Universes in explicit mathematics

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Abstract

This paper deals with universes in explicit mathematics. After introducing some basic definitions, the limit axiom and possible ordering principles for universes are discussed. Later, we turn to least universes, strictness and name induction. Special emphasis is put on theories for explicit mathematics with universes which are proof-theoretically equivalent to Feferman's T_0.

1 Introduction

In some form or another, universes play an important role in many systems of set theory and higher order arithmetic, in various formalizations of constructive mathematics and in logics for computation. One aspect of universes is that they expand the set or type formation principles in a natural and perspicuous way and provide greater expressive power and proof-theoretic strength.

The general idea behind universes is quite simple: suppose that we are given a formal system Th comprising certain set (or type) existence principles which are justified on specific philosophical grounds. Then it may be argued that there should also exist a collection of sets (or types) – a so-called universe – satisfying these closure conditions. This process can be iterated, thus establishing stronger and stronger extensions of Th.

In classical set theory this process is related to what is inherent in the usual reflection principles yielding the existence of certain large cardinals (cf. e.g. Drake [4]). In theories for iterated admissible sets, admissibles act as universes and provide for recursive analogues of large cardinals (cf. e.g. Jäger [11]). Universes in Martin-Löf type theory are generated by specific introduction and (sometimes) elimination rules and can be regarded as the constructive versions of certain regular cardinals. See Martin-Löf [21], Palmgren [25], Rathjen [26] and Setzer [27] for more information about this approach.

In the framework of explicit mathematics, universes have first been considered by Feferman [7] in connection with Hancock's conjecture and by

Marzetta [22] for designing an explicit analogue of Friedman's theory ATR_0 of arithmetic transfinite recursion (cf. e.g. Friedman, McAloon and Simpson [9] and Simpson [28]) and Jäger's theory KPI^0 of iterated admissible sets without foundation (cf. e.g. Jäger [10, 11]). More about universes in explicit mathematics can be found, for example, in Jäger and Strahm [15] and Strahm [29], always in connection with theories of predicative or metapredicative strength. Universes are also crucial for dealing with Mahloness in explicit mathematics, as shown in the forthcoming paper Jäger and Studer [16]. In Kahle [18], universes are studied for Frege structures, i.e. truth theories corresponding to explicit mathematics.

The purpose of this article is to clarify several principle aspects of universes in explicit mathematics and to present them in compact form. After introducing some basic definitions, the limit axiom and possible ordering principles for universes are discussed. Later we turn to least universes, strictness and name induction. Special emphasis is put on theories for explicit mathematics with universes which are proof-theoretically equivalent to Feferman's T_0.

2 Explicit mathematics

Explicit mathematics has been introduced in Feferman [5] as a framework for Bishop style constructive mathematics. The relationship between explicit mathematics, other formalizations of constructive mathematics and subsystems of analysis and an interesting interplay between set-theoretic and recursion-theoretic models of explicit mathematics have first been studied in Feferman [5, 6].

In the following, we do not work with Feferman's original formalization of systems of explicit mathematics. Instead, we treat them as theories of types and names as developed in Jäger [12].

Our theories of types and names are formulated in the second order language $\mathbb{L}$ for individuals and types. It comprises individual variables $a, b, c, f, u, v, w, x, y, z, \ldots$ as well as type variables $S, T, U, V, W, X, Y, Z, \ldots$, both possibly with subscripts. $\mathbb{L}$ also includes the individual constants k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projections), 0 (zero), s_N (successor), p_N (predecessor), d_N (definition by numerical cases). There are additional individual constants, called *generators*, which will be used for the uniform naming of types, namely nat (natural numbers), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image), j (join), i (inductive generation) and ℓ (universe generator). There is one binary function symbol $\cdot$ for (partial) application of individuals to individuals. Further, $\mathbb{L}$ has unary relation symbols $\downarrow$ (defined)

and N (natural numbers) as well as the three binary relation symbols $\in$ (membership), $=$ (equality) and $\Re$ (naming, representation).

The *individual terms* $(r, s, t, r_1, s_1, t_1, \dots)$ of $\mathbb{L}$ are built up from individual variables and individual constants by means of our function symbol $\cdot$ for application. In the following, we often abbreviate $(s \cdot t)$ simply as (st) or st and adopt the convention of association to the left so that $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 \cdot s_2)\dots s_n)$. Further we put $t' := \mathsf{s_N} t$.

Usually, we write (s, t) instead of $\mathsf{p}st$ and define general n tupling by induction on n as follows:

$$(s_1) := s_1, \qquad (s_1, \dots, s_{n+1}) := ((s_1, \dots, s_n), s_{n+1}).$$

The atomic formulas of $\mathbb{L}$ are the formulas $\mathsf{N}(s)$, $s{\downarrow}$, $s = t$, $s \in U$ and $\Re(s, U)$. Since we work with a logic of partial terms, it is not guaranteed that all terms have values, and $s{\downarrow}$ is read as s *is defined* or s *has a value*. Moreover, $\mathsf{N}(s)$ says that s is a natural number, and the formula $\Re(s, U)$ is used to express that the individual s *represents* the type U or is a *name* of U.

The *formulas* of $\mathbb{L}$ $(A, B, C, A_1, B_1, C_1, \dots)$ are generated from the atomic formulas by closing against the usual connectives as well as quantification in both sorts. The following table contains a useful list of abbreviations:

$$\begin{array}{rcl}
s \simeq t & := & s{\downarrow} \vee t{\downarrow} \to s = t, \\
s \in \mathsf{N} & := & \mathsf{N}(s), \\
(\exists x \in \mathsf{N})A(x) & := & (\exists x)(x \in \mathsf{N} \wedge A(x)), \\
(\forall x \in \mathsf{N})A(x) & := & (\forall x)(x \in \mathsf{N} \to A(x)), \\
U \subset V & := & (\forall x)(x \in U \to x \in V), \\
U = V & := & U \subset V \wedge V \subset U, \\
s \mathrel{\dot\in} t & := & (\exists X)(\Re(t, X) \wedge s \in X), \\
U \mathrel{\widetilde\in} V & := & (\exists x)(\Re(x, U) \wedge x \in V), \\
(\exists x \mathrel{\dot\in} s)A(x) & := & (\exists x)(x \mathrel{\dot\in} s \wedge A(x)), \\
(\forall x \mathrel{\dot\in} s)A(x) & := & (\forall x)(x \mathrel{\dot\in} s \to A(x)), \\
s \doteq t & := & (\exists X)[\Re(s, X) \wedge \Re(t, X)], \\
s \mathrel{\dot\subset} t & := & (\exists X, Y)[\Re(s, X) \wedge \Re(t, Y) \wedge X \subset Y], \\
\Re(s) & := & (\exists X)\Re(s, X).
\end{array}$$

The vector notation $\vec{U}$ and $\vec{s}$ is sometimes used to denote finite sequences of type variables $U_1, \dots, U_m$ and individual terms $s_1, \dots, s_n$, respectively,

whose lengths are given by the context. For example, for $\vec{U} = U_1, \ldots, U_n$ and $\vec{s} = s_1, \ldots, s_n$ we write:

$$\begin{array}{rcl} \Re(\vec{s}, \vec{U}) & := & \Re(s_1, U_1) \wedge \cdots \wedge \Re(s_n, U_n), \\ \Re(\vec{s}) & := & \Re(s_1) \wedge \cdots \wedge \Re(s_n). \end{array}$$

The logic of systems of explicit mathematics is Beeson's classical *logic of partial terms* (cf. Beeson [1] or Troelstra and van Dalen [30]) for individuals and classical logic for types.

Now we introduce the theory EETJ which provides a framework for explicit elementary types with join. The nonlogical axioms of EETJ can be divided into the following groups.

I. Applicative axioms. These axioms formalize that the individuals form a partial combinatory algebra, that we have pairing and projection and the usual closure conditions on the natural numbers, as well as definition by numerical cases.

(1) $\mathsf{k}ab = a$,

(2) $\mathsf{s}ab{\downarrow} \wedge \mathsf{s}abc \simeq ac(bc)$,

(3) $\mathsf{p}_0(a,b) = a \wedge \mathsf{p}_1(a,b) = b$,

(4) $0 \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(x' \neq 0 \wedge \mathsf{p}_{\mathsf{N}}(x') = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \rightarrow \mathsf{p}_{\mathsf{N}}x \in \mathsf{N} \wedge (\mathsf{p}_{\mathsf{N}}x)' = x)$,

(7) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \rightarrow \mathsf{d}_{\mathsf{N}}xyab = x$,

(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \rightarrow \mathsf{d}_{\mathsf{N}}xyab = y$.

