

# Realization of analysis into Explicit Mathematics

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## Abstract

We define a novel interpretation  $\mathcal{R}$  of second order arithmetic into Explicit Mathematics. As a difference from standard  $\mathcal{D}$ -interpretation, which was used before and was shown to interpret only subsystems proof-theoretically weaker than  $\mathbf{T}_0$ , our interpretation can reach the full strength of  $\mathbf{T}_0$ . The  $\mathcal{R}$ -interpretation is an adaptation of Kleene's recursive realizability, and is applicable only to intuitionistic theories.

## Introduction

Systems of Explicit Mathematics were introduced by S. Feferman in the 70-es as a logical framework for Bishop-style constructive mathematics (see [Fef75], [Fef79]). In [Fef79] he gave an embedding of the basic theory  $\mathbf{T}_0$  into a subsystem  $\Delta_2^1\text{-CA}+\mathbf{BI}$  of 2-nd order arithmetic and conjectured that the converse also holds. In [Jä83] G. Jäger carried out a necessary well-ordering proof in  $\mathbf{T}_0$ , which together with [JP82] completed its proof-theoretical analysis and established proof-theoretic equivalence of the system of Explicit Mathematics  $\mathbf{T}_0$ , system of analysis  $\Delta_2^1\text{-CA}+\mathbf{BI}$ , and the set theory  $\mathbf{KPi}$ . However, up to now, there were no direct embeddings of strong conventional theories, e.g. analysis or set theory of the strength of  $\mathbf{T}_0$  and higher, into Explicit Mathematics. This also yielded that the only method for establishing proof-theoretic lower bounds for  $\mathbf{T}_0$  and stronger systems of Explicit Mathematics remained to be well-ordering proofs carried out directly in those theories.

The situation is quite different with Martin-Löf type theories, where, in addition to well-ordering proofs (see [Se98]), we also have direct embeddings of constructive set theory  $\mathbf{CZF}$ , [Acz78], and its extensions, [Acz86, RaCZFM], or a subsystem of analysis  $\mathbf{IARI}$ , [GR94]. The possibility of such an embedding is often considered as an evidence for constructivity of a given theory. The obstacle for similar embeddings into Explicit Mathematics was its specific nature, where intuitionistic and classical principles, set-theoretic and recursion-theoretic intuition can be combined. It is sufficiently straightforward to do for “weak” theories (essentially up to  $\Pi_1^1\text{-CA}$ ); however, for stronger systems with mathematical meaning, where adding the law of excluded middle often results in dramatic increase in proof-theoretic strength (see, e.g., [RGP98]), the distinction classical/intuitionistic must have played a prominent role. The price for this universality of Explicit Mathematics is that, while in  $\mathbf{ML}$  type theories derivability simply means Kleene-type realizability, in  $\mathbf{EM}$  these notions are different. The reason for this is that a lot of realizable formulas, e.g.

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Church's thesis and axiom of choice in analysis, are incompatible with classical logic without damaging consistency or consistency strength.

In this paper we develop *a realizability interpretation into Explicit Mathematics*. We have chosen here the simplest example, *realization of analysis*, which keeps the amount of technical details at minimum, and demonstrates the method most distinctly. For constructive set theory **CZF** and its extensions, one combines realizability with other methods of interpreting set theory. This is reserved for another publication ([TurCZFM]).

The paper is organised as follows. For reader's convenience in Sections 1 and 2 we briefly introduce the theory  $\mathbf{T}_0$  and subsystems of analysis we are interested in. In Section 3 we define two interpretations of analysis into  $\mathbf{T}_0$ , a *direct* interpretation  $\mathcal{D}$  and a *realizability* interpretation  $\mathcal{R}$ . A *direct* interpretation  $\mathcal{D}$  means simply that variables are interpreted as ranging over natural numbers and sets of natural numbers and the meaning of logical connectives does not change. It's this interpretation which was used before, e.g. in [Fef79, Ch.V] and [BFPS, Ch.II, §1]. The drawback of this translation is that it does not really exploit the axiom of Join of Explicit Mathematics, the consequence of which being that the only systems which have been interpreted via  $\mathcal{D}$  are proof-theoretically weaker than  $\mathbf{T}_0$ .

Alternatively, we define a *realizability* interpretation  $\mathcal{R}$ , which is a variant of Kleene 1945 recursive realizability. The general setting for realizing one language into another was given already in [Fef79]; however, that paper studies in detail only realization of Explicit Mathematics into itself. As to relationships between the interpretations  $\mathcal{D}$  and  $\mathcal{R}$ , we prove that they are equivalent over an applicative part **App** of  $\mathbf{T}_0$  for first-order negative formulas, Theorem 2, and  $\mathcal{R}(F)$  implies  $\mathcal{D}(F)$  for  $F$  from a certain CC-class, Theorem 3. Thus  $\mathcal{R}$ -interpretation automatically transfers proof-theoretic upper bounds from Explicit Mathematics to analysis, and lower bounds vice versa. Axiom of Choice, on the contrary, is an example of a formula for which  $\mathcal{D}$  does not follow from  $\mathcal{R}$ , and is much stronger in presence of the law of excluded middle, Theorem 6. In Section 4 we finally build realizations of various axioms, giving together the theory **IARI** of [GR94], which has the same proof-theoretic strength as  $\mathbf{T}_0$ .

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## 1 Explicit Mathematics. The theory $\mathbf{T}_0$

We follow essentially the original type-free two-sorted formulation of Explicit Mathematics from [Fef75]. Alternative formulations are given in [Be85] and [Jä88].

**Language  $\mathcal{L}_{EM}$ .** The theory  $\mathbf{T}_0$  is formulated in a two-sorted language: operations (individuals) and names (classifications). Names are thought of as a special kind of operations, coding sets of operations. We use **variables**  $a, b, c, \dots$  as ranging over operations, and  $\alpha, \beta, \gamma, \dots$  as ranging over names. The operation **constants** of the theory are the following: combinators  $k, s$ , pairing  $p$  and projections  $p_0, p_1$ , zero  $0$ , successor  $s_N$  and predecessor  $p_N$ , distinction by cases on natural numbers  $d_N$ , join  $j$  and inductive generation  $i$ . Additionally we have the following 8 operation **constants** called *name generators*:  $\text{nat}$ ,  $\text{id}$ ,  $\text{inv}$ ,  $\text{and}$ ,  $\text{or}$ ,  $\text{imp}$ ,  $\text{all}$ ,  $\text{ex}$ . **Terms** are built from variables and constants by the following application clause: if  $s$  and  $t$  are *terms* then  $s \cdot t$  is a *term*, so that the *application* function symbol  $\cdot$  accepts arguments of both sorts and returns an operation. **Atomic formulas** are  $s = t$  ( $s$  coincides with  $t$ ) and  $s \varepsilon t$  ( $s$  belongs to the set named by  $t$ ,  $s$  is classified under  $t$ ), where  $s$  and  $t$  are terms. **Formulas** are built from atomic formulas by  $\wedge, \vee, \rightarrow$  and two types of quantifiers, over operations and over names, e.g.  $\forall a, \exists a, \forall \alpha, \exists \alpha$ . Finally, **expression** is a term or a formula.

**Abbreviations.** We use the following standard abbreviations:

$$\begin{aligned}
\neg F &:\Leftrightarrow F \rightarrow 0 = \mathfrak{s}_N \cdot 0; \\
F_0 \leftrightarrow F_1 &:\Leftrightarrow (F_0 \rightarrow F_1) \wedge (F_1 \rightarrow F_0); \\
t \downarrow &:\Leftrightarrow \exists x(t = x); \\
\mathcal{N}[t] &:\Leftrightarrow \exists \alpha(t = \alpha); \\
s \simeq t &:\Leftrightarrow (s \downarrow \vee t \downarrow) \rightarrow s = t; \\
s \dot{\subset} t &:\Leftrightarrow \forall x \varepsilon s(x \varepsilon t); \quad s \dot{=} t :\Leftrightarrow s \dot{\subset} t \wedge t \dot{\subset} s; \\
r : s \mapsto t &\text{ for } \forall x \varepsilon s(rx \varepsilon t); \\
r : s^1 \mapsto t &\text{ for } r : s \mapsto t, \quad r : s^{m+1} \mapsto t \text{ for } \forall x \varepsilon s(rx : s^m \mapsto t); \\
t' \text{ for } \mathfrak{s}_N \cdot t; \quad 1 \text{ for } 0'; \quad st &\text{ for } s \cdot t; \quad t(s_1, \dots, s_n) \text{ for } (\dots (ts_1) \dots s_n); \quad \langle s, t \rangle \text{ for } (\mathfrak{p}s)t; \quad s \neq t \\
&\text{ for } \neg s = t, \text{ etc.}
\end{aligned}$$

**Syntactical conventions.**

1. We use  $e[*]$  for an expression  $e$ , possibly containing occurrences of a variable  $*$  (of appropriate sort). In this context by  $e[t]$  we mean the result of substituting expression  $t$  for all occurrences of  $*$  in  $e$ .
2. Parentheses in terms are assumed to be associated to the left: e.g.,  $s \cdot t \cdot u$  is read as  $(s \cdot t) \cdot u$ .
3. We adopt the following priority among propositional connectives and their abbreviations:  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ . For example,  $F_1 \vee \neg F_2 \wedge F_3 \rightarrow F_4 \leftrightarrow F_5$  has to be read as  $((F_1 \vee ((\neg F_2) \wedge F_3)) \rightarrow F_4) \leftrightarrow F_5$ .

**Logic.** Intuitionistic 2-sorted logic of partial terms with equality. See, e.g., [Be85, Ch.VI, 1] or [Tr98, 1.3].

**Axioms.** The axioms are divided in 6 groups, according to their nature.

**I. Applicative axioms.** These axioms formalise that operations form a partial combinatory algebra, that we have pairing and projections, usual closure conditions on natural numbers, as well as definition by numerical cases:

- (1)  $kab = a$ ;
- (2)  $sab \downarrow \wedge sabc \simeq ac(bc)$ ;
- (3)  $pab \downarrow \wedge p_0a \downarrow \wedge p_1a \downarrow \wedge p_0(pab) = a \wedge p_1(pab) = b$ ;
- (4)  $0 \varepsilon \text{nat} \wedge \forall x \varepsilon \text{nat}(\mathfrak{s}_N x \varepsilon \text{nat})$ ;
- (5)  $\forall x \varepsilon \text{nat}(\mathfrak{s}_N x \neq 0 \wedge \mathfrak{p}_N(\mathfrak{s}_N x) = x)$ ;
- (6)  $\forall x \varepsilon \text{nat}(x \neq 0 \rightarrow \mathfrak{p}_N x \varepsilon \text{nat} \wedge \mathfrak{s}_N(\mathfrak{p}_N x) = x)$ ;
- (7)  $a \varepsilon \text{nat} \wedge b \varepsilon \text{nat} \rightarrow (a = b \rightarrow \mathfrak{d}_N xyab = x) \wedge (a \neq b \rightarrow \mathfrak{d}_N xyab = y)$ .

