The Modal μ-Calculus and the Logic of Common Knowledge

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Preface

"A mathematician is a device for turning coffee into theorems." *(P. Erdös)*

This thesis contains two parts; one about the modal μ-calculus and one about the logic of common knowledge. Apart from one section in the second part both can be read independently. The first chapter of both parts is an introduction to the subject and furthermore each section begins with an abstract of its content. After the acknowledgements in German we give a 'How to read this thesis' diagram.

Ich danke...
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How to read this thesis:

		1			7		
	3		2	8.3		8.1/8.2	
5.1		4	6			9	
	5.2				10		11

Contents

Part I

The Modal μ-Calculus

Chapter 1

Introducing the Modal μ-Calculus

"Aller Anfang ist leicht, und die letzten Stufen werden am seltensten erstiegen." (J. W. von Goethe)

Modal μ-calculus is an extension of modal logics, with least and greatest fixpoint constructors. The term 'μ-calculus' and the idea of extending modal logics with fixpoint constructors appeared for the first time in a paper by Scott and De Bakker [44], and was further developed, see for example, Hitchcock and Park [24], Park [39], De Bakker and De Roever [6] and De Roever [42]. Nowadays, the term 'modal μ-calculus' stands for the formal system introduced by Kozen in [31]. It is the system we study in this part of the thesis.

As a modal logic, the semantics of the μ-calculus is usually given by transition systems, or equivalently Kripke-models. There are also other possible interpretations of modal μ-formulae, such as in a topological context or in an algebraic context [10, 3]. In this thesis, we deal with the usual semantics given by transition systems.

Transition systems play an important role in computer science. If we analyse systems or processes operationally, hence we are working in an operational semantics, it can be very convenient to model the 'real world' with transition systems. In order to specify properties, mainly liveness and fairness properties, of these transition systems various kinds of logics have been introduced. Many of them belong to the family of modal logics (or sometimes temporal logics, process logics), such as Floyd-Hoare Logics , HML (Hennessey-Miller Logic), PDL (Propositional Dynamic Logic), CTL (Computation Tree Logic), CTL* and PLTL (Propositional Linear Time Temporal Logic) [41, 25, 21, 22].Many of these logics contain modalities which can be interpretated as a least or greatest fixpoint of an operator based on other, more basic, modalities. Therefore, the necessity of having a logic which reasons about fixpoints in this modal context on a sufficiently abstract level arised. The modal μ-calculus provides a formal framework where we can study the role of fixpoints in transition systems, that is *the role of fixpoints in operational semantics*.

Another example of modal logic where you have modalities which are interpretated as fixpoints is the logic of common knowledge which will be studied in detail in the second part of this thesis.

If we look to transition systems operationally, that is as a representation of the behavior of a process, then we often do not want to distinguish between bisimilar transition systems. This follows from the fact that bisimilarity is often regarded as an observational, and hence operational, equivalence. Thus, a specification language should not distinguish between bisimilar transition systems. First order logic can sometimes make this distinction whereas modal logic can not. In fact, van Benthem proved in [48] that *exactly* the first order properties on transition systems which are invariant under bisimulation are expressible in modal logic. Thus, all first order properties which make sense in operational semantics are modal properties. Unfortunately, many operational properties, such as liveness and safety properties, are not expressible in first order language but only in a monadic second order (MSO) language. Janin and Walukiewicz in [26] proved that the MSO properties, invariant under bisimulation, are exactly the ones which are expressible by the modal μ-calculus. Thus, *the modal μ-calculus is the 'operationally relevant' part of MSO logic, which itself, in most cases, is powerful enough in expressiveness.*

The modal μ-calculus is further strongly connected with another important area of computer science, the theory of automata (for an introduction see Thomas [47]). The idea of translating temporal logics to automata, in order to get a nice framework to study the algorithms, has a long tradition in system verification. For example PLTL can be translated into Büchi automata, since for each PLTL-formula we can find an equivalent Büchi automaton. Thus, many model checking algorithms for PLTL are based on algorithmic tools provided by Büchi automata (see Vardi and Wolper [49]). The modal μ-calculus is equivalent to alternating tree automata. Also here, many algorithmic problems, such as the model checking problem or the satisfiability problem, are solved efficiently by using this equivalence.

Recapitulating we get the following features which make the study of modal μ-calculus both interesting and important:

- The modal μ-calculus is an abstract framework to reason about fixpoints in modal logic. Thus, it is a 'meta' formal system for many logics used in computer science.
- From the point of view of operational semantics it is the 'correct' weakening of second order logics, which itself has enough expressibility for applications.
- The connection with another lively and successful area of computer science: the theory of automata.

Finally, the reason which lead me to the subject:

- The beauty of its theory, the natural basic concepts and the variety of results.

We conclude with an overview of the first part of the thesis. At the beginning of each chapter a detailed abstract can be found, so here we just highlight the important contents and results.

- In Chapter 2 we introduce basic syntactical and semantical notions and results of the modal μ-calculus, including the 'fundamental theorem'.
- In Chapter 3 the alternating tree automata are introduced and their connection with the theory of parity games is highlighted.
- In Chapter 4 the first main result of the thesis is proven: The equivalence of alternating tree automata and the modal μ-calculus.
- In Chapter 5 we use the results of the two previous chapters to give a proof for the strictness of the modal μ-calculus hierarchy.
- In Chapter 6 we deal with the completeness of Kozen's Axiomatisation and provide two partial completeness results.

Chapters 4 and 5 are based the paper Alberucci [1].

Chapter 2

Basic Definitions and Results

"Where do you want to go today?" (Microsoft)

The first section introduces the syntax of the modal μ-calculus. We begin with the language and the formulae. The language does not contain labels for the modalities, that is, it has only one kind of modality, as opposed to many, all determined by a given set of labels. A non-labeled language is enough for our purpose since we are mainly interested in the theoretical aspects of the modal μ-calculus. On the other hand, if you are interested in applications, mainly in a multi-agent context, in most cases a labeled language is required since you have to distinguish between the single agents. After, we define the syntactical modal μ-calculus hierarchy. An other chapter of this thesis will then talk of the semantical side of this hierarchy and prove its strictness. The section concludes by presenting the axiomatisation proposed by Kozen in [31] for the calculus.

The second section deals with the semantics of the modal μ-calculus. Our semantics is given by transition systems. It is the most common, and probably intuitive, semantics for this calculus. The semantical counterpart of the modal μ-calculus hierarchy is then introduced, and some well known results of fixpoint theory are applied to our context.

The final section presents Streett and Emerson's [45] fundamental theorem, an important result about modal μ-calculus. Its importance is due to the fact that it helps understanding the structure of the models of a given formula and that it will be very useful to prove some results later on in this thesis.

2.1 Syntax

Let us first define the set of formulae of the modal μ-calculus. As mentioned above, we do not introduce a syntax with labels, since for our, mainly theoretical, purposes it is enough to have an non-labeled language.

We start from a set of *propositional variables* $\mathsf{P} = \{p, q, \ldots, X, Y, Z, \ldots\}$, and the symbols $\top, \bot, \wedge, \vee, \neg, \Box, \Diamond, \mu$ and ν. The class of all *μ-formulae* $\mathcal{L}_\mu$, denoted by $\varphi, \psi, \alpha, \beta, \gamma, \ldots$, is defined to be the smallest set such that

$$\mathsf{P} \cup \{\top, \bot\} \subseteq \mathcal{L}_\mu,$$

and further that:

- if $\varphi, \psi \in \mathcal{L}_\mu$ then $(\varphi \wedge \psi) \in \mathcal{L}_\mu$ and $(\varphi \vee \psi) \in \mathcal{L}_\mu$,
- if $\varphi \in \mathcal{L}_\mu$ then $\neg\varphi \in \mathcal{L}_\mu, \Box\varphi \in \mathcal{L}_\mu$ and $\Diamond\varphi \in \mathcal{L}_\mu$.

In addition, for each $\varphi \in \mathcal{L}_\mu$ such that each appearance of a propositional variable X is in the scope of an even number of negations (that is X appears only positively in φ), we have:

- if $\varphi \in \mathcal{L}_\mu$ then $\nu X.\varphi \in \mathcal{L}_\mu$ and $\mu X.\varphi \in \mathcal{L}_\mu$.

We omit the parentheses if no danger of confusion arises and often abbreviate $\neg\alpha \vee \beta$ by $\alpha \to \beta$. The set of *subformulae* of a formula φ consists of all formulae which are needed in the inductive definition of φ given above (including φ itself). Given a formula of the form $\sigma X.\varphi$, where σ is either μ or ν, we say X is *bounded by* $\sigma X.\varphi$. If σ is ν we say that X is a *ν-variable*, in the other case it is a *μ-variable*. A variable X is *bound in* φ if it is bounded by a subformula of φ, otherwise X is *free in* φ. By $\mathsf{Free}(\varphi)$ we denote all the free propositional variables and by $\mathsf{Bound}(\varphi)$ the set of all bound propositional variables in φ. A formula is called *fixpoint-free* if it does not contain a subformula of the form $\sigma X.\varphi$ ($\sigma \in \{\mu, \nu\}$).

Remark 1 *Some authors introduce a language which, instead of having just propositional variables, distinguishes between variables ($X, Y, \ldots$) and primitive propositions ($p, q, \ldots$). In this case, you are only allowed to bound variables and not primitive propositions. The situation is similar to the one in first order logic where you have constants and variables. We do not make this distinction since it is not necessary from a purely technical point of view.*

However, our notation is such that it denotes a propositional variable with $p, q, \ldots$ when it is used like a primitive proposition, and with $X, Y \ldots$ when used like a variable.

$\varphi(X_1, \ldots, X_n)$ means that each $X_i \in \{X_1, \ldots, X_n\}$ is not bound in φ. For $\varphi(X_1, \ldots, X_n)[\psi_1/X_1, \ldots, \psi_n/X_n]$ we then often write $\varphi(\psi_1, \ldots, \psi_n)$.

The *negation normal form* $\mathsf{nnf}(\varphi)$ of φ is defined by shifting negations inside the formula φ as follows:

- $\mathsf{nnf}(p) \equiv \mathsf{nnf}(\neg\neg p) \equiv p$ and $\mathsf{nnf}(\neg p) \equiv \neg p$ for all $p \in \mathsf{P}$,
- $\mathsf{nnf}(\neg\Box\varphi) \equiv \Diamond\mathsf{nnf}(\neg\varphi)$ and $\mathsf{nnf}(\neg\Diamond\varphi) \equiv \Box\mathsf{nnf}(\neg\varphi)$,
- $\mathsf{nnf}(\neg\mu X.\varphi) \equiv \nu X.\mathsf{nnf}(\neg\varphi[\neg X/X])$ and
- $\mathsf{nnf}(\neg\nu X.\varphi) \equiv \mu X.\mathsf{nnf}(\neg\varphi[\neg X/X])$.

A formula φ is *well-named* if every variable is bounded at most once in φ and free variables are distinct from bound. $\mathsf{wn}(\varphi)$ is a formula obtained from φ by renaming the bounded variables, such that the requirements mentioned above are fulfilled (for wn we choose one arbitrary but fixed renaming). Note that for a variable X bound in a well-named formula φ there exists exactly one subformula of φ of the form $\sigma X.\psi$. A formula is in *normal form* if it is both in negation normal form and well-named. Let φ be a formula in normal form with X and Y two bound variables in it. We say that X is *higher* than Y if and only if Y is bounded by a subformula of the one bounding X.

We introduce a hierarchy on the μ-formulae. It will be similar to the one introduced on, so-called, fixpoint algebras by Niwinski in [37]. The various levels of the hierarchy capture the alternation depth of a μ-formula, whereby the alternation depth measures how many 'relevant' nestings of μ and ν occur in the formula.

We first introduce two operators μ and ν on classes of μ-formulae. Let Φ be a class of μ-formulae. We define $\mu(\Phi)$ to be smallest class of formulae such that the following requirements are fulfilled:

1. $\Phi \subseteq \mu(\Phi)$ and $\neg\Phi \subseteq \mu(\Phi)$, where $\neg\Phi := \{\neg\varphi \,|\, \varphi \in \Phi\}$.
2. If $\psi \in \mu(\Phi)$ then $\mu X.\psi \in \mu(\Phi)$ (provided that each appearance of X in ψ is positive).

3. If $\psi, \varphi \in \mu(\Phi)$ then $\psi \wedge \varphi, \psi \vee \varphi, \Box\psi, \Diamond\psi \in \mu(\Phi)$.
4. If $\psi, \varphi \in \mu(\Phi)$ and $X \notin \mathsf{Free}(\psi)$, then $\varphi[\psi/X] \in \mu(\Phi)$.

$\nu(\Phi)$ is defined analogously to $\mu(\Phi)$ with the only difference that line 2. is substituted by:

2'. If $\psi \in \nu(\Phi)$ then $\nu X.\psi \in \nu(\Phi)$ (provided that each appearance of X in ψ is positive).

The *modal μ-calculus hierarchy* on formulae consists of all Π_n^μ and Σ_n^μ, which are defined inductively for all natural numbers n as follows:

- Σ_0^μ and Π_0^μ are equal and consist of all fixpoint-free μ-formulae.
- $\Sigma_{n+1}^\mu = \mu(\Pi_n^\mu)$.
- $\Pi_{n+1}^\mu = \nu(\Sigma_n^\mu)$.

It is obvious that we have

$$\mathcal{L}_\mu = \bigcup_{n\in\omega} \Sigma_n^\mu = \bigcup_{n\in\omega} \Pi_n^\mu.$$

And moreover from the definitions above, we can easily prove

$$(\Sigma_n^\mu \cup \Pi_n^\mu) \subsetneq \Pi_{n+1}^\mu, \quad \text{and} \quad (\Sigma_n^\mu \cup \Pi_n^\mu) \subsetneq \Sigma_{n+1}^\mu.$$

This clearly shows that the hierarchy is strict on the syntactical side, the strictness of the semantical counterpart, first proven by Bradfield in [12], will be a subject of Chapter 5. If a formula φ is in $\Sigma_n^\mu \cup \Pi_n^\mu$ but not in $\Sigma_{n-1}^\mu \cup \Pi_{n-1}^\mu$ then the *alternation depth of* φ, $\mathsf{ad}(\varphi)$, is equal to n.

We present *Kozen's axiomatisation* for the modal μ-calculus KOZ, which was introduced by him in [31]. The axiomatisation is introduced in the Hilbert style.

Axioms:

KOZ includes the axioms of the classical propositional calculus, the *distribution axiom*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),$$

the *fixpoint axiom*

$$\nu X.\alpha \leftrightarrow \alpha(\nu X.\alpha)$$

and the *duality axioms*

$$\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi \quad \text{and} \quad \neg\mu X.\neg\varphi[X/\neg X] \leftrightarrow \nu X.\varphi$$

which are necessary since we did not define $\Diamond$ as $\neg\Box\neg$, and similarly for μ and ν.

Inference rules:

In addition to the classical *modus ponens* (MP), we have, as usual in modal logic, the *necessitation rule* (Nec) and, for dealing with fixpoints, the *induction rule* (Ind). All the rules are described below as schemes:

$$\frac{\psi \to \varphi \quad \psi}{\varphi}\ (MP) \qquad\qquad \frac{\varphi}{\Box\varphi}\ (Nec)$$

$$\frac{\varphi \to \alpha(\varphi)}{\varphi \to \nu X.\alpha}\ (Ind)$$

As usual if a formula φ is provable in KOZ, we write $\mathsf{KOZ} \vdash \varphi$, if φ is provable without the use of the induction rule (Ind), we write $\mathsf{KOZ}^{-(Ind)} \vdash \varphi$.

2.2 Semantics

In this section we introduce the 'usual' semantics for the modal μ-calculus. As for all modal logics the semantics is given by transition systems, which sometimes are called Kripke-Models.

A *transition system* $\mathcal{S}$ is a tupel of the form (S, R, λ), where:

- S is the *set of states*,
- $R \subseteq S \times S$ is a binary relation on S,
- $\lambda : \mathsf{P} \to \mathcal{P}(S)$ is the *valuation*, which assigns a subset of S to each propositional variable p.

We sometimes want to specify an initial point in the transition system and so introduce *pointed transition systems*. They are of the form $(\mathcal{S}, s_{\mathsf{I}})$, where $\mathcal{S}$ is a transition system and $s_{\mathsf{I}} \in S$ a state of it.

Given a valuation λ, a propositional variable $p \in \mathsf{P}$ and a set of states S' $\lambda[p \mapsto S']$ is defined as follows:

$$\lambda[p \mapsto S'](p') = \begin{cases} S' & \text{if } p' = p \\ \lambda(p') & \text{if } p' \neq p. \end{cases}$$

If $\mathcal{S} = (S, R, \lambda)$ is a transition system then $\mathcal{S}[p \mapsto S']$ denotes the transition system $(S, R, \lambda[p \mapsto S'])$.

Given a μ-formula φ and a transition system $\mathcal{S}$, the set $\|\varphi\|_{\mathcal{S}} \subseteq S$ denotes the states where φ holds, and is called the *denotation of* φ *in* $\mathcal{S}$. This is defined inductively on the structure of φ simultaneously for all transition systems $\mathcal{S}$, as follows:

- $\|p\|_{\mathcal{S}} = \lambda(p)$ for all $p \in \mathsf{P}$,
- $\|\neg\alpha\|_{\mathcal{S}} = S - \|\alpha\|_{\mathcal{S}}$,
- $\|\alpha \wedge \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cap \|\beta\|_{\mathcal{S}}$,
- $\|\alpha \vee \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cup \|\beta\|_{\mathcal{S}}$,
- $\|\Box\alpha\|_{\mathcal{S}} = \{s \in S \mid (\forall t \in R(s))\ \ t \in \|\alpha\|_{\mathcal{S}}\}$,
- $\|\Diamond\alpha\|_{\mathcal{S}} = \{s \in S \mid (\exists t \in R(s))\ \ t \in \|\alpha\|_{\mathcal{S}}\}$,
- $\|\nu X.\alpha\|_{\mathcal{S}} = \bigcup\{S' \subseteq S \mid S' \subseteq \|\alpha(X)\|_{\mathcal{S}[X \mapsto S']}\}$,
- $\|\mu X.\alpha\|_{\mathcal{S}} = \bigcap\{S' \subseteq S \mid \|\alpha(X)\|_{\mathcal{S}[X \mapsto S']} \subseteq S'\}$.

Given a transition system $\mathcal{S}$ and a state s in it. If $s \in \|\varphi\|_{\mathcal{S}}$ and if it is clear that we are referring to the transition system $\mathcal{S}$ we often write $s \models \varphi$, and say φ is *valid in* s. We write $\mathcal{S} \models \varphi$, and say φ is *valid in* $\mathcal{S}$, if it is valid in all states of $\mathcal{S}$. Furthermore, we write $\models \varphi$, and say φ is *valid*, if it is in all transition systems. If $(\mathcal{S}, s_{\mathsf{I}})$ is a pointed transition system we write $(\mathcal{S}, s_{\mathsf{I}}) \in \|\varphi\|$ if $s_{\mathsf{I}} \in \|\varphi\|_{\mathcal{S}}$, and $\|\varphi\|$ denotes the class of all pointed transition systems $(\mathcal{S}, s_{\mathsf{I}})$ such that $(\mathcal{S}, s_{\mathsf{I}}) \in \|\varphi\|$.

We are now able to define a semantical counterpart of the modal μ-calculus hierarchy. It is an hierarchy on the class of all pointed transition systems, which is named as **TR**. The semantical hierarchy consists of all $\Sigma_n^{\mu\mathbf{TR}}$ and $\Pi_n^{\mu\mathbf{TR}}$, which are defined as

$$\Sigma_n^{\mu\mathbf{TR}} = \{\|\varphi\| \mid \varphi \in \Sigma_n^{\mu}\} \quad \text{and} \quad \Pi_n^{\mu\mathbf{TR}} = \{\|\varphi\| \mid \varphi \in \Pi_n^{\mu}\}.$$

The following lemma connects the Σ and $\Pi-$levels of the semantical hierarchy above.

Lemma 2 *For all natural numbers n the following holds:*

1. $\Sigma_n^{\mu\mathbf{TR}} = \{\|\neg\varphi\| \mid \varphi \in \Pi_n^\mu\}$ *and* $\Sigma_n^{\mu\mathbf{TR}} = \{\mathbf{TR} - \|\varphi\| \mid \varphi \in \Pi_n^\mu\}$;
2. $\Pi_n^{\mu\mathbf{TR}} = \{\|\neg\varphi\| \mid \varphi \in \Sigma_n^\mu\}$ *and* $\Pi_n^{\mu\mathbf{TR}} = \{\mathbf{TR} - \|\varphi\| \mid \varphi \in \Sigma_n^\mu\}$.

Proof. By Lemma 98 in the appendix we have for all transition system $\mathcal{S}$ and all formulae φ

$$\|\neg\nu X.\varphi\|_{\mathcal{S}} = \|\mu X.\neg\varphi[\neg X/X]\|_{\mathcal{S}} \quad \text{and} \quad \|\neg\mu X.\varphi\|_{\mathcal{S}} = \|\nu X.\neg\varphi[\neg X/X]\|_{\mathcal{S}}.$$

From this fact the lemma is easily proved. □

We continue by stating two propositions. The first establishes the equivalence of a given formula with its negation normal form, and the existence of an equivalent well-named formula. These follow easily from the definition of the denotation. The second is the correctness result of KOZ which is proven by induction on the length of the derivation.

Proposition 3 *For all formulae φ and all transition systems $\mathcal{S}$ we have:*

1. $\|\varphi\|_{\mathcal{S}} = \|\mathsf{nnf}(\varphi)\|_{\mathcal{S}}$,
2. $\|\varphi\|_{\mathcal{S}} = \|\mathsf{wn}(\varphi)\|_{\mathcal{S}}$.

Proposition 4 *For all formulae φ we have*

$$\mathsf{KOZ} \vdash \varphi \quad \Rightarrow \quad \models \varphi.$$

Let $\varphi(X_1, \ldots, X_n)$ be a μ-formula and let $S_1, \ldots, S_n$ be sets of states of a transition system $\mathcal{S}$. We define

$$\|\varphi(S_1, \ldots, S_n)\|_{\mathcal{S}} = \|\varphi(X_1, \ldots, X_n)\|_{\mathcal{S}[X_1 \mapsto S_1, \ldots, X_n \mapsto S_n]}.$$

Furthermore, $\|\varphi(X_1, \ldots, X_n)\|_{\mathcal{S}}$ denotes the functional from $(\mathcal{P}(S))^n$ to $\mathcal{P}(S)$, which is defined as

$$\|\varphi(X_1, \ldots, X_n)\|_{\mathcal{S}} : (S_1, \ldots, S_n) \mapsto \|\varphi(S_1, \ldots, S_n)\|_{\mathcal{S}}.$$

It can easily be seen, that if all occurrences of a variable X_i are positive, then the functional is monotone in this variable. Hence, we know that for all transition systems $\mathcal{S}$, and formulae $\varphi(X)$, with X appearing only positively, the functional $\|\varphi(X)\|_{\mathcal{S}}$ has a greatest fixpoint, $\mathsf{GFP}(\|\varphi(X)\|_{\mathcal{S}})$, and a least fixpoint, $\mathsf{LFP}(\|\varphi(X)\|_{\mathcal{S}})$.

The following proposition is a reformulation of the well-known Tarski-Knaster Theorem 95, which is proven in the appendix.

Proposition 5 *For all formulae $\varphi(X)$, where X appears only positively, and for all transition systems $\mathcal{S}$ we have:*

1. $\mathsf{GFP}(\|\varphi(X)\|_{\mathcal{S}}) = \|\nu X.\varphi(X)\|_{\mathcal{S}}$,
2. $\mathsf{LFP}(\|\varphi(X)\|_{\mathcal{S}}) = \|\mu X.\varphi(X)\|_{\mathcal{S}}$.

Greatest fixpoints can be approximated from above and least fixpoints from below. To do that, we need to define the notion of *approximant.* Let $\varphi(X)$ be a formula, with X appearing only positively, and let $\mathcal{S} = (S, R, \lambda)$ by a transition system, we define for all ordinals α, the *α-approximant from above* $\|\varphi^\alpha(S)\|_{\mathcal{S}}$ and the *α-approximant from below* $\|\varphi^\alpha(\emptyset)\|_{\mathcal{S}}$ as follows:

For $\alpha = 0$,

$$\|\varphi^0(\emptyset)\|_{\mathcal{S}} = \emptyset \quad \text{and} \quad \|\varphi^0(S)\|_{\mathcal{S}} = S.$$

For $\alpha = \beta + 1$, a successor ordinal,

$$\|\varphi^\alpha(\emptyset)\|_{\mathcal{S}} = \|\varphi(\|\varphi^\beta(\emptyset)\|_{\mathcal{S}})\|_{\mathcal{S}} \quad \text{and} \quad \|\varphi^\alpha(S)\|_{\mathcal{S}} = \|\varphi(\|\varphi^\beta(S)\|_{\mathcal{S}})\|_{\mathcal{S}}.$$

For α a limit ordinal,

$$\|\varphi^\alpha(\emptyset)\|_{\mathcal{S}} = \bigcup_{\beta<\alpha}\{\|\varphi^\beta(\emptyset)\|_{\mathcal{S}}\} \quad \text{and} \quad \|\varphi^\alpha(S)\|_{\mathcal{S}} = \bigcap_{\beta<\alpha}\{\|\varphi^\beta(S)\|_{\mathcal{S}}\}.$$

The following lemma is a translation of the Lemmata 96 and 97 in the appendix.

Lemma 6 *Let $\mathcal{S} = (S, R, \lambda)$ be a transition systems of cardinality κ. For all formulae $\varphi(X)$, with X appearing only positively, we have:*

1. $\|\mu X.\varphi(X)\|_{\mathcal{S}} = \|\varphi^\kappa(\emptyset)\|_{\mathcal{S}}$,
2. $\|\nu X.\varphi(X)\|_{\mathcal{S}} = \|\varphi^\kappa(S)\|_{\mathcal{S}}$.

Remark 7 *The approximation done in Lemma 6 can be done in a pure syntactical way for finite transition systems. This follows from the following observations, which can easily be verified. Let $\varphi^0(X)$ be defined as X and for all natural numbers let $\varphi^{n+1}(X)$ be $\varphi[X/\varphi^n(X)]$. Then we have for all transition systems $\mathcal{S}$ and natural numbers n:*

$$\|\varphi^n(X)[X/\top]\|_{\mathcal{S}} = \|\varphi^n(S)\|_{\mathcal{S}} \quad \text{and} \quad \|\varphi^n(X)[X/\bot]\|_{\mathcal{S}} = \|\varphi^n(\emptyset)\|_{\mathcal{S}}.$$

2.3 The Fundamental Theorem

This section reviews the fundamental theorem due to Emerson and Streett [45]. We use a slightly different notation and nomenclature than those introduced by the two authors. The label 'fundamental' was, in my knowledge, first used by Bradfield and Stirling in [11].

In the sequel we will assume that all the formulae are in normal form, that is, in negation normal form and are well-named. This is no restriction by Proposition 3. To define the notion of well-founded pre-model, we fist need some preliminary definitions.

Let $\mathcal{S} = (S, R, \lambda)$ be a transition system and φ a formula:

1. An *annotated structure for φ and $\mathcal{S}$* consists of states of the transition system $\mathcal{S}$ annotated with subformulae of φ. If $s \in S$ is annotated with α we write $\alpha@s$. Furthermore, we must have $\varphi@s$ for at least one state s.

2. A *quasi-model (for φ and $\mathcal{S}$)* is an annotated structure (for φ and $\mathcal{S}$), which fulfills the following local consistency conditions.

   - $\alpha \wedge \beta@s \;\Rightarrow\; \alpha@s$ and $\beta@s$,
   - $\alpha \vee \beta@s \;\Rightarrow\; \alpha@s$ or $\beta@s$,
   - $\Box\alpha@s \;\Rightarrow\; \alpha@t$ for all $t \in R(s)$,
   - $\Diamond\alpha@s \;\Rightarrow\; \alpha@t$ for a $t \in R(s)$,
   - $p@s \;\Rightarrow\; s \in \lambda(p)$, if p is a free propositional variable in φ,
   - $\neg p@s \;\Rightarrow\; s \notin \lambda(p)$, if p is a free propositional variable in φ,
   - $X@s \;\Rightarrow\; \sigma X.\alpha@s$, if X is a propositional variable bounded in φ by $\sigma X.\alpha$ (where $\sigma \in \{\mu, \nu\}$).

Notice that the case $\neg X@s$, where X is a bounded variable, is not possible, since X appears only positively.

3. A *choice function on an annotated structure* is a partial function f, such that either $f(\alpha \vee \beta@s) = \alpha$ or $f(\alpha \vee \beta@s) = \beta$ and such that $f(\Diamond\beta@s) = t$ for a $t \in R(s)$.

4. A *pre-model (for φ and $\mathcal{S}$)* is a quasi-model (for φ and $\mathcal{S}$) with a choice function.

To define the notion of well-founded pre-model we moreover have to define a *dependency relation* on annotated states of a pre-model. Given a pre-model for φ and $\mathcal{S}$ with choice function f, we define the dependency relation $\succ_1$ on the annotated states of the pre-model as follows.

- $\alpha \wedge \beta@s \ \succ_1 \ \alpha@s$ and $\alpha \wedge \beta@s \ \succ_1 \ \beta@s$,
- $\alpha \vee \beta@s \ \succ_1 \ f(\alpha \vee \beta@s)@s$,
- $\Box\alpha@s \ \succ_1 \ \alpha@t$ for all $t \in R(s)$,
- $\Diamond\alpha@s \ \succ_1 \ \alpha@f(\Diamond\alpha@s)$,
- $\sigma X.\alpha@s \ \succ_1 \ \alpha@s$ where σ is either μ or ν,
- $X@s \ \succ_1 \ \sigma X.\alpha@s$ where X is bounded by $\sigma X.\alpha$ in φ.

A *branch of a pre-model* is a maximal chain of dependencies. A formula ψ occurs in a branch if it contains an annotated state of the form $\psi@s$. In order to define well-foundedness of a pre-model we need a lemma.

Lemma 8 *Let φ be a formula and $\mathcal{S}$ a transition system. For each infinite branch of a pre-model for φ and $\mathcal{S}$ there is a variable $X \in \mathsf{Bound}(\varphi)$, occurring infinitely often, which is higher than all the other bound variables occurring infinitely often.*

Proof. It can easily be seen that in an infinite branch there must be variables occurring infinitely often. So, only the uniqueness has to be proven. We prove it by contraposition. Suppose there is a branch with two variables $X, Y \in \mathsf{Bound}(\varphi)$, occurring infinitely often, such that all the other bound variables occurring infinitely often are not higher than these. Furthermore, suppose that X is bounded by $\sigma X.\alpha$ in φ and Y by $\sigma Y.\beta$. By definition of higher we know that $\sigma X.\alpha$ is not a subformula of $\sigma Y.\beta$ and vice versa, and hence we know that both X does not appear in $\sigma Y.\beta$ and Y does not appear in $\sigma X.\alpha$. For, if for example X appears free in $\sigma Y.\beta$ then $\sigma Y.\beta$ would be a subformula of $\sigma X.\alpha$ and if X appears bounded in $\sigma Y.\beta$ then $\sigma X.\alpha$ would be a subformula of $\sigma Y.\beta$. Without loss of generality suppose that $\sigma X.\alpha$ appears earlier than $\sigma Y.\beta$ in the infinite branch. Since Y can not appear in $\sigma X.\alpha$ in this branch we will never reach an annotated state of the form $\sigma Y.\beta @ s'$. This is a contradiction to the fact that Y occurs infinitely often. $\Box$

A branch of a pre-model is *closed*, if the highest variable occurring infinitely often is a ν-variable. The dependency relation of a pre-model is well-founded if every branch is closed. A *well-founded pre-model* is then a pre-model, where the dependency relation is well-founded.

As the nomenclature suggests, a (well-founded) pre-model for φ and $\mathcal{S}$ can be described as a tree with root $\varphi @ s$, the other vertices are all the $\psi @ t$ depending on $\varphi @ s$ (that is there is a branch containing $\psi @ s$) and if $\psi @ t \succ_1 \psi' @ t'$, then there is an edge from $\psi @ t$ to $\psi' @ t'$. This graph is called the *(well-founded) pre-model for $\mathcal{S}$ with root $\varphi @ s$.*

We now state the *fundamental theorem* of the μ-calculus, the proof can be found in the paper of Street and Emerson [45].

Theorem 9 *Let $\mathcal{S}$ be a transition system. For all states s in $\mathcal{S}$ and formulae φ the following two conditions are equivalent:*

1. $s \in \|\varphi\|_{\mathcal{S}}$.
2. *There is a well-founded pre-model for $\mathcal{S}$ with root $\varphi @ s$.*

Chapter 3

Alternating Tree Automata

"Lerne die Regeln, damit du sie richtig brechen kannst." (Dalai Lama)

Alternating tree automata where first introduced by Muller and Schupp in [36] as an extension of automata on infinite trees, a subject previously well studied, see for example Rabin in [40]. This chapter deals with the alternating tree automata model introduced by Wilke in [53].

In Section 1 we first define our alternating tree automata model. Further, we introduce the notion of run of an automaton over a transition system, which is crucial for the definition of acceptance of a transition system by an automaton. We then introduce a hierarchy of automata, based on the notion of index of an automaton. The section ends with some lemmas and an optional definition of acceptance.

The second section begins with some basic definitions of the theory of parity games, such as strategy tree, game, strategy, winning strategy. The existence of a winning strategy on a certain strategy tree is then shown to be equivalent to the acceptance of a transition system by a given automaton. The section ends with the construction of a an automaton which is able to test the existence of a winning strategy. As a consequence, we can reduce the problem of acceptance of a transition system by a given automaton to the problem of acceptance of a strategy tree by such a test automaton.

The final section states some decidability and complexity results about the emptiness problem and the model checking problem for alternating tree automata.

3.1 Basic Definitions

In this section we introduce alternating tree automata following the model introduced by Wilke in [53]. Then we define the notions of acceptance of a transition system by an alternating tree automaton. It will be done by introducing the notion of run of an automaton over a transition system. We first describe informally, what alternating tree automata are:

Alternating tree automata consist of a finite set of states. They accept or reject a given pointed transition system. A run of an alternating tree automaton on a pointed transition system can be seen as a computation tree. The computation begins in the initial state of the automaton. At this stage the automaton reads the emphasized state of the pointed transition system, and then according to the, so-called, transition function, the automaton changes, possibly in a non-deterministic way, its internal state and the state of the transition system. The computation continues by reading the new state of the transition system while being in its new internal state. Since the computation may last forever, we need a priority function to define acceptance of the run produced by it. The priority function is a partial function with finite range from the states of the automaton to the natural numbers. An infinite branch of a run is accepted, if the maximal priority appearing is even, and a run is accepted if all infinite branches are. Formally:

An *alternating tree automaton* $\mathcal{A}$ is a tuple $\mathcal{A} = (Q, \mathsf{P}, q_{\mathsf{I}}, \delta, \Omega)$, where:

- Q is a finite set of *states*,
- P is a set of *propositional variables*,
- $q_{\mathsf{I}} \in Q$ is the *initial state*,
- $\Omega : Q \to \omega$ is a (partial) *priority function* and
- $\delta : Q \to TC^{Q \cup P}$ is a *transition function*, where $TC^{Q \cup P}$ is the set of all *transition conditions over* $Q \cup P$ defined inductively as:
  - $\bot, \top \in TC^{Q \cup P}$,
  - $p, \neg p \in TC^{Q \cup P}$ for all $p \in P$,
  - if $q \in Q$ then $q, \Box q, \Diamond q \in TC^{Q \cup P}$,
  - if $q, q' \in Q$ then $q \wedge q'$, $q \vee q' \in TC^{Q \cup P}$.

Remark 10 *Notice that a transition condition $\delta(q)$ can be interpreted as a μ-formula over propositional variables in $Q \cup P$. We sometimes write $\delta_q(q_1, \ldots, q_n)$ if $\delta(q)$ can be interpreted as a μ-formula whose variables are among $\{q_1, \ldots, q_n\} \cup \mathsf{P}$.*

The definition of *acceptance of a pointed transition system* $(\mathcal{S}, s_\mathsf{I})$ by an automaton $\mathcal{A}$ needs the notion of *accepting run* of an automaton over a transition system.

