The proof-theoretic analysis of Σ^1_1 transfinite dependent choice *

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March 3, 2003

Abstract

This article provides an ordinal analysis of Σ^1_1 transfinite dependent choice.

1 Introduction

Σ^1_1-TDC_0 (Σ^1_1 Transfinite Dependent Choice) is a natural strengthening of Σ^1_1-DC_0. Both are subsystems of analysis and assure the existence of implicitly Σ^1_1 definable sequences (of sets). In Σ^1_1-DC_0, the length of these sequences is ω, whereas in Σ^1_1-TDC_0 we can choose these sequences along an arbitrary well-ordering. Σ^1_1-DC_0 has proof-theoretic strength $\varphi\omega 0$ (cf. [2]), it is a predicative theory. On the other hand, the proof-theoretic strength of Σ^1_1-TDC_0 is $\varphi\omega 00$. If we add complete induction for arbitrary formulas, then the corresponding proof-theoretic ordinals are $\varphi\varepsilon_0 0$ and $\varphi\varepsilon_0 00$.

The theory Σ^1_1-TDC_0 and its proof-theoretic analysis typically belong to the new area of so-called *metapredicative proof-theory*. Metapredicative systems have proof-theoretic ordinals beyond Γ_0 but can still be treated by methods of predicative proof-theory only. Recently, numerous interesting metapredicative systems have been characterized. For previous work in metapredicativity the reader is referred to Jäger [5], Jäger, Kahle, Setzer and Strahm [7], Jäger and Strahm [8, 9], Kahle [10], Rathjen [12], Rüede [13, 14] and Strahm [19, 20, 21]. A central result of [14] is that (Σ^1_1-TDC)

*This paper comprises a part of the author's Ph.D. dissertation [13] is equivalent over ACA_0 to Π^1_2 reflection on ω-models of Σ^1_1-DC. We will use this equivalence for the determination of the upper bound of Σ^1_1-TDC_0. The underlying idea of this proof-theoretic analysis is closely related to the determination of the upper proof-theoretic bound of metapredicative Mahlo (cf. [9]). On the other hand, we carry-through the well-ordering proof directly in the theory Σ^1_1-TDC_0. (The proof of the equivalence of (Σ^1_1-TDC_0) and Π^1_2 reflection on ω-models of Σ^1_1-DC given in [14] uses a pseudohierarchy argument. This argument is needed to prove Π^1_2 reflection on ω-models of Σ^1_1-DC assuming (Σ^1_1-TDC). The other direction is proved without the method of pseudohierarchies.)

The plan of this article is as follows. In the next section we introduce the notation and definitions. The well-ordering proof is given in section 3. In sections 4, 5 and 6 we discuss semi-formal systems needed for the determination of the upper bound of Π^1_2 reflection on ω-models of Σ^1_1-DC. In some sense, these semi-formal systems can be seen as analogues of systems for n-(hyper)inaccessibles (cf. [9]). The interpretation of Π^1_2 reflection on ω-models of Σ^1_1-DC into these semi-formal systems is given in section 7.

2 Preliminaries

In this section we fix notation and abbreviations and introduce some subsystems of analysis, in particular Σ^1_1-TDC and $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$.

We let $\mathcal{L}_2$ denote the language of second order arithmetic. $\mathcal{L}_2$ includes *number variables* (denoted by small letters, except r, s, t), *set variables* (denoted by capital letters), symbols for all primitive recursive functions and relations, the symbol $\in$ for element-hood between numbers and sets as well as equality in the first sort. We write $\mathcal{L}_1$ for the first order part of $\mathcal{L}_2$. The *number terms* r, s, t of $\mathcal{L}_2$ and the *formulas* $\varphi, \psi, \theta, \ldots$ of $\mathcal{L}_2$ are defined as usual.

An $\mathcal{L}_2$ formula is called *arithmetic*, if it does not contain bound set variables (but possibly free set variables). We write Π^1_0 for the collection of these formulas and Σ^1_1 for the collection of all arithmetic formulas and of all $\mathcal{L}_2$ formulas $\exists X \varphi(X)$ with $\varphi(X)$ from Π^1_0. Σ^1_k and Π^1_k are defined similarly.

In the following $\langle \ldots \rangle$ denotes a primitive recursive coding function for n-tuples $\langle t_1, \ldots, t_n \rangle$ with associated projections $(\cdot)_1, \ldots, (\cdot)_n$. Seq_n is the primitive recursive set of sequence numbers of length n. Seq denotes the primitive recursive set of sequence numbers. We write $s \in (X)_t$ for $\langle s, t \rangle \in X$, i.e. $(X)_t = \{x : \langle x, t\rangle \in X\}$, and $\vec{X}$ for $X_1, ..., X_n$. Occasionally we use the following abbreviations.

$$\begin{array}{rcl} x \in X \oplus Y & := & \mathsf{Seq}_2 x \wedge \\ & & [((x)_0 \in X \wedge (x)_1 = 1) \vee ((x)_0 \in Y \wedge (x)_1 = 2)], \\ X = Y & := & (\forall x)(x \in X \leftrightarrow x \in Y), \\ X \neq Y & := & \neg(X = Y), \\ X \mathrel{\dot{\in}} Y & := & (\exists k)(\forall x)(x \in X \leftrightarrow \langle x, k\rangle \in Y), \\ (\exists Y \mathrel{\dot{\in}} Z)\varphi(Y) & := & (\exists k)\varphi((Y)_k), \\ (\forall Y \mathrel{\dot{\in}} Z)\varphi(Y) & := & (\forall k)\varphi((Y_k), \\ \vec{X} \mathrel{\dot{\in}} Y & := & X_1 \mathrel{\dot{\in}} Y \wedge \ldots \wedge X_n \mathrel{\dot{\in}} Y. \end{array}$$

By $\varphi[\vec{x}, \vec{X}]$ we indicate that the variables $\vec{x}$, $\vec{X}$ really occur in φ, i.e., the free variables are $\{\vec{x}, \vec{X}\}$. $\varphi(\vec{x}, \vec{X})$ just means that $\vec{x}$, $\vec{X}$ may occur in φ. $\varphi[\vec{x}\backslash\vec{t}, \vec{X}\backslash\vec{S}]$ is obtained from $\varphi[\vec{x}, \vec{X}]$ by replacing all occurrences of x_i and X_j by t_i and S_j. Similarly we define $\varphi(\vec{x}\backslash\vec{t}, \vec{X}\backslash\vec{S})$. We adopt the standard notation φ^X for the relativization of the formula φ to X (for example $(\forall Y \varphi(Y))^X := (\forall Y \mathrel{\dot{\in}} X)\varphi^X(Y))$.

In a next step we introduce some well-known subsystems of analysis which we shall need. We use the following abbreviations.

$$\begin{array}{rcl} \mathsf{WO}(X) & := & \text{formalization of “X codes a non-reflexive well-ordering”}, \\ x \in \mathsf{field}(X) & := & (\exists y)(\langle x, y\rangle \in X \vee \langle y, x\rangle \in X), \\ x \in (Y)_{Za} & := & \mathsf{Seq}_2 x \wedge x \in Y \wedge \langle (x)_1, a\rangle \in Z. \end{array}$$

$(Y)_{Za}$ is the disjoint union of all projections $(Y)_b$ with b less than a w.r.t. Z. For a well-ordering Z we let 0_Z denote the Z-least element in $\mathsf{field}(Z)$ and for $a \in \mathsf{field}(Z)$ we let $a +_Z 1$ denote the Z-successor of a. Sometimes we write $\prec$ for our well-ordering. Then e.g. $(X)_{Za}$ is written as $(X)_{\prec a}$.

All subsystems are based on the usual axioms and rules for the two-sorted predicate calculus. Often we write (T) for the central axiom of a theory T.

The theory ACA includes defining axioms for all primitive recursive functions and relations, the induction scheme for arbitrary formulas of $\mathcal{L}_2$ and the scheme

(ACA) For all arithmetic $\mathcal{L}_2$ formulas $\varphi(x)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x))$.

The theory Σ^1_1-AC extends ACA by the scheme

(Σ^1_1-AC) For all $\mathcal{L}_2$ formulas $\varphi(x, X)$ in Σ^1_1:
$(\forall x)(\exists X)\varphi(x, X) \rightarrow (\exists X)(\forall x)\varphi(x, (X)_x)$.

The theory ATR extends ACA by the scheme

(ATR) For all arithmetic $\mathcal{L}_2$ formulas $\varphi(x, X)$:
$\mathsf{WO}(Z) \rightarrow (\exists Y)(\forall a \in \mathsf{field}(Z))(\forall x)(x \in (Y)_a \leftrightarrow \varphi(x, (Y)_{Za}))$.

The theory Σ^1_1-DC extends ACA by the scheme

(Σ^1_1-DC) For all $\mathcal{L}_2$ formulas $\varphi(X, Y)$ in Σ^1_1:
$(\forall X)(\exists Y)\varphi(X, Y) \rightarrow (\exists Z)[(Z)_0 = X \wedge (\forall u)\varphi((Z)_u, (Z)_{u+1})]$.

$\mathsf{Ax}_{\mathsf{ACA}}$ denotes a finite axiomatization of the arithmetical comprehension scheme (ACA) (cf. [18] Lemma VIII.1.5 for such a finite axiomatization). Using these notations, we formulate the theory Π^1_{n+1}-RFN. It extends ACA by the scheme

(Π^1_{n+1}-RFN) For all $\mathcal{L}_2$ formulas $\varphi[\vec{x}, \vec{Z}]$ in Π^1_{n+1}:
$\varphi[\vec{x}, \vec{Z}] \rightarrow (\exists X)(\vec{Z} \dot{\in} X \wedge (\mathsf{Ax}_{\mathsf{ACA}})^X \wedge \varphi^X[\vec{x}, \vec{Z}])$.

Next we introduce for each natural number n predicates I_n.

$$\begin{aligned} \mathsf{I}_0(M) &:= (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^M \\ \mathsf{I}_{n+1}(M) &:= (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge (\forall X \dot{\in} M)(\exists Y \dot{\in} M)(X \dot{\in} Y \wedge \mathsf{I}_n(Y)). \end{aligned}$$

We have written $\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}}$ for a finite axiomatization of (Σ^1_1-DC) + (ACA) and $\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}}$ for a finite axiomatization of (Σ^1_1-AC) + (ACA). We refer to Lemma VIII.1.5 [18] for a finite axiomatization of (ACA). And using the fact that (Σ^1_1-DC) and (Π^0_1-DC) ((Σ^1_1-AC) and (Π^0_1-AC), resp.) are equivalent over ACA_0, the finite axiomatization of the mentioned axiom schemes can easily be found (cf. e.g. [13]). Using these predicates I_n we can define the theories I_n-RFN. I_n-RFN extends ACA by the axiom

(I_n-RFN) $(\forall X)(\exists Y)(X \dot{\in} Y \wedge \mathsf{I}_n(Y))$.

Finally we introduce the basic subsystems of analysis of this paper: Σ^1_1 transfinite dependent choice and Π^1_2 reflection on ω-models of Σ^1_1-DC. The theory Σ^1_1-TDC is the theory ACA extended by the scheme of Σ^1_1 Transfinite Dependent Choice.

(Σ^1_1-TDC) For all Σ^1_1 formulas φ:

$$(\forall X)(\exists Y)\varphi(X,Y) \land \mathsf{WO}(Z) \\ \to (\exists Y)(\forall a \in \mathsf{field}(Z))\varphi((Y)_{Za},(Y)_a).$$

The theory $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ extends ACA by the scheme

($(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$) For all Π^1_2 formulas $\varphi[\vec{z},\vec{Z}]$:

$$\varphi[\vec{z},\vec{Z}] \to (\exists M)[\vec{Z} \,\dot{\in}\, M \land (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^M \land \varphi^M].$$

T_0 denotes the theory T with set-induction instead of the induction scheme for arbitrary formulas.

In the following we will measure the proof-theoretic strength of formal theories in terms of their proof-theoretic ordinals. As usual we set for all primitive recursive relations $\prec$ and all formulas φ:

$$\begin{aligned} \mathsf{Prog}(\varphi,\prec) &:= (\forall x)[(\forall y)(y \prec x \to \varphi(y)) \to \varphi(x)], \\ \mathsf{TI}(\varphi,\prec) &:= \mathsf{Prog}(\varphi,\prec) \to (\forall x \in \mathsf{field}(\prec))\varphi(x). \end{aligned}$$

We say that an ordinal α is provable in T, if there is a primitive recursive well-ordering $\prec$ of order type α so that $\mathsf{T} \vdash (\forall X)\mathsf{TI}(X,\prec)$. The proof-theoretic ordinal of T, denoted by $|\mathsf{T}|$, is the least ordinal which is not provable in T.

3 A well-ordering proof for Σ^1_1-TDC_0

In this section we show that Σ^1_1-TDC_0 (Σ^1_1-TDC) proves transfinite induction for each initial segment of the ordinal $\varphi\omega 00$ ($\varphi\varepsilon_0 00$). The proof and the presentation is inspired by [19, 21]. The ordinal notation system which we use here is based on n-ary φ functions (cf. e.g. [7]). These φ functions correspond to Schütte‘s Klammersymbole [15].

We have mentioned that we do not use Π^1_2 reflection on ω-models in the well-ordering proof of Σ^1_1-TDC_0. Nevertheless, in [13] we have given a well-ordering proof of Σ^1_1-TDC_0 using Π^1_2 reflection on ω-models. That well-ordering proof is nearly the same as for KPm_0 (cf. [21]).

In the sequel we presuppose the same ordinal-theoretic facts as given in section 2 of [7]. Namely, we let Φ_0 denote the least ordinal greater than 0 which is closed under all n-ary φ functions, and we assume that a standard notation system of order type Φ_0 is given in a straightforward manner. We write $\prec$ for the corresponding primitive recursive wellordering. We assume without loss of generality that the field of $\prec$ is the set of all natural numbers and that 0 is the least element with respect to $\prec$. Hence, each natural number codes an ordinal less than Φ_0. When working in Σ^1_1-$\mathsf{TDC_0}$ in this section, we let $a, b, c, \ldots$ range over the field of $\prec$, and ℓ denotes limit notations. There exist primitive recursive functions acting on the codes of this notation system which corresponds to the usual operations on ordinals. In the sequel it is often convenient in order to simplify notation to use ordinals and ordinal operations instead of their codes and primitive recursive analogues. Then (for example) ω and $\omega + \omega$ stand for the natural numbers whose order type with respect to $\prec$ are ω and $\omega + \omega$. Finally, let us put as usual

$$\begin{aligned} \mathsf{Prog}(\varphi) &:= \mathsf{Prog}(\varphi, \prec), \\ \mathsf{TI}(\varphi, a) &:= \mathsf{TI}(\varphi, \prec\restriction a). \end{aligned}$$

If we want to stress the relevant induction variable of a formula φ, we sometimes write $\mathsf{Prog}(\lambda a.\varphi(a))$ instead of $\mathsf{Prog}(\varphi)$. If S is a set term, then $\mathsf{Prog}(S)$ and $\mathsf{TI}(S, a)$ have their obvious meanings.

We assign fundamental sequences $(\ell[n])_{n \geq 0}$ to each limit ordinal ℓ. We can assume $\ell[u] \prec \ell[u+1]$ and $0 \prec \ell[u]$ for all u. We choose $\ell^-[u]$ to denote the unique ordinal such that $\ell[u] + \ell^-[u] = \ell[u+1]$. Moreover, we use for each natural number $n > 0$ the following abbreviations.

$$\begin{aligned} \mathsf{K}_1(M) &:= (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^M, \\ \mathsf{K}_{n+1}(M) &:= (\mathsf{Ax}_{\mathsf{ACA}})^M \wedge \\ &\quad\ [(\forall Q)(\exists Y)(\forall a)(WO(\prec\restriction a) \to \mathsf{Hier}_n(a, Q, (Y)_a))]^M, \\ \mathsf{Hier}_n(a, Q, Y) &:= (\forall c \prec a)((Y)_{\prec c} \mathrel{\dot\in} (Y)_c \wedge Q \mathrel{\dot\in} (Y)_c \wedge \mathsf{K}_n((Y)_c)), \\ \mathsf{TI}^{\prec c}(Y, a) &:= (\forall b \prec c)(\forall X \mathrel{\dot\in} (Y)_b)\mathsf{TI}(X, a), \\ a \uparrow b &:= (\exists c, \ell)(b = c + a \cdot \ell), \qquad (\ell \text{ denotes limit notations}) \end{aligned}$$

$$\mathsf{Main}^n_{c,Y}(a) \quad := \quad (\forall b)(\forall e \preceq c)[\omega^{1+a} \uparrow e \wedge \mathsf{TI}^{\prec e}(Y, b) \to \mathsf{TI}^{\prec e}(Y, \varphi nab)].$$

Next we specify the steps of the well-ordering proof in the following lemmas. We first collect some basic facts which we will often use tacitly in the following.

Lemma 1 *The following holds.*

a) *(*ATR*) and* $(\forall X)(\exists Y)(X \dot\in Y \wedge (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$ *are equivalent over* ACA_0*.*

b) *We have for each natural number* $n > 1$ *and for each instance* φ *of* (ATR)

$$\mathsf{ACA}_0 \vdash \mathsf{K}_n(M) \to (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^M \wedge \varphi^M.$$

c) *We have for each natural number* n

$$\mathsf{ACA}_0 \vdash \mathsf{Hier}_n(a, Q, Y) \to (\forall b \prec a)(\exists Z \dot\in (Y)_b)(\forall x)(x \in Z \leftrightarrow \mathsf{TI}^{\prec b}(Y, x)).$$

Proof. We first prove a). It can be proved in ACA_0 by induction on the wellordering Z that for each arithmetic formula ψ the following holds

$$\begin{aligned}&(\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^D \wedge Z, \vec{X} \dot\in D \wedge \mathsf{WO}(Z)\\ &\quad \to (\exists Y \dot\in D)(\forall a \in \mathsf{field}(Z))(\forall x)(x \in (Y)_a \leftrightarrow \psi[x, a, \vec{z}, (Y)_{Za}, \vec{X}]).\end{aligned}$$

Hence $(\forall X)(\exists Y)(X \dot\in Y \wedge (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$ implies (ATR). The converse direction follows from Theorem VIII.3.15 [18] and Lemma VIII.4.19 [18]. Next we prove b). Since (ATR) is equivalent over ACA_0 to

$$(\forall X)(\exists Y)(X \dot\in Y \wedge (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^Y)$$

(cf. a)) and since ATR_0 proves (Σ^1_1-AC) (cf. [18], Theorem V.8.3), we have to prove only

$$\mathsf{ACA}_0 \vdash \mathsf{K}_n(M) \to (\forall X \dot\in M)(\exists Y \dot\in M)(X \dot\in Y \wedge (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^Y).$$

This can be proved by an easy (meta-)induction on $n > 1$. We omit the details. We now discuss c). We argue in ACA_0 and assume that $\mathsf{Hier}_n(a, Q, Y)$ holds. Furthermore, we choose a $b \prec a$. Note that we have

$$(\forall x)(\mathsf{TI}^{\prec c}(Y, x) \leftrightarrow (\forall c \prec b)(\forall X \dot\in ((Y)_{\prec b})_c)\mathsf{TI}(X, x)).$$

Thus $\mathsf{TI}^{\prec b}(Y,x)$ is arithmetic in $(Y)_{\prec b}$. Hence assertion b) – in the case $n=1$ we know by definition that $(\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_b}$ holds – and $(Y)_{\prec b} \mathrel{\dot\in} (Y)_b$ imply the claim. □

In the next lemma we prove in Σ^1_1-TDC_0 the existence of sets M with $\mathsf{K}_n(M)$.

