{MUT}^=)^T$ we prove partial cut elimination for $(\mathsf{MUT}^=_0)^T$ and embedding of $\mathsf{MUT}^=_0$ into $(\mathsf{MUT}^=_0)^T$. Notice that all lengths are finite.

3. We introduce the corresponding translation $\varphi^{m,n,k}$ $(m,n,k \in \mathbb{N})$ and prove a corresponding asymmetric interpretation theorem where we need only finitely many universes.

We collect all results in the following corollary.

Corollary 87 *We have for all arithmetic sentences φ the following reductions:*

a) $\mathsf{MUT}^=_0 \vdash \varphi \implies$ *There is a $k \in \mathbb{N}$ and a $\gamma < \varepsilon_0$ with $\mathsf{T}^0_k \,|\!\frac{\gamma}{1}\, \varphi$.*

b) $\mathsf{MUT}^= \vdash \varphi \implies$ *There is an $\alpha < \varepsilon_0$ and a $\gamma < \varepsilon_0$ with $\mathsf{T}^0_\alpha \,|\!\frac{\gamma}{1}\, \varphi$.*

4.2 An interpretation of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ into T^n_α

In this section we reduce $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to T^n_α. First we will reduce $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to $\bigcup_{n\in\mathbb{N}}(\mathsf{I}_n\text{-}\mathsf{RFN}_0)$ by a symmetric interpretation. Secondly we will use an asymmetric interpretation for the reduction of $\bigcup_{n\in\mathbb{N}}(\mathsf{I}_n\text{-}\mathsf{RFN}_0)$ to $\bigcup_{n\in\mathbb{N}}\mathsf{T}^n_\alpha$.

We let $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ denote a Tait-style reformulation of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$. Notice that in this Tait-calculus $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ is formulated as a rule. Furthermore, we have no ω-rule, but the $\forall x$-rule and furthermore the set-induction axiom. And we define the cut rank of a formula φ as 0 iff φ is a Σ^1_1 or a Π^1_1 formula.

Of course we have an embedding of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ into its Tait-calculus and corresponding to the definition of the cut-rank we have partial cut elimination. We now formulate the reduction of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to $\bigcup_{n\in\mathbb{N}}(\mathsf{I}_n\text{-}\mathsf{RFN}_0)$. It is a symmetric interpretation. For an analogous reduction in the context of set theory we refer to [18].

Theorem 88 *For all finite sets $\Gamma \subset \Sigma^1_1$ of closed $\mathcal{L}_2$ formulas, all arithmetic sentences φ and all $n \in \mathbb{N}$ we have:*

a) $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \,|\!\frac{n}{1}\, \Gamma[\vec{Z}] \implies \mathsf{ACA}_0 \vdash \mathsf{I}_{n+1}(D) \wedge \vec{Z} \,\dot{\in}\, D \to \Gamma^D[\vec{Z}]$.

b) $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \implies \bigcup_{n\in\mathbb{N}}(\mathsf{I}_n\text{-}\mathsf{RFN}_0) \vdash \varphi$.

Proof. Assertion b) follows from assertion a), since in I_n-RFN_0 we have sets D with $\mathsf{I}_n(D)$ and since we can embed $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ into its Tait-calculus. Thus, we have to show a). The proof is by induction on n. We discuss only the case where Γ is the conclusion of the $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$-rule. Hence, assume that $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ proves with deduction length $n > 0$

$$\Gamma[\vec{P}], (\exists M)(\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1\text{-}\mathsf{DC}})^M \wedge \varphi^M), \tag{4.9}$$

where φ is of the form $(\forall X)(\exists Y)\psi(X, Y, \vec{Z})$ and all free set parameters of $\psi \in \Pi^1_0$ are among $X, Y, \vec{Z}$. We have to prove in ACA_0

$$\mathsf{I}_{n+1}(D) \wedge \vec{P}, \vec{Z} \,\dot{\in}\, D \rightarrow \Gamma^D[\vec{P}] \vee (\exists M \,\dot{\in}\, D)(\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M).$$

First, we notice that we also have

$$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \,\frac{\,n-1\,}{1}\, \Gamma[\vec{P}], (\forall X)(\exists Y)\psi(X, Y, \vec{Z}).$$

We can prove $(\forall X)$-inversion in $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$. This is, we have

$$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \,\frac{\,n-1\,}{1}\, \Gamma[\vec{P}], (\exists Y)\psi(V, Y, \vec{Z}),$$

V a fresh variable. Now we apply the induction hypothesis and get

$$\mathsf{ACA}_0 \vdash \mathsf{I}_n(D) \wedge \vec{Z}, \vec{P}, V \,\dot{\in}\, D \rightarrow \Gamma^D[\vec{P}] \vee (\exists Y \,\dot{\in}\, D)\psi(V, Y, \vec{Z}). \tag{4.10}$$

From now on we argue within ACA_0. Choose a set C with $\mathsf{I}_{n+1}(C)$ and $\vec{P}, \vec{Z} \in C$. We have to show

$$\Gamma^C[\vec{P}] \vee (\exists M \,\dot{\in}\, C)(\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M). \tag{4.11}$$

Since we have $\mathsf{I}_{n+1}(C)$, there is an M in C with $\vec{P}, \vec{Z} \,\dot{\in}\, M$ and $\mathsf{I}_n(M)$. Therefore, we conclude with (4.10)

$$V \,\dot{\in}\, M \rightarrow \Gamma^M[\vec{P}] \vee (\exists Y \,\dot{\in}\, M)\psi(V, Y, \vec{Z}).$$

That is $\Gamma^M[\vec{P}] \vee \varphi^M$. But Γ is a disjunction of Σ^1_1 formulas. Therefore, we have also $\Gamma^C[\vec{P}] \vee \varphi^M$. Furthermore, we have $n > 0$ and hence $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$. Thus

$$\Gamma^C[\vec{P}] \vee (\exists M \,\dot{\in}\, C)(\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M).$$

But this is exactly (4.11). □

In a next step we interpret I_n-RFN_0 into $\bigcup_{k \in \mathbb{N}} \mathsf{T}^n_k$. The procedure resembles strongly the asymmetric interpretation of $\mathsf{MUT}^=_0$ into $\bigcup_{k \in \mathbb{N}} \mathsf{T}^0_k$. Therefore, we only sketch this reduction.