As usual, a theorem about λ abstraction and a form of the recursion theorem can be derived from axioms (1) and (2).

II. Explicit representation and equality. The following axioms state that each type has a name, that there are no homonyms and that $\Re$ respects the extensional equality of types.

(1) $\exists x \Re(x, U)$,

(2) $\Re(a, U) \wedge \Re(a, V) \rightarrow U = V$,

(3) $U = V \wedge \Re(s, U) \rightarrow \Re(s, V)$.

III. Basic type existence axioms. In the following we provide a finite axiomatization of uniform elementary comprehension plus join.

Natural numbers

$\Re(\mathsf{nat}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{nat} \leftrightarrow \mathsf{N}(x))$.

Identity

$\Re(\mathsf{id}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{id} \leftrightarrow (\exists y)(x = (y, y)))$.

Complements

$\Re(a) \rightarrow \Re(\mathsf{co}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{co}(a) \leftrightarrow x \mathrel{\dot\notin} a)$.

Intersections

$\Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{int}(a, b)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{int}(a, b) \leftrightarrow x \mathrel{\dot\in} a \wedge x \mathrel{\dot\in} b)$.

Domains

$\Re(a) \rightarrow \Re(\mathsf{dom}(a)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{dom}(a) \leftrightarrow \exists y((x, y) \mathrel{\dot\in} a))$.

Inverse images

$\Re(a) \rightarrow \Re(\mathsf{inv}(a, f)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{inv}(a, f) \leftrightarrow fx \mathrel{\dot\in} a)$.

Joins

$\Re(a) \wedge (\forall x \mathrel{\dot\in} a)\Re(fx) \rightarrow \Re(\mathsf{j}(a, f)) \wedge \Sigma(a, f, \mathsf{j}(a, f))$.

In this last axiom, the formula $\Sigma(a, f, b)$ expresses that b names the disjoint union of f over a, i.e.

$$\Sigma(a, f, b) := \forall x(x \mathrel{\dot\in} b \leftrightarrow \exists y \exists z(x = (y, z) \wedge y \mathrel{\dot\in} a \wedge z \mathrel{\dot\in} fy)).$$

An $\mathbb{L}$ formula A is called *elementary* if it contains neither the relation symbol $\Re$ nor bound type variables. In the original formulation of explicit mathematics, elementary comprehension is not dealt with by a finite axiomatization, but directly as an infinite axiom schema. According to a theorem in Feferman and Jäger [8], reformulated in Lemma 1 below, this schema of uniform elementary comprehension is provable from our finite axiomatization. Join is not needed for this argument.

In the following we assume that $z1, z2, \ldots$ and $Z1, Z2, \ldots$ are arbitrary but fixed enumerations of the individual and type variables of $\mathbb{L}$, respectively. If A is an elementary $\mathbb{L}$ formula with no other individual variables than $z1, \ldots, zm$ and no other type variables than $Z1, \ldots, Zn$ and if $\vec{a} = a_1, \ldots, a_m$ and $\vec{S} = S_1, \ldots, S_n$, then we write $A[\vec{a}, \vec{S}]$ for the $\mathbb{L}$ formula

which results from A by a simultaneous replacement of zi by a_i and Zj by S_j $(1 \leq i \leq m, 0 \leq j \leq n)$.

Lemma 1 (Elementary comprehension) *Let A be an elementary $\mathbb{L}$ formula with no individual variables other than $z1, \ldots, z(m+1)$ and no type variables other than $Z1, \ldots, Zn$. Then there exists a closed individual term t of $\mathbb{L}$, depending on A, so that* EETJ *proves for all $\vec{a} = a_1, \ldots, a_m$, $\vec{b} = b_1, \ldots, b_n$ and $\vec{S} = S_1, \ldots, S_n$:*

1. $\Re(\vec{b}, \vec{S}) \rightarrow \Re(t(\vec{a}, \vec{b}))$,
2. $\Re(\vec{b}, \vec{S}) \rightarrow \forall x(x \mathrel{\dot\in} t(\vec{a}, \vec{b}) \leftrightarrow A[x, \vec{a}, \vec{S}])$.

We often informally write $\{x : B(x)\}$ for the collection of all individuals c such that $B(c)$. Hence, the previous lemma implies that for elementary $\mathbb{L}$ formulas $A[u, \vec{v}, \vec{W}]$ one has:

(i) $\{x : A[x, \vec{a}, \vec{S}]\}$ is a type;

(ii) this type can be named, via a closed individual term t of $\mathbb{L}$, in a uniform way depending on its individual parameters and the names of its type parameters.

For many applications, however, this formulation of elementary comprehension is too restricted. Below, we therefore present a modified form. Before doing this, however, we introduce some further convenient shorthand notations.

Let $\vec{U} = U_1, \ldots, U_n$ and $\vec{s} = s_1, \ldots, s_n$ be sequences of type variables and individual terms of $\mathbb{L}$, respectively, and let $A(\vec{U})$ be an elementary $\mathbb{L}$ formula. Then we write $A(\vec{s})$ for the $\mathbb{L}$ formula which results from $A(\vec{U})$ by replacing for $i = 1, \ldots, n$ each occurrence of $t \in U_i$ by $t \mathrel{\dot\in} s_i$. In addition, given a sequence $\vec{r} = r_1, \ldots, r_m$ of individual terms of $\mathbb{L}$, then $\vec{r}(\vec{s})$ stands for the sequence of individual terms $r_1(\vec{s}), \ldots, r_m(\vec{s})$ of $\mathbb{L}$.

Lemma 2 (Modified elementary comprehension) *Let A be an elementary $\mathbb{L}$ formula with no individual variables other than $z1, \ldots, z(m+1)$ and no type variables other than $Z1, \ldots, Zn$, and let $\vec{s} = s_1, \ldots, s_n$ be a sequence of closed individual terms of $\mathbb{L}$. Then there exists a closed individual term t of $\mathbb{L}$, depending on A and $\vec{s}$, so that* EETJ *proves for all $\vec{a} = a_1, \ldots, a_m$:*

1. $\Re(\vec{s}(\vec{a})) \rightarrow \Re(t(\vec{a}))$,
2. $\Re(\vec{s}(\vec{a})) \rightarrow \forall x(x \mathrel{\dot\in} t(\vec{a}) \leftrightarrow A[x, \vec{a}, \vec{s}(\vec{a})])$.

In the following we employ two forms of induction on the natural numbers, type induction and formula induction. Type induction is the axiom

$(\mathsf{T}\text{-}\mathsf{I}_\mathsf{N})$ $\quad \forall X(0 \in X \wedge (\forall x \in \mathsf{N})(x \in X \to x' \in X) \to (\forall x \in \mathsf{N})(x \in X)).$

Formula induction, on the other hand, is the schema

$(\mathbb{L}\text{-}\mathsf{I}_\mathsf{N})$ $\qquad A(0) \wedge (\forall x \in \mathsf{N})(A(x) \to A(x')) \to (\forall x \in \mathsf{N})A(x)$

for each $\mathbb{L}$ formula $A(u)$. Sometimes, we also want additional axioms which guarantee that different generators create different names. This can be achieved by adding, for example, axioms of the following kind.

Uniqueness of generators with respect to $\mathbb{L}$ is given by the collection $(\mathbb{L}\text{-}\mathsf{UG})$ of the following axioms for all syntactically different generators r_0 and r_1 and arbitrary generators s and t of $\mathbb{L}$:

(1) $r_0 \neq r_1$,

(2) $\forall x(sx \neq \mathsf{nat} \wedge sx \neq \mathsf{id})$,

(3) $\forall x \forall y(sx = ty \to s = t \wedge x = y)$.

3 The limit axiom and basic properties of universes

Now we are going to introduce universes in explicit mathematics. In short, a universe is a type U so that: (i) U is closed under elementary comprehension and join; (ii) all elements of U are names. This second condition (ii) is crucial to avoid universes from being trivial since otherwise, for example, the universal type $\mathbb{V} = \{x : x = x\}$ could act as the topmost universe.

