Universes in metapredicative analysis *

Christian Rüede

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Abstract

In this paper we introduce theories of universes in analysis. We discuss a non-uniform, a uniform and a minimal variant. An analysis of the proof-theoretic bounds of these systems is given, using only methods of predicative proof-theory. It turns out that all introduced theories are of proof-theoretic strength between Γ_0 and $\varphi 1 \varepsilon_0 0$.

1 Introduction

From an abstract point of view a *universe* is a collection of objects which is closed under certain constructions. The idea which leads to this concept of a universe is the following. Given some principles and operations which are (philosophically) justified, we should also accept a collection of objects satisfying these closure conditions. Consequently, this process can – maybe has to – be iterated, leading to universes of universes etc.

The concept of universes is frequently studied in constructive mathematics. In admissible set theory admissibles can be regarded as universes (cf. e.g. [5]). In Martin-Löf type theory a universe is a type of types closed under certain type constructions (cf. e.g. [11, 12]). In explicit mathematics a universe is a type of names closed under some name formation operations (cf. e.g. [9, 22]). It is the aim of this article to discuss the concept of universes in metapredicative analysis.

Metapredicativity is a new term in proof-theory. Metapredicative systems have proof-theoretic ordinals beyond Γ_0 but can still be treated by methods

*This paper is a part of the author's Ph.D. dissertation [15] of predicative proof-theory only. Recently, numerous interesting metapredicative systems have been characterized. For previous work in metapredicativity the reader is referred to Jäger [4], Jäger, Kahle, Setzer and Strahm [6], Jäger and Strahm [7, 8], Kahle [10], Rathjen [14], Rüede [15] and Strahm [22, 23, 24]. A central result of [15] is that in the context of metapredicative analysis the notions of hierarchy, reflection and universe are very natural and fruitful. In [16] we have discussed hierarchies and reflections, here we are concerned with universes.

We present three different theories of universes, a non-uniform (NUT), a uniform (UUT) and a minimal (MUT) variant. In NUT, a limit axiom asserts the existence of universes. In UUT, we can build universes using a universe operator. Finally, in MUT, we can choose minimal universes with respect to a linear ordering on the universes. Moreover we determine the proof-theoretic strength of all these theories. The proof-theoretic ordinals of the theories which we consider in this paper are most easily expressed by making use of a ternary Veblen or φ function (cf. e.g. [6]). They generate an initial section of the notation system given by Schütte's Klammersymbole [18].

	non-uniform	uniform	minimal
with set induction	$\varphi 100$	$\varphi 100$	$\varphi 100$
with formula induction	$\varphi 10\varepsilon_0$	$\varphi 1\varepsilon_0 0$	$\varphi 1\varepsilon_0 0$

2 Preliminaries

In this section we present the languages, classes of formulas, notations and abbreviations. Furthermore, we introduce some well-known subsystems of analysis.

Languages, terms, formulas and special classes of formulas

The language $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ includes *number variables* (denoted by small letters, except r, s, t), *set variables* (denoted by capital letters, except R, S, T), symbols for all primitive recursive functions and relations, the symbol $\in$ for elementhood between numbers and sets, as well as equality in the first sort and a symbol $\sim$ for forming negative literals. Furthermore, there is a unary relation symbol U for being a universe and a unary universe operator $\mathcal{U}$.

The *number terms* r, s, t of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ are defined as usual; the *set terms* R, S, T are the set variables and all expressions $\mathcal{U}(X)$, $\mathcal{U}(\mathcal{U}(X))$, ... The *positive literals* of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ are all expressions $(s = t)$, $K(s_1, \ldots, s_n)$, $s \in S$, $\mathsf{U}(S)$ for K a symbol for an n-ary primitive recursive relation. The *negative literals* of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ have the form $(\sim E)$ so that E is a positive literal. We often write $(s \neq t)$ and $(s \notin S)$ instead of $\sim(s = t)$ and $\sim(s \in X)$. The *true literals* of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ are all literals $(s = t)$, $K(s_1, \ldots s_n)$ such that $(s = t)$, $K(s_1, \ldots, s_n)$ ist true respectively. The *formulas* $\varphi, \psi, \theta, \ldots$ of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ are generated from the positive and negative literals of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ by closing against disjunction, conjunction, existential and universal number and set quantification. The *negation* $\neg\varphi$ of an $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula φ is defined by making use of De Morgan's laws and the law of double negation.

An $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula is called *arithmetic*, if it does not contain bound set variables (but possibly free set variables); for the collection of these formulas we write $\Pi^1_0(\mathsf{U},\mathcal{U})$. For the collection of all arithmetic formulas and of all $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formulas $\exists X \varphi(X)$ with $\varphi(X)$ from $\Pi^1_0(\mathsf{U},\mathcal{U})$ we write $\Sigma^1_1(\mathsf{U},\mathcal{U})$. The definitions of $\Sigma^1_k(\mathsf{U},\mathcal{U})$ and $\Pi^1_k(\mathsf{U},\mathcal{U})$ are analogous.