**II. Induction on nat.**

$$\varphi[0] \wedge \forall x(\varphi[x] \rightarrow \varphi[\mathfrak{s}_N x]) \rightarrow \forall x \varepsilon \text{nat} \varphi[x]$$

for each formula  $\varphi$ .

The following lemmas 1.1 and 1.2 are provable using only applicative axioms I; Lemma 1.3 in addition calls for induction on natural numbers II (see, for example, [Fef79], [Be85], or a review [JKS99]).

**Lemma 1.1**  $\lambda$ -abstraction

For every term  $t[x]$  there exists a term  $\lambda x.t[x]$  such that  $\lambda x.t[x] \downarrow$  and for every term  $s$

$$s \downarrow \rightarrow (\lambda x.t[x])s \simeq t[s].$$

**Lemma 1.2** Recursion Theorem

There exists a closed term  $\text{rec}$  such that

$$\text{rec}f \downarrow \wedge \text{rec}fx \simeq f(\text{rec}f)x.$$

**Lemma 1.3** Primitive recursion on natural numbers

There exists a closed term  $\text{prim}$  such that

$$f : \text{nat} \mapsto \text{nat} \wedge g : \text{nat}^3 \mapsto \text{nat} \wedge x \varepsilon \text{nat} \wedge y \varepsilon \text{nat} \longrightarrow \\ \text{prim}fg : \text{nat}^2 \mapsto \text{nat} \wedge \text{prim}fgx0 = fx \wedge \text{prim}fgx(\text{s}_{\text{N}}y) = gxy(\text{prim}fgxy).$$

**III. Explicit representation.** This axiom states that each name is an operation:

$$\exists x(x = \alpha).$$

**IV. Elementary comprehension (ECA).** These axiomatise *name generators*:

- (1)  $\mathcal{N}[\text{nat}]$ ;
- (2)  $\mathcal{N}[\text{id}] \wedge \forall x(x \varepsilon \text{id} \leftrightarrow x = \langle \mathbf{p}_0x, \mathbf{p}_1x \rangle \wedge \mathbf{p}_0x = \mathbf{p}_1x)$ ;
- (3)  $\mathcal{N}[\text{inv}(f, \alpha)] \wedge \forall x(x \varepsilon \text{inv}(f, \alpha) \leftrightarrow fx \varepsilon \alpha)$ ;
- (4)  $\mathcal{N}[\text{and}(\alpha, \beta)] \wedge \forall x(x \varepsilon \text{and}(\alpha, \beta) \leftrightarrow x \varepsilon \alpha \wedge x \varepsilon \beta)$ ;
- (5)  $\mathcal{N}[\text{or}(\alpha, \beta)] \wedge \forall x(x \varepsilon \text{or}(\alpha, \beta) \leftrightarrow x \varepsilon \alpha \vee x \varepsilon \beta)$ ;
- (6)  $\mathcal{N}[\text{imp}(\alpha, \beta)] \wedge \forall x(x \varepsilon \text{imp}(\alpha, \beta) \leftrightarrow x \varepsilon \alpha \rightarrow x \varepsilon \beta)$ ;
- (7)  $\mathcal{N}[\text{all}\alpha] \wedge \forall x(x \varepsilon \text{all}\alpha \leftrightarrow \forall y(\langle x, y \rangle \varepsilon \alpha))$ ;
- (8)  $\mathcal{N}[\text{ex}\alpha] \wedge \forall x(x \varepsilon \text{ex}\alpha \leftrightarrow \exists y(\langle x, y \rangle \varepsilon \alpha))$ .

**Definition 1.1** Elementary formula

A formula is elementary iff it's constructed from  $s = t$  and  $t \varepsilon \alpha$  by means of  $\wedge, \vee, \rightarrow, \forall x, \exists x$  only. (No occurrences of  $t \varepsilon s$  with  $s$  not a name variable and name quantifiers are allowed.)

The following lemma is an intuitionistic analogue of reducing Elementary Comprehension as stated in [Fef75] to name generators  $\text{nat}$ ,  $\text{id}$ ,  $\text{co}$ ,  $\text{int}$ ,  $\text{dom}$  and  $\text{inv}$ , which holds in classical setting (see [FJ96]); its proof requires only axioms I, III and IV. For alternative intuitionistic reductions of Elementary Comprehension to a finite number of its instances see [GR94, Sect.1] and [Tat98, Sect.3].

**Lemma 1.4** ECA

If a formula  $F := F[x; \bar{a}; \bar{\alpha}]$  is elementary then there exists a term  $\mathbf{t}_F^x$  such that  $\text{FV}(\mathbf{t}_F^x) = \text{FV}(F) \setminus \{x\}$  and

$$\mathcal{N}[\mathbf{t}_F^x] \wedge \forall x(x \varepsilon \mathbf{t}_F^x \leftrightarrow F).$$

**Proof.** The term  $\mathbf{t}_F^x$  is built by recursion on  $F$ :

$$\mathbf{t}_F^x := \begin{cases} \text{inv}(\lambda x. \langle s[x], t[x] \rangle, \text{id}) & \text{if } F \text{ is } s[x] = t[x]; \\ \text{inv}(\lambda x. s[x], \alpha) & \text{if } F \text{ is } s[x] \varepsilon \alpha; \\ \text{inv}(\lambda x. \langle s[x], s[x] \rangle, \text{id}) & \text{if } F \text{ is } s[x] \downarrow; \\ \text{and}(\mathbf{t}_{F_0[x]}^x, \mathbf{t}_{F_1[x]}^x) & \text{if } F \text{ is } F_0[x] \wedge F_1[x]; \\ \text{or}(\mathbf{t}_{F_0[x]}^x, \mathbf{t}_{F_1[x]}^x) & \text{if } F \text{ is } F_0[x] \vee F_1[x]; \\ \text{imp}(\mathbf{t}_{F_0[x]}^x, \mathbf{t}_{F_1[x]}^x) & \text{if } F \text{ is } F_0[x] \rightarrow F_1[x]; \\ \text{allt}_{G[p_0z, p_1z]}^z & \text{if } F \text{ is } \forall yG[x, y]; \\ \text{ext}_{G[p_0z, p_1z]}^z & \text{if } F \text{ is } \exists yG[x, y]. \end{cases}$$

Now the property of  $\mathbf{t}_F^x$  is proved by induction on  $F$ . □

**V. Join (J).** This axiom states that if  $f$  is an operation from a set named by  $\alpha$ , each value of which is a name, then  $\text{j}(\alpha, f)$  names a disjoint union of all  $fx$  for  $x \varepsilon \alpha$ :

$$\forall x \varepsilon \alpha \mathcal{N}[fx] \rightarrow (\mathcal{N}[\text{j}(\alpha, f)] \wedge \forall z(z \varepsilon \text{j}(\alpha, f) \leftrightarrow \exists x \varepsilon \alpha \exists y(z = \langle x, y \rangle \wedge y \varepsilon fx)).$$

**VI. Inductive Generation (IG).** The first part of this axiom states that  $i(\alpha, \beta)$  names a well-founded part of a set named by  $\alpha$  along an ordering named by  $\beta$ ; the second part allows induction over that set for an arbitrary formula:

$$\mathcal{N}[i(\alpha, \beta)] \wedge \forall x \varepsilon \alpha (\forall y (\langle y, x \rangle \varepsilon \beta \rightarrow y \varepsilon i(\alpha, \beta)) \rightarrow x \varepsilon i(\alpha, \beta)) \\ \wedge (\forall x \varepsilon \alpha (\forall y (\langle y, x \rangle \varepsilon \beta \rightarrow \phi[y]) \rightarrow \phi[x]) \rightarrow \forall x \varepsilon i(\alpha, \beta) \phi[x]),$$

where  $\phi \in \mathcal{L}_{EM}$  is an arbitrary formula.

The theory **App** is the one containing only applicative axioms I; **EON** has axioms I–II. The theory **EONN** has axioms of the groups I–III. **EET** is **EONN** + **ECA**, **EETJ** is **EET** + **J** and **T<sub>0</sub>** is **EETJ** + **IG**<sup>1</sup>.

By **TND** (*tetrium non datur*), both in Explicit Mathematics and analysis, we mean a schema consisting of all instances of the Law of Excluded Middle.

## 2 Subsystems of analysis

The basic theory **EHA** (*Elementary Heyting Analysis*) is formulated in a two-sorted language  $\mathcal{L}_2$ : numbers and sets of numbers. We use **variables**  $a, b, c, \dots$  as ranging over numbers, and  $A, B, C, \dots$  as ranging over sets. There is only one **individual constant** 0. The **function constants** are: successor  $'$ , pairing  $(*, *)$  and projections  $(*)_0, (*)_1$ , and also countably many function constants  $f_1, f_2, \dots$  for primitive recursive functions. **Terms** are built as usual. **Atomic formulas** are of the kinds  $s = t$  and  $s \in A$  ( $s$  and  $t$  are terms). **Formulas** are built from atomic formulas by  $\wedge, \vee, \rightarrow$  and two types of quantifiers, over numbers and over sets, e.g.  $\forall x, \exists x, \forall X, \exists X$ . By  $FV(e)$  we denote the set of free variables occurring in an expression  $e$ , and by  $FV_0(e)$  and  $FV_1(e)$  respectively the set of 1-st and 2-nd order free variables of  $e$ . A formula is called *negative* iff it doesn't contain  $\vee$  or  $\exists$ . A formula is *first-order* iff it doesn't contain second-order variables. A formula is *arithmetical* iff it doesn't contain second-order quantifiers. We use the same **abbreviations** and **syntactical conventions** as in the previous Section. The **logic** is intuitionistic 2-sorted logic. Axioms are the following: **equality axioms**, **Peano axioms**, **prim.-rec. definitions for function symbols**  $(*, *), (* )_0, (* )_1, f_1, f_2, \dots$  and **mathematical induction schema**. **Note** that we have no comprehension in **EHA**, thus **EHA** being a conservative extension of Heyting arithmetic.

