Extending the system $\mathsf{T_0}$ of explicit mathematics: the limit and Mahlo axioms *

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Abstract

In this paper we discuss extensions of Feferman's theory $\mathsf{T_0}$ for explicit mathematics by the so-called limit and Mahlo axioms and present a novel approach to constructing natural recusion-theoretic models for (fairly strong) systems of explicit mathematics which is based on nonmonotone inductive definitions.

1 Introduction

The purpose of this paper is twofold: our first aim is to discuss the extensions of Feferman's theory $\mathsf{T_0}$ for explicit mathematics by the so-called limit and Mahlo axioms; secondly, we want to present a novel approach to constructing natural recusion-theoretic models for (fairly strong) systems of explicit mathematics which is based on nonmonotone inductive definitions.

Subsystems of second order arithmetic and set theory dealing with or describing a recursively inaccessible or Mahlo universe play an important role in proof theory for quite some time. In this context we have to mention the theories $(\Delta^1_2\text{-}\mathsf{CA}) + (\mathsf{BI})$ and KPi (cf. e.g. Jäger [8]) whose least standard models are the structures $\mathsf{L}(\iota_0)_\mathbb{N} \cap P(\mathbb{N})$ and $\mathsf{L}(\iota_0)_\mathbb{N}$, respectively, with ι_0 being the first recursively inaccessible ordinal. The theory KPm, on the other hand, reflects a recursively Mahlo universe and thus has the least standard model $\mathsf{L}(\rho_0)_\mathbb{N}$ provided that ρ_0 is the least recursively Mahlo ordinal. The proof-theoretic analysis of KPm is carried through in Rathjen [14].

On the more constructive side, Feferman's theory $\mathsf{T_0}$, introduced in [2], is of particular interest. Originally designed as a framework for Bishop-style constructive mathematics, it contains axioms which make it possible to reach

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the recursively inaccessible "from below". By results of Feferman, Jäger and Pohlers in [4, 7, 12] we also know that T_0 with classical or intuitionistic logic is proof-theoretically equivalent to $(\Delta^1_2\text{-}\mathsf{CA}) + (\mathsf{BI})$ and KPi.

Martin-Löf type theories for treating the constructive analogue of recursively inaccessible sets are presented in Griffor and Rathjen [6] and in Setzer [18]. A variant of Mahlo à la Martin-Löf has first been given in Setzer [17]; later Rathjen [13] developed an alternative type system in the spirit of Martin-Löf also dealing with aspects of Mahloness.

In this article we study the limit axiom and Mahlo axioms tailored for explicit inaccessibility and Mahloness. They are conceptually and syntactically very simple and perspicuous and permit a recursive as well as a classical interpretation along the lines of the *marriage of convenience* in Feferman [3].

For establishing the upper proof-theoretic bounds of T_0 plus limit and Mahlo axioms, we generate models by means of specific nonmonotone inductive definitions introduced in Richter [16]. The lower bounds are either obvious from the literature or will be treated elswhere.

All systems considered in this paper are based on classical logic. However, their variants obtained by working with intuitionistic logic are presently studied by Tupailo [20], and it seems that they are proof-theoretically equivalent. Thus, Tupailo's results, results about constructive set theory and the results of this paper provide a constructive justification in the sense of reductive proof theory of the classical theories studied below as well as of their intuitionistic counterparts.

The plan of this paper is as follows: In the next section we introduce the general framework for explicit mathematics and present Feferman's theory T_0 as well as some of its subsystems. We also turn to the notion of universe in explicit mathematics, make some remarks about natural ordering principles for universes and formulate the so called limit axiom. Section 3 is dedicated repeating the basic facts about those first order theories for inductive definitions of Jäger [10] which will be needed later for our model constructions.

The first such model construction is carried through in section 4 and provides a model for T_0 plus the limit axiom and the two ordering axiom which claim connectivity and linearity of normal universes. In section 5 we turn to strict universes, the limit axiom for strict universes and name induction and show that the model of the previous section also validates these principles.

The Mahlo axioms for explicit mathematics build the core of section 6. First we give their exact formulation and consider some immediate consequences.

Then we show how to build a model of $\mathsf{T_0}$ plus these axioms by shifting from a $[\mathsf{POS}, \mathsf{QF}]$ to a $[\mathsf{POS}, \Pi^0_1]$ nonmonotone inductive definition. The proof-theoretic upper bounds for $\mathsf{T_0}$ plus Mahlo and its subsystems which are obtained by restricting induction finally follow from the results about the involved first order theories for nonmonotone $[\mathsf{POS}, \Pi^0_1]$ inductive definitions.

2 Explicit mathematics

Explicit mathematics has been introduced in Feferman [2] and further studied in Feferman [3, 4]. However, in the following we do not work with Feferman's original formalization of systems of explicit mathematics; instead we treat them as theories of types and names as developed in Jäger [9]. This section is very much like in related papers. Nevertheless we decided to include it in order to make our article also accessible for a reader who is not a specialist in explicit mathematics.

2.1 Basic notions

The theories of types and names which we will consider in the following are formulated in the second order language $\mathbb{L}$ about individuals and types. It comprises individual variables $a, b, c, f, u, v, w, x, y, z, \dots$ as well as type variables $U, V, W, X, Y, Z, \dots$ (both possibly with subscripts). $\mathbb{L}$ also includes the individual constants k, s (combinators), $\mathsf{p}, \mathsf{p_0}, \mathsf{p_1}$ (pairing and projections), 0 (zero), $\mathsf{s_N}$ (successor), $\mathsf{p_N}$ (predecessor), $\mathsf{d_N}$ (definition by numerical cases). There are additional individual constants, called *generators*, which will be used for the uniform naming of types, namely nat (natural numbers), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image), j (join), i (inductive generation) and ℓ as well as m (universe generators). There is one binary function symbol $\cdot$ for (partial) application of individuals to individuals. Further, $\mathbb{L}$ has two unary relation symbols $\downarrow$ (defined) and N (natural numbers) as well as the two binary relation symbols $\in$ (membership) and $\Re$ (naming, representation).

The *individual terms* $(r, s, t, r_1, s_1, t_1, \dots)$ of $\mathbb{L}$ are built up from individual variables and individual constants by means of our function symbol $\cdot$ for application. In the following we often abbreviate $(s \cdot t)$ simply as (st), st or sometimes also as $s(t)$; the context will always assure that no confusion arises. We further adopt the convention of association to the left so that $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 \cdot s_2) \dots s_n)$. We also set $t' := \mathsf{s_N} t$. Finally, we define general n tupling by induction on $n \geq 2$ as follows:

$$(s_1, s_2) := \mathsf{p} s_1 s_2, \qquad (s_1, \dots, s_{n+1}) := ((s_1, \dots, s_n), s_{n+1}).$$

The atomic formulas of $\mathbb{L}$ are the formulas $s\downarrow$, $\mathsf{N}(s)$, $s = t$, $s \in U$ and $\Re(s, U)$. Since we work with a logic of partial terms, it is not guaranteed that all terms have values, and $s\downarrow$ is read as *s is defined* or *s has a value.* Moreover, $\mathsf{N}(s)$ says that s is a natural number, and the formula $\Re(s, U)$ is used to express that the individual s *represents* the type U or is a *name* of U.

The *formulas* $(A, B, C, A_1, B_1, C_1, \dots)$ of $\mathbb{L}$ are generated from the atomic formulas by closing against the usual propositional connectives as well as quantification in both sorts. The following table contains a list of useful abbreviations:

$$
\begin{array}{rcl}
s \simeq t & := & s\downarrow \vee t\downarrow \;\rightarrow\; s = t, \\
s \in \mathsf{N} & := & \mathsf{N}(s), \\
(\exists x \in \mathsf{N})A(x) & := & \exists x(x \in \mathsf{N} \wedge A(x)), \\
(\forall x \in \mathsf{N})A(x) & := & \forall x(x \in \mathsf{N} \rightarrow A(x)), \\
U \subset V & := & \forall x(x \in U \rightarrow x \in V), \\
U = V & := & U \subset V \wedge V \subset U, \\
s \,\dot{\in}\, t & := & \exists X(\Re(t, X) \wedge s \in X), \\
s \,\dot{\subset}\, t & := & (\forall x \,\dot{\in}\, s)(x \,\dot{\in}\, t), \\
s \doteq t & := & s \,\dot{\subset}\, t \wedge t \,\dot{\subset}\, s, \\
(\exists x \,\dot{\in}\, s)A(x) & := & \exists x(x \,\dot{\in}\, s \wedge A(x)), \\
(\forall x \,\dot{\in}\, s)A(x) & := & \forall x(x \,\dot{\in}\, s \rightarrow A(x)), \\
\Re(s) & := & \exists X \Re(s, X).
\end{array}
$$

The vector notation $\vec{\mathcal{Z}}$ is sometimes used to denote finite sequences $\mathcal{Z}_1, \dots \mathcal{Z}_n$ of expressions. The length of such a sequence $\vec{\mathcal{Z}}$ is then either given by the context or irrelevant. For example, for $\vec{U} = U_1, \dots, U_n$ and $\vec{s} = s_1, \dots, s_n$ we write

$$
\begin{array}{rcl}
\Re(\vec{s}, \vec{U}) & := & \Re(s_1, U_1) \wedge \dots \wedge \Re(s_n, U_n), \\
\Re(\vec{s}) & := & \Re(s_1) \wedge \dots \wedge \Re(s_n).
\end{array}
$$

Now we introduce the theory EETJ which provides a framework for explicit elementary types with join. Its logic is Beeson's classical *logic of partial terms* (cf. Beeson [1] or Troelstra and Van Dalen [19]) for individuals and classical logic for types. The nonlogical axioms of EETJ can be divided into the following groups.

$\mathsf{I.\ Applicative\ axioms.}$ These axioms formalize that the individuals form a partial combinatory algebra, that we have paring and projection and the usual closure conditions on the natural numbers as well as definition by numerical cases.

(1) $\mathsf{k}ab = a$,

(2) $\mathsf{s}ab\downarrow \wedge \mathsf{s}abc \simeq ac(bc)$,

(3) $\mathsf{p}_0(a,b) = a \wedge \mathsf{p}_1(a,b) = b$,

(4) $0 \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(x' \in \mathsf{N})$,

(5) $(\forall x \in \mathsf{N})(x' \neq 0 \wedge \mathsf{p}_{\mathsf{N}} x' = x)$,

(6) $(\forall x \in \mathsf{N})(x \neq 0 \rightarrow \mathsf{p}_{\mathsf{N}} x \in \mathsf{N} \wedge (\mathsf{p}_{\mathsf{N}} x)' = x)$,

(7) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a = b \rightarrow \mathsf{d}_{\mathsf{N}} uvab = u$,

(8) $a \in \mathsf{N} \wedge b \in \mathsf{N} \wedge a \neq b \rightarrow \mathsf{d}_{\mathsf{N}} uvab = v$.