Let $\mathcal{A}$ be an automaton containing a state q_0 and let $\mathcal{S}$ be a transition system containing a state s_0. We define ϱ to be a *q_0-run on s_0 of $\mathcal{A}$ on $\mathcal{S}$* if ϱ is a $(S \times Q)$-vertex-labeled tree of the form (V, E, ℓ), where V is the set of *vertices*, E is a binary relation on V, and $\ell : V \to (S \times Q)$ is the *labeling function*. If v_0 is the root of V, then $\ell(v_0)$ must be (s_0, q_0). Further, for all vertices $v \in V$, with label (s, q), the following requirements must be fulfilled:

- $\delta(q) \neq \bot$,
- if $\delta(q) = p$ then $s \in \|p\|_\mathcal{S}$, and if $\delta(q) = \neg p$ then $s \notin \|p\|_\mathcal{S}$,
- if $\delta(q) = q'$ then there is a $v' \in E(v)$ such that $\ell(v') = (s, q')$,
- if $\delta(q) = \Diamond q'$, then there is a $v' \in E(v)$ such that $\ell(v') = (s', q')$ where $s' \in R(s)$,
- if $\delta(q) = \Box q'$ then for all $s' \in R(s)$ there is a $v' \in E(v)$ such that $\ell(v') = (s', q')$,
- if $\delta(q) = q' \vee q''$ then there is a $v' \in E(v)$ such that $\ell(v') = (s, q')$ or $\ell(v') = (s, q'')$,
- if $\delta(q) = q' \wedge q''$ then there are $v', v'' \in E(v)$ such that $\ell(v') = (s, q')$ and $\ell(v'') = (s, q'')$.

An *infinite branch* of a run *is accepting* if the highest priority which appears infinitely often is even. That is:

Suppose we have an infinite branch of the form $(s_0, q_0), (s_1, q_1), \ldots, (s_i, q_i), \ldots$. Define S to be the set consisting of all natural numbers n such there are infinitely many pairs (s_j, q_j) with $\Omega(q_j) = n$. If S is empty the branch is not accepting otherwise $\max(S)$ must be even. Notice that for non-empty S,

$\max(S)$ always exists. A *run is accepting* if all infinite branches are accepting. Finally, an automaton $\mathcal{A}$ *accepts a pointed transition system* $\mathcal{S} = (S, s_{\mathsf{I}})$ if there is an accepting q_{I}-run on s_{I} of $\mathcal{A}$ on $\mathcal{S}$ (where q_{I} is the initial state of the automaton).

Our definition of transition conditions is restrictive since it does not allow 'complex' transition conditions. Nevertheless, this restriction can be circumvented by adding new states and extending the transition function appropriately. For example, suppose we want a transition condition φ which intuitively acts as: Change the inner state to q_1 if p is true, otherwise change the inner state to q_2. Formally, we represent this by

$$\varphi \equiv (q_1 \wedge p) \vee (q_2 \wedge \neg p).$$

Clearly, $\varphi \notin \mathrm{TC}^{\mathsf{Q} \cup \mathsf{P}}$. On the other hand, by introducing the new states q_φ, $q_{(q_1 \wedge p)}$, $q_{(q_2 \wedge \neg p)}$, q_p, $q_{\neg p}$, and extending the transition function δ such that

$$\begin{aligned} \delta(q_\varphi) &= q_{(q_1 \wedge p)} \vee q_{(q_2 \wedge \neg p)}, \\ \delta(q_{(q_1 \wedge p)}) &= q_1 \wedge q_p, \\ \delta(q_{(q_2 \wedge \neg p)}) &= q_2 \wedge q_{\neg p}, \\ \delta(q_p) &= p, \\ \delta(q_{\neg p}) &= \neg p, \end{aligned}$$

we get a new automaton, with restricted transition conditions, which meets the intended intuition. Using this method we can talk about automata with complex transition conditions.

Let $\mathcal{A}$ be an automaton. $\|\mathcal{A}\|_{\mathcal{S}}$ is the set of all states s of the transition system $\mathcal{S}$ such that $\mathcal{A}$ accepts $(\mathcal{S}, s)$. $\|\mathcal{A}\|$ is the class of all pointed transition systems accepted by an automaton $\mathcal{A}$.

The index of an automaton is a measure of its complexity. Before we define it, we first need the definition of *transition graph* of the automaton. The transition graph of an automaton $\mathcal{A}$ has its states as vertices, and there is an edge (q, q') if q' appears in $\delta(q)$. A *strongly connected* component of this graph is a subset of the graph where all the vertices are pairwise reachable.

The *index*, $\mathrm{ind}(\mathcal{A})$, of an automaton $\mathcal{A}$ is then defined as

$$\mathrm{ind}(\mathcal{A}) = \max\Big(\{|\Omega(Q')| \mid Q' \subseteq Q, Q' \text{ is strongly connected}\} \cup \{0\}\Big).$$

We introduce a *syntactical hierarchy* for automata, consisting of the following classes of alternating automata:

- $\Sigma_0 = \Pi_0$ consists of all automata of index 0.
- $\Sigma_n (n > 0)$ contains $\Sigma_{n-1} \cup \Pi_{n-1}$ and all automata of index n where the maximal priority on a strongly connected component of the transition graph of the automaton is odd.
- $\Pi_n (n > 0)$ contains $\Sigma_{n-1} \cup \Pi_{n-1}$ and all automata of index n where the maximal priority on a strongly connected component of the transition graph of the automaton is even.

Clearly this syntactical hierarchy strict. The *semantical* counterpart of this *hierarchy* is defined naturally as follows:

$$\Sigma_n^{\mathbf{TR}} = \{\|\mathcal{A}\| \mid \mathcal{A} \in \Sigma_n\} \quad \text{and} \quad \Pi_n^{\mathbf{TR}} = \{\|\mathcal{A}\| \mid \mathcal{A} \in \Pi_n\}.$$

Strictness of the semantical hierarchy will be proved in the sequel.

A Σ_n-automaton is in *normal form* if the range of the priority function is Ω_{Σ_n} and a Π_n-automaton is in normal form if the range is Ω_{Π_n}, whereby Ω_{Σ_n} and Ω_{Π_n} are defined by case distinction on n:

- If $n = 0$ we have
$$\Omega_{\Sigma_n} = \Omega_{\Pi_n} = \emptyset.$$
- If n is an even positive natural number then
$$\Omega_{\Sigma_n} = \{0, \ldots, n-1\} \quad \text{and} \quad \Omega_{\Pi_n} = \{1, \ldots, n\}.$$
- If n is an odd natural number then
$$\Omega_{\Sigma_n} = \{1, \ldots, n\} \quad \text{and} \quad \Omega_{\Pi_n} = \{0, \ldots, n-1\}.$$

The following lemma claims the existence of an equivalent automaton in normal form for all automata, and can easily be proven.

Lemma 11 *For each automaton $\mathcal{A} \in \Sigma_n (\in \Pi_n)$ there is an automaton $\mathcal{A}' \in \Sigma_n (\in \Pi_n)$ in normal form, such that*

$$\|\mathcal{A}\| = \|\mathcal{A}'\|.$$

The next lemma is the *complementation theorem* for alternating tree automata. It claims that for all automata there is another that accepts exactly the pointed transition systems, that were not accepted by the first. The proof can be found in Kirsten [30].

Lemma 12 *For each automaton $\mathcal{A} \in \Sigma_n (\in \Pi_n)$ there is an automaton $\hat{\mathcal{A}} \in \Pi_n (\in \Sigma_n)$ such that*

$$\|\hat{\mathcal{A}}\| = \mathbf{TR} - \|\mathcal{A}\|.$$

We finish the section by giving alternative definition of run. This needs the interpretation of transition conditions as μ-formulae over $\mathsf{P} \cup Q$.

Let $\mathcal{A} = (Q, P, q_1, \delta, \Omega)$ be an automaton, $\mathcal{S} = (S, R, \lambda)$ a transition system and $\varrho = (V, E, \ell)$ a $(S \times Q)$-vertex-labeled tree. For all $v \in V$ and $q \in Q$ we define

$$S_{E(v)|q} = \{s \in S \mid (\exists v' \in E(v))\ (\ell(v) = (s, q))\}.$$

The following lemma can be proven by unwinding the definitions.

Lemma 13 *Let $\mathcal{A} = (Q, P, q_{\mathsf{I}}, \delta, \Omega)$ be an automaton, $\mathcal{S} = (S, R, \lambda)$ a transition system and $\varrho = (V, E, \ell)$ a $(S \times Q)$-vertex-labeled tree with root v_0. For all $s_0 \in S$ and $q_0 \in Q$ the following two sentences are equivalent:*

- *$\varrho = (V, E, \ell)$ is a q_0-run on s_0 of $\mathcal{A}$ on $\mathcal{S}$,*
- *$\ell(v_0) = (s_0, q_0)$ and for all vertices v which are labeled by (s, q) we have $s \in \|\delta_q(S_{E(v)|q_1}, \ldots, S_{E(v)|q_n})\|_{\mathcal{S}}$.*

3.2 Reduction of Acceptance

In this section we reduce the problem of acceptance of a transition system by an automaton to the problem of acceptance of an other transition system by a test automaton. This new transition system can be seen as a strategy tree in a parity game. We avoid to introduce the whole formalism of parity games, although we will borrow its terminology. For a detailed introduction in the theory of parity games see Wilke [53].

Let $(\mathcal{S}, s_{\mathsf{I}})$ be a pointed transition system, where $\mathcal{S} = (S, R, \lambda)$, and let $\mathcal{A}$ be the automaton $(Q, \mathsf{P}, q_{\mathsf{I}}, \delta, \Omega)$. The *strategy tree* $\mathsf{G}_{\mathcal{A},\mathcal{S}}$ of $\mathcal{A}$ on $\mathcal{S}$ is defined as follows:

The root v_{I} of the tree is $(s_{\mathsf{I}}, q_{\mathsf{I}})$. All the vertices v are finite sequences of the form

$$v \equiv (s_{\mathsf{I}}, q_{\mathsf{I}}), (s_1, q_1), \ldots, (s_n, q_n)$$

where $s_i \in S$ and $q_i \in Q$ for all $i \in \{1, \ldots, n\}$. Let $v \equiv (s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_n, q_n)$ be a vertex. We define the successors of v by case distinction on $\delta(q_n)$:

1. If $\delta(q_n) = \bot, \top, p, \neg p$ then v has no successors,
2. if $\delta(q_n) = q'$ then v has as successor the sequence $v, (s_n, q')$,
3. if $\delta(q_n) = q \vee q'$ then v has the successors $v, (s_n, q)$ and $v, (s_n, q')$,
4. if $\delta(q_n) = \Diamond q$ then v has a successor $v, (s, q)$ for all $s \in R(s_n)$,
5. if $\delta(q_n) = q \wedge q'$ then v has the successors $v, (s_n, q)$ and $v, (s_n, q')$,
6. if $\delta(q_n) = \Box q$ then v has a successor $v, (s, q)$ for all $s \in R(s_n)$.

If $\delta(q_n)$ falls under cases (2), (5),(6) and (7), then we call v a *disjunctive vertex (d-node)*, whereas if $\delta(q_n)$ is one of (7) and (8), then we call v a *conjunctive vertex (c-node)*. Clearly, if a vertex has a successor in the strategy tree then it must be either a c-node or a d-node.

Remark 14 *Any run of $\mathcal{A}$ on $(\mathcal{S}, s_{\mathsf{I}})$ can be seen as a 'subtree' of the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. This is due to the fact that, starting from the strategy tree, we can construct any run by taking all successors of the c-nodes we reach and by selecting the adequate successor, that is the one which chooses the run, at every d-node we reach. Thus, we can use the strategy tree to define acceptance.*

Let $(\mathcal{S}, s_{\mathsf{I}})$ be a pointed transition system and let $\mathcal{A}$ be the automaton $(Q, \mathsf{P}, q_{\mathsf{I}}, \delta, \Omega)$. We informally define a *parity game on the strategy tree* $\mathsf{G}_{\mathcal{A},\mathcal{S}}$:

We have two players, the disjunctive player (player D) and the conjunctive player (player C). The game begins at the root v of the strategy tree; if v is a d-node, player D chooses a successor in the tree; if it is a c-node, player C chooses one. Thus the game continues: Whenever a c-node is reached D plays, whenever a c-node is reached C plays. Either the game continues forever or a node is reached which has no successor. Let us define when player D wins: If the game is finite we have a last position $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_n, q_n)$. Player

D wins if either the last position is a c-node, or if we have $s_n \in \|\delta(q_n)\|_{\mathcal{S}}$. If the game lasts forever, we have an infinite branch of the strategy tree $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_i, q_i) \ldots$. Player D wins if the maximum priority appearing infinitely often in the branch is even; that is, the maximum value appearing infinitely often in the sequence of all $\Omega(q_i)$, where q_i appears in the branch, must be even. In all the other cases player C wins.

A *strategy* for player D is a function that assigns to every d-node a successor in the strategy tree. A winning strategy for player D is a strategy such that, whenever D follows the strategy, no matter how C plays, D wins the game. Similar concepts are defined for player C.

The following theorem relates winning strategies of player D to acceptance.

Theorem 15 *Let $\mathcal{A}$ be an automaton and $(\mathcal{S}, s_{\mathsf{I}})$ a pointed transition system. $\mathcal{A}$ accepts $(\mathcal{S}, s_{\mathsf{I}})$ if and only if player D has a winning strategy on $\mathsf{G}_{\mathcal{A},\mathcal{S}}$.*

Proof. By Remark 14 a run of $\mathcal{A}$ on $(\mathcal{S}, s_{\mathsf{I}})$ and a strategy tree of $\mathcal{A}$ on $(\mathcal{S}, s_{\mathsf{I}})$ only differ in the fact, that there is branching at d-nodes. If we want to construct an accepting run of $\mathcal{A}$ on $\mathcal{S}$, we can choose a successor only when we are at a d-node. In this sense, we have the same choice nodes as player D. On the other hand, at a c-node, also the run is branching and we have to test for acceptance all the branches, that is, no matter which branch we take, it must be accepting. In this sense, the adversary, player C, can choose an arbitrary branch which in every case must be accepted. Since the winning conditions of player D are exactly the conditions of acceptance in a branch of a run, the result follows naturally. □

A strategy for player D is called *memoryless* if the choice of the next position in a d-node of the form $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_i, q_i)$ depends only on the last pair (s_i, q_i). Formally: Suppose we have two d-nodes $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_i, q_i)$ and $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s'_j, q'_j)$. According to his memoryless strategy, player D moves to the successors $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_i, q_i), (s, q)$ and $(s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s'_j, q'_j), (s', q')$. A memoryless strategy is such that, if we have $(s_i, q_i) \equiv (s'_j, q'_j)$ then we must have $(s, q) \equiv (s', q')$.

The next theorem claims the existence of a memoryless winning strategy whenever there is any winning strategy, it is proven by Emerson and Jutla in [13].

Theorem 16 *Let $\mathsf{G}_{\mathcal{A},\mathcal{S}}$ be a strategy tree. Let Player D has a winning strategy on $\mathsf{G}_{\mathcal{A},\mathcal{S}}$ if and only if it has a memoryless winning strategy on $\mathsf{G}_{\mathcal{A},\mathcal{S}}$.*

Before we apply this result to accepting runs let us introduce the notion of *memoryless run*: Let $\mathcal{A}$ be an automaton and $\mathcal{S} = (S, R, \lambda)$ a transition system. A run (V, E, ℓ) is called memoryless if any two vertices $v, v' \in V$ with $\ell(v) = \ell(v') = (s, q)$ satisfy the following: If $\delta(q) = \Diamond q'$ or $\delta(q) = q' \vee q''$ then for all $\bar{v} \in E(v)$ there is a $\bar{v'} \in E(v')$ such that and $\ell(\bar{v}) = \ell(\bar{v'})$, and vice versa.

Corollary 17 *Let $\mathcal{A}$ be an automaton and $(\mathcal{S}, s_\mathsf{I})$ a pointed transition system. There is an accepting run of $\mathcal{A}$ on $(\mathcal{S}, s_\mathsf{I})$ if and only if there is a memoryless accepting run of $\mathcal{A}$ on $(\mathcal{S}, s_\mathsf{I})$.*

Proof. The "if" direction is obvious. The "only if" direction can be seen as follows. Suppose there is an accepting run. Then by Theorem 15 there is a winning strategy for player D on the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. By Theorem 16 there is a memoryless winning strategy for player D on $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. Now we can convert $\mathsf{G}_{\mathcal{A},\mathcal{S}}$ in an accepting run by choosing at all d-nodes the successor resulting from the move of player D. The run must by memoryless, since player D follows a memoryless strategy. □

To end this section we now do the *reduction of acceptance*, that is we reduce the problem of acceptance of a pointed transition system $(\mathcal{S}, s_\mathsf{I})$ by an automaton $\mathcal{A}$ to the problem of acceptance of the strategy tree, seen as a pointed transition system, by a test automaton.

Let $\mathcal{A} = (Q, \mathsf{P}, q_\mathsf{I}, \delta, \Omega)$ be an automaton and $(\mathcal{S}, s_\mathsf{I})$ a pointed transition system where $\mathcal{S} = (S, R, \lambda)$. We define $\mathsf{T}_\mathcal{A}$, the *test automaton for $\mathcal{A}$*, which tests the existence of a winning strategy on the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. Since an automaton acts only on transition systems, starting from the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$, we also have to define a transition system $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ on which the test automaton acts. We define

$$\mathsf{T}_\mathcal{A} = (Q^\mathsf{T}, \mathsf{P}^\mathsf{T}, q_u^\mathsf{T}, \delta^\mathsf{T}, \Omega^\mathsf{T}),$$

where

- $Q^\mathsf{T} = \{q_i^\mathsf{T} \mid i \in \Omega(Q)\} \cup \{q_u^\mathsf{T}\}$,
- $\mathsf{P}^\mathsf{T} = \{c_u, c_1, c_2, \ldots, c_i, \ldots\} \cup \{d_u, d_1, d_2, \ldots, d_i, \ldots\}$, where u is a new symbol,
- $\Omega^\mathsf{T}(q_j^\mathsf{T}) = j$ if $j \neq u$, and $\Omega^\mathsf{T}(q_u^\mathsf{T}) \uparrow$ and

- for all states $q_j^{\mathsf{T}} \in Q^{\mathsf{T}}$ we have:

$$\delta^{\mathsf{T}}(q_j^{\mathsf{T}}) = \bigvee_{i \in Q^{\mathsf{T}}} (c_i \wedge \Box q_i^{\mathsf{T}}) \vee \bigvee_{i \in Q^{\mathsf{T}}} (d_i \wedge \Diamond q_i^{\mathsf{T}}).$$

$\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ is of the form $(\mathsf{G}_{\mathcal{A},\mathcal{S}}, E, \lambda')$, where E is the edge relation on the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$; and λ' evaluates the set P^{T} of variables (where P^{T} is defined as above), such that validity at a vertex $v \equiv (s_{\mathsf{I}}, q_{\mathsf{I}}), \ldots, (s_n, q_n)$ is defined as follows:

- If v has a successor in the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$

$$v \in \lambda'(p) \quad \text{iff} \quad \begin{cases} v \text{ is disjunctive, } \Omega(q_n) \uparrow \text{ and } p \equiv d_u, \\ v \text{ is disjunctive, } \Omega(q_n) = i \text{ and } p \equiv d_i, \\ v \text{ is conjunctive, } \Omega(q_n) \uparrow \text{ and } p \equiv c_u, \\ v \text{ is conjunctive, } \Omega(q_n) = i \text{ and } p \equiv c_i. \end{cases}$$

- If v is a leaf of $\mathsf{G}_{\mathcal{A},\mathcal{S}}$ for all $p \in \mathsf{P}^{\mathsf{T}}$ we have

$$v \in \lambda'(p) \quad \text{iff} \quad s_n \in \|\delta(q_n)\|_{\mathcal{S}}.$$

Remark 18 *As the notation suggests, the construction of $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ depends on the automaton $\mathcal{A}$ and the transition system $\mathcal{S}$, whereas $\mathsf{T}_{\mathcal{A}}$ depends only on $\mathcal{A}$. In fact, from the definition it can be seen that $\mathsf{T}_{\mathcal{A}}$ depends only on the range of the priority function of the automaton $\mathcal{A}$. By Lemma 11 all automata have a normal form where the range of the priority function is fixed by their membership to a class in the hierarchy. So, if we assume all automata of being in normal form, then we can talk of the* Σ_n-test automaton T_{Σ_n} *and the* Π_n-test automaton T_{Π_n}.

In the following the main theorem of this section, which shows us how the reduction of acceptance is done.

Theorem 19 *Let $\mathcal{A}$ be an automaton with initial state q_{I} and $(\mathcal{S}, s_{\mathsf{I}})$ a pointed transition system:*

1. *$\mathsf{T}_{\mathcal{A}}$ accepts $(\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}}), (s_{\mathsf{I}}, q_{\mathsf{I}}))$ if and only if there is a winning strategy of player D in $\mathsf{G}_{\mathcal{A},\mathcal{S}}$.*

2. $\mathsf{T}_{\mathcal{A}}$ *accepts* $(\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}}), (s_{\mathsf{I}}, q_{\mathsf{I}}))$ *iff* $\mathcal{A}$ *accepts* $(\mathcal{S}, s_{\mathsf{I}})$*.*

Proof. Part 1. is a consequence to the fact that an accepting run of $\mathsf{T}_{\mathcal{A}}$ on $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ gives player D a winning strategy, and vice versa. Let us informally explain this:

Let us analyze how the automaton $\mathsf{T}_{\mathcal{A}}$ works through $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ (which basically is $\mathsf{G}_{\mathcal{A},\mathcal{S}}$). A run of $\mathsf{T}_{\mathcal{A}}$ must have the following structure: If it reaches a c-node, which by construction is labeled with a c-variable, then by the definition of the transition function δ^{T}, it has a branching to all successors in $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. If it reaches a d-node, which by construction is labeled with a d-variable, then by the definition of the transition function δ^{T}, the run goes to one successor of the d-node in $\mathsf{G}_{\mathcal{A},\mathcal{S}}$. In this sense in the construction of an accepting run we have the same choice nodes as player D. Since the run is accepting if all branches are, and since the branching happens exactly at the c-nodes, the choice nodes of player C, the acceptance of the run corresponds to the existence of a winning strategy of player D.

2. Follows from part 1. of the theorem and from Theorem 15. □

3.3 Decidability and Complexity

This section states some results concerning the decidability and complexity of the model checking problem and the emptiness problem for alternating tree automata. For all definitions of the complexity classes we refer to the literature (for example Papadimitriou [38]). Let us first formulate the two problems.

The model checking problem:
Given an automaton $\mathcal{A}$ and a pointed transition system $(\mathcal{S}, s_{\mathsf{I}})$, does $\mathcal{A}$ accept $(\mathcal{S}, s_{\mathsf{I}})$?

The emptiness problem:
Given an automaton $\mathcal{A}$, is there a pointed transition system $(\mathcal{S}, s_{\mathsf{I}})$, such that $\mathcal{A}$ accepts $(\mathcal{S}, s_{\mathsf{I}})$?

The following theorem gives the solution to the model checking problem and shows its complexity. It has first been proven by Jurdzinski in [27] with methods of game theory. Another nice proof is given by Wilke in [53]. Notice that UP is the complexity class of all computations which can be done by a nondeterministic Turing machine in polynomial time, such that for any input x there is at most one accepting computation. Note that $\mathsf{UP} \subseteq \mathsf{NP}$.

Theorem 20 *Let $\mathcal{A} = (Q, \mathsf{P}, q_{\mathsf{I}}, \delta, \Omega)$ be an alternating tree automaton and $(\mathcal{S}, s_{\mathsf{I}})$ be a finite pointed transition system, with set of states S and transition relation R. Furthermore, let d be the index of $\mathcal{A}$. Then:*

1. *There is an algorithm which computes whether $\mathcal{A}$ accepts $(\mathcal{S}, s_{\mathsf{I}})$ in time*
$$\mathcal{O}\left(d\,|Q|\,\Big(|R|+1\Big)\left(\frac{|Q||S|+1}{\lceil d/2\rceil}\right)^{\lceil d/2\rceil}\right)$$
and in space
$$\mathcal{O}\left(d\,|Q|\,|S|\,\log\left(|Q|\,|S|\right)\right).$$
2. *The problem whether $\mathcal{A}$ accepts $(\mathcal{S}, s_{\mathsf{I}})$ is in $\mathsf{UP} \cap \mathsf{co}\text{-}\mathsf{UP}$.*

Finally, the solution to the emptiness problem due to Emerson and Jutla [14].

Theorem 21 *The emptiness problem for alternating tree automata is in* EXP, *where* EXP *is the class of all exponential time decidable problems.*

Chapter 4

Equivalence of μ-Calculus and Automata

"Translation as a way of life." *(J. van Benthem)*

This chapter deals with the equivalence of alternating tree automata and modal μ-calculus. For each modal μ-formula φ there is an automaton which accepts exactly those pointed transition systems where φ is true; and vice versa.

In the first section we discuss a translation of the modal μ-calculus to automata. One possible translation has been presented by Wilke in [53], where a direct proof that the constructed automaton is equivalent to the original formula is given. We present an alternative proof method by applying the fundamental theorem of the modal μ-calculus.

In the second section we translate automata to the modal μ-calculus. A very similar result has been proven by Niwinski in [37]. He introduces automata on semi-algebras and shows the equivalence with certain fixpoint terms on, so-called, powerset algebras. By using the fact that on binary structures the μ-calculus corresponds to a certain powerset algebra, this result can then be applied to the modal μ-calculus to obtain the equivalence of alternating automata to the calculus, on binary transition systems. By applying the same proof techniques of Niwinski to the alternating tree automata of Wilke in [53] we give a direct translation of automata to μ-formulae.[1] The translation gives us for every automaton a μ-formula which is equivalent to it on all transition

[1] At this point I thank Prof. Wilke for the hint.

systems. In this sense, the new result we get is a generalization of the original one by Niwinski, since we are not restricting ourselves to binary transition systems.

The last section deals with the decidability and complexity of the satisfiability and the model checking problem for the modal μ-calculus. Our results follow from the analogous results for alternating tree automata using the previously proven equivalence.

4.1 From μ-Calculus to Automata

In this section we assume that all μ-formulae are in normal form. Remember that in this case all the negations appear in front of propositional variables and that all bounded variables are distinct.

Let φ be a μ-formula in normal form. We construct an equivalent alternating tree automaton $\mathcal{A}_\varphi$, that is, for all pointed transition systems $(\mathcal{S}, s_\mathsf{I})$ the automaton $\mathcal{A}_\varphi$ satisfies

$$(\mathcal{S}, s_\mathsf{I}) \in \|\varphi\| \quad \text{iff} \quad (\mathcal{S}, s_\mathsf{I}) \in \|\mathcal{A}_\varphi\|.$$

Let us first introduce $\mathcal{A}_\varphi$ informally: The structure of the automaton $\mathcal{A}_\varphi$ reflects the one of φ in the following sense. For each subformula ψ of φ the automaton has a state denoted by $\langle\psi\rangle$. The initial state is $\langle\varphi\rangle$ itself. A state $\langle\alpha\rangle$ occurs in the transition condition $\delta(\langle\psi\rangle)$ of the state $\langle\psi\rangle$ if and only if α is a maximal subformula of ψ. In addition, the transition function reflects the outermost connective of ψ. For example, $\delta(\langle\psi_1 \wedge \psi_2\rangle) = \langle\psi_1\rangle \wedge \langle\psi_2\rangle$ and $\delta(\langle\Diamond\psi\rangle) = \Diamond\langle\psi\rangle$. In the case that $\psi = p$ for a $p \in \mathsf{Free}(\varphi)$ the automaton has simply to check whether in the current state p is true. Thus, $\delta(\langle p\rangle) = p$. More interesting is the case where we have a bounded variable X in φ. Let $\varphi_X = \sigma X.\psi$ be the subformula of φ that bounds X. Then $\delta(\langle X\rangle) = \langle\varphi_X\rangle$. The difference between the least and the greatest fixpoints will be expressed by the priority function. Formally:

Let φ be a μ-formula in normal form over the set of propositional variables P. For each $X \in \mathsf{Bound}(\varphi)$ let φ_X be the subformula of φ which bounds X. We define the alternating tree automaton $\mathcal{A}_\varphi$ as follows:

$$\mathcal{A}_\varphi = (Q, \mathsf{P}, q_\mathrm{I}, \delta, \Omega),$$

where:

- $Q := \{\langle\psi\rangle \mid \psi$ is subformula of $\varphi\}$,
- $q_{\mathrm{I}} := \langle\varphi\rangle$,
- $\delta : Q \to \mathrm{TC}^{Q \cup \mathsf{P}}$ is defined by:

$$\delta(\langle\bot\rangle) = \bot, \qquad \delta(\langle\top\rangle) = \top,$$
$$\delta(\langle(\neg)p\rangle) = (\neg)p, \text{ where } p \in \mathsf{Free}(\varphi),$$
$$\delta(\langle X\rangle) = \varphi_X, \text{ where } X \in \mathsf{Bound}(\varphi),$$
$$\delta(\langle\psi \wedge \chi\rangle) = \langle\psi\rangle \wedge \langle\chi\rangle, \qquad \delta(\langle\psi \vee \chi\rangle) = \langle\psi\rangle \vee \langle\chi\rangle,$$
$$\delta(\langle\Box\psi\rangle) = \Box\langle\psi\rangle, \qquad \delta(\langle\Diamond\psi\rangle) = \Diamond\langle\psi\rangle,$$
$$\delta(\langle\mu X.\psi\rangle) = \langle\psi\rangle, \qquad \delta(\langle\nu X.\psi\rangle) = \langle\psi\rangle.$$

- The priority function $\Omega : Q \to \omega$ is defined only on states of the form $\langle\sigma X.\psi\rangle$ in the following way:

  $\Omega(\langle\mu X.\psi\rangle) =$ the smallest odd number greater or equal to $\mathsf{ad}(\psi)$,
  $\Omega(\langle\nu X.\psi\rangle) =$ the smallest even number greater or equal to $\mathsf{ad}(\psi)$.

Remark 22 *It follows from the construction, that $\mathcal{A}_\varphi \in \Sigma_n$ if $\varphi \in \Sigma_n^\mu$, and that $\mathcal{A}_\varphi \in \Pi_n$ if $\varphi \in \Pi_n^\mu$.*

The next theorem proves the equivalence of the automaton $\mathcal{A}_\varphi$ to the formula φ.

Theorem 23 *For all modal μ-formulae φ we have*

$$\|\varphi\| = \|\mathcal{A}_\varphi\|.$$

Proof. By Theorem 9, $(\mathcal{S}, s_\mathsf{I}) \in \|\varphi\|$ is equivalent to the fact that there is a well-founded pre-model for $\mathcal{S}$ with root $\varphi@s_\mathsf{I}$. And $(\mathcal{S}, s_\mathsf{I}) \in \|\mathcal{A}_\varphi\|$ means that there is an accepting q_I-run on s_I of $\mathcal{A}_\varphi$ on $\mathcal{S}$, where q_I is the initial state of $\mathcal{A}_\varphi$. So, for the equivalence, it is enough to show that an accepting q_I-run on s_I of $\mathcal{A}_\varphi$ on $\mathcal{S}$ can be transformed into a well-founded pre-model for $\mathcal{S}$ with root $\varphi@s_\mathsf{I}$; and vice versa.

Let us first convert accepting runs into well-founded pre-models. Suppose we have an accepting q_I-run on s_I of $\mathcal{A}_\varphi$ on $\mathcal{S}$, where $\mathcal{S}$ is of the form (S, R, λ). By

Corollary 17 we have a memoryless accepting run, which is a $(S \times Q)$-vertex-labeled tree of the form (V, E, ℓ). From this run we construct a well-founded pre-model as follows:

For each vertex $v \in V$, where $\ell(v) = (s, \langle\psi\rangle)$, we take an annotated state $\psi@s$ in the pre-model. The relation $\succ_1$ between the annotated states and the choice function are then defined, by case distinction on ψ, as follows (p denotes a free propositional variable in φ, and X a bounded one):

- $\psi \equiv (\neg)p$: In this case the annotated state $(\neg)p@s$ has no successor.
- $\psi \equiv X$: Suppose that X is bounded by φ_X in φ. In this case in the accepting run we have a successor $(s, \langle\varphi_X\rangle)$, and so, by construction of the pre-model, we have an annotated state $\varphi_X@s$. In this case we set $X@s \succ_1 \varphi_X@s$.
- $\psi \equiv \alpha \wedge \beta$: In the accepting run we have two successors $(s, \langle\alpha\rangle)$ and $(s, \langle\alpha\rangle)$. So we set $\alpha \wedge \beta@s \succ_1 \alpha@s$ and $\alpha \wedge \beta@s \succ_1 \beta@s$.
- $\psi \equiv \alpha \vee \beta$: We have a successor $(s, \langle\alpha\rangle)$ or $(s, \langle\beta\rangle)$. Suppose the successor is $(s, \langle\alpha\rangle)$. In this case we set $\alpha \vee \beta@s \succ_1 \alpha@s$. Obviously, the choice function f must be defined such that $f(\alpha \vee \beta@s) = \alpha$.
- $\psi \equiv \Box\alpha$: For all $s' \in R(s)$ we have a successor $(s', \langle\alpha\rangle)$ in the accepting run. In this case we set $\Box\alpha@s \succ_1 \alpha@s'$ for all these $s' \in R(s)$.
- $\psi \equiv \Diamond\alpha$: There is a $s' \in R(s)$ such that we have a successor $(s', \langle\alpha\rangle)$ in the accepting run. In this case we set $\Diamond\alpha@s \succ_1 \alpha@s'$ for this $s' \in R(s)$. The choice function f has to be defined such that $f(\Diamond\alpha@s) = s'$.
- $\psi \equiv \mu X.\alpha$: We have a successor $(s, \langle\alpha\rangle)$. So we set $\mu X.\alpha@s \succ_1 \alpha@s$.
- $\psi \equiv \nu X.\alpha$: Goes exactly as the case $\psi \equiv \mu X.\alpha$.

Notice that the choice function f is well defined since we assumed a memoryless accepting run.

Now it remains to show, that the constructed pre-model is well-founded. Remember, that we started from an accepting run of $\mathcal{A}_\varphi$ on $(\mathcal{S}, s_\mathsf{I})$. Hence, the local consistency conditions of a pre-model are all fulfilled (see definition of a quasi-model). So it remains to show that the pre-model is well-founded. Now, this is clear since all infinite branches of the pre-model were constructed by an infinite branch of the accepting run. And, since the priority function of $\mathcal{A}_\varphi$ was only defined on vertices of the form $(s, \langle X \rangle)$, such that it is even if X is a ν-variable and odd if it is a μ variable, the accepting condition of the run corresponds exactly to the well-foundedness condition of the pre-model.

The conversion of a well-founded pre-model of φ on $\mathcal{S}$ to an accepting run of $\mathcal{A}_\varphi$ on $\mathcal{S}$ is basically the construction above made backwards and is left to the reader. $\square$

4.2 From Automata to μ-Calculus

In this section we translate alternating tree automata into the modal μ calculus. We assign to every automaton a μ-formula which is valid in exactly the pointed transition systems accepted by the automaton.

Let us begin with a lemma which deals with simultaneous fixpoints of monotone functionals. It is a reformulation of the Theorems 101 and 102 in the Appendix, in terms of the modal μ-calculus.