In order to give the definition of universe in greater detail, we introduce some auxiliary notation and let $\mathcal{C}(S, a)$ be the closure condition which is the disjunction of the following $\mathbb{L}$ formulas:

(1) $a = \mathsf{nat} \vee a = \mathsf{id}$,

(2) $\exists x(a = \mathsf{co}(x) \wedge x \in S)$,

(3) $\exists x \exists y(a = \mathsf{int}(x, y) \wedge x \in S \wedge y \in S)$,

(4) $\exists x(a = \mathsf{dom}(x) \wedge x \in S)$,

(5) $\exists f \exists x(a = \mathsf{inv}(f, x) \wedge x \in S)$,

(6) $\exists x \exists f[a = \mathsf{j}(x,f) \wedge x \in S \wedge (\forall y \mathrel{\dot\in} x)(fy \in S)]$.

Thus the formula $\forall x(\mathcal{C}(S,x) \to x \in S)$ describes that S is a type which is closed under the type constructions of EETJ, i.e. elementary comprehension and join. A universe is a type which consists of names only and satisfies this closure condition.

Definition 3 *1. We write* $\mathsf{U}(S)$ *to express that the type* S *is a* universe,

$$\mathsf{U}(S) := \forall x(\mathcal{C}(S,x) \to x \in S) \wedge (\forall x \in S)\Re(x).$$

2. $\mathcal{U}(t)$ *means that the individual* t *is a* name of a universe,

$$\mathcal{U}(t) := \exists X(\Re(t,X) \wedge \mathsf{U}(X)).$$

It follows immediately from this definition that one can work within universes as in EETJ; in particular, there is an analogue of Lemma 1 relativized to all universes.

Lemma 4 (Modified elementary comprehension in universes) *Let* A *be an elementary* $\mathbb{L}$ *formula with no individual variables other than* $z1, \ldots, z(m+1)$ *and no type variables other than* $Z1, \ldots, Zn$, *and let* $\vec{s} = s_1, \ldots, s_n$ *be a sequence of closed individual terms of* $\mathbb{L}$. *Then there exists a closed individual term* t *of* $\mathbb{L}$, *depending on* A *and* $\vec{s}$, *so that* EETJ *proves for all* $\vec{a} = a_1, \ldots, a_m$:

1. $\mathsf{U}(S) \wedge \vec{s}(\vec{a}) \in S \;\to\; t(\vec{a}) \in S$,
2. $\Re(\vec{s}(\vec{a})) \;\to\; \forall x(x \mathrel{\dot\in} t(\vec{a}) \leftrightarrow A[x, \vec{a}, \vec{s}(\vec{a})])$.

We now observe that universes do not contain their names; for a proof see Marzetta [22]. This property of universes corresponds in a certain sense to the set-theoretic fact that admissibles do not contain themselves, even if $\in$ foundation is not available.

Lemma 5 *In* EETJ, *one can prove that*

$$\mathsf{U}(S) \wedge \Re(a,S) \to a \notin S.$$

Note that in explicit mathematics, the names of a type do not form a type. This is proved in various places, for example in Cantini and Minari [3], Jäger [13] and Jansen [17]; join is not needed for this argument. In connection with universes, a stronger result is possible: each type has so many names that not all of them can be contained in a single universe, or, in other words, no universe is large enough to contain all names of a given type (see also Minari [24]).

Lemma 6 *In* EETJ*, one can prove that*

$$\mathsf{U}(S) \to \exists x(\Re(x,T) \wedge x \notin S).$$

Proof Let S be a universe and choose a name a of S. Then $\mathsf{j}(a, \lambda x.x)$ is a name of the type

$$U = \{(x,y) : x \in S \wedge y \mathrel{\dot\in} x\}. \tag{1}$$

The next step is to prove the equivalence

$$\Re(b,T) \leftrightarrow \forall x(x \mathrel{\dot\in} b \leftrightarrow x \in T) \tag{2}$$

for all $b \in S$. The direction from left to right is obvious. To establish the converse direction, let b be an element of S. Then b is a name of a type V since all elements of universes are names. Hence we have $\Re(b,V)$, and the right hand side of (2) yields $T = V$. Thus we conclude $\Re(b,T)$.
For all $b \in S$, we derive from (1) and (2) that

$$\Re(b,T) \leftrightarrow \forall x((b,x) \in U \leftrightarrow x \in T).$$

Since the right hand side of this equivalence is elementary, elementary comprehension gives the type

$$W = \{x : x \in S \wedge \Re(x,T)\}.$$

If all names of T were contained in S, then W would be the type of all names of T. But, in view of the remark above, this is not possible. □

The theory EETJ does not prove the existence of universes. However, as in the case of theories for admissible sets (cf. e.g. Jäger [11]), a so-called *limit axiom* can easily be added. By making use of the generator ℓ, one assigns to each name x the name ℓx of a universe containing x, i.e.

(Lim) $$\forall x(\Re(x) \to \mathcal{U}(\ell x) \wedge x \mathrel{\dot\in} \ell x).$$

The standard model constructions of Jäger and Strahm [15] for metapredicative and Jäger and Studer [16] for impredicative Mahlo provide natural models for (Lim). The proof-theoretic strengths of (Lim) in the context of elementary comprehension and join plus type or formula induction on the natural numbers have been analyzed in Kahle [19] and Strahm [29]. Although, in many situations, (Lim) is proof-theoretically equivalent to its obvious non-uniform version as studied in Marzetta [22] and Marzetta and Strahm [23],

sometimes there are subtle differences between (Lim) and its nonuniform version, which will be discussed elsewhere.

There are, of course, many universes which contain a given name a. The universe named by ℓa can be regarded as the standard or normal universe and ℓa as its normal name.

Definition 7 *We write $\mathcal{U}_\ell(t)$ to express that the individual t is a* normal name *of a universe,*

$$\mathcal{U}_\ell(t) := \exists x(\Re(x) \wedge t = \ell x).$$

A first simple observation concerning the generator ℓ says that for all names a, the type named by a and the type named by ℓa have to be different.

Lemma 8 *In* EETJ + (Lim)*, one can prove that*

$$\forall x(\Re(x) \to x \mathrel{\dot\neq} \ell x).$$

Proof Let a be a name. Because of (Lim), we know that ℓa is a name of a universe S which contains a. According to Lemma 5, this ℓa cannot be an element of S. Hence $a \mathrel{\dot\neq} \ell a$. □

Simple generators like co and int are extensional in the sense that $a \doteq b$ and $u \doteq v$ imply $\mathsf{co}(a) \doteq \mathsf{co}(b)$ and $\mathsf{int}(a, u) \doteq \mathsf{int}(b, v)$. The following lemma shows that such a form of extensionality is not the case for the generator ℓ.

Lemma 9 *In* EETJ + (Lim)*, one can prove that*

$$\exists x \exists y(\Re(x) \wedge \Re(y) \wedge x \doteq y \wedge \ell x \mathrel{\dot\neq} \ell y).$$

Proof Choose an arbitrary type T and a name a of T. Then ℓa is a name of a universe S which contains a. Because of Lemma 6, there exists a name b of T which does not belong to S, i.e. $b \mathrel{\dot\notin} \ell a$. Now consider ℓa and ℓb. Both are names of universes, but since $b \mathrel{\dot\in} \ell b$ and $b \mathrel{\dot\notin} \ell a$ we have $\ell a \mathrel{\dot\neq} \ell b$. On the other hand, $a \doteq b$ since both are names of T. □

Now we turn to possible "ordering principles" for universes. Motivated by the familiar set-theoretic situation, we begin with considering linearity, transitivity and connectivity of universes which are formulated in our context as follows:

(U-Lin) $$\forall X \forall Y(\mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\widetilde\in} Y \vee X = Y \vee Y \mathrel{\widetilde\in} X),$$

(U-Tran) $$\forall X \forall Y(\mathsf{U}(X) \wedge \mathsf{U}(Y) \wedge X \mathrel{\widetilde\in} Y \to X \subset Y),$$

(U-Con) $\forall X \forall Y (\mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \subset Y \vee Y \subset X).$

Although these three assertions may appear natural, they are problematic in our context. For example, they are not valid in the standard model of Jäger and Studer [16] and incompatible, as we will see now, with uniqueness of generators.

In the proof of the following theorem, we exploit the fact that suitably constructed universes remain universes if certain elements are taken out. For this sort of argument, it is important that we have a criterion for testing whether a type is a universe. If universes were introduced by an implicit definition, such an argument would hardly work.