As usual a theorem about λ abstraction and a form of the recursion theorem can be derived from axioms (1) and (2).

II. Explicit representation and equality. The following axioms state that each type has a name, that there are no homonyms and that $\Re$ respects the extensional equality of types.

(1) $\exists x \Re(x, U)$,

(2) $(\Re(s, U) \wedge \Re(s, V)) \rightarrow U = V$,

(3) $(U = V \wedge \Re(s, U)) \rightarrow \Re(s, V)$.

III. Basic type existence axioms. In the following we provide a finite axiomatization of uniform elementary comprehension plus join.

Natural numbers

$\Re(\mathsf{nat}) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{nat} \leftrightarrow \mathsf{N}(x))$.

Identity

$\Re(\mathsf{id}) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{id} \leftrightarrow \exists y(x = (y, y)))$.

Complements

$\Re(a) \rightarrow \Re(\mathsf{co}(a)) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{co}(a) \leftrightarrow x \mathrel{\dot{\notin}} a)$.

Intersections

$\Re(a) \wedge \Re(b) \;\rightarrow\; \Re(\mathsf{int}(a,b)) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{int}(a,b) \leftrightarrow x \mathrel{\dot{\in}} a \wedge x \mathrel{\dot{\in}} b).$

Domains

$\Re(a) \;\rightarrow\; \Re(\mathsf{dom}(a)) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{dom}(a) \leftrightarrow \exists y((x,y) \mathrel{\dot{\in}} a)).$

Inverse images

$\Re(a) \;\rightarrow\; \Re(\mathsf{inv}(a,f)) \wedge \forall x(x \mathrel{\dot{\in}} \mathsf{inv}(a,f) \leftrightarrow fx \mathrel{\dot{\in}} a).$

Joins

$\Re(a) \wedge (\forall x \mathrel{\dot{\in}} a)\Re(fx) \;\rightarrow\; \Re(\mathsf{j}(a,f)) \wedge \Sigma(a,f,\mathsf{j}(a,f)).$

In this last axiom the formula $\Sigma(a,f,b)$ expresses that b names the disjoint union of f over a, i.e.

$$\Sigma(a,f,b) \;:=\; \forall x(x \mathrel{\dot{\in}} b \leftrightarrow \exists y \exists z(x = (y,z) \wedge y \mathrel{\dot{\in}} a \wedge z \mathrel{\dot{\in}} fy)).$$

An $\mathbb{L}$ formula A is called *elementary* if it contains neither the relation symbol $\Re$ nor bound type variables. In the original formulation of explicit mathematics elementary comprehension is not dealt with by a finite axiomatization but directly as an infinite axiom schema. According to a theorem in Feferman and Jäger [5], the usual schema of uniform elementary comprehension is provable from our finite axiomatization; join is not needed for this argument.

In the following we employ two forms of induction on the natural numbers, formula induction and type induction. Formula induction is the schema

$$(\mathbb{L}\text{-}\mathsf{I_N}) \qquad A(0) \wedge (\forall x \in \mathsf{N})(A(x) \rightarrow A(x')) \;\rightarrow\; (\forall x \in \mathsf{N})A(x)$$

for all $\mathbb{L}$ formulas $A(u)$. Type induction, on the other hand, is the restriction of formula induction to types, i.e. the axiom

$$(\mathsf{T}\text{-}\mathsf{I_N}) \qquad \forall X(0 \in X \wedge (\forall x \in \mathsf{N})(x \in X \rightarrow x' \in X) \;\rightarrow\; (\forall x \in \mathsf{N})(x \in X)).$$

The most famous system of explicit mathematics is the theory $\mathsf{T_0}$ introduced in Feferman [2]. It is obtained from $\mathsf{EETJ} + (\mathbb{L}\text{-}\mathsf{I_N})$ by adding the principle of inductive generation (IG). As a helpful abbreviation we write

$$\mathsf{Closed}(a,b,S) \;:=\; (\forall x \mathrel{\dot{\in}} a)((\forall y \mathrel{\dot{\in}} a)((y,x) \mathrel{\dot{\in}} b \rightarrow y \in S) \;\rightarrow\; x \in S).$$

Consider b as the code of a binary relation. Then this definition means that S is a type which contains a $c \mathrel{\dot{\in}} a$ if all predecessors of c in a with respect to b belong to S. *Inductive generation* (IG) is now given by the following axioms

$$(\mathsf{IG}.1) \qquad \Re(a) \wedge \Re(b) \rightarrow \exists X(\Re(\mathsf{i}(a,b),X) \wedge \mathsf{Closed}(a,b,X)),$$

$$(\mathsf{IG.2}) \qquad \Re(a) \land \Re(b) \land \mathsf{Closed}(a, b, A) \;\rightarrow\; (\forall x \,\dot{\in}\, \mathsf{i}(a, b))A(x)$$

for all $\mathbb{L}$ formulas $A(u)$. Thus (IG), i.e. $(\mathsf{IG.1}) + (\mathsf{IG.2})$, states the existence of accessible parts, and again everything is uniform in the corresponding names. As mentioned before, Feferman's $\mathsf{T_0}$ is given by

$$\mathsf{T_0} \quad := \quad \mathsf{EETJ} + (\mathsf{IG}) + (\mathbb{L}\text{-}\mathsf{I_N}).$$

$\mathsf{T_0^w}$ is the system $\mathsf{T_0}$ where inductive generation is restricted to types. If in addition complete induction on the natural numbers is restricted to types, then we call the system $\mathsf{T_0^r}$.

Sometimes we also include additional axioms which guarantee that different generators create different names. This can be achieved, for example, by adding axioms of the following kind.

Uniqueness of generators with respect to $\mathbb{L}$ is given by the collection $(\mathbb{L}\text{-}\mathsf{UG})$ of the following axioms for all syntactically different generators r_0 and r_1 and arbitrary generators s and t of $\mathbb{L}$:

(1) $r_0 \neq r_1$,

(2) $\forall x(sx \neq \mathsf{nat} \;\land\; sx \neq \mathsf{id})$,

(3) $\forall x \forall y(sx = tx \;\rightarrow\; s = t \land x = y)$.

Let us point out already now, that uniqueness of generators can be easily established in all natural models of explicit mathematics.

2.2 Universes

The next step is to introduce the concept of a universe in explicit mathematics. To put it very simple, a universe is supposed to be a type which consists of names only and reflects the theory EETJ.

For the detailed definition of a universe we introduce some auxiliary notation and let $\mathcal{C}(W, a)$ be the closure condition which is the disjunction of the following $\mathbb{L}$ formulas:

(1) $a = \mathsf{nat} \lor a = \mathsf{id}$,

(2) $\exists x(a = \mathsf{co}(x) \land x \in W)$,

(3) $\exists x \exists y(a = \mathsf{int}(x, y) \land x \in W \land y \in W)$,

(4) $\exists x(a = \mathsf{dom}(x) \land x \in W)$,

(5) $\exists f \exists x (a = \mathsf{inv}(f, x) \wedge x \in W)$,

(6) $\exists x \exists f (a = \mathsf{j}(x, f) \wedge x \in W \wedge (\forall y \mathrel{\dot\in} x)(fy \in W))$.

Thus the formula $\forall x(\mathcal{C}(W, x) \to x \in W)$ describes that W is a type which is closed under the type constructions of EETJ, i.e. elementary comprehension and join. A universe is a type which consists of names only and satisfies this closure condition.

Definition 1 *1. We write* $\mathsf{U}(W)$ *to express that the type* W *is a* universe*,*

$$\mathsf{U}(W) \;:=\; \forall x(\mathcal{C}(W, x) \to x \in W) \;\wedge\; (\forall x \in W)\Re(x).$$

2. $\mathcal{U}(t)$ *means that the individual* t *is a* name of a universe*,*

$$\mathcal{U}(t) \;:=\; \exists X(\Re(t, X) \;\wedge\; \mathsf{U}(X)).$$

The theory EETJ does not prove the existence of universes. However, as in the case of theories for admissible sets (cf. e.g. Jäger [8]) a so-called *limit axiom* can easily be added. By making use of the generator ℓ, one assigns to each name a the name ℓa of a universe containing a, i.e.

$$(\mathsf{Lim}) \qquad \forall x(\Re(x) \;\to\; \mathcal{U}(\ell x) \wedge x \mathrel{\dot\in} \ell x).$$

It is an interesting theme to see what kind of ordering principles for universes can be consistently added to the previous axioms. This question is discussed at full length in Jäger, Kahle and Studer [11], and it is shown there that one must not be too liberal. As a consequence of these considerations we do not claim linearity and connectivity for arbitrary universes, but only for so-called *normal universes*, i.e. universes which are named by means of the type generators ℓ and m,

$$\mathcal{U}_{no}(t) \;:=\; \exists x(t = \ell x) \;\vee\; \exists x \exists f(t = \mathsf{m}(x, f)).$$

Of course the second disjunction in this definition does not play a role yet since so far no axioms about the generator m have been formulated. However, they will follow in Section 6.

Linearity and *connectivity* for normal (names of) universes can be now expressed by the following two axioms:

$$(\mathcal{U}_{no}\text{-}\mathsf{Lin}) \qquad \forall x \forall y(\mathcal{U}_{no}(x) \wedge \mathcal{U}_{no}(y) \;\to\; x \mathrel{\dot\in} y \vee x \doteq y \vee y \mathrel{\dot\in} x),$$

$$(\mathcal{U}_{no}\text{-}\mathsf{Con}) \qquad \forall x \forall y(\mathcal{U}_{no}(x) \wedge \mathcal{U}_{no}(y) \;\to\; x \mathrel{\dot\subset} y \vee y \mathrel{\dot\subset} x).$$

It is shown in [11] that connectivity of normal universes also implies transitivity of normal universes in its most general form.

Universes in explicit mathematics are discussed from a much broader perspective in Jäger, Kahle and Studer [11]. Among other things we also discuss the relationship between inductive generation and the existence of least universes and a form of name induction (see also Section 5 below). In this article we confine ourselves to establishing upper proof-theoretic bounds.

3 First order inductive definitions

Before introducing the Mahlo axioms into explicit mathematics we present a new and powerful method for constructing natural models for the theories presented so far. This method can then be easily extended to coping with Mahlo as well. The use of certain nonmonotone first order inductive definitions is crucial for our approach.