**EHA** is the basic theory of analysis in this paper. Additionally, we will consider extensions of **EHA** by the following axioms.

**Arithmetic comprehension (ACA):**

$$\exists X \forall x (x \in X \leftrightarrow \psi[x])$$

for  $\psi$  arithmetical.

**Axiom of Choice (AC):**

$$\forall x \exists Y \phi[x, Y] \rightarrow \exists Z \forall x \phi[x, Z_x]$$

for all formulas  $\phi$ .

**Replacement (RP):**

$$\forall X (\forall x \in X \exists ! Y \phi[x, Y] \rightarrow \exists Z \forall x \in X \phi[x, Z_x])$$

for all formulas  $\phi$ , where  $\phi[x, Z_x]$  arises from  $\phi[x, Z]$  by replacing each occurrence of  $s \in Z$  by  $(x, s) \in Z$ .

**Inductive Generation (IGA):**

$$\forall X \forall Y \exists Z (WP_Y[X, Z] \wedge TI_Y[X, Z, \phi])$$

for all formulas  $\phi$ , where we adopt the following abbreviations:

<sup>1</sup>In the literature the names **EET** and **EETJ** are also used for theories as defined here, but with restricted induction II.

$WP_Y[X, Z]$	denotes	$\text{Prog}_Y[X, Z] \wedge \forall U (\text{Prog}_Y[X, U] \rightarrow Z \subseteq U)$
$\text{Prog}_Y[X, Z]$	"–	$\forall x \in X (\forall y (y <_Y x \rightarrow y \in Z) \rightarrow x \in Z)$
$\text{Prog}_Y[X, \phi]$	"–	$\forall x \in X (\forall y (y <_Y x \rightarrow \phi[y]) \rightarrow \phi[x])$
$\text{TI}_Y[X, Z, \phi]$	"–	$\text{Prog}_Y[X, \phi] \rightarrow \forall z \in Z \phi[z]$
$y <_Y x$	"–	$(y, x) \in Y.$

**IARI** of [GR94] is the theory **EHA** + **ACA** + **RP** + **IGA**. It's shown there that **IARI** is directly interpretable in Martin-Löf type theory  $\mathbf{ML}_{1w}$  and has the same proof-theoretic strength as  $\mathbf{ML}_{1w}$  and  $\mathbf{T}_0$ .

### 3 Interpretations into Explicit Mathematics

In this section we define two interpretations of analysis into Explicit Mathematics, a *direct* interpretation  $\mathcal{D}$  and a *realizability* interpretation  $\mathcal{R}$ , and study relationships between them.

First, for each individual and function constant  $f \in \mathcal{L}_2$  by Lemma 1.3 we can define an operation  $\mathbf{N}(f)$  presenting the same primitive-recursive function as  $f$  and having the following property: if  $n$  is the arity of  $f$  then **EET** proves

$$\bigwedge_{i=1}^n x_i \varepsilon \text{nat} \rightarrow \mathbf{N}(f)x_1 \dots x_n \varepsilon \text{nat}.$$

We may assume that  $\mathbf{N}(0)$  is 0 and  $\mathbf{N}(s_N)$  is  $'$ . Now terms of  $\mathcal{L}_2$  are translated as follows:

**Definition 3.1**  $\mathbf{N}(t)$

$$\begin{cases} \mathbf{N}(x) := x; \\ \mathbf{N}(ft_1 \dots t_n) := \mathbf{N}(f)\mathbf{N}(t_1) \dots \mathbf{N}(t_n). \end{cases}$$

For each second-order variable  $A \in \mathcal{L}_2$  we assume a name variable  $\alpha_A \in \mathcal{L}_{EM}$ . A direct interpretation  $\mathcal{D} : \mathcal{L}_2 \mapsto \mathcal{L}_{EM}$  was introduced in [Fef79] and used later on (see, for example, [BFPS, Ch.II] and [G193]). It's defined as follows:

**Definition 3.2**  $\mathcal{D}$ -interpretation

$$\begin{cases} \mathcal{D}(s = t) := \mathbf{N}(s) = \mathbf{N}(t); \\ \mathcal{D}(t \in A) := \mathbf{N}(t) \varepsilon \alpha_A; \\ \mathcal{D}(F_0 \circ F_1) := \mathcal{D}(F_0) \circ \mathcal{D}(F_1), \text{ for } \circ \in \{\wedge, \vee, \rightarrow\}; \\ \mathcal{D}(QxG) := Qx \varepsilon \text{nat} \mathcal{D}(G), \text{ for } Q \in \{\forall, \exists\}; \\ \mathcal{D}(QXG) := Q\alpha_X \dot{\varepsilon} \text{nat} \mathcal{D}(G), \text{ for } Q \in \{\forall, \exists\}. \end{cases}$$

The following lemma is straightforward (see [BFPS, Ch.II, §1]):

**Lemma 3.1** For each theorem  $F$  of the theory **EHA** + **ACA**  $\mathcal{D}(F)$  is provable in **EET**.

Alternatively, we define a *realizability interpretation*  $\mathcal{R} : \mathcal{L}_2 \mapsto \mathcal{L}_{EM}$ .

**Definition 3.3**  $\mathcal{N}^N$

$$\mathcal{N}^N[t] := \mathcal{N}[t] \wedge \forall z \varepsilon t (z = \langle p_0z, p_1z \rangle \wedge p_0z \varepsilon \text{nat}).$$

**Definition 3.4**  $\tau$  realizes  $F$ ,  $\tau \underline{\text{rn}} F$

For each formula  $F \in \mathcal{L}_2$  we define a formula  $\tau \underline{\text{rn}} F \in \mathcal{L}_{EM}$ .  $\tau$  will always be treated as a new free individual variable. The definition is given by the table below:

$F$	$\mathfrak{r} \underline{\text{rn}} F$
$s = t$	$\mathbf{N}(s) = \mathbf{N}(t)$
$t \in A$	$\langle \mathbf{N}(t), \mathfrak{r} \rangle \varepsilon \alpha_A$
$F_0 \wedge F_1$	$\mathfrak{p}_0 \mathfrak{r} \underline{\text{rn}} F_0 \wedge \mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} F_1$
$F_0 \vee F_1$	$\mathfrak{p}_0 \mathfrak{r} \varepsilon \text{nat} \wedge \begin{cases} (\mathfrak{p}_0 \mathfrak{r} = 0 \rightarrow \mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} F_0) \\ (\mathfrak{p}_0 \mathfrak{r} \neq 0 \rightarrow \mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} F_1) \end{cases}$
$F_0 \rightarrow F_1$	$\forall \mathfrak{r} (\mathfrak{r} \underline{\text{rn}} F_0 \rightarrow \mathfrak{r} \mathfrak{r} \downarrow \wedge \mathfrak{r} \mathfrak{r} \underline{\text{rn}} F_1)$
$\forall x G[x]$	$\forall x \varepsilon \text{nat} (\mathfrak{r} x \downarrow \wedge \mathfrak{r} x \underline{\text{rn}} G[x])$
$\exists x G[x]$	$\mathfrak{p}_0 \mathfrak{r} \varepsilon \text{nat} \wedge \mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} G[\mathfrak{p}_0 \mathfrak{r}]$
$\forall X G[X]$	$\forall \alpha_X (\mathcal{N}^{\mathbf{N}}[\alpha_X] \rightarrow \mathfrak{r} \alpha_X \downarrow \wedge \mathfrak{r} \alpha_X \underline{\text{rn}} G[\alpha_X])$
$\exists X G[X]$	$\mathcal{N}^{\mathbf{N}}[\mathfrak{p}_0 \mathfrak{r}] \wedge \mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} G[\mathfrak{p}_0 \mathfrak{r}]$

**Remark.** According to our notation for substitution, p. 3, in the previous definition  $\mathfrak{p}_1 \mathfrak{r} \underline{\text{rn}} G[\mathfrak{p}_0 \mathfrak{r}]$  in the last clause, for example, stands for  $(\mathfrak{r} \underline{\text{rn}} G[X])_{\mathfrak{r}, \alpha_X}^{\mathfrak{p}_1 \mathfrak{r}, \mathfrak{p}_0 \mathfrak{r}}$ .

**Definition 3.5**  $\mathcal{R}$ -interpretation

For each  $F \in \mathcal{L}_2$  we define  $\mathcal{R}(F) := \exists \mathfrak{r} (\mathfrak{r} \underline{\text{rn}} F)$ .

**Remark.** An important difference of  $\mathcal{R}$ -interpretation from  $\mathcal{D}$ -interpretation is that sets are translated not as (names of) sets of natural numbers, but as (names of) sets of pairs, only first elements of which are natural numbers (see the clause for  $t \in A$ ). This is a standard effect in realizability interpretations of analysis, see e.g. [Tr98, Sect.7.2]. The second element  $\mathfrak{r}$  of a pair  $\langle \mathbf{N}(t), \mathfrak{r} \rangle$  can be thought of as a “proof” that  $t \in A$ .

**Syntactical convention.** We will often use the fractur font  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$  to stress that a given term plays a role as *realization*. Formally, this is *not* a new type of objects; it’s just a substitution for  $a, b, c, \dots$  used for better readability.

**Abbreviation.**  $\mathfrak{t} \downarrow \underline{\text{rn}} F$  will be used for  $\mathfrak{t} \downarrow \wedge \mathfrak{t} \underline{\text{rn}} F$ .

**Definition 3.6** Realization, realizable

1. A term  $\mathfrak{t} \in \mathcal{L}_{\text{EM}}$  is called realization for a formula  $F \in \mathcal{L}_2$  in a theory  $\mathcal{T} \in \mathcal{L}_{\text{EM}}$ , **App**  $\subset \mathcal{T}$ , iff

$$\text{FV}(\mathfrak{t}) \subset \text{FV}_0(F) \cup \{\alpha_A \mid A \in \text{FV}_1(F)\}$$

and

$$\mathcal{T} \vdash \bigwedge_{a \in \text{FV}_0(F)} a \varepsilon \text{nat} \wedge \bigwedge_{A \in \text{FV}_1(F)} \mathcal{N}^{\mathbf{N}}[\alpha_A] \rightarrow \mathfrak{t} \underline{\text{rn}} F.$$

2. If there exists such a term  $\mathfrak{t}$  then  $F$  is called realizable in  $\mathcal{T}$ . We call a theory  $\mathbb{T}_A$  realizable in  $\mathcal{T}$  iff every theorem of  $\mathbb{T}_A$  is realizable in  $\mathcal{T}$ .