Lemma 2 *We have for each natural number $n > 0$*

$$\Sigma^1_1\text{-}\mathsf{TDC}_0 \vdash (\exists M)(P \mathrel{\dot\in} M \land \mathsf{K}_n(M))$$

Proof. The proof is by (meta-)induction on $n > 0$. For $n = 1$ the claim follows from the fact that Σ^1_1-TDC_0 proves (ATR) and that ATR_0 proves the existence of ω-models of Σ^1_1-AC, as mentioned in the preceding lemma. Hence we can assume $n > 1$. The induction hypothesis is

$$(\forall P)(\exists M)(P \mathrel{\dot\in} M \land \mathsf{K}_{n-1}(M)).$$

We apply (Σ^1_1-TDC_0) and obtain

$$(\forall a)(\exists Y)(WO(\prec\restriction a) \to \mathsf{Hier}_{n-1}(a,Q,Y)).$$

An application of (Σ^1_1-AC) leads to

$$(\forall Q)(\exists Y)(\forall a)(WO(\prec\restriction a) \to \mathsf{Hier}_{n-1}(a,Q,(Y)_a)). \tag{1}$$

Using (Σ^1_1-AC) we can show that this formula is equivalent to a Π^1_2 formula ψ (cf. e.g. [18], Lemma VIII.6.2). Note that for the proof of this equivalence we need (Σ^1_1-AC) only for the implication (1)$\to \psi$. The other direction needs only (ACA). Here, we need the "strong" direction; i.e. we need in fact (Σ^1_1-AC). In the argument below we need the "weak" direction, i.e. only (ACA).

In [18], Theorem VIII.5.12, the equivalence of (Σ^1_1-DC) and (Π^1_2-RFN) over ACA_0 is proved. Since (Σ^1_1-TDC) implies (Σ^1_1-DC), we can use (Π^1_2-RFN) and obtain a set M such that

$$P \mathrel{\dot\in} M \land (\mathsf{Ax}_{\mathsf{ACA}})^M \land$$
$$((\forall Q)(\exists Y)(\forall a)(\mathsf{WO}(\prec\restriction a) \to \mathsf{Hier}_{n-1}(a,Q,(Y)_a)))^M$$

holds. This is just the claim. □

Our well-ordering proof is in some sense an iteration of the well-ordering proof for $\widehat{ID}_\alpha$. Roughly spoken, the next lemma corresponds to the beginning of the iteration. The statements are adaptions of Lemma 5, 6 and 7 in [7] to our situation. (Lemma 3 and some further technical lemmas in this article are adaptions of corresponding lemmas proven in the literature. We agree with the referee that there should be abstract lemmas from which the arguments in question follows. But this will be done in a different article.)

Lemma 3 *The following holds*

a) $\mathsf{ACA}_0 \vdash \mathsf{Hier}_1(\ell, Q, Y) \land \mathsf{TI}^{\prec \ell}(Y, a) \to \mathsf{TI}^{\prec \ell}(Y, \varphi a0)$.

b) $\mathsf{ACA}_0 \vdash \mathsf{Hier}_1(\ell, Q, Y) \to \mathsf{Prog}(\lambda a.\mathsf{TI}^{\prec \ell}(Y, \varphi 10a))$.

c) $\mathsf{ACA}_0 \vdash \mathsf{Hier}_1(c, Q, Y) \to \mathsf{Prog}(\lambda a.\mathsf{Main}^1_{c,Y}(a))$.

Proof. The proof of a) is standard. The relevant arguments can easily be extracted from [16], pp. 184 ff., or [3], Lemma 5.3.1 ff.. b) is an immediate consequence of a). Assertion c) corresponds to Main Lemma I in [7]. Since the proof of c) is very much the same as the proof given in [7] – we have to change only the underlying theories –, we omit it here. □

The induction step is given in the next three lemmas.

Lemma 4 ACA_0 *proves for each natural number* $n > 0$

$$\mathsf{K}_{n+1}(M) \land [(\forall Q, Y, c)(\mathsf{Hier}_n(c, Q, Y) \to \mathsf{Prog}(\lambda a.\mathsf{Main}^n_{c,Y}(a)))]^M \\ \to (\forall a)[(\forall X \mathrel{\dot\in} M)\mathsf{TI}(X, a) \to (\forall X \mathrel{\dot\in} M)\mathsf{TI}(X, \varphi na0)].$$

Proof. We argue in ACA_0 and assume

$$\mathsf{K}_{n+1}(M) \land [(\forall Q, Y, c)(\mathsf{Hier}_n(c, Q, Y) \to \mathsf{Prog}(\lambda a.\mathsf{Main}^n_{c,Y}(a)))]^M \tag{2}$$

Choose a such that $(\forall X \mathrel{\dot\in} M)\mathsf{TI}(X, a)$ holds and let X be a set in M. We have to show $\mathsf{TI}(X, \varphi na0)$. Since M is a ω-model of (ACA), we have $(\forall X \mathrel{\dot\in} M)\mathsf{TI}(X, \omega^{1+a} \cdot \omega)$, too. The definition of $\mathsf{K}_{n+1}(M)$ now implies the existence of a set P in M such that $\mathsf{Hier}_n(\omega^{1+a} \cdot \omega, X, P)$ holds. Using (2), we conclude that

$$\mathsf{Prog}(\lambda b.\mathsf{Main}^n_{\omega^{1+a}\cdot\omega,P}(b)) \tag{3}$$

holds. Since P is in M, the set $\{b : \mathsf{Main}^n_{\omega^{1+a}\cdot\omega,P}(b)\}$ is in M too. Hence $(\forall X \mathrel{\dot\in} M)\mathsf{TI}(X, a)$ and (3) imply $\mathsf{Main}^n_{\omega^{1+a}\cdot\omega,P}(a)$. It follows $\mathsf{TI}(X, \varphi na0)$, the claim. $\square$

The following lemma follows by

$$\begin{aligned}\varphi(n+1)00 &= \sup\{(\lambda x.\varphi n x 0)^m(0) \mid m \in \mathbb{N}\},\\ \varphi(n+1)0(a+1) &= \sup\{(\lambda x.\varphi n x 0)^m(\varphi(n+1)0a+1) \mid m \in \mathbb{N}\},\\ \varphi(n+1)0\ell &= \sup\{\varphi(n+1)0x \mid x \prec \ell\}.\end{aligned}$$

Lemma 5 ACA_0 *proves for each natural number* $n > 0$

$$\mathsf{K}_{n+1}(M) \wedge (\forall a)((\forall X \dot{\in} M)\mathsf{TI}(X,a) \to (\forall X \dot{\in} M)\mathsf{TI}(X,\varphi n a 0)) \\ \to \mathsf{Prog}(\lambda a.(\forall X \dot{\in} M)\mathsf{TI}(X,\varphi(n+1)0a)).$$

Lemma 6 ACA_0 *proves for each natural number* $n > 0$

$$\mathsf{Hier}_n(c,Q,Y) \wedge (\forall M)(\mathsf{K}_n(M) \to \mathsf{Prog}(\lambda a.(\forall X \dot{\in} M)\mathsf{TI}(X,\varphi n 0 a))) \\ \to \mathsf{Prog}(\lambda a.\mathsf{Main}^n_{c,Y}(a)).$$

Proof. We argue in ACA_0 and assume

$$\mathsf{Hier}_n(c,Q,Y) \wedge (\forall M)(\mathsf{K}_n(M) \to \mathsf{Prog}(\lambda a.(\forall X \dot{\in} M)\mathsf{TI}(X,\varphi n 0 a))) \quad (4)$$

We break the proof of $\mathsf{Prog}(\lambda a.\mathsf{Main}^n_{c,Y}(a))$ into three cases by showing

(a) $\mathsf{Main}^n_{c,Y}(0)$,

(b) $\mathsf{Main}^n_{c,Y}(a) \to \mathsf{Main}^n_{c,Y}(a+1)$,

(c) $\mathsf{Lim}(a) \wedge (\forall w)\mathsf{Main}^n_{c,Y}(a[w]) \to \mathsf{Main}^n_{c,Y}(a)$.

The proof of (b) and (c) corresponds to the proof of (b) and (c) in the proof of Main Lemma I [7]. There is only one relevant difference: Main Lemma I deals with fundamental sequences for e.g. $\varphi 1pq$ and not with fundamental sequences for e.g. φnpq. However, there is no difficulty to give corresponding fundamental sequences for φnpq. Hence we prove here only (a). Let us assume

$$e \preceq c \wedge \omega \uparrow e \wedge \mathsf{TI}^{\prec e}(Y,b).$$

We have to prove $\mathsf{TI}^{\prec e}(Y,\varphi n 0 b)$. There is a limit notation ℓ such that $e = e_0 + \omega \cdot \ell$ for an e_0. We set $e_u := e_0 + \omega \cdot \ell[u]$. It is sufficient to verify $\mathsf{TI}^{\prec e_u}(Y,\varphi n 0 b)$ for each u. We fix a u and a $d \prec e_u$. Then we have to prove

$$(\forall X \dot{\in} (Y)_d)\mathsf{TI}(X,\varphi n 0 b).$$

Since $\mathsf{K}_n((Y)_d)$ holds, it follows from (4)

$$\mathsf{Prog}(\lambda a.(\forall X \dot{\in} (Y)_d)\mathsf{TI}(X, \varphi n0a)).$$

Hence $\mathsf{TI}^{\prec e}(Y, b)$ implies $(\forall X \dot{\in} (Y)_d)\mathsf{TI}(X, \varphi n0b)$. □

The iteration of the preceding lemmas leads to the following lemma.

Lemma 7 ACA_0 *proves for each natural number* $n > 0$

a) $(\forall M)[(\mathsf{Ax}_{\mathsf{ACA}})^M \rightarrow$
$[(\forall Q, Y, c)(\mathsf{Hier}_n(c, Q, Y) \rightarrow \mathsf{Prog}(\lambda a.\mathsf{Main}^n_{c,Y}(a)))]^M]$,

b) $(\forall M)(\mathsf{K}_{n+1}(M) \rightarrow$
$(\forall a)((\forall X \dot{\in} M)\mathsf{TI}(X, a) \rightarrow (\forall X \dot{\in} M)\mathsf{TI}(X, \varphi na0)))$,

c) $(\forall M)(\mathsf{K}_{n+1}(M) \rightarrow \mathsf{Prog}(\lambda a.(\forall X \dot{\in} M)\mathsf{TI}(X, \varphi(n+1)0a)))$.

Proof. We first prove that a) implies b) and c). Since by Lemma 5 assertion b) implies c), we have to prove only a)⇒b). We argue in Σ^1_1-TDC_0 and assume a) and $\mathsf{K}_{n+1}(M)$. Furthermore, we choose an ordinal notation a such that we have $(\forall X \dot{\in} M)\mathsf{TI}(X, a)$. Let X be a set in M. We have to prove $\mathsf{TI}(X, \varphi na0)$. By $(\mathsf{Ax}_{\mathsf{ACA}})^M$ we conclude that $(\forall X \dot{\in} M)\mathsf{TI}(X, \omega^{1+a} \cdot \omega)$ holds. Using $\mathsf{K}_{n+1}(M)$ we obtain a set P in M such that $\mathsf{Hier}_n(\omega^{1+a} \cdot \omega, X, P)$ holds. Now a) implies

$$\mathsf{Prog}(\lambda d.\mathsf{Main}^n_{\omega^{1+a}\cdot\omega,P}(d)).$$

Since P is in M we obtain by $(\forall X \dot{\in} M)\mathsf{TI}(X, a)$

$$\mathsf{Main}^n_{\omega^{1+a}\cdot\omega,P}(a).$$

This implies $\mathsf{TI}(X, \varphi na0)$. Hence a)⇒b) is shown. Now we prove a) by (meta-)induction on n. For $n = 1$ this is just Lemma 3c). For $n > 1$ the claim follows from the induction hypothesis and Lemma 6. □

The following theorem follows immediately from Lemma 2 and Lemma 7b).

Theorem 8 Σ^1_1-TDC_0 *proves for each natural number* n

$$(\forall X)\mathsf{TI}(X, \varphi n00).$$

Corollary 9 $\varphi\omega00 \leq |\Sigma^1_1\text{-}\mathsf{TDC}_0|$.

We end this section with a discussion of the lower bound of Σ^1_1-TDC. In Σ^1_1-TDC we have induction for arbitrary formulas. The lower bound computation given in this section for Σ^1_1-TDC_0 can be extended in order to yield $\varphi\varepsilon_0 00$ as a proof-theoretic lower bound of Σ^1_1-TDC. The aim is to introduce within Σ^1_1-TDC for all ordinals $\alpha < \varepsilon_0$ the notion of a set X with "$\mathsf{K}_\alpha(X)$" and to show the existence of such sets. The formulas K_n (n a natural number) are arithmetic. The formulas K_α (α an ordinal) will be Σ^1_1. Hence we will need formula induction in order to prove that this Σ^1_1 formula serves the right role and that sets X with "$\mathsf{K}_\alpha(X)$" exists.

The main modification is that we do not speak about all sets X with $\mathsf{K}_n(X)$ but that we speak only about all sets X in Y with $\mathsf{K}_n(X)$. For each set Y we will define a characteristic function F with

$$k \in (F)_\alpha \leftrightarrow \text{“}\mathsf{K}_\alpha((Y)_k)\text{”}.$$

These functions F can be constructed inductively by using formula induction. We give first an informal description where K_α and Hier_α should be understood informally too. The formula φ_K means: F is the desired characteristic function on the sets in Y.

$$\begin{array}{llcl}
\multicolumn{4}{l}{\varphi_\mathsf{K}(F,Y,\alpha) :=} \\
\multicolumn{4}{l}{\text{for all } b \preceq \alpha\text{:}} \\
\text{if } b=0\text{:} & x \in (F)_0 & \leftrightarrow & (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_x} \\
\text{if } Suc(b)\text{:} & x \in (F)_b & \leftrightarrow & (\mathsf{Ax}_{\mathsf{ACA}})^{(Y)_x} \wedge \\
 & & & [(\forall Q)(\exists P)(\forall a)(WO(\prec\restriction a) \\
 & & & \quad \to \mathsf{Hier}_{b-1}(a,Q,(P)_a))]^{(Y)_x} \\
\text{if } Lim(b)\text{:} & x \in (F)_b & \leftrightarrow & (\mathsf{Ax}_{\mathsf{ACA}})^{(Y)_x} \wedge (\forall c \prec b)\mathsf{K}_c((Y)_x)
\end{array}$$

The exact definition of φ_K is: ($\alpha \in \Phi_0$)

$$\begin{array}{llcl}
\multicolumn{4}{l}{\varphi_\mathsf{K}(F,Y,\alpha) :=} \\
\multicolumn{4}{l}{(\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^Y \wedge} \\
\multicolumn{4}{l}{(\forall b \preceq \alpha)(\forall x)[} \\
(b=0 \to & (x \in (F)_0 & \leftrightarrow & (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^{(Y)_x}))\wedge \\
(Suc(b) \to & (x \in (F)_b & \leftrightarrow & ((\mathsf{Ax}_{\mathsf{ACA}})^{(Y)_x} \wedge \\
 & & & [(\forall Q)(\exists P)(\forall a)(WO(\prec\restriction a) \\
 & & & \quad \to \mathsf{Hier}_{b-1}(a,Q,(P)_a))]^{(Y)_x}))\wedge \\
(Lim(b) \to & (x \in (F)_b & \leftrightarrow & ((\mathsf{Ax}_{\mathsf{ACA}})^{(Y)_x} \wedge (\forall c \prec b)(x \in (F)_c))))]
\end{array}$$

where $\mathsf{Hier}_{b-1}(a, Q, (P)_a)$ is the following formula:

$$(\forall c \prec a)[((P)_a)_{\prec c} \mathrel{\dot\in} ((P)_a)_c \wedge Q \mathrel{\dot\in} ((P)_a)_c \\ \wedge(\exists j \in (F)_{b-1})(((P)_a)_c = (Y)_j)].$$

Using the formula φ_{K} we can define "K_α":

$$\mathsf{K}(\alpha, P) := (\exists F, Y)(\varphi_{\mathsf{K}}(F, Y, \alpha) \wedge (\exists x \in (F)_\alpha)(Y)_x = P).$$

Following the lines in [13], section 4.4, we can prove for each ordinal α less than ε_0

$$\Sigma^1_1\text{-}\mathsf{TDC} \vdash (\forall b \preceq \alpha)(\forall Q)(\exists P)(Q \mathrel{\dot\in} P \wedge \mathsf{K}(b, P))$$

and

$$\Sigma^1_1\text{-}\mathsf{TDC} \vdash (\forall X)\mathsf{TI}(\alpha, X).$$

Hence

$$\varphi\varepsilon_0 00 \leq |\Sigma^1_1\text{-}\mathsf{TDC}|. \tag{5}$$

4 The semi-formal systems T^n_α and E^n_α

Our next goal is to establish the upper bound of $\Sigma^1_1\text{-}\mathsf{TDC}_0$. Since $(\Sigma^1_1\text{-}\mathsf{TDC})$ and $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ are equivalent over ACA_0 (cf. [14]) it is sufficient to determine the upper bound of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$. And since we will reduce $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to $\bigcup_{n\in\mathbb{N}} \mathsf{I}_n\text{-}\mathsf{RFN}_0$, it will be sufficient to determine the upper bound of $\mathsf{I}_n\text{-}\mathsf{RFN}_0$. In this section we introduce for each $n \in \mathbb{N}$ and each ordinal $\alpha \in \Phi_0$ semi-formal systems T^n_α and E^n_α which we will need for the determination of the upper bound of $\mathsf{I}_n\text{-}\mathsf{RFN}_0$. In T^n_α we have constants D^n_β for each $\beta < \alpha$ such that $\mathsf{I}_n(\mathsf{D}^n_\beta)$ and $\mathsf{D}^n_\gamma \mathrel{\dot\in} \mathsf{D}^n_\beta$ holds for $\gamma < \beta$. Hence there is a hierarchy $\mathsf{D}^n_{<\alpha}$ up to α such that $\mathsf{I}_n((\mathsf{D}^n_{<\alpha})_\beta)$ holds for $\beta < \alpha$. E^n_α is a first order reformulation of T^n_α. The introduction of E^n_α is for technical reasons.