3.1 Combined operator forms

Let Φ be an arbitrary operator on the natural numbers, i.e. a function from $Pow(\mathbb{N}^k)$ to $Pow(\mathbb{N}^k)$ for some natural number k. As usual in the theory of (not necessarily monotone) inductive definitions, the stages of the inductive definition generated by Φ are defined for all ordinals σ as

$$\Phi^{\sigma} \ := \ \Phi^{<\sigma} \cup \Phi(\Phi^{<\sigma}) \qquad \text{and} \qquad \Phi^{<\sigma} \ := \ \bigcup \{\Phi^{\tau} : \tau < \sigma\}.$$

Standard cardinality arguments then yield the existence of a least ordinal σ_0 so that

$$\Phi^{<\sigma_0} \ = \ \Phi^{<\sigma_0} \cup \Phi(\Phi^{<\sigma_0}) \ = \ \bigcup \{\Phi^{\tau} : \tau \text{ ordinal}\}.$$

This ordinal σ_0 is called the *closure ordinal* $cl(\Phi)$ of the operator Φ; the set $\Phi^{<\sigma_0}$ is the subset of $\mathbb{N}^k$ inductively defined by Φ.

Richter [16] describes an interesting method to generate a new operator on the natural numbers from two given operators Φ and Ψ, which map $Pow(\mathbb{N}^k)$ to $Pow(\mathbb{N}^k)$. It is called $[\Phi, \Psi]$ an defined by

$$[\Phi, \Psi](S) \ := \ \begin{cases} \Phi(S) & \text{if} \ \ \Phi(S) \not\subset S, \\ \Psi(S) & \text{if} \ \ \Phi(S) \subset S \end{cases}$$

for all $S \subset \mathbb{N}^k$. Hence, the first operator is applied whenever something new is added; the second is only active on sets which are closed under the first.

Let $\mathcal{L}$ be some standard language of first order arithmetic with number variables $a, b, c, e, f, u, v, w, x, y, z, \dots$ (possibly with subscripts), a constant 0 and symbols for all primitive recursive functions and relations. The terms $(r, s, t, r_1, s_1, t_1, \dots)$ and formulas $(A, B, C, A_1, B_1, C_1, \dots)$ of $\mathcal{L}$ as well as all the other relevant syntactic notions are defined as usual.

In particular we make use of a conventional primitive recursive coding machinery in $\mathcal{L}$: $\langle \dots \rangle$ is a standard primitive recursive function for forming n-tuples $\langle t_1, \dots, t_n \rangle$; $\mathsf{lh}(t)$ denotes the length of (the sequence number coded by) t; $(t)_i$ is the ith component of (the sequence coded by) t if $i < \mathsf{lh}(t)$, i.e. $t = \langle (t)_0, \dots, (t)_{\mathsf{lh}(t) \dot{-} 1} \rangle$ if t is a sequence number.

Now let P be a fresh k-ary relation symbol and write $\mathcal{L}(P)$ for the extension of $\mathcal{L}$ by P. An $\mathcal{L}(P)$ formula which contains at most $\vec{u} = u_1, \dots, u_k$ free is called an *k-ary operator form*, and we let $\mathfrak{A}(P, \vec{u}), \mathfrak{B}(P, \vec{u}), \mathfrak{C}(P, \vec{u}), \dots$ range over such forms. Each k-ary operator form $\mathfrak{A}(P, \vec{u})$ gives rise to an operator $\Phi_{\mathfrak{A}}$ from $Pow(\mathbb{N}^k)$ to $Pow(\mathbb{N}^k)$ by setting

$$\Phi_{\mathfrak{A}}(S) \;:=\; \{(n_1, \dots, n_k) \in \mathbb{N}^k : \mathbb{N} \models \mathfrak{A}(S, n_1, \dots, n_k)\}$$

for all $S \subset \mathbb{N}^k$. To simplify the notation we write $cl(\mathfrak{A})$ for the closure ordinal $cl(\Phi_{\mathfrak{A}})$ of the operator associated to $\mathfrak{A}$. Moreover, if $\mathcal{K}$ is a class of operator forms, then

$$cl(\mathcal{K}) \;:=\; \sup\{cl(\mathfrak{A}) : \mathfrak{A} \in \mathcal{K}\}.$$

For our model constructions we will be mainly interested in the following classes of operator forms: POS comprises all operator forms $\mathfrak{A}(P, \vec{u})$ with P occurring only positively. An operator form $\mathfrak{A}(P, \vec{u})$ belongs to QF if it does not contain any quantifiers. Π^0_1 consists of all operator forms $\forall x \mathfrak{A}(P, \vec{u}, x)$ where all quantifiers occurring in $\mathfrak{A}$ are bounded.

For two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of k-ary operator forms we define $[\mathcal{K}_1, \mathcal{K}_2]$ to be the class of all operator forms

$$\mathfrak{A}(P, \vec{u}) \;:=\; \mathfrak{A}_0(P, \vec{u}) \,\vee\, (\forall \vec{x}(\mathfrak{A}_0(P, \vec{x}) \rightarrow P(\vec{x})) \,\wedge\, \mathfrak{A}_1(P, \vec{u}))$$

so that $\mathfrak{A}_0(P, \vec{u})$ belongs to $\mathcal{K}_1$ and $\mathfrak{A}_1(P, \vec{u})$ to $\mathcal{K}_2$. Obviously, this definition follows the pattern of the combination of operators à la Richter [16]. The following theorem is also taken from [16].

Theorem 2 *If ι_0 is the first recursively inaccessible ordinal and ρ_0 the first recursively Mahlo ordinal, then we have*

$$[\mathsf{POS}, \mathsf{QF}] \;=\; \iota_0 \qquad \textit{and} \qquad [\mathsf{POS}, \Pi^0_1] \;=\; \rho_0.$$

This theorem provides some motivation for taking operator forms from the classes $[\mathsf{POS}, \mathsf{QF}]$ and $[\mathsf{POS}, \Pi^0_1]$ for building models of systems for explicit mathematics which reflect the recursive inaccessible and recursive Mahlo.

3.2 Theories for first order inductive definitions

The theories $\mathsf{FID}(\mathcal{K})$ for first order nonmonotone inductive definitions have been introduced in Jäger [10]. They provide an appropriate framework for modeling theories of explicit mathematics and establishing proof-theoretic upper bounds for them.

Let $\mathcal{K}$ be a collection of operator forms. Then we extend $\mathcal{L}$ to the language $\mathcal{L}_{\mathcal{K}}$ by adding ordinal variables $\alpha, \beta, \gamma, \zeta, \eta, \xi, \ldots$ (possibly with subscripts), a binary relation symbol $<$ for the less relation on the ordinals and a $(k{+}1)$-ary relation symbol $P_{\mathfrak{A}}$ for each k-ary operator form $\mathfrak{A} \in \mathcal{K}$. The number terms $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}_{\mathcal{K}}$ are the terms of $\mathcal{L}$, the ordinal terms of $\mathcal{L}_{\mathcal{K}}$ are the ordinal variables. The atomic formulas of $\mathcal{L}_{\mathcal{K}}$ are the atomic formulas of $\mathcal{L}$ plus all expressions $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_{\mathfrak{A}}(\alpha, \vec{s})$ for each k-ary operator form $\mathfrak{A}(P, \vec{u})$ from $\mathcal{K}$.

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1 \ldots)$ of $\mathcal{L}_{\mathcal{K}}$ are generated from the atomic formulas of $\mathcal{L}_{\mathcal{K}}$ by closing under the propositional connectives, quantification over natural numbers and quantification over the ordinals. An $\mathcal{L}_{\mathcal{K}}$ formula is called $\Delta_0^{\mathbb{O}}$ if it does not contain unbounded ordinal quantifiers.

From now on we will write $P_{\mathfrak{A}}^{\alpha}(\vec{s})$ instead of $P_{\mathfrak{A}}(\alpha, \vec{s})$ and use the following abbreviations:

$$P_{\mathfrak{A}}^{<\alpha}(\vec{s}) \;:=\; (\exists \xi < \alpha) P_{\mathfrak{A}}^{\xi}(\vec{s}) \qquad \text{and} \qquad P_{\mathfrak{A}}(\vec{s}) \;:=\; \exists \xi P_{\mathfrak{A}}^{\xi}(\vec{s}).$$

Now we are ready to present the theory $\mathsf{FID}(\mathcal{K})$ for first order inductive definitions with definition clauses from $\mathcal{K}$. It is formulated in the language $\mathcal{L}_{\mathcal{K}}$ and based on classical two sorted predicate logic with equality in both sorts. Its nonlogical axioms comprise the following five groups.

I. Number-theoretic axioms. The axioms of Peano arithmetic PA with exception of complete induction on the natural numbers.

II. Linearity axioms.

$$\alpha \not< \alpha \;\wedge\; (\alpha < \beta \wedge \beta < \gamma \;\rightarrow\; \alpha < \gamma) \;\wedge\; (\alpha < \beta \;\vee\; \alpha = \beta \;\vee\; \beta < \alpha).$$

III. Operator axioms. For all operator forms $\mathfrak{A}(P, \vec{u})$ from $\mathcal{K}$:

(OP.1) $$P_{\mathfrak{A}}^{\alpha}(\vec{s}) \;\leftrightarrow\; P_{\mathfrak{A}}^{<\alpha}(\vec{s}) \vee \mathfrak{A}(P_{\mathfrak{A}}^{<\alpha}, \vec{s}),$$

(OP.2) $$\mathfrak{A}(P_{\mathfrak{A}}, \vec{s}) \;\rightarrow\; P_{\mathfrak{A}}(\vec{s}).$$

IV. Induction on the natural numbers. For all $\mathcal{L}_{\mathcal{K}}$ formula $A(u)$:

$$A(0) \wedge \forall x(A(x) \rightarrow A(x{+}1)) \;\rightarrow\; \forall x A(x).$$

V. Induction on the ordinals. For all $\mathcal{L}_\mathcal{K}$ formula $A(\alpha)$:

$$\forall\xi((\forall\eta < \xi)A(\eta) \rightarrow A(\xi)) \;\rightarrow\; \forall\xi A(\xi).$$

Natural subsystems of $\mathsf{FID}(\mathcal{K})$ are obtained by weakening its induction principles: $\mathsf{FID}^\mathsf{w}(\mathcal{K})$ results from $\mathsf{FID}(\mathcal{K})$ by restricting induction on the ordinals to $\Delta_0^\mathbb{O}$ formulas of $\mathcal{L}_\mathcal{K}$; the theory $\mathsf{FID}^\mathsf{r}(\mathcal{K})$ is obtained from $\mathsf{FID}(\mathcal{K})$ by restricting induction on the natural numbers and induction on the ordinals to $\Delta_0^\mathbb{O}$ formulas of $\mathcal{L}_\mathcal{K}$.

The interpretation of the languages $\mathcal{L}_\mathcal{K}$ into the language of admissible set theory with urelements is straightforward. Thus theories for (iterated) admissible sets can be used in order to establish upper bounds of theories $\mathsf{FID}(\mathcal{K})$ for several operator classes $\mathcal{K}$.