**Note.** If  $F$  is closed and realizable in  $\mathcal{T}$  then  $\mathcal{T} \vdash \mathcal{R}(F)$ .

**Theorem 1** Each theorem of **EHA** is realizable in **EONN**.

The **proof** is standard and can be found, for example, in [TD88, Ch.IV, Sect.4].

**Note.** According to Theorem 1, to prove realizability of a theory  $T_A \in \mathcal{L}_2$ ,  $\mathbf{EHA} \subset T_A$ , it's sufficient to construct realizing terms for additional axioms of  $T_A$ . This is what we do in Section 4.

Now we turn to the relationship between  $\mathcal{D}$  and  $\mathcal{R}$ -interpretations. For first-order negative formulas we can define *canonical realizers* as in [Tr98, Lemma 1.10].

**Definition 3.7** Canonical realization,  $\text{can}_F$

For  $F \in \mathcal{L}_2$  first-order negative we define a term  $\text{can}_F \in \mathcal{L}_{\text{EM}}$  (canonical realization of  $F$ ) in the following way:

$$\text{can}_F := \begin{cases} 0 & \text{if } F \text{ is } s = t; \\ \langle \text{can}_{F_0}, \text{can}_{F_1} \rangle & \text{if } F \text{ is } F_0 \wedge F_1; \\ \lambda x. \text{can}_{F_1} & \text{if } F \text{ is } F_0 \rightarrow F_1; \\ \lambda x. \text{can}_{G[x]} & \text{if } F \text{ is } \forall x G[x]. \end{cases}$$

**Note** that for every  $F$   $\text{can}_F$  is closed and  $\mathbf{App} \vdash \text{can}_F \downarrow$ .

**Theorem 2**  $\mathcal{D}(F) \leftrightarrow \mathcal{R}(F)$

For  $F \in \mathcal{L}_2$  being first-order negative in  $\mathbf{App}$  we have:

- (i)  $\exists x (\tau \underline{\text{rn}} F) \rightarrow \mathcal{D}(F)$ ;
- (ii)  $\mathcal{D}(F) \rightarrow \text{can}_F \underline{\text{rn}} F$ ;
- (iii)  $\mathcal{D}(F) \leftrightarrow \mathcal{R}(F)$ .

**Proof.** We prove (i) and (ii) by simultaneous induction on  $F$ .

If  $F$  is atomic then it is of the form  $s = t$  and both  $\mathcal{D}(F)$  and  $\tau \underline{\text{rn}} F$  are of the form  $\mathbf{N}(s) = \mathbf{N}(t)$ .

Assume  $F$  is  $F_0 \wedge F_1$  and the claim holds for  $F_0$  and  $F_1$ .

Assume  $\tau \underline{\text{rn}} (F_0 \wedge F_1)$ . Then we have  $\text{p}_0 \tau \underline{\text{rn}} F_0 \wedge \text{p}_1 \tau \underline{\text{rn}} F_1$ . By IH(i) we get  $\mathcal{D}(F_0) \wedge \mathcal{D}(F_1)$ , i.e.  $\mathcal{D}(F)$ .

Assume  $\mathcal{D}(F_0 \wedge F_1)$ . Then we have  $\mathcal{D}(F_0) \wedge \mathcal{D}(F_1)$ . By IH(ii) we get  $\text{can}_{F_0} \underline{\text{rn}} F_0 \wedge \text{can}_{F_1} \underline{\text{rn}} F_1$ , which by Definition 3.7 gives  $\text{can}_F \underline{\text{rn}} F$ .

Assume  $F$  is  $F_0 \rightarrow F_1$  and the claim holds for  $F_0$  and  $F_1$ .

Assume  $\tau \underline{\text{rn}} (F_0 \rightarrow F_1)$  and assume  $\mathcal{D}(F_0)$ . By IH(ii) we have  $\text{can}_{F_0} \underline{\text{rn}} F_0$ . Then  $\tau \text{can}_{F_0} \downarrow \underline{\text{rn}} F_1$  and by IH(i)  $\mathcal{D}(F_1)$ .

Assume  $\mathcal{D}(F_0 \rightarrow F_1)$  and assume  $\tau \underline{\text{rn}} F_0$ . By IH(i)  $\mathcal{D}(F_0)$  and therefore  $\mathcal{D}(F_1)$ . By IH(ii)  $\text{can}_{F_1} \underline{\text{rn}} F_1$ , which by Definition 3.7 gives  $\text{can}_F \underline{\text{rn}} F$ .

Assume  $F$  is  $\forall x G[x]$  and the claim holds for  $G[x]$ .

Assume  $\underline{\text{rn}}(\forall x G[x])$ . Then  $\forall x \varepsilon \text{nat}(\tau x \downarrow \underline{\text{rn}} G[x])$ . By IH(i) this yields  $\forall x \varepsilon \text{nat} \mathcal{D}(G[x])$ , i.e.  $\mathcal{D}(F)$ .

Assume  $\mathcal{D}(\forall x G[x])$ . Then  $\forall x \varepsilon \text{nat} \mathcal{D}(G[x])$ . By IH(ii) this yields  $\forall x \varepsilon \text{nat}(\text{can}_{G[x]} \underline{\text{rn}} G[x])$ , which by Definition 3.7 gives  $\text{can}_F \underline{\text{rn}} F$ .

(iii) is an immediate consequence of (i) and (ii). □

**Definition 3.8** CC-class, cf. [Tr98, Sect.1.14]

A first-order formula  $F \in \mathcal{L}_2$  belongs to the CC-class iff for every subformula  $G \rightarrow H$  of it  $G$  is negative.

**Theorem 3**  $\mathcal{R}(F) \rightarrow \mathcal{D}(F)$

If  $F \in \text{CC}$  then  $\mathbf{App} \vdash \mathcal{R}(F) \rightarrow \mathcal{D}(F)$ .

**Proof.** The claim is proved by induction on  $F$ .

If  $F$  is atomic then it is of the form  $s = t$  and both  $\mathcal{D}(F)$  and  $\tau \underline{\text{rn}} F$  are of the form  $\mathbf{N}(s) = \mathbf{N}(t)$ .

Assume  $F$  is  $F_0 \wedge F_1$  and  $\mathbf{r} \underline{\mathbf{r}} (F_0 \wedge F_1)$ . Then we have  $\mathbf{p}_0 \mathbf{r} \underline{\mathbf{r}} F_0 \wedge \mathbf{p}_1 \mathbf{r} \underline{\mathbf{r}} F_1$ . By IH we get  $\mathcal{D}(F_0) \wedge \mathcal{D}(F_1)$ , i.e.  $\mathcal{D}(F)$ .

Assume  $F$  is  $F_0 \vee F_1$  and  $\mathbf{r} \underline{\mathbf{r}} (F_0 \vee F_1)$ . Then we have  $\mathbf{p}_0 \mathbf{r} \varepsilon \mathbf{nat} \wedge (\mathbf{p}_0 \mathbf{r} = 0 \rightarrow \mathbf{p}_1 \mathbf{r} \underline{\mathbf{r}} F_0) \wedge (\mathbf{p}_0 \mathbf{r} \neq 0 \rightarrow \mathbf{p}_1 \mathbf{r} \underline{\mathbf{r}} F_1)$ . In the case  $\mathbf{p}_0 \mathbf{r} = 0$  by IH  $\mathcal{D}(F_0)$ ; if  $\mathbf{p}_0 \mathbf{r} \neq 0$  then similarly  $\mathcal{D}(F_1)$ . In both cases  $\mathcal{D}(F)$ .

Assume  $F$  is  $F_0 \rightarrow F_1$  and  $\mathbf{r} \underline{\mathbf{r}} (F_0 \rightarrow F_1)$ . Then  $F_0$  is first-order negative. Assume  $\mathcal{D}(F_0)$ . By the previous Theorem(ii)  $\mathbf{can}_{F_0} \underline{\mathbf{r}} F_0$ . Then  $\mathbf{tcan}_{F_0} \downarrow \underline{\mathbf{r}} F_1$ . Now by IH  $\mathcal{D}(F_1)$ .

Assume  $F$  is  $\forall x G[x]$  and  $\mathbf{r} \underline{\mathbf{r}} (\forall x G[x])$ . Then  $\forall x \varepsilon \mathbf{nat} (\mathbf{t} x \downarrow \underline{\mathbf{r}} G[x])$ . By IH this yields  $\forall x \varepsilon \mathbf{nat} \mathcal{D}(G[x])$ , i.e.  $\mathcal{D}(F)$ .

Assume  $F$  is  $\exists x G[x]$  and  $\mathbf{r} \underline{\mathbf{r}} (\exists x G[x])$ . Then we have  $\mathbf{p}_0 \mathbf{r} \varepsilon \mathbf{nat} \wedge \mathbf{p}_1 \mathbf{r} \underline{\mathbf{r}} G[\mathbf{p}_0 \mathbf{r}]$ . By IH  $\mathcal{D}(G[\mathbf{p}_0 \mathbf{r}])$ , which implies  $\mathcal{D}(F)$ . □

**Remarks about proof-theoretic strength.** We assume here that  $\mathbf{T}_A$  is realizable in  $\mathcal{T}$ .

1) Note that  $\text{Consis}(\mathbf{T})$  is  $\Pi_1^0$ -formula for any theory  $\mathbf{T} \in \mathcal{L}_2$  with a decidable predicate  $\text{Prf}_{\mathbf{T}}(a, b)$ . Therefore, if  $\mathbf{T}_A \vdash \text{Consis}(\mathbf{T})$  then  $\mathcal{T} \vdash \mathcal{D}(\text{Consis}(\mathbf{T}))$ .

2) Note that prenex formulas, in particular  $\Pi_2^0$ -formulas, are CC. Therefore, if  $\mathbf{T}_A$  proves totality of a function  $f$ , then so does  $\mathcal{T}$ .