Theorem 10 *1. In* EETJ*, one can prove that*

$$(\mathsf{U\text{-}Con}) \to (\mathsf{U\text{-}Tran}).$$

2. In $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$*, one can prove that*

$$\neg(\mathsf{U\text{-}Lin}) \wedge \neg(\mathsf{U\text{-}Tran}) \wedge \neg(\mathsf{U\text{-}Con}).$$

Proof For the proof of the first assertion, take two universes S and T with $S \mathrel{\widetilde{\in}} T$. Then $T \not\subset S$ since T contains a name of S which cannot be an element of S by Lemma 5. Therefore, (U-Con) implies $S \subset T$, completing the proof of the first assertion.

Now we work in the theory $\mathsf{EETJ}+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$. In order to show $\neg(\mathsf{U\text{-}Lin})$, we let S be the universe named by $\ell(\ell(\mathsf{nat}))$ and T the type $S \setminus \{\ell(\mathsf{nat})\}$. Then T is properly contained in S. Moreover, because of the uniqueness of generators, T is a universe. Now we apply Lemma 5 and derive $S \mathrel{\widetilde{\notin}} T$ from $T \subset S$. It only remains to check that $T \mathrel{\widetilde{\notin}} S$, or equivalently,

$$(\forall x \in S) \neg \Re(x, T).$$

If a is in S, then $\mathsf{co}(\mathsf{co}(a))$ is also an element of S and $a \doteq \mathsf{co}(\mathsf{co}(a))$. The uniqueness of generators and the definition of T therefore yield $\mathsf{co}(\mathsf{co}(a)) \in T$. We apply Lemma 5 again and conclude that $\mathsf{co}(\mathsf{co}(a))$ is not a name of T. But a and $\mathsf{co}(\mathsf{co}(a))$ name the same type so that a cannot be a name of T. Hence we have $T \mathrel{\widetilde{\notin}} S$ and therefore also $\neg(\mathsf{U\text{-}Lin})$.

The proof of $\neg(\mathsf{U\text{-}Tran})$ follows the same pattern. In this case we choose R to be the universe named by $\ell(\ell(\mathsf{nat}))$, S to be the universe named by $\ell(\ell(\ell(\mathsf{nat})))$ and T to be the type $S \setminus \{\ell(\mathsf{nat})\}$. It is $\ell(\mathsf{nat}) \neq \ell(\ell(\mathsf{nat}))$ according to Lemma 8. Hence $\ell(\ell(\mathsf{nat})) \in T$. Therefore $R \mathrel{\widetilde{\in}} T$. In addition,

we have $\ell(\mathsf{nat}) \in R$ which implies $R \not\subset T$. Therefore, $\neg(\mathsf{U\text{-}Tran})$ is proved. Owing to the first assertion of this theorem, we also have $\neg(\mathsf{U\text{-}Con})$. □

This lemma makes it clear that there are too many universes – universes that are not generated by ℓ – which violate linearity, transitivity and connectivity. As a consequence, we claim linearity, transitivity and connectivity only for normal (names of) universes. These restricted versions are natural, sufficient for all practical purposes and justified by the standard model construction of Jäger and Studer [16]. Therefore our "official" formulations are:

$(\mathcal{U}_\ell\text{-}\mathsf{Lin})$ $\quad \forall x \forall y (\mathcal{U}_\ell(x) \wedge \mathcal{U}_\ell(y) \to x \mathrel{\dot\in} y \vee x \doteq y \vee y \mathrel{\dot\in} x),$

$(\mathcal{U}_\ell\text{-}\mathsf{Tran})$ $\quad \forall x \forall y (\mathcal{U}_\ell(x) \wedge \mathcal{U}_\ell(y) \wedge x \mathrel{\dot\in} y \to x \mathrel{\dot\subset} y),$

$(\mathcal{U}_\ell\text{-}\mathsf{Con})$ $\quad \forall x \forall y (\mathcal{U}_\ell(x \wedge \mathcal{U}_\ell(y) \to x \mathrel{\dot\subset} y \vee y \mathrel{\dot\subset} x).$

According to the following lemma, $(\mathcal{U}_\ell\text{-}\mathsf{Tran})$ is provable in every theory of the form $\mathsf{Th} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$, provided that Th comprises EETJ; therefore, it need not be included in the list of axioms. To be more precise: $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$ proves $(\mathcal{U}_\ell\text{-}\mathsf{Tran})$, and $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Lin})$ proves the equivalence of $(\mathcal{U}_\ell\text{-}\mathsf{Con})$ and $(\mathcal{U}_\ell\text{-}\mathsf{Tran})$.

Lemma 11 *In* $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$*, one can prove that*

$$\forall x \forall y \forall z (\mathcal{U}_\ell(x) \wedge \mathcal{U}_\ell(y) \wedge z \doteq x \wedge z \mathrel{\dot\in} y \to x \mathrel{\dot\subset} y).$$

Since this formula is a (useful) generalization of $(\mathcal{U}_\ell\text{-}\mathsf{Tran})$*,* $(\mathcal{U}_\ell\text{-}\mathsf{Tran})$ *is also provable in* $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$*.*

Proof Assume that a is a normal name of the universe S, b a normal name of the universe T, $a \doteq c$ and $c \in T$. Then c is also a name of S and, therefore, we have $c \notin S$ according to Lemma 5. Hence T is not contained in S. Because of $(\mathcal{U}_\ell\text{-}\mathsf{Con})$, we thus have $S \subset T$. □

The most famous system of explicit mathematics is the theory T_0 introduced in Feferman [5]. It is obtained from $\mathsf{EETJ} + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N})$ by adding the principle of inductive generation (IG). As a helpful abbreviation, we write

$$\mathsf{Closed}(a, b, S) := (\forall x \mathrel{\dot\in} a)[(\forall y \mathrel{\dot\in} a)((y, x) \mathrel{\dot\in} b \to y \in S) \to x \in S].$$

Consider b as the code of a binary relation. Then this definition means that S is a type which contains a $c \mathrel{\dot\in} a$ if all predecessors of c in a with respect

to b belong to S. *Inductive generation* (IG) is now given by the following axioms:

($\mathsf{IG}.1$) $\quad \Re(a) \wedge \Re(b) \to \exists X(\Re(\mathsf{i}(a,b), X) \wedge \mathsf{Closed}(a,b,X)),$

($\mathsf{IG}.2$) $\quad \Re(a) \wedge \Re(b) \wedge \mathsf{Closed}(a,b,A) \to (\forall x \mathrel{\dot\in} \mathsf{i}(a,b))A(x)$

for all $\mathbb{L}$ formulas $A(u)$. Thus (IG), i.e. ($\mathsf{IG}.1$) + ($\mathsf{IG}.2$), states the existence of accessible parts and, again, everything is uniform in the corresponding names. As mentioned before, Feferman's T_0 is given by

$$\mathsf{T}_0 \;:=\; \mathsf{EETJ} + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N}) + (\mathsf{IG}).$$

Space does not permit us to discuss the semantics of theories for explicit mathematics with universes and to present some standard model constructions. This issue is treated in some detail in Jäger and Studer [16] for impredicative systems and in Jäger and Strahm [15] for their (meta)predicative variants. These articles also contain the proof-theoretic analysis of a series of theories for explicit mathematics with universes.

Some relevant results are listed in the following theorem. Parts one and two follow from Jäger and Studer [16]. For parts three and four, see Strahm [29] and Kahle [19], respectively. The fixed point theory $\widehat{\mathsf{ID}}_{<\omega}$ is studied in Feferman [7]. Transfinitely iterated fixed point theories are introduced and analyzed in Jäger, Kahle, Setzer and Strahm [14].

Theorem 12 *1. The theory* $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$ *is consistent and of the same proof-theoretic strength as* T_0.

2. This proof-theoretic equivalence remains true if, on both sides, inductive generation or complete induction on the natural numbers plus inductive generation are restricted to types.

3. The theory $\mathsf{EETJ} + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con}) + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N})$ *is proof-theoretically equivalent to* $\widehat{\mathsf{ID}}_{<\varepsilon_0}$.

4. Moreover, if complete induction on the natural numbers in the previous system is restricted to types, then the resulting theory is proof-theoretically equivalent to $\widehat{\mathsf{ID}}_{<\omega}$ *and* ATR_0.