1. We write again $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ for the Tait-style reformulation of $\mathsf{I}_n\text{-}\mathsf{RFN}_0$. In this reformulation we have a set-induction axiom but no ω-rule – but again we have the $\forall x$-rule. There is an embedding of $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ into $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ where the deduction lengths are finite.

   We have partial cut elimination by defining the cut-rank of a formula φ as 0 iff φ is an ess-Σ^1_1 or an ess-Π^1_1 formula.

2. We introduce a translation. For each $\mathcal{L}_2$ formula φ we define a $\mathcal{L}^n_{\max(k,l)+1}$ formula $\varphi^{k,l}$ analogous to the definition 83. For example $(\exists X\psi)^{k,l}$ is the formula $(\exists X \mathrel{\dot\in} \mathsf{D}^n_l)\psi^{k,l}$ if ψ is not of the form $X \mathrel{\dot\in} Y \wedge \theta$ for all θ, Y. Then we can prove the asymmetric interpretation theorem:

$$\begin{aligned}&(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T \mathrel{\vdash^{k}_{1}} \Gamma[\vec{x}, \vec{X}]\\ &\quad\Longrightarrow \text{ For all } i,k,l \text{ with } i+2^k < l \text{ we have for all closed number terms } t\\ &\qquad\quad \mathsf{T}^n_l \mathrel{\vdash^{\omega^{\omega^{i+\omega^k}}}_{<\omega}} \vec{X} \mathrel{\dot\notin} \mathsf{D}^n_i, \Gamma^{i,i+2^k}[\vec{t}, \vec{X}].\end{aligned}$$

We get the following theorem:

Theorem 89 *We have for all arithmetic sentences φ*

a) $\mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash \varphi \implies$ *There is an m with* $\mathsf{T}^n_m \mathrel{\vdash^{<\varepsilon_0}_{<\omega}} \varphi$,

b) $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \implies$ *There is an n and an m with* $\mathsf{T}^n_m \mathrel{\vdash^{<\varepsilon_0}_{<\omega}} \varphi$.

4.3 The proof-theoretic strength of MUT and Σ^1_1-TDC_0

This section is a collection of results which lead finally to the proof-theoretic strength of MUT and Σ^1_1-TDC_0.

Theorem 90 *We have the following proof-theoretic ordinals:*

a) $|\mathsf{NUT}_0| = |\mathsf{UUT}_0| = |\mathsf{MUT}_0| = |\mathsf{MUT}_0| = \Gamma_0$.

b) $|\mathsf{NUT}| = \Gamma_{\varepsilon_0}$.

c) $|\mathsf{MUT}| = |\mathsf{UUT}| = |\mathsf{MUT}^=| = \varphi 1 \varepsilon_0 0$.

Proof.

a) $|\mathsf{NUT}_0| = \Gamma_0$ is stated in corollary 9a). It is trivial that the lower bound of UUT_0, MUT_0 and $\mathsf{MUT}_0^=$ is Γ_0 and that we have $|\mathsf{MUT}_0| \leq |\mathsf{MUT}_0^=|$. Corollary 12a) yields $|\mathsf{UUT}_0| \leq |\mathsf{MUT}_0^=|$ and corollary 87a) together with theorem 77 $|\mathsf{MUT}_0^=| \leq \Gamma_0$.

b) This is corollary 9b).

c) With $|\mathsf{MUT}| \leq |\mathsf{MUT}^=|$ and corollary 21 we conclude that $\varphi 1 \varepsilon_0 0$ is the lower bound of MUT, UUT, $\mathsf{MUT}^=$. Together with corollary 12b) we get $\varphi 1 \varepsilon_0 0 \leq |\mathsf{UUT}| \leq |\mathsf{MUT}| \leq |\mathsf{MUT}^=|$. Finally corollary 87b) and theorem 77 lead to the proof-theoretic upper bound $\varphi 1 \varepsilon_0 0$ of $\mathsf{MUT}_0^=$. □

Theorem 91 *We have the following proof-theoretic ordinals:*

a) $|\mathsf{I}_n\text{-}\mathsf{RFN}_0| = \varphi(n+1)00$.

b) $|(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}| = |\Sigma^1_1\text{-}\mathsf{TDC}_0| = \varphi\omega 00$.

Proof.

a) From corollary 57a) we take $\varphi(n+1)00 \leq |\mathsf{I}_n\text{-}\mathsf{RFN}_0|$. Theorem 89a) and theorem 78 give the upper bound $|\mathsf{I}_n\text{-}\mathsf{RFN}_0| \leq \varphi(n+1)00$.

b) From theorem 47 we know that $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ and $\Sigma^1_1\text{-}\mathsf{TDC}_0$ are equivalent. Especially we have $|\Sigma^1_1\text{-}\mathsf{TDC}_0| = |(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}|$. The lower bound of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ is stated in corollary 57b). And from theorem 89b) and theorem 78 we can take the upper bound. □

4.4 Proof-theoretic ordinal of Σ^1_1-TDC

In [18] not only the proof-theoretic ordinal of KPm^0 is stated but also the proof-theoretic ordinal of KPm^0 plus formula induction on the natural numbers. Theorem 14 in [18] establishes the proof-theoretic ordinal $\varphi\varepsilon_0 00$ for this strengthened theory. In this section we sketch that $\varphi\varepsilon_0 00$ is the proof-theoretic ordinal of $\Sigma^1_1\text{-}\mathsf{TDC}$ too. We do not give a rigorous proof. We describe only the ideas which lead to the desired result.