3) Let  $f_{\prec}$  be a characteristic function of a standard prim.-rec. ordering  $\prec$  on natural numbers (e.g. an initial part of some fixed standard ordering), i.e.  $y \prec x \Leftrightarrow f_{\prec}(y, x) = 0$ . For every first-order negative formula  $F[x] \in \mathcal{L}_2$  we set:  
 $\text{Prog}(\prec, F) \Leftrightarrow \forall x (\forall y (f_{\prec}(y, x) = 0 \rightarrow F[y]) \rightarrow F[x])$ ,  
 $\text{Fund}(\prec, F) \Leftrightarrow \text{Prog}(\prec, F) \rightarrow \forall x F[x]$ .

We say that a theory  $\mathbf{T}_A$  proves well-foundedness of  $\prec$  iff  $\mathbf{T}_A \vdash \text{Fund}(\prec, F)$  for every first-order negative formula  $F$ . Since  $\text{Fund}(\prec, F)$  also is a negative formula, by Theorem 2 we have that if  $\mathbf{T}_A$  proves well-foundedness of  $\prec$ , then so does  $\mathcal{T}$  as well.

## 4 Realizing subsystems of analysis

In this section we provide realizing terms for additional axioms of analysis listed in the Section 2.

**Theorem 4** Arithmetic comprehension  
*Every instance of ACA is realizable in EET.*

**Proof.** Assume a formula  $\psi[a] \in \mathcal{L}_2$  to be arithmetical. By ECA there exists a term  $t$  such that

$$\mathcal{N}^{\mathbf{N}}[t] \wedge \forall x \varepsilon \mathbf{nat} \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon t \leftrightarrow \mathbf{r} \underline{\mathbf{r}} \psi[x]). \quad (4.1)$$

We are to prove now that the pair  $\langle t, \lambda x. \langle \lambda \mathbf{r}. \mathbf{r}, \lambda \mathbf{r}. \mathbf{r} \rangle \rangle$  is a realization of an instance of ACA

$$\exists X \forall x (x \in X \leftrightarrow \psi[x]).$$

Indeed,

$$\begin{aligned} \langle t, \lambda x. \langle \lambda \mathbf{r}. \mathbf{r}, \lambda \mathbf{r}. \mathbf{r} \rangle \rangle \underline{\mathbf{r}} \exists X \forall x (x \in X \leftrightarrow \psi[x]) &\equiv \\ \lambda x. \langle \lambda \mathbf{r}. \mathbf{r}, \lambda \mathbf{r}. \mathbf{r} \rangle \underline{\mathbf{r}} \forall x (x \in t \leftrightarrow \psi[x]) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\langle \lambda \mathbf{r}. \mathbf{r}, \lambda \mathbf{r}. \mathbf{r} \rangle \underline{\mathbf{r}} x \in t \leftrightarrow \psi[x]) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\lambda \mathbf{r}. \mathbf{r} \underline{\mathbf{r}} (x \in t \rightarrow \psi[x]) \wedge \lambda \mathbf{r}. \mathbf{r} \underline{\mathbf{r}} (\psi[x] \rightarrow x \in t)), & \end{aligned} \quad (4.2)$$

which follows from 4.1. □

**Theorem 5** Axiom of Choice  
*Every instance of **AC** is realizable in **EETJ**.*

**Proof.** Assume  $\tau \underline{\text{rn}} \forall x \exists Y \phi[x, Y]$ . We then have

$$\tau \underline{\text{rn}} \forall x \exists Y \phi[x, Y] \equiv \forall x \varepsilon \text{nat} (\tau x \downarrow \wedge \mathcal{N}^{\text{N}}[\mathbf{p}_0(\tau x)] \wedge \mathbf{p}_1(\tau x) \underline{\text{rn}} \phi[x, \mathbf{p}_0(\tau x)]). \quad (5.1)$$

By **ECA** and **J** (over **nat**) let  $t := t[\tau]$  be such that

$$\mathcal{N}^{\text{N}}[t] \wedge (\langle (x, x_1), \mathbf{r}_1 \rangle \varepsilon t \leftrightarrow \langle x_1, \mathbf{r}_1 \rangle \varepsilon \mathbf{p}_0(\tau x)). \quad (5.2)$$

**Claim.** *For every formula  $\psi[x, Y]$ ,*

$$\forall x \varepsilon \text{nat} \forall u (\mathbf{u} \underline{\text{rn}} \psi[x, \mathbf{p}_0(\tau x)] \leftrightarrow \mathbf{u} \underline{\text{rn}} \psi[x, t_x]). \quad (5.3)$$

*Proof:* First we prove the most important case when  $\psi[x, Y]$  is of the form  $s \in Y$ . We have

$$\begin{aligned} \mathbf{u} \underline{\text{rn}} s \in t_x &\equiv \mathbf{u} \underline{\text{rn}} (x, s) \in t \equiv \langle (x, s), \mathbf{u} \rangle \varepsilon t \stackrel{5.2}{\equiv} \\ &\langle s, \mathbf{u} \rangle \varepsilon \mathbf{p}_0(\tau x) \equiv \mathbf{u} \underline{\text{rn}} s \in \mathbf{p}_0(\tau x). \end{aligned} \quad (5.4)$$

Now, the proof is completed by straightforward induction on  $\psi$ . ◇

Using this claim, we have, for  $x \varepsilon \text{nat}$ ,  $\mathbf{p}_1(\tau x) \underline{\text{rn}} \phi[x, t_x]$ . Therefore  $\langle t[\tau], \lambda x. \mathbf{p}_1(\tau x) \rangle$  is a realization of the conclusion and  $\lambda \tau. \langle t[\tau], \lambda x. \mathbf{p}_1(\tau x) \rangle$  is a realization of the instance of **AC**

$$\forall x \exists Y \phi[x, Y] \rightarrow \exists Z \forall x \phi[x, Z].$$

□

**Corollary.** **EHA + ACA + AC** is realizable in **EETJ** and has proof-theoretic strength bounded by  $\varphi(\varepsilon_0, 0)^2$ .

**Proof.** Realizability follows from Theorems 1, 4 and 5. The bound for proof-theoretic strength follows from the Remarks in the end of Section 3 and the fact  $|\mathbf{EETJ}| = |\Sigma_1^1 - \mathbf{AC} \text{ classical}| = \varphi(\varepsilon_0, 0)$  (see, e.g., [Fef79, Section V]). □

The following theorem gives an example where  $\mathcal{D}$ - and  $\mathcal{R}$ -interpretations are essentially different. While, according to the Theorem 5, **T<sub>0</sub>** proves  $\mathcal{R}(\mathbf{AC})$ , it fails to prove  $\mathcal{D}(\mathbf{AC})$ ; the latter in the presence of the Law of Excluded Middle is at least as strong as full second order arithmetic.

**Theorem 6** (**T<sub>0</sub>**  $\not\vdash \mathcal{D}(\mathbf{AC})$ )

1) **T<sub>0</sub>**  $\not\vdash \mathcal{D}(\mathbf{AC})$ ;

2) **EET + TND + D(AC)** has the strength of at least full analysis.

**Proof.** Obviously

$$\mathbf{EHA} + \mathbf{ACA} + \mathbf{TND} \text{ is } \mathcal{D}\text{-interpretable in } \mathbf{EET} + \mathbf{TND} \quad (6.1)$$

(see, e.g. [Gl93, Section 2]). Then we have

$$\begin{aligned} \mathbf{EHA} + \mathbf{ACA} + \mathbf{TND} + \mathbf{AC} \text{ is } \mathcal{D}\text{-interpretable} \\ \text{in } \mathbf{EET} + \mathbf{TND} + \mathcal{D}(\mathbf{AC}). \end{aligned} \quad (6.2)$$

But **ACA + TND + AC** implies full comprehension, so **EHA + ACA + TND + AC** is full analysis. By 6.2 we have 2).

For 1), assume **T<sub>0</sub>**  $\vdash \mathcal{D}(\mathbf{AC})$ . Then **T<sub>0</sub> + TND**  $\vdash \mathcal{D}(\mathbf{AC})$  and **T<sub>0</sub> + TND + D(AC) = T<sub>0</sub> + TND**. By 2) **T<sub>0</sub> + TND** is at least as strong as full analysis, contradiction, since **T<sub>0</sub> + TND** is known to have the strength of  $\Delta_2^1 - \mathbf{CA} + \mathbf{BI}$  (classical) (see [Fef79, Jä83]). □

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<sup>2</sup>In fact, as shown in [Be85, Ch.XIII,§2-3], this bound is exact.

**Theorem 7** Replacement

Every instance of **RP** is realizable in **EETJ**.

**Proof.** Assume  $\mathcal{N}^{\mathbb{N}}[\alpha]$ . Assume also  $\mathbf{r} \underline{\mathbf{r}} \mathbf{m} \forall x \in \alpha \exists ! Y \phi[x, Y]$ . By Definition 3.4 we have