The model constructions in Kahle [19] and Strahm [29] employed for establishing the proof-theoretic upper bounds of $\mathsf{EETJ}+(\mathsf{Lim})+(\mathcal{U}_\ell\text{-}\mathsf{Lin})+(\mathcal{U}_\ell\text{-}\mathsf{Con})$ plus type or formula induction on the natural numbers can easily be adapted to satisfying ($\mathbb{L}$-UG) as well. Hence, the addition of uniqueness of generators with respect to $\mathbb{L}$ to these two theories does not increase their respective proof-theoretic strength.

4 Least universes

The limit axiom (Lim) claims that every name a is contained in a normal universe named by ℓa. It does not claim, however, that this universe is a minimal or least universe containing a. In this section we want more and introduce the theory LUN which requires each name to be element of a least universe. Then we deal with consequences of the existence of least universes.

In the following, we make a careful distinction between the normal universes considered in the previous section and the *least* universes to be generated now. Accordingly, LUN is formulated in the language $\mathbb{L}'$ which is the variant of $\mathbb{L}$ using the generator lt instead of the generator ℓ; the generator i is not needed in $\mathbb{L}'$. The $\mathbb{L}'$ formulas and other syntactic categories of $\mathbb{L}'$ are defined in analogy to those of $\mathbb{L}$.

The axioms of LUN are the axioms of EETJ formulated for $\mathbb{L}'$, uniqueness of generators ($\mathbb{L}'$-UG) with respect to the language $\mathbb{L}'$, the schema ($\mathbb{L}'$-$\mathsf{I_N}$) of complete induction on the natural numbers for all $\mathbb{L}'$ formulas plus the following *leastness axioms*:

$$\forall x(\Re(x) \to \mathcal{U}(\mathsf{lt}(x)) \wedge x \mathrel{\dot\in} \mathsf{lt}(x)), \tag{L.1}$$

$$\forall x[\Re(x) \wedge \forall y(\mathcal{C}(A, y) \to A(y)) \wedge A(x) \to (\forall y \mathrel{\dot\in} \mathsf{lt}(x))A(y)] \tag{L.2}$$

for all $\mathbb{L}'$ formulas $A(u)$. The schema (L.2) is an induction principle establishing that there are no definable proper subcollections of the type with name $\mathsf{lt}(a)$ with the closure properties of a universe and a as an element.

The proof of the following lemma, which lists some further properties of the generator lt, is straightforward and left to the reader. The uniqueness of generators ($\mathbb{L}'$-UG) with respect to $\mathbb{L}'$ is used several times.

Lemma 13 *In* LUN*, one can prove:*

1. $\Re(a) \wedge \Re(\mathsf{lt}(a), S) \to \forall x[(\mathcal{C}(S, x) \vee x = a) \leftrightarrow x \in S]$,
2. $\Re(a) \wedge \mathsf{lt}(b) \mathrel{\dot\in} \mathsf{lt}(a) \to \mathsf{lt}(b) = a$,
3. $a \doteq b \to \mathsf{lt}(a) \mathrel{\dot\notin} \mathsf{lt}(b)$,
4. $\Re(a) \to a \mathrel{\dot\neq} \mathsf{lt}(a)$,
5. $\Re(a) \wedge \mathsf{j}(b, f) \mathrel{\dot\in} \mathsf{lt}(a) \wedge a \neq \mathsf{j}(b, f) \to b \mathrel{\dot\in} \mathsf{lt}(a) \wedge (\forall x \mathrel{\dot\in} b)(fx \mathrel{\dot\in} \mathsf{lt}(a))$.

Taking up the arguments of the previous section one immediately sees that (U-Lin), (U-Tran) and (U-Con) are inconsistent with LUN. However, the situation is even worse in LUN with reference to our ordering principles for universes: even linearity, transitivity and connectivity for normal names are inconsistent. In analogy to EETJ + (Lim), normal (names of) universes are defined in LUN by

$$\mathcal{U}_{\mathsf{lt}}(x) := \exists y(\Re(y) \wedge x = \mathsf{lt}(y)).$$

Linearity ($\mathcal{U}_{\mathsf{lt}}$-Lin), transitivity ($\mathcal{U}_{\mathsf{lt}}$-Tran) and connectivity ($\mathcal{U}_{\mathsf{lt}}$-Con) of normal names of universes are then formulated as expected.

Theorem 14 *In* LUN*, one can prove that*

$$\neg(\mathcal{U}_{\mathsf{lt}}\text{-Lin}) \wedge \neg(\mathcal{U}_{\mathsf{lt}}\text{-Tran}) \wedge \neg(\mathcal{U}_{\mathsf{lt}}\text{-Con}).$$

Proof Let a be the name $\mathsf{lt}(\mathsf{nat})$ and S the universe named by a. Now choose a different name b of S. Then $\mathsf{lt}(a) \dot{\notin} \mathsf{lt}(b)$ and $\mathsf{lt}(b) \dot{\notin} \mathsf{lt}(a)$ follow from the third part of Lemma 13. Since a is the term $\mathsf{lt}(\mathsf{nat})$, the second part of this lemma yields $a \dot{\notin} \mathsf{lt}(b)$. Hence, we also have $\mathsf{lt}(a) \dot{\neq} \mathsf{lt}(b)$, and $\neg(\mathcal{U}_{\mathsf{lt}}\text{-Lin})$ is proved.

We continue by defining c to be the name $\mathsf{lt}(a)$. This implies $\mathsf{lt}(a) \dot{\in} \mathsf{lt}(c)$. Furthermore, in view of the second and the fourth part of Lemma 13, it is also true that $\mathsf{lt}(\mathsf{nat}) \dot{\notin} \mathsf{lt}(c)$. From $\mathsf{lt}(\mathsf{nat}) = a \dot{\in} \mathsf{lt}(a)$, we thus conclude that $\mathsf{lt}(a) \dot{\not\subset} \mathsf{lt}(c)$. Hence $\neg(\mathcal{U}_{\mathsf{lt}}\text{-Tran})$.

To finish the proof of this theorem, we can proceed as in the proof of Theorem 10 and derive $\neg(\mathcal{U}_{\mathsf{lt}}\text{-Con})$ from the just shown $\neg(\mathcal{U}_{\mathsf{lt}}\text{-Tran})$ and the fact that ($\mathcal{U}_{\mathsf{lt}}$-Con) implies ($\mathcal{U}_{\mathsf{lt}}$-Tran). □

Our next aim is to show that inductive generation (IG) can be handled in LUN. The basic idea is to make use of the induction schema (L.2) of LUN for dealing with the induction schema of inductive generation. Please keep in mind the definition of the formula $\mathsf{Closed}(a, b, X)$ in the previous section.

Theorem 15 *There exists a closed individual term* acc *of* $\mathbb{L}'$ *so that* LUN *proves for arbitrary* $\mathbb{L}'$ *formulas* $A(u)$*:*

1. $\Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{acc}(a, b))$,
2. $\Re(a) \wedge \Re(b) \rightarrow \mathsf{Closed}(a, b, \mathsf{acc}(a, b))$,
3. $\Re(a) \wedge \Re(b) \wedge \mathsf{Closed}(a, b, A) \rightarrow (\forall x \dot{\in} \mathsf{acc}(a, b))A(x)$.

PROOF We begin by introducing some notation. Given two types U and V, we write $U \uplus V$ for the disjoint union of U and V and $\mathsf{Pred}(U, V, w)$ for the type of the predecessors of w in U with respect to V,

$$U \uplus V := \{(0, x) : x \in U\} \cup \{(1, x) : x \in V\},$$
$$\mathsf{Pred}(U, V, w) := \{x : x \in U \wedge (x, w) \in V\}.$$

Elementary comprehension shows that $U \uplus V$ and $\mathsf{Pred}(U, V, w)$ are types. Moreover, because of Lemma 4, there are even closed terms du and pd for which LUN proves:

$$\Re(u, U) \wedge \Re(v, V) \to \Re(\mathsf{du}(u, v), U \uplus V), \tag{1}$$
$$\mathsf{U}(W) \wedge u \in W \wedge v \in W \to \mathsf{du}(u, v) \in W, \tag{2}$$
$$\Re(u, U) \wedge \Re(v, V) \to \Re(\mathsf{pd}(u, v, w), \mathsf{Pred}(U, V, w)), \tag{3}$$
$$\mathsf{U}(W) \wedge \Re(u) \wedge \Re(v) \wedge \mathsf{du}(u, v) \in W \to \mathsf{pd}(u, v, w) \in W. \tag{4}$$

Because of the uniqueness of generators du can even be chosen so that $\mathsf{du}(u, v)$ is different from $\mathsf{j}(w, f)$ for arbitrary u, v, w and f.