First we discuss the lower bound. The wellordering proof of $\Sigma^1_1\text{-}\mathsf{TDC}$ is similar to the wellordering proof of metapredicative Mahlo (cf. [32]). We remember that we can prove in $\Sigma^1_1\text{-}\mathsf{TDC}_0$ the existence of sets X with $\mathsf{I}_n(X)$ for all natural numbers n. The aim is to introduce within $\Sigma^1_1\text{-}\mathsf{TDC}$ for all ordinals $\alpha < \varepsilon_0$ the notion of a set X with "$\mathsf{I}_\alpha(X)$" and to show the existence of such sets. The formulas I_n (n natural number) are arithmetic. In $\Sigma^1_1\text{-}\mathsf{TDC}$ we will define formulas I_α (α an ordinal) and these formulas will be Σ^1_1. We will need formula induction in order to prove that this Σ^1_1 formula serves the right role and that there are sets X with $\mathsf{I}_\alpha(X)$.

The main modification is that we do not speak about all sets P with $\mathsf{I}_n(P)$ but that we speak only about all sets P in Y with $\mathsf{I}_n(P)$. For each Y we will define a characteristic function X with

$$k \in (X)_\alpha \leftrightarrow \text{“}\mathsf{I}_\alpha((Y)_k)\text{”}.$$

These functions X can be constructed inductively by using formula induction. We give first an informal description where I_α should be understood informally too.

$$
\begin{array}{lll}
\varphi_J(X,Y,\alpha) := & & \\
\quad \text{for all } b \preceq \alpha : & & \\
\quad\quad \text{if } b = 0 : & & x \in (X)_0 \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_x} \\
\quad\quad \text{if } Suc(b) : & & x \in (X)_b \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^{(Y)_x} \wedge [(\forall P)(\exists Q)(P \mathrel{\dot\in} Q \wedge \mathsf{I}_{b-1}(Q))]^{(Y)_x} \\
\quad\quad \text{if } Lim(b) : & & x \in (X)_b \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^{(Y)_x} \wedge (\forall c \prec b)\mathsf{I}_c((Y)_x).
\end{array}
$$

The exact definition of φ_J is: $(\alpha \in \Phi_0)$

$$
\begin{array}{l}
\varphi_J(X,Y,\alpha) := (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^Y \wedge \\
\quad (\forall b \preceq \alpha)(\forall x)[\\
\quad b = 0 \rightarrow (x \in (X)_0 \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_x}) \wedge \\
\quad Suc(b) \rightarrow (x \in (X)_b \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^{(Y)_x} \wedge \\
\qquad\qquad (\forall k)(\exists l)(((Y)_x)_k \mathrel{\dot\in} ((Y)_x)_l \wedge (\exists j \in (X)_{b-1})(((Y)_x)_l = (Y)_j)) \wedge \\
\quad Lim(b) \rightarrow (x \in (X)_b \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^{(Y)_x} \wedge (\forall c \prec b)(x \in (X)_c))]
\end{array}
$$

Notice that the formula φ_J only defines a predicate “I_b” ($b \prec \alpha, \alpha \in \Phi_0$) on $\{(Y)_n : n \in \omega\}$, namely X, and not a hierarchy. In order to proof the existence of X we need formula induction. The following lemma can be proved with formula induction on b; we omit the proof, since the arguments are standard.

Lemma 92 *For each ordinal α less than ε_0 Σ^1_1-AC proves*

a) $\varphi_J(X,Y,b) \wedge \varphi_J(Z,Y,b) \wedge b \prec \alpha \rightarrow (\forall c \preceq b)(X)_c = (Y)_c$.

b) $b \prec \alpha \rightarrow (\exists X)\varphi_J(X,Y,b)$.

In the following lemma we list some properties of the formula φ_J which can be proved by induction on b. All these properties are kinds of compatibility properties. They confirm that the chosen definition of φ_J is a good one. We need some of these properties in the proof of lemma 94 and 96. Again we do not give the proofs.

Lemma 93 *For each ordinal α less than ε_0 ACA_0 proves*

a) $\varphi_J(X,Y,\alpha) \wedge \varphi_J(Z,M,\alpha) \wedge Y \dot{\in} M \wedge x \in (X)_b \wedge b \preceq \alpha \to (\exists y \in (Z)_b)(Y)_x = (M)_y$.
"If Y is in M, then each b-inaccessible in Y is a b-inaccessible in M too."

b) $\varphi_J(X,Y,\alpha) \wedge z \in (X)_b \wedge b \preceq \alpha \wedge (Y)_z = (Y)_y \to y \in (X)_b$.
"If Z is in Y and is equal to a b-inaccessible in Y, then Z is a b-inaccessible in Y too."

c) $\varphi_J(X,Y,b) \wedge x \in (X)_b \wedge c \preceq b \preceq \alpha \to x \in (X)_c$.
"If Z is a b-inaccessible in Y and $c \prec b$, then Z is a c-inaccessible in Y too."

d) $\varphi_J(X,Y,b) \wedge \varphi_J(Z,M,b) \wedge x \in (X)_b \wedge (Y)_x = (M)_z \wedge b \preceq \alpha \to z \in (Z)_b$.
"If Z is a b-inaccessible in Y and Z is in M, then Z is a b-inaccessible in M too."

Lemma 92 ensures the existence of an X with $\varphi_J(X,Y,b)$ for a given Y and a given b with $b \prec \alpha, \alpha < \varepsilon_0$. Now we have to prove the existence of sets X *and* Y with $\varphi_J(X,Y,b)$ and $(\exists x)(x \in (X)_b)$. This is done in the next lemma.