In order to deal with the class operator forms $[\mathsf{POS}, \mathsf{QF}]$, we employ the theory KPi, described, for example, in Jäger [8], which formalizes a recursively inaccessible universe. Operator forms from $[\mathsf{POS}, \Pi^0_1]$ are interpreted into the system KPm. It is the variant of the theory KPM for a recursively Mahlo universe, analyzed in Rathjen [14, 15], in which the natural numbers are permitted as urelements. The theories KPi^w, KPi^r, KPm^w and KPm^r are the expected subsystems of KPi and KPm which are obtained by restricting inductions. In particular the following result can be obtained.

Theorem 3 *For the class of operator forms* $[\mathsf{POS}, \mathsf{QF}]$ *we have:*

1. $\mathsf{FID}^\mathsf{r}([\mathsf{POS}, \mathsf{QF}])$ *is contained in* KPi^r.
2. $\mathsf{FID}^\mathsf{w}([\mathsf{POS}, \mathsf{QF}])$ *is contained in* KPi^w.
3. $\mathsf{FID}([\mathsf{POS}, \mathsf{QF}])$ *is contained in* KPi.

This theorem as well as the following theorem are proved in Jäger [10]. Together with the proof-theoretic analysis of the theories KPi and KPm and their subsystems KPi^w, KPi^r, KPm^w and KPm^r they yield the desired upper proof-theoretic bounds.

Theorem 4 *For the class of operator forms* $[\mathsf{POS}, \Pi^0_1]$ *we have:*

1. $\mathsf{FID}^\mathsf{r}([\mathsf{POS}, \Pi^0_1])$ *is contained in* KPm^r.
2. $\mathsf{FID}^\mathsf{w}([\mathsf{POS}, \Pi^0_1])$ *is contained in* KPm^w.
3. $\mathsf{FID}([\mathsf{POS}, \Pi^0_1])$ *is contained in* KPm.

We dispense with mentioning the proof-theoretic ordinals of theses theories. Of course they are well-known and can be found in the articles quoted above.

4 Modeling $\mathsf{T}_0 + (\mathsf{Lim})$

Finally we can turn to building a model of $\mathsf{T}_0 + (\mathsf{Lim})$ in $\mathsf{FID}([\mathsf{POS}, \mathsf{QF}])$. As we will see, this model also satisfies the uniqueness of generators ($\mathbb{L}$-UG) and the ordering principles ($\mathcal{U}_{no}$-Lin) and ($\mathcal{U}_{no}$-Con) for normal universes.

We interpret application $\cdot$ of $\mathbb{L}$ in the sense of ordinary recursion theory so that $(a{\cdot}b)$ in $\mathbb{L}$ is translated into $\{a\}(b)$ in $\mathcal{L}$, where $\{n\}$ for $n = 0, 1, 2, 3, \dots$ is a standard enumeration of the partial recursive functions. Then it is possible to assign pairwise different numerals to the constants k, s, p, p_0, p_1, s_N, p_N and d_N so that the applicative axioms (1)–(8) of T_0 are satisfied. We also require that the constant 0 of $\mathbb{L}$ is interpreted as the 0 of $\mathcal{L}$ and the term $\mathsf{s}_\mathsf{N} a$ of $\mathbb{L}$ as $a + 1$ in $\mathcal{L}$. In addition, we let pairing and projections of $\mathbb{L}$ go over into the primitive recursive pairing and unpairing machinery introduced above.

For each $\mathbb{L}$ term t there also exists an $\mathcal{L}$ formula $\mathsf{Val}_t(a)$ expressing that a is the value of t under the interpretation described above. Accordingly, the atomic formulas $t\downarrow$, $s = t$ and $\mathsf{N}(t)$ are given their obvious interpretations in $\mathcal{L}$ with the translation of N ranging over all natural numbers.

For dealing with the generators we choose, again by ordinary recursion theory, numerals $\underline{\mathsf{nat}}$, $\underline{\mathsf{id}}$, $\underline{\mathsf{co}}$, $\underline{\mathsf{int}}$, $\underline{\mathsf{dom}}$, $\underline{\mathsf{inv}}$, $\underline{\mathsf{j}}$, $\underline{\mathsf{i}}$, $\underline{\ell}$ and $\underline{\mathsf{m}}$ so that we have the properties

$$
\begin{array}{lll}
\underline{\mathsf{nat}} = \langle 0, 0\rangle, & \underline{\mathsf{id}} = \langle 1, 0\rangle, & \{\underline{\mathsf{co}}\}(a) = \langle 2, a\rangle, \\
\{\underline{\mathsf{int}}\}(\langle a, b\rangle) = \langle 3, a, b\rangle, & \{\underline{\mathsf{dom}}\}(a) = \langle 4, a\rangle, & \{\underline{\mathsf{inv}}\}(\langle a, b\rangle) = \langle 5, a, b\rangle, \\
\{\underline{\mathsf{j}}\}(\langle a, b\rangle) = \langle 6, a, b\rangle, & \{\underline{\mathsf{i}}\}(\langle a, b\rangle) = \langle 7, a, b\rangle, & \{\underline{\ell}\}(a) = \langle 8, a\rangle, \\
\{\underline{\mathsf{m}}\}(\langle a, b\rangle) = \langle 9, a, b\rangle, & \{e_0\}(a) \neq e_1 &
\end{array}
$$

for all natural numbers a, b and all e_0 and e_1 from the set ranging over $\underline{\mathsf{nat}}$, $\underline{\mathsf{id}}$, $\underline{\mathsf{co}}$, $\underline{\mathsf{int}}$, $\underline{\mathsf{dom}}$, $\underline{\mathsf{inv}}$, $\underline{\mathsf{j}}$, $\underline{\mathsf{i}}$, $\underline{\ell}$ and $\underline{\mathsf{m}}$.

The crucial step in our model construction is to find adequate codes for the types of explicit mathematics and to properly deal with the fact that the individual b is an element of the type coded by a. For achieving this we make use of an operator form $\mathfrak{A}(P, a, b, c)$ from $[\mathsf{POS}, \mathsf{QF}]$ and the corresponding relation symbol $P_\mathfrak{A}$. Then our interpretation will be so that

$$P_\mathfrak{A}(a, 0, 0) \ \text{ stands for } \ \Re(a),$$

$$P_\mathfrak{A}(a, b, 1) \ \text{ for } \ \Re(a) \land b \,\dot{\in}\, a \qquad \text{and} \qquad P_\mathfrak{A}(a, b, 2) \ \text{ for } \ \Re(a) \land b \,\dot{\notin}\, a.$$

We first need an operator form $\mathfrak{A}_0(P, a, b, c)$ from POS which is the disjunction of the following formulas (1)–(21):

(1) $a = \langle 0, 0\rangle \ \wedge\ b = 0 \ \wedge\ c = 0,$

(2) $a = \langle 0, 0\rangle \ \wedge\ c = 1,$

(3) $a = \langle 1, 0\rangle \ \wedge\ b = 0 \ \wedge\ c = 0,$

(4) $a = \langle 1, 0\rangle \ \wedge\ \exists x(b = \langle x, x\rangle) \ \wedge\ c = 1,$

(5) $a = \langle 1, 0\rangle \ \wedge\ \forall x(b \neq \langle x, x\rangle) \ \wedge\ c = 2,$

(6) $\exists u[a = \langle 2, u\rangle \wedge P(u, 0, 0)] \ \wedge\ b = 0 \ \wedge\ c = 0,$

(7) $\exists u[a = \langle 2, u\rangle \wedge P(u, 0, 0) \wedge P(u, b, 2)] \ \wedge\ c = 1,$

(8) $\exists u[a = \langle 2, u\rangle \wedge P(u, 0, 0) \wedge P(u, b, 1)] \ \wedge\ c = 2,$

(9) $\exists u\exists v[a = \langle 3, u, v\rangle \wedge P(u, 0, 0) \wedge P(v, 0, 0)] \ \wedge\ b = 0 \ \wedge\ c = 0,$

(10) $\exists u\exists v(a = \langle 3, u, v\rangle \wedge P(u, 0, 0) \wedge P(v, 0, 0) \wedge P(u, b, 1) \wedge P(v, b, 1) \ \wedge$
$c = 1,$

(11) $\exists u\exists v(a = \langle 3, u, v\rangle \wedge P(u, 0, 0) \wedge P(v, 0, 0) \wedge (P(u, b, 2) \vee P(v, b, 2))] \ \wedge$
$c = 2,$

(12) $\exists u[a = \langle 4, u\rangle \wedge P(u, 0, 0)] \ \wedge\ b = 0 \ \wedge\ c = 0,$

(13) $\exists u[a = \langle 4, u\rangle \wedge P(u, 0, 0) \wedge \exists x P(u, \langle b, x\rangle, 1)] \ \wedge\ c = 1,$

(14) $\exists u[a = \langle 4, u\rangle \wedge P(u, 0, 0) \wedge \forall x P(u, \langle b, x\rangle, 2)] \ \wedge\ c = 2,$

(15) $\exists u\exists f[a = \langle 5, u, f\rangle \wedge P(u, 0, 0)] \ \wedge\ b = 0 \ \wedge\ c = 0,$

(16) $\exists u\exists f[a = \langle 5, u, f\rangle \wedge P(u, 0, 0) \wedge \exists x(x = \{f\}(b) \wedge P(u, x, 1))] \ \wedge\ c = 1,$

(17) $\exists u\exists f[a = \langle 5, u, f\rangle \wedge P(u, 0, 0) \wedge \forall x(x \neq \{f\}(b) \vee P(u, x, 2))] \ \wedge\ c = 2,$

(18) $\exists u\exists f[a = \langle 6, u, f\rangle \wedge P(u, 0, 0) \wedge \forall x(P(u, x, 2) \vee P(\{f\}(x), 0, 0))] \ \wedge$
$b = 0 \ \wedge\ c = 0,$

(19) $\exists u\exists f[a = \langle 6, u, f\rangle \wedge P(u, 0, 0) \wedge \forall x(P(u, x, 2) \vee P(\{f\}(x), 0, 0)) \wedge$
$\exists y\exists z(b = \langle y, z\rangle \wedge P(u, y, 1) \wedge P(\{f\}(y), z, 1))] \ \wedge\ c = 1,$

(20) $\exists u\exists f[a = \langle 6, u, f\rangle \wedge P(u, 0, 0) \wedge \forall x(P(u, x, 2) \vee P(\{f\}(x), 0, 0)) \wedge$
$\forall y\forall z(b \neq \langle y, z\rangle \vee P(u, y, 2) \vee P(\{f\}(y), z, 2))] \ \wedge\ c = 2,$

(21) $\exists u\exists v[a = \langle 7, u, v\rangle \wedge P(u, 0, 0) \wedge P(v, 0, 0) \wedge P(u, b, 1) \wedge$
$\forall x(P(u, x, 2) \vee P(v, \langle x, b\rangle, 2) \vee P(a, x, 1))] \ \wedge\ c = 1.$

This operator form takes care of the generators nat, id, co, int, dom, inv and j. Moreover, clause (21) also builds up the elements of the accessible part which is named $\mathsf{i}(u, v)$ in the language $\mathbb{L}$. Observe, however, that the code for this accessible part is not introduced by this operator form; and it also does not say which elements do not belong to its extension.