Lemma 24 *Let $\delta_1(s_1, \ldots, s_k), \ldots, \delta_k(s_1, \ldots, s_k)$ be μ-formulae contained in a class of formulae Φ such that all s_j $(j = 1, \ldots, k)$ appear only positively. Further, define for all transition systems $\mathcal{S} = (S, R, \lambda)$ a functional $\mathcal{F}_\mathcal{S}$ from $(\mathcal{P}(S))^k$ to $(\mathcal{P}(S))^k$ as*

$$\mathcal{F}_\mathcal{S} : (S_1, \ldots, S_k) \mapsto (\|\delta_1(S_1, \ldots, S_k)\|_\mathcal{S}, \ldots, \|\delta_k(S_1, \ldots, S_k)\|_\mathcal{S}).$$

There are μ-formulae $\tau_1, \ldots, \tau_k$ in $\nu(\Phi)$ and $\rho_1, \ldots, \rho_k$ in $\mu(\Phi)$ such that for all transition systems $\mathcal{S}$ we have

$$\mathsf{GFP}(\mathcal{F}_\mathcal{S}) = (\|\tau_1\|_\mathcal{S}, \ldots, \|\tau_k\|_\mathcal{S})$$

and

$$\mathsf{LFP}(\mathcal{F}_\mathcal{S}) = (\|\rho_1\|_\mathcal{S}, \ldots, \|\rho_k\|_\mathcal{S}).$$

Example 25 *We illustrate how we can construct these simultaneous fixpoints in the case $k = 2$, that is we have $\delta_1(X, Y)$ and $\delta_2(X, Y)$.*

- $\tau_1 \equiv \nu X.\delta_1(X, Y)[\nu Y.\delta_2(X, Y)/Y]$,
- $\tau_2 \equiv \nu Y.\delta_2(X, Y)[\nu X.\delta_1(X, Y)/X]$,

- $\rho_1 \equiv \mu X.\delta_1(X,Y)[\mu Y.\delta_2(X,Y)/Y]$,
- $\rho_2 \equiv \mu Y.\delta_2(X,Y)[\mu X.\delta_1(X,Y)/X]$.

Clearly, an automaton of index 0 can only have finite accepting runs. The next lemma gives an equivalent μ-formula to an automaton which has a finite accepting run. Thus, it gives already a translation of alternating automata of index 0 into the modal μ-calculus.

Lemma 26 *Let $\mathcal{A}$ be an alternating automaton. There is a μ-formula ρ in Σ_1^μ such that, for any pointed transition system $(\mathcal{S}, s_\mathsf{I})$, we have*

$$s_\mathsf{I} \in \|\rho\|_\mathcal{S} \quad \text{iff} \quad \textit{there is a finite run of } \mathcal{A} \textit{ on } s_\mathsf{I}.$$

Proof. Suppose the automaton has states $Q = \{q_1, \ldots, q_k\}$ and transition function δ. For all states q_i we abbreviate $\delta(q_i)$ by δ_i. Now, remember that all δ_i can be interpreted as μ-formulae which can contain propositional variables among $q_1, \ldots, q_k$, that is, they are of the form $\delta_i(q_1, \ldots, q_k)$. By definition of the transition conditions we see that all δ_i are in Π_0^μ . So, by Lemma 24 for all $i \in \{1, \ldots, k\}$ there are formulae $\rho_i \in \Sigma_1^\mu$ such that for each transition system $\mathcal{S} = (S, R, \lambda)$ the functional

$$\mathcal{F}_\mathcal{S} : (S_1, \ldots, S_k) \mapsto (\|\delta_1(S_1, \ldots, S_k)\|_\mathcal{S}, \ldots, \|\delta_k(S_1, \ldots, S_k)\|_\mathcal{S})$$

has the least fixpoint

$$\mathsf{LFP}(\mathcal{F}_\mathcal{S}) = (\|\rho_1\|_\mathcal{S}, \ldots, \|\rho_k\|_\mathcal{S}).$$

For all transition systems $\mathcal{S} = (S, R, \lambda)$ and all $i = 1, \ldots, k$, let $\mathcal{A}_i^{\mathcal{S}*}$ denote the set of all $s \in S$ such that there is a finite q_i-run on s of $\mathcal{A}$. To complete the proof the lemma it is enough to show

$$(\mathcal{A}_1^{\mathcal{S}*}, \ldots, \mathcal{A}_k^{\mathcal{S}*}) = (\|\rho_1\|_\mathcal{S}, \ldots, \|\rho_k\|_\mathcal{S}).$$

By the Tarski-Knaster Theorem 95 and since $\mathsf{LFP}(\mathcal{F}_\mathcal{S}) = (\|\rho_1\|_\mathcal{S}, \ldots, \|\rho_k\|_\mathcal{S})$, this can be shown by proving the two following things:

(i) $\mathcal{F}_\mathcal{S}(\mathcal{A}_1^{\mathcal{S}*}, \ldots, \mathcal{A}_k^{\mathcal{S}*}) \subseteq (\mathcal{A}_1^{\mathcal{S}*}, \ldots, \mathcal{A}_k^{\mathcal{S}*})$ and

(ii) for all $(S_1, \ldots, S_k) \subseteq S^k$ such that $\mathcal{F}_\mathcal{S}(S_1, \ldots, S_k) \subseteq (S_1, \ldots, S_k)$ we have

$$(\mathcal{A}_1^{\mathcal{S}*}, \ldots, \mathcal{A}_k^{\mathcal{S}*}) \subseteq (S_1, \ldots, S_k).$$

Let us first prove (i). Suppose that $(s_1, \dots, s_k) \in \mathcal{F}_\mathcal{S}(\mathcal{A}_1^{\mathcal{S}*}, \dots, \mathcal{A}_k^{\mathcal{S}*})$, so all s_i are in $\|\delta_i(\mathcal{A}_1^{\mathcal{S}*}, \dots, \mathcal{A}_k^{\mathcal{S}*})\|_\mathcal{S}$. By Lemma 13 there is the 'beginning of a run' with root (s_i, q_i) and leaves of the form (s'_j, q_j), where $j \in \{1, \dots, k\}$ and $s'_j \in \mathcal{A}_j^{\mathcal{S}*}$. From this we get $s_i \in \mathcal{A}_i^{\mathcal{S}*}$, which proves (i).

To prove (ii), let $(S_1, \dots, S_k)$ satisfy the premise of (ii) and suppose that $(s_1, \dots, s_k) \in (\mathcal{A}_1^{\mathcal{S}*}, \dots, \mathcal{A}_k^{\mathcal{S}*})$. This means that for each s_i there is a finite q_i-run. We set d_i to be the minimal depth of all the finite q_i-runs on s_i. By induction on $d = \max\{d_i \mid 1 \leq k\}$ we prove $(s_1, \dots, s_k) \in (S_1, \dots, S_k)$.

- $d = 1$: Since, for each $i \in \{1, \dots, k\}$, the root of the accepting q_i-run on s_i has no successor we must have $s_i \in \|\delta_i(\emptyset, \dots, \emptyset)\|$ which gives us

  $$(s_1, \dots, s_k) \in \mathcal{F}_\mathcal{S}(\emptyset, \dots, \emptyset).$$

  Using the monotonicity of $\mathcal{F}_\mathcal{S}$, we get

  $$(s_1, \dots, s_k) \in \mathcal{F}_\mathcal{S}(S_1, \dots, S_n).$$

  and with the premise of (ii) the desired result.

- $d > 1$: For all $i \in \{1, \dots, k\}$, the root v_i of the finite accepting q_i-run on s_i is labeled by $\ell(v_i) = (s_i, q_i)$. Suppose, all the sons v'_j of v_i are labeled by $\ell(v'_j) = (s'_j, q_{l(v'_j)})$, where $l(v'_j) \in \{1 \dots, k\}$. By induction hypothesis we have

  $$s'_j \in S_{l(v'_j)}.$$

  So, by Lemma 13 we see, for all i, that $s_i \in \|\delta_i(S_1, \dots, S_k)\|_\mathcal{S}$. This means that s_i is in the i-component of $\mathcal{F}_\mathcal{S}(S_1, \dots, S_k)$. Since s_i was an arbitrary component of the tuple $(s_1, \dots, s_k)$ we get

  $$(s_1, \dots, s_k) \in \mathcal{F}_\mathcal{S}(S_1, \dots, S_k).$$

  And with the premise of (ii) we get the desired result.

So the proof is completed. □

Let us prove the main result of this section.

Theorem 27 *For any alternating automaton $\mathcal{A} = (Q, P, \delta, q_\mathsf{I}, \Omega)$ there is a μ-formula $\tau_\mathcal{A}$ over propositional variables $P \cup Q$ such that, for all pointed transition systems $(\mathcal{S}, s_\mathsf{I})$, we have*

$$\mathcal{A} \text{ accepts } (\mathcal{S}, s_\mathsf{I}) \quad \text{iff} \quad s_\mathsf{I} \in \|\tau_\mathcal{A}\|_\mathcal{S}.$$

Further, if $\mathcal{A}$ is Σ_n, then $\tau_{\mathcal{A}}$ can be chosen in Σ^μ_{n+1}; if $\mathcal{A}$ is Π_n, then $\tau_{\mathcal{A}}$ can be chosen in Π^μ_{n+1}.

Proof. The proof goes by induction on the index n of the automaton. We assume for all alternating automata $\mathcal{A}$ that the priority function is defined only on strongly connected components of the transition graph. Moreover, we assume that the priority function of any automaton of index n has a range of cardinality n. This is no real restriction since by Lemma 11 all automata are equivalent to one fulfilling these assumptions. There will be two cases for the induction step ($n > 0$):

Case 1: If the maximal priority m is even, we consider k auxiliary automata of index $\leq n-1$, in which the states of $\Omega^{-1}[m]$ are moved into variables. Then we apply the greatest fixpoint operator.

Case 2: If the maximal priority m is odd, we consider the complement $\hat{\mathcal{A}}$ of our automaton $\mathcal{A}$. By Lemma 12, $\hat{\mathcal{A}}$ can be chosen to have the same index as $\mathcal{A}$, but with maximal priority even. Thus, if we assume that the induction step for Case 1 has been made, we have a Π^μ_{n+1}-formula $\tau_{\hat{\mathcal{A}}}$ representing the complement. By Lemma 2 we know that there is a formula $\tau_{\mathcal{A}} \in \Sigma^\mu_{n+1}$ which is equivalent to $\neg\tau_{\hat{\mathcal{A}}}$. It is easy to check that $\tau_{\mathcal{A}}$ is the Σ^μ_{n+1}-formula fulfilling the requirements of the theorem. So, only the induction step for Case 1 has to be done.

The informal description above, shows that greatest fixpoints capture the automata with even maximal priority and the least fixpoints, as negations of greatest fixpoints, the automata with an odd maximal priority.

Before we do the induction, let us explain what means *"moving states into variables"*. We need to define two transformations for automata:

The first takes an automaton $\mathcal{A} = (Q, P, \delta, q_{\mathsf{I}}, \Omega)$ and a set $X \subsetneq Q$, such that $q_{\mathsf{I}} \notin X$ and defines a new automaton

$$\mathcal{A}_{free(X)} = (Q - X, P \cup X, \delta', q_{\mathsf{I}}, \Omega')$$

where δ' and Ω' are the restrictions of δ (resp. Ω) to $Q - X$.

This is the transformation which converts states of the automaton into variables. Notice that the runs of $\mathcal{A}_{free(X)}$ are like the "beginning" of a run of the automaton $\mathcal{A}$. If we reach a point (s, q), where $q \in X$ the run of $\mathcal{A}_{free(X)}$ stops, whereas if it was a run of the automaton $\mathcal{A}$ it would go on.

The second transformation on automata helps us to deal with the restriction $q_I \notin X$ we had on the first transformation. It takes an automaton as above, a state $q \in Q$ and a new symbol $\hat{q} \notin (Q \cup P)$ and defines a new automaton

$$\mathcal{A}_{start(q)} = (Q \cup \{\hat{q}\}, P, \delta'', \hat{q}, \Omega)$$

where δ'' is equal to δ on Q and $\delta''(\hat{q}) = \delta(q)$.

It is clear, that $\mathcal{A}_{start(q)}$ accepts the same pointed transition systems as $\mathcal{A}$ with initial state q. Moreover, note that for all $X \subseteq Q$, the introduction of $\hat{q}$ makes possible for all automata to do the operation $(\mathcal{A}_{start(q)})_{free(X)}$ (shorter $\mathcal{A}_{start(q)free(X)}$).

Let us do the induction on the index n of an automaton $\mathcal{A} = (Q, P, \delta, q_I, \Omega)$.

$n = 0$: In this case every run of an automaton must be finite, and so the theorem follows from Lemma 26.

$n > 0$: As shown before it is enough to do the induction step only for Case 1. We define U to be the set of states $\Omega^{-1}[m]$, where m is the maximal priority, assuming that $q_I \notin U$; otherwise we consider the semantically equivalent automaton $\mathcal{A}_{start(q_I)}$. Suppose $U = \{q_1, \ldots, q_k\}$. We consider the automata $\mathcal{A}_{free(U)}$ and $\mathcal{A}_{start(q_i)free(U)}$ for all $i = 1, \ldots, k$. It is easy to see that all these automata are of index $\leq n - 1$. So by induction hypothesis there are μ-formulae $\tau_0(\vec{q})$ and $\tau_1(\vec{q}), \ldots, \tau_k(\vec{q})$ in Σ_n^μ (where $\vec{q} \equiv (q_1, \ldots, q_k)$), such that for all pointed transition systems $(\mathcal{S}, s_I)$ we have

$$\mathcal{A}_{free(U)} \text{ accepts } (\mathcal{S}, s_I) \quad \Leftrightarrow \quad s_I \in \|\tau_0(\vec{q})\|_{\mathcal{S}}$$

and

$$\mathcal{A}_{start(q_i)free(U)} \text{ accepts } (\mathcal{S}, s_I) \quad \Leftrightarrow \quad s_I \in \|\tau_i(\vec{q})\|_{\mathcal{S}},$$

for all $i = 1, \ldots, k$.
Now consider the functionals $\mathcal{F}_{\mathcal{S}} : (\mathcal{P}(S))^k \to (\mathcal{P}(S))^k$ with

$$\mathcal{F}_{\mathcal{S}} : (S_1, \ldots, S_k) \mapsto (\|\tau_1(S_1, \ldots, S_k)\|_{\mathcal{S}}, \ldots, \|\tau_k(S_1, \ldots, S_k)\|_{\mathcal{S}}).$$

By Lemma 24 there are μ-formulae $\rho_1, \ldots, \rho_k$ in Π_{n+1}^μ such that for all transition systems $\mathcal{S}$, $(\|\rho_1\|_{\mathcal{S}}, \ldots, \|\rho_k\|_{\mathcal{S}})$ is the greatest fixpoint of $\mathcal{F}_{\mathcal{S}}$. In order to do the induction step let us make the following claim.

Claim:
For all pointed transition systems $(\mathcal{S}, s_I)$ and for all $i = 1. \ldots, k$ we have the two following facts:

1. $\mathcal{A}_{start(q_i)}$ accepts $(\mathcal{S}, s_\mathsf{I}) \quad \Leftrightarrow \quad s_\mathsf{I} \in \|\rho_i\|_\mathcal{S}$.
2. $\mathcal{A}$ accepts $(\mathcal{S}, s_\mathsf{I}) \quad \Leftrightarrow \quad s_\mathsf{I} \in \|\tau_0[\rho_1/q_1, \ldots, \rho_k/q_k]\|_\mathcal{S}$.

Since $\tau_0[\rho_1/q_1, \ldots, \rho_k/q_k] \in \Pi^\mu_{n+1}$ the claim completes the induction step for Case 1.

We prove the claim by first showing that 1 implies 2, and then by showing the correctness of 1.

- 1 implies 2: Let us remark, that by choice of τ_0 and by 1 we have

$$s_I \in \|\tau_0[\rho_1/q_1, \ldots, \rho_k/q_k]\|_\mathcal{S} \quad \Leftrightarrow \quad \mathcal{A}_{free(U)} \text{ accepts } (\mathcal{S}', s_\mathsf{I})$$

  where $\mathcal{S}' \equiv \mathcal{S}[q_1 \mapsto \mathcal{A}^\mathcal{S}_{start(q_1)}, \ldots, q_k \mapsto \mathcal{A}^\mathcal{S}_{start(q_k)}]$ and $\mathcal{A}^\mathcal{S}_{start(q_i)}$ is the set of states s in S such that $\mathcal{A}_{start(q_i)}$ accepts $(\mathcal{S}, s)$. So it is enough to show

$$\mathcal{A} \text{ accepts } (\mathcal{S}, s_\mathsf{I}) \quad \Leftrightarrow \quad \mathcal{A}_{free(U)} \text{ accepts } (\mathcal{S}', s_\mathsf{I}).$$

  To prove the "only if" direction let us assume that ϱ is a q_I-run on s_I of the automaton $\mathcal{A}$ on $\mathcal{S}$. We want to convert it into a q_I-run on s_I of the automaton $\mathcal{A}_{free(U)}$ on $\mathcal{S}'$. Let us do the conversion for every branch of ϱ. If we have a branch where there is no state of U, then we do not change anything; otherwise, by the first $q_i \in U$ appearing in the branch, we cut it up. The new end point we get is of the form (s, q_i), where by assumption $(\mathcal{S}, s)$ is accepted by $\mathcal{A}$ with new initial state q_i. Using the fact that this automaton is equivalent to $\mathcal{A}_{start(q_i)}$ and that q_i is now a variable, which by definition is true in $s \in S$ (under the new valuation for $\mathcal{S}'$), we get the desired result. The proof of the "if" direction follows similar arguments.

- 1: As before $\mathcal{A}^\mathcal{S}$ is the set of all points s in $\mathcal{S}$ such that $(\mathcal{S}, s)$ is accepted by $\mathcal{A}$. By definition of ρ_i it is enough to prove that the greatest fixpoint of $\mathcal{F}_\mathcal{S}$ is $(\mathcal{A}^\mathcal{S}_{start(q_1)}, \ldots, \mathcal{A}^\mathcal{S}_{start(q_k)})$, and so by Tarski-Knaster we have to prove:

  (i) $(\mathcal{A}^\mathcal{S}_{start(q_1)}, \ldots, \mathcal{A}^\mathcal{S}_{start(q_k)}) \subseteq \mathcal{F}_\mathcal{S}(\mathcal{A}^\mathcal{S}_{start(q_1)}, \ldots, \mathcal{A}^\mathcal{S}_{start(q_k)})$

  (ii) For all $(S_1, \ldots, S_k) \subseteq S^k$ such that $(S_1, \ldots, S_k) \subseteq \mathcal{F}_\mathcal{S}(S_1, \ldots, S_k)$ we have

$$(S_1, \ldots, S_k) \subseteq (\mathcal{A}^\mathcal{S}_{start(q_1)}, \ldots, \mathcal{A}^\mathcal{S}_{start(q_k)}).$$

We first prove (i). Let remind ourselves that the i-th component of the tuple $\mathcal{F}_{\mathcal{S}}(\mathcal{A}^{\mathcal{S}}_{start(q_1)}, \ldots, \mathcal{A}^{\mathcal{S}}_{start(q_k)})$ is of the form

$$\|\tau_i(\mathcal{A}^{\mathcal{S}}_{start(q_1)}/q_1, \ldots, \mathcal{A}^{\mathcal{S}}_{start(q_k)}/q_k)\|_{\mathcal{S}}.$$

Since τ_i was the formula equivalent to the automaton $\mathcal{A}_{start(q_i)free(U)}$ it is enough to show the following, for all states s in S

$$\mathcal{A}_{start(q_i)} \text{ accepts } (\mathcal{S}, s) \quad \Rightarrow \quad \mathcal{A}_{start(q_i)free(U)} \text{ accepts } (\mathcal{S}', s),$$

where $\mathcal{S}'$ is $\mathcal{S}[q_1 \mapsto \mathcal{A}^{\mathcal{S}}_{start(q_1)}, \ldots q_k \mapsto \mathcal{A}^{\mathcal{S}}_{start(q_k)}]$. Since a very similar result has been proven above, we omit the details.

We now prove (ii). Let $(S_1, \ldots, S_k)$ satisfy the premise of (ii), and let $s_i \in S_i$. Since $s_i \in \|\tau_i(S_1, \ldots, S_k)\|_{\mathcal{S}}$, by hypothesis about τ_i we have $s_i \in \mathcal{A}^{\mathcal{S}'}_{start(q_i)}$, where $\mathcal{S}'$ is $\mathcal{S}[q_1 \mapsto S_1, \ldots, q_k \mapsto S_k]$ (remember that τ_i is of the form $\tau_i(q_1, \ldots, q_k)$). So there is an accepting run of $\mathcal{A}_{start(q_i)free(U)}$ with the property that if it has a vertex (q_j, s_j) such that $q_j \in U$, then it is a leaf and we have $s_j \in S_j$. Hence we can re apply the premise of (ii) and construct a s_j-run of $\mathcal{A}_{start(q_j)free(U)}$, such that for all leaves of the form $(q_{j'}, s_{j'})$ with $q_{j'} \in U$ the premise of (ii) can be "re-"re applied. By the iteration of this process, in the limit case, we get an accepting run of $\mathcal{A}_{start(q_i)}$ for s_i, since the following holds for all branches. If the branch is finite, then its end point is of the form (σ, s_i), where $\sigma \notin \{q_1, \ldots, q_k\} = U$. By assumption we have $s_i \in \lambda(\sigma) = \lambda'(\sigma)$ (where λ' is the valuation of $\mathcal{S}'$ and λ the valuation of $\mathcal{S}$). For the infinite branches we have two cases. For the first case the infinite branch contains only finitely many states q, which are in U, thus it easily follows, that from the last appearance of a $q \in U$ on, this branch is the same as a branch of an accepting run of an automaton $\mathcal{A}_{start(q_l)free(U)}$. So the highest priority appearing infinitely often must be even, and the branch is accepted. For the other case, there are infinitely many states of U in the branch, and since U is the set where the priority function has its maximal value m and m is even, we have an accepted branch.

So the proof is completed. $\square$

We end this section by giving an example of an automaton and an equivalent μ-formula obtained with the construction described in the proof.

Example 28 *Given an automaton $\mathcal{A} = (\{q_0, q_1, q_2\}, \{p_1\}, \delta, q_0, \Omega)$ such that $\delta(q_0) = \Box q_1$, $\delta(q_1) = q_2 \vee \Diamond q_0$ and $\delta(q_2) = p_1 \wedge \Box q_1$, and such that $\Omega(q_0) \uparrow$ and $\Omega(q_1) = \Omega(q_2) = 0$. We construct an equivalent μ-formula, following the proof of Theorem 27 (we use trivial equivalences of μ-formulae to get more compact representations).*
We set $U = \{q_1, q_2\}$. By construction the formula φ equivalent to the automaton has the structure $\tau_0[\rho_1/q_1, \rho_2/q_2]$, where the formulae τ_0, ρ_1, ρ_2 are defined as follows:

- *τ_0 is equivalent to $\mathcal{A}_{free(U)}$,*
- *ρ_1, ρ_2 are formulae such that for all $\mathcal{S}$ we have $(\|\rho_1\|, \|\rho_2\|) = \mathsf{GFP}(\mathcal{F}_1)$, where $\mathcal{F}_1$ is the functional*

$$\mathcal{F}_1 : (S_1, S_2) \mapsto \|\tau_1(S_1, S_2), \tau_2(S_1, S_2)\|_{\mathcal{S}},$$

  where $\tau_1(q_1, q_2)$ is the formula equivalent to $\mathcal{A}_{start(q_1)free(U)}$ and $\tau_2(q_1, q_2)$ is the formula equivalent to $\mathcal{A}_{start(q_2)free(U)}$.

By construction we also have for all transition systems $\mathcal{S}$:

- *$\mathcal{A}_{free(U)}$ is equivalent to $\mathsf{LFP}(\mathcal{F}_2)$ with*

$$\mathcal{F}_2 : S \mapsto \|\Box q_1\|_{\mathcal{S}}.$$

- *$\mathcal{A}_{start(q_1)free(U)}$ is equivalent to the second component of $\mathsf{LFP}(\mathcal{F}_3)$ with*

$$\mathcal{F}_3 : (S_0, S_1) \mapsto \|(\Box q_1, q_2 \vee \Diamond S_0)\|_{\mathcal{S}}.$$

- *$\mathcal{A}_{start(q_2)free(U)}$ is equivalent to the second component of $\mathsf{LFP}(\mathcal{F}_4)$ with*

$$\mathcal{F}_4 : (S_0, S_2) \mapsto \|(\Box q_1, p_1 \wedge \Box q_1)\|_{\mathcal{S}}.$$

Putting all this together we obtain (Example 25 may be useful for a better understanding):

- *$\tau_0 \equiv \Box q_1$,*
- *$\tau_1 \equiv q_2 \vee \Diamond \Box q_1$,*
- *$\tau_2 \equiv p_1 \wedge \Box q_1$.*

So, we get

$$\mathcal{F}_1 : (S_1, S_2) \mapsto \|S_2 \vee \Diamond\Box S_1, p_1 \wedge \Box S_1\|_{\mathcal{S}}$$

which gives us

$$(\rho_1, \rho_2) \equiv (\nu X.((p_1 \wedge \Box X) \vee \Diamond\Box X), \nu Y.(p_1 \wedge \Box \nu X.(p_1 \wedge \Box X)))$$

and so we have

$$\varphi \equiv \Box(\nu X.((p_1 \wedge \Box X) \vee \Diamond\Box X)).$$

4.3 Exhausting the Equivalence

In this section we give a solution to the satisfiability problem and the model checking problem for the modal μ-calculus. Our solution uses the equivalence with alternating tree automata, proved previously in this section, and the strong connection of the satisfiability problem with the emptiness problem for automata. Let us first formulate the satisfiability problem for the modal μ-calculus.

The satisfiability problem:
Given a μ-formula φ. Is there a pointed transition system $(\mathcal{S}, s_\mathsf{I})$, such that $(\mathcal{S}, s_\mathsf{I}) \in \|\varphi\|$?

We first look to the model checking problem.

Theorem 29 *Let φ be a μ-formula with $\mathsf{ad}(\varphi) = d$ and length $|\varphi|$. And let $(\mathcal{S}, s_\mathsf{I})$ be a finite pointed transition system, with set of states S and transition relation R.*

1. *There is an algorithm which computes whether $\mathcal{A}$ accepts $(\mathcal{S}, s_\mathsf{I})$ in time*
$$\mathcal{O}\left(d\,|\varphi|\,\Big(|R|+1\Big)\Big(\frac{|\varphi||S|+1}{\lceil d/2\rceil}\Big)^{\lceil d/2\rceil}\right)$$
and in space
$$\mathcal{O}\left(d\,|\varphi|\,|S|\,\log\left(|\varphi|\,|S|\right)\right).$$
2. *The problem whether φ accepts $(\mathcal{S}, s_\mathsf{I})$ is in $\mathsf{UP} \cap \mathsf{co{-}UP}$.*

Proof. Given the μ-formula φ with $\mathsf{ad}(\varphi) = d$ we translate it in the equivalent alternating tree automaton $\mathcal{A}_\varphi$, which by Remark 22 has index d. Since $\mathcal{A}_\varphi$ has basically all subformulae as states, and since there are $\mathcal{O}(|\varphi|)$ many subformulae, the proof is a consequence of Theorem 20. □

The next theorem deals with the satisfiability problem.

Theorem 30 *The satisfiability problem, hence the problem if a formula φ is in* EXP*.*

Proof. The proof follows easily from Theorem 21, since the satisfiability for a μ-formula φ corresponds to the emptiness problem for $\mathcal{A}_\varphi$. $\square$

Chapter 5

Hierarchy Theorems

"Der moderne Klassenkampf spielt sich heute auf der linken Seite der Autobahn ab." (E. Schachtschnabel)

This chapter proves the strictness of the modal μ-calculus hierarchy. That is, the fact that increasing the syntactical complexity of μ-formulae gives us more expressiveness.

The strictness has first been proven by Bradfield in [12] by using methods of descriptive set theory. Simultaneously, Lenzi in [32] has proven a strictness theorem for the positive μ-calculus, that is, the fragment consisting of all formulae such that the propositional variables appear only positively. Our presentation follows the one of Arnold in [4] where the result follows from a hierarchy theorem for alternating tree automata by using the equivalence to the μ-calculus on binary structures established by Niwinski in [37].

In the first section we prove the strictness of the hierarchy on alternating tree automata induced by the indices. The theorem is proven with a diagonalisation argument on a certain fixpoint. It is the fixpoint of the mapping which assigns to an automaton and a transition system the corresponding strategy tree. Contrary to Arnold, who establishes its existence by applying the fixpoint theorem of Banach, we give a construction of it.

In the second section we apply the result previously established to the modal μ-calculus using the equivalence proved in Chapter 4. Some additional corollaries then give a nice picture of the structure of the modal μ-calculus hierarchy.

5.1 A Hierarchy Theorem for Automata

In this section we prove a hierarchy theorem for alternating tree automata. We assume that all automata are in normal form, which by Lemma 11 is no restriction. Since the automata are in normal form we have the same range of the priority function for two automata of the same complexity. That is, all Σ_n automata have Ω_{Σ_n} as the range of the priority function, and analogously for all Π_n automata the range is Ω_{Π_n}, where Ω_{Σ_n} and Ω_{Π_n} were introduced in Chapter 3. By Remark 18, if all automata are assumed to be in normal form then we just have to distinguish the complexity classes when we are introducing the test automaton. Let us repeat the notions of Σ_n-test automaton T_{Σ_n} and the Π_n-test automaton T_{Π_n} since they are crucial for this section. Contrary to Chapter 3 we introduce them directly.

Every Σ_n-*test automaton* T_{Σ_n} is of the form (where u is a new symbol)

$$\mathsf{T}_{\Sigma_n} = (Q_{\Sigma_n}, \mathsf{P}^{\mathsf{T}}, q_u, \delta_{\Sigma_n}, \Omega)$$

and every Π_n-*test automaton* T_{Π_n} is of the form

$$\mathsf{T}_{\Pi_n} = (Q_{\Pi_n}, \mathsf{P}^{\mathsf{T}}, q_u, \delta_{\Pi_n}, \Omega),$$

where:

- $Q_{\Sigma_n} = \{q_i \mid i \in \Omega_{\Sigma_n}\} \cup \{q_u\}$,
- $Q_{\Pi_n} = \{q_i \mid i \in \Omega_{\Pi_n}\} \cup \{q_u\}$,
- $\mathsf{P}^{\mathsf{T}} = \{c_u, c_1, c_2, \ldots, c_i, \ldots\} \cup \{d_u, d_1, d_2, \ldots, d_i, \ldots\}$,
- for all states $q_j \in Q_{\Sigma_n}$ we have
$$\delta_{\Sigma_n}(q_j) = \bigvee_{q_i \in Q_{\Sigma_n}} (c_i \wedge \Box q_i) \vee \bigvee_{q_i \in Q_{\Sigma_n}} (d_i \wedge \Diamond q_i),$$
- for all states $q_j \in Q_{\Pi_n}$ we have
$$\delta_{\Pi_n}(q_j) = \bigvee_{q_i \in Q_{\Pi_n}} (c_i \wedge \Box q_i) \vee \bigvee_{q_i \in Q_{\Pi_n}} (d_i \wedge \Diamond q_i),$$
- $\Omega(q_j) = j$ if $j \neq u$ and $\Omega(q_u) \uparrow$.

In the following we restrict ourselves to *binary transition systems*, that is systems whose underlying structure is a binary tree. Let us introduce a more compact notation for these systems: In the sequel the symbols $t_1, t_2, \ldots$ stand for binary trees, when no confusion arises we also use them to denote binary transition systems. ϵ is the trivial binary tree (or transition system), that is the one with no states. If t_1 and t_2 are two binary transition systems and a is a subset of the propositional variables then $a(t_1, t_2)$ denotes the binary transition system with root v such that exactly the variables in a are valid there and such that v has two sons v_1, v_2 which generate the sub-transition systems t_1 and t_2. If $a = \{p\}$ we write $p(t_1, t_2)$. If v has only one son (resp. no son) we write $a(t_1, \epsilon)$ (resp. $a(\epsilon, \epsilon)$). Obviously, for any nontrivial binary transition system there are a, t_1, t_2 such that it is of the form $a(t_1, t_2)$.

In Chapter 3 we saw that the acceptance of a transition system $\mathcal{S}$ by an automaton $\mathcal{A}$ can be reduced to the acceptance of a transition system $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$, deduced from the strategy tree $\mathsf{G}_{\mathcal{A},\mathcal{S}}$, by the test automaton $\mathsf{T}_{\mathcal{A}}$ (or if the automata are assumed to be in normal form: T_{Σ_n} or T_{Π_n}). For a binary transition system $\mathcal{S}$ and an automaton $\mathcal{A}$ with initial state q the construction of the strategy tree $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ can be described by a transformation $\mathcal{T}_{\mathcal{A},q}$ from binary trees to binary trees such that

$$\mathcal{T}_{\mathcal{A},q}(\mathcal{S}) \equiv \mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}}).$$

$\mathcal{T}_{\mathcal{A},q}$ is a procedure beginning at the root of a binary transition system which substitutes the nodes it reaches with new ones, depending on the 'actual' internal state of the automaton $\mathcal{A}$. It can be seen as an extension of rewriting rules for words, in the sense that we have binary trees instead of words and that the rewriting depends also on an internal state of the automaton.

Let $\mathcal{A} = (Q, \mathsf{P}, q_\mathsf{I}, \delta, \Omega)$ be an automaton, q a state of $\mathcal{A}$ and let $a(t_1, t_2)$ be a binary transition system. $\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2))$ is obtained by applying recursively the appropriate rule of the following given below:

- If $\delta(q) = q' \wedge q''$ and $\Omega(q) = i \in \omega$ then

$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_i(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \mathcal{T}_{\mathcal{A},q''}(a(t_1, t_2))),$$

- if $\delta(q) = q' \vee q''$ and $\Omega(q) = i \in \omega$ then

$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv d_i(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \mathcal{T}_{\mathcal{A},q''}(a(t_1, t_2))),$$

- if $\delta(q) = q'$ and $\Omega(q) = i \in \omega$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_i(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \epsilon),$$
- if $\delta(q) = \Diamond q'$, $\Omega(q) = i \in \omega$ and $t_1 \not\equiv \epsilon$ or $t_2 \not\equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv d_i(\mathcal{T}_{\mathcal{A},q'}(t_1), \mathcal{T}_{\mathcal{A},q''}(t_2)),$$
- if $\delta(q) = \Box q'$, $\Omega(q) = i \in \omega$ and $t_1 \not\equiv \epsilon$ or $t_2 \not\equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_i(\mathcal{T}_{\mathcal{A},q'}(t_1), \mathcal{T}_{\mathcal{A},q''}(t_2)),$$
- if $\delta(q) = q' \wedge q''$ and $\Omega(q) \uparrow$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_u(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \mathcal{T}_{\mathcal{A},q''}(a(t_1, t_2))),$$
- if $\delta(q) = q' \vee q''$ and $\Omega(q) \uparrow$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv d_u(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \mathcal{T}_{\mathcal{A},q''}(a(t_1, t_2))),$$
- if $\delta(q) = q'$ and $\Omega(q) \uparrow$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_u(\mathcal{T}_{\mathcal{A},q'}(a(t_1, t_2)), \epsilon),$$
- if $\delta(q) = \Diamond q'$, $\Omega(q) \uparrow$ and $t_1 \not\equiv \epsilon$ or $t_2 \not\equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv d_u(\mathcal{T}_{\mathcal{A},q'}(t_1), \mathcal{T}_{\mathcal{A},q''}(t_2)),$$
- if $\delta(q) = \Box q'$, $\Omega(q) \uparrow$ and $t_1 \not\equiv \epsilon$ or $t_2 \not\equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv c_u(\mathcal{T}_{\mathcal{A},q'}(t_1), \mathcal{T}_{\mathcal{A},q''}(t_2)),$$
- if $\delta(q) = \Diamond q'$ and $t_1 \equiv t_2 \equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv \emptyset,$$
- if $\delta(q) = \Box q'$ and $t_1 \equiv t_2 \equiv \epsilon$ then
$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv \mathsf{P}^{\mathsf{T}},$$

- if $\delta(q) = \top$, or $\delta(q) = p$ and $p \in a$ then

$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv \mathsf{P}^{\mathsf{T}},$$

- if $\delta(q) = \bot$, or $\delta(q) = p$ and $p \notin a$ then

$$\mathcal{T}_{\mathcal{A},q}(a(t_1, t_2)) \equiv \emptyset.$$

Let $\mathcal{S}$ be a binary transition system with root s and let s' be a state in $\mathcal{S}$ which generates the sub-transition system $\mathcal{S}'$. We say that $\mathcal{T}_{\mathcal{A},q}$ reaches s' in one step if there is a q' such that $\mathcal{T}_{\mathcal{A},q'}(\mathcal{S}')$ appears on the right hand side of the defining clause for $\mathcal{T}_{\mathcal{A},q}(\mathcal{S})$. Notice, that s' is either s or a son of s. All the *reachable* states are then given by the transitive closure of the 'reachable in one step' relation. The next lemma follows from the definitions above.