Lemma 94 *For each ordinal α less than ε_0 Σ^1_1-TDC proves*

$$b \prec \alpha \to (\forall Q)(\exists P,Y,X)(Q \dot{\in} P \dot{\in} Y \wedge \varphi_J(X,Y,b) \wedge (\exists x \in (X)_b)(Y)_x = P).$$

Proof. We work in Σ^1_1-TDC and prove the claim by formula induction on b. We distinguish the cases $b = 0$, $Suc(b)$ and $Lim(b)$. As an illustration we show the case $Lim(b)$. Choose a set Q. Remember that we also have $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$. We apply $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ to the induction hypothesis

$$(\forall c \prec b)(\forall K)(\exists B,I,E)(K \dot{\in} B \dot{\in} I \wedge \varphi_J(E,I,c) \wedge (\exists x \in (E)_c)(I)_x = B)$$

and obtain a set P with

$$Q \dot{\in} P \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge \tag{4.12}$$
$$[(\forall c \prec b)(\forall K)(\exists B,I,E)(K \dot{\in} B \dot{\in} I \wedge \varphi_J(E,I,c) \wedge (\exists x \in (E)_c)(I)_x = B)]^P.$$

A further application of $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ and an application of lemma 92 yields sets Y, X and an x with $(Y)_x = P$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^Y$ and $\varphi_J(X,Y,b)$. If we can prove $x \in (X)_b$ we are done, i.e we have to prove

$$(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge (\forall c \prec b)(x \in (X)_c).$$

We know $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P$ by construction. By induction on c we prove now

$$c \prec b \to x \in (X)_c.$$

Again we distinguish the three cases $c = 0$, $Suc(c)$, $Lim(c)$. The cases $c = 0$ and $Lim(c)$ can easily be proved. Here we only discuss the case $Suc(c)$. In this case we have to prove

$$(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge (\forall k)(\exists l)((P)_k \mathrel{\dot\in} (P)_l \wedge (\exists j \in (X)_c)(P)_l = (Y)_j).$$

Again we know $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P$ by assumption. It remains the second property. Fix a k. Because of (4.12) we can choose l, r, E, x with

$$(P)_k \mathrel{\dot\in} (P)_l \mathrel{\dot\in} (P)_r \wedge \varphi_J(E, (P)_r, c) \wedge x \in (E)_c \wedge ((P)_r)_x = (P)_l.$$

Now we prove $(\exists j)(Y)_j = (P)_l$. Then we are done. The idea is to apply lemma 93a). We have

$$\varphi_J(E, (P)_r, c) \wedge \varphi_J(X, Y, c) \wedge (P)_r \mathrel{\dot\in} Y \wedge x \in (E)_c.$$

Hence, we obtain from lemma 93a) a j with $j \in (X)_c$ and $((P)_r)_x = (Y)_j$. And with $((P)_r)_x = (P)_l$ we conclude $(Y)_j = (P)_l$. $\square$

The construction of hierarchies can now be formulated as a corollary. We define

$$\begin{aligned}
&H_J(Q, Y, \alpha, Z) := \\
&\quad (\forall c \in field(Z))[(Y)_{Zc} \mathrel{\dot\in} (Y)_c \wedge \\
&\qquad\qquad (\exists X, M)(Q \mathrel{\dot\in} (Y)_c \mathrel{\dot\in} M \wedge \varphi_J(X, M, \alpha) \wedge (\exists x \in (X)_\alpha)(M)_x = (Y)_c)]
\end{aligned}$$

and conclude from lemma 94 and (Σ^1_1-TDC)

Corollary 95 *For each ordinal α less than ε_0 Σ^1_1-TDC proves*

$$WO(Z) \to (\forall a \prec \alpha)(\exists Y)H_J(Q, Y, a, Z).$$

The stage is set up in order to define our predicate "I_α". Instead of "$\mathsf{I}_\alpha(P)$" we write $J(\alpha, P)$. It is the following Σ^1_1 formula:

$$J(\alpha, P) := (\exists Y, X)(\varphi_J(X, Y, \alpha) \wedge (\exists x \in (X)_\alpha)(Y)_x = P).$$

In the next lemma we prove that $J(\alpha, P)$ has the desired properties.

Lemma 96 *For each ordinal α less than ε_0 Σ^1_1-TDC proves*

$$\begin{aligned}
&b + 1, \ell \prec \alpha \to \\
&\quad (J(0, P) \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^P) \wedge \\
&\quad (J(b + 1, P) \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge [(\forall N)(\exists M)(N \mathrel{\dot\in} M \wedge J(b, M))]^P) \wedge \\
&\quad (J(\ell, P) \leftrightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge (\forall b \prec \ell)J(b, P)).
\end{aligned}$$

Proof. Choose $b+1$ and ℓ less than $\alpha < \varepsilon_0$ and a set P. We have in Σ^1_1-TDC the following equivalences:

$$\begin{array}{rcl} J(0,P) & \leftrightarrow & (\exists Y,X)(\varphi_J(X,Y,0) \wedge (\exists x \in (X)_0)(Y)_x = P) \\ & \leftrightarrow & (Ax_{\Sigma^1_1\text{-AC}})^P. \end{array}$$

This proves the first property. We now prove the second property, the property of $J(b+1,P)$. We have to prove

$$\begin{array}{l} (\exists Y,X,x)[(Y)_x = P \wedge \varphi_J(X,Y,b) \wedge (Ax_{\Sigma^1_1\text{-DC}})^Y \wedge (Ax_{\Sigma^1_1\text{-DC}})^P \wedge \\ \qquad (\forall k)(\exists l)((P)_k \dot\in (P)_l \wedge (\exists z \in (X)_b)(P)_l = (Y)_z)] \\ \leftrightarrow \\ (Ax_{\Sigma^1_1\text{-DC}})^P \wedge \\ [(\forall N)(\exists M)(N \dot\in M \wedge (\exists Y,X)(\varphi_J(X,Y,b) \wedge (\exists x \in (X)_b)(Y)_x = M))]^P. \end{array} \tag{4.13}$$