In the end, however, we will work with combined operators, and therefore a second operator form $\mathfrak{A}_1(P, a, b, c)$ from QF is employed for the codes of accessible parts and and for those element not belonging to their extension. As soon as closure under $\mathfrak{A}_0(P, a, b, c)$ is achieved, we know that all elements of the relevant accessible parts have been generated. Then we switch to $\mathfrak{A}_1(P, a, b, c)$ to obtain the codes of these accessible parts and their complements and continue with $\mathfrak{A}_0(P, a, b, c)$. The limit axiom is treated accordingly.

Hence, let $\mathfrak{A}_1(P, a, b, c)$ be the operator form from QF which is the disjunction of the following formulas (22)–(26):

(22) $a = \langle 7, (a)_1, (a)_2 \rangle \,\wedge\, P((a)_1, 0, 0) \,\wedge\, P((a)_2, 0, 0) \,\wedge\, b = 0 \,\wedge\, c = 0,$

(23) $a = \langle 7, (a)_1, (a)_2 \rangle \,\wedge\, P((a)_1, 0, 0) \,\wedge\, P((a)_2, 0, 0) \,\wedge\, \neg P(a, b, 1) \,\wedge\, c = 2,$

(24) $a = \langle 8, (a)_1 \rangle \,\wedge\, \neg P(a, 0, 0) \,\wedge\, P((a)_1, 0, 0) \,\wedge\, b = 0 \,\wedge\, c = 0,$

(25) $a = \langle 8, (a)_1 \rangle \,\wedge\, \neg P(a, 0, 0) \,\wedge\, P((a)_1, 0, 0) \,\wedge\, P(b, 0, 0) \,\wedge\, c = 1,$

(26) $a = \langle 8, (a)_1 \rangle \,\wedge\, \neg P(a, 0, 0) \,\wedge\, P((a)_1, 0, 0) \,\wedge\, \neg P(b, 0, 0) \,\wedge\, c = 2.$

Using the first operator form $\mathfrak{A}_0(P, a, b, c)$ from POS and the second operator form $\mathfrak{A}_1(P, a, b, c)$ from QF we can define the combined operator form $\mathfrak{A}(P, a, b, c)$ from $[\mathsf{POS}, \mathsf{QF}]$ as

$$\mathfrak{A}_0(P, a, b, c) \,\vee\, [\forall x \forall y \forall z (\mathfrak{A}_0(P, x, y, z) \to P(x, y, z)) \,\wedge\, \mathfrak{A}_1(P, a, b, c)].$$

By easy $\Delta_0^{\mathbb{O}}$ induction on the ordinals we can now show that the extension of the code of a type is determined as soon as the code is created. In such cases also the complementarity of $P_{\mathfrak{A}}(\ldots, 1)$ and $P_{\mathfrak{A}}(\ldots, 2)$ are guaranteed.

Lemma 5 *The following assertions are provable in* $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS}, \mathsf{QF}])$*:*

1. $P_{\mathfrak{A}}^{\alpha}(a, 0, 0) \wedge \alpha \leq \beta \,\to\, \forall x (P_{\mathfrak{A}}^{\alpha}(a, x, 1) \leftrightarrow P_{\mathfrak{A}}^{\beta}(a, x, 1)).$
2. $P_{\mathfrak{A}}^{\alpha}(a, 0, 0) \wedge \alpha \leq \beta \,\to\, \forall x (P_{\mathfrak{A}}^{\alpha}(a, x, 2) \leftrightarrow P_{\mathfrak{A}}^{\beta}(a, x, 2)).$
3. $P_{\mathfrak{A}}^{\alpha}(a, 0, 0) \wedge \alpha \leq \beta \,\to\, \forall x (\neg P_{\mathfrak{A}}^{\alpha}(a, x, 1) \leftrightarrow P_{\mathfrak{A}}^{\beta}(a, x, 2)).$

Before turning to the interpretation of the types, the $\in$ relation and the naming relation we introduce the following two definitions:

$$\mathsf{Rep}(a) \ := \ P_{\mathfrak{A}}(a,0,0) \qquad \text{and} \qquad \mathsf{E}(b,a) \ := \ P_{\mathfrak{A}}(a,b,1).$$

In our embedding of $\mathsf{T}_0^{\mathsf{r}}$ into $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}])$ we first assume that the number and type variables of $\mathbb{L}$ are mapped into the number variables of the language of $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}])$ so that no conflicts arise; to simplify the notation we often identify the type variables of $\mathbb{L}$ with their translations. Then we let the type variables of $\mathbb{L}$ range over Rep and the translation of the atomic formulas of $\mathbb{L}$ involving types is as follows:

$$\begin{aligned}
\Re(t,U)^* \ &:= \ \exists x[\mathsf{Val}_t(x) \ \wedge \ \mathsf{Rep}(x) \ \wedge \ \forall y(\mathsf{E}(y,x) \leftrightarrow \mathsf{E}(y,U))],\\
(t \in U)^* \ &:= \ \exists x(\mathsf{Val}_t(x) \wedge \mathsf{E}(x,U)).
\end{aligned}$$

On the basis of these initial cases the translation of arbitrary $\mathbb{L}$ formulas A into formulas A^* of $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}])$ should be obvious. The embedding of T_0 (and its subsystems) into $\mathsf{FID}([\mathsf{POS},\mathsf{QF}])$ (and its subsystems) is given by the following theorem.

Theorem 6 *We have for all $\mathbb{L}$ formulas $A(\vec{U},\vec{a})$ with at most the variables $\vec{U}$ and $\vec{a}$ free:*

1. *If the theory* $\mathsf{T}_0^{\mathsf{r}}+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})+(\mathcal{U}_{no}\text{-}\mathsf{Lin})+(\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \ \rightarrow \ A^*(\vec{U},\vec{a}).$$

2. *If the theory* $\mathsf{T}_0^{\mathsf{w}}+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})+(\mathcal{U}_{no}\text{-}\mathsf{Lin})+(\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID}^{\mathsf{w}}([\mathsf{POS},\mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \ \rightarrow \ A^*(\vec{U},\vec{a}).$$

3. *If the theory* $\mathsf{T}_0+(\mathsf{Lim})+(\mathbb{L}\text{-}\mathsf{UG})+(\mathcal{U}_{no}\text{-}\mathsf{Lin})+(\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID}([\mathsf{POS},\mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \ \rightarrow \ A^*(\vec{U},\vec{a}).$$

Proof If it is an applicative axiom or an axiom concerning the uniqueness of generators, then its translation is provable in $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}])$ by our assumptions about the coding of the first order part of $\mathsf{T}_0^{\mathsf{r}}$. The translations of the axioms about explicit representation and equality as well as linearity

($\mathcal{U}_{no}$-Lin) and connectivity ($\mathcal{U}_{no}$-Con) of normal universes are easily verified. In the case of the basic type existence axioms we confine ourselves to showing the translation of the axioms about *Intersection*.

Assume we are given two natural numbers a and b so that $\mathsf{Rep}(a)$ and $\mathsf{Rep}(b)$. Then we immediately obtain $\mathsf{Rep}(\langle 3, a, b\rangle)$ by closure under $\mathfrak{A}_0$. The same argument yields $\mathsf{E}(c, \langle 3, a, b\rangle)$ if $\mathsf{E}(c, a)$ and $\mathsf{E}(c, b)$ hold. To show the other direction assume $\mathsf{E}(c, \langle 3, a, b\rangle)$. Then there exists an ordinal α such that $P^{\alpha}_{\mathfrak{A}}(\langle 3, a, b\rangle, c, 1)$ but not $P^{<\alpha}_{\mathfrak{A}}(\langle 3, a, b\rangle, c, 1)$. By the definition of $\mathfrak{A}$ we therefore must have $P^{<\alpha}_{\mathfrak{A}}(a, c, 1)$ as well as $P^{<\alpha}_{\mathfrak{A}}(b, c, 1)$. Thus the definition of E yields $\mathsf{E}(c, a)$ and $\mathsf{E}(c, b)$.

Hence, the intersection axiom of $\mathsf{T}^{\mathsf{r}}_0$ is verified. The other basic type existence axioms are either trivial or treated accordingly; their exact discussion can therefore be omitted.

Before treating inductive generation, we turn to the verification of the limit axiom (Lim). For this purpose take an arbitrary a so that $\mathsf{Rep}(a)$. Then there exists an α with $P^{\alpha}_{\mathfrak{A}}(a, 0, 0)$ and $\neg P^{<\alpha}_{\mathfrak{A}}(a, 0, 0)$. Furthermore, there must be a least ordinal β strictly greater than α such that $P^{<\beta}_{\mathfrak{A}}$ is closed under $\mathfrak{A}_0$. It follows that $P^{\beta}_{\mathfrak{A}}(\langle 8, a\rangle, 0, 0)$ and $P^{\beta}_{\mathfrak{A}}(\langle 8, a\rangle, a, 1)$; hence we have $\mathsf{Rep}(\langle 8, a\rangle)$ and $\mathsf{E}(a, \langle 8, a\rangle)$.

From the definition of the operator form $\mathfrak{A}_1$ we can further deduce that for every natural number c the assertion $P^{\beta}_{\mathfrak{A}}(\langle 8, a\rangle, c, 1)$ is equivalent to $P^{<\beta}_{\mathfrak{A}}(c, 0, 0)$ and the assertion $P^{\beta}_{\mathfrak{A}}(\langle 8, a\rangle, c, 2)$ equivalent to $\neg P^{<\beta}_{\mathfrak{A}}(c, 0, 0)$. Because of Lemma 5 we therefore have for all c that

$$\mathsf{E}(c, \langle 8, a\rangle) \;\leftrightarrow\; P^{<\beta}_{\mathfrak{A}}(c, 0, 0).$$

So we know that $\langle 8, a\rangle$ codes a type whose extension contains a and consists of codes of types only.

It still has to be shown that the extension of $\langle 8, a\rangle$ is closed under the generators nat, id, co, int, dom, inv and j. But this follows directly from the fact that $P^{<\beta}_{\mathfrak{A}}$ is closed under $\mathfrak{A}_0$. As illustration, we consider closure under *Intersection.* So assume $\mathsf{E}(b, \langle 8, a\rangle)$ and $\mathsf{E}(c, \langle 8, a\rangle)$. Then we know $P^{<\beta}_{\mathfrak{A}}(b, 0, 0)$ as well as $P^{<\beta}_{\mathfrak{A}}(c, 0, 0)$ and obtain $\mathfrak{A}_0(P^{<\beta}_{\mathfrak{A}}, \langle 3, b, c\rangle, 0, 0)$. Hence the closure of $P^{<\beta}_{\mathfrak{A}}$ under $\mathfrak{A}_0$ implies $P^{<\beta}_{\mathfrak{A}}(\langle 3, b, c\rangle, 0, 0)$, and we conclude $\mathsf{E}(\langle 3, b, c\rangle, \langle 8, a\rangle)$.