Lemma 31 *Let $\mathcal{S}$ be a binary transition system whose underlying binary tree has depth m and let $\mathcal{A}$ be an automaton with state q. If the procedure $\mathcal{T}_{\mathcal{A},q}$ reaches at least one leaf of $\mathcal{S}$ then $\mathcal{T}_{\mathcal{A},q}(\mathcal{S})$ has at least depth m.*

By definition, for all binary transition systems $\mathcal{S}$, we see that $\mathcal{T}_{\mathcal{A},q}(\mathcal{S})$ is isomorphic to the transition system $\mathcal{S}(\mathsf{G}_{\mathcal{A},\mathcal{S}})$ if q is the initial state of $\mathcal{A}$. The next lemma is just a reformulation of Theorem 19.

Lemma 32 *For any binary transition system $\mathcal{S}$ and any automaton $\mathcal{A}$ with initial state q, we have:*

- *If $\mathcal{A} \in \Sigma_n$:*
$$\mathcal{S} \in \|\mathcal{A}\| \quad \Leftrightarrow \quad \mathcal{T}_{\mathcal{A},q}(\mathcal{S}) \in \|T_{\Sigma_n}\|.$$
- *If $\mathcal{A} \in \Pi_n$:*
$$\mathcal{S} \in \|\mathcal{A}\| \quad \Leftrightarrow \quad \mathcal{T}_{\mathcal{A},q}(\mathcal{S}) \in \|T_{\Pi_n}\|.$$

The following example illustrates how this transformation on binary transition systems works.

Example 33 *Figure 5.1 shows $\mathcal{S}$, $\mathcal{T}_{\mathcal{A},q_1}(\mathcal{S})$ and $\mathcal{T}_{\mathcal{A},q_1}(\mathcal{T}_{\mathcal{A},q_1}(\mathcal{S})) =: \mathcal{T}^2_{\mathcal{A},q_1}(\mathcal{S})$. $\mathcal{S}$ is a binary transition system over a set of propositional variables $\{p_1, p_2\}$ of the form $p_1(t_1, t_2)$, where $t_1 \equiv p_1(\epsilon, \epsilon)$ and $t_2 \equiv p_2(\epsilon, \epsilon)$. Furthermore $\mathcal{A} = (\{q, q_2, q_3, q_4\}, \{p_1, p_2\}, \delta, q_1, \Omega)$ is an alternating Π_2-automaton with:*

- $\delta(q_1) = \Box q_2$,

- $\delta(q_2) = q_4 \vee q_3$,
- $\delta(q_3) = p_1$,
- $\delta(q_4) = \Diamond q_1$,
- $\Omega(q_1) = \Omega(q_4) = 1$ *and*
- $\Omega(q_2) = \Omega(q_3) = 2$.

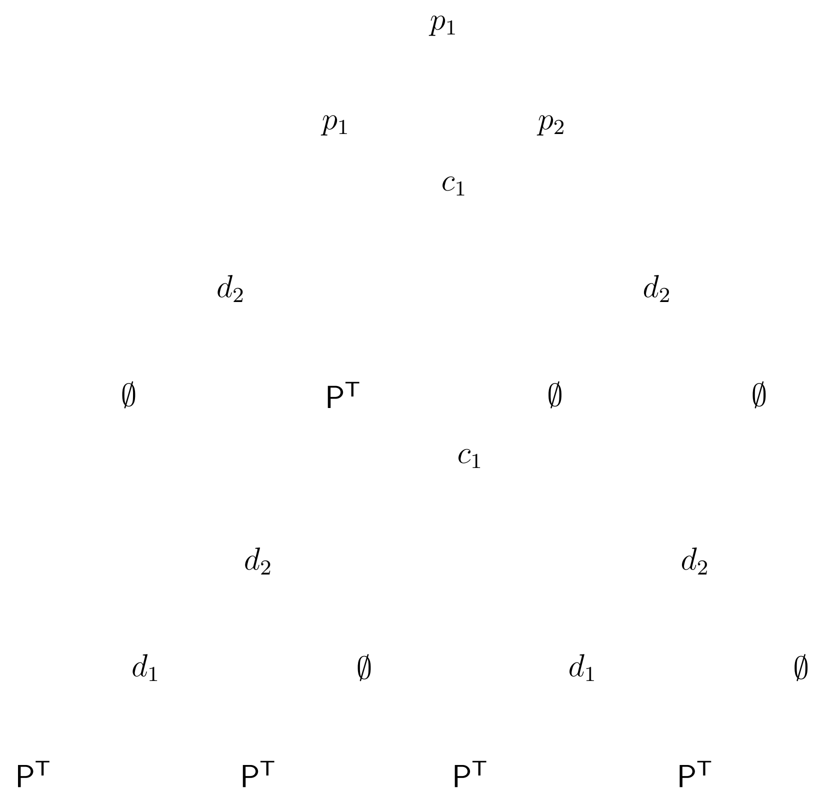

Figure 5.1: $\mathcal{S}$, $\mathcal{T}_{\mathcal{A},q_1}(\mathcal{S})$ and $\mathcal{T}^2_{\mathcal{A},q_1}(\mathcal{S})$

The proof of the next lemma needs the notion of *limit tree* . Suppose we have a sequence of trees $(t_n)_{n\in\omega}$ which is monotone, that is, the following holds:

For all $m \in \omega$ there is a $n(m) \in \omega$ such that for all $n', n'' \geq n(m)$ the trees $t_{n'}$ and $t_{n''}$ are identical up to depth m.

In that case we can define the limit tree $\lim((t_n)_{n\in\omega})$ of the sequence $(t_n)_{n\in\omega}$ such that for all natural numbers m the limit tree is identical to $t_{n(m)}$ up to depth m. Notice that $\lim((t_n)_{n\in\omega})$ is well defined since $(t_n)_{n\in\omega}$ is monotone.

Lemma 34 *Let $\mathcal{A} \in \Sigma_n (\in \Pi_n)$ be an automaton. There is an automaton $\mathcal{A}' \in \Sigma_n (\in \Pi_n)$ with initial state q' and a transition system $F_{\mathcal{A}',q}$ such that*

$$\|\mathcal{A}\| = \|\mathcal{A}'\| \quad \text{and} \quad \mathcal{T}_{\mathcal{A}',q'}(F_{\mathcal{A}',q'}) \equiv F_{\mathcal{A}',q'}.$$

Proof. Let $\mathcal{A}$ be an automaton of the form $(Q, \mathsf{P}, q_{\mathsf{I}}, \delta, \Omega)$. For the semantically equivalent automaton $\mathcal{A}'$ we take a new state q' and set

$$\mathcal{A}' = (Q \cup \{q'\}, \mathsf{P}, q', \delta', \Omega')$$

where δ' is equal to δ on Q and $\delta'(q') = q_{\mathsf{I}} \wedge q_{\mathsf{I}}$; and where Ω' is equal to Ω on Q and $\Omega'(q') \uparrow$. It can easily be seen that $\|\mathcal{A}\| = \|\mathcal{A}'\|$ and that if $\mathcal{A} \in \Sigma_n (\in \Pi_n)$ then $\mathcal{A}' \in \Sigma_n (\in \Pi_n)$.

Claim: Given two binary transition systems $\mathcal{S}$ and $\mathcal{S}'$, which are identical upto depth m, then $\mathcal{T}_{\mathcal{A}',q'}(\mathcal{S})$ and $\mathcal{T}_{\mathcal{A}',q'}(\mathcal{S}')$ are identical up to depth $m+1$.

Let us prove the claim. The procedure $\mathcal{T}_{\mathcal{A}',q'}$ works down the two transition systems beginning from the root, and since both systems are equal until depth m, the same sub procedures are executed until then. If the procedure does not reach a node with depth m then $\mathcal{T}_{\mathcal{A}',q'}(\mathcal{S})$ and $\mathcal{T}_{\mathcal{A}',q'}(\mathcal{S}')$ are identical. And and for this case we get the claim. In the second case the procedure reaches a node with depth m. Then, by Lemma 31 we know that for the original automaton $\mathcal{A}$ the trees $\mathcal{T}_{\mathcal{A},q}(\mathcal{S})$ and $\mathcal{T}_{\mathcal{A},q}(\mathcal{S}')$ are identical up to depth m. Further, if we look to the construction of $\mathcal{A}'$ we see that for any transition system of the form $a(t_1, t_2)$

$$\mathcal{T}_{\mathcal{A}',q'}(a(t_1,t_2)) \equiv c_u(\mathcal{T}_{\mathcal{A},q}(a(t_1,t_2)), \mathcal{T}_{\mathcal{A},q}(a(t_1,t_2))).$$

Putting the last two remarks together we also get the claim for the second case.

Let us now construct the fixpoint $F_{\mathcal{A}',q'}$. We first define a monotone sequence $(t_n)_{n\in\omega}$ of binary transition systems: t_1 is the binary transition system $\mathcal{S}$ of the form $c_u(\epsilon, \epsilon)$ and

$$t_{n+1} \equiv \mathcal{T}_{\mathcal{A}',q'}(t_n).$$

By induction on n, with the help of the claim, we can easily prove that for all n the trees t_n and t_{n+1} are identical up to depth $n+1$. From that, the monotonicity of $(t_n)_{n\in\omega}$ easily follows. We set

$$F_{\mathcal{A}',q'} \equiv \lim((t_n)_{n\in\omega}).$$

By definition of the limit tree we see that $F_{\mathcal{A}',q'}$ is a fixpoint of $\mathcal{T}_{\mathcal{A}',q'}$, and this completes the proof. □

We now prove the hierarchy theorem for alternating tree automata.

Theorem 35 *For all natural numbers n we have:*

1. $\Sigma^{\mathbf{TR}}_{n+1} \neq \Sigma^{\mathbf{TR}}_{n}$,
2. $\Pi^{\mathbf{TR}}_{n+1} \neq \Pi^{\mathbf{TR}}_{n}$.

Proof. 1. We prove the contrapositive. Suppose $\Sigma^{\mathbf{TR}}_{n+1} = \Sigma^{\mathbf{TR}}_{n}$, by definition it follows that $\Pi^{\mathbf{TR}}_{n} \subseteq \Sigma^{\mathbf{TR}}_{n}$. With Lemma 12 we get

$$\mathbf{TR} - \|T_{\Sigma_n}\| \in \Sigma^{\mathbf{TR}}_{n}.$$

So, there exists a Σ_n-automaton $\mathcal{A}$ such that $\mathbf{TR} - \|T_{\Sigma_n}\| = \|\mathcal{A}\|$. By Lemma 32 and Lemma 34 there is a semantically equivalent automaton $\mathcal{A}' \in \Sigma_n$ and a transition system $F_{\mathcal{A}'}$ such that

$$F_{\mathcal{A}'} \in \|T_{\Sigma_n}\| \quad \Leftrightarrow \quad F_{\mathcal{A}'} \in \|\mathcal{A}'\|.$$

Since $\|\mathcal{A}'\| = \|\mathcal{A}\| = \mathbf{TR} - \|T_{\Sigma_n}\|$ we get

$$F_{\mathcal{A}'} \in \|T_{\Sigma_n}\| \quad \Leftrightarrow \quad F_{\mathcal{A}'} \in \mathbf{TR} - \|T_{\Sigma_n}\|$$

and hence a contradiction, which proves part 1 of the theorem.

2. can be proven similarly to part 1. □

5.2 A Hierarchy Theorem for the μ-Calculus

We apply Theorem 35 to the modal μ-calculus, by using the fact that the index of an automaton corresponds to the alternation depth of a μ-formula, and we get the following theorem.

Theorem 36 *For all natural numbers n we have:*

1. $\Sigma^{\mu\mathbf{TR}}_{n} \neq \Sigma^{\mu\mathbf{TR}}_{n+1}$,
2. $\Pi^{\mu\mathbf{TR}}_{n} \neq \Pi^{\mu\mathbf{TR}}_{n+1}$.

Proof. 1. We prove the contrapositive. Suppose $\Sigma_n^{\mu\mathbf{TR}} = \Sigma_{n+1}^{\mu\mathbf{TR}}$, since by Lemma 2 we have for all natural numbers m

$$\Pi_m^{\mu\mathbf{TR}} = \{\mathbf{TR} - \|\varphi\| \mid \|\varphi\| \in \Sigma_m^{\mu\mathbf{TR}}\}$$

we get

$$\Pi_n^{\mu\mathbf{TR}} = \Pi_{n+1}^{\mu\mathbf{TR}}.$$

Then, since $\Sigma_{n+2}^{\mu} = \mu(\Pi_{n+1}^{\mu})$ and $\Sigma_{n+1}^{\mu} = \mu(\Pi_n^{\mu})$ we can derive $\Sigma_{n+2}^{\mu\mathbf{TR}} = \Sigma_{n+1}^{\mu\mathbf{TR}}$ and we get

$$\Sigma_n^{\mu\mathbf{TR}} = \Sigma_{n+1}^{\mu\mathbf{TR}} = \Sigma_{n+2}^{\mu\mathbf{TR}}.$$

Let $\mathcal{A}$ be an automaton such that $\|\mathcal{A}\| \in \Sigma_{n+1}^{\mathbf{TR}}$ by Theorem 27 there is μ-formula φ such that $\|\mathcal{A}\| = \|\varphi\| \in \Sigma_{n+2}^{\mu\mathbf{TR}}$ and with the equivalence shown above $\|\mathcal{A}\| = \|\varphi\| \in \Sigma_n^{\mu\mathbf{TR}}$. By Theorem 23 there is a Σ_n-automaton $\mathcal{A}'$ equivalent to φ and so we get $\|\mathcal{A}\| = \|\varphi\| = \|\mathcal{A}'\| \in \Sigma_n^{\mathbf{TR}}$. Since $\mathcal{A}$ was chosen arbitrary we get

$$\Sigma_{n+1}^{\mathbf{TR}} \subseteq \Sigma_n^{\mathbf{TR}}.$$

Using the fact that the other inclusion is trivial we have

$$\Sigma_n^{\mathbf{TR}} = \Sigma_{n+1}^{\mathbf{TR}}$$

which contradicts Theorem 35.
2. is proven similarly to part 1. $\square$

The theorem shows us that no finite part of the modal μ-hierarchy has the expressiveness of the hole calculus. In this sense, it can be seen as the evidence that the modal μ-calculus hierarchy is strict. Let us prove two corollaries before we illustrate the modal μ-calculus hierarchy.

Corollary 37 *For all natural numbers $n > 0$ we have*

$$\Sigma_n^{\mu\mathbf{TR}} \neq \Pi_n^{\mu\mathbf{TR}}.$$

Proof. We prove the contrapositive. Suppose that we have $\Sigma_n^{\mu\mathbf{TR}} = \Pi_n^{\mu\mathbf{TR}}$ for an $n > 0$. Now, let $\|\varphi\| \in \Sigma_{n+1}^{\mu\mathbf{TR}}$. So there is a $\psi \in \Sigma_{n+1}^{\mu}$ such that $\|\varphi\| = \|\psi\|$. Since $\Sigma_{n+1}^{\mu} = \mu(\Pi_n^{\mu})$, by definition of the operator μ there are formulae $\psi_1, \ldots, \psi_m, \neg\psi_{m+1}, \ldots, \neg\psi_{m+k}$ such that all $\psi_i \in \Pi_n^{\mu}$ and such that ψ is obtained from these formulae using $\wedge, \vee, \mu, \Box, \Diamond$ and substitution. Using this representation of ψ we show that the formula is equivalent to a formula

$\psi' \in \Sigma_n^\mu$. Hence we have $\psi \in \Sigma_n^{\mu\mathbf{TR}}$, which is a contradiction to Theorem 36, since we have $\|\varphi\| = \|\psi\|$.

So, let us show the equivalence of ψ to a $\psi' \in \Sigma_n^\mu$. In the construction of ψ we started from formulae $\psi_1, \ldots, \psi_m, \neg\psi_{m+1}, \ldots, \neg\psi_{m+k}$ such that all $\psi_i \in \Pi_n^\mu$. Since by assumption $\Sigma_n^{\mu\mathbf{TR}} = \Pi_n^{\mu\mathbf{TR}}$ for all $i \in \{1, \ldots, m\}$ there are formulae $\psi_i' \in \Sigma_n^\mu$ which are equivalent to ψ_i. Further, by Lemma 2 for all $i \in \{m+1, \ldots, m+k\}$ there are formulae $\psi_i' \in \Sigma_n^\mu$ equivalent to $\neg\psi_i$. Hence ψ is equivalent to a formula constructed analogously starting from formulae $\psi_1', \ldots, \psi_m', \psi_{m+1}', \ldots, \psi_{m+k}'$, where all $\psi_i' \in \Sigma_n^\mu$, that is ψ is obtained from the ψ_i' by using $\wedge, \vee, \mu, \Box, \Diamond$ and substitution. Since $n > 0$ we have $\Sigma_n^\mu = \mu(\Pi_{n-1}^\mu)$, and, by definition of the operator μ, Σ_n^μ is closed under composition with $\wedge, \vee, \mu, \Box, \Diamond$ and substitution. That means that $\psi' \in \Sigma_n^\mu$. Since ψ' is equivalent to ψ the proof is completed. $\Box$

Corollary 38 *For all natural numbers n we have:*

1. $\Sigma_n^{\mu\mathbf{TR}} \subsetneq \Pi_{n+1}^{\mu\mathbf{TR}}$,
2. $\Pi_n^{\mu\mathbf{TR}} \subsetneq \Sigma_{n+1}^{\mu\mathbf{TR}}$.

Proof. 1. We prove the contrapositive. Suppose that $\Sigma_n^{\mu\mathbf{TR}} \subsetneq \Pi_{n+1}^{\mu\mathbf{TR}}$ does not hold. Since it is clear that $\Sigma_n^{\mu\mathbf{TR}} \subseteq \Pi_{n+1}^{\mu\mathbf{TR}}$ holds we then have $\Sigma_n^{\mu\mathbf{TR}} = \Pi_{n+1}^{\mu\mathbf{TR}}$. Now, suppose we have $\|\varphi\| \in \Sigma_{n+1}^{\mu\mathbf{TR}}$, by Lemma 2 we have $\|\neg\varphi\| \in \Pi_{n+1}^{\mu\mathbf{TR}}$ and with our assumption we get $\|\neg\varphi\| \in \Sigma_n^{\mu\mathbf{TR}}$ and by Lemma 2 $\|\varphi\| \in \Pi_n^{\mu\mathbf{TR}}$. Since φ was arbitrary we have $\Pi_n^{\mu\mathbf{TR}} = \Sigma_{n+1}^{\mu\mathbf{TR}}$. All together, this gives to us

$$\Sigma_n^{\mu\mathbf{TR}} = \Pi_{n+1}^{\mu\mathbf{TR}} \quad \text{and} \quad \Pi_n^{\mu\mathbf{TR}} = \Sigma_{n+1}^{\mu\mathbf{TR}}.$$

But then we easily get

$$\Sigma_{n+1}^{\mu\mathbf{TR}} \subseteq \Pi_{n+1}^{\mu\mathbf{TR}} \quad \text{and} \quad \Pi_{n+1}^{\mu\mathbf{TR}} \subseteq \Sigma_{n+1}^{\mu\mathbf{TR}}$$

which is not the case by Corollary 37.

2. is proven similarly. $\Box$

Figure 5.2 illustrates the structure of the modal μ-calculus hierarchy.

$$
\begin{tikzcd}
 & \mathcal{L}_\mu^{\mathbf{TR}} & \\
\Pi_3^{\mu\mathbf{TR}} & \neq & \Sigma_3^{\mu\mathbf{TR}} \\
\Pi_2^{\mu\mathbf{TR}} \arrow[u] & \neq & \Sigma_2^{\mu\mathbf{TR}} \arrow[u] \\
\Pi_1^{\mu\mathbf{TR}} \arrow[u] & \neq & \Sigma_1^{\mu\mathbf{TR}} \arrow[u] \\
 & \Sigma_0^{\mu\mathbf{TR}} = \Pi_0^{\mu\mathbf{TR}} &
\end{tikzcd}
$$

Figure 5.2: The modal μ-calculus hierarchy. Notice that the arrows stand for strict inclusion and that $\mathcal{L}_\mu^{\mathbf{TR}} = \{\|\varphi\| \mid \varphi \in \mathcal{L}_\mu\}$.

Chapter 6

Completeness

“I have a dream...” (Martin Luther King)

In this section we prove the completeness of Kozen’s axiomatisation KOZ for two fragments of the modal μ-calculus. In this sense we supply two partial completeness results.

When Kozen introduced his axiomatisation KOZ in [31] he proved completeness for the fragment of the aconjunctive formulae. Completeness for the whole μ-calculus remained as an open problem.

Ten years later Walukiewicz in [51] proposed another deduction system for the modal μ-calculus and proved its completeness. And recently, in [52], he proved the completeness of KOZ using deep results of automata theory. Nevertheless, the question if there is a direct proof, that is, a proof which does not use automata theory, still remained open.

In this chapter we present an attempt of finding a ‘more direct’ completeness proof in form of two partial completeness results.

6.1 Partial Completeness Results for KOZ

In this section we prove completeness for two fragments of modal μ-calculus. The results are established by construction of finite canonical models, extending the proof method used by Fagin, Halpern, Moses and Vardi in [15] to prove completeness of the modal logic with a common knowledge operator.

Let us first define the two fragments:

1. The fragment $\mathcal{F}^1_\mu$ consists of all formulae φ, with the property that if φ has a subformula of the form $\nu X.\alpha$ then $\mathsf{nnf}(\alpha)$ has no subformula of the form $\Diamond\beta$ (that is $\mathsf{nnf}(\alpha)$ has no diamonds).

2. The fragment $\mathcal{F}^2_\mu$ consists of all formulae φ, such that $\mathsf{nnf}(\varphi)$ doesn't contain a subformula of the form $\mu X.\psi$.

The following three examples show some basic differences of our fragments:

1. $\nu X.\mu Y.((\Diamond X \wedge \Box Y) \vee P) \notin \mathcal{F}^1_\mu, \mathcal{F}^2_\mu$

2. $\nu X.(\Box X \vee (\nu Y.\Diamond Y \vee \neg\mu X.\Box X)) \notin \mathcal{F}^1_\mu \in \mathcal{F}^2_\mu$

3. $\mu X.(\Box\varphi \vee \nu Y.(Y \vee \Box(X \vee Q))) \vee \Diamond P \in \mathcal{F}^1_\mu \notin \mathcal{F}^2_\mu$

The following corollary of Theorem 9 will be used to prove the results.

Corollary 39 *Let $\mathcal{S}$ be a transition system of the form (S, R, λ) and $S' \subseteq S$. Given a pre-model for $\mathcal{S}$ with root $\alpha(X)[\beta/X]@s$ such that for each branch we have: If it contains a point $X[\beta/X]@s'$, then $s' \in S'$, if it does not contain a point of the form $X[\beta/X]@s'$, then it is closed. In this case we have*

$$s \in \|\alpha(S')\|_{\mathcal{S}}.$$

Proof. We define a new formula $\alpha(X)[p'/X]$ where p' is a new propositional variable. If we set

$$\lambda(p') = S',$$

it is easy to see, that we can construct a well-founded pre-model with root $\alpha(X)[p'/X]$, hence $s \in \|\alpha(X)[p'/X]\|_{\mathcal{S}}$ and so by definition $s \in \|\alpha(S')\|_{\mathcal{S}}$. □

We are now able to prove the completeness theorems. First for the fragment $\mathcal{F}^1_\mu$ and then for the fragment $\mathcal{F}^2_\mu$.

6.1.1 Completeness for the Fragment $\mathcal{F}^1_\mu$

For the completeness of this fragment, we prove that for each consistent formula $\varphi \in \mathcal{F}^1_\mu$, that is a φ such that $\mathsf{KOZ} \nvdash \neg\varphi$, there is a model and a state in the model which satisfies φ, by contraposition we get for all formulae $\varphi \in \mathcal{F}^1_\mu$

$$\models \varphi \;\;\Rightarrow\;\; \mathsf{KOZ} \vdash \varphi.$$

Before we do the model construction, we introduce some basic notions: A formula φ is called *consistent* (for the calculus KOZ) if KOZ $\not\vdash \neg\varphi$. A *set of formulae S is consistent* if for all finite subsets $\{\varphi_1, \ldots, \varphi_n\} \subseteq S$ the formula $\bigvee_{i\in\{1,\ldots,n\}} \varphi_i$ is consistent. A set of formulae is *maximal consistent*, if it is not a proper subset of an other consistent set.

Given a consistent set, the existence of a maximal consistent superset follows from the following lemma.

Lemma 40 *Given a consistent set of formulae M, there is a maximal consistent set M' such that $M \subseteq M'$.*

Proof. First we fix an enumeration $\{\alpha_i \mid i \in \omega\}$ of all formulae in $\mathcal{L}_\mu$. Then, we construct a (non strict) ascending chain $\{M_i \mid i \in \omega \cup \{0\}\}$ of consistent sets of formulae such that $M_0 = M$ and for all $i \in \omega \cup \{0\}$

$$M_{i+1} = \begin{cases} M_i \cup \{\alpha_{i+1}\} & \text{if } M_i \cup \{\alpha_{i+1}\} \text{ is consistent,} \\ M_i & \text{if } M_i \cup \{\alpha_{i+1}\} \text{ is not consistent.} \end{cases}$$

Let us now show that $M' = \bigcup_{i\in\omega} M_i$ is a maximal consistent set fulfilling the requirements. First, we show that M' is consistent: This follows from the fact that each finite subset M'' of M' is consistent since M'' is already contained in an M_i for an $i \in \omega$. Secondly, we show that M' is maximal: For if we assume that M' is not maximal there is a formula $\beta \notin M'$ such that $M' \cup \{\beta\}$ is consistent. Now, since we have an enumeration of the formulae, there is an $i \in \omega$ such that $\beta \equiv \alpha_i$. By construction of M' we must have that $M_{i-1} \cup \{\alpha_i\}$ is not consistent, hence also M' cannot be consistent. Which is a contradiction. This proves maximality and the lemma. □

The following folklore lemma states basic facts about maximal consistent sets of formulae.

Lemma 41 *Let M be a maximal consistent set of formulae. For all formulae φ and ψ we have:*

- $\varphi \vee \psi \in M \;\Leftrightarrow\; \varphi \in M$ or $\psi \in M$,
- $\varphi \wedge \psi \in M \;\Leftrightarrow\; \varphi \in M$ and $\psi \in M$,
- $\varphi \to \psi \in M$ and $\varphi \in M \;\Rightarrow\; \psi \in M$,
- KOZ $\vdash \varphi \;\Rightarrow\; \varphi \in M$,
- $\varphi \in M$ *or* $\neg\varphi \in M$,
- $\varphi \in M \;\Leftrightarrow\; \neg\varphi \notin M$.

The definition of the Fischer-Ladner closure is very close to the original one, given by the two authors in [16]. Given a formula φ, we define the *Fischer-Ladner closure* $FL(\varphi)$ to be the smallest set containing φ such that:

- If $\psi \in FL(\varphi)$ and α is a subformula of ψ then $\alpha \in FL(\varphi)$,
- if $\alpha \in FL(\varphi)$ and $\alpha \not\equiv \neg\beta$ for any β then $\neg\alpha \in FL(\varphi)$,
- if $\nu X.\alpha \in FL(\varphi)$ then $\alpha(\nu X.\alpha) \in FL(\varphi)$,
- if $\mu X.\alpha \in FL(\varphi)$ then $\alpha(\mu X.\alpha) \in FL(\varphi)$,
- if $\alpha \in FL(\varphi)$ then $\mathsf{nnf}(\alpha) \in FL(\varphi)$.

The cardinality of the Fischer-Ladner closure $|FL(\varphi)|$ is in $\mathcal{O}(|\varphi|)$ where $|\varphi|$ is the length of the formula, that is, the number of symbols in φ. Given a consistent formula $\varphi \in \mathcal{F}^1_\mu$ we define a model, *the canonical model for* φ, which satisfies it.

The canonical model $\mathcal{CM}_\varphi = (S, R, \lambda)$ for φ is given by S, R and λ defined as follows:

- $S = \{M \cap FL(\varphi) \mid M \text{ is a maximal consistent set of formulae}\}$,
- $R = \{(M, M') \mid M/\Box \subseteq M'\}$, where $M/\Box = \{\psi \mid \Box\psi \in M\}$,
- $\lambda(p) = \{M \mid p \in M\}$.

Since $FL(\varphi)$ is a finite set of formulae, each state consists of finitely many formulae and so the constructed model consists of finitely many states M. In the following for all states M we define φ_M as

$$\varphi_M \equiv \bigwedge_{\psi \in M} \psi.$$

Further, if S' is a set of states of the canonical model then $\phi_{S'}$ is defined as

$$\phi_{S'} \equiv \bigvee_{M \in S'} \varphi_M.$$

Now, some basic properties of the canonical model.

Lemma 42 *Let $\mathcal{CM}_\varphi = (S, R, \lambda)$ be the canonical model for a formula φ. We have:*

1.

$$\mathsf{KOZ} \vdash \phi_S$$

2.

$$\mathsf{KOZ} \vdash \varphi_M \to \Box\phi_{R(M)}.$$

Proof.1. Suppose $\mathsf{KOZ} \nvdash \phi_S$. Since $\phi_S \equiv \bigvee_{M\in S}\varphi_M$ we have that $\bigwedge_{M\in S}\neg\varphi_M$ is consistent. Let M^{max} be a maximal consistent set containing $\bigwedge_{M\in S}\neg\varphi_M$. By Lemma 41, for all $M \in S$, M^{max} contains $\neg\psi$, for a $\psi \in M$. But this contradicts the fact that S consists of all intersections of a maximal consistent set and $FL(\varphi)$.

2. First we prove for all $M'' \notin R(M)$

$$\mathsf{KOZ} \vdash \varphi_M \to (\Box\neg\varphi_{M''}). \quad (*)$$

Let $M'' \notin R(M)$, so there is a $\Box\psi \in M$ with $\psi \notin M''$, by Lemma 41 $\neg\psi \in M''$. So we get $\mathsf{KOZ} \vdash \bigwedge_{\psi\in M/\Box}\psi \to \neg\varphi_{M''}$. Hence

$$\mathsf{KOZ} \vdash \Box(\bigwedge_{\psi\in M/\Box}\psi) \to \Box\neg\varphi_{M''},$$

and since $\Box$ distributes over conjunction, we have shown $(*)$. So, we easily get

$$\mathsf{KOZ} \vdash \varphi_M \to (\Box\bigwedge_{M''\notin R(M)}\neg\varphi_{M''}).$$

Since by the first part of this lemma we have

$$\mathsf{KOZ} \vdash \bigwedge_{M''\notin R(M)}\neg\varphi_{M''} \to \bigvee_{M'\in R(M)}\varphi_{M'}$$

we get the desired result. □

The next result is used essentially in the completeness theorem for the fragment. It cannot be proven for the whole language.

Proposition 43 *Let $\mathcal{CM}_\varphi = (S, R, \lambda)$ be the canonical model of a formula φ, let $\alpha(X_1, \ldots, X_n)$ be a formula such that* $\mathsf{nnf}(\alpha)$ *has no diamonds and let $S_1, \ldots, S_n$ be subsets of S. If $M \in \|\alpha(S_1, \ldots, S_n)\|_{\mathcal{CM}_\varphi}$ then*

$$\mathsf{KOZ} \vdash \varphi_M \to \alpha(\phi_{S_1}, \ldots, \phi_{S_n}).$$

Proof. Let us first define for all formulae α a rank $\mathsf{rn}(\alpha)$. We assume that $\mathcal{CM}_\varphi$ has $N \in \omega$ many states. $\mathsf{rn}(\alpha)$ is defined inductively as follows:

- If $\alpha \equiv p, \top, \bot$, where p is a propositional variable, then $\mathsf{rn}(\alpha) = 0$,
- if $\alpha \equiv \beta \wedge \gamma, \beta \vee \gamma$ then $\mathsf{rn}(\alpha) = \max\{\mathsf{rn}(\beta), \mathsf{rn}(\gamma)\} + 1$,
- if $\alpha \equiv \Box\beta, \Diamond\beta$ then $\mathsf{rn}(\alpha) = \mathsf{rn}(\beta) + 1$,
- if $\alpha \equiv \nu X.\beta$ then $\mathsf{rn}(\alpha) = \mathsf{rn}(\bigwedge_{n\leq N} \beta^n(X)[\top/X]) + 1$,
- if $\alpha \equiv \mu X.\beta$ then $\mathsf{rn}(\alpha) = \mathsf{rn}(\bigvee_{n\leq N} \beta^n(X)[\bot/X]) + 1$.

The proposition is proved by induction on $\mathsf{rn}(\alpha)$. Since we have

$$\mathsf{KOZ} \vdash \alpha \leftrightarrow \mathsf{nnf}(\alpha)$$

we can assume, that α is in negation normal form.

$\mathsf{rn}(\alpha) = 0$: If $\alpha \equiv \bot, \top$ then the implication is trivial. If $\alpha \equiv p$, where p is a propositional variable, then $M \in S_i$, and trivially

$$\mathsf{KOZ} \vdash \varphi_M \rightarrow \phi_{S_i}.$$

$\mathsf{rn}(\alpha) > 0$: We do the induction step by case distinction on the structure of α:

- $\alpha \equiv \neg p, \beta \wedge \gamma, \beta \vee \gamma$: These cases go through straightforward and are left to the reader.
- $\alpha \equiv \Box\beta$: By definition of validity for all $\overline{M} \in R(M)$ we have $\overline{M} \in \|\beta(S_1, \dots, S_n)\|_{\mathcal{CM}_\varphi}$. Since $\mathsf{rn}(\beta) < \mathsf{rn}(\alpha)$ we can apply the induction hypothesis and we get

  $$\mathsf{KOZ} \vdash \varphi_{\overline{M}} \rightarrow \beta(\phi_{S_1}, \dots, \phi_{S_n})$$

  for all $\overline{M} \in R(M)$. That gives us

  $$\mathsf{KOZ} \vdash \phi_{R(M)} \rightarrow \beta(\phi_{S_1}, \dots, \phi_{S_n})$$

  and

  $$\mathsf{KOZ} \vdash \Box\phi_{R(M)} \rightarrow \Box\beta(\phi_{S_1}, \dots, \phi_{S_n}).$$

  With Lemma 42.2 we get the desired result.

- $\alpha \equiv \mu Y.\beta(X_1, \ldots, X_n)$: We have $M \in \|\mu Y.\beta(S_1, \ldots, S_n)\|_{\mathcal{CM}_\varphi}$. Since the canonical model has N many states by Lemma 24.2 we have a natural number $n \leq N$ such that $M \in \|\beta^n(Y)[\bot/Y](S_1, \ldots, S_n)\|_{\mathcal{CM}_\varphi}$. We have $\mathsf{rn}(\alpha) > \mathsf{rn}(\beta^n(Y)[\bot/Y])$ and so by induction hypothesis we get
$$\mathsf{KOZ} \vdash \varphi_M \to \beta^n(Y)(\phi_{S_1}, \ldots, \phi_{S_n})[\bot/Y].$$
Since for all formulae α, with X appearing only positively, and all $n \in \omega$, we can prove
$$\mathsf{KOZ} \vdash \alpha^n(Y)[\bot/Y] \to \mu Y.\alpha,$$
we get the desired result.