We now turn to the exact definition of the semi-formal systems T^n_α. T^n_α is based on the language $\mathcal{L}^n_\alpha$. $\mathcal{L}^n_\alpha$ is the extension of $\mathcal{L}_2$ by new unary relation symbols D^n_β for each $\beta < \alpha$ and new unary relation symbols $\mathsf{D}^n_{<\gamma}$ for each $\gamma \leq \alpha$. The *set terms* of $\mathcal{L}^n_\alpha$ are the set variables. The $\mathcal{L}^n_\alpha$ *formulas* are the $\mathcal{L}_2$ literals and all formulas $[\neg]\mathsf{D}^n_\beta(t)$, $[\neg]\mathsf{D}^n_{<\gamma}(t)$ for each set variable X, all number terms t and all ordinals $\beta < \alpha$, $\gamma \leq \alpha$. Furthermore, the class of $\mathcal{L}^n_\alpha$ formulas is closed under $\wedge, \vee, \forall x, \exists x, \exists X \mathrel{\dot\in} \mathsf{D}^n_\beta, \forall X \mathrel{\dot\in} \mathsf{D}^n_\beta, \exists X, \forall X$ for each $\beta < \alpha$. Note that T^n_α is formulated with bounded second order quantifiers

$\exists X \dot{\in} \mathsf{D}^n_\beta$ and $\forall X \dot{\in} \mathsf{D}^n_\beta$ for $\beta < \alpha$. The exact meaning of the bounded second order quantifiers will be given in the definition of T^n_α.

In the following we write for instance $t \in \mathsf{D}^n_\beta$ for $\mathsf{D}^n_\beta(t)$, $t \in \mathsf{D}^n_{<\beta}$ for $\mathsf{D}^n_{<\beta}(t)$ etc. Analogously we use $\mathsf{D}^n_\beta \dot{\in} X$, $X \dot{\in} \mathsf{D}^n_\beta$, ... Finally, we fix for each $n, m \in \mathbb{N}$ a universal Π^0_1 predicate $\pi^0_1[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$. We take as $\mathcal{L}^n_\alpha$ *formulas of* T^n_α the $\mathcal{L}^n_\alpha$ formulas without free number variables.

We let $\Gamma, \Lambda, \ldots$ range over finite sets of $\mathcal{L}^n_\alpha$ formulas; we often write (for instance) Γ, φ for the union of Γ and $\{\varphi\}$. The Tait-calculus T^n_α is an extension of the classical Tait-calculus [17]. It contains the following axioms and rules of inference:

T^n_α-1 Ontological axioms I. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$, all set variables X and all $\beta < \alpha, \gamma \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X, s \notin X$$

$$\text{and} \qquad \Gamma, t \in \mathsf{D}^n_\beta, s \notin \mathsf{D}^n_\beta \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^n_{<\gamma}, s \notin \mathsf{D}^n_{<\gamma}.$$

T^n_α-2 Propositional rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α and all $\mathcal{L}^n_\alpha$ formulas φ and ψ of T^n_α:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

T^n_α-3 Quantifier rules. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta < \alpha$, all $\mathcal{L}^n_\alpha$ formulas φ and ψ of T^n_α, all closed number terms s, all set variables Y:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)},$$

$$\frac{\Gamma, \psi(Y)}{\Gamma, (\exists X)\psi(X)}, \qquad \frac{\Gamma, \psi(Y)}{\Gamma, (\forall X)\psi(X)} \quad (vc),$$

$$\frac{\Gamma, Y \dot{\in} \mathsf{D}^n_\beta \wedge \psi(Y)}{\Gamma, (\exists X \dot{\in} \mathsf{D}^n_\beta)\psi(X)}, \qquad \frac{\Gamma, Y \dot{\in} \mathsf{D}^n_\beta \rightarrow \psi(Y)}{\Gamma, (\forall X \dot{\in} \mathsf{D}^n_\beta)\psi(X)} \quad (vc),$$

By (vc) we indicate that the rule has to respect the usual variable conditions. That is, Y must not occur the conclusion.

T^n_α**-4 Ontological axioms II**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta \leq \alpha$, all closed terms s so that $\mathsf{Seq}_2 s$ is false, all closed terms t such that $\mathsf{Seq}_2 t$, $\mathsf{Seq}_2 (t)_0$ and $\beta \preceq (t)_1$ is true:

$$\Gamma, s \notin \mathsf{D}^n_{<\beta} \qquad \text{and} \qquad \Gamma, t \notin \mathsf{D}^n_{<\beta}.$$

T^n_α**-5 Ontological rules III**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta \leq \alpha$, $\gamma < \beta$, all closed terms t so that $\mathsf{Seq}_2 t$ and $(t)_1 = \gamma$ is true:

$$\frac{\Gamma, (t)_0 \in \mathsf{D}^n_\gamma}{\Gamma, t \in \mathsf{D}^n_{<\beta}}, \qquad \frac{\Gamma, (t)_0 \notin \mathsf{D}^n_\gamma}{\Gamma, t \notin \mathsf{D}^n_{<\beta}}.$$

T^n_α**-6 Closure axioms**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all closed number terms e, r, all set variables U, V and all $\beta < \alpha$:

$$\Gamma, (U, V \mathrel{\dot{\notin}} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}^n_\beta)(X = U \oplus V),$$
$$\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}^n_\beta)(\forall x)(x \in X \leftrightarrow \pi^0_1[e, x, r, U, \mathsf{D}^n_{<\beta}]).$$

T^n_α**-7 Closure rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all closed number terms e, r, all $\beta < \alpha$, all set variables U, V and if $n = 0$:

$$\frac{\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}^0_\beta), (\forall x)(\exists X \mathrel{\dot{\in}} \mathsf{D}^0_\beta)\pi^0_1[e, x, r, X, U, \mathsf{D}^0_{<\beta}]}{\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}^0_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}^0_\beta)(\forall x)\pi^0_1[e, x, r, (X)_x, U, \mathsf{D}^0_{<\beta}]},$$

and if $n > 0$:

$$\frac{\Gamma, (U, V \mathrel{\dot{\notin}} \mathsf{D}^n_\beta), (\forall X \mathrel{\dot{\in}} \mathsf{D}^n_\beta)(\exists Y \mathrel{\dot{\in}} \mathsf{D}^n_\beta)\pi^0_1[e, r, X, Y, V, \mathsf{D}^n_{<\beta}]}{\Gamma, (U, V \mathrel{\dot{\notin}} \mathsf{D}^n_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}^n_\beta)[(X)_0 = U \wedge (\forall u)\pi^0_1[e, r, (X)_u, (X)_{u+1}, V, \mathsf{D}^n_{<\beta}]]}.$$

T^n_α**-8 Reflection axioms**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α, all $\beta < \alpha$, all set variables U and if $n > 0$:

$$\Gamma, U \mathrel{\dot{\notin}} \mathsf{D}^n_\beta, (\exists X \mathrel{\dot{\in}} \mathsf{D}^n_\beta)(U \mathrel{\dot{\in}} X \wedge \mathsf{I}_{n-1}(X)).$$

T^n_α**-9 Cut rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of T^n_α and for all $\mathcal{L}^n_\alpha$ formulas φ of T^n_α:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In order to prove a partial cut elimination, we have to introduce a cut rank. Choose an $\mathcal{L}^n_\alpha$ formula φ of T^n_α. We set $rk(\varphi) = 0$ iff in φ there are no unbounded second order quantifiers $\exists X, \forall X$. Otherwise we set

1. If φ is a formula $\psi \wedge \theta$ or $\psi \vee \theta$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.
2. If φ is a formula $\exists x\psi$, $\forall x\psi$, $\exists X\psi$ or $\forall X\psi$, then $rk(\varphi) := rk(\psi) + 1$.
3. If φ is a formula $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\gamma)\psi$ or $(\forall X \mathrel{\dot\in} \mathsf{D}^n_\gamma)\psi$, then $rk(\varphi) := rk(\psi) + 2$ $(\gamma < \alpha)$.

The notion $\mathsf{T}^n_\alpha \overset{\beta}{\underset{m}{\vdash}} \Gamma$ is used to express that Γ is provable in T^n_α by a proof of depth less than or equal to β and so that all its cut formulas have ranks less than m. We write $\mathsf{T}^n_\alpha \overset{<\beta}{\underset{<m}{\vdash}} \Gamma$ if there exists a $\gamma < \beta$ and a $k < m$ with $\mathsf{T}^n_\alpha \overset{\gamma}{\underset{k}{\vdash}} \Gamma$. We write $\mathsf{T}^n_\alpha \overset{<\beta}{\underset{<\omega}{\vdash}} \Gamma$ if there exists a $\gamma < \beta$ and a k with $\mathsf{T}^n_\alpha \overset{\gamma}{\underset{k}{\vdash}} \Gamma$. Finally we write $\mathsf{T}^n_\alpha \overset{\beta}{\underset{<\omega}{\vdash}} \Gamma$ if there exists a k with $\mathsf{T}^n_\alpha \overset{\beta}{\underset{k}{\vdash}} \Gamma$. Since all main formulas of the conclusions of T^n_α-4 – T^n_α-8 have rank 0 we can prove partial cut elimination for T^n_α. The proof is standard and hence omitted. We set $\omega_0(\gamma) := \gamma$ and $\omega_{k+1}(\gamma) := \omega^{\omega_k(\gamma)}$.

Lemma 10 $\quad \mathsf{T}^n_\alpha \overset{\gamma}{\underset{k+1}{\vdash}} \Gamma \quad \Longrightarrow \quad \mathsf{T}^n_\alpha \overset{\omega_k(\gamma)}{\underset{1}{\vdash}} \Gamma.$

Next we introduce the systems E^n_α; they are first order reformulations of T^n_α. We formulate E^n_α in the first order part of $\mathcal{L}^n_\alpha$. The *formulas of* E^n_α are the formulas of T^n_α in which no set variables occur. We now give the definition of the Tait-calculus E^n_α.

E^n_α**-1 Ontological axioms I**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all $\beta < \alpha, \gamma \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^n_\beta, s \notin \mathsf{D}^n_\beta \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^n_{<\gamma}, s \notin \mathsf{D}^n_{<\gamma}.$$

E^n_α**-2 Propositional rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α and all $\mathcal{L}^n_\alpha$ formulas φ and ψ of E^n_α:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

E^n_α**-3 Quantifier rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta < \alpha$, all closed number terms s and all $\mathcal{L}^n_\alpha$ formulas φ and ψ of E^n_α:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)}.$$

E^n_α**-4 Ontological axioms II**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta \leq \alpha$, all closed terms s so that $\mathsf{Seq}_2 s$ is false, all closed terms t such that $\mathsf{Seq}_2 t$, $\mathsf{Seq}_2 (t)_0$ and $\beta \preceq (t)_1$ is true:

$$\Gamma, s \notin \mathsf{D}^n_{<\beta} \qquad \text{and} \qquad \Gamma, t \notin \mathsf{D}^n_{<\beta}.$$

E^n_α**-5 Ontological rules III**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all $\beta \leq \alpha$, $\gamma < \beta$, all closed terms t so that $\mathsf{Seq}_2 t$ and $(t)_1 = \gamma$ is true:

$$\frac{\Gamma, (t)_0 \in \mathsf{D}^n_\gamma}{\Gamma, t \in \mathsf{D}^n_{<\beta}}, \qquad \frac{\Gamma, (t)_0 \notin \mathsf{D}^n_\gamma}{\Gamma, t \notin \mathsf{D}^n_{<\beta}}.$$

E^n_α**-6 Closure axioms**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms e, r, s, t and all $\beta < \alpha$:

$$\Gamma, (\exists k)(\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_t \oplus (\mathsf{D}^n_\beta)_s,$$
$$\Gamma, (\exists k)(\forall x)(x \in (\mathsf{D}^n_\beta)_k \leftrightarrow \pi^0_1[e, x, r, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]).$$

E^n_α**-7 Closure rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms e, r, s, t, all $\beta < \alpha$ and if $n = 0$:

$$\frac{\Gamma, (\forall x)(\exists k)\pi^0_1[e, x, r, (\mathsf{D}^0_\beta)_k, (\mathsf{D}^0_\beta)_t, \mathsf{D}^0_{<\beta}]}{\Gamma, (\exists k)(\forall x)\pi^0_1[e, x, r, ((\mathsf{D}^0_\beta)_k)_x, (\mathsf{D}^0_\beta)_t, \mathsf{D}^0_{<\beta}]},$$

and if $n > 0$:

$$\frac{\Gamma, (\forall k)(\exists l)\pi^0_1[e, r, (\mathsf{D}^n_\beta)_k, (\mathsf{D}^n_\beta)_l, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]}{\Gamma, (\exists k)[((\mathsf{D}^n_\beta)_k)_0 = (\mathsf{D}^n_\beta)_s \wedge (\forall u)\pi^0_1[e, r, ((\mathsf{D}^n_\beta)_k)_u, ((\mathsf{D}^n_\beta)_k)_{u+1}, (\mathsf{D}^n_\beta)_t, \mathsf{D}^n_{<\beta}]]}.$$

E^n_α**-8 Reflection axioms**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α, all closed number terms t, all $\beta < \alpha$ and if $n > 0$:

$$\Gamma, (\exists k)((\mathsf{D}^n_\beta)_t \mathrel{\dot{\in}} (\mathsf{D}^n_\beta)_k \wedge \mathsf{I}_{n-1}((\mathsf{D}^n_\beta)_k)).$$

E^n_α**-9 Cut rules**. For all finite sets Γ of $\mathcal{L}^n_\alpha$ formulas of E^n_α and for all $\mathcal{L}^n_\alpha$ formulas φ of E^n_α:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In a next step we give a partial cut elimination for E^n_α. The situation here is more complicated than for T^n_α. We have in E^n_α, for instance, that the formula $(\exists k)\varphi((\mathsf{D}^n_\beta)_k)$ corresponds to $(\exists X \dot\in \mathsf{D}^n_\beta)\varphi(X)$. The problem is that we want to characterize formulas $(\exists k)\varphi(k)$ with subformulas of type $\langle s, k\rangle \in \mathsf{D}^n_\beta$ $(k \notin FV(s))$ but not with, e.g., a subformula of type $k \in \mathsf{D}^n_\beta$. In order to define such an appropriate class of formulas we introduce (nominal) symbols $*_i$ $(i \in \mathbb{N})$ which are different from all symbols in $\mathcal{L}^n_\alpha$. We now define the classes ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$.

Definition 11 We fix an $\alpha \in \Phi_0$, a $\beta < \alpha$ and an $n \in \mathbb{N}$. The classes ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ are inductively defined as follows:

1. For all number terms $\vec{s}, t$ of $\mathcal{L}_1$, all $\gamma < \beta$, all primitive recursive relation symbols K of $\mathcal{L}_1$ and all $*_i$ the following expressions are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ and ess-$\Pi^1_1(\mathsf{D}^n_\beta)$: $[\neg]K\vec{s}$, $[\neg]t \in \mathsf{D}^n_\gamma$, $[\neg]t \in \mathsf{D}^n_{<\gamma}$, $[\neg]t \in (\mathsf{D}^n_\beta)_{*_i}$, $[\neg]t \in \mathsf{D}^n_{<\beta}$. (We write $t \in (\mathsf{D}^n_\beta)_{*_i}$ for $\mathsf{D}^n_\beta(\langle t, *_i\rangle)$.)

2. If φ, ψ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.), then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (resp. ess-$\Pi^1_1(\mathsf{D}^n_\beta)$).

3. If φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.), then $\exists x\varphi$ and $\forall x\varphi$ are in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.).

4. If $\varphi(*_i)$ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.), then $\exists x\varphi[*_i\backslash x]$ ($\forall x\varphi[*_i\backslash x]$, resp.) is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.). Here we write $\varphi[*_i\backslash x]$ for the expression φ where all occurrences of $*_i$ are substituted by x.

There is one point worth mentioning. If φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ or in ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ and of the form $t \in \mathsf{D}^n_\gamma$, then γ is strict less than β. And if φ is in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ or in ess-$\Pi^1_1(\mathsf{D}^n_\beta)$ and of the form $t \in (\mathsf{D}^n_\gamma)_{*_i}$, then γ is (syntactically) equal to β.

Further we define that the class ess-$\Sigma^1_1(\mathsf{D}^n_\beta)^c$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)^c$, resp.) is the subset of all expressions in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)$ (ess-$\Pi^1_1(\mathsf{D}^n_\beta)$, resp.) which have no free number variables. For a given φ in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)^c$ or in ess-$\Pi^1_1(\mathsf{D}^n_\beta)^c$ and for $\vec{*} = *_1, \ldots, *_k$ we write $\varphi[\vec{*}]$ if all $*_i$ occurring in φ are among $*_1, \ldots, *_k$. Often we write only $\varphi[\vec{t}]$ for $\varphi[\vec{*}][\vec{*}\backslash\vec{t}]$. Notice that $\varphi[\vec{t}]$ is an $\mathcal{L}^n_\alpha$ formula of E^n_α. Analogously we write $\Gamma[\vec{*}]$ if all $*_i$ occurring in a φ in Γ are listed in $\vec{*}$ and if Γ is a finite subset of ess-$\Sigma^1_1(\mathsf{D}^n_\beta)^c \cup ess$-$\Pi^1_1(\mathsf{D}^n_\beta)^c$. And again we write $\Gamma[\vec{t}]$ for $\Gamma[\vec{*}][\vec{*}\backslash\vec{t}]$.

We can now define the rank $rk(\varphi)$ of a $\mathcal{L}^n_\alpha$ formula φ of E^n_α. We set $rk(\varphi) = 0$ iff there is a $\vec{t}$ and a $\psi[\vec{*}]$ in ess-$\Sigma^1_1(\mathsf{D}^n_\beta)^c$ or ess-$\Pi^1_1(\mathsf{D}^n_\beta)^c$ with $\beta < \alpha$ such that $\varphi \equiv \psi[\vec{t}]$. Otherwise we set

1. If φ is a formula $t \in \mathsf{D}^n_{<\alpha}$, $t \notin \mathsf{D}^n_{<\alpha}$, $t \in \mathsf{D}^n_\beta$ or a formula $t \notin \mathsf{D}^n_\beta$ $(\beta + 1 = \alpha)$, then $rk(\varphi) := 1$.

2. If φ is a formula $\psi \wedge \theta$ or $\psi \vee \theta$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.

3. If φ is a formula $\exists x \psi$ or $\forall x \psi$, then $rk(\varphi) := rk(\psi) + 1$.

Concerning clause 1 of this rank definition of E^n_α, we give some explanations. First, assume that α is a limit number. Then each $t \in \mathsf{D}^n_\beta$ with $\beta < \alpha$ has rank 0, since $t \in \mathsf{D}^n_\beta$ is an element of $ess\text{-}\Sigma^1_1(\mathsf{D}^n_{\beta+1})^c$ and $\beta + 1 < \alpha$. $t \in \mathsf{D}^n_{<\alpha}$ has rank 1 for each term t. Secondly, we assume that α is a successor ordinal. We write $\alpha - 1$ for the predecessor of α. Then each $t \in \mathsf{D}^n_\beta$ with $\beta < \alpha - 1$ has rank 0 and $\langle r, s\rangle \in \mathsf{D}^n_{\alpha-1}$ has rank 0 too. If t is a term different of all terms $\langle r, s\rangle$, then $t \in \mathsf{D}^n_{\alpha-1}$ has rank 1. Again the rank of $t \in \mathsf{D}^n_{<\alpha}$ is 1 and the rank of $t \in \mathsf{D}^n_{<\beta}$ is 0 for $\beta < \alpha$.

The notion $\mathsf{E}^n_\alpha \mathrel{\vert\!\frac{\beta}{m}} \Gamma$ is defined as for T^n_α but now with the above cut ranks. The rank of the main formulas in E^n_α-4 – E^n_α-8 is 0. Hence one immediately realizes that the axioms and rules of E^n_α are tailored in such a way that one can prove partial cut elimination in a straightforward manner. This observation is stated in the following lemma.