$$\begin{aligned}
\mathbf{r} \underline{\mathbf{r}} \mathbf{m} \forall x \in \alpha \exists ! Y \phi[x, Y] &\equiv \mathbf{r} \underline{\mathbf{r}} \mathbf{m} \forall x (x \in \alpha \rightarrow \exists ! Y \phi[x, Y]) \equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \underline{\mathbf{r}} \mathbf{m} (x \in \alpha \rightarrow \exists ! Y \phi[x, Y])) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\mathbf{r} \underline{\mathbf{r}} \mathbf{m} x \in \alpha \rightarrow \mathbf{r} x \mathbf{r} \downarrow \underline{\mathbf{r}} \mathbf{m} \exists ! Y \phi[x, Y])) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow \mathbf{r} x \mathbf{r} \downarrow \underline{\mathbf{r}} \mathbf{m} \exists ! Y \phi[x, Y])) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow & \\
\mathbf{r} x \mathbf{r} \downarrow \underline{\mathbf{r}} \mathbf{m} (\exists Y (\phi[x, Y] \wedge \forall Z (\phi[x, Z] \rightarrow Y = Z)))) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow (\mathbf{r} x \mathbf{r} \downarrow \wedge \mathcal{N}^{\mathbb{N}}[\mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\mathbf{p}_1(\mathbf{r} x \mathbf{r}) \underline{\mathbf{r}} \mathbf{m} (\phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge \forall Z (\phi[x, Z] \rightarrow \mathbf{p}_0(\mathbf{r} x \mathbf{r}) = Z)))) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow (\mathcal{N}^{\mathbb{N}}[\mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\mathbf{p}_0(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \underline{\mathbf{r}} \mathbf{m} \phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\mathbf{p}_1(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \underline{\mathbf{r}} \mathbf{m} \forall Z (\phi[x, Z] \rightarrow \mathbf{p}_0(\mathbf{r} x \mathbf{r}) = Z)))) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow (\mathcal{N}^{\mathbb{N}}[\mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\mathbf{p}_0(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \underline{\mathbf{r}} \mathbf{m} \phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\forall \gamma (\mathcal{N}^{\mathbb{N}}[\gamma] \rightarrow (\mathbf{p}_1(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \gamma \downarrow \underline{\mathbf{r}} \mathbf{m} (\phi[x, \gamma] \rightarrow \mathbf{p}_0(\mathbf{r} x \mathbf{r}) = \gamma)))) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{r} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow (\mathcal{N}^{\mathbb{N}}[\mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge & \\
\mathbf{p}_0(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \underline{\mathbf{r}} \mathbf{m} \phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})] \wedge \forall \gamma (\mathcal{N}^{\mathbb{N}}[\gamma] \rightarrow & \\
\mathbf{p}_1(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \gamma \downarrow \wedge & \\
\forall \mathbf{q} (\mathbf{q} \underline{\mathbf{r}} \mathbf{m} \phi[x, \gamma] \rightarrow \mathbf{p}_1(\mathbf{p}_1(\mathbf{r} x \mathbf{r})) \gamma \mathbf{q} \downarrow \underline{\mathbf{r}} \mathbf{m} \mathbf{p}_0(\mathbf{r} x \mathbf{r}) = \gamma)))) & .
\end{aligned} \tag{7.1}$$

Continuing 7.1,

$$\begin{aligned}
\mathbf{r}_1 \underline{\mathbf{r}} \mathbf{m} \mathbf{p}_0(\mathbf{r} x \mathbf{r}) = \gamma &\equiv \\
\mathbf{r}_1 \underline{\mathbf{r}} \mathbf{m} \forall x_1 ((x_1 \in \mathbf{p}_0(\mathbf{r} x \mathbf{r}) \rightarrow x_1 \in \gamma) \wedge (x_1 \in \gamma \rightarrow x_1 \in \mathbf{p}_0(\mathbf{r} x \mathbf{r}))) &\equiv \\
\forall x_1 \varepsilon \text{nat} (\mathbf{r}_1 x_1 \downarrow \wedge (\forall \mathbf{r}_0 (\langle x_1, \mathbf{r}_0 \rangle \varepsilon \mathbf{p}_0(\mathbf{r} x \mathbf{r}) \rightarrow \langle x_1, \mathbf{p}_0(\mathbf{r}_1 x_1) \mathbf{r}_0 \rangle \varepsilon \gamma) \wedge & \\
\forall \mathbf{r}_1 (\langle x_1, \mathbf{r}_1 \rangle \varepsilon \gamma \rightarrow \langle x_1, \mathbf{p}_1(\mathbf{r}_1 x_1) \mathbf{r}_1 \rangle \varepsilon \mathbf{p}_0(\mathbf{r} x \mathbf{r})))) & .
\end{aligned} \tag{7.2}$$

Also,

$$\begin{aligned}
\mathbf{u} \underline{\mathbf{r}} \mathbf{m} \forall x \in \alpha \phi[x, C_x] &\equiv \mathbf{u} \underline{\mathbf{r}} \mathbf{m} \forall x (x \in \alpha \rightarrow \phi[x, C_x]) \equiv \\
\forall x \varepsilon \text{nat} (\mathbf{u} x \downarrow \underline{\mathbf{r}} \mathbf{m} (x \in \alpha \rightarrow \phi[x, C_x])) &\equiv \\
\forall x \varepsilon \text{nat} (\mathbf{u} x \downarrow \wedge \forall \mathbf{r} (\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow \mathbf{u} x \mathbf{r} \downarrow \underline{\mathbf{r}} \mathbf{m} \phi[x, C_x])) & .
\end{aligned} \tag{7.3}$$

By **ECA** there exists a term  $t := t[\alpha, \mathbf{r}]$  s.t.

$$\begin{aligned}
\mathcal{N}^{\mathbb{N}}[t[\alpha, \mathbf{r}]] \wedge (\langle (x, x_1), \mathbf{r} \rangle \varepsilon t[\alpha, \mathbf{r}] \leftrightarrow & \\
\mathbf{r} = \langle \mathbf{p}_0 \mathbf{r}, \mathbf{p}_1 \mathbf{r} \rangle \wedge \langle \langle x, \mathbf{p}_0 \mathbf{r} \rangle, \langle x_1, \mathbf{p}_1 \mathbf{r} \rangle \rangle \varepsilon \mathbf{j}(\alpha, \lambda y. \mathbf{p}_0(\mathbf{r}(\mathbf{p}_0 y))(\mathbf{p}_1 y))) & .
\end{aligned} \tag{7.4}$$

If we had  $\langle x, \mathbf{r} \rangle \varepsilon \alpha \rightarrow (\mathbf{r}_2 \underline{\mathbf{r}} \mathbf{m} \phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})] \leftrightarrow \mathbf{r}_2 \underline{\mathbf{r}} \mathbf{m} \phi[x, t_x])$ , this would provide us with a realization of the Axiom of Choice on  $\alpha$  (instead of Replacement) (cf. proof of Theorem 5). While this is not the case, by making use of uniqueness part of 7.1 we obtain a pair of operations, which map realizations of  $\phi[x, \mathbf{p}_0(\mathbf{r} x \mathbf{r})]$  and  $\phi[x, t_x]$  into each other. This pair of operations is represented by a term  $\mathbf{r}_\phi^Y$  defined below, and is sufficient to build up a realization of Replacement.

For each formula  $\varphi := \varphi[x, Y] \in \mathcal{L}_2$  such that  $\mathbf{r} \notin \text{FV}_0(\varphi)$  and a predicate variable  $Y \in \mathcal{L}_2$  we define a term  $\mathbf{r}_\varphi^Y := \mathbf{r}_\varphi^Y[\mathbf{r}, x, \mathbf{r}, \text{FV}_0(\varphi)]$  by recursion on  $\varphi$  in the following way:

$$\mathfrak{r}_\varphi^Y := \left\{ \begin{array}{ll} \langle \lambda \mathfrak{r}_0 \cdot \mathfrak{r}_0, \lambda \mathfrak{r}_1 \cdot \mathfrak{r}_1 \rangle & \text{if } \varphi \text{ is } s_1 = s_2; \\ \langle \lambda \mathfrak{r}_0 \cdot \mathfrak{r}_0, \lambda \mathfrak{r}_1 \cdot \mathfrak{r}_1 \rangle & \text{if } \varphi \text{ is } s \in D \text{ and } D \text{ is not } Y; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{r}, \mathfrak{r}_0 \rangle, \lambda \mathfrak{r}_1 \cdot \mathfrak{p}_1(\mathfrak{r}_1[\mathfrak{r}, x, \mathfrak{r}, \mathfrak{r}_1]s[x])(\mathfrak{p}_1 \mathfrak{r}_1) \rangle & \left\{ \begin{array}{l} \text{if } \varphi \text{ is } s[x] \in Y, \text{ where} \\ \mathfrak{r}_1 := \mathfrak{p}_1(\mathfrak{p}_1(\mathfrak{r}x\mathfrak{r}))\gamma\mathfrak{q}, \\ \gamma := \mathfrak{p}_0(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1)), \\ \mathfrak{q} := \mathfrak{p}_0(\mathfrak{p}_1(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1))); \end{array} \right. \\ \left. \begin{array}{l} \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_{\varphi_0}^Y(\mathfrak{p}_0 \mathfrak{r}_0), \mathfrak{p}_0 \mathfrak{r}_{\varphi_1}^Y(\mathfrak{p}_1 \mathfrak{r}_0) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_1 \mathfrak{r}_{\varphi_0}^Y(\mathfrak{p}_0 \mathfrak{r}_1), \mathfrak{p}_1 \mathfrak{r}_{\varphi_1}^Y(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \rangle \} \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{d}_N(\mathfrak{p}_0 \mathfrak{r}_0, 0, \mathfrak{r}_{\varphi_0}^Y, \mathfrak{r}_{\varphi_1}^Y)(\mathfrak{p}_1 \mathfrak{r}_0) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{d}_N(\mathfrak{p}_0 \mathfrak{r}_1, 0, \mathfrak{r}_{\varphi_0}^Y, \mathfrak{r}_{\varphi_1}^Y)(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \rangle \} & \text{if } \varphi \text{ is } \varphi_0 \wedge \varphi_1; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{d}_N(\mathfrak{p}_0 \mathfrak{r}_0, 0, \mathfrak{r}_{\varphi_0}^Y, \mathfrak{r}_{\varphi_1}^Y)(\mathfrak{p}_1 \mathfrak{r}_0) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{d}_N(\mathfrak{p}_0 \mathfrak{r}_1, 0, \mathfrak{r}_{\varphi_0}^Y, \mathfrak{r}_{\varphi_1}^Y)(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \rangle \} & \text{if } \varphi \text{ is } \varphi_0 \vee \varphi_1; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{r}_{\varphi_1}^Y(\mathfrak{r}_0(\mathfrak{p}_1 \mathfrak{r}_{\varphi_0}^Y z)) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_1 \mathfrak{r}_{\varphi_1}^Y(\mathfrak{r}_1(\mathfrak{p}_0 \mathfrak{r}_{\varphi_0}^Y z)) \rangle \} & \text{if } \varphi \text{ is } \varphi_0 \rightarrow \varphi_1; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{r}_{\psi[k]}^Y(\mathfrak{r}_0 k) \rangle, \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_1 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{r}_{\psi[k]}^Y(\mathfrak{r}_1 k) \rangle \rangle & \text{if } \varphi \text{ is } \forall k \psi[k]; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{r}_{\psi[\mathfrak{p}_0 \mathfrak{r}_0]}^Y(\mathfrak{p}_1 \mathfrak{r}_0) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{r}_{\psi[\mathfrak{p}_0 \mathfrak{r}_1]}^Y(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \rangle \} & \text{if } \varphi \text{ is } \exists k \psi[k]; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{r}_{\psi[K]}^Y(\mathfrak{r}_0 \kappa) \rangle, \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_1 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{r}_{\psi[K]}^Y(\mathfrak{r}_1 \kappa) \rangle \rangle & \text{if } \varphi \text{ is } \forall K \psi[K]; \\ \langle \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_0, \mathfrak{p}_0 \mathfrak{r}_{\psi[\mathfrak{p}_0 \mathfrak{r}_0]}^Y(\mathfrak{p}_1 \mathfrak{r}_0) \rangle, \\ \lambda \mathfrak{r}_1 \cdot \langle \mathfrak{p}_0 \mathfrak{r}_1, \mathfrak{p}_1 \mathfrak{r}_{\psi[\mathfrak{p}_0 \mathfrak{r}_1]}^Y(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \rangle \} & \text{if } \varphi \text{ is } \exists K \psi[K]. \end{array} \right.$$