First we show the "$\rightarrow$" direction of the equivalence. So, choose sets Y,X with

$$\begin{array}{l} (Y)_x = P \wedge \varphi_J(X,Y,b) \wedge (Ax_{\Sigma^1_1\text{-DC}})^Y \wedge (Ax_{\Sigma^1_1\text{-DC}})^P \wedge \\ (\forall k)(\exists l)((P)_k \dot\in (P)_l \wedge (\exists z \in (X)_b)(P)_l = (Y)_z) \end{array} \tag{4.14}$$

Moreover, we choose an N with $N = (P)_k$. Because of (4.14) there exist l,z with

$$(P)_k \dot\in (P)_l \wedge z \in (X)_b \wedge (P)_l = (Y)_z.$$

We set $M := (P)_l$ and have to prove

$$(\exists I, E \dot\in P)(\varphi_J(I,E,b) \wedge (\exists d \in (I)_b)(E)_d = (P)_l).$$

Two applications of (4.14) yield sets $(P)_g$, $(P)_h$ and z,y,u in $(X)_b$ with

$$\begin{array}{ccccccc} (P)_k & \dot\in & (P)_l & \dot\in & (P)_g & \dot\in & (P)_h \\ \| & & \| & & & & \\ N & & M & & \| & & \| \\ & & \| & & & & \\ & & (Y)_z & & (Y)_y & & (Y)_u \end{array}$$

Notice that we have $(Ax_{\Sigma^1_1\text{-DC}})^P$. Since lemma 92 is provable in Σ^1_1-DC, we obtain a set Q in P with $\varphi_J(Q,(P)_g,b)$. From lemma 93d) we conclude

$$\varphi_J(Q,(P)_g,b) \wedge \varphi_J(X,Y,b) \wedge z \in (X)_b \wedge (Y)_z = ((P)_g)_k \rightarrow k \in (Q)_b.$$

We also have $(Y)_z = (P)_l$ and therefore $(P)_l = ((P)_g)_k$. If we set $I := Q$ and $E := (P)_g$ and $d := k$ we are done. It remains to prove the "$\leftarrow$" direction in (4.13). An application of $((\Pi^1_2\text{-RFN})^{\Sigma^1_1\text{-DC}})$ yields a set Y and an application of lemma 92 yields a set X with

$$(Y)_x = P \wedge \varphi_J(X,Y,b) \wedge (Ax_{\Sigma^1_1\text{-DC}})^P \wedge (Ax_{\Sigma^1_1\text{-DC}})^Y.$$

Now we fix a k. By assumption there exists an l with

$$(P)_k \dot{\in} (P)_l \wedge (\exists I, E \dot{\in} P)(\varphi_J(E, I, b) \wedge (\exists x \in (E)_b)(I)_x = (P)_l).$$

Once more we use lemma 93a):

$$\varphi_J(E, I, b) \wedge \varphi_J(X, Y, b) \wedge I \dot{\in} Y \wedge x \in (E)_b \rightarrow (\exists z \in (X)_b)(I)_x = (Y)_z.$$

The equality $(I)_x = (P)_l$ leads to the claim. We now discuss the limit case. We have to prove

$$\begin{aligned}
&(\exists Y, X, x)((Y)_x = P \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^Y \wedge (\forall b \prec l)(\varphi_J(X, Y, b) \wedge x \in (X)_b) \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P) \\
&\leftrightarrow \\
&(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge (\forall b \prec l)(\exists Y, X)(\varphi_J(X, Y, b) \wedge (\exists x \in (X)_b)(Y)_x = P).
\end{aligned}$$

The "$\rightarrow$" direction of the equivalence can easily be proved. Hence, we only discuss the "$\leftarrow$" direction. We apply $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ to

$$(\forall b \prec l)(\exists Y, X)(\varphi_J(X, Y, b) \wedge (\exists x \in (X)_b)(Y)_x = P)$$

and obtain a set Y (and with lemma 92 a set X) and an x with

$$\begin{aligned}
&(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^P \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^Y \wedge (Y)_x = P \wedge \varphi_J(X, Y, \ell) \wedge \\
&(\forall b \prec \ell)(\exists I, E \dot{\in} Y)(\varphi_J(E, I, b) \wedge (\exists z \in (E)_b)(I)_z = P).
\end{aligned}$$

It remains to show $x \in (X)_b$ for $b \prec \ell$. Therefore, choose a $b \prec \ell$ and $z \in (E)_b$, $I, E \dot{\in} Y$ with $\varphi_J(E, I, b)$ and $(I)_z = P$. From lemma 93a) we conclude that

$$\varphi_J(E, I, b) \wedge \varphi_J(X, Y, \ell) \wedge I \dot{\in} Y \wedge z \in (E)_b \rightarrow (\exists y \in (X)_b)(I)_z = (Y)_y.$$

We know $(I)_z = P$ and $P = (Y)_x$. Hence,

$$(\exists y \in (X)_b)(Y)_y = (Y)_x.$$

And from lemma 93b) we conclude $x \in (X)_b$. $\square$

Lemma 96 shows that $J(\alpha, P)$ is our desired definition. From lemma 94 we conclude immediately that for each ordinal α less than ε_0 we have

$$(\forall b \preceq \alpha)(\forall Q)(\exists P)(Q \dot{\in} P \wedge J(b, P))$$

and corollary 95 ensures the existence of the corresponding hierarchies. This definition of the formula $J(\alpha, P)$ and the proof of the properties of this formula is the hard part in the determination of the lower bound of $\Sigma^1_1\text{-}\mathsf{DC}$. Now, the tedious part begins. It has to be proved that for each ordinal α less than ε_0 we have

$$\Sigma^1_1\text{-}\mathsf{TDC} \vdash (\forall X) TI(\alpha, X).$$

We do not carry out this step but notice that a straightforward generalization and extension of the methods that led to the lower bound of $\Sigma^1_1\text{-}\mathsf{TDC}_0$ lead to the following theorem (cf. [32])

Theorem 97 $|\Sigma^1_1\text{-}\mathsf{TDC}| \geq \varphi\varepsilon_0 00$.