The next step is an application of the recursion theorem for placing a closed term t at our disposal with

$$t(u, v, w) \simeq \mathsf{j}(\mathsf{pd}(u, v, w), \lambda z.t(u, v, z)) \tag{5}$$

for all u, v and w. Making use of this term t and the generator lt, we can now apply modified elementary comprehension (cf. Lemma 2) in order to obtain a closed term acc so that $\mathsf{acc}(u, v)$ uniformly names the type

$$\{x : x \mathrel{\dot\in} u \wedge t(u, v, x) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(u, v))\},$$

provided that u and v are names, i.e.

$$\Re(u) \wedge \Re(v) \to \Re(\mathsf{acc}(u, v)), \tag{6}$$
$$\Re(u) \wedge \Re(v) \to \forall x[x \mathrel{\dot\in} \mathsf{acc}(u, v) \leftrightarrow x \mathrel{\dot\in} u \wedge t(u, v, x) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(u, v))]. \tag{7}$$

Thus, the first assertion of our theorem is obviously satisfied. To deal with the second assertion, assume $\Re(a)$, $\Re(b)$ and $\Re(\mathsf{acc}(a, b), S)$; we have to show $\mathsf{Closed}(a, b, S)$. To this end, take an individual $c \mathrel{\dot\in} a$ with the property

$$(\forall x \mathrel{\dot\in} a)((x, c) \mathrel{\dot\in} b \to x \in S).$$

From this we conclude that

$$\forall x(x \mathrel{\dot\in} \mathsf{pd}(a, b, c) \to t(a, b, x) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(a, b))).$$

Together with (4), the closure properties of universes and equation (5) we obtain

$$t(a,b,c) \simeq \mathsf{j}(\mathsf{pd}(a,b,c), \lambda z.t(a,b,z)) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(a,b)).$$

Therefore, c is an element of S, and the proof of $\mathsf{Closed}(a,b,S)$ is complete. Hence, the second assertion of our theorem is established as well.

Before turning to the proof of the third assertion, which requires a bit more effort, we show two auxiliary assertions (A) and (B).

(A) $\Re(u) \wedge \Re(v) \wedge w \mathrel{\dot\in} \mathsf{acc}(u,v) \to (\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,w))(x \mathrel{\dot\in} \mathsf{acc}(u,v))$.

Proof of (A). Let u and v be names. Then $w \mathrel{\dot\in} \mathsf{acc}(u,v)$ implies in view of equation (5) and property (7) that

$$\mathsf{j}(\mathsf{pd}(u,v,w), \lambda z.t(u,v,z)) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(u,v)).$$

Remember that $\mathsf{du}(u,v)$ is different from $\mathsf{j}(\mathsf{pd}(u,v,w), \lambda z.t(u,v,z))$ according to our choice of du. Hence, assertion 5 of Lemma 13 yields

$$(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,w))(t(u,v,x) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(u,v))).$$

Thus, we have $(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,w))(x \mathrel{\dot\in} \mathsf{acc}(u,v))$, and the proof of the first auxiliary assertion (A) is complete.

Depending on the closed terms t and acc and the parameters u and v, we now define for each $\mathbb{L}'$ formula $A(w)$ an $\mathbb{L}'$ formula $B_A(u,v,w)$ which helps to reduce the closure principle of inductive generation to the closure condition for universes,

$$B_A(u,v,w) := \forall y(y \mathrel{\dot\in} \mathsf{acc}(u,v) \wedge w = t(u,v,y) \to A(y)).$$

(B) $\Re(u) \wedge \Re(v) \wedge \mathsf{Closed}(u,v,A) \wedge \mathcal{C}(B_A(u,v,.),w) \to B_A(u,v,w)$.

Proof of (B). Assuming the left hand side of this implication, we have to show that $A(c)$ follows from

$$c \mathrel{\dot\in} \mathsf{acc}(u,v) \wedge w = t(u,v,c) \tag{8}$$

for all c. So we also assume (8). Then equation (5) implies

$$w = \mathsf{j}(\mathsf{pd}(u,v,c), \lambda z.t(u,v,z)).$$

Hence, the uniqueness of generators and $\mathcal{C}(B_A(u,v,.),w)$ yield

$$(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,c))B_A(u,v,t(u,v,x)),$$

and the definition of B_A therefore implies

$$(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,c))\forall y(y \mathrel{\dot\in} \mathsf{acc}(u,v) \wedge t(u,v,x) = t(u,v,y) \to A(y)).$$

From this we immediately obtain

$$(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,c))(x \mathrel{\dot\in} \mathsf{acc}(u,v) \to A(x)).$$

Because of $c \mathrel{\dot\in} \mathsf{acc}(u,v)$, applying (A) gives $(\forall x \mathrel{\dot\in} \mathsf{pd}(u,v,c))A(x)$ and, therefore, $A(c)$ follows from $\mathsf{Closed}(u,v,A)$. Thus (B) is proved.

Now we are ready for the third assertion of our theorem. Take an arbitrary $\mathbb{L}'$ formula $A(u)$ and assume $\Re(a)$, $\Re(b)$, $\Re(\mathsf{acc}(a,b),S)$ and $\mathsf{Closed}(a,b,A)$. We apply the auxiliary assertion (B) and obtain

$$\forall x(\mathcal{C}(B_A(a,b,.),x) \to B_A(a,b,x)).$$

Because of the uniqueness of generators, we also have $B_A(a,b,\mathsf{du}(a,b))$. Thus, the leastness principle (L.2) yields $(\forall x \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(a,b)))B_A(a,b,x)$. Hence, by the definition of B_A, we conclude that

$$(\forall x \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(a,b)))\forall y(y \mathrel{\dot\in} \mathsf{acc}(a,b) \wedge x = t(a,b,y) \to A(y)).$$

Since $\mathsf{acc}(a,b)$ is a name of S and $t(a,b,c) \mathrel{\dot\in} \mathsf{lt}(\mathsf{du}(a,b))$ for all elements c of S, it follows $(\forall x \in S)A(x)$. This finishes the proof of the third assertion of our theorem. □

This theorem provides the desired reduction of T_0 to LUN: (i) the language $\mathbb{L}$ is translated into the language $\mathbb{L}'$ by interpreting the generator i of $\mathbb{L}$ as the closed individual term acc of $\mathbb{L}'$ and leaving the remaining vocabulary unchanged; (ii) then the translations of all instances of inductive generation obtained in this way are provable in LUN according to the previous theorem; (iii) the (translations of the) remaining axioms of T_0 are obviously provable in LUN. A careful inspection of the previous proof also establishes the second part of the following corollary.

Corollary 16 *The theory* T_0 *is contained in* LUN. *Moreover, the subsystems of* T_0 *which are obtained by restricting inductive generation or complete induction on the natural numbers plus inductive generation to types are contained in the corresponding subsystems of* LUN.

5 Embedding of LUN into $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$

In the previous section, we have shown how T_0 can be embedded into LUN. Therefore, we have a lower bound for the proof-theoretic strength of LUN. It

remains to be proved that this bound is sharp. This aim will be achieved by interpreting LUN into the extension $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$ of T_0 and by exploiting a result of Jäger and Studer [16] implying the proof-theoretic equivalence of T_0 and $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$.

The crucial step in the interpretation of LUN into $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$ is to construct a closed term lst of the language $\mathbb{L}$ so that for each name a the term $\mathsf{lst}(a)$ names, provable in $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$, the least universe containing a. In general, the generator ℓ will not do this job since the universe denoted by ℓa may be too big. And we know more: according to Theorem 14, it is inconsistent with $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$ and linearity, transitivity or connectivity of normal names to assume that each ℓa is a name of the least universe containing a.

For defining the closed term lst we proceed as follows: given a name a, we go over to the normal universe provided by ℓa. Then we use inductive generation on this universe in order to single out those names which are absolutely needed for a universe containing a. This means we employ a binary relation on the universe (named by) ℓa according to the "date of generation" of the respective names. Because of the uniqueness of generators, the history of the elements of ℓa relevant for this construction is well-determined in the theory $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})$.