Lemma 12 $\quad \mathsf{E}^n_\alpha \mathrel{\vert\!\frac{\gamma}{k+1}} \Gamma \quad \Longrightarrow \quad \mathsf{E}^n_\alpha \mathrel{\vert\!\frac{\omega_k(\gamma)}{1}} \Gamma.$

In a next step we embed T^n_α into E^n_α. In order to achieve this, we inductively define for each $\mathcal{L}^n_\alpha$ formula φ of T^n_α an $\mathcal{L}^n_\alpha$ formula φ^* of E^n_α. If in φ there is no occurrence of $\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta$ and of $\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta$ for all $\beta < \alpha$, then we set $\varphi^* := \varphi$. Otherwise we set

1. If φ is of the form $\theta \vee \psi$ ($\theta \wedge \psi$ respectively), then we set $\varphi^* := \theta^* \vee \psi^*$ ($\theta^* \wedge \psi^*$ respectively).

2. If φ is of the form $\exists x \psi$ ($\forall x \psi$, $\exists X \psi$, $\forall X \psi$ respectively), then we set $\varphi^* := \exists x \psi^*$ ($\forall x \psi^*$, $\exists X \psi^*$, $\forall X \psi^*$ respectively).

3. If φ is of the form $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)$ ($(\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)\psi(X)$, resp.), then we set $\varphi^* := (\exists k)\psi^*((\mathsf{D}^n_\beta)_k)$ ($(\forall k)\psi^*((\mathsf{D}^n_\beta)_k)$, resp.) for $\beta < \alpha$.

This translation leads to the following embedding. For $\vec{t} = t_1, \ldots, t_n$ we write $(\mathsf{D}^n_{<\alpha})_{\vec{t}}$ for $(\mathsf{D}^n_{<\alpha})_{t_1}, \ldots, (\mathsf{D}^n_{<\alpha})_{t_n}$. $\Gamma[(\mathsf{D}^n_{<\alpha})_{\vec{t}}]$ is a shorthand for $\Gamma[\vec{X}][\vec{X} \backslash (\mathsf{D}^n_{<\alpha})_{\vec{t}}]$. (The bound in Lemma 13 is not optimal, we have chosen it for technical reasons.)

Lemma 13 *Assume that Γ is a set of T^n_α formulas without occurrences of unbounded set quantifiers $\exists X, \forall X$. Then there exists an integer m such that we have for all closed number terms $\vec{t}$*

$$\mathsf{T}^n_\alpha \,\frac{\gamma}{1}\, \Gamma[\vec{X}] \quad \Longrightarrow \quad \mathsf{E}^n_\alpha \,\frac{\omega^{\omega^\gamma}}{m}\, \Gamma^*[(\mathsf{D}^n_{<\alpha})_{\vec{t}}].$$

Proof. The proof is by induction on γ. If Γ is an axiom of T^n_α-1 or T^n_α-4, the claim is immediate. If Γ is the conclusion of a propositional rule T^n_α-2, of an ontological rule III T^n_α-5 or of a cut rule T^n_α-9, the claim follows immediately from the induction hypothesis. We now discuss the quantifier rules T^n_α-3. By assumption we do not have to deal with the $(\exists X)$- and $(\forall X)$-rule. The $(\exists x)$- and $(\forall x)$-rule follows immediately from the induction hypothesis. There remain the cases of the bounded second order quantifiers. First we discuss the $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule. We assume that $\Gamma[\vec{X}]$ is the conclusion of the $(\exists X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule $(\beta < \alpha)$. Then there exists a $\gamma_0 < \gamma$ and a set variable Z with

$$\mathsf{T}^n_\alpha \,\frac{\gamma_0}{1}\, \Gamma[\vec{X}], Z \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi(Z). \tag{6}$$

The induction hypothesis yields an integer m with

$$\mathsf{E}^n_\alpha \,\frac{\omega^{\omega^{\gamma_0}}}{m}\, \Gamma^*[(\mathsf{D}^n_{<\alpha})_{\vec{t}}], (\mathsf{D}^n_{<\alpha})_r \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_{<\alpha})_r)$$

for all closed number terms $r, \vec{t}$ such that $X_i \equiv Z$ implies $t_i \equiv r$. An application of the $(\exists x)$-rule leads to

$$\mathsf{E}^n_\alpha \,\frac{<\omega^{\omega^\gamma}}{m}\, \Gamma^*[(\mathsf{D}^n_{<\alpha})_{\vec{t}}], (\exists k)((\mathsf{D}^n_{<\alpha})_k \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_{<\alpha})_k)).$$

We prove now

$$\mathsf{E}^n_\alpha \,\frac{<\omega}{k}\, \neg(\exists k)((\mathsf{D}^n_{<\alpha})_k \mathrel{\dot\in} \mathsf{D}^n_\beta \wedge \psi^*((\mathsf{D}^n_{<\alpha})_k)), (\exists k)\psi^*((\mathsf{D}^n_\beta)_k) \tag{7}$$

for an integer k. Then a cut implies the claim. Notice that there are integers l_1, l_2 such that we have for all closed terms t, r

$$\mathsf{E}^n_\alpha \,\frac{l_1}{l_2}\, (\mathsf{D}^n_{<\alpha})_t \neq (\mathsf{D}^n_\beta)_r, \neg\psi^*((\mathsf{D}^n_{<\alpha})_t), \psi^*((\mathsf{D}^n_\beta)_r).$$

Hence there are integers l_3, l_4 such that

$$\mathsf{E}^n_\alpha \,\frac{l_3}{l_4}\, (\mathsf{D}^n_{<\alpha})_t \mathrel{\dot\in} \mathsf{D}^n_\beta \to \neg\psi^*((\mathsf{D}^n_{<\alpha})_t), (\exists k)\psi^*((\mathsf{D}^n_\beta)_k)$$

holds for all closed terms t. Now the $(\forall x)$-rule implies (7). Next we discuss the $(\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule. Hence we assume that $\Gamma[\vec{X}]$ is the conclusion of the $(\forall X \mathrel{\dot\in} \mathsf{D}^n_\beta)$-rule $(\beta < \alpha)$. Then there exists a $\gamma_0 < \gamma$ and a set variable Y which does not occur in $\Gamma[\vec{X}]$ with

$$\mathsf{T}^n_\alpha \mathrel{|\!\frac{\gamma_0}{1}} \Gamma[\vec{X}], Y \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi(Y).$$

The induction hypothesis yields an integer m such that we have

$$\mathsf{E}^n_\alpha \mathrel{|\!\frac{\omega^{\omega^{\gamma_0}}}{m}} \Gamma^*[(\mathsf{D}^n_{<\alpha})_{\vec{t}}], (\mathsf{D}^n_{<\alpha})_r \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi^*((\mathsf{D}^n_{<\alpha})_r) \tag{8}$$

for all closed terms $\vec{t}, r$. Since we can prove for $\beta < \alpha$

$$\mathsf{E}^n_\alpha \mathrel{|\!\frac{l_1}{l_2}} \neg(\forall k)((\mathsf{D}^n_{<\alpha})_k \mathrel{\dot\in} \mathsf{D}^n_\beta \to \psi^*((\mathsf{D}^n_{<\alpha})_k)), (\forall k)\psi^*((\mathsf{D}^n_\beta)_k),$$

for suitable integers l_1, l_2, a cut together with the $(\forall x)$-rule implies the claim. There remain the closure and reflection properties. We first prove closure under disjoint union. We have to prove for all $\beta < \alpha$ the existence of integers l_1, l_2 such that

$$\mathsf{E}^n_\alpha \mathrel{|\!\frac{l_1}{l_2}} (\mathsf{D}^n_{<\alpha})_t \mathrel{\dot\notin} \mathsf{D}^n_\beta, (\mathsf{D}^n_{<\alpha})_s \mathrel{\dot\notin} \mathsf{D}^n_\beta, (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_{<\alpha})_t \oplus (\mathsf{D}^n_{<\alpha})_s)$$

for all closed terms s, t. Let us fix a $\beta < \alpha$. Since we have closure under disjoint union in E^n_α too, we have

$$\mathsf{E}^n_\alpha \mathrel{|\!\frac{0}{0}} (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_{r_1} \oplus (\mathsf{D}^n_\beta)_{r_2})$$

for all closed terms r_1 and r_2 and hence there are integers l_3, l_4 such that

$$\mathsf{E}^n_\alpha \mathrel{|\!\frac{l_3}{l_4}} (\mathsf{D}^n_{<\alpha})_t \neq (\mathsf{D}^n_\beta)_{r_1}, (\mathsf{D}^n_{<\alpha})_s \neq (\mathsf{D}^n_\beta)_{r_2}, (\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_{<\alpha})_t \oplus (\mathsf{D}^n_{<\alpha})_s)$$

for all closed terms r_1, r_2. The $(\forall x)$-rule implies the claim. Similarly we can prove the remaining axioms and rules of T^n_α-6, T^n_α-7 and T^n_α-8 using the corresponding properties in E^n_α. This is straightforward, hence omitted. $\square$

The following lemma will be used in the asymmetric interpretation. It states that in $\mathsf{E}^0_{\alpha+1}$ the projections $(\mathsf{D}^0_\alpha)_t$ are first order analogues of the second order variables X. Usual second order systems have a substitution property: If they prove $\Gamma[\vec{X}]$, then they prove $\Gamma[\vec{Y}]$ too. We prove in Lemma 14 the corresponding property for the system $\mathsf{E}^0_{\alpha+1}$: If we can prove $\Gamma[\vec{t}]$ (as mentioned we write $\Gamma[\vec{t}]$ for $\Gamma[\vec{*}][\vec{*}\backslash\vec{t}]$) we can also prove $\Gamma[\vec{s}]$ (for $t_i = t_j \Rightarrow s_i = s_j$). Of course we can not prove this for arbitrary sets Γ of formulas; but only for formulas which have a second order analogue. That is, we prove this substitution property for formulas in $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$. In fact, it would be possible to prove the substitution property for a larger class of such second order analogue but we do not want to introduce further classes of formulas. We also refer to Lemma 13. There it is proved that free set variables (in $\mathsf{T}^n_{\alpha+1}$) are represented by projections (in $\mathsf{E}^n_{\alpha+1}$).

Note that this substitution property reflects a typical quality of countable coded ω-models. Assume that M is such a countable coded ω-model, e.g. of ACA. Then the projections $(M)_k$ are the sets in M. The number variable k is the index of the set $(M)_k$ in M. We know absolutely nothing about this index. If there is given an index k we have no more information than the fact "k is an index". Perhaps, this can serve as motivation for the following lemma. We write only $s = t$ for "$s = t$ is true" (s, t closed number terms).

Lemma 14 *Assume that* $\Gamma[\vec{*}]$ *is a finite subset of* $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$. *We assume that*

$$\mathsf{E}^0_{\alpha+1} \,|\!\frac{\gamma}{1}\, \Gamma[\vec{t}].$$

Then we have for all n-tuples $\vec{s}$ of closed terms s_i $(1 \leq i \leq n)$ such that for all i, j $(1 \leq i, j \leq n)$ $t_i = t_j$ implies $s_i = s_j$ that

$$\mathsf{E}^0_{\alpha+1} \,|\!\frac{\gamma}{1}\, \Gamma[\vec{s}].$$

Proof. The proof is by induction on γ. The case of the closure axioms $\mathsf{E}^n_{\alpha+1}$-6 and rules $\mathsf{E}^n_{\alpha+1}$-7 follows immediately from the induction hypothesis. If Γ is the conclusion of a propositional rule $\mathsf{E}^n_{\alpha+1}$-2, of an ontological rule III $\mathsf{E}^n_{\alpha+1}$-5 or of a cut rule $\mathsf{E}^n_{\alpha+1}$-9, the claim is immediate from the induction hypothesis. The case of the ontological axioms II $\mathsf{E}^n_{\alpha+1}$-4 is also trivial. There remain the cases of the ontological axioms I $\mathsf{E}^n_{\alpha+1}$-1 and of the quantifier rules $\mathsf{E}^n_{\alpha+1}$-3. Let us discuss the ontological axioms I. Here we have only to discuss the case of the following axioms, since the other cases are trivial. Assume

$$\Lambda[\vec{t}], r_1 \in (\mathsf{D}^0_\alpha)_{t_{n+1}}, r_2 \notin (\mathsf{D}^0_\alpha)_{t_{n+2}}$$

such that $\vec{t} = (t_1, \ldots, t_n)$ and $t_{n+1} = t_{n+2}$, $r_1 = r_2$. Choose an n-tuple $\vec{s}$ and s_{n+1}, s_{n+2} such that $t_i = t_j$ implies $s_i = s_j$ $(1 \leq i, j \leq n+2)$. We have to prove

$$\Lambda[\vec{s}], r_1 \in (\mathsf{D}^0_\alpha)_{s_{n+1}}, r_2 \notin (\mathsf{D}^0_\alpha)_{s_{n+2}}$$

But this is again an axiom, since $s_{n+1} = s_{n+2}$. We now discuss the quantifier rules. Γ is a subset of $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$. First, we assume that Γ is the conclusion of the $(\exists x)$-rule. Then the main formula of the conclusion is of type $\exists k \varphi(k)$. If there occur no $(\mathsf{D}^0_\alpha)_k$ in φ the claim follows immediately from the induction hypothesis. If there occur a $(\mathsf{D}^0_\alpha)_k$ in φ, then k occurs in φ only in $(\mathsf{D}^0_\alpha)_k$. Hence there are a $\gamma_0 < \gamma$ and a closed number term r such that

$$\mathsf{E}^0_{\alpha+1} \mathrel{|}\!\frac{\gamma_0}{1}\; \Gamma[\vec{t}], \varphi[\vec{t}, r].$$

We fix $\vec{s}$ such that $t_i = t_j$ implies $s_i = s_j$. Then the induction hypothesis yields

$$\mathsf{E}^0_{\alpha+1} \mathrel{|}\!\frac{\gamma_0}{1}\; \Gamma[\vec{s}], \varphi[\vec{s}, r'].$$

We have written r' instead of r, since it is possible that the application of the induction hypothesis changes r too. Now the $(\exists x)$-rule implies the claim. Finally we discuss the $(\forall x)$-rule. Here the main formula of the conclusion is of type $\forall k \varphi(k)$. Again we discuss only the case where $(\mathsf{D}^0_\alpha)_k$ occurs in φ. Then there are $\gamma_r < \gamma$ such that

$$\mathsf{E}^0_{\alpha+1} \mathrel{|}\!\frac{\gamma_r}{1}\; \Gamma[\vec{t}], \varphi[\vec{t}, r]$$

for all closed number terms r. We fix an $\vec{s}$ such that $t_i = t_j$ implies $s_i = s_j$ $(1 \leq i, j \leq n)$. Choose an r such that $r \neq t_i$ for all i $(1 \leq i \leq n)$. Then an application of the induction hypothesis leads to

$$\mathsf{E}^0_{\alpha+1} \mathrel{|}\!\frac{\gamma_{r_2}}{1}\; \Gamma[\vec{s}], \varphi[\vec{s}, r].$$

for all closed terms r. Then the $(\forall x)$-rule gives the claim. $\square$

5 Finite reduction

In this and the next section the proof-theoretic analysis of E^n_α is given.

5.1 Reduction of $\mathsf{E}^0_{\alpha+1}$ to E^0_α

In some sense our reductions are adaptions of the reductions presented in [2]. Thus we introduce further semi-formal systems $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ in which we have in addition iterated arithmetical comprehension up to $\nu \in \Phi_0$. We will prove an asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. The next step will be the elimination of "H_ν" in $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. To achieve this we introduce a system RA_α of ramified analysis. The first order part of RA_α essentially corresponds to E^0_α. We can embed $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ into RA_α. There is also a partial (second) cut elimination in RA_α. Finally, we will embed the first order fragment of RA_α into E^0_α. This will yield the desired reduction of $\mathsf{E}^0_{\alpha+1}$ to E^0_α.

The class of *arithmetic* $\mathcal{L}^0_\alpha$ *formulas of* E^0_α contains all $\mathcal{L}^0_\alpha$ formulas φ such that no quantifier $\exists X \dot{\in} \mathsf{D}^0_\gamma$, $\forall X \dot{\in} \mathsf{D}^0_\gamma$, $\exists X$, $\forall X$ occurs in φ ($\gamma < \alpha$).

Definition of the Tait-calculus $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ is formulated in $\mathcal{L}^0_\alpha$. The *formulas of* $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ are those of T^0_α which do not contain bounded second order quantifiers. In particular we allow unbounded second order quantifiers. $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ includes all axioms and rules of E^0_α extended to formulas of $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. In addition there are quantifier rules for unbounded second order quantification, as well as the following scheme.

Iterated arithmetical comprehension. For all finite sets Γ of $\mathcal{L}^0_\alpha$ formulas of $\mathsf{H}_\nu\mathsf{E}^0_\alpha$, all arithmetic $\mathcal{L}^0_\alpha$ formulas $\varphi[x, y, Z, Y]$ of E^0_α and all set variables Y:

$$\Gamma, (\exists X)(\forall x)(\forall c \prec \nu)(x \in (X)_c \leftrightarrow \varphi[x, c, (X)_{\prec c}, Y]).$$

In $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ we need a rank definition for the definition of the notion of deduction $\frac{\delta}{k}$ which is defined as before. For simplicity we set $rk(\varphi) := 0$ iff there are either no unbounded second order universal quantifiers $\forall X$ or no unbounded second order existence quantifiers $\exists X$ in φ.

We can now define in $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ the hyperarithmetical hierarchy $(\mathcal{H}^S_a)_{a \prec \nu}$ and predicates $(\mathsf{I}^S_a)_{a \prec \nu}$. We fix a Π^0_1 complete predicate j and define

1. $\mathsf{H}^S_0 := \{x : x \in S\}$.
   $\mathsf{H}^S_{a+1} := \{x : \mathsf{j}(\mathsf{H}^S_a, x)\}$.
   $\mathsf{H}^S_\ell := \{\langle x, a\rangle : a \prec \ell \wedge x \in \mathsf{H}^S_a\}$.
2. $\mathcal{H}^S_a := \{Y : Y$ is recursive in a H^S_b with $b \preceq a\}$,
   $:= \{\langle x, \langle e, b\rangle\rangle : b \preceq a \wedge (\forall y)(\exists z)(\{e\}^{\mathsf{H}^S_b}(y) = z) \wedge \{e\}^{\mathsf{H}^S_b}(x) = 0\}$.
3. $\mathsf{I}^S_a := \{\langle e, b\rangle : \langle e, b\rangle$ is an index of an element of $\mathcal{H}^S_a\}$,
   $:= \{\langle e, b\rangle : b \preceq a \wedge (\forall y)(\exists z)(\{e\}^{\mathsf{H}^S_b}(y) = z)\}$.

In the following we will prove an asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. It corresponds essentially to the asymmetric interpretation of Σ^1_1-AC into $(\Pi^1_0\text{-}\mathsf{CA})_{<\varepsilon_0}$ in [2]. The only difference is that our situation is more complicated. We first give a translation.