The pattern of the argument for the upper bound of Σ^1_1-TDC is as follows: Instead of Σ^1_1-TDC we discuss $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. Then $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ is interpreted into the system $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})^T$, a Tait-calculus of $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. In particular, there is an ω-rule in this Tait-reformulation. The deduction lengths will be infinite, since we also have to interpret full complete induction. Again we prove partial cut elimination and this leads to

$$(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \quad \Longrightarrow \quad ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mid\!\frac{<\varepsilon_0}{1}\, \varphi$$

for each arithmetic sentence φ. From now on we can proceed as before, but always with families $(\mathsf{T}^\alpha_\beta)_{\alpha<\varepsilon_0}$ instead of families $(\mathsf{T}^n_\beta)_{n\in\mathbb{N}}$. Carrying through everything in detail leads (together with theorem 97) to

Theorem 98 $|\Sigma^1_1\text{-}\mathsf{TDC}| = \varphi\varepsilon_0 00$.

4.5 Conclusions

We have given the proof-theoretic strength of Σ^1_1-TDC: $\varphi\varepsilon_0 00$. This is the strongest system discussed in this thesis. The question is: "Are there 'metapredicative subsystems of analysis' stronger than $\varphi\varepsilon_0 00$?" Of course, the answer is "yes". But, what do such systems look like?

One way is, to extend the ideas developed in chapter 2. First, we build models of Σ^1_1-TDC. For instance, it should be possible to prove

$$|\mathsf{ACA}_0 + (\forall X)(\exists Y)(X \,\dot{\in}\, Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{TDC}})^Y)| = \varphi 1000.$$

Notice that also here the rule "the 'T' in Σ^1_1-TDC gives a '0' more" is satisfied, since from theorem VIII.4.20 [29] we conclude that

$$|\mathsf{ACA}_0 + (\forall X)(\exists Y)(X \,\dot{\in}\, Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^Y)| = \varphi 100.$$

Secondly, we build hierarchies of models of Σ^1_1-TDC (this will lead to $\varphi 2000$). Then we discuss models in which we can build hierarchies of models of Σ^1_1-TDC and so on. Probably the theory $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{TDC}}_0$ will be a limit of all these theories; analogous to $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0$ which is a limit of the theories $\mathsf{I}_n\text{-}\mathsf{RFN}_0$. Probably the proof-theoretic ordinal of $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{TDC}}_0$ is $\varphi\omega 000$. Perhaps there are again equivalences of certain hierarchies and certain model reflections. May be, we can iterate this procedure up to Φ_0.

Another question is: "Are there metapredicative subsystems of analysis which correspond (in some sense) to impredicative subsystems of analysis or set theory?" One possibility is, to imitate certain kinds of cardinals. For instance, our theory $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{TDC}}$ corresponds to KPm_0. We refer also to corollary 95 where we have shown that we can build in Σ^1_1-TDC hierarchies of "α-inacessibles" for $\alpha < \varepsilon_0$. Perhaps it would be helpful to discuss the notion of "$|Z|$-inacessibles" for Z a wellordering (provable in the theory). In this connection the conjecture is that the following two principles are equivalent over ACA_0

1. $(\forall X)(\exists Y)(X \dot{\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{TDC}})^Y)$.
2. $(WO(Z) \wedge WO(W)$
   $\rightarrow$ "there is a hierarchy along Z of $|W|$-inaccessibles,
   starting with Q".

We think that there must be further natural metapredicative subsystems of analysis. And it seems also possible to approximate with such subsystems the Howard-Bachmann ordinal from below. We are sure that this would lead to new insights of the notion "impredicativity".

List of symbols

The following list of symbols is divided into three separate tables: theories and systems, axioms and rules, and other symbols. The symbols in all three tables are given in the order of their appearance in the text.

Theories and Systems

ACA, 18	arithmetical comprehension
Σ^1_1-AC, 18	arithmetical comprehension and Σ^1_1 axiom of choice
ATR, 18	arithmetical comprehension and arithmetical transfinite recursion
Σ^1_1-DC, 18	arithmetical comprehension and Σ^1_1 axiom of dependent choice
Π^1_{n+1}-RFN, 19	arithmetical comprehension and reflection of Π^1_{n+1} formulas on models of ACA
NUT, 20	theory of non-uniform universes
MUT, 21	theory of minimal universes
UUT, 21	theory of uniform universes
$\mathsf{MUT}^{=}_0$, 33	strengthened MUT
UFP, 44	uniform fixed point theory
FTR, 50	arithmetical comprehension and fixed point transfinite recursion
$\mathsf{K}_n\mathsf{TR}$, 51	arithmetical comprehension and K_n transfinite recursion
$(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})$-$\mathsf{RFN}$, 51	arithmetical comprehension and $(\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC})$ reflection
I_n-RFN, 66	arithmetical comprehension and I_n reflection
$\mathsf{I}_n\mathsf{TR}$, 67	arithmetical comprehension and I_n transfinite recursion
$(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$, 68	arithmetical comprehension and Π^1_2 reflection on models of Σ^1_1-DC

Σ^1_1-TDC, 68	arithmetical comprehension and Σ^1_1 transfinite dependent choice
$\bar{\mathsf{T}}^n_\alpha$ 81	theory corresponding to T^n_α
T^n_α, 83	semi-formal system for $\bar{\mathsf{T}}^n_\alpha$
E^n_α, 86	semi-formal system for $\bar{\mathsf{T}}^n_\alpha$
$H_\nu \mathsf{E}^0_\alpha$, 92	arithmetical comprehension up to ν over E^0_α
RA_α, 98	ramified analysis over E^0_α
$\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots \mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$, 101	iterated E^n_α
$(\mathsf{MUT}^=)^T$, 110	Tait-calculus of $\mathsf{MUT}^=$
$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$, 118	Tait-calculus of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$