Therefore, also the limit axiom (Lim) is established in our model, and it only remains to verify inductive generation (IG) and the induction principles of our theories.

For the axiom (IG.1) of inductive generation we assume $\mathsf{Rep}(a)$ and $\mathsf{Rep}(b)$ for two given natural numbers a and b. As a consequence of the second operator axiom we obtain $\mathsf{Rep}(\langle 7,a,b\rangle)$. Furthermore, we find an ordinal α such that $P^{\alpha}_{\mathfrak{A}}(a,0,0)$, $P^{\alpha}_{\mathfrak{A}}(b,0,0)$ and $P^{\alpha}_{\mathfrak{A}}(\langle 7,a,b\rangle,0,0)$. Now assume

$$\mathsf{E}(c,a) \ \wedge\ \forall x[\mathsf{E}(x,a) \wedge \mathsf{E}(\langle x,c\rangle,b) \ \rightarrow\ \mathsf{E}(x,\langle 7,a,b\rangle)]$$

for some natural number c. In view of the three assertions of Lemma 5 this implies

$$P^{\alpha}_{\mathfrak{A}}(a,c,1) \ \wedge\ \forall x[P^{\alpha}_{\mathfrak{A}}(a,x,2) \ \vee\ P^{\alpha}_{\mathfrak{A}}(b,\langle x,c\rangle,2) \ \vee\ P^{\alpha}_{\mathfrak{A}}(\langle 7,a,b\rangle,x,1)]$$

and therefore, by closure under $\mathfrak{A}_0$, we have $\mathsf{E}(x,\langle 7,a,b\rangle)$. Hence ($\mathsf{IG}$.1) is settled as well.

So far everything has been the same for all three assertions of our theorem; distinctions occur in connection with the induction principles. In order to verify full (IG.2) we choose an arbitrary formula $A(u)$ and set

$$B_A(a,b) \ :=\ \forall x[\mathsf{E}(x,a) \ \wedge\ \forall y(\mathsf{E}(y,a) \wedge \mathsf{E}(\langle y,x\rangle,b) \ \rightarrow\ A(y)) \ \rightarrow\ A(x)].$$

Now suppose $\mathsf{Rep}(a)$, $\mathsf{Rep}(b)$ and $B_A(a,b)$. Working in $\mathsf{FID}([\mathsf{POS},\mathsf{QF}])$, we prove

$$\forall\alpha\forall x(P^{\alpha}_{\mathfrak{A}}(\langle 7,a,b\rangle,x,1) \rightarrow A(x))$$

by full induction on the ordinals: Let u be a natural number such that $P^{\alpha}_{\mathfrak{A}}(\langle 7,a,b\rangle,u,1)$. Then for every natural number v we know that $\mathsf{E}(v,a)$ and $\mathsf{E}(\langle v,u\rangle,b)$ imply $P^{<\alpha}_{\mathfrak{A}}(\langle 7,a,b\rangle,v,1)$. Hence we obtain $A(v)$ by the induction hypothesis. Together with $B(a,b)$ this implies $A(u)$. Hence we have proved for all formulas $A(u)$ that

$$\mathsf{Rep}(a) \wedge \mathsf{Rep}(b) \wedge B_A(a,b) \ \rightarrow\ \forall x(\mathsf{E}(x,\langle 7,a,b\rangle) \rightarrow A(x)).$$

This implies that each instance of full (IG.2) can be dealt with in the theory $\mathsf{FID}([\mathsf{POS},\mathsf{QF}])$. In the case of ($\mathsf{IG}$.2) restricted to types we have to be a bit more careful. For arbitrary a, b and c we set

$$B_c(a,b) \ :=\ \forall x[\mathsf{E}(x,a) \ \wedge\ \forall y(\mathsf{E}(y,a) \wedge \mathsf{E}(\langle y,x\rangle,b) \ \rightarrow\ \mathsf{E}(y,c)) \ \rightarrow\ \mathsf{E}(x,c)].$$

Now assume $\mathsf{Rep}(a)$, $\mathsf{Rep}(b)$, $\mathsf{Rep}(c)$ and $B_c(a,b)$. It is also clear that there exists an ordinal γ so that we have $P^{\gamma}_{\mathfrak{A}}(a,0,0)$, $P^{\gamma}_{\mathfrak{A}}(b,0,0)$ and $P^{\gamma}_{\mathfrak{A}}(c,0,0)$. From Lemma 5 we can therefore conclude that $B_c(a,b)$ is equivalent to the $\Delta^{\mathbb{O}}_0$ formula $B^{\gamma}_c(a,b)$,

$$B^{\gamma}_c(a,b) \ :=$$
$$\forall x[\mathsf{E}^{\gamma}(x,a) \ \wedge\ \forall y(\mathsf{E}^{\gamma}(y,a) \wedge \mathsf{E}^{\gamma}(\langle y,x\rangle,b) \ \rightarrow\ \mathsf{E}^{\gamma}(y,c)) \ \rightarrow\ \mathsf{E}^{\gamma}(x,c)]$$

where $\mathsf{E}^\gamma(u,v)$ is shorthand for $P^\gamma_{\mathfrak{A}}(v,u,1)$. Similar to before we obtain from our assumptions that

$$\forall\alpha\forall x(P^\alpha_{\mathfrak{A}}(\langle 7,a,b\rangle,x,1) \to \mathsf{E}^\gamma(x,c)).$$

However, we only need $\Delta_0^{\mathbb{O}}$ induction on the ordinals instead of full induction to carry through this argument. This means that the restricted theory $\mathsf{FID}^\mathsf{r}([\mathsf{POS},\mathsf{QF}])$ proves

$$\mathsf{Rep}(a) \wedge \mathsf{Rep}(b) \wedge \mathsf{Rep}(c) \wedge B_c(a,b) \;\to\; \forall x(\mathsf{E}(x,\langle 7,a,b\rangle) \to \mathsf{E}(x,c)).$$

Consequently the (translation of the) restriction of (IG.2) to types is provable in $\mathsf{FID}^\mathsf{r}([\mathsf{POS},\mathsf{QF}])$.

Obviously every instance of full induction ($\mathbb{L}$-$\mathsf{I_N}$) on the natural number in $\mathsf{T_0}$ translates into an instance of full induction on the natural numbers in the theory $\mathsf{FID}([\mathsf{POS},\mathsf{QF}])$. In addition, following the pattern of the treatment of (IG.2), it can be easily verified that type induction (T-$\mathsf{I_N}$) of $\mathsf{T_0^r}$ can be handled via $\Delta_0^{\mathbb{O}}$ induction on the natural numbers in $\mathsf{FID}^\mathsf{r}([\mathsf{POS},\mathsf{QF}])$.

So far we have show that the three assertions of our theorem are valid for all respective axioms. Otherwise we just have to proceed by induction on the length of the derivation of the formula $A(\vec{U},\vec{a})$. $\Box$

It is immediate consequence of the previous theorem, Theorem 3 and well-known results about $\mathsf{T_0}$ and KPi that adding (Lim), ($\mathbb{L}$-UG), ($\mathcal{U}_{no}$-Lin) and ($\mathcal{U}_{no}$-Con) does not increase the proof-theoretic strength of $\mathsf{T_0}$ or its subsystems $\mathsf{T_0^w}$ and $\mathsf{T_0^r}$. In the next section we will see that our model construction can be used to establish further interesting proof-theoretic equivalences.

5 Strict universes and name induction

Remember that a type W is called a universe if it consists of names only and if we have $\forall x(\mathcal{C}(W,x) \to x \in W)$ for the closure condition $\mathcal{C}$ introduced in Section 2.2. Hence, if W is a universe and a an element of W, then $\mathsf{co}(a)$ has to be an element of W as well. On the other hand, if $\mathsf{co}(a)$ is in the universe W, then it may happen that a does not belong to W. Such situations are ruled out for strict universes.

The notion of *strictness of universes* has been introduced in Jäger, Kahle and Studer [11] and is discussed there in greater detail. We write $\mathsf{Str}(W)$ for the conjunction of the following formulas:

(1) $\forall x(\mathsf{co}(x) \in W \;\to\; x \in W)$,

(2) $\forall x \forall y(\mathsf{int}(x,y) \in W \rightarrow x \in W \wedge y \in W)$,

(3) $\forall x(\mathsf{dom}(x) \in W \rightarrow x \in W)$,

(4) $\forall f \forall x(\mathsf{inv}(f,x) \in W \rightarrow x \in W)$,

(5) $\forall x \forall f(\mathsf{j}(x,f) \in W \rightarrow x \in W \wedge (\forall y \dot{\in} x)(fy \in W))$,

(6) $\forall x \forall y(\mathsf{i}(x,y) \in W \rightarrow x \in W \wedge y \in W)$,

(7) $\forall x(\ell(x) \in W \rightarrow x \in W)$,

(8) $\forall x \forall f(\mathsf{m}(x,f) \in W \rightarrow x \in W \wedge (\forall y \in W)(fy \in W))$.

Accordingly, a type W is called a strict universe if it is a universe and if it satisfies the condition $\mathsf{Str}(W)$.

Definition 7 *1. We write* $\mathsf{SU}(W)$ *to express that the type* W *is a* strict universe*,*

$$\mathsf{SU}(W) \;:=\; \mathsf{U}(W) \wedge \mathsf{Str}(W).$$

2. We write $\mathcal{SU}(t)$ *to express that the individual* t *is a* name of a strict universe*,*

$$\mathcal{SU}(t) \;:=\; \exists X(\Re(t,X) \wedge \mathsf{SU}(X)).$$

The limit axiom (Lim) postulates that every name a belongs to a universe which is named ℓa. In the context strictness, this axiom is now replaced by the corresponding limit axiom for strict universes

$$(\mathsf{sLim}) \qquad \forall x(\Re(x) \rightarrow \mathcal{SU}(\ell x) \wedge x \dot{\in} \ell x).$$

Studying our model construction of the previous section we can easily check that it satisfies the strict limit axiom. Hence, have the following theorem.