- $\alpha \equiv \nu Y.\beta(X_1, \ldots, X_n)$: We abbreviate $\|\nu Y.\beta(S_1, \ldots, S_n)\|_{\mathcal{CM}_\varphi}$ with S_0. By assumption we have $M \in S_0$. By definition we get
$$M \in \|\beta(S_1, \ldots, S_n, S_0)\|_{\mathcal{CM}_\varphi}.$$
Since $\mathsf{rn}(\alpha) > \mathsf{rn}(\beta(X_1, \ldots, X_n, Y))$ for all $M'' \in S_0$ we have by induction hypothesis
$$\mathsf{KOZ} \vdash \varphi_{M''} \to \beta(\phi_{S_1}, \ldots, \phi_{S_n}, \phi_{S_0})$$
and so
$$\mathsf{KOZ} \vdash \phi_{S_0} \to \beta(\phi_{S_1}, \ldots, \phi_{S_n}, \phi_{S_0}).$$
An application of the induction rule gives us
$$\mathsf{KOZ} \vdash \phi_{S_0} \to \nu Y.\beta(\phi_{S_1}, \ldots, \phi_{S_n})$$
and so
$$\mathsf{KOZ} \vdash \varphi_M \to \nu Y.\beta(\phi_{S_1}, \ldots, \phi_{S_n}).$$

□

We are now ready to prove the completeness theorem for the fragment $\mathcal{F}^1_\mu$.

Theorem 44 *Let φ be a formula of $\mathcal{F}^1_\mu$. We have*
$$\models \varphi \quad \Leftrightarrow \quad \mathsf{KOZ} \vdash \varphi$$

Proof. As mentioned at the beginning of this chapter it is enough to show, that all consistent φ of $\mathcal{F}^1_\mu$ are satisfied in the canonical model for φ. To do that, we show for all $\psi \in FL(\varphi)$ and all states M in the canonical model by induction on the structure of ψ

$$\psi \in M \quad \Leftrightarrow \quad M \models \psi.$$

$\psi \equiv P$: In this case the equivalence follows from the definition of the valuation λ of the canonical model $\mathcal{CM}_\varphi$.

$\psi \equiv \alpha \wedge \beta$: Suppose we have $\alpha \wedge \beta \in M$ by Lemma 41 this is equivalent to $\alpha \in M$ and $\beta \in M$, since $\alpha \wedge \beta \in FL(\varphi)$. By induction hypothesis this is equivalent to $M \models \alpha$ and $M \models \beta$, which is equivalent to $M \models \alpha \wedge \beta$.

$\psi \equiv \alpha \vee \beta$: This case is dual to the case where $\psi \equiv \alpha \wedge \beta$.

$\psi \equiv \neg\alpha$: By Lemma 41 and since $\neg\alpha, \alpha \in FL(\varphi)$ we have

$$\neg\alpha \in M \quad \Leftrightarrow \quad \alpha \notin M.$$

Since by definition we have

$$M \models \neg\alpha \quad \Leftrightarrow \quad M \not\models \alpha$$

this proves the equivalence.

$\psi \equiv \Box\alpha$: First suppose $\Box\alpha \in M$, hence for all $M' \in R(M)$ by construction of the canonical model we have $\alpha \in M'$. With the induction hypothesis we get $M' \models \alpha$ for all $M' \in R(M)$ and so also $M \models \Box\alpha$.

For the other direction we show the contrapositive. Suppose $\Box\alpha \notin M$, hence by Lemma 41 we have $\neg\Box\alpha \in M$. We claim that $\{\neg\alpha\} \cup M/\Box$ is consistent. If the claim is true then there exists a maximal consistent set of formulae such that its intersection with $FL(\varphi)$ contains $\{\neg\alpha\} \cup M/\Box$. Now, this intersection yields a world $M' \in R(M)$ and by induction hypothesis $M' \not\models \alpha$. Hence, $M \not\models \Box\alpha$. So, it remains to prove the claim: It follows from the fact that if we have

$$\mathsf{KOZ} \vdash \bigwedge_{\psi \in M/\Box} \psi \to \alpha$$

then by necessitation rule and since $\Box$ distributes over conjunction and implication we have

$$\mathsf{KOZ} \vdash \bigwedge_{\psi \in M} \psi \to \Box\alpha.$$

And from that we could deduce the inconsistency of M since by Lemma 41 from the implication proved above we can get $\Box\alpha \in M$. So, we have $\mathsf{KOZ} \not\vdash \bigwedge_{\psi \in M} \psi \to \Box\alpha$ and thus $\mathsf{KOZ} \not\vdash \bigwedge_{\psi \in M/\Box} \psi \to \alpha$. And this is equivalent to the fact that $\{\neg\alpha\} \cup M/\Box$ is consistent.

$\psi \equiv \Diamond\alpha$: This case is dual to the case where $\psi \equiv \Box\alpha$.

$\psi \equiv \nu X.\alpha$: We first prove

$$\nu X.\alpha \in M \quad \Rightarrow \quad M \models \nu X.\alpha.$$

Since $\nu X.\alpha \in M \;\; \Rightarrow \;\; \nu X.\mathsf{nnf}(\alpha) \in M$, by Lemma 41 we can assume that α is in negation normal form. First a claim:

Claim:
If we define $S_{\nu X.\alpha} = \{M' \in S \mid \nu X.\alpha \in M'\}$, we have

$$M \in \|\alpha(S_{\nu X.\alpha})\|_{\mathcal{CM}_\varphi}.$$

Let us first prove the claim. We have the following two facts:

$$\Box\beta \in M \text{ and } M' \in R(M) \quad \Rightarrow \quad \beta \in M'$$

and, since $\{\beta\} \cup M/\Box$ is consistent, for all M there is a $M' \in R(M)$ such that

$$\Diamond\beta \in M \quad \Rightarrow \quad \beta \in M'.$$

With the observations made above, using the induction hypothesis and the fact that $\alpha(\nu X.\alpha) \in M$ we can construct a pre-model such that each branch fulfills one of the following two conditions: If it contains a point of the form $\nu X.\alpha@M''$ then $M'' \in S_{\nu X.\alpha}$, and if it does not contain a point of the form $\nu X.\alpha@M''$, then the branch is closed. Since in this case we can apply Corollary 39 we get the desired result. And so the claim is proved.

Since the claim holds for all $M \in S_{\nu X.\alpha}$ we have

$$S_{\nu X.\alpha} \subseteq \|\alpha(S_{\nu X.\alpha})\|_{\mathcal{CM}_\varphi}.$$

So we know, that M is in a pre-fixpoint of the functional $\|\alpha(X)\|_{\mathcal{CM}_\varphi}$, so M is in the greatest fixpoint of the functional $\|\alpha(X)\|_{\mathcal{CM}_\varphi}$, hence $M \models \nu X.\alpha$.

We now prove

$$M \models \nu X.\alpha \quad \Rightarrow \quad \nu X.\alpha \in M.$$

For all $M' \in \|\alpha(\|\nu X.\alpha\|_{\mathcal{CM}_\varphi})\|_{\mathcal{CM}_\varphi}$ since by assumption α must be diamond free, by Proposition 43 we get

$$\mathsf{KOZ} \vdash \varphi_{M'} \to \alpha(\phi_{\|\nu X.\alpha\|_{\mathcal{CM}_\varphi}}),$$

and since $\|\alpha(\|\nu X.\alpha\|_{\mathcal{CM}_\varphi})\|_{\mathcal{CM}_\varphi} = \|\nu X.\alpha\|_{\mathcal{CM}_\varphi}$

$$\mathsf{KOZ} \vdash \phi_{\|\nu X.\alpha\|_{\mathcal{CM}_\varphi}} \to \alpha(\phi_{\|\nu X.\alpha\|_{\mathcal{CM}_\varphi}}).$$

We now apply the induction rule and get

$$\mathsf{KOZ} \vdash \phi_{\|\nu X.\alpha\|_{\mathcal{CM}_\varphi}} \to \nu X.\alpha$$

and so, since $M \in \|\nu X.\alpha\|_{\mathcal{CM}_\varphi}$ we get

$$\mathsf{KOZ} \vdash \varphi_M \to \nu X.\alpha.$$

Now, if $\neg\nu X.\alpha \in M$, then M would not be consistent, so we have $\nu X.\alpha \in M$.
$\square$

6.1.2 Completeness for the Fragment $\mathcal{F}_\mu^2$

As we did for the fragment $\mathcal{F}_\mu^1$ it is enough to show, that if φ is consistent, there is a model and a state in the model, which satisfies φ. The model construction goes analogously to the one made for $\mathcal{F}_\mu^1$.

The definitions of consistent formula must be adapted to the deduction system without induction. So, φ is *consistent* if $\mathsf{KOZ}^{-(\mathsf{Ind})} \nvdash \neg\varphi$. Further, the definitions for consistent and maximal consistent set of formulae are similar to the definitions made for KOZ. The *Fischer-Ladner closure* of φ $FL(\varphi)$ and the *canonical model* for a consistent formula are defined as before. It is now obvious, that the (new) maximal consistent sets of formulae fulfill the conditions stated in Lemma 41.

We can now state the completeness theorem.

Theorem 45 *Let φ be a formula of $\mathcal{F}_\mu^2$. We have*

$$\models \varphi \quad \Rightarrow \quad \mathsf{KOZ}^{-(Ind)} \vdash \varphi.$$

Proof. We show that all consistent φ of $\mathcal{F}^2_\mu$ are satisfied in the canonical model for φ. To do that, we show for all $\psi \in FL(\varphi)$ and all states M in the canonical model by induction on ψ

$$\psi \in M \quad \Rightarrow \quad M \models \psi.$$

Since $\psi \in M \;\Rightarrow\; \mathsf{nnf}(\psi) \in M$, we can assume that ψ is in negation normal form. For the induction, the cases where $\psi \equiv P, \neg P, \alpha \wedge \beta, \Box\alpha$ or $\nu X.\alpha$ are treated like the part of Theorem 67, where we prove: $\psi \in M \quad \Rightarrow \quad M \models \psi$. In addition to that we have to prove the cases where $\psi \equiv \alpha \vee \beta$ and where $\psi \equiv \Diamond\alpha$. The first case goes with Lemma 41, the second follows from the fact, that if $M \cup \{\Diamond\alpha\}$ is consistent, so is $M/\Box \cup \{\alpha\}$. $\Box$

Part II

The Logic of Common Knowledge

Chapter 7

Introducing Common Knowledge

"So kann also die Mathematik definiert werden als diejenige Wissenschaft, in der wir niemals das kennen, worüber wir sprechen, und niemals wissen, ob das, was wir sagen, wahr ist." (B. Russell)

The idea to formalize reasoning about knowledge in modal logic goes at least back to the work of Wright [50] in the early fifties. Ten years later with Hintikka's seminal work, *Knowledge and Belief* [23], the logic of knowledge became an important source of interest mainly for philosophers. The major task was in trying to capture the inherent properties of knowledge and/or belief. Axioms for knowledge were suggested, attacked and defended. More recently, researchers in other areas have become interested in this area. Thus, the first formal approaches to the notion of common knowledge came from different fields, such as philosophy (see Lewis [33]), artificial intelligence and theoretical computer science (see McCarthy, Sato, Hayashi and Igarishi [35]) and economics (see Aumann [5]).

In this thesis we work with the formal framework, based on multi-modal logic, introduced by Halpern and Moses in [20] and developed in the well-known book, *Reasoning about Knowledge*, by Fagin, Halpern, Moses and Vardi [15]. In this framework common knowledge is introduced as the iteration of the more basic notion of 'everybody knows'. Another possibility to formalize the notion of common knowledge is given by Barwise's Situation Semantics (see [9, 8]) where common knowledge is introduced as the greatest fixpoint of an operator, also based on the notion of 'everybody knows'. It then comes out that in the Situation Semantics the fixpoint and the iterative approach differ whereas in our multi-modal framework they coincide. More about the relationship between these two formal frameworks can be found in Graf [19] and Lismont [34].

There are many 'real life' examples, such as the unfaithful wives (see Gamow and Stern [18]) or the muddy children puzzle (see Barwise [7]), which illustrate the subtleties of common knowledge. Hence, we shall consider the following card game:

We have two players Alice and Bob and the dealer. Both Alice and Bob get exactly one card from the dealer, the task of the game is to find out whether the other player holds an ace or not. The dealer gives both of them an ace. Let p stand for the proposition: "At least Alice or Bob holds an ace." Of course, we have that both Alice knows p, $\mathsf{K}_A p$, and Bob knows p, $\mathsf{K}_B p$. Let us analyze two scenarios:

Scenario 1: The dealer says nothing. Now, the dealer asks Bob in such a way that everybody can hear it: "Do you know if Alice holds an ace?" Bob denies, since he can not know. Then the dealer asks Alice whether she knows if Bob holds an ace, and, of course, Alice also must deny. The dealer could continue asking this question and he never will get a positive answer since both have no chance to find out whether the other holds an ace or not.

Scenario 2: The dealer says p, that is, he says: "At least Alice or Bob holds an ace." At first sight the situation does not change since the dealer says something that they already know. Now, the dealer asks Bob in such a way that everybody can hear it: "Do you know whether Alice holds an ace?" Bob must deny. Then the dealer asks Alice if she knows if Bob has an ace in his hands. Alice replies: "Yes!"

What is the difference between scenarios 1 and 2? In the first scenario we have $\mathsf{K}_A p$ and $\mathsf{K}_B p$, and in the second the dealer makes a public announcement of p, that is, he says what Alice and Bob already know. The situation changes because the public announcement of p implies that p becomes common knowledge ($\mathsf{C}p$), and thus: 'Alice knows that Bob knows p' and 'Bob knows that Alice knows that Bob knows p', and so on. Thus, since Bob answers negatively to the first question of the dealer Alice now knows that he must hold an ace. This follows from the fact that she now has information concerning the knowledge of Bob. Hence she reasons that if Bob would not hold an ace he would answer with 'yes', since he knows that at least one of them holds an ace.

This example shows that two apparently similar scenarios can evolve differently depending on whether there is common knowledge or not. How can common knowledge, which raises from the public announcement, be formalized? As previously said there are two main approaches: the iterative and the fixpoint approach. The iterative one is based on the fact that once we make an announcement of p, we immediately get that 'everybody knows p', $\mathsf{E}p$, that 'everybody knows that everybody knows p', $\mathsf{EE}p$, and so on, such as $\mathsf{EEE}p$. Formally, one could say:

$$\mathsf{C}p \equiv \bigwedge_{i \geq 1} \mathsf{E}^i p.$$

The fixpoint approach is based on the fact that common knowledge of p, $\mathsf{C}p$, is equivalent to the fact that 'everybody knows p' and that 'everybody knows that p is common knowledge', $\mathsf{E}(\mathsf{C}p) \wedge \mathsf{E}p$. Hence, common knowledge is a fixpoint of the operator $\mathsf{E}(X) \wedge \mathsf{E}p$. Barwise, in [8], argues that indeed it is the greatest fixpoint. Formally one could say:

$$\mathsf{C}p \equiv \nu X.(\mathsf{E}(X) \wedge \mathsf{E}p).$$

The relevance of common knowledge in computer science is mainly due to its connection with coordination of agents in a system. For example, it is common knowledge that a red traffic light means 'stop' and a green one means 'go'. Thus, most drivers feel safe when they cross a green traffic light. Suppose that this fact is not common knowledge. Even if every car driver knows that he can go when the light is green and must stop when it is red, he will not feel safe any more. How can he know that the other drivers know that too? Thus, a very safe driver will never pass a light when there are other cars at a crossroad; there would be no coordination. This example illustrates that common knowledge plays an important role for coordination, and simultaneous actions, in a multi agent system. In fact, Fagin, Halpern, Moses and Vardi in [15] prove formally that common knowledge and simultaneous actions are strongly related, in the sense that common knowledge is a prerequisite for simultaneous actions.

In this thesis we study some proof-theoretic aspects of the multi-modal formalization of common knowledge. Let us highlight the main results; more detailed abstracts can be found at the beginning of each chapter.

- In Chapter 8 we introduce the syntax and semantics of the logics of common knowledge in the multi-modal framework. We show, with an embedding into the modal μ-calculus, that the iterative and the fixpoint approach coincide.

- In Chapter 9 we introduce a Tait-style calculus for the logic of common knowledge and provide our first completeness result.

- In Chapter 10 we present cut elimination results for various logics of common knowledge. These results lead to proof systems which satisfy a subformula property and are therefore convenient for decision procedures.

- In Chapter 11 we introduce an infinitary cut-free calculus and show its completeness. Further, we present partial finitisation results by providing finite cut-free systems for the positive and the negative fragments.

Chapters 9 to 11 are based on joint work with Jäger [2].

Chapter 8

Basic Definitions and Results

"For all p: $\mathsf{K}_I \neg \mathsf{K}_I p$*" (Socrates)*

In the first section we introduce the syntax of the logic of common knowledge. We begin by introducing the class of formulae whereby the negation is defined in such a way that each formula is in negation normal form. Further, we introduce complete axiomatic systems in the Hilbert-style for the logics of common knowledge over $\mathbf{K}, \mathbf{T}, \mathbf{S4}$ and $\mathbf{S5}$, these axiomatic systems slightly differ from the original ones introduced by Fagin, Halpern, Moses and Vardi in [15] since we replace the fixpoint axiom with the co-closure axiom. The last result shows that both axiomatisations, the one with fixpoint and the one with co-closure axiom, are equivalent.

In the second section we introduce the semantics, given by labeled transition systems which in an epistemic context are called Kripke-models. In our multi-modal framework common knowledge of a fact φ, $\mathsf{C}\varphi$, is the iteration of the fact that everybody knows φ, that is, 'everybody knows φ' ($\mathsf{E}\varphi$), and 'everybody knows that everybody knows φ' ($\mathsf{EE}\varphi$), and so on. After some basic results we state the completeness of the various logics of common knowledge with respect to the corresponding classes of Kripke-models. These completeness results are due to Fagin, Halpern, Moses and Vardi in [15].

In the last section, we compare the iterative approach, which was introduced in the previous section, with the, so-called, fixpoint approach and establish the equivalence of them in our semantics. In order to do that we embed the logics of common knowledge into the modal μ-calculus by interpreting $\mathsf{C}\varphi$ as $\nu X.(\mathsf{E}X \wedge \mathsf{E}\varphi)$.

8.1 Syntax

To define the formulae of the logic of common knowledge we start from a set of *primitive propositions* $\mathsf{P} = \{p, q, \ldots\}$, the *propositional connectives* $\wedge$ and $\vee$, the *epistemic operators* $\mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_n$ and the *common knowledge operator* C; further, to define the negation on primitive propositions and on epistemic operators, we introduce the connective $\sim$. The *class of formulae* $\mathcal{L}_\mathsf{C}^\mathsf{n}$, denoted by $\alpha, \beta, \gamma, \varphi, \psi, \ldots$, then is defined inductively as follows:

- $p, \sim p \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ for all $p \in \mathsf{P}$,
- if $\alpha, \beta \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ then $(\alpha \wedge \beta) \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ and $(\alpha \vee \beta) \in \mathcal{L}_\mathsf{C}^\mathsf{n}$,
- if $\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ then $\mathsf{K}_i\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ and $\sim\mathsf{K}_i\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$,
- if $\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ then $\mathsf{C}\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ and $\sim\mathsf{C}\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$.

We often omit the parentheses if there is no danger of confusion. With the help of de Morgan's laws and the law of double negation for each formula φ we define recursively a negation $\neg\varphi$ as follows:

- If φ is primitive proposition p then $\neg\alpha$ is the the formula $\sim p$; if φ is $\sim p$ then $\neg\alpha$ is p.
- If φ is $\alpha \wedge \beta$ then $\neg\alpha$ is $(\neg\alpha \vee \neg\beta)$; if φ is $\alpha \vee \beta$ then $\neg\alpha$ is $(\neg\alpha \wedge \neg\beta)$.
- If φ is $\mathsf{K}_i\alpha$ then $\neg\varphi$ is $\sim\mathsf{K}_i\alpha$; if φ is $\sim\mathsf{K}_i\alpha$ then $\neg\varphi$ is $\mathsf{K}_i\alpha$.
- If φ is $\mathsf{C}\alpha$ then $\neg\varphi$ is $\sim\mathsf{C}\alpha$; if φ is $\sim\mathsf{C}\alpha$ then $\neg\varphi$ is $\mathsf{C}\alpha$.

Instead of $\neg\alpha \vee \beta$ we often write $\alpha \to \beta$. The formula $\mathsf{K}_i\varphi$ will be interpreted as 'agent i knows φ'. To write efficiently statements as 'everybody knows φ' we introduce the abbreviation E defined as

$$\mathsf{E}\alpha \equiv \mathsf{K}_1\alpha \wedge \ldots \wedge \mathsf{K}_n\alpha.$$

To express things such as 'everybody knows that everybody knows α' we introduce for all natural numbers m the iteration E^m of E as

$$\mathsf{E}^0\alpha \equiv \alpha \quad \text{and} \quad \mathsf{E}^{m+1}\alpha \equiv \mathsf{E}\mathsf{E}^m\alpha.$$

We end this section by presenting axiomatisatic system our multi-modal logics, such as $\mathbf{K}, \mathbf{T}, \mathbf{S4}$ and $\mathbf{S5}$, with a common knowledge operator. The systems are presented in the Hilbert-style, the axioms and rules are presented as schemes. We start with *logic of common knowledge over* **K** , whose presented axiomatic system is called $\mathsf{H}_{\mathsf{K}_n^C}$.

Axioms of $\mathsf{H}_{\mathsf{K}_n^C}$:

$\mathsf{H}_{\mathsf{K}_n^C}$ includes the axioms of the classical propositional calculus, for each K_i, the *distribution axiom*

$$\mathsf{K}_i(\varphi \to \psi) \to (\mathsf{K}_i\varphi \to \mathsf{K}_i\psi),$$

and the *co-closure axiom*

$$\mathsf{C}\varphi \to (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi).$$

Inference rules of $\mathsf{H}_{\mathsf{K}_n^C}$:

In addition to the classical *modus ponens* (MP), we have the *necessitation rule* (Nec) and the *induction rule* (Ind). All the rules are described below as schemes:

$$\frac{\psi \to \varphi \quad \psi}{\varphi}\ (MP) \qquad\qquad \frac{\varphi}{\mathsf{K}_i\varphi}\ (Nec)$$

$$\frac{\varphi \to (\mathsf{E}\varphi \wedge \mathsf{E}\psi)}{\varphi \to \mathsf{C}\psi}\ (Ind)$$

The calculus $\mathsf{H}_{\mathsf{T}_n^C}$ for the *logic of common knowledge over* **T** is obtained from $\mathsf{H}_{\mathsf{K}_n^C}$ by adding the axiom scheme

$$\mathsf{K}_i\varphi \to \varphi.$$

The calculus $\mathsf{H}_{\mathsf{S4}_n^C}$ for the *logic of common knowledge over* **S4** is obtained from $\mathsf{H}_{\mathsf{T}_n^C}$ by adding the axiom scheme for *positive introspection*

$$\mathsf{K}_i\varphi \to \mathsf{K}_i\mathsf{K}_i\varphi.$$

The calculus $\mathsf{H}_{\mathsf{S5}_n^C}$ for the *logic of common knowledge over* **S5** is obtained from $\mathsf{H}_{\mathsf{S4}_n^C}$ by adding the axiom scheme for *negative introspection*

$$\neg\mathsf{K}_i\varphi \to \mathsf{K}_i\neg\mathsf{K}_i\varphi.$$

Let H_* be one of the theories $\mathsf{H}_{\mathsf{K}_n^C}, \mathsf{H}_{\mathsf{T}_n^C}, \mathsf{H}_{\mathsf{S4}_n^C}$ or $\mathsf{H}_{\mathsf{S5}_n^C}$. If a formula φ is provable in H_* , we write

$$\mathsf{H}_* \vdash \varphi.$$

We end the section with a proposition which shows us that the co-closure axiom could be substituted by a fixpoint axiom.

Proposition 46 *Let H_* be one of the theories $\mathsf{H}_{\mathsf{K}_n^C}, \mathsf{H}_{\mathsf{T}_n^C}, \mathsf{H}_{\mathsf{S4}_n^C}$ or $\mathsf{H}_{\mathsf{S5}_n^C}$. For all formulae φ we have*

$$\mathsf{H}_* \vdash \mathsf{C}\varphi \leftrightarrow (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi).$$

Proof. Since one implication corresponds to the co-closure axiom we just have to prove

$$\mathsf{H}_* \vdash (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi) \rightarrow \mathsf{C}\varphi.$$

Let us prove this implication. It can easily be seen that we can prove

$$\mathsf{H}_* \vdash \mathsf{C}\varphi \rightarrow \mathsf{E}\varphi \quad \text{and} \quad \mathsf{H}_* \vdash \mathsf{C}\varphi \rightarrow \mathsf{EC}\varphi.$$

With the rules (Nec) and with the distribution axiom we then get

$$\mathsf{H}_* \vdash \mathsf{EC}\varphi \rightarrow \mathsf{EE}\varphi \quad \text{and} \quad \mathsf{H}_* \vdash \mathsf{EC}\varphi \rightarrow \mathsf{EEC}\varphi$$

and thus

$$\mathsf{H}_* \vdash \mathsf{EC}\varphi \rightarrow (\mathsf{EE}\varphi \wedge \mathsf{EEC}\varphi).$$

With the distribution axiom we get

$$\mathsf{H}_* \vdash \mathsf{EC}\varphi \rightarrow \mathsf{E}(\mathsf{E}\varphi \wedge \mathsf{EC}\varphi)$$

and thus

$$\mathsf{H}_* \vdash (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi) \rightarrow \mathsf{E}(\mathsf{E}\varphi \wedge \mathsf{EC}\varphi).$$

Since we have the propositional axiom

$$\mathsf{H}_* \vdash (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi) \rightarrow \mathsf{E}\varphi$$

we get

$$\mathsf{H}_* \vdash (\mathsf{E}\varphi \wedge \mathsf{EC}\varphi) \rightarrow (\mathsf{E}(\mathsf{E}\varphi \wedge \mathsf{EC}\varphi) \wedge \mathsf{E}\varphi).$$

With an application of the induction rule we get the desired result. □

8.2 Semantics

The semantics of the logics of common knowledge is given by Kripke-models, which are the same as labeled transition systems.

A *Kripke-model* $\mathcal{M}$ is of the form $(W, R_1, \ldots, R_n, \lambda)$, where:

- W is a nonempty set of *worlds*,
- all R_i are binary relations on W and
- $\lambda : \mathsf{P} \to \mathcal{P}(W)$ is the *valuation*, which assigns to each primitive proposition p a subset $\lambda(p)$ of W.

Given a formula φ and a Kripke-model $\mathcal{M} = (W, R_1, \ldots, R_n, \lambda)$, the set $\|\varphi\|_{\mathcal{M}} \subseteq W$ denotes the states where φ holds, and is called the *denotation of φ in $\mathcal{M}$*. It is defined inductively on the structure of φ, as follows:

- $\|p\|_{\mathcal{M}} = \lambda(p)$ and $\| \sim p\|_{\mathcal{M}} = W - \|p\|_{\mathcal{M}}$ for all $p \in \mathsf{P}$,
- $\|\alpha \wedge \beta\|_{\mathcal{M}} = \|\alpha\|_{\mathcal{M}} \cap \|\beta\|_{\mathcal{M}}$,
- $\|\alpha \vee \beta\|_{\mathcal{M}} = \|\alpha\|_{\mathcal{M}} \cup \|\beta\|_{\mathcal{M}}$,
- $\|\mathsf{K}_i\alpha\|_{\mathcal{M}} = \{w \in W \mid (\forall w' \in R_i(w))\ \ w' \in \|\alpha\|_{\mathcal{M}}\}$,
- $\|{\sim}\mathsf{K}_i\alpha\|_{\mathcal{M}} = W - \|\mathsf{K}_i\alpha\|_{\mathcal{M}}$,
- $\|\mathsf{C}\alpha\|_{\mathcal{M}} = \bigcap_{i \geq 1} \|\mathsf{E}^i\alpha\|_{\mathcal{M}}$,
- $\|{\sim}\mathsf{C}\alpha\|_{\mathcal{M}} = W - \|\mathsf{C}\alpha\|_{\mathcal{M}}$.

Let $\mathcal{M}$ be a Kripke-model and w a world in it. If $w \in \|\varphi\|_{\mathcal{M}}$ and if it is clear from the context that we are referring to the model $\mathcal{M}$ we often write $w \models \varphi$, and say φ is *valid in w*. We write $\mathcal{M} \models \varphi$, and say φ is *valid in $\mathcal{M}$*, if it is valid in all worlds of $\mathcal{M}$.

We now introduce for all modal logics $\mathbf{K}, \mathbf{T}, \mathbf{S4}$ and $\mathbf{S5}$ the corresponding classes of models $C^{\mathbf{K}}, C^{\mathbf{T}}, C^{\mathbf{S4}}$ and $C^{\mathbf{S5}}$:

1. $C^{\mathbf{K}}$ consists of all Kripke-models.
2. $C^{\mathbf{T}}$ consists of all Kripke-models, where the relations are reflexive.

3. $C^{\mathbf{S4}}$ consists of all Kripke-models, where the relations are reflexive and transitive.

4. $C^{\mathbf{S5}}$ consists of all Kripke-models, where the relations are reflexive, transitive and symmetric.

Let C^* be one of the classes $C^{\mathbf{K}}, C^{\mathbf{T}}, C^{\mathbf{S4}}$ or $C^{\mathbf{S5}}$. We write $C^* \models \varphi$, if φ is valid in all models in C^*.

Given two worlds in a model w and w', we say w' is *accessible in one step* from w if there is an $i \in \{1, \ldots, n\}$, such that $(w, w') \in R_i$. The accessibility in n steps is the defined recursively as follows: w' is *accessible in* $n+1$ *steps* from w, if there is a world w'' accessible in n steps from w, such that w' is accessible in one step from w''. Finally, w' is *accessible* from w if there is a natural number n, such that w' is accessible in n steps from w. $\mathcal{A}(w)$ denotes the set of all such w'. The next lemma can easily be proven with an inductive argument.

Lemma 47 *For all formulae φ, Kripke-models and worlds w we have*

$$w \models \mathsf{C}\varphi \quad \Leftrightarrow \quad w' \models \varphi \text{ for all } w' \in \mathcal{A}(w).$$

We end this section with a completeness and correctness result. The correctness part can easily be proven by induction on the proof-length, for the completeness part we refer to Fagin, Halpern, Moses and Vardi [15].

Theorem 48 *Let H_* be one of the theories $\mathsf{H}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}, \mathsf{H}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}}, \mathsf{H}_{\mathsf{S4}^{\mathsf{C}}_{\mathsf{n}}}$ or $\mathsf{H}_{\mathsf{S5}^{\mathsf{C}}_{\mathsf{n}}}$, and let C^* be the corresponding class among $C^{\mathbf{K}}, C^{\mathbf{T}}, C^{\mathbf{S4}}$ or $C^{\mathbf{S5}}$. For all formulae φ we have*

$$\mathsf{H}_* \vdash \varphi \quad \Leftrightarrow \quad C^* \models \varphi.$$

8.3 Common Knowledge as a Fragment of μ-Calculus

In this section we embed the logics of common knowledge into the modal μ-calculus. This embedding uses the fact that the common knowledge operator can be seen as a greatest fixpoint of the basic modalities.

Let $\mathcal{M} = (W, R_1, \ldots, R_n, \lambda)$ be a Kripke-model, $W' \subseteq W$ a subset of W and $\varphi(p)$ a formula in $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ containing a primitive proposition p. $\|\varphi(W')\|_{\mathcal{M}}$

denotes the set of worlds $\|\varphi(p)\|_{\mathcal{M}'}$ where $\mathcal{M}' = (W, R_1, \ldots, R_n, \lambda')$ and λ' maps p to W' and otherwise is equal to λ. For each formula φ we define an operator $O_{\mathcal{M},\varphi}$ from $\mathcal{P}(W)$ to $\mathcal{P}(W)$ such that for each $W' \subseteq W$ we have

$$O_{\mathcal{M},\varphi}(W') = \|\mathsf{E}(\varphi) \wedge \mathsf{E}(W')\|_{\mathcal{M}}.$$

For each ordinal α we can define recursively a subset $O^{\alpha}_{\mathcal{M},\varphi}$ of the set of worlds W as follows:

- $O^{0}_{\mathcal{M},\varphi} = W$,
- $O^{\alpha+1}_{\mathcal{M},\varphi} = O_{\mathcal{M},\varphi}(O^{\alpha}_{\mathcal{M},\varphi})$ and
- $O^{\lambda}_{\mathcal{M},\varphi} = \bigcap_{\alpha<\lambda} O^{\alpha}_{\mathcal{M},\varphi}$ for each limes ordinal λ.

Lemma 49 *For all Kripke-models $\mathcal{M} = (W, R_1, \ldots, R_n, \lambda)$, formulae φ and natural numbers $n \geq 1$ we have*

$$O^{n}_{\mathcal{M},\varphi} = \|\mathsf{E}\varphi \wedge \ldots \wedge \mathsf{E}^{n}\varphi\|_{\mathcal{M}}.$$

Proof. The proof goes by induction on n.

$n = 1$: $O^{1}_{\mathcal{M},\varphi} = \|\mathsf{E}\varphi \wedge \mathsf{E}(W)\|_{\mathcal{M}} = \|\mathsf{E}\varphi\|_{\mathcal{M}}$.

$n = m + 1$: $O^{m+1}_{\mathcal{M},\varphi}$ by definition is equal to $O_{\mathcal{M},\varphi}(O^{m}_{\mathcal{M},\varphi})$. By induction hypothesis this is equal

$$\|\mathsf{E}\varphi \wedge \mathsf{E}(\mathsf{E}\varphi \wedge \ldots \wedge \mathsf{E}^{m}\varphi)\|_{\mathcal{M}}.$$

Since E distributes over conjunction we get the equality with

$$\|\mathsf{E}\varphi \wedge \mathsf{E}\mathsf{E}\varphi \wedge \ldots \wedge \mathsf{E}^{m+1}\varphi\|_{\mathcal{M}}.$$

□

It can easily be seen that $O_{\mathcal{M},\varphi}$ is a monotone operator on the powerset of the worlds of $\mathcal{M}$. The following lemma shows that the greatest fixpoint of this operator, which exists by the Tarski-Knaster Theorem 95, is always reached after ω-many approximations.

Lemma 50 *For each Kripke-model $\mathcal{M}$ and formula φ we have*

$$\mathsf{GFP}(O_{\mathcal{M},\varphi}) = \bigcap_{\alpha \in ON} O^{\alpha}_{\mathcal{M},\varphi} = O^{\omega}_{\mathcal{M},\varphi}.$$

Proof. We show

$$\mathsf{GFP}(O_{\mathcal{M},\varphi}) \subseteq \bigcap_{\alpha \in ON} O^{\alpha}_{\mathcal{M},\varphi} \subseteq O^{\omega}_{\mathcal{M},\varphi} \subseteq \mathsf{GFP}(O_{\mathcal{M},\varphi}).$$

The first inclusion follows from Tarski-Knaster Theorem 95 and the second from the definition. To get the third one we show that $O^{\omega}_{\mathcal{M},\varphi}$ is a fixpoint of $O_{\mathcal{M},\varphi}$, that is

$$O_{\mathcal{M},\varphi}(O^{\omega}_{\mathcal{M},\varphi}) = O^{\omega}_{\mathcal{M},\varphi}.$$

$\supseteq$: The following four facts are equivalent:

1. $w \in O^{\omega}_{\mathcal{M},\varphi}$,
2. $w \in \bigcap_{n \in \omega} O^{n}_{\mathcal{M},\varphi}$,
3. $\forall n \in \omega \;\; w \in \|\mathsf{E}\varphi \wedge \ldots \wedge \mathsf{E}^n\varphi\|_{\mathcal{M}}$,
4. $\forall w' \in \mathcal{A}(w) \;\; w' \in \|\varphi\|_{\mathcal{M}}$.

The equivalence of 1. and 2. follows from definition, the one of 2. and 3. by Lemma 49 and the equivalence of 3. and 4. can easily be verified.