Axioms and rules

(ACA), 18	arithmetical comprehension
$(\Sigma^1_1\text{-}\mathsf{AC})$, 18	Σ^1_1 choice
(ATR), 18	arithmetical transfinite recursion
$(\Sigma^1_1\text{-}\mathsf{DC})$, 19	Σ^1_1 dependent choice
$(\Pi^1_{n+1}\text{-}\mathsf{RFN})$, 19	reflection of Π^1_{n+1} formulas on models of ACA
$(rel\text{-}\Pi^1_0(\mathsf{U})\text{-}\mathsf{CA})$, 20	$rel\text{-}\Pi^1_0(\mathsf{U})$ comprehension
$(rel\text{-}\Sigma^1_1(\mathsf{U})\text{-}\mathsf{AC})$, 20	$rel\text{-}\Sigma^1_1(\mathsf{U})$ choice
$(rel\text{-}\Pi^1_0(\mathsf{U},\mathcal{U})\text{-}\mathsf{CA})$, 22	$rel\text{-}\Pi^1_0(\mathsf{U},\mathcal{U})$ comprehension
$(rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})\text{-}\mathsf{AC})$, 22	$rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$ choice
$(\mathrm{Lin}^=)$, 33	strenghtened linearity axiom
(Lin), 33	linearity axiom
$(\Sigma^1_1(\mathcal{F}^\mathcal{A})\text{-}\mathsf{AC})$, 45	$\Sigma^1_1(\mathcal{F}^\mathcal{A})$ choice
(FTR), 50	fixed point transfinite recursion
$(\mathsf{K}_n\mathsf{TR})$, 51	K_n transfinite recursion
$((\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})\text{-}\mathsf{RFN})$, 51	$(\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC})$ reflection
$(\mathsf{I}_n\mathsf{TR})$, 67	I_n transfinite recursion
$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$, 68	Π^1_2 reflection on models of $\Sigma^1_1\text{-}\mathsf{DC}$
$(\Sigma^1_1\text{-}\mathsf{TDC})$, 68	Σ^1_1 transfinite dependent choice
$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$, 73	Π^1_2 reflection on models of $\Sigma^1_1\text{-}\mathsf{AC}$
$(weak\ \Sigma^1_1\text{-}\mathsf{TDC})$, 74	weak Σ^1_1 transfinite dependent choice

Other Symbols

$\mathcal{L}_1$, 15	language of first order arithmetic

Q, 15	unary free predicate
$\sim$, 15	symbol for forming negative literals
$\mathcal{L}_2$, 16	language of second order arithmetic
Π^1_0, 16	arithmetic $\mathcal{L}_2$ formulas
Σ^1_k, Π^1_k 16	Σ^1_k and Π^1_k formulas of $\mathcal{L}_2$
$\mathcal{L}_2(\mathsf{U})$, 16	language of NUT and MUT
U, 16	universe predicate
$\mathcal{L}_2(\mathsf{U},\mathcal{U})$, 16	language of UUT
$\mathcal{U}$, 16	universe operator
$\Pi^1_0(\mathsf{U})$, 17	arithmetic $\mathcal{L}_2(\mathsf{U})$ formulas
$\Pi^1_0(\mathsf{U},\mathcal{U})$, 17	arithmetic $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formulas
$\Sigma^1_1(\mathsf{U})$, 17	Σ^1_1 formulas of $\mathcal{L}_2(\mathsf{U})$
$\Sigma^1_1(\mathsf{U},\mathcal{U})$, 17	Σ^1_1 formulas of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$
$\langle \ldots \rangle$, 17	primitive recursive coding function for n-tuples
$(\cdot)_i$, 17	associated pojection of $\langle \ldots \rangle$
Seq, 17	primitive recursive set of sequence numbers
$s \in (S)_t$, 17	s is an element of the projection $(S)_t$
$S \oplus T$, 17	disjoint sum
$S \dot{\in} T$, 17	S is a projection of T
$S \doteq T$, 17	the projections of S and T are the same
$\varphi[\vec{x}\backslash\vec{t}, \vec{X}\backslash\vec{S}]$, 17	substitution of $\vec{x}, \vec{X}$ by $\vec{t}, \vec{S}$ in φ
$\lvert\mathsf{T}\rvert$, 17	proof-theoretic ordinal of T
WO, 18	(reflexive) well-ordering
$field$, 18	field of a linear ordering
$(Y)_{Za}$, 18	disjoint union of all projections $(Y)_b$ with bZa
0_Z, 18	least element of the well-ordering Z
$a +_Z 1$, 18	successor of a w.r.t. Z
aZb, 18	a is less than b w.r.t. Z
Ax_{ACA}, 19	finite axiomatization of (ACA)
φ^X, 19	relativization of φ to X
T_0, 19	theory T with set-induction instead of full induction
rel-$\Pi^1_0(\mathsf{U})$, 19	relativ arithmetic $\mathcal{L}_2(\mathsf{U})$ formulas
rel-$\Pi^1_k(\mathsf{U})$, rel-$\Sigma^1_k(\mathsf{U})$ 19	relativ $\Pi^1_k(\mathsf{U}), \Sigma^1_k(\mathsf{U})$ formulas of $\mathcal{L}_2(\mathsf{U})$
rel-$\Pi^1_k(\mathsf{U},\mathcal{U})$, rel-$\Sigma^1_k(\mathsf{U},\mathcal{U})$, 19	relativ $\Pi^1_k(\mathsf{U},\mathcal{U}), \Sigma^1_k(\mathsf{U},\mathcal{U})$ formulas of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$
rel-$\Delta^1_1(\mathsf{U})$, 30	relativ $\Delta^1_1(\mathsf{U})$ formulas of $\mathcal{L}_2(\mathsf{U})$
rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$, 30	relativ $\Delta^1_1(\mathsf{U},\mathcal{U})$ formulas of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$
$Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$, 31	finite axiomatization of Σ^1_1-$\mathsf{AC}) + (\mathsf{ACA})$
$\pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$, 31	universal Π^0_1 formula of $\mathcal{L}_2$
φ^{Ax}, 32	formula φ with $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$ instead of $\mathsf{U}(X)$
$minU(x, X)$, 34	x is in the minimal universe over X
φ^{min}, 35	min-translation of φ