**Claim.** For every formula  $\varphi := \varphi[x, Y] \in \mathcal{L}_2$  we have

$$\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow \mathfrak{r}_\varphi^Y[\mathfrak{r}, x, \mathfrak{r}] \underline{\text{rn}} (\varphi[x, \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r})] \leftrightarrow \varphi[x, t_x]). \quad (7.5)$$

*Proof:* We check only the most important case (the only one which requires use of uniqueness in the premise of Replacement axiom), when  $\varphi$  is of the form  $s[x] \in Y$ . Other cases are proved by routine induction on  $\varphi$ .

By Definition 3.4 we have

$$\mathfrak{r}_0 \underline{\text{rn}} s[x] \in \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r}) \equiv \langle s[x], \mathfrak{r}_0 \rangle \varepsilon \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r}), \quad (7.6)$$

$$\begin{aligned} \mathfrak{r}_1 \underline{\text{rn}} s[x] \in t_x &\equiv \mathfrak{r}_1 \underline{\text{rn}} (x, s[x]) \in t \equiv \langle (x, s[x]), \mathfrak{r}_1 \rangle \varepsilon t \stackrel{7.4}{\Leftrightarrow} \\ &\langle \langle x, \mathfrak{p}_0 \mathfrak{r}_1 \rangle, \langle s[x], \mathfrak{p}_1 \mathfrak{r}_1 \rangle \rangle \varepsilon \text{j}(\alpha, \lambda y. \mathfrak{p}_0(\mathfrak{r}(\mathfrak{p}_0 y)(\mathfrak{p}_1 y))). \end{aligned} \quad (7.7)$$

By **J** 7.6 and 7.7 immediately yield

$$\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow \lambda \mathfrak{r}_0 \cdot \langle \mathfrak{r}, \mathfrak{r}_0 \rangle \underline{\text{rn}} (s[x] \in \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r}) \rightarrow s[x] \in t_x). \quad (7.8)$$

For the opposite direction, assume  $\langle x, \mathfrak{r} \rangle \varepsilon \alpha$  and  $\mathfrak{r}_1 \underline{\text{rn}} s[x] \in t_x$ , i.e.  $\langle \langle x, \mathfrak{p}_0 \mathfrak{r}_1 \rangle, \langle s[x], \mathfrak{p}_1 \mathfrak{r}_1 \rangle \rangle \varepsilon \text{j}(\alpha, \lambda y. \mathfrak{p}_0(\mathfrak{r}(\mathfrak{p}_0 y)(\mathfrak{p}_1 y)))$ . Then by **J**

$$\langle x, \mathfrak{p}_0 \mathfrak{r}_1 \rangle \varepsilon \alpha \wedge \langle s[x], \mathfrak{p}_1 \mathfrak{r}_1 \rangle \varepsilon \mathfrak{p}_0(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1)). \quad (7.9)$$

By 7.1 we have

$$\mathcal{N}^N[\mathfrak{p}_0(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1))] \wedge \mathfrak{p}_0(\mathfrak{p}_1(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1))) \underline{\text{rn}} \phi[x, \mathfrak{p}_0(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1))]. \quad (7.10)$$

Now by the uniqueness part of 7.1 we obtain

$$\mathfrak{p}_1(\mathfrak{p}_1(\mathfrak{r}x\mathfrak{r}))\gamma\mathfrak{q} \downarrow \underline{\text{rn}} \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r}) = \gamma, \quad (7.11)$$

where  $\gamma := \mathfrak{p}_0(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1))$  and  $\mathfrak{q} := \mathfrak{p}_0(\mathfrak{p}_1(\mathfrak{r}x(\mathfrak{p}_0 \mathfrak{r}_1)))$ . Taking  $\mathfrak{r}_1 := \mathfrak{p}_1(\mathfrak{p}_1(\mathfrak{r}x\mathfrak{r}))\gamma[\mathfrak{r}, x, \mathfrak{r}_1]\mathfrak{q}[\mathfrak{r}, x, \mathfrak{r}_1]$ , from 7.2 and 7.9 we have

$$\langle s[x], \mathfrak{p}_1(\mathfrak{r}_1 s[x])(\mathfrak{p}_1 \mathfrak{r}_1) \rangle \varepsilon \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r}). \quad (7.12)$$

Equation 7.12 shows that

$$\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow \lambda \mathfrak{r}_1 \cdot \mathfrak{p}_1(\mathfrak{r}_1[\mathfrak{r}, x, \mathfrak{r}, \mathfrak{r}_1]s[x])(\mathfrak{p}_1 \mathfrak{r}_1) \underline{\text{rn}} (s[x] \in t_x \rightarrow s[x] \in \mathfrak{p}_0(\mathfrak{r}x\mathfrak{r})), \quad (7.13)$$

q.e.d. ◇

The previous Claim and equation 7.1 together prove that

$$\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow \left( \mathbf{p}_0 \mathbf{r}_{\phi[x, Y]}^Y[\mathfrak{r}, x, \mathfrak{r}] (\mathbf{p}_0 (\mathbf{p}_1 (\mathbf{r} x \mathfrak{r}))) \downarrow \underline{\mathbf{r}m} \phi[x, (t[\alpha, \mathfrak{r}])_x] \right), \quad (7.14)$$

which is to say that

$$\mathbf{r} \underline{\mathbf{r}m} \forall x \in \alpha \exists ! Y \phi[x, Y] \rightarrow \lambda x \lambda \mathfrak{r}. \mathbf{p}_0 \mathbf{r}_{\phi[x, Y]}^Y[\mathfrak{r}, x, \mathfrak{r}] (\mathbf{p}_0 (\mathbf{p}_1 (\mathbf{r} x \mathfrak{r}))) \underline{\mathbf{r}m} \forall x \in \alpha \phi[x, (t[\alpha, \mathfrak{r}])_x]. \quad (7.15)$$

Last equation shows that an operation  $\lambda x \lambda \mathfrak{r}. (t[\alpha, \mathfrak{r}], \lambda x \lambda \mathfrak{r}. \mathbf{p}_0 \mathbf{r}_{\phi[x, Y]}^Y[\mathfrak{r}, x, \mathfrak{r}] (\mathbf{p}_0 (\mathbf{p}_1 (\mathbf{r} x \mathfrak{r}))))$  is a realization for an instance of **RP**

$$\forall X (\forall x \in X \exists ! Y \phi[x, Y] \rightarrow \exists Z \forall x \in X \phi[x, Z_x]).$$

□

### Theorem 8 Inductive Generation

Every instance of **IGA** is realizable in **EET** + **IG**.

**Proof.** Assume  $\mathcal{N}^N[\alpha] \wedge \mathcal{N}^N[\beta]$ . Assume also  $\mathbf{r} \underline{\mathbf{r}m} \forall x \varepsilon \alpha (\forall y (y <_\beta x \rightarrow \phi[y]) \rightarrow \phi[x])$ . By Definition 3.4 we have:

$$\begin{aligned} \mathbf{r} \underline{\mathbf{r}m} \forall x \varepsilon \alpha (\forall y (y <_\beta x \rightarrow \phi[y]) \rightarrow \phi[x]) &\equiv \\ \mathbf{r} \underline{\mathbf{r}m} \forall x (x \varepsilon \alpha \rightarrow (\forall y (y <_\beta x \rightarrow \phi[y]) \rightarrow \phi[x])) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\mathbf{r} x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow & \\ \mathbf{r} x \mathfrak{r} \downarrow \underline{\mathbf{r}m} (\forall y (y <_\beta x \rightarrow \phi[y]) \rightarrow \phi[x])) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\mathbf{r} x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathbf{r} x \mathfrak{r} \downarrow \wedge & \\ \forall \mathbf{u} (\mathbf{u} \underline{\mathbf{r}m} \forall y (y <_\beta x \rightarrow \phi[y]) \rightarrow \mathbf{r} x \mathfrak{r} \mathbf{u} \downarrow \underline{\mathbf{r}m} \phi[x])) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\mathbf{r} x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathbf{r} x \mathfrak{r} \downarrow \wedge \forall \mathbf{u} (\forall y \varepsilon \mathbf{nat} (\mathbf{u} y \downarrow \wedge & \\ \forall \mathbf{v} (\mathbf{v} \underline{\mathbf{r}m} (y, x) \varepsilon \beta \rightarrow \mathbf{u} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y])) \rightarrow \mathbf{r} x \mathfrak{r} \mathbf{u} \downarrow \underline{\mathbf{r}m} \phi[x])) &\equiv \\ \forall x \varepsilon \mathbf{nat} (\mathbf{r} x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathbf{r} x \mathfrak{r} \downarrow \wedge \forall \mathbf{u} (\forall y \varepsilon \mathbf{nat} (\mathbf{u} y \downarrow \wedge & \\ \forall \mathbf{v} (\langle (y, x), \mathbf{v} \rangle \varepsilon \beta \rightarrow \mathbf{u} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y])) \rightarrow \mathbf{r} x \mathfrak{r} \mathbf{u} \downarrow \underline{\mathbf{r}m} \phi[x])) & \end{aligned} \quad (8.1)$$