Hence, if $w \in O^{\omega}_{\mathcal{M},\varphi}$ then we have

$$\text{for all } w' \in \mathcal{A}(w) \;\; w' \in \|\varphi\|_{\mathcal{M}}.$$

From this we get that for all w' which are accessible in one step from w we have

$$\text{for all } w'' \in \mathcal{A}(w') \;\; w'' \in \|\varphi\|_{\mathcal{M}}.$$

Using the equivalences above we get for all such w'

$$w' \in O^{\omega}_{\mathcal{M},\varphi}.$$

This gives to us $w \in \|\mathsf{E}(O^{\omega}_{\mathcal{M},\varphi})\|_{\mathcal{M}}$. Further, using the equivalences above, it can easily be seen that $w \in O^{\omega}_{\mathcal{M},\varphi}$ implies $w \in \|\mathsf{E}\varphi\|_{\mathcal{M}}$. With this we get

$$w \in O^{\omega}_{\mathcal{M},\varphi} \quad \Rightarrow \quad w \in \|\mathsf{E}\varphi \wedge \mathsf{E}(O^{\omega}_{\mathcal{M},\varphi})\|_{\mathcal{M}}$$

which proves the inclusion.

$\subseteq$: We show, by induction, for all ordinals α

$$O_{\mathcal{M},\varphi}(O^{\alpha}_{\mathcal{M},\varphi}) \subseteq O^{\alpha}_{\mathcal{M},\varphi}.$$

- $\alpha = 0$: Follows from the definition.

- $\alpha = \beta + 1$: $O_{\mathcal{M},\varphi}(O^{\alpha}_{\mathcal{M},\varphi})$ is equal to $O_{\mathcal{M},\varphi}(O_{\mathcal{M},\varphi}(O^{\beta}_{\mathcal{M},\varphi}))$ by definition. By induction hypothesis and since $O_{\mathcal{M},\varphi}$ is monotone we get

$$O_{\mathcal{M},\varphi}(O_{\mathcal{M},\varphi}(O^{\beta}_{\mathcal{M},\varphi})) \subseteq O_{\mathcal{M},\varphi}(O^{\beta}_{\mathcal{M},\varphi}).$$

  And hence

$$O_{\mathcal{M},\varphi}(O^{\alpha}_{\mathcal{M},\varphi}) \subseteq O^{\alpha}_{\mathcal{M},\varphi}.$$

- $\alpha = \lambda$: By monotonicity for all ordinal $\gamma < \alpha$ we have

$$O_{\mathcal{M},\varphi}(\bigcap_{\beta<\alpha} O^{\beta}_{\mathcal{M},\varphi}) \subseteq O_{\mathcal{M},\varphi}(O^{\gamma}_{\mathcal{M},\varphi}).$$

  And so we get for all $\gamma < \alpha$

$$O_{\mathcal{M},\varphi}(\bigcap_{\beta<\alpha} O^{\beta}_{\mathcal{M},\varphi}) \subseteq O^{\gamma+1}_{\mathcal{M},\varphi}$$

  and thus

$$O_{\mathcal{M},\varphi}(\bigcap_{\beta<\alpha} O^{\beta}_{\mathcal{M},\varphi}) \subseteq \bigcap_{\beta<\alpha} O^{\beta}_{\mathcal{M},\varphi}$$

  which is the desired inclusion.

□

We are now able to embed the logics of common knowledge into the modal μ-calculus. Notice that the modal μ-calculus is over a multi-modal language with modalities $\mathsf{K}_1, \ldots, \mathsf{K}_n$. The embedding $\mathcal{E}$ takes a formula of the logics of common knowledge and assigns to it a μ-formula. It is defined recursively on the structure of the formula as follows:

1. $\mathcal{E}(q) = q$ and $\mathcal{E}(\sim q) = \neg q$,

2. $\mathcal{E}(\alpha \wedge \beta) = \mathcal{E}(\alpha) \wedge \mathcal{E}(\beta)$,

3. $\mathcal{E}(\alpha \vee \beta) = \mathcal{E}(\alpha) \vee \mathcal{E}(\beta)$,

4. $\mathcal{E}(\mathsf{K}_i\alpha) = \mathsf{K}_i\mathcal{E}(\alpha)$,

5. $\mathcal{E}(\sim\mathsf{K}_i\alpha) = \neg\mathsf{K}_i\mathcal{E}(\alpha)$,

6. $\mathcal{E}(\mathsf{C}\alpha) = \nu X.\mathcal{E}(\mathsf{E}\alpha \wedge \mathsf{E}X)$ (X is a new variable) and
7. $\mathcal{E}(\sim\mathsf{C}\alpha) = \neg\nu X.\mathcal{E}(\mathsf{E}\alpha \wedge \mathsf{E}X)$ (X is a new variable).

Notice, that except for the common knowledge operator $\mathcal{E}$ preserves the structure of the formula. The following theorem holds.

Theorem 51 *For all the Kripke-models $\mathcal{M}$ and formulae $\varphi \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ we have*

$$\|\varphi\|_{\mathcal{M}} = \|\mathcal{E}(\varphi)\|_{\mathcal{M}}.$$

Proof. The proof goes by induction on the structure of φ. The only case which is not straightforward is when φ is of the form $\mathsf{C}\alpha$. In this case we have by definition $\|\mathsf{C}\alpha\|_{\mathcal{M}} = \bigcap_{i\geq 1} \|\mathsf{E}^i\alpha\|_{\mathcal{M}}$ and with Lemma 49 we get

$$\|\mathsf{C}\alpha\|_{\mathcal{M}} = O^{\omega}_{\mathcal{M},\alpha}.$$

With Lemma 50 we can deduce

$$\|\mathsf{C}\alpha\|_{\mathcal{M}} = \mathsf{GFP}(O_{\mathcal{M},\alpha}).$$

Remember that $O_{\mathcal{M},\alpha}(W')$ was defined by $\|\mathsf{E}\alpha \wedge \mathsf{E}(W')\|_{\mathcal{M}}$ for all set of worlds W'. Using the induction hypothesis we can easily see that $\mathsf{GFP}(O_{\mathcal{M},\alpha})$ is the greatest fixpoint of the function which maps a set of worlds W' to $\|\mathcal{E}(\mathsf{E}\alpha \wedge \mathsf{E}(W'))\|_{\mathcal{M}}$. This gives us

$$\|\mathsf{C}\alpha\|_{\mathcal{M}} = \|\nu X.\mathcal{E}(\mathsf{E}\alpha \wedge \mathsf{E}X)\|_{\mathcal{M}} = \|\mathcal{E}(\mathsf{C}\alpha)\|_{\mathcal{M}}.$$

□

Chapter 9

A Tait-Style Reformulation of $\mathsf{H}_{\mathsf{K}_\mathsf{n}^\mathsf{C}}$

“ ‘Obviously’ is the most dangerous word in mathematics.” (E. T. Bell)

In this chapter we introduce the Tait-style calculus $\mathsf{T}_{\mathsf{K}_\mathsf{n}^\mathsf{C}}$ for the logic of common knowledge over $\mathbf{K}$. We fix the notation for Tait-style calculi which will be also used in the following chapters. After, we prove that the introduced calculus with general cut is equivalent to the complete Hilbert-style calculus introduced in the previous chapter and so immediately get the completeness for $\mathsf{T}_{\mathsf{K}_\mathsf{n}^\mathsf{C}} + (\mathsf{G}{-}\mathsf{Cut})$. We end the chapter with the proof that our Tait-style calculus does not allow complete cut elimination.

9.1 Definition and Completeness

As usual $p, q, r, \ldots$ stand for primitive propositions and small Greek letters for arbitrary formulae. Further, the capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$ (possibly with subscripts) stand for finite subsets of $\mathcal{L}_\mathsf{C}^\mathsf{n}$ which are called *sequents*. For any sequents Γ, Δ and formulae α, β the sequent $\Gamma \cup \Delta \cup \{\alpha\} \cup \{\delta\}$ is denoted by $\Gamma, \Delta, \alpha, \beta$. Let Γ be the sequent $\{\alpha_1, \ldots, \alpha_m\}$, we often use the following convenient abbreviations:

- $\bigvee \Gamma = \{\alpha_1 \vee \ldots \vee \alpha_m\}$,
- $\neg\Gamma = \{\neg\alpha_1, \ldots, \neg\alpha_m\}$,
- $\neg\mathsf{K}_i\Gamma = \{\neg\mathsf{K}_i\alpha_1, \ldots, \neg\mathsf{K}_i\alpha_m\}$,

- $\mathsf{K}_i\Gamma = \{\mathsf{K}_i\alpha_1, \dots, \mathsf{K}_i\alpha_m\}$,
- $\neg\mathsf{E}\Gamma = \{\neg\mathsf{E}\alpha_1, \dots, \neg\mathsf{E}\alpha_m\}$,
- $\neg\mathsf{C}\Gamma = \{\neg\mathsf{C}\alpha_1, \dots, \neg\mathsf{C}\alpha_m\}$ and
- $\Gamma/\mathsf{K}_i = \{\alpha \mid \mathsf{K}_i\alpha \in \Gamma\}$.

Further, for each formula α we define inductively a *complexity measure* $\mathsf{me}(\alpha)$:

- $\mathsf{me}(p) = \mathsf{me}(\sim p) = 0$ for all $p \in \mathsf{P}$,
- $\mathsf{me}(\alpha \wedge \beta) = \mathsf{me}(\alpha \vee \beta) = \max\{\mathsf{me}(\alpha), \mathsf{me}(\beta)\} + 1$,
- $\mathsf{me}(\mathsf{K}_i\alpha) = \mathsf{me}(\sim\mathsf{K}_i\alpha) = \mathsf{me}(\alpha) + 1$,
- $\mathsf{me}(\mathsf{C}\alpha) = \mathsf{me}(\sim\mathsf{C}\alpha) = \mathsf{me}(\alpha) + n + 1$.

Let us introduce the Tait-style calculus $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}$ for the logic of common knowledge over **K**. All the rules are represented as schemes.

Axiom of $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}$:

$$\frac{}{\Gamma, p, \neg p}\ (ID)$$

Basic inference rules of $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}$:

$$\frac{\Gamma, \alpha, \beta}{\Gamma, \alpha \vee \beta}\ (\vee) \qquad\qquad \frac{\Gamma, \alpha \quad \Gamma, \beta}{\Gamma, \alpha \wedge \beta}\ (\wedge)$$

$$\frac{\neg\mathsf{C}\Delta, \neg\Gamma, \alpha}{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \mathsf{K}_i\alpha, \Sigma}\ (\mathsf{K}_i)$$

C-rules of $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}$:

$$\frac{\neg\mathsf{C}\Gamma, \mathsf{E}\alpha}{\neg\mathsf{C}\Gamma, \mathsf{C}\alpha, \Sigma}\ (\mathsf{C}.1) \qquad\qquad \frac{\Gamma, \neg\mathsf{E}\alpha}{\Gamma, \neg\mathsf{C}\alpha}\ (\neg\mathsf{C})$$

The induction rule of $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}}$:

$$\frac{\neg\alpha, \mathsf{E}\alpha \quad \neg\alpha, \mathsf{E}\beta}{\neg\alpha, \mathsf{C}\beta, \Sigma}\ (Ind)$$

The *designated formula* of the $(\wedge)$-rule is $\alpha \wedge \beta$, for the $(\vee)$-rule it is $\alpha \vee \beta$, for the (K_i)-rule it is $\mathsf{K}_i\alpha$, for the $(\mathsf{C}.1)$- and the (Ind)-rules it is $\mathsf{C}\alpha$ and for the $(\neg\mathsf{C})$-rule it is $\neg\mathsf{C}\alpha$. We did not introduce any cut rules since we want to distinguish $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with various additional cuts. Hence, we always mention explicitly which cut rules are admitted. Let us introduce the most general cut scheme, the *general cut rule.*

General cut:

$$\frac{\Gamma, \alpha \quad \Gamma, \neg\alpha}{\Gamma} \ (\mathsf{G{-}Cut})$$

In this case the designated formulae α and $\neg\alpha$ are called the *cut formulae* of $(\mathsf{G{-}Cut})$.

For any inference rule (ρ) presented above, the sequents over the separating line are called *premises of* (ρ) and the sequent under the separating line is called *conclusion of* (ρ). The *derivability in n steps* is defined as usual: The conclusions of (ID) are derivable in arbitrary many steps, and if all the premises of an inference rule are derivable in n steps then the conclusion is derivable in $n+1$ steps. We write

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (*_1{-}\mathsf{Cut}) + \ldots + (*_{\mathsf{m}}{-}\mathsf{Cut}) \vdash^n \Gamma$$

if Γ is derivable in n steps in the calculus $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with possible additional use of cuts from $(*_1{-}\mathsf{Cut}), \ldots, (*_{\mathsf{m}}{-}\mathsf{Cut})$. If Γ is derivable in less than n steps we write

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (*_1{-}\mathsf{Cut}) + \ldots + (*_{\mathsf{m}}{-}\mathsf{Cut}) \vdash^{<n} \Gamma.$$

If there is a natural number n such that Γ is derivable in n steps we write

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (*_1{-}\mathsf{Cut}) + \ldots + (*_{\mathsf{m}}{-}\mathsf{Cut}) \vdash \Gamma.$$

The next proposition states the correctness of $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with general cut and can be proven by induction on the length of the proof.

Proposition 52 *For all sequents Γ we have*

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \Gamma \quad \Rightarrow \quad C^{\mathbf{K}} \models \bigvee \Gamma.$$

Proposition 52 and Theorem 48 give us the following corollary.

Corollary 53 *For all sequents Γ we have*

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \Gamma \quad \Rightarrow \quad \mathsf{H}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \bigvee \Gamma.$$

Lemma 54 *For all formulae $\alpha \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ and, possibly empty, sequents Γ we have:*

1. $\mathsf{T}_{\mathsf{K}_n^C} \vdash \Gamma, \alpha, \neg\alpha$.

2. $\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{C}\alpha, \mathsf{E}\alpha \wedge \mathsf{EC}\alpha$.

Proof. The proof of the first assertion goes by induction on $\mathsf{me}(\alpha)$ and is omitted. Let us prove the second one. By the first part of the lemma we have

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{E}\alpha, \mathsf{E}\alpha$$

and so an application of $(\neg\mathsf{C})$ gives us

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{C}\alpha, \mathsf{E}\alpha. \tag{1}$$

Again by the first part we have

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{C}\alpha, \mathsf{C}\alpha$$

and so by applying all the (K_i)-rules and the $(\wedge)$ rule we get

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{C}\alpha, \mathsf{EC}\alpha. \tag{2}$$

An application of $(\wedge)$ to (1) and (2) gives the desired result. □

The next lemma can easily be proven with the help of the induction rule (Ind) and some additional formula manipulations in the calculus.

Lemma 55 *For any two formulae $\alpha, \beta \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ such that*

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G{-}Cut}) \vdash \neg\beta, \mathsf{E}\alpha \wedge \mathsf{E}\beta$$

we have

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G{-}Cut}) \vdash \neg\beta, \mathsf{C}\alpha.$$

Theorem 56 *For all formulae $\alpha \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ we have*

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G{-}Cut}) \vdash \alpha \quad \Leftrightarrow \quad \mathsf{H}_{\mathsf{K}_n^C} \vdash \alpha.$$

Proof. The direction from left to right follows from Corollary 53. For the other direction we first observe that all the axioms of propositional logic and the distribution axiom can be proven in $\mathsf{T}_{\mathsf{K}_n^C}$. Since the co-closure axiom corresponds to Lemma 54.2 all the axioms of $\mathsf{H}_{\mathsf{K}_n^C}$ are provable in $\mathsf{T}_{\mathsf{K}_n^C}$. Further, observe that the (MP)-rule corresponds to $(\mathsf{G}{-}\mathsf{Cut})$, the (Nec)-rule to (K_i) and that the induction rule corresponds to Lemma 55. Hence by a simple induction on the proof length in $\mathsf{H}_{\mathsf{K}_n^C}$ we can prove the direction from right to left. □

In combination with Theorem 48 we immediately get the following corollary.

Corollary 57 *For all formulae $\alpha \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ we have*

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G}{-}\mathsf{Cut}) \vdash \alpha \quad \Leftrightarrow \quad C^{\mathbf{K}} \models \alpha.$$

The previous theorem shows that $\mathsf{T}_{\mathsf{K}_n^C}$ with general cut is complete. The question if $\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G}{-}\mathsf{Cut})$ admits complete cut elimination is answered negatively by the next proposition.

Proposition 58 *Assume we have a language with two agents $\{1, 2\}$ and let α be the formula*

$$\neg\mathsf{K}_1(q \wedge \mathsf{C}p) \vee \neg\mathsf{K}_2(p \wedge \mathsf{C}q) \vee \mathsf{C}(p \vee q).$$

We have

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G}{-}\mathsf{Cut}) \vdash \alpha \quad \textit{and} \quad \mathsf{T}_{\mathsf{K}_n^C} \nvdash \alpha.$$

Proof. It can easily be seen that α is valid, and so by Corollary 57 we have

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathsf{G}{-}\mathsf{Cut}) \vdash \alpha.$$

On the other hand we do not have

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg\mathsf{K}_1(q \wedge \mathsf{C}p) \vee \neg\mathsf{K}_2(p \wedge \mathsf{C}q) \vee \mathsf{C}(p \vee q).$$

For, if this was the case then in the proof of α the subformula $\mathsf{C}(p \vee q)$ must have been the distinguished formula of either $(\mathsf{C}.1)$ or of (Ind). If it was the distinguished formula of $(\mathsf{C}.1)$ then we must have

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \mathsf{E}(p \vee q)$$

which can not be the case by Proposition 52 since $\mathsf{E}(p \vee q)$ is not valid. If it was the distinguished formula of (Ind) then we have two cases

1. The conclusion of this application of (Ind) was

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg \mathsf{K}_1(q \wedge \mathsf{C}p), \neg \mathsf{K}_2(p \wedge \mathsf{C}q), \mathsf{C}(p \vee q),$$

2. The conclusion of this application of (Ind) was

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg \mathsf{K}_1(q \wedge \mathsf{C}p) \vee \neg \mathsf{K}_2(p \wedge \mathsf{C}q), \mathsf{C}(p \vee q).$$

In the first case one of premises of (Ind) is either

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg \mathsf{K}_1(q \wedge \mathsf{C}p), \mathsf{E}(\mathsf{K}_1(q \wedge \mathsf{C}p)) \quad \text{or} \quad \mathsf{T}_{\mathsf{K}_n^C} \vdash \neg \mathsf{K}_2(p \wedge \mathsf{C}q), \mathsf{E}(\mathsf{K}_2(p \wedge \mathsf{C}q)).$$

Since both sequents are not valid with Proposition 52 get a contradiction. In the second case one premise is

$$\mathsf{T}_{\mathsf{K}_n^C} \vdash \neg \mathsf{K}_1(q \wedge \mathsf{C}p) \vee \neg \mathsf{K}_2(p \wedge \mathsf{C}q), \mathsf{E}(\mathsf{K}_1(q \wedge \mathsf{C}p) \wedge \mathsf{K}_2(p \wedge \mathsf{C}q)).$$

Since this sequent is not valid, too, with Proposition 52 get a contradiction and thus the proof. □

Chapter 10

Fischer-Ladner Cuts

"Die Kompliziertheit treibt uns an, die Einfachheit voran." (E. Hablé)

The last chapter has shown that $\mathsf{T}_{\mathsf{K}_n^C}$ does not admit complete cut elimination. In this chapter we provide partial cut elimination results for the logics of common knowledge over $\mathbf{K}, \mathbf{T}, \mathbf{S4}$ and $\mathbf{S5}$.

In the first section we prove the partial cut elimination result for the Tait-style calculus $\mathsf{T}_{\mathsf{K}_n^C}$ introduced in the previous chapter. We show that a formula φ is provable in $\mathsf{T}_{\mathsf{K}_n^C}$ with general cut if and only if it is provable with the use of cuts whose cut formulae represent subsets of the Fischer-Ladner closure of φ. Thus, we get a proof system where proof-search becomes decidable. The result is proven by adapting the proof-methods used by Fagin, Halpern, Moses and Vardi in [15] for the Hilbert-style system to our context.

In the second section we extend the method developed in the first one to prove partial cut elimination results for the logics of common knowledge over $\mathbf{T}, \mathbf{S4}$ and $\mathbf{S5}$.

10.1 Partial Cut Elimination for $\mathsf{T}_{\mathsf{K}_n^C}$

In order to define the notion of Fischer-Ladner cuts let us first introduce the more general notion of Π-cut.

Let $\Pi \subseteq \mathcal{L}_C^n$ be a set of formulae which is closed under negation, that is, we have $\neg\Pi = \Pi$. Then the Π-*cuts* are all cuts

$$\frac{\Gamma, \alpha \quad \Gamma, \neg\alpha}{\Gamma} \ (\Pi\text{−Cut})$$

such that the cut formula α belongs to Π. The next lemma can easily be proven by induction on the proof-length.

Lemma 59 *Let $\Pi \subseteq \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a set of formulae closed under negation. For all sequents Γ and Δ we have*

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma \quad \Rightarrow \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma, \Delta.$$

Lemma 60 *Let $\Pi \subseteq \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a set of formulae closed under negation. For all sequents Γ and all formulae $\alpha_1 \wedge \alpha_2 \in \Pi$ and $\alpha \vee \beta \in \Pi$ we have*

1. $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma, \alpha \vee \beta \quad \Rightarrow \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma, \alpha, \beta,$
2. $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma, \alpha_1 \wedge \alpha_2 \quad \Rightarrow \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \vdash \Gamma, \alpha_i.$

Proof. In $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ we can easily derive

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \neg\alpha \wedge \neg\beta, \alpha, \beta \quad \text{and} \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \neg\alpha_1 \vee \neg\alpha_2, \alpha_i$$

for $i \in \{1, 2\}$. Since both $\neg\alpha \wedge \neg\beta$ and $\neg\alpha_1 \vee \neg\alpha_2$ belong to Π with two applications of Π-cut we get the desired results. □

We define what a Π-consistent set of formulae is. Contrary to the notion of consistent set we always distinguish the cuts we admit in the proofs.

1. A set of $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formulae M is Π-*consistent* if for each finite subset $\Gamma \subseteq M$ we have
$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−Cut}) \nvdash \neg\Gamma.$$
2. A set of $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formulae M is *maximal* Π-*consistent* if it is Π-consistent and if no proper superset is Π-consistent, too.

The following lemma assures the existence of maximal consistent supersets for each consistent set. The proof goes exactly as the one of Lemma 40.

Lemma 61 *Given a Π-consistent set of formulae M, there is a maximal Π-consistent set M' such that $M \subseteq M'$.*

Lemma 62 *Let $\Pi \subseteq \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a set closed under negation and let M be maximal Π-consistent. For all sequents Γ and formulae α we have:*

1. $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$ or $\neg\alpha \in M$.

2. $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$ iff $\neg\alpha \notin M$.

3. $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−}\mathsf{Cut}) \vdash \neg\Gamma, \alpha$ *and* $\Gamma \subseteq M$ *and* $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$.

Proof. Let us first prove assertion one. Suppose $\alpha \notin M$ and $\neg\alpha \notin M$. By the maximality of M there are finite subsets Γ, Δ of M such that

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−}\mathsf{Cut}) \vdash \neg\Gamma, \neg\alpha \quad \text{and} \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−}\mathsf{Cut}) \vdash \neg\Delta, \alpha.$$

With Lemma 59 and an application of $(\Pi\text{−}\mathsf{Cut})$ to α we get

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−}\mathsf{Cut}) \vdash \neg\Gamma, \neg\Delta.$$

Hence $\Gamma, \Delta \subset M$ is not consistent and so M is not consistent, too. This is a contradiction to the assumption. The second assertion immediately follows from the first one and from the consistency of M. For the third assertion, suppose $\alpha \notin M$. By part (1) we have $\neg\alpha \in M$. But then, since M is consistent, we would have

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi\text{−}\mathsf{Cut}) \nvdash \neg\Gamma, \alpha$$

which is not the case. □

Before we introduce the set Π which is relevant for our purposes we have to define the *Fischer-Ladner closure* $\mathsf{FL}_{\mathsf{K}}(\alpha)$ of a formula $\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$. It was first introduced by Fischer and Ladner in [16] and is defined to be the smallest set such that:

- α belongs to $\mathsf{FL}_{\mathsf{K}}(\alpha)$,
- if $\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$, then $\neg\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$,
- if $\beta \vee \gamma \in \mathsf{FL}_{\mathsf{K}}(\alpha)$, then $\beta, \gamma \in \mathsf{FL}_{\mathsf{K}}(\alpha)$,
- if $\beta \wedge \gamma \in \mathsf{FL}_{\mathsf{K}}(\alpha)$, then $\beta, \gamma \in \mathsf{FL}_{\mathsf{K}}(\alpha)$,
- if $\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$, then $\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$,
- if $\mathsf{C}\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$, then $\mathsf{E}\beta \wedge \mathsf{EC}\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha)$.

According to [16], the number of elements of $\mathsf{FL}_\mathsf{K}(\alpha)$ is of order $\mathcal{O}(|\alpha|)$, where $|\alpha|$ denotes the length of α.

The set Π, we want to use, is the *conjunctive closure of* $\mathsf{FL}_\mathsf{K}(\alpha)$, introduced as follows:

Given a formula α we fix an arbitrary enumeration

$$\delta_1, \delta_2, \ldots, \delta_n$$

of the elements of $\mathsf{FL}_\mathsf{K}(\alpha)$. Each subset M of $\mathsf{FL}_\mathsf{K}(\alpha)$ can then be written as

$$\{\delta_{s(1)}, \delta_{s(2)}, \ldots, \delta_{s(|M|)}\}$$

such that $1 \leq s(1) < s(2) < \ldots < s(|M|) \leq |M|$. And so for each subset M we can define exactly one formula φ_M representing it

$$\varphi_M \equiv (\ldots(\delta_{s(1)} \wedge \delta_{s(2)}) \ldots \wedge \delta_{s(|M|)}).$$

The conjunctive closure $\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\alpha)$ is now defined to be the set

$$\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\alpha) = \{\varphi_M \mid M \subseteq \mathsf{FL}_\mathsf{K}(\alpha)\} \cup \{\neg\varphi_M \mid M \subseteq \mathsf{FL}_\mathsf{K}(\alpha)\}.$$

Lemma 63 *Let α be a $\mathcal{L}^\mathsf{n}_\mathsf{C}$-formula and M be a maximal $\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\alpha)$-consistent set of formulae. For all $\beta \vee \gamma, \beta \wedge \gamma \in \mathsf{FL}_\mathsf{K}(\alpha)$ we have:*

1. *If $\beta \vee \gamma \in M$ then $\beta \in M$ or $\gamma \in M$.*
2. *If $\beta \wedge \gamma \in M$ then $\beta \in M$ and $\gamma \in M$.*

Proof. It can easily be seen that we have

$$\mathsf{T}_{\mathsf{K}^\mathsf{C}_\mathsf{n}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\alpha)\text{–}\mathsf{Cut}) \vdash \neg(\beta \vee \gamma), \beta, \gamma.$$

Since $\gamma \in \mathsf{FL}_\mathsf{K}(\alpha) \subset \mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\alpha)$ by Lemma 62.1 we have that either $\gamma \in M$ or $\neg\gamma \in M$. If $\gamma \in M$ then we are finished. If $\neg\gamma \in M$ then, since $\beta \vee \gamma, \neg\gamma \in M$, by Lemma 62.3 we have that $\beta \in M$. This gives us the first part. The second part is proven similarly. □

For each formula $\varphi \in \mathcal{L}^\mathsf{n}_\mathsf{C}$ we now introduce the crucial notion of *canonical $\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)$-model*. The canonical $\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)$-model

$$\mathcal{M}^{\mathsf{FL}_\mathsf{K}}_\varphi = (W, R_1, \ldots, R_n, \lambda)$$

is given by W, $R_i \subseteq W \times W$ and λ, defined as follows:

- $W = \{M \cap \mathsf{FL}_{\mathsf{K}}(\varphi) \mid M \text{ is maximal } \mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi)\text{-consistent}\}$,
- $R_i = \{(M, M') \mid M/\mathsf{K}_i \subseteq M'\}$,
- $\lambda(p) = \{M \mid p \in M\}$.

Clearly, any canonical $\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi)$-model has only finitely many worlds. Thus, for all sets of worlds $W' \subseteq W$ we can define a formula $\phi_{W'}$ as follows:

We fix an enumeration

$$M_1, M_2, \ldots, M_{|W|}$$

of all worlds $M \in W$. So, each set of worlds $W' \subseteq W$ can be written as

$$\{M_{t(1)}, M_{t(2)}, \ldots, M_{t(N)}\}$$

such that $1 \leq t(1) < t(2) < \ldots < t(N) \leq |W|$. For each subset W' we can define exactly one formula $\phi_{W'}$ representing it

$$\phi_{W'} \equiv (\ldots(\varphi_{M_{t(1)}} \vee \varphi_{M_{t(2)}}) \vee \ldots) \vee \varphi_{M_{t(N)}}).$$

This representation of subsets of worlds has the substantial disadvantage that we do not have $\phi_{W'} \in \mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi)$. However, we can get rid of this disadvantage by introducing for each set of worlds W' the formula $\phi_{\bigcap W'}$ as follows:

First we define for all $W' \subseteq W$ a set $\bigcap W' \subseteq \mathsf{FL}_{\mathsf{K}}(\varphi)$ as

$$\bigcap W' = \{\alpha \mid (\forall M \in W')\ \alpha \in M\}.$$

By using the enumeration $\{\delta_1, \ldots, \delta_n\}$ of $\mathsf{FL}_{\mathsf{K}}(\varphi)$, $\bigcap W'$ can be written as

$$\bigcap W' = \{\delta_{s(1)}, \delta_{s(2)}, \ldots, \delta_{s(|\bigcap W'|)}\}$$

such that $1 \leq s(1) < s(2) < \ldots < s(|\bigcap W'|) \leq |\bigcap W'|$. For each subset W' we can define exactly one formula $\phi_{\bigcap W'}$ representing it

$$\phi_{\bigcap W'} \equiv (\ldots(\delta_{s(1)} \wedge \delta_{s(2)}) \wedge \ldots) \wedge \delta_{s(|\bigcap W'|)}).$$

Let $\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{K}}}$ be a canonical model and let Γ be an arbitrary sequent. By $\hat{\Gamma}$ we denote the sequent which is obtained by substituting, in all $\alpha \in \Gamma$, the subformulae of the form $\phi_{W'}$, where W' is a set of worlds in $\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{K}}}$, by $\phi_{\bigcap W'}$.

The next lemma shows that the provability of a sequent Γ implies the provability of $\hat{\Gamma}$. In this sense it allows us to reduce the complexity of a provable sequent in such a way that we can cut off formulae of the form $\phi_{W'}$ although they are not in the conjunctive closure. Remark 65, after the lemma, explains this fact.

Lemma 64 *Let $\mathcal{M}_\varphi^{\mathsf{FL_K}}$ be a canonical model. For all sequents $\Gamma \subset \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ we have*

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash^n \Gamma \quad \Rightarrow \quad \mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \hat{\Gamma}.$$

Proof. The proof goes by induction on n. The case where $n = 0$ is trivial. If $n > 0$ we do the induction step by case distinction on the last inference in the proof of Γ, this goes through straightforward except for the case where we have a last inference of the form

$$\frac{\Gamma, \phi_{W''}, \varphi_M}{\Gamma, \phi_{W'}}$$

where $\phi_{W'} \equiv \phi_{W''} \vee \varphi_M$ and $W' = W'' \cup \{M\}$. In this case we have

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash^{<n} \Gamma, \phi_{W''}, \varphi_M$$

and by induction hypothesis (notice that $\varphi_M \equiv \phi_{\{M\}}$)

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \hat{\Gamma}, \phi_{\bigcap W''}, \varphi_M.$$

Since it can easily be shown that for all subsets $\bar{W} \subseteq W$ and all $\alpha \in \bigcap \bar{W}$ we have

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \neg\phi_{\bigcap \bar{W}}, \alpha$$

and since $\phi_{\bigcap \bar{W}} \in \mathcal{C}_{\mathsf{FL_K}}(\varphi)$, with Lemma 59 and $(\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut})$, we get for all formulae $\alpha_i \in \bigcap W''$ and $\alpha_j \in M$

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \hat{\Gamma}, \alpha_i, \alpha_j.$$

Thus, for all $\alpha_i \in \bigcap W' = \bigcap W'' \cap M$ we have

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \hat{\Gamma}, \alpha_i.$$

And so we easily can derive

$$\mathsf{T}_{\mathsf{K_n^C}} + (\mathcal{C}_{\mathsf{FL_K}}(\varphi){-}\mathsf{Cut}) \vdash \hat{\Gamma}, \phi_{\bigcap W'}$$

which completes the induction and proves the lemma. □

Remark 65 *The following simple example explains how the previous lemma can be used. Suppose that the sequents $\Gamma, \phi_{W'}$ and $\Gamma, \neg\phi_{W'}$ are provable, and suppose that $\Gamma = \hat{\Gamma}$. Since, in general, $\phi_{W'} \notin \mathcal{C}_{\mathsf{FL_K}}(\varphi)$ we can not immediately infer Γ. But with the Lemma 64 we get that $\Gamma, \phi_{\bigcap W'}$ and $\Gamma, \neg\phi_{\bigcap W'}$ are provable and, since $\phi_{\bigcap W'} \in \mathcal{C}_{\mathsf{FL_K}}(\varphi)$, we can infer Γ.*

Lemma 66 *Let φ be a $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formula. For all formulae $\alpha \in \mathsf{FL}_{\mathsf{K}}(\varphi)$ and all worlds M of $\mathcal{M}^{\mathsf{FL}_{\mathsf{K}}}_{\varphi}$ we have*

$$\alpha \in M \quad \Rightarrow \quad M \models \alpha.$$

Proof. We prove the lemma by induction on $\mathsf{me}(\alpha)$.

$\alpha \equiv (\neg)p$: Follows directly from the definition.

$\alpha \equiv \beta \wedge \gamma, \alpha \equiv \beta \vee \gamma$: These two cases follow by Lemma 63.

$\alpha \equiv \mathsf{K}_i\beta$: This fact follows easily from the induction hypothesis and the definition of the accessibility relation R_i.

$\alpha \equiv \neg\mathsf{K}_i\beta$: If $\neg\mathsf{K}_i\beta \in M$ then, since M is consistent, we know that

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \nvdash \{\neg\mathsf{K}_i\gamma \mid \mathsf{K}_i\gamma \in M\}, \mathsf{K}_i\beta$$

and since we have that from

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \vdash \{\neg\gamma \mid \mathsf{K}_i\gamma \in M\}, \beta$$

we can derive

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \vdash \{\neg\mathsf{K}_i\gamma \mid \mathsf{K}_i\gamma \in M\}, \mathsf{K}_i\beta$$

we get

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \nvdash \{\neg\gamma \mid \mathsf{K}_i\gamma \in M\}, \beta.$$

So, by Lemma 61 there exists a $\bar{M}$ which is a maximal consistent extension of $\{\gamma \mid \mathsf{K}_i\gamma \in M\}, \neg\beta$. Define $M' = \mathsf{FL}_{\mathsf{K}}(\varphi) \cap \bar{M}$. By the induction hypothesis we get $M' \not\models \beta$ and since, by construction, $(M, M') \in R_i$ we get $M \not\models \mathsf{K}_i\beta$.

$\alpha \equiv \mathsf{C}\beta$: By Lemma 54 and Lemma 60 we have

$$\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \vdash \neg\mathsf{C}\beta, \mathsf{EC}\beta \quad \text{and} \quad \mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{K}}}(\varphi){-}\mathsf{Cut}) \vdash \neg\mathsf{C}\beta, \mathsf{E}\beta.$$

By Lemma 62.3. $\mathsf{EC}\beta \in M$ and $\mathsf{E}\beta \in M$. Thus, if M' is accessible from M in one step we get $\mathsf{C}\beta \in M'$ and $\beta \in M'$. Since $\mathsf{C}\beta \in M'$ for all these M', we get the same result for all worlds accessible in two steps. Inductively we get $\beta \in M''$ for all worlds accessible from M, and by induction hypothesis all these worlds fulfill β. This gives $M \models \mathsf{C}D$.