Definition 15 For each expression φ in $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)$ or in $ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)$ we inductively define $\varphi^{\beta,\gamma,\nu}$ as follows:

1. If there is no occurrence of D^0_α in φ, then $\varphi^{\beta,\gamma,\nu} := \varphi$.

2. $(t \in (\mathsf{D}^0_\alpha)_{*_i})^{\beta,\gamma,\nu} := t \in (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_{*_i}$ and $(t \notin (\mathsf{D}^0_\alpha)_{*_i})^{\beta,\gamma,\nu} := t \notin (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_{*_i}$.

3. If φ is of the form $\theta \wedge \psi$ ($\theta \vee \psi$, resp.), then $\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu} \wedge \psi^{\beta,\gamma,\nu}$ ($\theta^{\beta,\gamma,\nu} \vee \psi^{\beta,\gamma,\nu}$, resp.).

4. If φ is of the form $\exists k\psi(k)$ ($\forall k\psi(k)$, resp.) such that there is no $(\mathsf{D}^0_\alpha)_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := \exists k\psi^{\beta,\gamma,\nu}(k)$ ($\forall k\psi^{\beta,\gamma,\nu}(k)$, resp.).

5. If φ is of the form $\exists k\psi((\mathsf{D}^0_\alpha)_k)$ (($\forall k\psi((\mathsf{D}^0_\alpha)_k)$, resp.) such that there is a $(\mathsf{D}^0_\alpha)_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := (\exists k \in \mathsf{I}_\gamma^{\mathsf{D}^0_{<\alpha}})\psi^{\beta,\gamma,\nu}((\mathcal{H}_\gamma^{\mathsf{D}^0_{<\alpha}})_k)$ ($(\forall k \in \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}})\psi^{\beta,\gamma,\nu}((\mathcal{H}_\beta^{\mathsf{D}^0_{<\alpha}})_k)$, resp.).

In clause 2 we have given a translation of $t \in (\mathsf{D}^0_\alpha)_{*_i}$. In the following we set

$$(t \in (\mathsf{D}^0_\alpha)_s)^{\beta,\gamma,\nu} := (t \in (\mathsf{D}^0_\alpha)_{*_i})^{\beta,\gamma,\nu}[*_i \backslash s].$$

We extend this translation to all expressions $\varphi[\vec{*}]$ in $ess\text{-}(\Sigma^1_1\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$ by setting $\varphi[\vec{t}]^{\beta,\gamma,\nu} := (\varphi[\vec{*}]^{\beta,\gamma,\nu})[\vec{*}\backslash\vec{t}]$. Notice that for s a closed number term the formulas $t \in (\mathsf{D}^0_\alpha)_s$ and $t \notin (\mathsf{D}^0_\alpha)_s$ are interpreted symmetrically, whereas the quantifiers $\exists k\psi((\mathsf{D}^0_\alpha)_k)$, $\forall k\psi((\mathsf{D}^0_\alpha)_k)$ are interpreted asymmetrically.

We will give an asymmetric interpretation. It is typical for such situations that there is a persistency property. Notice that we have defined $(\mathcal{H}^S_a)_{a \prec \nu+1}$ and $(\mathsf{I}^S_a)_{a \prec \nu+1}$ in such a way that we can prove in $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$ with finite deduction length

$$k \notin \mathsf{I}_\gamma^{\mathsf{D}^0_{<\alpha}}, k \in \mathsf{I}_{\gamma'}^{\mathsf{D}^0_{<\alpha}}$$

and

$$k \notin \mathsf{I}_\gamma^{\mathsf{D}^0_{<\alpha}}, (\mathcal{H}_\gamma^{\mathsf{D}^0_{<\alpha}})_k = (\mathcal{H}_{\gamma'}^{\mathsf{D}^0_{<\alpha}})_k$$

for all ordinals $\gamma \leq \gamma'$. Hence there is a persistency lemma. We omit the proof, since it is proved by straightforward induction on the deduction length δ.

Lemma 16 *For all finite sets $\Gamma[\vec{*}] \cup \{\varphi[\vec{*}]\}$ of expressions in $ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha)^c$ and for all ordinals $\nu, \rho, \rho', \gamma, \gamma', \delta$ with $\nu > \rho > \rho'$, $\gamma < \gamma' < \nu$ there are integers k, m such that we have for all closed number terms $\vec{t}$*

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \mathrel{\substack{\delta \\ \vdash \\ k}} \Gamma[\vec{t}], \varphi[\vec{t}]^{\rho,\gamma,\nu} \quad \Longrightarrow \quad \mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \mathrel{\substack{\delta+m \\ \vdash \\ k}} \Gamma[\vec{t}], \varphi[\vec{t}]^{\rho',\gamma',\nu}.$$

The asymmetric interpretation is established in the following theorem.

Theorem 17 *For all finite subsets* $\Gamma[\vec{*}]$ *of ess-*$\Sigma^1_1(\mathsf{D}^0_\alpha)^c \cup$ *ess-*$\Pi^1_1(\mathsf{D}^0_\alpha)^c$ *and for all ordinals* $\beta, \gamma, \nu \in \Phi_0$ *with* $\beta + \omega^\gamma < \nu$ *there is an integer* k *such that we have for all closed number terms* $\vec{t}$

$$\mathsf{E}^0_{\alpha+1} \mathrel{|\!\frac{\gamma}{1}} \Gamma[\vec{t}] \Longrightarrow \mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \mathrel{|\!\frac{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{k}} \vec{t} \notin \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma,\nu}.$$

Proof. For technical reasons we first introduce a formal theory M. We tailor M in such a way that the semi-formal system $\mathsf{H}_{\nu+1}\mathsf{E}_\alpha$ is a (first order) Tait-style version of M. M is formulated in $\mathcal{L}^0_\alpha$ and based on the usual axioms and rules for the two-sorted predicate calculus. We have defining axioms for all primitive recursive functions and relations and

(1) *ontological properties for* $\gamma < \beta < \alpha$

$$(\forall x)(x = \gamma \rightarrow (\mathsf{D}^0_{<\beta})_x = \mathsf{D}^0_\gamma),$$

(2) *closure conditions for all* D^0_β *(*$\beta < \alpha$*)*

$$\begin{aligned}
&(2.1) && Y, Z \mathrel{\dot\in} \mathsf{D}^0_\beta \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(X = Y \oplus Z),\\
&(2.2) && Z \mathrel{\dot\in} \mathsf{D}^0_\beta \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)(\forall x)(x \in X \leftrightarrow \pi^0_1[e, x, z, Z, \mathsf{D}^0_{<\beta}]),\\
&(2.3) && Z \mathrel{\dot\in} \mathsf{D}^0_\beta \wedge (\forall x)(\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)\pi^0_1[e, x, z, X, Z, \mathsf{D}^0_{<\beta}]\\
&&& \quad \rightarrow (\exists X \mathrel{\dot\in} \mathsf{D}^0_\beta)\pi^0_1[e, x, z, (X)_x, Z, \mathsf{D}^0_{<\beta}],
\end{aligned}$$

(3) *iterated arithmetical comprehension up to* $\nu + 1$ *for all formulas arithmetic in* $\mathsf{D}^0_{<\alpha}$*,*

(4) *set-induction up to* $\nu + 1$

$$(\forall X)\mathsf{TI}(X, \nu + 1).$$

The whole point of introducing this extra theory M is that we can carry out more easily certain proofs in M and then interpret them into the more complicated framework $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$. Let us formulate this embedding of M into $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$. For all formulas φ of $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$ there are integers k, n such that

$$\mathsf{M} \vdash \varphi \quad\Longrightarrow\quad \mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \mathrel{|\!\frac{\omega^{\nu+1}+n}{k}} \varphi \tag{9}$$

holds. Furthermore we can prove a tautology lemma for $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$. For each formula $\varphi[\vec{s}]$ of $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$ there exists an integer k such that we have for all closed number terms $\vec{t}$

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_{\alpha+1} \vdash^k_0 \varphi[\vec{t}], \neg\varphi[\vec{t}]. \tag{10}$$

We now start to prove the claim by induction on γ. We have to discuss $\mathsf{E}^0_{\alpha+1}$-1 – $\mathsf{E}^0_{\alpha+1}$-9. If Γ is an axiom of $\mathsf{E}^0_{\alpha+1}$-1, the claim follows immediately, since we can prove in $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$ $\neg\varphi, \varphi$ with finite deduction length (cf. (10)). Since $\wedge$ and $\vee$ commute with $(\cdot)^{\delta,\varepsilon,\nu}$, we immediately get the claim in the case of $\mathsf{E}^0_{\alpha+1}$-2. Notice that there is no $\mathsf{D}^0_{<\alpha+1}$ in Γ, hence the cases $\mathsf{E}^0_{\alpha+1}$-4 and $\mathsf{E}^0_{\alpha+1}$-5 are immediate. If the last rule is a cut rule $\mathsf{E}^0_{\alpha+1}$-9 we can argue as in similar asymmetric interpretations, cf. e.g. [2], Theorem 2.5. And since $\mathsf{E}^0_{\alpha+1}$ does not contain $\mathsf{E}^0_{\alpha+1}$-8 there remain $\mathsf{E}^0_{\alpha+1}$-3, $\mathsf{E}^0_{\alpha+1}$-6, $\mathsf{E}^0_{\alpha+1}$-7. Let us write in this proof $\varphi^{\delta,\varepsilon}$ for $\varphi^{\delta,\varepsilon,\nu}$.

$\mathsf{E}^0_{\alpha+1}$-3. We have only to deal with the $(\forall x)$-rule and the $(\exists x)$-rule. We first discuss the $(\exists x)$-rule. Hence, assume that $\Gamma[\vec{t}]$ is the conclusion of the $(\exists x)$-rule. There is a $\gamma_0 < \gamma$ and a closed term t_{n+1} such that

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_0}_1 \Gamma[\vec{t}], \varphi(t_{n+1})[\vec{t}].$$

If no $(\mathsf{D}^0_\alpha)_{t_{n+1}}$ occurs in φ, the claim follows easily from the induction hypothesis. Therefore, we assume that $(\mathsf{D}^0_\alpha)_{t_{n+1}}$ occurs in φ. Thus we have

$$\mathsf{E}^0_{\alpha+1} \vdash^{\gamma_0}_1 \Gamma[\vec{t}], \varphi((\mathsf{D}^0_\alpha)_{t_{n+1}})[\vec{t}, t_{n+1}].$$

We prefer here – and sometimes also later on – to write $\varphi((\mathsf{D}^0_\alpha)_{t_{n+1}})[\vec{t}, t_{n+1}]$ instead of $\varphi[\vec{t}, t_{n+1}]$, since later on we have also to control the ordinal ε in $\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_\varepsilon$. Using Lemma 14 and the induction hypothesis, we obtain an integer k such that for all closed terms $\vec{s} = (s_1, \ldots, s_n)$ and s_{n+1} such that $t_i = t_j$ implies $s_i = s_j$ $(1 \le i, j \le n+1)$

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \vdash^{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}}_k \vec{s}, s_{n+1} \notin \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta, \Gamma[\vec{s}]^{\beta,\beta+\omega^{\gamma_0}}, \varphi((\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_\nu)_{s_{n+1}})[\vec{s}, s_{n+1}]^{\beta,\beta+\omega^{\gamma_0}}$$

holds. We can prove with finite deduction length for all s_{n+1} $(s_{n+1} \notin \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta$, $s_{n+1} \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^\gamma})$. Then we use the $\wedge$-rule, the $(\exists k)$-rule and persistency. Hence, there is an integer j such that

$$\begin{aligned}\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \ \vdash^{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}+j}_k \ & \vec{s}, s_{n+1} \notin \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta, \Gamma[\vec{s}]^{\beta,\beta+\omega^\gamma}, \\ & (\exists k \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^\gamma})\varphi((\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^\gamma})_k)[\vec{s}]^{\beta,\beta+\omega^\gamma}\end{aligned}$$

holds for all $\vec{s}, s_{n+1}$ which satisfy the condition above. If there is a t_i $(1 \leq i \leq n)$ with $t_i = t_{n+1}$ we can set $\vec{s} := \vec{t}, s_{n+1} := t_i$ and we are done. If there is no t_i with $t_{n+1} = t_i$ we distinguish two cases: If $n \geq 1$, we set $\vec{s} := \vec{t}$ and $s_{n+1} := t_1$. If $n = 0$ we use the $(\forall x)$-rule and obtain

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \ \left|\frac{\ <\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}\ }{k}\right. \ (\forall k)(k \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma^{\beta,\beta+\omega^\gamma}, (\exists k \in \mathsf{I}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})\varphi^{\beta,\beta+\omega^\gamma}((\mathcal{H}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k).$$

We can show with finite deduction length $\neg(\forall k)(k \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}})$. Hence, a cut implies the claim.

Now, we discuss the $(\forall x)$-rule. We assume that $\Gamma[\vec{t}]$ is the conclusion of the $(\forall x)$-rule. Hence there is for each closed term r a $\gamma_r < \gamma$ such that

$$\mathsf{E}^0_{\alpha+1} \left|\frac{\gamma_r}{1}\right. \Gamma[\vec{t}], \varphi(r)[\vec{t}].$$

If no $(\mathsf{D}^0_\alpha)_r$ occurs in φ, the claim follows easily from the induction hypothesis. Therefore, we assume that $(\mathsf{D}^0_\alpha)_r$ occurs in φ. Thus we have

$$\mathsf{E}^0_{\alpha+1} \left|\frac{\gamma_r}{1}\right. \Gamma[\vec{t}], \varphi((\mathsf{D}^0_\alpha)_r)[\vec{t}, r]$$

for all closed terms r. We apply the induction hypothesis and obtain with the aid of persistency integers j, k such that

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \left|\frac{\ \omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_r}}+j\ }{k}\right. \vec{t}, r \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma}, \varphi((\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_r)[\vec{t}, r]^{\beta,\beta+\omega^\gamma}.$$

holds for all closed terms r. The $\vee$-rule and $(\forall x)$-rule imply

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \left|\frac{\ <\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}\ }{k}\right. \vec{t} \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma}, (\forall k \in \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}})\varphi((\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_k)[\vec{t}]^{\beta,\beta+\omega^\gamma}.$$

Since we can prove with finite deduction length

$$t \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t = (\mathcal{H}_\beta^{\mathsf{D}^0_{<\alpha}})_t,$$

we can prove with finite deduction length

$$\neg(\forall k \in \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}})\varphi((\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_k)[\vec{t}]^{\beta,\beta+\omega^\gamma}, (\forall k \in \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}})\varphi((\mathcal{H}_\beta^{\mathsf{D}^0_{<\alpha}})_k)[\vec{t}]^{\beta,\beta+\omega^\gamma}$$

and a cut implies the claim.

$\mathsf{E}^0_{\alpha+1}$-6. We discuss the second axioms, the first are proved with similar arguments. We have to prove

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \;\Big|\!\frac{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{n}\; t \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}, \\ (\exists k \in \mathsf{I}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)(x \in (\mathcal{H}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k \\ \leftrightarrow \pi^0_1[e, x, r, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]).$$

for an integer n. Recall that we have in M iterated arithmetical comprehension up to $\nu+1$ and set induction up to $\nu+1$. Using this iterated arithmetical comprehension we can build in M the hyperarithmetical hierarchy $(\mathcal{H}_a^{\mathsf{D}^0_{<\alpha}})_{a\prec\nu+1}$ and the sets $(\mathsf{I}_a^{\mathsf{D}^0_{<\alpha}})_{a\prec\nu+1}$. We now fix an index t in $\mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}$ and integers e and r. Since $(\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t = (\mathcal{H}_\beta^{\mathsf{D}^0_{<\alpha}})_t$ holds, the set

$$\{x : \pi^0_1[e, x, r, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]\}$$

is recursive in $(\mathcal{H}_{\beta+1}^{\mathsf{D}^0_{<\alpha}})_t$. It follows that there is an index $d \in \mathsf{I}_{\beta+1}^{\mathsf{D}^0_{<\alpha}}$ such that

$$(\forall x)(x \in (\mathcal{H}_{\beta+1}^{\mathsf{D}^0_{<\alpha}})_d \leftrightarrow \pi^0_1[e, x, r, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}])$$

holds. Hence we can prove in M

$$t \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}} \vee (\exists k \in \mathsf{I}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)(x \in (\mathcal{H}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k \leftrightarrow \pi^0_1[e, x, r, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_t, \mathsf{D}^0_{<\alpha}]). \quad (11)$$

Using the embedding given in (9), we obtain the claim.

$\mathsf{E}^0_{\alpha+1}$-7. We know

$$\mathsf{E}^0_{\alpha+1} \,\Big|\!\frac{\gamma}{1}\, \Gamma[\vec{t}], (\exists k)(\forall x)\pi^0_1[e, x, r, ((\mathsf{D}^0_\alpha)_k)_x, (\mathsf{D}^0_\alpha)_r, \mathsf{D}^0_{<\alpha}]$$

and have to prove

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \;\Big|\!\frac{\omega^{\nu+1}+\omega^{\beta+\omega^\gamma}}{n}\; (\vec{t}, r \notin \mathsf{I}_\beta^{\mathsf{D}^0_{<\alpha}}), \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma}, \\ (\exists k \in \mathsf{I}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})(\forall x)\pi^0_1[e, x, r, ((\mathcal{H}_{\beta+\omega^\gamma}^{\mathsf{D}^0_{<\alpha}})_k)_x, (\mathcal{H}_\nu^{\mathsf{D}^0_{<\alpha}})_r, \mathsf{D}^0_{<\alpha}]$$

for an integer n. We know that there exists a $\gamma_0 < \gamma$ with

$$\mathsf{E}^0_{\alpha+1} \,\Big|\!\frac{\gamma_0}{1}\, \Gamma[\vec{t}], (\forall x)(\exists k)\pi^0_1[e, x, r, (\mathsf{D}^0_\alpha)_k, (\mathsf{D}^0_\alpha)_r, \mathsf{D}^0_{<\alpha}].$$

Applying the induction hypothesis yields an integer n such that

$$\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha \ \stackrel{\omega^{\nu+1}+\omega^{\beta+\omega^{\gamma_0}}}{\overline{\quad\ n\quad}} \ (\vec{t}, r \notin \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta), \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma_0}}, \\ (\forall x)(\exists k \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma_0}})\pi^0_1[e, x, r, (\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma_0}})_k, (\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_\nu)_r, \mathsf{D}^0_{<\alpha}].$$

Arguing as in $\mathsf{E}^0_{\alpha+1}$-6, it is sufficient to prove in M

$$(\vec{t}, r \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta \\ \rightarrow (\Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma_0}} \vee (\forall x)(\exists k \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma_0}})\pi^0_1[e, x, r, (\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma_0}})_k, (\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_\nu)_r, \mathsf{D}^0_{<\alpha}])) \\ \rightarrow (\vec{t}, r \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_\beta \\ \rightarrow (\Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}} \vee (\exists k \in \mathsf{I}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma}})(\forall x)\pi^0_1[e, x, r, ((\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_{\beta+\omega^{\gamma}})_k)_x, (\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_\nu)_r, \mathsf{D}^0_{<\alpha}])).$$

Again we build in M the hyperarithmetical hierarchy $(\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_a)_{a\prec\nu+1}$. And again we use the fundamental properties of this set hierarchy in order to prove in M the formula above. Since the proof is standard – the relevant arguments can be extracted from e.g. Case 1 of the proof of Theorem 2.5 in [2] –, we omit it. □

In a next step we reduce $\mathsf{H}_{\nu+1}\mathsf{E}^0_\alpha$ to E^0_α. This reduction together with the asymmetric interpretation of Theorem 17 will lead to an interpretation of $\mathsf{E}^0_{\alpha+1}$ into E^0_α. As mentioned we introduce a semi-formal system RA_α. RA_α is essentially an extension of $\mathsf{RA}^\star$ of Schütte (cf. [16]) by E^0_α. The language $\mathcal{L}_{RA_\alpha}$ of RA_α is similar to $\mathcal{L}^0_\alpha$. We have set variables $X^\beta, Y^\beta, Z^\beta, \dots$ for all $\beta \in \Phi_0$, and we have all predicates of $\mathcal{L}^0_\alpha$. The *number terms* of $\mathcal{L}_{RA_\alpha}$ are those of $\mathcal{L}_2$. The *set terms* $R, S, T, \dots$ of $\mathcal{L}_{RA_\alpha}$ are defined simultaneously with the *formulas* of $\mathcal{L}_{RA_\alpha}$.