Theorem 8 *We have for all* $\mathbb{L}$ *formulas* $A(\vec{U},\vec{a})$ *with at most the variables* $\vec{U}$ *and* $\vec{a}$ *free:*

1. *If the theory* $\mathsf{T}_0^{\mathsf{r}}$+($\mathsf{sLim}$)+($\mathbb{L}$-$\mathsf{UG}$)+($\mathcal{U}_{no}$-$\mathsf{Lin}$)+($\mathcal{U}_{no}$-$\mathsf{Con}$) *proves* $A(\vec{U},\vec{a})$*, then*

$$\mathsf{FID}^{\mathsf{r}}([\mathsf{POS},\mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \rightarrow A^*(\vec{U},\vec{a}).$$

2. *If the theory* $\mathsf{T}_0^{\mathsf{w}} + (\mathsf{sLim}) + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U}, \vec{a})$, *then*

$$\mathsf{FID}^{\mathsf{w}}([\mathsf{POS}, \mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \;\rightarrow\; A^*(\vec{U}, \vec{a}).$$

3. *If the theory* $\mathsf{T}_0 + (\mathsf{sLim}) + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U}, \vec{a})$, *then*

$$\mathsf{FID}([\mathsf{POS}, \mathsf{QF}]) \vdash \mathsf{Rep}(\vec{U}) \;\rightarrow\; A^*(\vec{U}, \vec{a}).$$

In Jäger, Kahle and Studer [11] the principle of name induction is introduced as an alternative to inductive generation or least universes. This axiom schema claims that the elements of $\Re$ are built up by the use of generators only. In a certain sense it can be understood as an intensional version of $\in$ induction.

Since the generators i and m do not play a role in the form of name induction studied here we introduce the auxiliary language $\mathbb{L}^-$ which is exactly as $\mathbb{L}$ but with i and m omitted. The following theory NAI is formulated in $\mathbb{L}^-$.

In order to state the axiom schema of name induction we introduce the closure condition $\widehat{\mathcal{C}}(U, a)$ which extends $\mathcal{C}(U, a)$ by a new clause for the universe generator ℓ,

$$\widehat{\mathcal{C}}(U, a) \;:=\; \mathcal{C}(U, a) \,\vee\, \exists x(a = \ell x \wedge x \in U).$$

The schema of *name induction* with respect to $\mathbb{L}^-$ is the principle that there are no definable subcollections of the names with this closure property. It is given by

$$(\mathbb{L}^-\text{-}\mathsf{I}_{\Re}) \qquad \forall x(\widehat{\mathcal{C}}(A, x) \rightarrow A(x)) \;\rightarrow\; \forall x(\Re(x) \rightarrow A(x))$$

for all $\mathbb{L}^-$ formulas $A(u)$. This form of name induction will be considered now in the context of EETJ with the strict limit axiom, uniqueness of generators and the schema of complete induction on the natural numbers. We set

$$\mathsf{NAI} \;\;:=\;\; \mathsf{EETJ} + (\mathsf{sLim}) + (\mathbb{L}^-\text{-}\mathsf{UG}) + (\mathbb{L}^-\text{-}\mathsf{I}_{\mathsf{N}}) + (\mathbb{L}^-\text{-}\mathsf{I}_{\Re}).$$

Of course the formulation of (sLim), in particular the formulation of strictness has been adjusted in the obvious way to the language $\mathbb{L}^-$.

Again $\mathsf{NAI}^{\mathsf{w}}$ is the system NAI with name induction restricted to types and $\mathsf{NAI}^{\mathsf{r}}$ is obtained by restricting name induction and complete induction on the natural numbers to types.

We can adapt our model construction such that it satisfies name induction. For this aim we let the operator form $\mathfrak{B}_0(P, a, b, c)$ be the disjunction of the clauses (1)–(20) of the definition of the operator form $\mathfrak{A}_0(P, a, b, c)$ in Section 4. $\mathfrak{B}_1(P, a, b, c)$ is the disjunction of the clauses (24)–(26) of the definition of the operator form $\mathfrak{A}_1(P, a, b, c)$. Then the new operator form $\mathfrak{B}(P, a, b, c)$ from $[\mathsf{POS}, \mathsf{QF}]$ is defined as

$$\mathfrak{B}_0(P, a, b, c) \vee [\forall x \forall y \forall z(\mathfrak{B}_0(P, x, y, z) \to P(x, y, z)) \wedge \mathfrak{B}_1(P, a, b, c)].$$

This means that $\mathfrak{B}(P, a, b, c)$ results from the operator form $\mathfrak{A}(P, m, n, k)$ by dropping all cases which deal with inductive generation.

For the rest of this section let A° be the translation of the $\mathbb{L}^-$ formula A into the language of $\mathsf{FID}([\mathsf{POS}, \mathsf{QF}])$ which is defined as the translation A^* in Section 4 with the only difference that all subformulas $\exists \xi P^\xi_{\mathfrak{A}}(\dots)$ are replaced by $\exists \xi P^\xi_{\mathfrak{B}}(\dots)$. $\mathsf{Rep}^\circ(a)$ stands for $P_{\mathfrak{B}}(a, 0, 0)$. In this sense the operator form $\mathfrak{B}(P, a, b, c)$ from $[\mathsf{POS}, \mathsf{QF}]$ provides for a model of explicit mathematics with name induction.

Theorem 9 *We have for all $\mathbb{L}^-$ formulas $A(\vec{U}, \vec{a})$ with at most the variables $\vec{U}$ and $\vec{a}$ free:*

1. *If the theory $\mathsf{NAI}^\mathsf{r} + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ proves $A(\vec{U}, \vec{a})$, then*

$$\mathsf{FID}^\mathsf{r}([\mathsf{POS}, \mathsf{QF}]) \vdash \mathsf{Rep}^\circ(\vec{U}) \ \to \ A^\circ(\vec{U}, \vec{a}).$$

2. *If the theory $\mathsf{NAI}^\mathsf{w} + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ proves $A(\vec{U}, \vec{a})$, then*

$$\mathsf{FID}^\mathsf{w}([\mathsf{POS}, \mathsf{QF}]) \vdash \mathsf{Rep}^\circ(\vec{U}) \ \to \ A^\circ(\vec{U}, \vec{a}).$$

3. *If the theory $\mathsf{NAI} + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ proves $A(\vec{U}, \vec{a})$, then*

$$\mathsf{FID}([\mathsf{POS}, \mathsf{QF}]) \vdash \mathsf{Rep}^\circ(\vec{U}) \ \to \ A^\circ(\vec{U}, \vec{a}).$$

Proof Of course this proof runs along the same lines as the proofs of Theorem 6 and Theorem 8. Here we confine ourselves to looking at name induction only. So we work in $\mathsf{FID}([\mathsf{POS}, \mathsf{QF}])$, take an arbitrary $\mathbb{L}^-$ formula $B(u)$ and assume $\forall x(\widehat{\mathcal{C}}(B, x) \to B(x))^\circ$. Now we prove

$$\forall \xi \forall x(P^\xi_{\mathfrak{B}}(x, 0, 0) \to B^\circ(x))$$

by full induction on the ordinals. Hence, choose an ordinal α and a natural number a so that $P^\alpha_{\mathfrak{B}}(a, 0, 0)$. It follows $P^{<\alpha}_{\mathfrak{B}}(a, 0, 0)$ or $\mathfrak{B}(P^{<\alpha}_{\mathfrak{B}}, a, 0, 0)$. From

$P_{\mathfrak{B}}^{<\alpha}(a, 0, 0)$ we obtain $B^\circ(a)$ immediately from the induction hypothesis. If we have $\mathfrak{B}(P_{\mathfrak{B}}^{<\alpha}, a, 0, 0)$, then we carry through a distinction by cases with respect to a. Since $B^\circ(u)$ is closed under $\widehat{\mathcal{C}}^\circ$ by assumption, the induction hypothesis always implies $B^\circ(a)$.

This settles full name induction. Again we easily see that name induction restricted to types can be handled by $\Delta_0^{\mathbb{O}}$ induction on the ordinals. Therefore it is possible to interpret name induction for types into $\mathsf{FID}^{\mathsf{r}}([\mathsf{POS}, \mathsf{QF}])$ and $\mathsf{FID}^{\mathsf{w}}([\mathsf{POS}, \mathsf{QF}])$. □

This theorem and results from Jäger, Kahle and Studer [11] show that the theories NAI, $\mathsf{NAI}^{\mathsf{w}}$ and $\mathsf{NAI}^{\mathsf{r}}$ with our without ($\mathcal{U}_{no}$-Lin) and ($\mathcal{U}_{no}$-Con) are proof-theoretically equivalent to T_0, $\mathsf{T}_0^{\mathsf{w}}$ and $\mathsf{T}_0^{\mathsf{r}}$, respectively.

6 Mahlo

In classical set theory an ordinal κ is called a *Mahlo ordinal* if it is a regular cardinal and if for every ordinal μ less than κ and every normal function $\mathcal{F}$ from κ to κ there exists a regular cardinal ν less than κ so that $\mu < \nu$ and $\{\mathcal{F}(\xi) : \xi < \nu\} \subset \nu$. It is well-known that the existence of Mahlo ordinals cannot be proved in theories like ZFC, provided that they are consistent.

To obtain Mahlo for explicit mathematics, we work with universes and let them take over the role of regular cardinals. Then our Mahlo axioms will require that for each name a and for each operation f mapping names to names there exists a universe with name $\mathsf{m}(a, f)$ which contains a and is closed under f.

In the recursive interpretation of explicit mathematics, universes correspond to admissible sets, and the fact that an operation maps names to names is a Π_2 assertion which has to be reflected on admissibles in the shape of universes.

The following shorthand notations are useful for obtaining a compact form of our Mahlo axiom:

$$
\begin{aligned}
f \in (\Re \to \Re) &:= \forall x(\Re(x) \to \Re(fx)), \\
f \in (r \to r) &:= (\forall x \,\dot{\in}\, r)(fx \,\dot{\in}\, r).
\end{aligned}
$$

Obviously, $f \in (\Re \to \Re)$ and $f \in (r \to r)$ mean that f maps names to names and elements of (the type named by) r to elements of (the type named by) r, respectively. Mahloness in explicit mathematics is now expressed by the axioms

$$(\mathsf{Mahlo}.1) \qquad \Re(a) \wedge f \in (\Re \to \Re) \;\to\; \mathcal{U}(\mathsf{m}(a,f)) \wedge a \mathrel{\dot{\in}} \mathsf{m}(a,f),$$

$$(\mathsf{Mahlo}.2) \qquad \Re(a) \wedge f \in (\Re \to \Re) \;\to\; f \in (\mathsf{m}(a,f) \to \mathsf{m}(a,f)).$$

The extension of the system $\mathsf{T_0}$ of explicit mathematics by the Mahlo axioms is now given by

$$\mathsf{T_0(M)} \;:=\; \mathsf{EETJ} + (\mathsf{IG}) + (\mathbb{L}\text{-}\mathsf{I_N}) + (\mathsf{Mahlo}.1) + (\mathsf{Mahlo}.2).$$

If inductive generation is restricted to types, then we call the obtained system $\mathsf{T_0^w(M)}$, and if additionally induction on the natural numbers is restricted to types the system is called $\mathsf{T_0^r(M)}$.