Now we define an $\mathbb{L}$ formula $\sqsubset_a$ which says that b and c are elements of the universe ℓa and b comes before c in the inductive build up of the least universe containing a. Let $\mathsf{Bef}(a,b,c)$ be the disjunction of the following formulas:

(1) $c=\mathsf{co}(b)$,

(2) $\exists x(c=\mathsf{int}(b,x)\vee c=\mathsf{int}(x,b))$,

(3) $c=\mathsf{dom}(b)$,

(4) $\exists f(c=\mathsf{inv}(f,b))$,

(5) $\exists f(c=\mathsf{j}(b,f))$,

(6) $\exists x\exists y\exists f(c=\mathsf{j}(x,f)\wedge(x,y)\mathrel{\dot\in}\mathsf{j}(\ell a,\lambda z.z)\wedge b=fy)$.

Then we set

$$b\sqsubset_a c:=b\mathrel{\dot\in}\ell a\wedge c\mathrel{\dot\in}\ell a\wedge c\neq a\wedge\mathsf{Bef}(a,b,c).$$

Remember that the subformula $(x,y)\mathrel{\dot\in}\mathsf{j}(\ell a,\lambda z.z)$ in clause (6) is equivalent to $x\mathrel{\dot\in}\ell a\wedge y\mathrel{\dot\in}x$. Hence (6) is the same as the more familiar

$$\exists x\exists y\exists f(c=\mathsf{j}(x,f)\wedge x\mathrel{\dot\in}\ell a\wedge y\mathrel{\dot\in}x\wedge b=fy),$$

saying that b is one of the "predecessors" of c in the case that c is generated by join. The name a itself is considered as an urelement of the least universe containing a; therefore we have the condition $c \neq a$ in the definition of $b \sqsubset_a c$ for ruling out the possibility that a has $\sqsubset_a$ predecessors.

The candidates for the least universe containing a are a itself, the constants nat as well as id and all names b with at least one element before b and all such belonging to ℓa,

$$\mathsf{Cand}(a,b) := \begin{cases} (b = a \vee b = \mathsf{nat} \vee b = \mathsf{id}) \vee \\ (\exists x \mathsf{Bef}(a,x,b) \wedge \forall x(\mathsf{Bef}(a,x,b) \rightarrow x \mathrel{\dot\in} \ell a)). \end{cases}$$

All other individuals cannot belong to the least universe containing a. When applying inductive generation, this can be achieved by postulating that the corresponding accessibility relation is reflexive on the non-candidates. A candidate, on the other hand, goes into the intended inductively generated type whenever all its $\sqsubset_a$ predecessors are elements of this type.

From Lemma 2 about modified elementary comprehension, we conclude that, for every name a, there exists a type coding the intended accessibility relation,

$$\mathsf{Ar}(a) := \{(x,y) : (x = y \wedge \neg \mathsf{Cand}(a,y)) \vee (x \sqsubset_a y \wedge \mathsf{Cand}(a,y))\}.$$

This lemma also implies that in EETJ, there is a closed individual term ar of $\mathbb{L}$ which uniformly describes this assignment of the type $\mathsf{Ar}(a)$ to the name a, i.e. EETJ proves

$$\Re(a) \rightarrow \Re(\mathsf{ar}(a), \mathsf{Ar}(a)).$$

We finish the uniform construction of the least universe containing the name a in $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$ by carrying through inductive generation on the type with name ℓa along the relation coded by $\mathsf{Ar}(a)$,

$$\mathsf{lst} := \lambda z.\mathsf{i}(\ell z, \mathsf{ar}(z)).$$

Thus $\mathsf{lst}(a)$ is $\mathsf{i}(\ell a, \mathsf{ar}(a))$. The following theorem shows that the closed term lst produces for each name a the least universe containing a.

Theorem 17 *For each $\mathbb{L}$ formula $A(u)$ one can prove in T_0+(Lim)+($\mathbb{L}$-UG):*

1. $\Re(a) \rightarrow \mathcal{U}(\mathsf{lst}(a)) \wedge a \mathrel{\dot\in} \mathsf{lst}(a)$,
2. $\Re(a) \wedge \forall x(\mathcal{C}(A,x) \rightarrow A(x)) \wedge A(a) \rightarrow (\forall x \mathrel{\dot\in} \mathsf{lst}(a))A(x)$.

PROOF In view of the preceding remarks, the proof of the first assertion should be more or less obvious. For showing the second assertion, suppose $\Re(a)$, $\forall x(\mathcal{C}(A,x) \to A(x))$ and $A(a)$. Then some intermediate calculations yield $\mathsf{Closed}(\ell a, \mathsf{ar}(a), A)$. From this we conclude $(\forall x \mathrel{\dot\in} \mathsf{lst}(a))A(x)$ by the induction principle of inductive generation. □

We simply translate the language $\mathbb{L}'$ of LUN into the language $\mathbb{L}$ of $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$ by interpreting the generator lt of $\mathbb{L}'$ by the closed term lst of $\mathbb{L}$. Hence, the previous theorem yields the translation of the axioms (L.1) and (L.2). The treatment of the other axioms of LUN in $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$ is unproblematic.

Corollary 18 *The theory* LUN *is contained in* $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$. *Moreover, the subsystems of* LUN *which are obtained by restricting* (L.2) *or complete induction on the natural numbers plus* (L.2) *to types are contained in the corresponding subsystems of* $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathbb{L}\text{-}\mathsf{UG})$.

6 Name strictness

When considering the predicate $\Re$ in the theory $\mathsf{EETJ} + (\mathsf{Lim})$, we see that names are built up by the use of the generators nat, id, co, int, dom, inv, j and the universe generator ℓ. However, there is no restriction on whether other terms can belong to $\Re$ as well. For example, if $\mathsf{co}(a)$ is a name, then there is in general no need for a being a name, too.

In this section, we discuss the notion of *name strictness* of types stating that the (appropriate) arguments $s_1, \ldots, s_n$ of a generator r have to be elements of the name strict type W, provided that $r(s_1, \ldots, s_n)$ belongs to W. This notion is analogue to the strictness of definedness, implemented in the logic of partial terms, and the so-called N-strictness for the natural numbers, discussed in Kahle [20]. By reflecting name strictness on universes, one obtains *name strict universes*. In the next section, we introduce a form of name induction saying that all names have to be constructed by the use of generators. Adding name induction to the theory of name strict universes proves inductive generation and yields, thereby, an alternative to LUN.

Name strictness depends on the generators which are available in the underlying language. But since we discuss name strictness only in connection with the language $\mathbb{L}$, we do not mention this dependence and simply write $\mathsf{Str}(W)$ for the conjunction of the following formulas:

(1) $\forall x(\mathsf{co}(x) \in W \to x \in W)$,

(2) $\forall x \forall y(\mathsf{int}(x,y) \in W \to x \in W \wedge y \in W)$,

(3) $\forall x(\mathsf{dom}(x) \in W \to x \in W)$,

(4) $\forall f \forall x(\mathsf{inv}(f,x) \in W \to x \in W)$,

(5) $\forall x \forall f(\mathsf{j}(x,f) \in W \to x \in W \wedge (\forall y \mathrel{\dot\in} x)(fy \in W))$,

(6) $\forall x \forall y(\mathsf{i}(x,y) \in W \to x \in W \wedge y \in W)$,

(7) $\forall x(\ell(x) \in W \to x \in W)$.

Accordingly, a type W is called a strict universe if it is a universe and if it satisfies the condition $\mathsf{Str}(W)$.

Definition 19 *1. We write* $\mathsf{SU}(W)$ *to express that the type* W *is a* name strict universe*,*

$$\mathsf{SU}(W) := \mathsf{U}(W) \wedge \mathsf{Str}(W).$$

2. We write $\mathcal{SU}(t)$ *to express that the individual* t *is a* name of a name strict universe*,*

$$\mathcal{SU}(t) := \exists X(\Re(t,X) \wedge \mathsf{SU}(X)).$$

Our old limit axiom (Lim) postulates that every name a belongs to a universe which is named ℓa. In the context of name strictness, this axiom is now replaced by the corresponding limit axiom for name strict universes,

(sLim) $$\forall x(\Re(x) \to \mathcal{SU}(\ell(x)) \wedge x \mathrel{\dot\in} \ell(x)).$$

Our definition of universe is too general for requiring that all universes are name strict. Following the pattern of the proof of Theorem 10 it is easy to see that, at least in the presence of ($\mathbb{L}$-UG), there are universes that are not name strict.