1. Each X^β is a set term.

2. If φ is a $\mathcal{L}_{RA_\alpha}$ formula, then $\{x : \varphi\}$ is a set term.

3. $[\neg]K\vec{t}$, $[\neg]t \in \mathsf{D}^0_\beta$, $[\neg]t \in \mathsf{D}^0_{<\gamma}$ are $\mathcal{L}_{RA_\alpha}$ formulas for K a primitive recursive relation symbol and $\beta < \alpha$, $\gamma \leq \alpha$.

4. $[\neg]t \in T$ are $\mathcal{L}_{RA_\alpha}$ formulas for number terms t and set terms T.

5. $\mathcal{L}_{RA_\alpha}$ formulas are closed under $\wedge, \vee, \exists x, \forall x, \exists X^\beta, \forall X^\beta$ for $\beta > 0$.

The *level* of a set term and the level of a formula φ is defined by

$$\begin{aligned} lev(T) &:= \max(\{0\} \cup \{\alpha : X^\alpha \text{ occurs in } T\}), \\ lev(\varphi) &:= \max(\{0\} \cup \{\alpha : X^\alpha \text{ occurs in } \varphi\}). \end{aligned}$$

Definition 18 The rank $rk(\varphi)$ of an $\mathcal{L}_{RA_\alpha}$ formula φ and of RA_α is inductively defined as follows: If in φ there is no occurrence of an X^β or a $\{x : \psi\}$, then $rk(\varphi) := 0$. Otherwise:

1. If φ is a formula $t \in X^\beta$ or $t \notin X^\beta$, then $rk(\varphi) := \max\{1, \omega \cdot \beta\}$.
2. If φ is a formula $t \in \{x : \psi\}$ or $t \notin \{x : \psi\}$, then $rk(\varphi) := rk(\psi) + 1$.
3. If φ is a formula $\psi \vee \theta$ or $\psi \wedge \theta$, then $rk(\varphi) := \max(rk(\psi), rk(\theta)) + 1$.
4. If φ is a formula $(\exists x \psi)$ or $(\forall x \psi)$, then $rk(\varphi) := rk(\psi) + 1$.
5. If φ is a formula $(\exists X^\beta)\psi(X^\beta)$ or $(\forall X^\beta)\psi(X^\beta)$, then $rk(\varphi) := \max(\omega \cdot lev(\varphi), rk(\psi(X^0)) + 1)$.

Notice that $rk(\varphi) = rk(\neg\varphi)$. We make the following observations.

1. If $lev(\varphi) = \gamma$, then $\omega\gamma \leq rk(\varphi) < \omega(\gamma + 1)$.
2. If $lev(T) < \gamma$, then $rk(\varphi(T)) < rk(\exists X^\gamma \varphi(X^\gamma))$.

RA_α is defined as a Tait-calculus ($\alpha \in \Phi_0$). The axioms and rules are given below. Notice that these rank properties will lead to a partial cut elimination lemma.

1. Logical axioms. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all set variables X^β, all true $\mathcal{L}_1$ literals φ, all closed number terms s, t with identical value and all ordinals γ, δ with $\gamma < \alpha, \delta \leq \alpha$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X^\beta, s \notin X^\beta$$

$$\text{and} \qquad \Gamma, t \in \mathsf{D}^0_\gamma, s \notin \mathsf{D}^0_\gamma \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^0_{<\delta}, s \notin \mathsf{D}^0_{<\delta}.$$

2. Propositional rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas and all $\mathcal{L}_{RA_\alpha}$ formulas φ and ψ:

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

3. Set term rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all $\mathcal{L}_{RA_\alpha}$ formulas φ and all closed number terms t:

$$\frac{\Gamma, \varphi(t)}{\Gamma, t \in \{x : \varphi(x)\}}, \qquad \frac{\Gamma, \neg\varphi(t)}{\Gamma, t \notin \{x : \varphi(x)\}}.$$

4. Quantifier rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, all set terms T, all closed number terms s and all $\mathcal{L}_{RA_\alpha}$ formulas $\varphi(s), \psi(T)$:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)},$$

$$\frac{\Gamma, \psi(T)}{\Gamma, (\exists X^\beta)\psi(X^\beta)} \ \ lev(T) < \beta, \quad \frac{\Gamma, \psi(T) \quad \text{for all set terms } T \text{ with } lev(T) < \beta}{\Gamma, (\forall X^\beta)\psi(X^\beta)}.$$

5. E^0_α axioms and rules. For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas, for all axioms Λ_1 and all rules $\frac{\Lambda_2}{\Lambda_3}$ of the ontological axioms II and rules III and closure axioms Λ_1 and rules $\frac{\Lambda_2}{\Lambda_3}$ of E^0_α:

$$\Gamma, \Lambda_1 \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

6. Cut rules. For all finite sets Γ of closed $\mathcal{L}_{RA_\alpha}$ formulas and for all $\mathcal{L}_{RA_\alpha}$ formulas φ:

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

In the following theorem we collect the main results about RA_α. For the formulation we need the notion of a γ-instance.

Definition 19 Take an $\mathcal{L}^0_\alpha$ formula φ of $\mathsf{H}_\nu\mathsf{E}^0_\alpha$ (notice that there are no bounded second order quantifiers in φ). The $\mathcal{L}_{RA_\alpha}$ formula φ^γ is a γ-instance of φ if φ^γ is obtained from φ by

- free set variables are replaced by set terms of $\mathcal{L}_{RA_\alpha}$ with $lev < \gamma$.
- bound set variables get the superscript γ.

Theorem 20 *The following holds.*

a) *For all finite sets Γ of $\mathcal{L}_{RA_\alpha}$ formulas we have*

$$\mathsf{RA}_\alpha \mathrel{\Big|\!\!\frac{\ \gamma\ }{1+\beta+\omega^\delta}} \Gamma \quad \Longrightarrow \quad \mathsf{RA}_\alpha \mathrel{\Big|\!\!\frac{\ \varphi\delta\gamma\ }{1+\beta}} \Gamma.$$

b) *For all finite sets* Γ *of* $\mathcal{L}^0_\alpha$ *formulas of* $H_\nu \mathsf{E}^0_\alpha$*, we have for all* $\omega^{\nu+1}$*-instances* $\Gamma^{\omega^{\nu+1}}$ *of* Γ

$$H_\nu \mathsf{E}^0_\alpha \mid\!\frac{\gamma}{1}\, \Gamma \quad \Longrightarrow \quad \mathsf{RA}_\alpha \mid\!\frac{\omega^{\omega^{\nu+3}+\omega^\gamma}}{\omega^{\omega^{\nu+3}+\omega^\gamma}}\, \Gamma^{\omega^{\nu+1}}.$$

c) *For all finite sets* Γ *of* $\mathcal{L}_{RA_\alpha}$ *formulas without set terms* X^β*,* $\{x : \varphi(x)\}$ *we have*

$$\mathsf{RA}_\alpha \mid\!\frac{\gamma}{1}\, \Gamma \quad \Longrightarrow \quad \mathsf{E}^0_\alpha \mid\!\frac{\gamma}{<\omega}\, \Gamma.$$

Proof. The proof of the partial (second) cut elimination a) is standard and hence omitted (cf. for instance [11] Theorem 18.4). The proof of b) is by induction on γ. All cases beside the iterated arithmetical comprehension can be shown by standard arguments and some calculations of bounds. The relevant arguments for the embedding of iterated arithmetical comprehension in RA_α can be extracted from [4], Proposition 9. Finally, an easy induction on γ shows c). □

In Corollary 21 we write $\varepsilon(\gamma)$ for the next epsilon number above γ.

Corollary 21 *For all finite sets* $\Gamma \subset (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\alpha))^c \cup (ess\text{-}\Pi^1_1(\mathsf{D}^0_\alpha))^c$ *without an occurrence of* D^0_α *we have*

$$\mathsf{E}^0_{\alpha+1} \mid\!\frac{\gamma}{1}\, \Gamma \quad \Longrightarrow \quad \mathsf{E}^0_\alpha \mid\!\frac{<\varphi\varepsilon(\gamma)0}{1}\, \Gamma.$$

Proof. We assume that $\mathsf{E}^0_{\alpha+1} \mid\!\frac{\gamma}{1}\, \Gamma$. By Theorem 17 there exist ordinals ν, ξ less than $\varepsilon(\gamma)$ with

$$\mathsf{H}_\nu \mathsf{E}^0_\alpha \mid\!\frac{\xi}{<\omega}\, \Gamma.$$

We conclude from Theorem 20a) and 20b)

$$\mathsf{RA}_\alpha \mid\!\frac{<\varphi\varepsilon(\gamma)0}{1}\, \Gamma.$$

And from Theorem 20c) and Lemma 12

$$\mathsf{E}^0_\alpha \mid\!\frac{<\varphi\varepsilon(\gamma)0}{1}\, \Gamma.$$

□

5.2 The semi-formal systems $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$

In Theorem 17 we have interpreted $\mathsf{E}^0_{\alpha+1}$ into "Iterated arithmetical comprehension over E^0_α". In the following we give an asymmetric interpretation of $\mathsf{E}^{n+1}_{\alpha+1}$ into "E^n_ν over E^{n+1}_α". We will introduce in this subsection e.g. a semi-formal system $\mathsf{E}^{n,n+1}_{\nu,\alpha}$, which corresponds to "$\mathsf{E}^n_\nu$ over E^{n+1}_α".

We write $\vec{l} = n, n+1, \ldots, n+k$ where $n, n+1, \ldots, n+k$ are natural numbers and $\vec{\alpha} = \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k}$ where $\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k} \in \Phi_0$. The language $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ is an extension of $\mathcal{L}_1$ by the predicates $\mathsf{D}^i_{\beta_i}$, $\mathsf{D}^i_{<\gamma_i}$ for each i with $n \le i \le n+k$ and all ordinals β_i, γ_i with $\beta_i < \alpha_i$, $\gamma_i \le \alpha_i$. The *formulas* of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ are built in analogy to E^m_α: All $\mathcal{L}_1$ literals and $[\neg]t \in \mathsf{D}^i_{\beta_i}$, $[\neg]t \in \mathsf{D}^i_{<\gamma_i}$ are formulas of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ for $n \le i \le n+k$, $\beta_i < \alpha_i$, $\gamma_i \le \alpha_i$. Moreover, the formulas of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ are closed under $\wedge$, $\vee$, $\exists x$, $\forall x$. We take as *$\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$* the $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas without free number variables. The semi-formal system $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ corresponds to "$\mathsf{E}^n_{\alpha_n}$ over $\mathsf{E}^{n+1}_{\alpha_{n+1}}$ over ... over $\mathsf{E}^{n+k}_{\alpha_{n+k}}$". Hence its Tait-calculus contains the following axioms and rules of inference.

1. Ontological axioms I. For all finite sets Γ of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$, all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all $\beta_i < \alpha_i$, $\gamma_i \le \alpha_i$, $1 \le i \le n$:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^i_{\beta_i}, s \notin \mathsf{D}^i_{\beta_i} \qquad \text{and} \qquad \Gamma, t \in \mathsf{D}^i_{<\gamma_i}, s \notin \mathsf{D}^i_{<\gamma_i}.$$

2. Propositional and quantifier rules. Rules for $\wedge$, $\vee$, $\exists x$, $\forall x$ (ω-rule).

3. Ontological axioms II and rules III. For all finite sets Γ of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ and for all ontological axioms II Λ_1 and ontological rules III $\frac{\Lambda_2}{\Lambda_3}$ of the systems $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$:

$$\Gamma, \Lambda_1, \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

4. $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$ axioms and rules. For all finite sets Γ of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$, for all closure and reflection axioms Λ_1 and for all closure rules $\frac{\Lambda_2}{\Lambda_3}$ of the systems $\mathsf{E}^n_{\alpha_n}, \ldots, \mathsf{E}^{n+k}_{\alpha_{n+k}}$:

$$\Gamma, \Lambda_1, \qquad \text{and} \qquad \frac{\Gamma, \Lambda_2}{\Gamma, \Lambda_3}.$$

5. Inclusion axioms. For all finite sets Γ of $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formulas of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$, all i with $n \leq i < j \leq n+k$ and all ordinals $\beta_i < \alpha_i$:

$$\Gamma, (\exists k)((\mathsf{D}^i_{\beta_i})_k = \mathsf{D}^j_{<\alpha_j}).$$

6. Cut rules. The usual cut rules.

For $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ we introduce classes corresponding to $ess\text{-}\Sigma^1_1(\mathsf{D}^n_\beta)$ and $ess\text{-}\Pi^1_1(\mathsf{D}^n_\beta)$ with respect to $\beta < \alpha_n$.

Definition 22 We fix $\vec{l} = n, n+1, \ldots, n+k$, $\vec{\alpha} = \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k}$, $\vec{\beta} = \beta, \alpha_{n+1}, \ldots, \alpha_{n+k}$, $\beta < \alpha_n$. The classes $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ and $ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ are inductively defined as follows:

1. For all number terms $t, \vec{s}$, all primitive recursive relation symbols K, all $*_i$, all ordinals $\gamma_j \leq \alpha_j$ $(n < j \leq n+k)$ and all ordinals $\delta < \beta$ the following expressions are in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ and $ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$: $[\neg]K\vec{s}$, $[\neg]t \in \mathsf{D}^j_{\gamma_j}$, $[\neg]t \in \mathsf{D}^j_{<\gamma_j}$, $[\neg]t \in \mathsf{D}^n_\delta$, $[\neg]t \in \mathsf{D}^n_{<\delta}$, $[\neg]t \in (\mathsf{D}^n_\beta)_{*_i}$, $[\neg]t \in \mathsf{D}^n_{<\beta}$.
2. If φ, ψ are in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.), then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.).
3. If φ is in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.), then $\exists x\varphi$ and $\forall x\varphi$ are in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.).
4. If $\varphi(*_i)$ is in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.), then $\exists x\varphi[*_i\backslash x]$ ($\forall x\varphi[*_i\backslash x]$, resp.) is in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.). Here we write $\varphi[*_i\backslash x]$ for the expression φ where all occurrences of $*_i$ are substituted by x.

As in section 4 we define: $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})^c$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})^c$, resp.) is the subset of all expressions in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$ ($ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})$, resp.) which have no free number variables. The rank $rk(\varphi)$ of an $\mathcal{L}^{\vec{l}}_{\vec{\alpha}}$ formula φ of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ is now defined as follows: $rk(\varphi) = 0$ iff there is a closed term $\vec{t}$ and a $\psi[\vec{*}]$ in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})^c$ or $ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\beta}})^c$ with $\vec{\beta} = \beta, \alpha_{n+1}, \ldots \alpha_{n+k}$ and $\beta < \alpha_n$ and such that $\varphi \equiv \psi[\vec{t}]$. In order to achieve a finite reduction, we extend the methods of the preceding subsections, which led to Theorem 17, to $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$

Definition 23 Fix $\vec{l} = n+1, n+2, \dots, n+k$ and $\vec{\alpha} = \alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_{n+k}$. For each expression φ in $ess\text{-}\Sigma^1_1(\mathsf{D}^{\vec{l}}_{\vec{\alpha}})$ or in $ess\text{-}\Pi^1_1(\mathsf{D}^{\vec{l}}_{\vec{\alpha}})$ we inductively define $\varphi^{\beta,\gamma,\nu}$ as follows:

1. If there is no occurrence of $\mathsf{D}^{n+1}_{\alpha_{n+1}}$ in φ, then $\varphi^{\beta,\gamma,\nu} := \varphi$.
2. $(t \in (\mathsf{D}^{n+1}_{\alpha_{n+1}})_{*_i})^{\beta,\gamma,\nu} := t \in (\mathsf{D}^n_\nu)_{*_i}$ and $(t \notin (\mathsf{D}^{n+1}_{\alpha_{n+1}})_{*_i})^{\beta,\gamma,\nu} := t \notin (\mathsf{D}^n_\nu)_{*_i}$.
3. If φ is of the form $\theta \wedge \psi$ ($\theta \vee \psi$, resp.), then $\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu} \wedge \psi^{\beta,\gamma,\nu}$ ($\varphi^{\beta,\gamma,\nu} := \theta^{\beta,\gamma,\nu} \vee \psi^{\beta,\gamma,\nu}$, resp.).
4. If φ is of the form $\exists x\psi$ ($\forall x\psi$, resp.) such that there is no $(\mathsf{D}^{n+1}_{\alpha_{n+1}})_x$ in φ, then $\varphi^{\beta,\gamma,\nu} := \exists x\psi^{\beta,\gamma,\nu}$ ($\varphi^{\beta,\gamma,\nu} := \forall x\psi^{\beta,\gamma,\nu}$, resp.).
5. If φ is of the form $(\exists k)\psi((\mathsf{D}^{n+1}_{\alpha_{n+1}})_k)$ ($(\forall k)\psi((\mathsf{D}^{n+1}_{\alpha_{n+1}})_k)$, resp.) such that there is a $(\mathsf{D}^{n+1}_{\alpha_{n+1}})_k$ in ψ, then $\varphi^{\beta,\gamma,\nu} := (\exists k)\psi^{\beta,\gamma,\nu}((\mathsf{D}^n_\gamma)_k)$ ($\varphi^{\beta,\gamma,\nu} := (\forall k)\psi^{\beta,\gamma,\nu}((\mathsf{D}^n_\beta)_k)$, resp.).

We now formulate the asymmetric interpretation. It corresponds to the asymmetric interpretation of $\mathsf{E}^0_{\alpha+1}$ into $\mathsf{H}_\nu\mathsf{E}^0_\alpha$. We write in this interpretation $(\exists\vec{k})(\mathsf{D}^n_\beta)_{\vec{k}} = (\mathsf{D}^n_\beta)_{\vec{t}}$ for $(\exists k)(\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_{t_1}, \dots (\exists k)(\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\beta)_{t_r}$.