By recursion theorem for a function  $f := \lambda z \lambda x \lambda \mathfrak{r}. \mathbf{r} x \mathfrak{r} (\lambda y \lambda \eta. z y \eta)$  there exists a term  $\mathbf{R} := \mathbf{rec} f$  such that

$$\mathbf{R} x \mathfrak{r} \simeq \mathbf{r} x \mathfrak{r} (\lambda y \lambda \mathbf{v}. \mathbf{R} y \mathbf{v}). \quad (8.2)$$

By **ECA** there exists a term  $\beta^* := \beta^*[\beta]$  s.t.

$$\begin{aligned} \mathcal{N}[\beta^*] \wedge (\mathbf{u} \varepsilon \beta^* \leftrightarrow \mathbf{u} = \langle \mathbf{p}_0 \mathbf{u}, \mathbf{p}_1 \mathbf{u} \rangle \wedge & \\ \mathbf{p}_0 \mathbf{u} = \langle \mathbf{p}_0 \mathbf{p}_0 \mathbf{u}, \mathbf{p}_1 \mathbf{p}_0 \mathbf{u} \rangle \wedge \mathbf{p}_1 \mathbf{u} = \langle \mathbf{p}_0 \mathbf{p}_1 \mathbf{u}, \mathbf{p}_1 \mathbf{p}_1 \mathbf{u} \rangle \wedge & \\ \mathbf{p}_0 \mathbf{p}_0 \mathbf{u} \varepsilon \mathbf{nat} \wedge \mathbf{p}_0 \mathbf{p}_1 \mathbf{u} \varepsilon \mathbf{nat} \wedge \langle (\mathbf{p}_0 \mathbf{p}_0 \mathbf{u}, \mathbf{p}_0 \mathbf{p}_1 \mathbf{u}), \mathbf{p}_1 \mathbf{p}_0 \mathbf{u} \rangle \varepsilon \beta) & \end{aligned} \quad (8.3)$$

We want to prove  $\text{Prog}_{\beta^*}(\alpha, \mathbf{R} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y])$ , i.e.

$$\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\forall y \varepsilon \mathbf{nat} \forall \mathbf{v} (\langle (y, \mathbf{v}), \langle x, \mathfrak{r} \rangle \rangle \varepsilon \beta^* \rightarrow \mathbf{R} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y]) \rightarrow \mathbf{R} x \mathfrak{r} \downarrow \underline{\mathbf{r}m} \phi[x]). \quad (8.4)$$

Assume  $\langle x, \mathfrak{r} \rangle \varepsilon \alpha \wedge \forall y \varepsilon \mathbf{nat} \forall \mathbf{v} (\langle (y, \mathbf{v}), \langle x, \mathfrak{r} \rangle \rangle \varepsilon \beta^* \rightarrow \mathbf{R} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y])$ . Then by 8.3 we have

$$\forall y \varepsilon \mathbf{nat} \forall \mathbf{v} (\langle (y, x), \mathbf{v} \rangle \varepsilon \beta \rightarrow \mathbf{R} y \mathbf{v} \downarrow \underline{\mathbf{r}m} \phi[y]). \quad (8.5)$$

Therefore for the operation  $\mathbf{u} := \lambda y \lambda \mathbf{v}. \mathbf{R} y \mathbf{v}$  by 8.1 we have  $\mathbf{r} x \mathfrak{r} \mathbf{u} \downarrow \underline{\mathbf{r}m} \phi[x]$ , i.e.  $\mathbf{r} x \mathfrak{r} (\lambda y \lambda \mathbf{v}. \mathbf{R} y \mathbf{v}) \downarrow \underline{\mathbf{r}m} \phi[x]$ . From this fact and equation 8.2 we obtain 8.4.

By **IG** we obtain

$$\forall x \varepsilon \mathbf{nat} \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon i(\alpha, \beta^*) \rightarrow \mathbf{R} x \mathfrak{r} \downarrow \underline{\mathbf{r}m} \phi[x]), \quad (8.6)$$

or

$$\lambda x \lambda \mathfrak{r}. \mathbf{R}x\mathfrak{r} \underline{\mathbf{r}} \forall x \in i(\alpha, \beta^*) \phi[x]. \quad (8.7)$$

This shows that an operation  $\lambda \mathfrak{r} \lambda x \lambda \mathfrak{r}. \mathbf{R}x\mathfrak{r}$  is a realization of a “part” of **IGA**

$$\forall x \varepsilon \alpha (\forall y (y <_{\beta} x \rightarrow \phi[y]) \rightarrow \phi[x]) \rightarrow \forall z \varepsilon i(\alpha, \beta^*) \phi[z]. \quad (8.8)$$

Applying the above considerations to the formula  $x \in U$  in place of  $\phi[x]$ , we see that  $\lambda \gamma \lambda \mathfrak{r} \lambda x \lambda \mathfrak{r}. \mathbf{R}x\mathfrak{r}$  is a realization of

$$\forall U (\text{Prog}_{\beta}[\alpha, U] \rightarrow i(\alpha, \beta^*) \subseteq U), \quad (8.9)$$

Last, we have to find a realization of

$$\text{Prog}_{\beta}[\alpha, i(\alpha, \beta^*)]. \quad (8.10)$$

By 8.1 with  $x \in i(\alpha, \beta^*)$  in place of  $\phi[x]$ , we have

$$\begin{aligned} \mathfrak{q} \underline{\mathbf{r}} \text{Prog}_{\beta}[\alpha, i(\alpha, \beta^*)] &\equiv \\ \mathfrak{q} \underline{\mathbf{r}} \forall x \varepsilon \alpha (\forall y (y <_{\beta} x \rightarrow y \in i(\alpha, \beta^*)) \rightarrow x \in i(\alpha, \beta^*)) &\equiv \\ \forall x \varepsilon \text{nat} (\mathfrak{q}x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathfrak{q}x\mathfrak{r} \downarrow \wedge \forall u (\forall y \varepsilon \text{nat}(uy \downarrow \wedge \\ \forall \mathfrak{v} (\langle \langle y, x \rangle, \mathfrak{v} \rangle \varepsilon \beta \rightarrow uy\mathfrak{v} \downarrow \underline{\mathbf{r}} y \in i(\alpha, \beta^*))) \rightarrow \\ \mathfrak{q}x\mathfrak{r}u \downarrow \underline{\mathbf{r}} x \in i(\alpha, \beta^*))) &\equiv \\ \forall x \varepsilon \text{nat} (\mathfrak{q}x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathfrak{q}x\mathfrak{r} \downarrow \wedge \forall u (\forall y \varepsilon \text{nat}(uy \downarrow \wedge \\ \forall \mathfrak{v} (\langle \langle y, x \rangle, \mathfrak{v} \rangle \varepsilon \beta \rightarrow \langle y, uy\mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*))) \rightarrow \\ \langle x, \mathfrak{q}x\mathfrak{r}u \rangle \varepsilon i(\alpha, \beta^*))) &\stackrel{8.3}{\Leftrightarrow} \\ \forall x \varepsilon \text{nat} (\mathfrak{q}x \downarrow \wedge \forall \mathfrak{r} (\langle x, \mathfrak{r} \rangle \varepsilon \alpha \rightarrow (\mathfrak{q}x\mathfrak{r} \downarrow \wedge \forall u (\forall y \varepsilon \text{nat}(uy \downarrow \wedge \\ \forall \mathfrak{v} (\langle \langle y, \mathfrak{v} \rangle, \langle x, \mathfrak{r} \rangle \rangle \varepsilon \beta^* \rightarrow \langle y, uy\mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*))) \rightarrow \\ \langle x, \mathfrak{q}x\mathfrak{r}u \rangle \varepsilon i(\alpha, \beta^*))) &). \end{aligned} \quad (8.11)$$

Assume now  $\langle x, \mathfrak{r} \rangle \varepsilon \alpha \wedge \forall y \varepsilon \text{nat}(uy \downarrow \wedge \forall \mathfrak{v} (\langle \langle y, \mathfrak{v} \rangle, \langle x, \mathfrak{r} \rangle \rangle \varepsilon \beta^* \rightarrow \langle y, uy\mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*)))$ . Note that from the definition 8.3 of  $\beta^*$  it follows that

$$\exists \mathfrak{v} (\langle \langle y_0, \mathfrak{v}_0 \rangle, \langle y_1, \mathfrak{v} \rangle \rangle \varepsilon \beta^*) \leftrightarrow \forall \mathfrak{v} (\langle \langle y_0, \mathfrak{v}_0 \rangle, \langle y_1, \mathfrak{v} \rangle \rangle \varepsilon \beta^*) \quad (8.12)$$

and therefore by **IG**

$$\exists \mathfrak{v} (\langle y, \mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*)) \leftrightarrow \forall \mathfrak{v} (\langle y, \mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*)). \quad (8.13)$$

So, we also have  $\forall y \varepsilon \text{nat} \forall \mathfrak{v} (\langle \langle y, \mathfrak{v} \rangle, \langle x, \mathfrak{r} \rangle \rangle \varepsilon \beta^* \rightarrow \langle y, \mathfrak{v} \rangle \varepsilon i(\alpha, \beta^*))$ . By **IG** we obtain  $\langle x, \mathfrak{r} \rangle \varepsilon i(\alpha, \beta^*)$ . This demonstrates that an operation  $\mathfrak{q} := \lambda x \lambda \mathfrak{r} \lambda u. \mathfrak{r}$  is a realization of 8.10.

Bringing realizations of 8.8–8.10 together shows that an operation

$$\lambda \alpha \lambda \beta. \langle i(\alpha, \beta^*[\beta]), \langle \langle \lambda x \lambda \mathfrak{r} \lambda u. \mathfrak{r}, \lambda \gamma \lambda \mathfrak{r} \lambda x \lambda \mathfrak{r}. \mathbf{R}x\mathfrak{r} \rangle, \lambda \mathfrak{r} \lambda x \lambda \mathfrak{r}. \mathbf{R}x\mathfrak{r} \rangle$$

is a realization of an instance of **IGA**

$$\forall X \forall Y \exists Z \left( \text{WP}_Y[X, Z] \wedge (\forall x \in X (\forall y (y <_Y x \rightarrow \phi[y]) \rightarrow \phi[x]) \rightarrow \forall z \in Z \phi[z]) \right).$$

□

**Corollary.** **IARI** is realizable in **T<sub>0</sub>**; its proof-theoretic strength is bounded by that of **T<sub>0</sub>**.

**Proof.** This follows from Theorems 1, 4, 7 and 8. □

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