Lemma 20 *In* $\mathsf{EETJ} + (\mathsf{sLim}) + (\mathbb{L}\text{-}\mathsf{UG})$*, one can prove that*

$$\exists X(\mathsf{U}(X) \wedge \neg\mathsf{SU}(X)).$$

Proof Let S be the universe which is named by $\ell(\ell(\ell(\mathsf{nat})))$, and let T be the type $S \setminus \{\ell(\mathsf{nat})\}$. As in the proof of Theorem 10, we realize that T is a universe. However, T obviously does not contain the element $\ell(\mathsf{nat})$, although it contains $\ell(\ell(\mathsf{nat}))$. Hence T is not name strict. □

The model construction of Jäger and Studer [16] shows that all proof-theoretic equivalences mentioned in the first two parts of Theorem 12 remain true if

we replace the limit axiom (Lim) by our new axiom (sLim). The same should be the case for parts three and four of that theorem.

We end this section with a simple example which illustrates the usefulness of name strict universes. Suppose that we want a universe which contains two given names a and b. In the presence of (sLim) we can proceed as follows. We first select the name $\mathsf{du}(a, b)$ of the disjoint union of the types named by a and b (cf. proof of Theorem 15). Then we form $\ell(\mathsf{du}(a, b))$. Because of name strictness, it is easy to check that the universe with this name contains a and b. If only the axiom (Lim) is available, then $\ell(\mathsf{du}(a, b))$ can be formed as well, but now we cannot conclude that $a \mathrel{\dot\in} \ell(\mathsf{du}(a, b))$ and $b \mathrel{\dot\in} \ell(\mathsf{du}(a, b))$. It merely follows that there are names $a', b' \mathrel{\dot\in} \ell(\mathsf{du}(a, b))$ so that $a' \doteq a$ and $b' \doteq b$.

7 Name induction

As an alternative to least universes, we can add name induction to the theory $\mathsf{EETJ} + (\mathsf{sLim}) + (\mathbb{L}\text{-}\mathsf{UG})$ to obtain inductive generation. Name induction claims that the elements of $\Re$ are built up by the use of generators only. In a certain sense it can be understood as an intensional version of $\in$ induction.

In order to state the axiom schema of name induction, we introduce the closure condition $\mathcal{C}_\ell(S, a)$ which extends $\mathcal{C}(S, a)$ by a new clause for the universe generator ℓ,

$$\mathcal{C}_\ell(S, a) := \mathcal{C}(S, a) \vee \exists x(a = \ell x \wedge x \in S).$$

The type existence axioms of $\mathsf{EETJ}+(\mathsf{Lim})$ and $\mathsf{EETJ}+(\mathsf{sLim})$ guarantee that the names are closed under this closure condition,

$$\forall x(\mathcal{C}_\ell(\Re, x) \to \Re(x)).$$

The schema of *name induction* on the other hand, is the principle that there are no definable subcollections of the names with this closure property. It is given by

$$(\mathbb{L}\text{-}\mathsf{I}_\Re) \qquad \forall x(\mathcal{C}_\ell(A, x) \to A(x)) \to \forall x(\Re(x) \to A(x))$$

for all $\mathbb{L}$ formulas $A(u)$. This form of name induction will be considered now in the context of EETJ with the strict limit axiom, uniqueness of generators and the schema of complete induction on the natural numbers,

$$\mathsf{NAI} \;:=\; \mathsf{EETJ} + (\mathsf{sLim}) + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N}) + (\mathbb{L}\text{-}\mathsf{I}_\Re).$$

As an immediate consequence of name induction we obtain the name strictness of the predicate $\Re(u)$. The proof of the following lemma is routine work; name strictness of the limit axiom and complete induction on the natural numbers are not needed.

Lemma 21 *In* NAI*, one can prove that* $\mathsf{Str}(\Re)$.

In the proof of Theorem 15 which provides for inductive generation in the theory LUN, we made essential use of the fifth assertion of Lemma 13. Working in the theory NAI, an even slightly stronger property follows immediately from the name strictness of normal universes.

Lemma 22 *In* NAI*, one can prove that*

$$\Re(a) \wedge \mathsf{j}(b, f) \mathrel{\dot\in} \ell a \to b \mathrel{\dot\in} \ell a \wedge (\forall x \mathrel{\dot\in} b)(fx \mathrel{\dot\in} \ell a).$$

With this lemma available, we can now simulate inductive generation in the theory NAI in the same way as we did it in LUN.

Theorem 23 *There exists a closed individual term* ig *of* $\mathbb{L}$ *so that* NAI *proves for arbitrary* $\mathbb{L}$ *formulas* $A(u)$*:*

1. $\Re(a) \wedge \Re(b) \to \Re(\mathsf{ig}(a, b))$,
2. $\Re(a) \wedge \Re(b) \to \mathsf{Closed}(a, b, \mathsf{ig}(a, b))$,
3. $\Re(a) \wedge \Re(b) \wedge \mathsf{Closed}(a, b, A) \to (\forall x \mathrel{\dot\in} \mathsf{ig}(a, b))A(x)$.

Proof This proof is literally the same as the proof of Theorem 15, provided that we make the following changes: instead of the generator lt, we use the generator ℓ and instead of the induction schema (L.2), we apply name induction ($\mathbb{L}$-$\mathsf{I}_{\Re}$). □

Since the generator i does not play any role in the theory NAI, it has no function for the embedding of $\mathsf{T}_0 + (\mathsf{sLim})$; the part of the generator i in $\mathsf{T}_0 + (\mathsf{sLim})$ is taken over by the just defined closed term ig. Hence, if A^* is the $\mathbb{L}$ formula which results from the $\mathbb{L}$ formula A by replacing all occurrences of i by ig, we see that NAI proves A^* for all axioms A of $\mathsf{T}_0 + (\mathsf{sLim})$.

Corollary 24 *The theory* $\mathsf{T}_0 + (\mathsf{sLim})$ *is contained in* NAI*. Moreover, the subsystems of* $\mathsf{T}_0 + (\mathsf{sLim})$ *which are obtained by restricting inductive generation or complete induction on the natural numbers plus inductive generation to types are contained in the corresponding subsystems of* NAI*.*

For determining the upper proof-theoretic bounds of NAI, we refer again to the model construction in Jäger and Studer [16]. It follows that NAI, together with the axioms $(\mathcal{U}_\ell\text{-}\mathsf{Lin})$ and $(\mathcal{U}_\ell\text{-}\mathsf{Con})$, is valid in this model. The results of [16] thus show that the proof-theoretic strength of $\mathsf{NAI} + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$ cannot be greater than that of T_0.

We conclude this article by recapitulating several results concerning theories of explicit mathematics with universes. One important aspect is that the addition of (Lim) or (sLim) plus certain ordering principles for normal universes does not increase the proof-theoretic strength of T_0; another observation says that inductive generation can be replaced by leastness of universes or name induction.

Conclusion 25 *The following theories with universes have the same proof-theoretic strength as the theory* T_0*:*

1. $\mathsf{T}_0 + (\mathsf{Lim})$ *and* $\mathsf{T}_0 + (\mathsf{Lim}) + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$,
2. $\mathsf{T}_0 + (\mathsf{sLim})$ *and* $\mathsf{T}_0 + (\mathsf{sLim}) + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$,
3. LUN, NAI *and* $\mathsf{NAI} + (\mathcal{U}_\ell\text{-}\mathsf{Lin}) + (\mathcal{U}_\ell\text{-}\mathsf{Con})$.

Name induction added to $\mathsf{EETJ} + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathbb{L}\text{-}\mathsf{I}_\mathsf{N})$ yields a theory of explicit mathematics which is proof-theoretically equivalent to ID_1. The lower bound is established by embedding the theory $\mathsf{ID}_1(\mathsf{acc})$ of accessible parts (cf. e.g. Buchholz, Feferman, Pohlers and Sieg [2]). For the upper bound, the treatment of EETJ in $\widehat{\mathsf{ID}}_1$, cf. Beeson [1] or Marzetta [22], can easily be modified using the leastness condition of ID_1 to handle name induction.

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