Theorem 24 *We set $\vec{l} = n+2, \dots, n+k$ and $\vec{\alpha} = \alpha_{n+2}, \dots, \alpha_{n+k}$ with $\alpha_{n+2}, \dots, \alpha_{n+k} \in \Phi_0$. For all finite subsets $\Gamma[\vec{*}]$ of*

$$ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1,\vec{l}}_{\alpha_{n+1},\vec{\alpha}})^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n+1,\vec{l}}_{\alpha_{n+1},\vec{\alpha}})^c,$$

for all ordinals $\alpha_{n+1}, \beta, \gamma, \nu \in \Phi_0$ with $\beta + \omega^\gamma < \nu$ there is a natural number m such that we have for all closed number terms $\vec{t}$

$$\mathsf{E}^{n+1,\vec{l}}_{\alpha_{n+1}+1,\vec{\alpha}} \overset{\gamma}{\underset{1}{\vdash}} \Gamma[\vec{t}] \quad \Longrightarrow \quad \mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}} \overset{\omega^{\beta+\omega^\gamma}}{\underset{m}{\vdash}} \neg(\exists\vec{k})(\mathsf{D}^n_\beta)_{\vec{k}} = (\mathsf{D}^n_\nu)_{\vec{t}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^\gamma,\nu}.$$

Proof. We only have to adapt the proof of Theorem 17. First, notice that we can prove for all semi-formal systems $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ properties corresponding to Lemma 14 ("substitution property") and Lemma 16 ("persistency"). Since the proof of these properties is straightforward, we omit it. We prove now the claim by induction on γ. Apart from the inclusion axioms and the $\mathsf{E}^{n+1}_{\alpha_{n+1}+1}$, $\mathsf{E}^{n+2}_{\alpha_{n+2}}, \dots,$ $\mathsf{E}^{n+k}_{\alpha_{n+k}}$ axioms and rules all axioms and rules are treated in a similar way as in Theorem 17. We first discuss the inclusion axioms. The only non trivial cases are

$$\mathsf{E}^{n+1,\vec{l}}_{\alpha_{n+1}+1,\vec{\alpha}} \vdash^{\gamma}_{1} \Gamma[\vec{t}], (\exists k)((\mathsf{D}^{n+1}_{\alpha_{n+1}})_k = \mathsf{D}^{j}_{<\alpha_j})$$

where $n+1 < j \leq n+k$. We have to prove

$$\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}} \vdash^{\omega^{\beta}+\omega^{\gamma}}_{m} \neg(\exists \vec{k})(\mathsf{D}^{n}_{\beta})_{\vec{k}} = (\mathsf{D}^{n}_{\nu})_{\vec{t}}, \Gamma[\vec{t}]^{\beta,\beta+\omega^{\gamma}}, (\exists k)((\mathsf{D}^{n}_{\beta+\omega^{\gamma}})_k = \mathsf{D}^{j}_{<\alpha_j}).$$

(We write $\varphi^{\delta,\varepsilon}$ for $\varphi^{\delta,\varepsilon,\nu}$.) Since this is an inclusion axiom of $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$, we are done. There remain the $\mathsf{E}^{n+1}_{\alpha_{n+1}+1}$, $\mathsf{E}^{n+2}_{\alpha_{n+2}}$, ..., $\mathsf{E}^{n+k}_{\alpha_{n+k}}$ axioms and rules. We assume that $\mathsf{D}^{n+1}_{\alpha_{n+1}}$ occurs in $\Gamma[\vec{t}]$ – the other cases are immediate. Recall that in the proof of Theorem 17 we have built the hyperarithmetical hierarchy $(\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_b)_{b \prec \nu}$. Here we have a hierarchy $(\mathsf{D}^n_\beta)_{\beta<\nu}$. In this asymmetric interpretation here, each D^n_β corresponds to $\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_b$ $(b = \beta)$ and vice versa. And since the properties of D^n_β are analogous to the properties of $\mathcal{H}^{\mathsf{D}^0_{<\alpha}}_b$ (in fact stronger), the argumentation is very similar as in Theorem 17. Hence the relevant arguments for the closure of $\mathsf{D}^{n+1}_{\alpha_{n+1}}$ under disjoint union and Π^1_0 comprehension can be extracted from Theorem 17 and the relevant arguments for the closure under Σ^1_1-DC can be extracted from Theorem 3.1 in [2] or from Theorem 13 in [8].

There remain the reflection axioms. It is sufficient to show that we can prove in $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$ with finite deduction length

$$\neg(\exists k)(\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\nu)_t, (\exists k)((\mathsf{D}^n_\nu)_t \mathrel{\dot{\in}} (\mathsf{D}^n_{\beta+\omega^\gamma})_k \wedge \mathsf{I}_n((\mathsf{D}^n_{\beta+\omega^\gamma})_k)).$$

We assume $n > 0$, the case $n = 0$ is immediate. As in the proof of Theorem 17 we introduce a theory M. M is formulated in $\mathcal{L}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$ and tailored in such a way that $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$ is the Tait-style version of M. In particular M is based on the usual axioms and rules of one-sorted predicate calculus, and M contains all axioms and rules corresponding to the axioms and rules 1, 3-6 of $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$. And again we argue in M and then embed into $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$. We have in M

$$(\forall l)(\exists k)((\mathsf{D}^n_\beta)_l \mathrel{\dot{\in}} (\mathsf{D}^n_\beta)_k \wedge \mathsf{I}_{n-1}((\mathsf{D}^n_\beta)_k)).$$

Since we also know $(\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^{\mathsf{D}^n_\beta}$, we conclude that $\mathsf{I}_n(\mathsf{D}^n_\beta)$ holds. Hence we can prove in M

$$(\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\nu)_t) \rightarrow ((\mathsf{D}^n_\nu)_t \mathrel{\dot{\in}} \mathsf{D}^n_\beta \wedge \mathsf{I}_n(\mathsf{D}^n_\beta)).$$

We know $\mathsf{D}^n_\beta \dot{\in} \mathsf{D}^n_{\beta+\omega^\gamma}$, thus

$$\neg(\exists k)((\mathsf{D}^n_\beta)_k = (\mathsf{D}^n_\nu)_t) \vee (\exists k)((\mathsf{D}^n_\nu)_t \dot{\in} (\mathsf{D}^n_{\beta+\omega^\gamma})_k \wedge \mathsf{I}_n((\mathsf{D}^n_{\beta+\omega^\gamma})_k)). \tag{12}$$

Notice that we do not have used induction in this argumentation. Hence we can prove (12) in $\mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}}$ with finite deduction length. This is the claim. $\square$

The following corollary is an immediate consequence.

Corollary 25 *We set $\vec{l} = n+2,\ldots,n+k$ and $\vec{\alpha} = \alpha_{n+2},\ldots,\alpha_{n+k}$ with $\alpha_{n+2},\ldots,\alpha_{n+k} \in \Phi_0$. For all finite subsets Γ of*

$$ess\text{-}\Sigma^1_1(\mathsf{D}^{n+1,\vec{l}}_{\alpha_{n+1},\vec{\alpha}})^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n+1,\vec{l}}_{\alpha_{n+1},\vec{\alpha}})^c,$$

without occurrences of $\mathsf{D}^{n+1}_{\alpha_{n+1}}$ and for all ordinals $\alpha_{n+1} \in \Phi_0$ there is a natural number m such that we have

$$\mathsf{E}^{n+1,\vec{l}}_{\alpha_{n+1}+1,\vec{\alpha}} \mathrel{\Big|\!\frac{\gamma}{<\omega}} \Gamma[\vec{t}] \quad \Longrightarrow \quad \text{there is an ordinal } \nu < \varepsilon(\gamma) \text{ with } \mathsf{E}^{n,n+1,\vec{l}}_{\nu,\alpha_{n+1},\vec{\alpha}} \mathrel{\Big|\!\frac{\varepsilon(\gamma)}{m}} \Gamma.$$

6 Transfinite reduction

The transfinite reductions in our context are very similar to the reduction of transfinitely many fixed points (cf. [7] Main Lemma II) or to the reduction of transfinitely many n-inaccessibles (cf. [9] Theorem 10). Roughly spoken, the hard part is the finite reduction, since usually for that we need asymmetric interpretations and embeddings and "back-embeddings". On the other hand, when we inspect the proofs of the transfinite reductions we see that nearly nothing happens: The initial step of the induction follows from the finite reduction, and the induction step essentially follows from the induction hypothesis. Again we distinguish two cases: E^0_α and $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$. We start with the first case.

6.1 Transfinite reduction of E^0_α

The following theorem corresponds to Main Lemma II in [7]. Also the proof is very similar.

Theorem 26 *Assume* $\mathsf{E}^0_{\beta+\omega^{1+\rho}} \overset{\alpha}{\underset{1}{\vdash}} \Gamma$ *for a finite set*

$$\Gamma \subset \bigcup_{\delta<\beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c).$$

Then we have for all ordinals ξ *less than* $\omega^{1+\rho}$

$$\Gamma \subset \bigcup_{\delta<\beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^0_\delta)^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^0_\delta)^c) \quad \Longrightarrow \quad \mathsf{E}^0_{\beta+\xi} \overset{\varphi 1\rho\alpha}{\underset{1}{\vdash}} \Gamma.$$

Proof. We follow the proof of Main Lemma II in [7]. We prove the claim by main induction on ρ and side induction on α. We distinguish the cases $\rho = 0$, ρ is a successor or ρ is a limit ordinal. Here we discuss only the case $\rho = 0$, since the relevant arguments for the other cases can easily be extracted from the corresponding cases in the proof of Main Lemma II in [7]. That proof and the proof here differ only in the underlying theories.

Let us assume that $\rho = 0$ and that Γ is a finite set of $\mathcal{L}^0_{\beta+n}$ formulas of $\mathsf{E}^0_{\beta+n}$ for some natural number n so that $\mathsf{E}^0_{\beta+\omega} \overset{\alpha}{\underset{1}{\vdash}} \Gamma$. If Γ is an axiom of $\mathsf{E}^0_{\beta+n}$, then the claim is trivial. Furthermore, if Γ is the conclusion of a rule different from the cut rule, the claim is immediate from the induction hypothesis. Hence, the only critical case comes up if Γ is the conclusion of a cut-rule. Then there exist a natural number $m \geq n$, ordinals $\alpha_0, \alpha_1 < \alpha$ and an $\mathcal{L}^0_{\beta+m}$ formula φ such that all $\mathsf{D}^0_\theta, \mathsf{D}^0_{<\lambda}$ in φ fulfill $\lambda, \theta < \beta + m$ and such that

$$\mathsf{E}^0_{\beta+\omega} \overset{\alpha_0}{\underset{1}{\vdash}} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^0_{\beta+\omega} \overset{\alpha_1}{\underset{1}{\vdash}} \Gamma, \neg\varphi.$$

By the induction hypothesis we can conclude that

$$\mathsf{E}^0_{\beta+m} \overset{\varphi 10\alpha_0}{\underset{1}{\vdash}} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^0_{\beta+m} \overset{\varphi 10\alpha_1}{\underset{1}{\vdash}} \Gamma, \neg\varphi$$

and an application of the cut-rule yields $\mathsf{E}^0_{\beta+m} \overset{\gamma}{\underset{<\omega}{\vdash}} \Gamma$ for

$$\gamma := \max(\varphi 10\alpha_0, \varphi 10\alpha_1) + 1.$$

Partial cut elimination (Lemma 12) gives $\mathsf{E}^0_{\beta+m} \overset{<\varepsilon(\gamma)}{\underset{1}{\vdash}} \Gamma$. If $m = n$, we are done. Otherwise, successive application of Corollary 21 (finite reduction) and partial cut elimination gives

$$\mathsf{E}^0_{\beta+n} \overset{\varphi 10\alpha}{\underset{1}{\vdash}} \Gamma.$$

□

Notice that we have proved in Corollary 21 a reduction of $\mathsf{E}^0_{\alpha+1}$ to E^0_α. Inspecting the arguments which led to Corollary 21, we see that we can adapt the arguments obtaining a reduction of $\mathsf{E}^{0,\vec{l}}_{\alpha+1,\vec{\alpha}}$ to $\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$. The only difference is that now we have $\mathsf{E}^0_{\alpha+1}$ over $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$ (and not only $\mathsf{E}^0_{\alpha+1}$). We can adapt Lemma 12 and Lemma 14 and introduce semi-formal systems $\mathsf{H}_\nu\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$ too. There are no problems to generalize the asymmetric interpretation (Theorem 17) to embed $\mathsf{H}_\nu\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$ into a system of ramified analysis over $\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$ and to back-embed the first order part into $\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$. We can use this reduction of $\mathsf{E}^{0,\vec{l}}_{\alpha+1,\vec{\alpha}}$ to $\mathsf{E}^{0,\vec{l}}_{\alpha,\vec{\alpha}}$ in order to obtain Theorem 27, a generalized version of Theorem 24. Since the proof is straightforward, we omit it.

Theorem 27 *Let $\vec{l} = 1, \ldots, k$ be a vector of natural numbers and let $\vec{\alpha} = \alpha_1, \ldots, \alpha_k$ be a vector of elements of Φ_0. Assume $\mathsf{E}^{0,\vec{l}}_{\beta+\omega^{1+\rho},\vec{\alpha}} \vdash^{\gamma}_{1} \Gamma$ for a finite set*

$$\Gamma \subset \bigcup_{\delta<\beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^{0,\vec{l}}_{\delta,\vec{\alpha}})^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{0,\vec{l}}_{\delta,\vec{\alpha}})^c).$$

Then we have for all ordinals ξ less than $\omega^{1+\rho}$

$$\Gamma \subset \bigcup_{\delta<\beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^{0,\vec{l}}_{\delta,\vec{\alpha}})^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{0,\vec{l}}_{\delta,\vec{\alpha}})^c) \quad\Longrightarrow\quad \mathsf{E}^{0,\vec{l}}_{\beta+\xi,\vec{\alpha}} \vdash^{\varphi 1\rho\gamma}_{1} \Gamma.$$

6.2 Transfinite reduction of $\mathsf{E}^{\vec{l}}_{\vec{\alpha}}$

We give in this subsection a kind of iteration of Theorem 27.

Theorem 28 *Let $\vec{l} = n+1, \ldots, n+k$ be a vector of natural numbers and let $\vec{\alpha} = \alpha_{n+1}, \ldots, \alpha_{n+k}$ be a vector of elements of Φ_0. Assume $\mathsf{E}^{n,\vec{l}}_{\beta+\omega^{1+\rho},\vec{\alpha}} \vdash^{\alpha}_{1} \Gamma$ for a finite subset*

$$\Gamma \subset \bigcup_{\delta<\beta+\omega^{1+\rho}} (ess\text{-}\Sigma^1_1(\mathsf{D}^{n,\vec{l}}_{\delta,\vec{\alpha}})^c) \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n,\vec{l}}_{\delta,\vec{\alpha}})^c).$$

Then we have for all ξ less than $\omega^{1+\rho}$

$$\Gamma \subset \bigcup_{\delta<\beta+\xi} (ess\text{-}\Sigma^1_1(\mathsf{D}^{n,\vec{l}}_{\delta,\vec{\alpha}})^c \cup ess\text{-}\Pi^1_1(\mathsf{D}^{n,\vec{l}}_{\delta,\vec{\alpha}})^c) \quad\Longrightarrow\quad \mathsf{E}^{n,\vec{l}}_{\beta+\xi,\vec{\alpha}} \vdash^{\varphi(n+1)\rho\alpha}_{1} \Gamma.$$

Proof. The proof is by meta-induction on n. The case $n = 0$ is exactly Theorem 27. It remains to prove the claim for $n > 0$. Therefore, we assume $n > 0$. We prove the claim by main induction on ρ and side induction on α. We distinguish the cases $\rho = 0$, ρ is a successor or ρ is a limit ordinal. Here we discuss again only the case $\rho = 0$, since for the other two cases we refer to the corresponding cases in the proof of Main Lemma II in [7].

Let us assume $\rho = 0$ and that Γ is a finite set of $\mathcal{L}^{n,\vec{l}}_{\beta+l,\vec{\alpha}}$ formulas of $\mathsf{E}^{n,\vec{l}}_{\beta+l,\vec{\alpha}}$ for some natural number l so that $\mathsf{E}^{n,\vec{l}}_{\beta+\omega,\vec{\alpha}} \vdash^{\alpha}_{1} \Gamma$. Again, the only critical case comes up if Γ is the conclusion of a cut-rule. Then there exist a natural number $m \geq l$, ordinals $\alpha_0, \alpha_1 < \alpha$ and a $\mathcal{L}^{n,\vec{l}}_{\beta+m,\vec{\alpha}}$ formula φ such that all $\mathsf{D}^n_\theta, \mathsf{D}^n_{<\lambda}$ in φ fulfill $\lambda, \theta < \beta + m$ and such that

$$\mathsf{E}^{n,\vec{l}}_{\beta+\omega,\vec{\alpha}} \vdash^{\alpha_0}_{1} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^{n,\vec{l}}_{\beta+\omega,\vec{\alpha}} \vdash^{\alpha_1}_{1} \Gamma, \neg\varphi.$$

By the induction hypothesis we can conclude that

$$\mathsf{E}^{n,\vec{l}}_{\beta+m,\vec{\alpha}} \vdash^{\varphi(n+1)0\alpha_0}_{1} \Gamma, \varphi \qquad \text{and}$$
$$\mathsf{E}^{n,\vec{l}}_{\beta+m,\vec{\alpha}} \vdash^{\varphi(n+1)0\alpha_1}_{1} \Gamma, \neg\varphi.$$

An application of the cut rule yields $\mathsf{E}^{n,\vec{l}}_{\beta+m,\vec{\alpha}} \vdash^{\gamma}_{<\omega} \Gamma$ for

$$\gamma := \max(\varphi(n+1)0\alpha_0, \varphi(n+1)0\alpha_1) + 1.$$

Partial cut elimination gives $\mathsf{E}^{n,\vec{l}}_{\beta+m,\vec{\alpha}} \vdash^{<\varepsilon(\gamma)}_{1} \Gamma$. (We have not proved partial cut elimination for $\mathsf{E}^{n-1,n,\vec{l}}_{\nu,\beta+m-1,\vec{\alpha}}$. But it is clear that we can do this as for E^{n-1}_{ν}, since the mainformulas of the added axioms and rules have cut rank 0.) If $m = l$, we are done. Otherwise an application of Corollary 25 yields

$$\mathsf{E}^{n-1,n,\vec{l}}_{\nu,\beta+m-1,\vec{\alpha}} \vdash^{<\varepsilon(\gamma)}_{<\omega} \Gamma$$

for a ν less than $\varepsilon(\gamma)$. Hence

$$\mathsf{E}^{n-1,n,\vec{l}}_{\nu,\beta+m-1,\vec{\alpha}} \vdash^{<\varepsilon(\gamma)}_{1} \Gamma.$$

Now, we use the meta-induction hypothesis and conclude that

$$\mathsf{E}^{n-1,n,\vec{l}}_{0,\beta+m-1,\vec{\alpha}} \vdash^{<\varphi n\varepsilon(\gamma)0}_{1} \Gamma.$$

Since $\mathsf{E}^{n-1,n,\vec{l}}_{0,\beta+m-1,\vec{\alpha}}$ is just $\mathsf{E}^{n,\vec{l}}_{\beta+m-1,\vec{\alpha}}$ we have $\mathsf{E}^{n,\vec{l}}_{\beta+m-1,\vec{\alpha}} \vdash^{<\varphi n\varepsilon(\gamma)0}_{1} \Gamma$. We do this again and again until we have $m-1=l$. Therefore

$$\mathsf{E}^{n,\vec{l}}_{\beta+l,\vec{\alpha}} \vdash^{\varphi(n+1)0\alpha}_{1} \Gamma.$$