$\alpha \equiv \neg\mathsf{C}\beta$: We have to show $M \models \neg\mathsf{C}\beta$. Equivalently, by Lemma 62, we show

$$M \models \mathsf{C}\beta \quad \Rightarrow \quad \mathsf{C}\beta \in M.$$

To do that it is enough to show for the set of worlds $W' = \|\mathsf{C}\beta\|_{\mathcal{M}^{\mathsf{FL}_{\mathsf{K}}}_{\varphi}}$:

1. $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\phi_{W'}, \mathsf{E}\beta$,
2. $\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\phi_{W'}, \mathsf{E}(\phi_{W'})$.

Let us first prove, that these two facts are enough to do the induction step. If they are proven then by the (Ind)-rule we have

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\phi_{W'}, \mathsf{C}\beta.$$

By Lemma 64 we have

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\phi_{\bigcap W'}, \mathsf{C}\beta$$

(notice that $\mathsf{C}\beta \in M \subseteq \mathsf{FL}_\mathsf{K}(\varphi)$ can not contain a subformula of the form $\phi_{W'}$). By Lemma 60, since $\neg\phi_{\bigcap W'} \in \mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)$, we get

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \{\neg\alpha \mid \alpha \in \bigcap W'\}, \mathsf{C}\beta.$$

Since $\bigcap W' \subset M'$ for all worlds $M' \in W'$ we easily can derive

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg M', \mathsf{C}\beta$$

for all such M'. By Lemma 62 we get $\mathsf{C}\beta \in M'$ for all M' fulfilling $\mathsf{C}\beta$, which completes the induction.

So, it remains to prove the two assertions. The first follows easily from the fact that for all $M \models \mathsf{C}\beta$ we have $M \models \mathsf{E}\beta$ and so by induction hypothesis and Lemma 62 we have $\mathsf{E}\beta \in M$. This gives us for all such M

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\varphi_M, \mathsf{E}\beta$$

and thus

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg\phi_{W'}, \mathsf{E}\beta.$$

Before we show the second assertion let us prove the following fact for all states M

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \vdash \neg(M/\mathsf{K}_i), \{\varphi_{M'} \mid M' \in R_i(M)\}.$$

Suppose not. Then we must have for all $M_i \in R_i(M)$ a formula $\delta_{M_i} \in M_i$ such that

$$\mathsf{T}_{\mathsf{K}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\text{--}\mathsf{Cut}) \nvdash \neg(M/\mathsf{K}_i), \{\delta_{M_i} \mid M_i \in R_i(M)\}.$$

Thus, $M/\mathsf{K}_i, \{\neg\delta_{M_i} \mid M_i \in R_i(M)\}$ is consistent. This is a contradiction, since in this case, by Lemma 61, there would be a maximal consistent extension, which is in $R_i(M)$ but which, by Lemma 62, is also different from all $M' \in R_i(M)$. So the fact is proven. From this fact we easily get

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\mathsf{-Cut}) \vdash \neg M, \mathsf{K}_i\phi_{R_i(M)}.$$

Since for all $M \models \mathsf{C}\beta$ and all i we have $R_i(M) \subseteq \|\mathsf{C}\beta\|_{\mathcal{M}_\varphi^{\mathsf{FL}_\mathsf{K}}} = W'$ with the previous assertion we get for all $M \models \mathsf{C}\beta$

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\mathsf{-Cut}) \vdash \neg M, \mathsf{E}\phi_{W'}.$$

And so also the second assertion, namely

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\mathsf{-Cut}) \vdash \neg\phi_{W'}, \mathsf{E}\phi_{W'}.$$

□

Theorem 67 *For all formulae $\varphi \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ we have*

$$C^{\mathbf{K}} \models \varphi \quad \Leftrightarrow \quad \mathsf{T}_{\mathsf{K}_n^C} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\mathsf{-Cut}) \vdash \varphi.$$

Proof. The direction from right to the left is given by Proposition 52. For the other direction we prove the contraposition. Suppose

$$\mathsf{T}_{\mathsf{K}_n^C} + (\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)\mathsf{-Cut}) \not\vdash \varphi,$$

thus, $\neg\varphi$ is $\mathcal{C}_{\mathsf{FL}_\mathsf{K}}(\varphi)$-consistent and, by Lemma 61, there is a state M in $\mathcal{M}_\varphi^{\mathsf{FL}_\mathsf{K}}$ containing $\neg\varphi$. By Lemma 66 we have $M \models \neg\varphi$ and so we have

$$C^{\mathbf{K}} \not\models \varphi$$

which proves the theorem. □

10.2 Not only $\mathsf{T}_{\mathsf{K}_n^C}$

In this section we apply the technique of Fischer-Ladner cuts to the logics of common knowledge over **T**, **S4** and **S5**. We will give a completeness result for the Tait-style calculi $\mathsf{T}_{\mathsf{T}_n^C}$, $\mathsf{T}_{\mathsf{S4}_n^C}$ and $\mathsf{T}_{\mathsf{S5}_n^C}$ with the respective Fischer-Ladner cuts.

10.2.1 Partial Cut Elimination for $\mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}}$

The calculus $\mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}}$ corresponds to $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with the only difference that we add the rule $(\neg\mathsf{K}_i)$ presented below.

$$\frac{\Gamma, \neg\alpha}{\Gamma, \neg\mathsf{K}_i\alpha}\ (\neg\mathsf{K}_i)$$

The notions of Π-consistent and maximal Π-consistent set of formulae are defined analogously as they were defined for the calculus $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$. The next lemma is a reformulation of Lemma 62.3.

Lemma 68 *Let $\Pi \subseteq \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a set closed under negation and let M be maximal Π-consistent. For all sequents Γ and formulae α we have*

$$\mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi{-}\mathsf{Cut}) \vdash \neg\Gamma, \alpha \quad \textit{and} \quad \Gamma \subseteq M \quad \textit{and} \quad \alpha \in \Pi \quad \Rightarrow \quad \alpha \in M.$$

Further, we define the *Fischer-Ladner closure* FL_T to be the same as FL_K and the *conjunctive closure* $\mathcal{C}_{\mathsf{FL}_\mathsf{T}}$ *of* FL_T to be the same as the conjunctive closure of FL_K, too. Given an arbitrary formula $\varphi \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ the *canonical* $\mathcal{C}_{\mathsf{FL}_\mathsf{T}}(\varphi)$*-model* is given by

$$\mathcal{M}^{\mathsf{FL}_\mathsf{T}}_{\varphi} = (W, R_1, \ldots, R_n, \lambda)$$

whereby W, $R_i \subseteq W \times W$ and λ are defined as follows:

- $W = \{M \cap \mathsf{FL}_\mathsf{T}(\varphi) \mid M \text{ is maximal } \mathcal{C}_{\mathsf{FL}_\mathsf{T}}(\varphi)\text{-consistent}\}$,
- $R_i = \{(M, M') \mid M/\mathsf{K}_i \subseteq M'\}$,
- $\lambda(p) = \{M \mid p \in M\}$.

The following lemma shows us that $\mathcal{M}^{\mathsf{FL}_\mathsf{T}}_{\varphi}$ is indeed a model in $C^{\mathbf{T}}$, that is, the accessibility relations are reflexive.

Lemma 69 *Let $\varphi \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be an arbitrary formula. We have*

$$\mathcal{M}^{\mathsf{FL}_\mathsf{T}}_{\varphi} \in C^{\mathbf{T}}.$$

Proof. We have to show that for each world M and each accessibility relation R_i of $\mathcal{M}^{\mathsf{FL}_\mathsf{T}}_{\varphi}$ we have $(M, M) \in R_i$ which is equivalent to

$$M/\mathsf{K}_i \subseteq M.$$

Suppose $\mathsf{K}_i\beta \in M$ since it can easily be seen that we have

$$\mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}} \vdash \neg\mathsf{K}_i\beta, \beta$$

by Lemma 68 we get $\beta \in M$ and thus the lemma. □

With the same techniques used in Lemma 66 we can prove the following lemma.

Lemma 70 *Let φ be a $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formula. For all formulae $\alpha \in \mathsf{FL}_{\mathsf{T}}(\varphi)$ and all worlds M of $\mathcal{M}^{\mathsf{FL}_{\mathsf{T}}}_{\varphi}$ we have*

$$\alpha \in M \quad \Rightarrow \quad M \models \alpha$$

Since Lemma 69 assures that $\mathcal{M}^{\mathsf{FL}_{\mathsf{T}}}_{\varphi}$ is in $C^{\mathbf{T}}$ we get the completeness.

Theorem 71 *For all formulae φ we have*

$$C^{\mathbf{T}} \models \varphi \quad \Leftrightarrow \quad \mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{T}}}(\varphi){-}\mathsf{Cut}) \vdash \varphi.$$

10.2.2 Partial Cut Elimination for $\mathsf{T}_{\mathsf{S4}^{\mathsf{C}}_{\mathsf{n}}}$

The calculus $\mathsf{T}_{\mathsf{S4}^{\mathsf{C}}_{\mathsf{n}}}$ corresponds to $\mathsf{T}_{\mathsf{T}^{\mathsf{C}}_{\mathsf{n}}}$ whereby we replace (K_i) by $(\mathsf{S4}_i)$.

$$\frac{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \alpha}{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \mathsf{K}_i\alpha, \Sigma}\ (\mathsf{S4}_i)$$

Again, we adapt the notions of Π-consistent and maximal Π-consistent to the calculus $\mathsf{T}_{\mathsf{S4}^{\mathsf{C}}_{\mathsf{n}}}$ and state a lemma which is a reformulation of Lemma 62.3.

Lemma 72 *Let $\Pi \subseteq \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a set closed under negation and let M be maximal Π-consistent. For all sequents Γ and formulae α we have*

$$\mathsf{T}_{\mathsf{S4}^{\mathsf{C}}_{\mathsf{n}}} + (\Pi{-}\mathsf{Cut}) \vdash \neg\Gamma, \alpha \quad \text{and} \quad \Gamma \subseteq M \quad \text{and} \quad \alpha \in \Pi \quad \Rightarrow \quad \alpha \in M.$$

For each formula $\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ the *Fischer-Ladner closure* $\mathsf{FL}_{\mathsf{S4}}$ is defined to be

$$\mathsf{FL}_{\mathsf{S4}}(\alpha) = \mathsf{FL}_{\mathsf{K}}(\alpha) \cup \{\mathsf{K}_i\mathsf{K}_i\beta \mid \mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha) \text{ and } 1 \leq i \leq n\}.$$

Starting from $\mathsf{FL}_{\mathsf{S4}}$ we use exactly the same techniques, as we used for FL_{K}, to define the *conjunctive closure* $\mathcal{C}_{\mathsf{FL}_{\mathsf{S4}}}$ *of* $\mathsf{FL}_{\mathsf{S4}}$. Given an arbitrary formula $\varphi \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ the *canonical* $\mathcal{C}_{\mathsf{FL}_{\mathsf{S4}}}(\varphi)$*-model* is given by

$$\mathcal{M}^{\mathsf{FL}_{\mathsf{S4}}}_{\varphi} = (W, R_1, \ldots, R_n, \lambda)$$

whereby W, $R_i \subseteq W \times W$ and λ are defined as follows:

- $W = \{M \cap \mathsf{FL}_{\mathsf{S4}}(\varphi) \mid M \text{ is maximal } \mathcal{C}_{\mathsf{FL}_{\mathsf{S4}}}(\varphi)\text{-consistent}\}$,
- $R_i = \{(M, M') \mid M/\mathsf{K}_i \subseteq M'\}$,
- $\lambda(p) = \{M \mid p \in M\}$.

Lemma 73 *Let $\varphi \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ be an arbitrary formula. We have*

$$\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{S4}}} \in C^{\mathbf{S4}}.$$

Proof. We have to show that each relation R_i is reflexive and transitive. The reflexivity is shown as in Lemma 69, for the transitivity it is enough to show

(1) $\quad \mathsf{K}_i\beta \in M \text{ and } (M, M'') \in R_i \text{ and } (M'', M') \in R_i \quad \Rightarrow \quad \beta \in M'.$

We distinguish two cases. In the first case we assume that $\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{K}}(\varphi)$ then by definition of $\mathsf{FL}_{\mathsf{S4}}(\varphi)$ we have $\mathsf{K}_i\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{S4}}(\varphi)$. It can easily be seen that we have

(2) $$\mathsf{T}_{\mathsf{S4}_{\mathsf{n}}^{\mathsf{C}}} \vdash \neg\mathsf{K}_i\beta, \mathsf{K}_i\mathsf{K}_i\beta$$

and with Lemma 72 we get

$$\mathsf{K}_i\mathsf{K}_i\beta \in M.$$

Thus, assertion (1) follows from the definition of R_i. For the second case we assume that $\mathsf{K}_i\beta \notin \mathsf{FL}_{\mathsf{K}}(\varphi)$ but $\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{S4}}(\varphi)$. By definition of $\mathsf{FL}_{\mathsf{S4}}(\varphi)$ we get that $\beta \equiv \mathsf{K}_i\gamma$. Suppose

$$\mathsf{K}_i\beta \in M \quad \text{and} \quad (M, M'') \in R_i,$$

by definition of R_i we get $\beta \in M''$. Since $\beta \equiv \mathsf{K}_i\gamma$ with Lemma 72 and assertion (2) we get

$$\mathsf{K}_i\beta \in M''$$

and then we easily get assertion (1). □

Again, we can adapt the proof of Lemma 66 to prove the following lemma.

Lemma 74 *Let φ be a $\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$-formula. For all formulae $\alpha \in \mathsf{FL}_{\mathsf{S4}}(\varphi)$ and all worlds M of $\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{S4}}}$ we have*

$$\alpha \in M \quad \Rightarrow \quad M \models \alpha$$

Since Lemma 73 assures that $\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{S4}}}$ is in $C^{\mathbf{S4}}$ we get the completeness.

Theorem 75 *For all formulae φ we have*

$$C^{\mathbf{S4}} \models \varphi \quad \Leftrightarrow \quad \mathsf{T}_{\mathsf{S4}_{\mathsf{n}}^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{S4}}}(\varphi){-}\mathsf{Cut}) \vdash \varphi.$$

10.2.3 Partial Cut Elimination for $\mathsf{T}_{\mathsf{S5}_n^C}$

The calculus $\mathsf{T}_{\mathsf{S5}_n^C}$ corresponds to $\mathsf{T}_{\mathsf{T}_n^C}$ whereby we replace (K_i) by $(\mathsf{S5}_i)$.

$$\frac{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \mathsf{K}_i\Omega, \alpha}{\neg\mathsf{C}\Delta, \neg\mathsf{K}_i\Gamma, \mathsf{K}_i\Omega, \mathsf{K}_i\alpha, \Sigma}\ (\mathsf{S5}_i)$$

Also for $\mathsf{T}_{\mathsf{S5}_n^C}$ we define the appropriate notions of Π-consistent and maximal Π-consistent sets of formulae and reformulate Lemma 62 as follows.

Lemma 76 *Let $\Pi \subseteq \mathcal{L}_C^n$ be a set closed under negation and let M be maximal Π-consistent. For all sequents Γ and formulae α we have:*

1. $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$ or $\neg\alpha \in M$.
2. $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$ iff $\neg\alpha \notin M$.
3. $\mathsf{T}_{\mathsf{S5}_n^C} + (\Pi\text{−}\mathsf{Cut}) \vdash \neg\Gamma, \alpha$ *and* $\Gamma \subseteq M$ *and* $\alpha \in \Pi \quad \Rightarrow \quad \alpha \in M$.

For each formula $\alpha \in \mathcal{L}_C^n$ the *Fischer-Ladner closure* $\mathsf{FL}_{\mathsf{S5}}$ is defined to be

$$\mathsf{FL}_{\mathsf{S5}}(\alpha) = S(\alpha) \cup \neg S(\alpha),$$

whereby

$$\begin{aligned} S(\alpha) = \ & \mathsf{FL}_{\mathsf{K}}(\alpha) \ \cup \\ & \{\mathsf{K}_i\neg\mathsf{K}_i\beta \mid \neg\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha) \text{ and } 1 \le i \le n\} \ \cup \\ & \{\mathsf{K}_i\mathsf{K}_i\beta \mid \mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{K}}(\alpha) \text{ and } 1 \le i \le n\}. \end{aligned}$$

Again, we define the *conjunctive closure* $\mathcal{C}_{\mathsf{FL}_{\mathsf{S5}}}$ *of* $\mathsf{FL}_{\mathsf{S5}}$ analogously as we did it for the other logics. Given an arbitrary formula $\varphi \in \mathcal{L}_C^n$ the *canonical* $\mathcal{C}_{\mathsf{FL}_{\mathsf{S5}}}(\varphi)$*-model* is given by

$$\mathcal{M}_\varphi^{\mathsf{FL}_{\mathsf{S5}}} = (W, R_1, \ldots, R_n, \lambda)$$

whereby W, $R_i \subseteq W \times W$ and λ are defined as follows:

- $W = \{M \cap \mathsf{FL}_{\mathsf{S5}}(\varphi) \mid M$ is maximal $\mathcal{C}_{\mathsf{FL}_{\mathsf{S5}}}(\varphi)$-consistent$\}$,
- $R_i = \{(M, M') \mid M/\mathsf{K}_i \subseteq M'\}$,
- $\lambda(p) = \{M \mid p \in M\}$.

Lemma 77 *Let $\varphi \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$ be an arbitrary formula. We have*

$$\mathcal{M}_{\varphi}^{\mathsf{FL}_{\mathsf{S5}}} \in C^{\mathbf{S5}}.$$

Proof. We have to show that each relation R_i is reflexive, transitive and symmetric. The reflexivity is shown like in Lemma 69 and the transitivity like in Lemma 73. To show symmetry we have to show that $(M, M') \in R_i$ implies $(M', M) \in R_i$. It is enough to show for all $(M, M') \in R_i$ and all $\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{S5}}(\varphi)$

$$\mathsf{K}_i\beta \in M' \quad \Rightarrow \quad \beta \in M.$$

By Lemma 76.2 this is equivalent to

$$\neg\beta \in M \quad \Rightarrow \quad \neg\mathsf{K}_i\beta \in M'.$$

We distinguish two cases. For the first case, suppose that $\mathsf{K}_i\neg\mathsf{K}_i\beta \in \mathsf{FL}_{\mathsf{S5}}(\varphi)$. If $(M, M') \in R_i$ and $\neg\beta \in M$ since we have

$$\mathsf{T}_{\mathsf{S5}_{\mathsf{n}}^{\mathsf{C}}} \vdash \neg\mathsf{K}_i\beta, \beta \tag{1}$$

by Lemma 76.3 we get $\neg\mathsf{K}_i\beta \in M$ and since we have

$$\mathsf{T}_{\mathsf{S5}_{\mathsf{n}}^{\mathsf{C}}} \vdash \mathsf{K}_i\beta, \mathsf{K}_i\neg\mathsf{K}_i\beta \tag{2}$$

by Lemma 76.3 we get

$$\mathsf{K}_i\neg\mathsf{K}_i\beta \in M.$$

By the definition of R_i we get the desired result. For the second case suppose that $\mathsf{K}_i\neg\mathsf{K}_i\beta \notin \mathsf{FL}_{\mathsf{S5}}(\varphi)$. By definition of $\mathsf{FL}_{\mathsf{S5}}(\varphi)$ we know that there is a γ such that

- $\mathsf{K}_i\beta \equiv \mathsf{K}_i\mathsf{K}_i\gamma$ or
- $\mathsf{K}_i\beta \equiv \mathsf{K}_i\neg\mathsf{K}_i\gamma$.

$\mathsf{K}_i\beta \equiv \mathsf{K}_i\mathsf{K}_i\gamma$: With assertion (2), Lemma 76.3 and the definition of R_i we get

$$\neg\mathsf{K}_i\gamma \in M \quad \Rightarrow \quad \neg\mathsf{K}_i\gamma \in M'.$$

With assertion (1) and Lemma 76.3 we obtain

$$\neg\mathsf{K}_i\mathsf{K}_i\gamma \in M'$$

and, since $\beta \equiv \mathsf{K}_i\gamma$ this case is proven.

$\mathsf{K}_i\beta \equiv \mathsf{K}_i\neg\mathsf{K}_i\gamma$: By definition of R_i

$$\mathsf{K}_i\gamma \in M \quad \Rightarrow \quad \gamma \in M'.$$

With assertion (1) and Lemma 76.3 we obtain

$$\neg\mathsf{K}_i\gamma \in M'.$$

With assertion (1) and Lemma 76.3 we obtain

$$\neg\mathsf{K}_i\neg\mathsf{K}_i\gamma \in M'$$

and, since $\beta \equiv \neg\mathsf{K}_i\gamma$, also this case is proven which gives us the lemma. □

Again, we can adapt the proof of Lemma 66 to prove the following lemma.

Lemma 78 *Let φ be a $\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}$-formula. For all formulae $\alpha \in \mathsf{FL}_{\mathsf{S5}}(\varphi)$ and all worlds M of $\mathcal{M}_\varphi^{\mathsf{FL}_{\mathsf{S5}}}$ we have*

$$\alpha \in M \quad \Rightarrow \quad M \models \alpha.$$

Since Lemma 77 assures that $\mathcal{M}_\varphi^{\mathsf{FL}_{\mathsf{S5}}}$ is in $C^{\mathbf{S5}}$ we get the completeness.

Theorem 79 *For all formulae φ we have*

$$C^{\mathbf{S5}} \models \varphi \quad \Leftrightarrow \quad \mathsf{T}_{\mathsf{S5}_n^{\mathsf{C}}} + (\mathcal{C}_{\mathsf{FL}_{\mathsf{S5}}}(\varphi){-}\mathsf{Cut}) \vdash \varphi.$$

Chapter 11

An Infinitary Calculus

"Zwei Dinge sind unendlich: Das Universum und die menschliche Dummheit. Aber bei dem Universum bin ich mir noch nicht ganz sicher." (A. Einstein)

In this chapter we follow the iterative approach by presenting the infinitary calculus $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ which reflects the fact that $\mathsf{C}\varphi$ can be interpreted as $\bigwedge_{i\geq 1} \mathsf{E}^i\varphi$.

In the first section, after introducing this infintary system for the logic of common knowledge over **K**, we prove the completeness for the system without any cuts. Hence, we provide a cut free system with the subformula property, where $\mathsf{E}^i\varphi$ counts as subformula of $\mathsf{C}\varphi$. This result is also interesting in connection with the work of Kaneko and Nagashima ([28, 29]). They introduce an other infinitary system and obtain a cut elimination result. However, the cut-free system they obtain does not have the subformula property.

In the second section we start from this infinitary cut-free system to get finitary ones for the positive and negative fragments. For the negative fragment this result follows immediately from the fact that any proof of a negative formula is finite. For the positive fragment we provide two results. The first one states that each provable positive formula has a proof with finite length. The second result is the completeness of a finite calculus obtained from the infinitary one.

11.1 Completeness of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$

The infinitary calculus $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ has the same axioms and basic inference rules as $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$. The C-rules consist of $(\neg\mathsf{C})$ and the infinitary (C^{ω})-rule.

$$\frac{\Gamma, \mathsf{E}^m\alpha \quad \text{for all } m \geq 1}{\Gamma, \mathsf{C}\alpha} \ (\mathsf{C}^\omega)$$

Further, we add a new rule to introduce the K_i operator, the (K_i^*)-rule. For any natural number m we have the following inference scheme.

$$\frac{\alpha}{\mathsf{K}_i\mathsf{E}^m\alpha, \Sigma} \ (\mathsf{K}_i^*)$$

Of course, from α we can derive the sequent $\vdash \mathsf{K}_i\mathsf{E}^m\alpha, \Sigma$ without using the (K_i^*)-rule. In this sense the (K_i^*)-rule seems to be superfluous. On the other hand the use of this rule changes substantially the length of the proofs.

It can easily be seen that the use of the (C^ω)-rule can lead to an infinite proof. Thus, the ordinals, denoted by the small Greek letters $\sigma, \tau, \eta, \xi, \ldots$ (possibly with subscripts), are necessary to characterize the proof-length. The *derivability in σ many steps* is then defined analogously to the derivability for the finitary proof system. Let us repeat that notion: The conclusions of (ID) are derivable in arbitrary many steps, and if all the premises of an inference rule are derivable in less than σ many steps then the conclusion is derivable in σ many steps. Again, we write

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma$$

if Γ is derivable in σ many steps, we write

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^{<\sigma} \Gamma$$

if it is derivable in less than σ many steps and we write

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash \Gamma$$

if there is an ordinal σ such that Γ is derivable in σ many steps. The following lemma can easily be proven by induction on the proof-length.

Lemma 80 *For all sequents $\Gamma \subset \mathcal{L}^\mathsf{n}_\mathsf{C}$, formulae $\alpha, \beta \in \mathcal{L}^\mathsf{n}_\mathsf{C}$ and ordinals σ we have:*

1. $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma, \alpha \vee \beta \quad \Rightarrow \quad \mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma, \alpha, \beta.$
2. $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma, \alpha \wedge \beta \quad \Rightarrow \quad \mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma, \alpha$ *and* $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma, \beta.$
3. $\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \mathsf{K}_i\alpha \quad \Rightarrow \quad \mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \alpha.$

Let $\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a formula, the *depth of* α, $\mathsf{depth}(\alpha)$, is defined by induction on the structure of the formula α:

- $\mathsf{depth}(P) = \mathsf{depth}(\sim P) = 0$,
- $\mathsf{depth}(\beta \wedge \gamma) = \mathsf{depth}(\beta \vee \gamma) = \max\{\mathsf{depth}(\beta), \mathsf{depth}(\gamma)\} + 1$,
- $\mathsf{depth}(\mathsf{K}_i\beta) = \mathsf{depth}(\sim \mathsf{K}_i\beta) = \mathsf{depth}(\beta) + 1$,
- $\mathsf{depth}(\mathsf{C}\beta) = \mathsf{depth}(\sim \mathsf{C}\beta) = \mathsf{sup}\{\mathsf{depth}(\mathsf{E}^m\beta) \mid m \geq 1\}$.

It can easily be seen that each formula containing a subformula of the form $\mathsf{C}\beta$ has an infinite depth. This definition emphasizes the fact that a formula $\mathsf{C}\beta$ can be seen as the infinite conjunction $\bigwedge_{m\geq 1} \mathsf{E}^m\beta$.

The following theorem, the correctness of the $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$, can be proven by induction on the possibly transfinite proof-length.

Theorem 81 *For all sequents $\Gamma \subset \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ and all ordinals σ we have*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash^{\sigma} \Gamma \quad \Rightarrow \quad C^{\mathbf{K}} \models \bigvee \Gamma.$$

The completeness of $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with general cuts can be proven by showing that each formula derivable in $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with general cut is derivable in the infinitary system. Let us prove that result before we establish a much stronger one, the completeness without cuts. First, we show that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ admits the induction rule.

Lemma 82 *Let α and β be $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formulae and suppose that*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \neg\alpha, \mathsf{E}\alpha \quad \textit{and} \quad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \neg\alpha, \mathsf{E}\beta.$$

Then we have

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \neg\alpha, \mathsf{C}\beta.$$

Proof. From the two premises we can deduce for all natural numbers m by several applications of all the (K_i)-rules and the $(\wedge)$-rule

$$(1) \qquad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \neg\mathsf{E}^m\alpha, \mathsf{E}^{m+1}\alpha$$

$$(2) \qquad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G{-}Cut}) \vdash \neg\mathsf{E}^m\alpha, \mathsf{E}^{m+1}\beta.$$

By applying the appropriate cuts to (1) and (2) and the two premises of the lemma we get for all natural numbers $m \geq 1$

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G}{-}\mathsf{Cut}) \vdash \neg\alpha, \mathsf{E}^m\beta.$$

Thus, by applying the (C^ω)-rule, we get the desired result. □

The lemma states that we can prove the induction (Ind) in our infinitary calculus where this rule is replaced by (C^ω). Thus every formula provable in $\mathsf{T}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ with general cut is provable in the infinitary calculus. From this fact, from Theorem 81 and from Corollary 57 we get the following theorem.

Theorem 83 *Let $\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ be a formula. We have*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} + (\mathsf{G}{-}\mathsf{Cut}) \vdash \alpha \quad \Leftrightarrow \quad C^{\mathbf{K}} \models \alpha.$$

As said before we can even prove the completeness of the calculus without the use of cuts. In order to do this completeness proof we need some new notions.
A sequent $\Gamma \subset \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ is called *saturated* if the following conditions are satisfied:

1. $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Gamma$.
2. For all formulae $\alpha \vee \beta \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ we have
$$\alpha \vee \beta \in \Gamma \quad \Rightarrow \quad \alpha \in \Gamma \text{ and } \beta \in \Gamma.$$
3. For all formulae $\alpha \wedge \beta \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ we have
$$\alpha \wedge \beta \in \Gamma \quad \Rightarrow \quad \alpha \in \Gamma \text{ or } \beta \in \Gamma.$$
4. For all formulae $\neg\mathsf{C}\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ we have
$$\neg\mathsf{C}\alpha \in \Gamma \quad \Rightarrow \quad \neg\mathsf{E}\alpha \in \Gamma.$$
5. For all formulae $\mathsf{C}\alpha \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ we have
$$\mathsf{C}\alpha \in \Gamma \quad \Rightarrow \quad \mathsf{E}^m\alpha \in \Gamma \text{ for some } m \geq 1.$$

For any non provable sequent Γ there is a saturated sequent $\Sigma \supseteq \Gamma$. It can be constructed by adding systematically formulae such that also the conditions 2 to 5 become satisfied. The next lemma assures that this process of adding formulae really works, that is, it terminates.

Lemma 84 *For each non provable sequent* $\Gamma \subset \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ *there is a saturated sequent* Σ *which contains* Γ.

Proof. We fix an enumeration $\delta_0, \delta_1, \ldots$ of all the $\mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$-formulae. If the formula α is the formula δ_i in this enumeration then we call i the *index* of α. Depending on this enumeration for each non-provable sequent Δ we define a sequent $\Delta' \supseteq \Delta$:

1. If Δ is saturated then $\Delta = \Delta'$.

2. If Δ is not saturated then we take the formula $\alpha \in \Delta$ with the smallest index for which one of the conditions 2 to 5 of the definition of saturated sequent is violated. Depending on the structure of this α we define Δ'.

2.1. If α is of the form $\beta \vee \gamma$ then

$$\Delta' = \Delta \cup \{\beta, \gamma\}.$$

2.2. If α is of the form $\beta \wedge \gamma$ then since Δ is not provable we know that

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Delta, \beta \quad \text{or} \quad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Delta, \gamma.$$

We set

$$\Delta' = \begin{cases} \Delta \cup \{\beta\} & \text{if } \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Delta, \beta, \\ \Delta \cup \{\gamma\} & \text{if } \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Delta, \gamma. \end{cases}$$

2.3. If α is of the form $\mathsf{C}\beta$. Since Δ is not provable we know that there is a natural number $m \geq 1$ such that

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \not\vdash \Delta, \mathsf{E}^m\beta.$$

We take the least such m and set

$$\Delta' = \Delta \cup \{\mathsf{E}^m\beta\}.$$

2.4. If α is of the form $\neg\mathsf{C}\beta$ then we set

$$\Delta' = \Delta \cup \{\neg\mathsf{E}\beta\}.$$

Observe that this construction assures that Δ' is not provable, too. We now show that with a finite iteration of this process for any non provable Γ we reach the saturated superset Σ. In order to show the termination of this process for each non provable Δ we introduce the notion of *deficiency-number* $\mathsf{dn}(\Delta)$.

If Δ is saturated then we set $\mathsf{dn}(\Delta) = 0$. Otherwise, let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the set of all elements of Δ which violate one of the conditions 2 to 5 of the definition of saturated sequent. In this case we set

$$\mathsf{dn}(\Delta) = \omega^{\mathsf{depth}(\alpha_1)} \ \# \ \omega^{\mathsf{depth}(\alpha_2)} \ \# \ldots \# \ \omega^{\mathsf{depth}(\alpha_k)}$$

where we make use of the natural sum of ordinals as introduced, for example, in Schütte [43].

Given a non-provable sequent Γ we define a sequence $\Gamma_0, \Gamma_1, \ldots$ of sequents such that

$$\Gamma_0 = \Gamma \quad \text{and} \quad \Gamma_{m+1} = \Gamma'_m$$

for all natural numbers m. Clearly, we have for all m:

- $\Gamma \subseteq \Gamma_m$,
- $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_n} \not\vdash \Gamma_m$,
- if $\mathsf{dn}(\Gamma_m) \neq 0$ then $\mathsf{dn}(\Gamma_{m+1}) < \mathsf{dn}(\Gamma_m)$.

Since there are no infinite decreasing sequences of ordinals there exists a number m such that $\mathsf{dn}(\Gamma_m) = 0$. If we set $\Sigma = \Gamma_m$ we get the saturated sequent we wanted. $\Box$

Let us now introduce the model $\mathcal{M}^\omega$ which will play an important role in the completeness proof for our infinitary calculus. $\mathcal{M}^\omega$ is of the form

$$(W^\omega, R^\omega_1, \ldots, R^\omega_n, \lambda)$$

where $W^\omega, R^\omega_i \subseteq W^\omega \times W^\omega$ and λ are specified as follows:

- W^ω consists of all saturated sequents.
- For any two saturated sequences Γ, Δ and all relation R^ω_i we have $(\Gamma, \Delta) \in R^\omega_i$ if and only if the following conditions hold:
  - $\{\neg\alpha \mid \neg\mathsf{K}_i\alpha \in \Gamma\} \subseteq \Delta$,
  - $\{\neg\mathsf{C}\alpha \mid \neg\mathsf{C}\alpha \in \Gamma\} \subseteq \Delta$.
- $\lambda(P) = \{\Gamma \mid P \notin \Gamma\}$.

Lemma 85 *For all formulae $\varphi \in \mathcal{L}^{\mathsf{n}}_{\mathsf{C}}$ and all $\Gamma \in W^{\omega}$ we have*

$$\varphi \in \Gamma \quad \Rightarrow \quad \Gamma \not\models \varphi.$$

Proof. The proof goes by induction on the depth of the formula φ, $\mathsf{depth}(\varphi)$.

$\varphi \equiv P$: This case follows from the definition of the valuation λ.

$\varphi \equiv \alpha \wedge \beta, \alpha \vee \beta, \mathsf{C}\alpha$: These cases follow from the the definition of saturated set with an application of the induction hypothesis.

$\varphi \equiv \mathsf{K}_i\alpha$: Since Γ is saturated we know that $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \nvdash \Gamma$. Hence, we have

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \nvdash \alpha, \{\neg\beta \mid \neg\mathsf{K}_i\beta \in \Gamma\}, \{\neg\mathsf{C}\beta \mid \neg\mathsf{C}\beta \in \Gamma\}$$

since the provability of this sequent would imply the provability of Γ by the (K_i)-rule. By Lemma 84 there is a saturated sequent Δ such that

$$\alpha, \{\neg\beta \mid \neg\mathsf{K}_i\beta \in \Gamma\}, \{\neg\mathsf{C}\beta \mid \neg\mathsf{C}\beta \in \Gamma\} \subseteq \Delta.$$

By induction hypothesis we have

$$\Delta \not\models \alpha$$

and by definition of R^{ω}_i we have

$$(\Gamma, \Delta) \in R^{\omega}_i.$$

With these two things we get the induction step.

$\varphi \equiv \neg\mathsf{K}_i\alpha$: By definition of R^{ω}_i, for all worlds Δ such that $(\Gamma, \Delta) \in R^{\omega}_i$ since $\neg\mathsf{K}_i\alpha \in \Gamma$ we have

$$\neg\alpha \in \Delta.$$

By induction hypothesis we get

$$\Delta \not\models \neg\alpha$$

for all $\Delta \in R^{\omega}_i(\Gamma)$. And thus each such Δ fulfills α and we get

$$\Gamma \models \mathsf{K}_i\alpha$$

which completes this induction step.