Please observe that the generator ℓ has no function in the theories $\mathsf{T_0(M)}$, $\mathsf{T_0^w(M)}$ and $\mathsf{T_0^r(M)}$. It is included into $\mathbb{L}$ for the formulation of the limit axiom (Lim) considered in the previous sections. If the Mahlo axioms are available, then it can be easily proved. For all natural numbers n we define closed $\mathbb{L}$ terms $\underline{hi}_n$ by recursion on n:

$$\underline{hi}_0 \;:=\; \lambda x.\mathsf{m}(x, \lambda y.y) \qquad \text{and} \qquad \underline{hi}_{n+1} \;:=\; \lambda x.\mathsf{m}(x, \underline{hi}_n).$$

It follows immediately from our first Mahlo axiom that $\underline{hi}_0$ is an operation which maps each name of a type on the name of a universe containing this name. Therefore it acts like the generator ℓ, and the limit axiom (with $\underline{hi}_0$ in place of ℓ) is provable in $\mathsf{T_0^r}$.

The terms $\underline{hi}_1, \underline{hi}_2, \underline{hi}_3, \ldots$ can be used to create universes with stronger closure conditions corresponding to (recursively) inaccessible ordinals, hyperinaccessible ordinals, hyperhyperinaccessible ordinals and the like. By choosing suitable wellorderings, this process can be iterated into the transfinite in a straightforward way.

Instead of pursuing this direction, however, we turn to the model construction for $\mathsf{T_0(M)}$ or, more precisely, to a slight extension of $\mathsf{T_0(M)}$. For its formulation we have to work with partial functions from names to names as well as with partial functions from (names of) types to (names of) types and introduce the following abbreviations:

$$\begin{aligned} f \in (\Re \curvearrowright \Re) &\;:=\; \forall x(\Re(x) \wedge fx{\downarrow} \;\to\; \Re(fx)), \\ f \in (r \curvearrowright r) &\;:=\; (\forall x \mathrel{\dot{\in}} r)(fx{\downarrow} \;\to\; fx \mathrel{\dot{\in}} r). \end{aligned}$$

It is clear that our Mahlo axioms $(\mathsf{Mahlo}.1)$ and $(\mathsf{Mahlo}.2)$ follow from the following axioms $(\mathsf{Mahlo}'.1)$ and $(\mathsf{Mahlo}'.2)$ in which we only require that the operation f is a partial function from names to names:

(Mahlo′.1) $\qquad \Re(a) \wedge f \in (\Re \curvearrowright \Re) \rightarrow \mathcal{U}(\mathsf{m}(a,f)) \wedge a \mathrel{\dot{\in}} \mathsf{m}(a,f),$

(Mahlo′.2) $\qquad \Re(a) \wedge f \in (\Re \curvearrowright \Re) \rightarrow f \in (\mathsf{m}(a,f) \curvearrowright \mathsf{m}(a,f)).$

The three theories $\mathsf{T_0(M')}$, $\mathsf{T_0^w(M')}$ and $\mathsf{T_0^r(M')}$ result from $\mathsf{T_0(M)}$, $\mathsf{T_0^w(M)}$ and $\mathsf{T_0^r(M)}$, respectively, if we replace the Mahlo axioms (Mahlo.1) and (Mahlo.2) by (Mahlo′.1) and (Mahlo′.2).

A universe with the name $\mathsf{m}(a,f)$ does not only reflect the basic type existence axioms but also the (partial or total) operation f from names to names. In order to build our model we must therefore check, before generating a code for the universe named $\mathsf{m}(a,f)$, whether f is an operation from names to names. This test can be carried through by a Π^0_1 formula in the second component of our combined operator form. So we work with a $[\mathsf{POS}, \Pi^0_1]$ operator form and formalize the construction of our model of $\mathsf{T_0(M')}$ in the system $\mathsf{FID}([\mathsf{POS}, \Pi^0_1])$.

As first component of our combined operator form $[\mathsf{POS}, \Pi^0_1]$ we take the operator form $\mathfrak{A}_0(P,a,b,c)$ introduced in Section 4. In addition we let $C(P,a)$ be the formula

$$a = \langle 9, (a)_1, (a)_2 \rangle \wedge \neg P(a,0,0) \wedge P((a)_1,0,0) \wedge$$
$$\forall x \forall y (P(x,0,0) \wedge \{(a)_2\}(x) = y \rightarrow P(y,0,0)).$$

Then $\mathfrak{A}_2(P,a,b,c)$ is the disjunction of the clauses (22) and (23) of the operator form $\mathfrak{A}_1(P,a,b,c)$ from Section 4 and of the following clauses (27)–(29):

(27) $C(P,a) \wedge b = 0 \wedge c = 0,$

(28) $C(P,a) \wedge P(b,0,0) \wedge c = 1,$

(29) $C(P,a) \wedge \neg P(b,0,0) \wedge c = 2.$

Hence $\mathfrak{A}_2(P,a,b,c)$ is (logically equivalent to) a Π^0_1 operator form. The combined operator form $\mathfrak{C}(P,a,b,c)$, with which we will work now, is defined as

$$\mathfrak{A}_0(P,a,b,c) \vee [\forall x \forall y \forall z (\mathfrak{A}_0(P,x,y,z) \rightarrow P(x,y,z)) \wedge \mathfrak{A}_2(P,a,b,c)].$$

$\mathfrak{C}(P,a,b,c)$ trivially belongs to $[\mathsf{POS}, \Pi^0_1]$. It is also easy to see that we can prove in $\mathsf{FID^r}([\mathsf{POS}, \Pi^0_1])$ the analogue of Lemma 5 for this operator form and the corresponding relation constant $P_{\mathfrak{C}}$.

To embed the theory $\mathsf{T_0(M)}$ into $\mathsf{FID}([\mathsf{POS},\Pi^0_1])$ we associate to each $\mathbb{L}$ formula A the formula A^+ from the language of $\mathsf{FID}([\mathsf{POS},\Pi^0_1])$. This translation is as the translation of A into A^* in Section 4 but with each occurrence of $\exists\xi P^\xi_{\mathfrak{A}}(\dots)$ replaced by $\exists\xi P^\xi_{\mathfrak{C}}(\dots)$. Accordingly, we let $\mathsf{Rep}^+(a)$ abbreviate $P_{\mathfrak{C}}(a,0,0)$, and $\mathsf{E}^+(b,a)$ stands for $P_{\mathfrak{C}}(a,b,1)$. The following theorem states that $\mathsf{T_0(M')}$ and thus also $\mathsf{T_0(M)}$ can be modeled in $\mathsf{FID}([\mathsf{POS},\Pi^0_1])$; we also have the obvious embeddings for the subtheories obtained by restricting induction.

Theorem 10 *We have for all $\mathbb{L}$ formulas $A(\vec{U},\vec{a})$ with at most the variables $\vec{U}$ and $\vec{a}$ free:*

1. *If the theory* $\mathsf{T^r_0(M')} + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID^r}([\mathsf{POS},\Pi^0_1]) \vdash \mathsf{Rep}^+(\vec{U}) \;\rightarrow\; A^+(\vec{U},\vec{a}).$$

2. *If the theory* $\mathsf{T^w_0(M')} + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID^w}([\mathsf{POS},\Pi^0_1]) \vdash \mathsf{Rep}^+(\vec{U}) \;\rightarrow\; A^+(\vec{U},\vec{a}).$$

3. *If the theory* $\mathsf{T_0(M')} + (\mathbb{L}\text{-}\mathsf{UG}) + (\mathcal{U}_{no}\text{-}\mathsf{Lin}) + (\mathcal{U}_{no}\text{-}\mathsf{Con})$ *proves* $A(\vec{U},\vec{a})$, *then*

$$\mathsf{FID}([\mathsf{POS},\Pi^0_1]) \vdash \mathsf{Rep}^+(\vec{U}) \;\rightarrow\; A^+(\vec{U},\vec{a}).$$

PROOF We only must verify the two Mahlo axioms; all other parts of the proof are as in the proof of Theorem 6. Hence let a and f be two natural numbers so that

$$\mathsf{Rep}^+(a) \quad \text{and} \quad \forall x \forall y(\mathsf{Rep}^+(x) \land \{f\}(x) = y \;\rightarrow\; \mathsf{Rep}^+(y)).$$

Since $P_{\mathfrak{C}}$ is closed under $\mathfrak{C}$, this yields $P_{\mathfrak{C}}(\langle 9,a,f\rangle,0,0)$. It follows that there exists an ordinal α so that $P^\alpha_{\mathfrak{C}}(\langle 9,a,f\rangle,0,0)$ but not $P^{<\alpha}_{\mathfrak{C}}(\langle 9,a,f\rangle,0,0)$. Hence, $P^{<\alpha}_{\mathfrak{C}}$ is closed under $\mathfrak{A}_0$, and we have

(1) $P^{<\alpha}_{\mathfrak{C}}(a,0,0)$,

(2) $\forall x \forall y(P^{<\alpha}_{\mathfrak{C}}(x,0,0) \land \{f\}(x) = y \;\rightarrow\; P^{<\alpha}_{\mathfrak{C}}(y,0,0))$.

As in the proof of Theorem 6 we can see that $\langle 9, a, f\rangle$ codes a universe and that $\mathsf{E}^+(a, \langle 9, a, f\rangle)$. This means that the conclusion of (Mahlo'.1) is established.

For (Mahlo'.2) we still have to check that $\langle 9, a, f\rangle$ reflects f. So assume that b and c are natural numbers satisfying

$$\mathsf{E}^+(b, \langle 9, a, f\rangle) \quad \text{and} \quad \{f\}(b) = c.$$

Because of $P^{\alpha}_{\mathfrak{C}}(\langle 9, a, f\rangle, 0, 0)$, the clauses (27)–(29) of the definition of $\mathfrak{A}_2$ and the analogue of Lemma 5 we also know

$$\forall x(\mathsf{E}^+(x, \langle 9, a, f\rangle) \leftrightarrow P^{<\alpha}_{\mathfrak{C}}(x, 0, 0)).$$

Together with (2) we obtain $P^{<\alpha}_{\mathfrak{C}}(c, 0, 0))$, i.e. $\mathsf{E}^+(c, \langle 9, a, f\rangle)$. This finishes the proof of our theorem. $\square$

Obviously (M') can be replaced by (M) in this theorem. Therefore Theorem 4 and Theorem 10 imply that the theories $\mathsf{T}^{\mathsf{r}}_0(\mathsf{M})$, $\mathsf{T}^{\mathsf{w}}_0(\mathsf{M})$ and $\mathsf{T}_0(\mathsf{M})$ plus the additional axioms mentioned in Theorem 10 are contained in $\mathsf{KPm}^{\mathsf{r}}$, $\mathsf{KPm}^{\mathsf{w}}$ and KPm, respectively. Work in progress of Tupailo should yield that these results are best possible modulo proof-theoretic strength.

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