□

6.3 Proof-theoretic upper bound of T^{n+1}_m and T^0_α

In this subsection we collect the results of the preceding subsections. Moreover we will present these results in such a form that we can directly apply them in the proof-theoretic analysis of our theories. We write $\mathsf{PA}^\star$ for a Tait-style reformulation (with ω-rule) of the Peano arithmetic PA. We can take E^0_0 as $\mathsf{PA}^\star$. Recall that E^0_0 is formulated in $\mathcal{L}_1$ extended by $\mathsf{D}^0_{<0}$. $\mathsf{D}^0_{<0}$ can be interpreted as the empty set. The Tait-calculus of E^0_0 is given by E^0_0-1, E^0_0-2, E^0_0-3, E^0_0-4, E^0_0-9. For a formula φ of $\mathcal{L}_1$ extended by $\mathsf{D}^0_{<0}$ we can define a (new) rank $rk(\varphi)$. We set $rk(\varphi)=0$ iff φ is an $\mathcal{L}_1$ literal or $t \in \mathsf{D}^0_{<0}$, $t \notin \mathsf{D}^0_{<0}$, t a closed number term. Hence we can prove full (predicative) cut elimination with respect to this rank definition (cf. e.g. [11])

$$\mathsf{E}^0_0 \vdash^{\alpha}_{k} \Gamma \quad\Longrightarrow\quad \mathsf{E}^0_0 \vdash^{\omega_k(\alpha)}_{0} \Gamma.$$

Theorem 29 *Assume that α is an ordinal less than Φ_0 given in the form*

$$\alpha = \omega^{1+\alpha_n} + \omega^{1+\alpha_{n-1}} + \ldots + \omega^{1+\alpha_1} + m$$

for ordinals $\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1$ and $m < \omega$. We set

$$(\alpha|0) := \varepsilon(\alpha) \qquad \textit{and} \qquad (\alpha|m+1) := \varphi(\alpha|m)0$$
$$\textit{and} \qquad \delta := \varphi 1\alpha_n(\varphi 1\alpha_{n-1}(\ldots\varphi 1\alpha_1(\alpha|m)\ldots)).$$

Then we have for all sentences φ of $\mathcal{L}_1$ and for all ordinals $\nu < \varepsilon(\alpha)$

$$\mathsf{T}^0_\alpha \vdash^{\nu}_{<\omega} \varphi \quad\Longrightarrow\quad \mathsf{PA}^\star \vdash^{<\delta}_{0} \varphi.$$

Proof. We assume $\mathsf{T}^0_\alpha \vdash^{\nu}_{<\omega} \varphi$. From Lemma 10, 12 and 13 we conclude that $\mathsf{E}^0_\alpha \vdash^{<\varepsilon(\nu)}_{1} \varphi$. Applying m-times Corollary 21 leads to

$$\mathsf{E}^0_{\omega^{1+\alpha_n}+\ldots+\omega^{1+\alpha_1}} \vdash^{<(\alpha|m)}_{1} \varphi.$$

We now use n-times Theorem 26 and conclude $\mathsf{E}^0_0 \vdash^{<\delta}_{1} \varphi$. We obtain the claim by predicative cut elimination (in PA^*). □

Theorem 30 *We set* $\gamma_{\nu,0} := \varepsilon(\nu)$ *and* $\gamma_{\nu,k+1} := \varphi n \gamma_{\nu,k} 0$ *for* $n > 0$. *Then we have for all sentences of* $\mathcal{L}_1$ *and for* $n > 0$

$$\mathsf{T}^n_m \vdash^{\nu}_{<\omega} \varphi \quad \Longrightarrow \quad \mathsf{PA}^\star \vdash^{\gamma_{\nu,m}}_{0} \varphi.$$

Proof. We assume $\mathsf{T}^n_m \vdash^{\nu}_{<\omega} \varphi$. Lemma 10 and Lemma 13 lead to $\mathsf{E}^n_m \vdash^{<\varepsilon(\nu)}_{1} \varphi$. By induction on m we prove

$$\mathsf{E}^n_m \vdash^{\beta}_{1} \varphi \quad \Longrightarrow \quad \mathsf{PA}^\star \vdash^{\gamma_{\beta,m}}_{0} \varphi$$

which implies the claim. If $m = 0$, we embed E^n_0 into $\mathsf{PA}^\star$ by interpreting all $\mathsf{D}^n_{<0}$ as $\mathsf{D}^0_{<0}$ and get the claim by predicative cut elimination. Now we assume $m > 0$. We conclude from Corollary 25 that there is an ordinal α less than $\varepsilon(\beta)$ with

$$\mathsf{E}^{n-1,n}_{\alpha,m} \vdash^{\alpha}_{<\omega} \varphi.$$

An application of Theorem 28 gives

$$\mathsf{E}^{n-1,n}_{0,m-1} \vdash^{\varphi n \alpha \alpha}_{1} \varphi.$$

This is $\mathsf{E}^n_{m-1} \vdash^{\varphi n \alpha \alpha}_{1} \varphi$. Now, the induction hypothesis implies the claim. □

7 Proof-theoretic strengths

In this section we finish the proof-theoretic analysis of $\Sigma^1_1\text{-}\mathsf{TDC}_0$. The lower bound is given in Corollary 9. It remains the determination of the upper bound. In order to achieve this, we use the equivalence of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ and $\Sigma^1_1\text{-}\mathsf{TDC}_0$ (cf. [14]). We first reduce $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to $\bigcup_{n\in\mathbb{N}} \mathsf{I}_n\text{-}\mathsf{RFN}_0$ by a symmetric interpretation. Secondly, using an asymmetric interpretation, we reduce $\bigcup_{n\in\mathbb{N}} \mathsf{I}_n\text{-}\mathsf{RFN}_0$ to $\bigcup_{n\in\mathbb{N}} \mathsf{T}^n_m$.

In the following we let $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ denote a Tait-style reformulation of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$. Note that in this Tait-calculus $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ is formulated as the rule

$$\frac{\Gamma, (\exists Y)\varphi[\vec{x}, X, Y, \vec{Z}]}{\Gamma, (\exists M)(\vec{Z} \mathrel{\dot\in} M \land (\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^M \land ((\forall X)(\exists Y)\varphi[\vec{x}, X, Y, \vec{Z}])^M)}$$

for all Π^1_0 formulas $\varphi[\vec{x}, X, Y, \vec{Z}]$. The arithmetic comprehension is formulated as

$$\Gamma, (\exists X)(\forall x)(x \in X \leftrightarrow \varphi[x, \vec{z}, \vec{Z}])$$

for each Π^1_0 formula $\varphi[x, \vec{z}, Z]$ and set induction has now the form

$$\Gamma, \neg 0 \in X, (\exists x)(x \in X \wedge \neg(x+1) \in X), (\forall x)(x \in X).$$

These mathematical axioms and rules are extended by rules for $\vee$, $\wedge$, $\exists x$, $\forall x$, $\exists X$, $\forall X$, by cut rules and by axioms Γ, φ and $\Gamma, t \in X, s \notin X$ for each true $\mathcal{L}_1$ literal φ and all closed number terms t, s with $t = s$. Of course these Tait-style reformulation of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ is tailored in such a way that we can embed $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ into it. For all $\mathcal{L}_2$ formulas φ there exists a natural number n such that

$$(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi[\vec{x}] \quad \Longrightarrow \quad ((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{\overset{n}{\vdash}} \varphi[\vec{t}]$$

holds for all closed number terms $\vec{t}$. Next we define the cut rank of a formula φ. We set $rk(\varphi) = 0$ iff φ is a Σ^1_1 or a Π^1_1 formula. Then one readily notes that the mainformulas of the mathematical axioms and rules of $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ have cut rank 0. As a consequence we obtain the following partial cut elimination

$$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{\overset{n}{\underset{k+1}{\vdash}}} \Gamma \quad \Longrightarrow \quad ((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{\overset{2^n}{\underset{1}{\vdash}}} \Gamma$$

and finally

$$(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \quad \Longrightarrow \quad ((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{\overset{<\omega}{\underset{1}{\vdash}}} \varphi$$

for each $\mathcal{L}_1$ sentence φ. Let us now formulate the reduction of $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ to $\bigcup_{n \in \mathbb{N}} \mathsf{I}_n\text{-}\mathsf{RFN}_0$. For an analogous reduction in the context of set theory we refer to [9].

Theorem 31 *For all finite sets $\Gamma \subset \Sigma^1_1$ of closed $\mathcal{L}_2$ formulas, all arithmetic sentences φ and all $n \in \mathbb{N}$ we have*

a) $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{\overset{n}{\underset{1}{\vdash}}} \Gamma[\vec{Z}] \quad \Longrightarrow \quad \mathsf{ACA}_0 \vdash \neg \mathsf{I}_{n+1}(D), \vec{Z} \mathrel{\dot{\notin}} D, \Gamma^D[\vec{Z}],$

b) $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \quad \Longrightarrow \quad \bigcup_{n \in \mathbb{N}} \mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash \varphi.$

Proof. Assertion b) follows from assertion a), since in $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ we have sets D with $\mathsf{I}_n(D)$ and since we can embed $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ into its Tait-calculus. Thus, we have to show a). The proof is by induction on n. We discuss only the case where Γ is the conclusion of the $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$-rule, since the other cases follow easily from the induction hypothesis and the definition of $\mathsf{I}_{n+1}(D)$. Hence, assume that $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ proves with deduction length $n > 0$

$$\Gamma[\vec{P}], (\exists M)(\vec{Z} \dot{\in} M \wedge (Ax_{\Sigma^1\text{-}\mathsf{DC}})^M \wedge \varphi^M), \tag{13}$$

where φ is of the form $(\forall X)(\exists Y)\psi(X, Y, \vec{Z})$ and all free set parameters of $\psi \in \Pi^1_0$ are among $X, Y, \vec{Z}$. We have to prove in ACA_0

$$\neg \mathsf{I}_{n+1}(D), \vec{P}, \vec{Z} \dot{\notin} D, \Gamma^D[\vec{P}], (\exists M \dot{\in} D)(\vec{Z} \dot{\in} M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M).$$

First, we notice that we have

$$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{|\!\frac{\,n-1\,}{1}} \Gamma[\vec{P}], (\forall X)(\exists Y)\psi(X, Y, \vec{Z}).$$

We can prove $(\forall X)$-inversion in $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$. Hence

$$((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T \mathrel{|\!\frac{\,n-1\,}{1}} \Gamma[\vec{P}], (\exists Y)\psi(V, Y, \vec{Z}),$$

V a fresh variable. Now we apply the induction hypothesis and obtain

$$\mathsf{ACA}_0 \vdash \neg \mathsf{I}_n(D), (\vec{Z}, \vec{P}, V \dot{\notin} D), \Gamma^D[\vec{P}], (\exists Y \dot{\in} D)\psi(V, Y, \vec{Z}). \tag{14}$$

From now on we argue within ACA_0. Choose a set C with $\mathsf{I}_{n+1}(C)$ and $\vec{P}, \vec{Z} \in C$. We have to show

$$\Gamma^C[\vec{P}], (\exists M \dot{\in} C)(\vec{Z} \dot{\in} M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M). \tag{15}$$

Since we have $\mathsf{I}_{n+1}(C)$, there is an M in C with $\vec{P}, \vec{Z} \dot{\in} M$ and $\mathsf{I}_n(M)$. Using (14), we obtain

$$V \dot{\notin} M, \Gamma^M[\vec{P}], (\exists Y \dot{\in} M)\psi(V, Y, \vec{Z}).$$

That is $\Gamma^M[\vec{P}], \varphi^M$. Notice that we have transitivity in C, i.e. $G \dot{\in} M \dot{\in} C$ implies $G \dot{\in} C$. This fact follows from $\mathsf{I}_{n+1}(C)$, in particular it follows from arithmetical comprehension in C. Since we know $G = (M)_k$ for a k, there is an F in C with

$$(\forall x)(x \in F \leftrightarrow \langle x, k \rangle \in M).$$

Hence $G \dot{\in} C$. Using this transitivity and the fact that Γ is a disjunction of Σ^1_1 formulas we have also $\Gamma^C[\vec{P}], \varphi^M$. Furthermore, we have $n > 0$ and hence $(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M$. Thus

$$\Gamma^C[\vec{P}], (\exists M \dot{\in} C)(\vec{Z} \dot{\in} M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M).$$

But this is exactly (15). □

In a next step we asymmetrically interpret $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ into $\bigcup_{k\in\mathbb{N}} \mathsf{T}^n_k$. Again we use a Tait-style reformulation $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ of $\mathsf{I}_n\text{-}\mathsf{RFN}_0$. $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ is the Tait-calculus $((\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ without the $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$-rule , but with

$$\Gamma, (\exists Y)(X \dot{\in} Y \wedge \mathsf{I}_n(Y))$$

In $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ we set $rk(\varphi) = 0$ iff φ is a Σ^1_1 or a Π^1_1 formula. Hence we can prove partial cut elimination and we can embed $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ into $(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T$ such that the deduction lengths are finite. Combining embedding and partial cut elimination we conclude that for all $\mathcal{L}_2$ formulas $\varphi[\vec{x}]$ there is a natural number k such that

$$\mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash \varphi[\vec{x}] \quad \Longrightarrow \quad (\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T \mathrel{\vert\!\frac{k}{1}} \varphi[\vec{t}]$$

holds for all closed number terms $\vec{t}$. Now, we introduce a translation. For each $\mathcal{L}_2$ formula φ we define a $\mathcal{L}^n_{\max(k,l)+1}$ formula $\varphi^{k,l}$. If there are no second order quantifiers $\exists X, \forall X$ in φ, we set $\varphi^{k,l} \equiv \varphi$. Otherwise we set $(\exists X \psi)^{k,l} = (\exists X \dot{\in} \mathsf{D}^n_l)\psi^{k,l}$ and $(\forall X \psi)^{k,l} = (\forall X \dot{\in} \mathsf{D}^n_k)\psi^{k,l}$. This is inductively extended to the whole class of $\mathcal{L}_2$ formulas. We now formulate the asymmetric interpretation.

Theorem 32 *For all i, j, l with $i + 2^j < l$, for all finite sets $\Gamma[\vec{x}, \vec{X}]$ of $\mathcal{L}_2$ formulas there exists a natural number k such that*

$$(\mathsf{I}_n\text{-}\mathsf{RFN}_0)^T \mathrel{\vert\!\frac{j}{1}} \Gamma[\vec{x}, \vec{X}] \implies \mathsf{T}^n_l \mathrel{\vert\!\frac{\omega^{\omega^{i+\omega^j}}}{k}} \vec{X} \dot{\notin} \mathsf{D}^n_i, \Gamma^{i,i+2^j}[\vec{t}, \vec{X}]$$

for all closed number terms $\vec{t}$.

Proof. This theorem is proved by induction on j. Apart from $(\mathsf{I}_n\text{-}\mathsf{RFN})$ all axioms and rules of inferences are treated as in similar asymmetric interpretations, cf. e.g. [2]. Now suppose that Γ is a reflection axiom $\Gamma, (\exists Y)(X \dot{\in} Y \wedge \mathsf{I}_n(Y))$. It is sufficient to prove that we can prove in T^n_l

$$X \dot{\notin} \mathsf{D}^n_i, (\exists Y \dot{\in} \mathsf{D}^n_{i+2^j})(X \dot{\in} Y \wedge \mathsf{I}_n(Y)) \tag{16}$$

with finite deduction length. We assume $n > 0$. (The case $n = 0$ is in fact easier.) Using T^n_l-8 we obtain

$$(\forall U \dot{\in} \mathsf{D}^n_i)(\exists V \dot{\in} \mathsf{D}^n_i)(U \dot{\in} V \wedge \mathsf{I}_{n-1}(V)).$$

And using T^n_l-6 and T^n_l-7, we obtain $(\mathsf{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^{\mathsf{D}^n_i}$. We conclude $\mathsf{I}_n(\mathsf{D}^n_i)$ – with finite deduction length. Hence

$$X \dot{\notin} \mathsf{D}^n_i, X \dot{\in} \mathsf{D}^n_i \wedge \mathsf{I}_n(\mathsf{D}^n_i).$$

Since we can prove in T^n_l that D^n_i is a set in $\mathsf{D}^n_{i+2^j}$, we can show (16) with finite deduction length. □

Finally we obtain the following theorem.

Theorem 33 *We have for all arithmetic sentences φ*

a) $\mathsf{I}_n\text{-}\mathsf{RFN}_0 \vdash \varphi \quad \Longrightarrow \quad$ *There is an m with* $\mathsf{T}^n_m \vdash^{<\varepsilon_0}_{<\omega} \varphi$

b) $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0 \vdash \varphi \quad \Longrightarrow \quad$ *There is an n and an m with* $\mathsf{T}^n_m \vdash^{<\varepsilon_0}_{<\omega} \varphi$.

Using the results of the preceding sections, we obtain the following theorem.

Theorem 34 $|(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0| = |\Sigma^1_1\text{-}\mathsf{TDC}_0| = \varphi\omega 00.$

Proof. From [14] we know that $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0$ and $\Sigma^1_1\text{-}\mathsf{TDC}_0$ are equivalent. In particular we have $|\Sigma^1_1\text{-}\mathsf{TDC}_0| = |(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0|$. The lower bound of $\Sigma^1_1\text{-}\mathsf{TDC}_0$ is stated in Corollary 9a). And from Theorem 33b) and Theorem 30 we can take the upper bound. □

In $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}_0$ complete induction is restricted to sets. The methods applied before also provide an upper bound for $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ where we have complete induction for arbitrary formula. The pattern of the argument is as follows. Let us write $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ for a Tait-calculus of $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$, where we have the ω-rule. We can embed $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ into this Tait-calculus, thus getting rid of full complete induction in favor of infinite derivation lengths. Hence

$$(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \quad \Longrightarrow \quad ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})^T \vdash^{<\varepsilon_0}_{1} \varphi$$

for each arithmetic sentence φ. Infinite derivations in $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})^T$ are modeled in infinite unions of theories $\mathsf{I}_\alpha\text{-}\mathsf{RFN}$. Following the proof of Theorem 31 we obtain

$$(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}} \vdash \varphi \quad \Longrightarrow \quad \bigcup_{\alpha<\varepsilon_0} \mathsf{I}_\alpha\text{-}\mathsf{RFN} \vdash^{<\varepsilon_0}_{1} \varphi$$

for each arithmetic sentence φ. From now on we can proceed as before, but always with families $(\mathsf{I}_\alpha : \alpha < \varepsilon_0)$ instead of families $(\mathsf{I}_n : n < \omega)$. Carrying-through everything in detail finally gives $\varphi\varepsilon_0 00$ as proof-theoretical upper bound for $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$. Using the equivalence of $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ and $(\Sigma^1_1\text{-}\mathsf{TDC})$ over ACA_0 and (5) we obtain

$$
\boxed{\begin{array}{lclcl}
|(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}| & = & |\Sigma^1_1\text{-}\mathsf{TDC}| & = & \varphi\varepsilon_0 00. \\
|(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}| & = & |\Sigma^1_1\text{-}\mathsf{TDC}_0| & = & \varphi\omega 00.
\end{array}}
$$

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