$\varphi \equiv \neg \mathsf{C}\alpha$: If $\neg\mathsf{C}\alpha \in \Gamma$ then by definition of the relations R_i^ω we have for all accessible Δ

$$\neg\mathsf{C}\alpha \in \Delta.$$

Since each of these Δ is saturated we get

$$\text{(1)} \qquad \neg\mathsf{E}\alpha \in \Delta \quad \text{and} \quad \neg\mathsf{K}_i\alpha \in \Delta \ \text{ for all } i$$

and since Γ is saturated we also get

$$\text{(2)} \qquad \neg\mathsf{E}\alpha \in \Gamma \quad \text{and} \quad \neg\mathsf{K}_i\alpha \in \Gamma \ \text{ for all } i.$$

With the same argument used in the case where $\varphi \equiv \neg\mathsf{K}_i\alpha$ applied to (1) and (2) we can easily see that for all accessible Δ we have

$$\neg\alpha \in \Delta.$$

With the induction hypothesis and Lemma 47 we get this induction step and thus the proof. □

Theorem 86 *For all formulae $\varphi \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ we have*

$$\mathsf{T}^\omega_{\mathsf{K}_\mathsf{n}^\mathsf{C}} \vdash \varphi \quad \Leftrightarrow \quad C^{\mathbf{K}} \models \varphi.$$

Proof. The direction from left to right follows from Theorem 81. For the other direction we prove the contrapositive. Suppose that φ is not provable by Lemma 84 there is a saturated sequent Γ containing φ. By Lemma 85 the world Γ of the model $\mathcal{M}^\omega$ does not satisfy φ, thus φ is not valid. □

The previous theorem and Theorem 83 immediately provide the following complete cut elimination result.

Corollary 87 *For all formulae $\varphi \in \mathcal{L}_\mathsf{C}^\mathsf{n}$ we have*

$$\mathsf{T}^\omega_{\mathsf{K}_\mathsf{n}^\mathsf{C}} \vdash \varphi \quad \Leftrightarrow \quad \mathsf{T}^\omega_{\mathsf{K}_\mathsf{n}^\mathsf{C}} + (\mathsf{G{-}Cut}) \vdash \varphi.$$

11.2 Partial Finitisation

In the previous section we have seen that by introducing infinitary derivation we get a cut-free deductive system for the logic of common knowledge over **K**. In this section we start from this infinitary system and show that both for the negative fragment and for the positive fragment a finite part of the deduction system still suffices. Before we introduce these two fragments we need a preliminary definition.

The *proof closure* $\mathsf{pc}(\Gamma)$ of a sequent Γ is the minimal set containing Γ such that:

- If $\alpha \wedge \beta, \alpha \vee \beta \in \mathsf{pc}(\Gamma)$ then $\alpha, \beta \in \mathsf{pc}(\Gamma)$,
- if $\mathsf{K}_i\alpha, \mathsf{C}\alpha \in \mathsf{pc}(\Gamma)$ then $\alpha \in \mathsf{pc}(\Gamma)$,
- if $\sim\mathsf{K}_i\alpha, \sim\mathsf{C}\alpha \in \mathsf{pc}(\Gamma)$ then $\neg\alpha \in \mathsf{pc}(\Gamma)$.

The *positive fragment* ${\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}}^{+}$ is the subclass of all formulae α such that $\mathsf{pc}(\alpha)$ does not contain a formula of the form $\sim\mathsf{C}\beta$; the *negative fragment* ${\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}}^{-}$ is the subclass of all formulae α such that $\mathsf{pc}(\alpha)$ does not contain a formula of the form $\mathsf{C}\beta$.

Remark 88 *It can easily be seen that in a derivation of a formula in the negative fragment no formula of the form* $\mathsf{C}\alpha$ *can appear. Thus, this derivation does not use the* (C^{ω})*-rule. Analogously, any derivation of a formula in the positive fragment can not use the* $(\neg\mathsf{C})$*-rule.*

Theorem 89 *For each sequent* $\Gamma \subset {\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}}^{-}$ *and all ordinals* σ *we have*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash^{\sigma} \Gamma \quad \Rightarrow \quad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash^{<\omega} \Gamma.$$

Proof. By Remark 88 we know that any derivation of a formula $\varphi \in {\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}}^{-}$ can not use the (C^{ω})-rule. Using this fact the proof can easily be done by induction on σ. $\Box$

Notice, that the previous theorem implies that the proofs for the negative fragment are finite, in the sense that not only the length of the proofs is bounded but also the branching of them. For the positive fragment we provide two finitisation results the first one provides a proofs of finite length for each provable positive formula, for the second one we slightly modify the calculus and provide a proof system where both the proof-length and the branching is finite.

For each formula $\varphi \in {\mathcal{L}_{\mathsf{C}}^{\mathsf{n}}}^{+}$ we define the measure $\partial(\varphi)$ inductively as follows:

- $\partial(p) = \partial(\sim p) = 0$,

- $\partial(\alpha \wedge \beta) = \partial(\alpha \vee \beta) = \max\{\partial(\alpha), \partial(\beta)\}$,
- $\partial(\mathsf{C}\alpha) = \partial(\mathsf{K}_i\alpha) = \partial(\alpha)$,
- $\partial(\sim\mathsf{K}_i\alpha) = \partial(\neg\alpha) + 1$.

For any sequent $\Gamma \subset \mathcal{L}_{\mathsf{C}}^{\mathsf{n}+}$ we then define

$$\partial(\Gamma) = \max\{\partial(\alpha) \mid \alpha \in \Gamma\}.$$

From the definition we immediately get for all sequents $\Gamma, \neg\Delta \subset \mathcal{L}_{\mathsf{C}}^{\mathsf{n}+}$

$$\neg\mathsf{K}_i\Delta \subseteq \Gamma \text{ and } \Delta \neq \emptyset \quad \Rightarrow \quad \partial(\neg\Delta) < \partial(\Gamma).$$

Lemma 90 *Let $\Gamma \subset \mathcal{L}_{\mathsf{C}}^{\mathsf{n}+}$ be a sequent and $\alpha \in \mathcal{L}_{\mathsf{C}}^{\mathsf{n}+}$ a formula. For all ordinals σ, all natural numbers l, m with $\partial(\Gamma) \leq l \leq m$ and all $i \in \{1, \ldots, n\}$ we have*

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash^{\sigma} \mathsf{K}_i\mathsf{E}^l\alpha, \Gamma \quad \Rightarrow \quad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash^{n\sigma} \mathsf{K}_i\mathsf{E}^m\alpha, \Gamma.$$

Proof. The proof goes by induction on σ. We distinguish the following cases for the last inference in the proof of $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$:

1. $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$ is an axiom. In this case also $\mathsf{K}_i\mathsf{E}^m\alpha, \Gamma$ is an axiom.

2. $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$ is the conclusion of $(\wedge), (\vee)$ or (C^{ω}). In these cases we just apply the induction hypothesis to the premise(s) and get the desired result.

3. $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$ is the conclusion of the (K_j^*)-rule. This case goes through straightforward since the premise is either

$$\alpha \quad \text{or} \quad \beta, \text{ where } \mathsf{K}_j\mathsf{E}^s\beta \in \Gamma.$$

In both cases we apply the (K_j^*)-rule with the appropriate weakening and get the desired result.

4. $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$ is of the form

$$\mathsf{K}_i\mathsf{E}^l\alpha, \mathsf{K}_j\beta, \neg\mathsf{K}_j\Delta, \Sigma$$

and was obtained by the (K_j)-rule applied to the premise

$$\beta, \neg\Delta.$$

In this case we apply the (K_j)-rule with the appropriate weakening and get the desired result.

5. $\mathsf{K}_i\mathsf{E}^l\alpha, \Gamma$ is of the form

$$\mathsf{K}_i\mathsf{E}^l\alpha, \neg\mathsf{K}_i\Delta, \Sigma$$

and was obtained by the (K_i)-rule applied to the premise

$$\mathsf{E}^l\alpha, \neg\Delta.$$

We distinguish two sub-cases: If $\Delta = \emptyset$ then by Lemma 80 there is an ordinal $\tau < \sigma$ such that

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\tau \alpha,$$

by applying the (K_i^*)-rule we get the desired result. For the second case we assume that $\Delta \neq \emptyset$. In this case by a previous remark we have $\partial(\neg\Delta) < \partial(\Gamma)$, and thus $l = k+1$ for some k. Further, by Lemma 80.2 there is a $\tau < \sigma$ such that for all $j \in \{1, \ldots, n\}$ we have

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\tau \mathsf{K}_j\mathsf{E}^k\alpha, \neg\Delta.$$

Since $k > \partial(\neg\Delta)$ we can apply the induction hypothesis and get

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^{n\tau} \mathsf{K}_j\mathsf{E}^r\alpha, \neg\Delta$$

for all natural numbers $r \geq k$ and all $j \in \{1, \ldots, n\}$. By applying $n-1$ times the $(\wedge)$-rule we conclude

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^{n\tau+(n-1)} \mathsf{E}^m\alpha, \neg\Delta$$

for all natural numbers $m \geq k+1 = l$. One application of the (K_i)-rule then gives (notice that $\neg\mathsf{K}_i\Delta \subset \Gamma$)

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^{n\tau+n} \mathsf{K}_i\mathsf{E}^m\alpha, \Gamma$$

for all natural $m \geq l$. Since $n\tau + n \leq n\sigma$ we have completed the induction step for this case and thus the proof. □

Theorem 91 *For all sequents $\Gamma \subset \mathcal{L}^{\mathsf{n}+}_{\mathsf{C}}$ and all ordinals σ we have*

$$\mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^\sigma \Gamma \quad \Rightarrow \quad \mathsf{T}^\omega_{\mathsf{K}^\mathsf{C}_\mathsf{n}} \vdash^{<\omega} \Gamma.$$

Proof. The proof goes by induction on σ. We distinguish the following cases for the last inference in the proof of Γ:

1. Γ is an axiom. In this case the assertion is trivial.

2. Γ is the conclusion of a rule $(\wedge), (\vee), (\mathsf{K}_i)$ or (K_i^*). For each of these inference rules we have only finitely many premises. By applying the induction hypothesis we get a finite derivation for each of these premises and since in each case there are only finitely many we easily get the desired result.

3. Γ is of the form

$$\mathsf{C}\alpha, \Delta$$

and was obtained with an application of (C^ω). In this cases there are ordinals $\sigma_1, \sigma_2, \dots$ smaller than σ such that for each $k \geq 1$ we have

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{\sigma_k} \mathsf{E}^k\alpha, \Delta.$$

Let l be the natural number $\partial(\Delta)$, by induction hypothesis for all $k \leq l+1$ we get natural numbers $r_1, \dots, r_{l+1}$ such that

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{r_k} \mathsf{E}^k\alpha, \Delta. \tag{1}$$

With Lemma 80 applied to the case where $k = l+1$ in addition we get for all $i \in \{1, \dots, n\}$

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{r_{l+1}} \mathsf{K}_i\mathsf{E}^l\alpha, \Delta.$$

An application of Lemma 90 then gives us for all natural numbers $m \geq l$ and all $i \in \{1, \dots, n\}$

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{nr_{l+1}} \mathsf{K}_i\mathsf{E}^m\alpha, \Delta.$$

By applying $n-1$ times the $(\wedge)$-rule we have for all natural numbers $m \geq l$

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{nr_{l+1}+(n-1)} \mathsf{E}^{m+1}\alpha, \Delta. \tag{2}$$

Define $r = \max\{r_1, \dots, r_{l+1}, nr_{l+1} + (n-1)\}$. With the equations (1) and (2) we can easily see that for all natural numbers $k \geq 1$ we have

$$\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}} \vdash^{r} \mathsf{E}^k\alpha, \Delta.$$

One application of the (C^ω)-rule gives us the desired result. $\Box$

We end this section by proving the completeness for the calculus $\mathsf{T}^{<\omega}_{\mathsf{K}_n^\mathsf{C}}$ which is obtained from $\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}}$ by restricting the (C^ω)-rule such that we have only a finite branching. Hence we always get finite proofs.

The calculus $\mathsf{T}^{<\omega}_{\mathsf{K}_n^\mathsf{C}}$ is equal to the calculus $\mathsf{T}^\omega_{\mathsf{K}_n^\mathsf{C}}$ where we replace the (C^ω)-rule by the $(\mathsf{C}^{<\omega})$-rule.

$$\frac{\Gamma, \mathsf{E}^m\alpha \quad \text{for all } m \in \{1, \dots, \partial(\Gamma)+1\}}{\Gamma, \mathsf{C}\alpha} \; (\mathsf{C}^{<\omega})$$

It can easily be seen that any sequent Γ provable in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$ is provable in $\mathsf{T}^{<\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$. For the positive fragment also the other direction holds.

Theorem 92 *For any sequent* $\Gamma \subset \mathcal{L}^{\mathsf{n}+}_{\mathsf{C}}$ *we have*

$$\mathsf{T}^{<\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Gamma \quad \Leftrightarrow \quad \mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Gamma.$$

Proof. The direction from right to left can easily be proven by induction on the proof-length in $\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$, for the other direction we do an induction on the proof-length in the calculus $\mathsf{T}^{<\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}}$. The only non trivial induction step is the one where the last inference rule was $(\mathsf{C}^{<\omega})$. In this case Γ is of the form

$$\Delta, \mathsf{C}\alpha$$

with premises

$$\Delta, \mathsf{E}^m\alpha$$

for all $m \in \{1, \dots, \partial(\Delta)+1\}$. By induction hypothesis we have for all these m

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Delta, \mathsf{E}^m\alpha. \tag{1}$$

With Lemma 80 we get

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Delta, \mathsf{K}_i\mathsf{E}^{\partial(\Delta)}\alpha,$$

thus, with Lemma 90 we get for all $l \geq \partial(\Delta)$ and all $i \in \{1, \dots, n\}$

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Delta, \mathsf{K}_i\mathsf{E}^l\alpha.$$

By applying the $(\wedge)$-rule we get for all $l > \partial(\Delta)$

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Delta, \mathsf{E}^l\alpha. \tag{2}$$

By equation (1) and (2) we get for all natural numbers $m \geq 1$

$$\mathsf{T}^{\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \Delta, \mathsf{E}^m\alpha$$

and one application of the (C^{ω})-rule gives the desired result. □

The following corollary follows immediately from the previous theorem and from Theorem 83.

Corollary 93 *For all formulae* $\alpha \in \mathcal{L}^{\mathsf{n}+}_{\mathsf{C}}$ *we have*

$$\mathsf{T}^{<\omega}_{\mathsf{K}^{\mathsf{C}}_{\mathsf{n}}} \vdash \alpha \quad \Leftrightarrow \quad C^{\mathbf{K}} \models \alpha.$$

Appendix A

Fixpoints

This chapter gives a short survey of basic fixpoint theory it is based on an article of Fritz [17]. The main theorems stated and proved are the Knaster-Tarski Theorem [46] (Theorem 95), and Theorems 101 and 102, the main theorems for the characterisation of simultaneous fixpoints.

A.1 Preliminaries

A *complete lattice* $\mathcal{L}$ is of the form $\mathcal{L} = (L, \leq, \top, \bot)$, where

1. L is a non-empty set,

2. $\leq \subseteq L \times L$ is a partial order on L such that every subset $M \subseteq L$ has a supremum $\mathsf{sup}(M)$ and an infimum $\mathsf{inf}(M)$,

3. $\top, \bot \in L$ are the greatest and least elements, respectively, of L, i.e., for every $x \in L$, $\bot \leq x \leq \top$ holds.

Note that $\mathsf{inf}(\emptyset) = \top$, $\mathsf{sup}(\emptyset) = \bot$. For us, the most interesting kind of complete lattice is the *power set lattice* $(\mathcal{P}(A), \subseteq, A, \emptyset)$ of an arbitrary set A. Note that for any subset $M \subseteq \mathcal{P}(A)$ we have $\mathsf{inf}(M) = \bigcap M$, $\mathsf{sup}(M) = \bigcup M$. If M is a sequence of the form $\{x_\beta \mid \beta \leq \alpha \;\wedge\; \beta, \alpha \in ON\}$ we will write $\mathsf{inf}_{\beta\leq\alpha} x_\beta$ resp. $\mathsf{sup}_{\beta\leq\alpha} x_\beta$ instead of $\mathsf{inf}(M)$ resp. $\mathsf{sup}(M)$.

Let ON be the class of ordinal numbers. For any set S, $\mathsf{card}(S)$ denotes its cardinality. For any ordinal number α, let α^+ be the least ordinal number such that $\mathsf{card}(\alpha^+)$ is greater than $\mathsf{card}(\alpha)$.

We define some basic notions referring to functions on complete lattices. Let $\mathcal{L} = (L, \leq, \top, \bot)$ be a complete lattice and $f : L \to L$ be a function on it.

1. $x \in L$ is a *fixpoint* of f iff $f(x) = x$.

2. x is the *least* (*greatest*) *fixpoint* of f iff x is a fixpoint of f and $x \leq y$ ($y \leq x$) holds for all fixpoints y of f.

3. f is *monotone* iff for all $x, y \in L$, $x \leq y$ implies $f(x) \leq f(y)$.

4. We inductively define a sequence $(f^\alpha)_{\alpha \in ON}$ of subsets of L by

   (1) $f^0 := \bot,$

   (2) $f^{\alpha+1} := f(f^\alpha),$

   (3) $f^\lambda := \mathsf{sup}_{\alpha<\lambda} f^\alpha$ for limit ordinals λ.

The following lemma connects these notions.

Lemma 94 *1. If f is a monotone function on L then for all ordinals α and β such that $\beta \leq \alpha$ we have*

$$f^\beta \subseteq f^\alpha.$$

2. If f is monotone, there is an $\alpha \in ON$ such that $\mathsf{card}(\alpha) \leq \mathsf{card}(L)$ and $f^{\alpha+1} = f^\alpha$ (i.e., f^α is a fixed point of f).

3. If $\mathcal{L}$ is the power set lattice of a set A and f is monotone, there is an $\alpha \in ON$ such that $\mathsf{card}(\alpha) \leq \mathsf{card}(A)$ and $f^{\alpha+1} = f^\alpha$ (i.e., f^α is a fixpoint of f).

Proof.

1. Follows easily from the monotonicity of f and from the definition of the sequence $(f^\alpha)_{\alpha \in ON}$ by induction on ordinals.

2. Assume that there is no such α. Then, for every $\alpha < \beta < \mathsf{card}(L^+)$, $f^\alpha \neq f^\beta$. But then the set $\{f^\alpha \in L \mid \alpha < (\mathsf{card}(L))^+\} \subseteq L$ and moreover has cardinality $\mathsf{card}(\mathsf{card}(L)^+) > \mathsf{card}(L)$, which is a contradiction.

3. Follows from 2.

□

The least $\alpha \in ON$ such that $f^{\alpha+1} = f^\alpha$ is called the *closure ordinal* $\mathsf{cl}(f)$ of f. Clearly, any monotone f has a closure ordinal. In the following for any monotone f we abbreviate $f^{\mathsf{cl}(f)}$ by f^{cl}.

A.2 Least and Greatest Fixpoints

The Knaster-Tarski Theorem [46] asserts the existence of a least and a greatest fixpoint of a monotone function on a complete lattice. More precisely, these fixpoints are the infimum and supremum, respectively, of certain subsets of the complete lattice and can be generated inductively.

Theorem 95 *Let $f : L \to L$ be monotone. Then there is a least fixpoint $\mathsf{LFP}(f)$ and a greatest fixpoint $\mathsf{GFP}(f)$ of f. Furthermore, we have*

$$\mathsf{LFP}(f) = \mathsf{inf}\{x \in L \mid f(x) \leq x\} \text{ and } \mathsf{GFP}(f) = \mathsf{sup}\{x \in L \mid x \leq f(x)\}.$$

Proof. We will only prove the equation for the least fixpoint, the proof for the greatest goes similarly. Let $\Phi := \{x \in L \mid f(x) \leq x\}$ and $y := \mathsf{inf}(\Phi)$. We first show that y is a fixpoint of f. To do that, we show $f(y) \leq y$ and $y \leq f(y)$. First $f(y) \leq y$. For all $x \in \Phi$, $y \leq x$ holds. Since f is monotone, using the definition of Φ, for all $x \in \Phi$ we have $f(y) \leq f(x) \leq x$. Thus $f(y) \leq \mathsf{inf}(\Phi) = y$. Now $y \leq f(y)$. Using $f(y) \leq y$ and the monotonicity of f, we have $f(f(y)) \leq f(y)$, that is, $f(y) \in \Phi$. Thus $y = \mathsf{inf}\Phi \leq f(y)$. Since y is the infimum of Φ, in particular $y \leq x$ holds for all $x \in L$ such that $f(x) = x$. Thus y is the least fixpoint of f. □

Now we show that the least fixpoint of a monotone $f : L \to L$ is contained in the sequence $(f^\alpha)_{\alpha \in ON}$ and can thus be generated inductively.

Lemma 96 *Let $f : L \to L$ be a monotone function. Then $\mathsf{LFP}(f) = f^{\mathsf{cl}}$.*

Proof. Again, let $\Phi := \{x \in L \mid f(x) \leq x\}$. By definition, f^{cl} is a fixpoint, thus $\mathsf{LFP}(f) \leq f^{\mathsf{cl}}$. For the reverse we show by induction on α that for all ordinals α and all $x \in \Phi$ we have

$$f^\alpha \leq x.$$

$\alpha = 0$: $f^0 = \bot \leq x$ for all $x \in L$.

$\alpha = \beta + 1$: Let $x \in \Phi$. By induction hypothesis, $f^\beta \leq x$. Thus we have $f^{\beta+1} = f(f^\beta) \leq f(x) \leq x$, using the monotonicity of f.

α is a limit ordinal: By induction hypothesis, $f^\beta \leq x$ holds for all $\beta < \alpha$, $x \in \Phi$, which implies $f^\alpha = \mathsf{sup}_{\beta<\alpha} f^\beta \leq x$. □

To generate the greatest fixpoint in the same fashion, we introduce a dual sequence $(\tilde{f}^\alpha)_{\alpha\in ON}$.

For a function $f : L \to L$, the sequence $(\tilde{f}^\alpha)_{\alpha\in ON}$ of elements $\tilde{f}^\alpha \in L$ is defined inductively as follows:

$$\begin{aligned}\tilde{f}^0 &= \top \\ \tilde{f}^{\alpha+1} &= f(\tilde{f}^\alpha) \\ \tilde{f}^\lambda &= \mathsf{inf}_{\alpha<\lambda}\tilde{f}^\alpha && \text{for limit ordinals } \lambda.\end{aligned}$$

Note that $(\tilde{f}^\alpha)_{\alpha\in ON}$ is a decreasing sequence for monotone f. We define $\tilde{f}^{\mathsf{cl}} := \tilde{f}^\alpha$ for the least α such that $\tilde{f}^{\alpha+1} = \tilde{f}^\alpha$.

Lemma 97 *Let $f : L \to L$ be monotone. Then $\mathsf{GFP}(f) = \tilde{f}^{\mathsf{cl}}$.*

Proof. Dual to the proof of Lemma 96. □

In the case of power set lattices – which is our main interest – we can exploit the duality of generation of least and greatest fixpoint by infering the greatest fixpoint from the least fixpoint (and vice versa). For the remainder of this section, let $\mathcal{L}$ be the power set lattice of a set A.

We define, for every function $f : \mathcal{P}(A) \to \mathcal{P}(A)$, a dual function f'. For every function $f : \mathcal{P}(A) \to \mathcal{P}(A)$, let $f' : \mathcal{P}(A) \to \mathcal{P}(A)$ be the *dual function* of f, defined by

$$f'(X) := \overline{f(\overline{X})}, \qquad \text{where } \overline{X} := A \setminus X$$

Note that $f'' = f$.

Lemma 98 *Let $f : \mathcal{P}(A) \to \mathcal{P}(A)$ be monotone. We have*

1. *f' is monotone, and*
2. *$\mathsf{GFP}(f) = \overline{\mathsf{LFP}(f')}$ and $\mathsf{LFP}(f) = \overline{\mathsf{GFP}(f')}$*

Proof.

1. Let $X, Y \in \mathcal{P}(A)$, $X \subseteq Y$. Thus $\overline{Y} \subseteq \overline{X}$, and $f(\overline{Y}) \subseteq f(\overline{X})$ by the monotonicity of f, which implies $\overline{f(\overline{X})} \subseteq \overline{f(\overline{Y})}$.

2. To prove the first claim, we show by induction that $f^\alpha = \overline{\widetilde{f'}^\alpha}$ holds for all $\alpha \in ON$.

   $\alpha = 0$: $f^0 = \emptyset = \overline{A} = \overline{\widetilde{f'}^0}$.

   $\alpha = \beta + 1$: We have

$$
\begin{aligned}
f^{\beta+1} &= f(f^\beta) & \\
&= f(\overline{\widetilde{f'}^\beta}) & \text{(Ind. Hyp.)} \\
&= \overline{f'(\widetilde{f'}^\beta)} & \text{(by Definition)} \\
&= \overline{\widetilde{f'}^{\beta+1}}. & \text{(by Definition)}
\end{aligned}
$$

   α a limit ordinal: Here we have

$$
\begin{aligned}
f^\alpha &= \bigcup_{\beta<\alpha} f^\beta & \\
&= \overline{\bigcap_{\beta<\alpha} \overline{f^\beta}} & \\
&= \overline{\bigcap_{\beta<\alpha} \widetilde{f'}^\beta} & \text{(Ind. Hyp.)} \\
&= \overline{\widetilde{f'}^\alpha}. &
\end{aligned}
$$

   The second claim is proven similarly.

□

A.3 Simultaneous Fixpoints

Let n be a natural number and let

$$\mathcal{L}_0 = (L_0, \leq_0, \top_0, \bot_0), \ldots, \mathcal{L}_{n-1} = (L_{n-1}, \leq_{n-1}, \top_{n-1}, \bot_{n-1})$$

be complete lattices. We define $L := L_0 \times \ldots \times L_{n-1}$ and

$$\mathcal{L} := (L, \leq, (\top_0, \ldots, \top_{n-1}), (\bot_0, \ldots, \bot_{n-1})),$$

where $\leq$ is defined by

$$(x_0, \ldots, x_{n-1}) \leq (y_0, \ldots, y_{n-1}) \quad \text{iff} \quad x_i \leq_i y_i \text{ for all } i \in \{0, \ldots, n-1\}.$$

It is easy to see that $\mathcal{L}$ is a complete lattice, the *product lattice* of $\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}$. Sometimes we write $\mathcal{L} = \mathcal{L}_0 \times \ldots \times \mathcal{L}_{n-1}$.

Now let $f_0 : L \to L_0, \ldots, f_{n-1} : L \to L_{n-1}$ be monotone functions. Obviously,

$$f : L \to L, (x_0, \ldots, x_{n-1}) \mapsto (f_0(x_0, \ldots, x_{n-1}), \ldots, f_{n-1}(x_0, \ldots, x_{n-1}))$$

is a monotone function (or functional) as well and thus has a least and a greatest fixpoint by Theorem 95. They are called the *simultaneous* (least and greatest) *fixpoints* of $f_0, \ldots, f_{n-1}$.

For the remainder of this chapter we will show how to compute the least and greatest fixpoints of f by generating nested fixpoints of monotone functions defined on the lattices $L_0, \ldots, L_{n-1}$. For the sake of brevity (and clarity), we restrict ourselves to the case $n = 2$ and the computation of the least fixpoint, but the generalization is straightforward. The exact statement an its proof in all its generality can be seen in [37].

Let $n = 2$, let $g : L \to L_0, h : L \to L_1$ be monotone functions, and let $f : L \to L, (x_0, x_1) \mapsto (g(x_0, x_1), h(x_0, x_1))$. Let $\mathsf{LFP}(f) := (f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) \in L_0 \times L_1$ denote the least fixpoint of f, that is, $f_i^{\mathsf{cl}} := \mathsf{pr}_{i+1} f^{\mathsf{cl}}$ $(i = 0, 1)$, where pr_j is the projection to the j-th component. For $\alpha \in ON$, we define $f_i^\alpha := \mathsf{pr}_{i+1} f^\alpha$ $(i = 0, 1)$.

For every $x \in L_0$, we define $h_x : L_1 \to L_1, y \mapsto h(x, y)$. The monotonicity of h implies the monotonicity of h_x, so we can generate, for every $x \in L_0$, the least fixpoint $\mathsf{LFP}(h_x) = h_x^{\mathsf{cl}} \in L_1$ (cf. Lemma 96).

Lemma 99 *Let $e : L_0 \to L_0, x \mapsto g(x, h_x^{\mathsf{cl}})$. e is monotone and $(e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}) = (f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}})$.*

Proof. We first show e to be monotone. To do that, it suffices to show that $x \mapsto h_x^{\mathsf{cl}}$ is monotone: If $x \mapsto h_x^{\mathsf{cl}}$ is monotone we get

$$x \leq_0 x' \quad \Rightarrow \quad e(x) = g(x, h_x^{\mathsf{cl}}) \leq_0 g(x', h_{x'}^{\mathsf{cl}}) = e(x')$$

since g is monotone.

Hence we show for all ordinals α and all $x, x' \in L_0$

$$(x \leq_0 x' \implies h_x^\alpha \leq_1 h_{x'}^\alpha)$$

by induction on α:

The case where $\alpha = 0$ is trivial.

$\alpha = \beta + 1$: Let $x \leq_0 x'$. We have $h_x^{\beta+1} = h(x, h_x^\beta) \leq_1 h(x', h_{x'}^\beta) = h_{x'}^{\beta+1}$.

α a limit ordinal: Let $x \leq_0 x'$. $h_x^\alpha = \mathsf{sup}_{\beta<\alpha} h_x^\beta \leq_1 \mathsf{sup}_{\beta<\alpha} h_{x'}^\beta = h_{x'}^\alpha$.

Next, we show that f_1^{cl} is a fixpoint of $h_{f_0^{\mathsf{cl}}}$, which implies $h_{f_0^{\mathsf{cl}}}^{\mathsf{cl}} \leq_1 f_1^{\mathsf{cl}}$. But that follows from the fact that $(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}})$ is a fixpoint of f and since the following holds

$$\begin{aligned} h_{f_0^{\mathsf{cl}}}(f_1^{\mathsf{cl}}) &= h(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) \\ &= \mathsf{pr}_2 f(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) \\ &= f_1^{\mathsf{cl}}. \end{aligned}$$

Now, we show $e^{\mathsf{cl}} \leq_0 f_0^{\mathsf{cl}}$. This implies $h_{e^{\mathsf{cl}}}^{\mathsf{cl}} \leq_1 h_{f_0^{\mathsf{cl}}}^{\mathsf{cl}}$, since $x \mapsto h_x^{\mathsf{cl}}$ is monotone.

Using $h_{f_0^{\mathsf{cl}}}^{\mathsf{cl}} \leq_1 f_1^{\mathsf{cl}}$, we have

$$\begin{aligned} e(f_0^{\mathsf{cl}}) &= g(f_0^{\mathsf{cl}}, h_{f_0^{\mathsf{cl}}}^{\mathsf{cl}}) \\ &\leq_0 g(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) && (g \text{ monotone}) \\ &= \mathsf{pr}_1 f(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) \\ &= f_0^{\mathsf{cl}}, \end{aligned}$$

i.e., $f_0^{\mathsf{cl}} \in \{x \in L_0 \mid e(x) \leq_0 x\}$. Now since $e^{\mathsf{cl}} = \mathsf{inf}\{x \in L_0 \mid e(x) \leq x\}$, we have $e^{\mathsf{cl}} \leq_0 f_0^{\mathsf{cl}}$.

Recapitulating, we have $h_{e^{\mathsf{cl}}}^{\mathsf{cl}} \leq_1 h_{f_0^{\mathsf{cl}}}^{\mathsf{cl}} \leq_1 f_1^{\mathsf{cl}}$ and $e^{\mathsf{cl}} \leq_0 f_0^{\mathsf{cl}}$.

To show that $f^{\mathsf{cl}} = (f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) \leq (e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}})$ and hence $(e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}) = \mathsf{LFP}(f)$, it suffices to establish for all ordinals α

$$f^\alpha \leq (e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}).$$

As usual, we do it by induction on α.

The case where $\alpha = 0$ is trivial.

$\alpha = \beta + 1$:

$$\begin{aligned} f^{\beta+1} &= f(f_0^\beta, f_1^\beta) \\ &= (g(f_0^\beta, f_1^\beta), h(f_0^\beta, f_1^\beta)) \\ &\leq (g(e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}), h(e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}})) && \text{(Ind. Hyp.)} \\ &= (e(e^{\mathsf{cl}}), h_{e^{\mathsf{cl}}}(h_{e^{\mathsf{cl}}}^{\mathsf{cl}})) && \text{(Defs. of } e^{\mathsf{cl}},\ h_{e^{\mathsf{cl}}}) \\ &= (e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}) \end{aligned}$$

α a limit ordinal:

$$\begin{aligned} f^\alpha &= \mathsf{sup}_{\beta<\alpha}(f_0^\beta, f_1^\beta) \\ &= (\mathsf{sup}_{\beta<\alpha} f_0^\beta, \mathsf{sup}_{\beta<\alpha} f_1^\beta) && \text{(Def. of } \leq) \\ &\leq (e^{\mathsf{cl}}, h_{e^{\mathsf{cl}}}^{\mathsf{cl}}) && \text{(Ind. Hyp.)} \end{aligned}$$

□

In other words we have

$$\mathsf{pr}_1 \mathsf{LFP}(f) = \mathsf{LFP}(x \mapsto g(x, \mathsf{LFP}(y \mapsto h(x, y)))).$$

In the same manner, we can show

Lemma 100 *Let $g_y : L_0 \to L_0, x \mapsto g(x, y)$, for every $y \in L_1$, and let $e : L_1 \to L_1, y \mapsto h(g_y^{\mathsf{cl}}, y)$. g_y and e are monotone, and $(f_0^{\mathsf{cl}}, f_1^{\mathsf{cl}}) = (g_{e^{\mathsf{cl}}}^{\mathsf{cl}}, e^{\mathsf{cl}})$.*

These lemmata imply

Theorem 101 *Let $\mathcal{L} = \mathcal{L}_0 \times \mathcal{L}_1$ be the product lattice of the lattices $\mathcal{L}_0 = (L_0, \leq_0, \top_0, \bot_0)$ and $\mathcal{L}_1 = (L_1, \leq_1, \top_1, \bot_1)$, and let $g : L \to L_0, h : L \to L_1$ be monotone functions. Let $f : L \to L, (x_0, x_1) \mapsto (g(x_0, x_1), h(x_0, x_1))$. Then*

$$\begin{aligned} \mathsf{pr}_1 \mathsf{LFP}(f) &= \mathsf{LFP}(x \mapsto g(x, \mathsf{LFP}(y \mapsto h(x, y)))), \\ \mathsf{pr}_2 \mathsf{LFP}(f) &= \mathsf{LFP}(y \mapsto h(\mathsf{LFP}(x \mapsto g(x, y)), y)). \end{aligned}$$

And analogously we get

Theorem 102 *Let $\mathcal{L} = \mathcal{L}_0 \times \mathcal{L}_1$ be the product lattice of the lattices $\mathcal{L}_0 = (L_0, \leq_0, \top_0, \bot_0)$ and $\mathcal{L}_1 = (L_1, \leq_1, \top_1, \bot_1)$, and let $g : L \to L_0, h : L \to L_1$ be monotone functions. Let $f : L \to L, (x_0, x_1) \mapsto (g(x_0, x_1), h(x_0, x_1))$. Then*

$$\begin{aligned} \mathsf{pr}_1 \mathsf{GFP}(f) &= \mathsf{GFP}(x \mapsto g(x, \mathsf{GFP}(y \mapsto h(x, y)))), \\ \mathsf{pr}_2 \mathsf{GFP}(f) &= \mathsf{GFP}(y \mapsto h(\mathsf{GFP}(x \mapsto g(x, y)), y)). \end{aligned}$$

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Symbols of Part 1

Notions of Part 1

Symbols of Part 2

Notions of Part 2