Φ_0, 37	least ordinal greater than 0 which is closed under all n-ary φ functions
$\prec$, 37	primitive recursive wellordering of order type Φ_0
ℓ, 37	limit notation
$Prog(\varphi)$, 37	Progression of the formula φ
$TI(\varphi, a)$, 37	transfinite induction of φ up to a
$Hier(S, H, a)$, 38	H is a hierarchy of universes above S along $\prec$ up to a
$H(X, Y, a)$, 40	Y is a hierarchy of minimal universes above X along $\prec$ up to a
$a \upharpoonright b$, 43	$(\exists c, \ell)(b = c + a \cdot \ell)$
$Main_\alpha(a)$, 43	formula expressing the possibility to extend transfinite induction
$I^c_{R,H}(a)$, 43	transfinite induction up to a for all sets in $(H)_b$ for $b \prec c$
$\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$, 44	language of UFP
$\mathcal{F}^{\mathcal{A}}$, 44	uniform fixed point operator
$\Pi^1_0(\mathcal{F}^{\mathcal{A}})$, 44	arithmetic $\mathcal{L}_2(\mathcal{F}^{\mathcal{A}})$ formulas
$En_S[e, z, x, y, Z]$, 45	enumeration predicate of $(\Sigma^1_1)^S$ formulas
φ^U, 45	U-translation of φ
K_n, 49	K_{n-1} reflecting predicate
$FHier_{\mathcal{A}}$, 50	fixed point hierarchy
$\mathsf{K}_n Hier$, 50	hierarchy with steps fulfilling K_n
$Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}$, 50	finite axiomatization of $\mathsf{ATR} + \Sigma^1_1\text{-}\mathsf{DC} + \mathsf{ACA}$
ω^Z, 57	closure of Z under ordinal addition
$\prec_{\omega^Z}$, 57	corresponding well-ordering on ω^Z
$+_{\omega^Z}$, 57	ordinal addition on ω^Z
$\ell[n]$, 58	fundamental sequence
$\ell^-[n]$, 58	unique ordinal with $\ell[n] + \ell^-[n] = \ell[n+1]$
$Hier_{\mathcal{A}}$, 58	fixed point hierarchy
$H_{\mathcal{A}}$, 58	extension of fixed point hierarchies
$\pi^0_1[x, X]$, 60	complete Π^0_1 predicate
$\overset{+}{\pi}[x, X, Y], \overset{-}{\pi}[x, X, Y]$, 60	"positive, negative" part of $\pi^0_1[x, X]$
$\prec^Q_a$, 60	in Q recursive well-ordering with index a
$Z \otimes Y$, 62	product of Z and Y
$4_<$, 62	canonical wellordering on $\{1, 2, 3, 4\}$
$Y_{\langle Zk,3\rangle}$, 62	disjoint union of all $(Y)_{\langle b,3\rangle}$ with $\langle b, k\rangle \in Z$ and $b \neq k$
I_n, 66	I_{n-1} reflecting predicate
$Ax_{\Sigma^1_1\text{-}\mathsf{DC}}$, 66	finite axiomatization of $(\Sigma^1_1\text{-}\mathsf{DC}) + (\mathsf{ACA})$
$\mathsf{I}_n Hier$, 67	hierarchy with steps fulfilling I_n
$Hier_{n,Q}$, 74	hierarchy with steps fulfilling I_n

$H_{n,Q}$, 74	extension of hierarchies with steps fulfilling I_n
$I_{e,Y}$, 74	transfinite induction up to a for all sets in $(Y)_d$ with $d \prec e$
$Main^n(a)$, 74	formula expressing the possibility to extend transfinite induction
$\mathcal{L}_2(\mathsf{D}^n)$, 81	language of $\bar{\mathsf{T}}^n_\alpha$
D^n, 81	relation symbol of $\bar{\mathsf{T}}^n_\alpha$
$\pi^0_1[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$, 81	short-hand for $\pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$
$\mathcal{L}^n_\alpha$, 83	language of T^n_α
D^n_β, $\mathsf{D}^n_{<\gamma}$, 83	relation symbols of T^n_α
$(\ldots)^\alpha$, 85	translation of $\mathcal{L}_2(\mathsf{D}^n)$ into $\mathcal{L}^n_\alpha$
$*_i$, 87	nominal symbol
ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$, ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, 87, 102	classes of expressions corresponding to ess-Σ^1_1 and ess-Π^1_1
ess-$\Sigma^1_1(\mathsf{D}^n_\beta)^c$, ess-$\Pi^1_1(\mathsf{D}^n_\beta)^c$, 88	ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$, ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ without free number variables
φ^*, 88	translation of φ in order to embed $\mathsf{T}^n_{\alpha+1}$ into $\mathsf{E}^n_{\alpha+1}$
I^S_c, E^S_c, 92	index predicate resp. enumeration predicate for the hyperarithmetical hierarchy
$\mathcal{L}_{RA_\alpha}$, 98	language of RA_α
$lev(T)$, $lev(\varphi)$, 98	level of set terms resp. formulas of RA_α
φ^γ, 100	γ instance of φ
$\mathcal{L}^{n,\ldots,n+k}_{\alpha_n,\ldots,\alpha_{n+k}}$, 101	language of $\mathsf{E}^n_{\alpha_n}[\mathsf{E}^{n+1}_{\alpha_{n+1}}[\ldots\mathsf{E}^{n+k}_{\alpha_{n+k}}]\ldots]]$
$essrel$-$\Sigma^1_1(\mathsf{U})$, $essrel$-$\Pi^1_1(\mathsf{U})$, 111	classes of formulas corresponding to ess-Σ^1_1 and ess-Π^1_1
φ_J, 122	formula needed for the definition of J
$J(\alpha, P)$, 124	P is an " α-inaccessible"

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