The language $\mathcal{L}_2(\mathsf{U})$, ($\mathcal{L}_2$, resp.) is $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ without $\mathcal{U}$ (without $\mathsf{U},\mathcal{U}$, resp.) and the language $\mathcal{L}_1$ ist $\mathcal{L}_2$ without set variables. The set terms, literals, formulas and classes of formulas of $\mathcal{L}_2(\mathsf{U})$, $\mathcal{L}_2$ and $\mathcal{L}_1$ are defined similarly.

Abbreviations, some subsystems of second order arithmetic and the proof-theoretic ordinal

In the following $\langle \ldots \rangle$ denotes a primitive recursive coding function for n-tuples $\langle t_1, \ldots, t_n \rangle$ with associated projections $(\cdot)_1, \ldots, (\cdot)_n$. Seq_n is the primitive recursive set of sequence numbers of length n. Seq denotes the primitive recursive set of sequence numbers. We write $s \in (S)_t$ for $\langle s, t \rangle \in S$ and $\vec{S}$ for $S_1, ..., S_n$.

By $\varphi[\vec{x}, \vec{X}]$ we indicate that the variables $\vec{x}$, $\vec{X}$ really occur in φ, i.e., the free variables are $\{x_1, \ldots, x_n, X_1, \ldots, X_m\}$. $\varphi(\vec{x}, \vec{X})$ just means that $\vec{x}$, $\vec{X}$ may occur in φ. $\varphi[\vec{x} \backslash \vec{t}, \vec{X} \backslash \vec{S}]$ is obtained from $\varphi[\vec{x}, \vec{X}]$ by replacing all occurrences of x_i and X_j by t_i and S_j. Similarly we define $\varphi(\vec{x} \backslash \vec{t}, \vec{X} \backslash \vec{S})$. If there is no danger of confusion we omit $\vec{x}$ and $\vec{X}$. Occasionally we use the abbreviations

$$
\begin{array}{rcl}
x \in S \oplus T & := & Seq_2 x \wedge \\
& & [((x)_1 = 1 \wedge (x)_0 \in S) \vee ((x)_1 = 2 \wedge (x)_0 \in T)], \\
S = T & := & (\forall x)(x \in S \leftrightarrow x \in T), \\
S \neq T & := & \neg S = T, \\
S \dot{\in} T & := & (\exists k)(\forall x)(x \in S \leftrightarrow \langle x, k \rangle \in T), \\
(\exists Y \dot{\in} S)\varphi(Y) & := & (\exists Y)(Y \dot{\in} S \wedge \varphi(Y)), \\
(\forall Y \dot{\in} S)\varphi(Y) & := & (\forall Y)(Y \dot{\in} S \rightarrow \varphi(Y)), \\
\vec{S} \dot{\in} T & := & S_1 \dot{\in} T \wedge \ldots \wedge S_n \dot{\in} T, \\
S \doteq T & := & (\forall X)(X \dot{\in} S \leftrightarrow X \dot{\in} T), \\
x \in field(X) & := & (\exists y)(\langle x, y \rangle \in X \vee \langle y, x \rangle \in X), \\
x \in (Y)_{Za} & := & Seq_2 x \wedge x \in Y \wedge \langle (x)_1, a \rangle \in Z.
\end{array}
$$

We often say "S is in T" for $S \dot{\in} T$. $(Y)_{Za}$ is the disjoint union of all projections $(Y)_b$ such that $\langle b, a \rangle \in Z$. For a well-ordering Z we let 0_Z denote the Z-least element in $field(Z)$ and for $a \in field(Z)$ we let $a +_Z 1$ denote the Z-successor of a. Sometimes we write aZb for $\langle a, b \rangle \in Z$.

We need some (well known) subsystems of second order arithmetic. All subsystems are based on the usual axioms and rules for two-sorted predicate calculus. The theory ACA includes defining axioms for all primitive recursive functions and relations, the induction scheme for arbitrary formulas of $\mathcal{L}_2$ and (ACA), an arithmetical comprehension axiom. The theory Σ^1_1-AC extends ACA by (Σ^1_1-AC), a Σ^1_1 choice axiom, the theory ATR extends ACA by the arithmetical transfinite recursion axiom (ATR) and the theory Σ^1_1-DC extends ACA by (Σ^1_1-DC), Σ^1_1 dependent choice. T_0 denotes the theory T with set-induction instead of the induction scheme for arbitrary formulas. More detailed descriptions of these subsystems can be found in [21].

In the following we will measure the proof-theoretic strength of formal theories in terms of their proof-theoretic ordinals. As usual we set for all primitive recursive relations $\prec$ and all formulas φ

$$
\begin{array}{rcl}
Prog(\prec, \varphi) & := & (\forall x)[(\forall y)(y \prec x \rightarrow \varphi(y)) \rightarrow \varphi(x)], \\
TI(\prec, \varphi) & := & Prog(\prec, \varphi) \rightarrow (\forall x \in field(\prec))\varphi(x).
\end{array}
$$

We say that an ordinal α is provable in T, if there is a primitive recursive well-ordering $\prec$ of order type α so that $\mathsf{T} \vdash (\forall X) TI(\prec, X)$. The proof-theoretic ordinal of T, denoted by $|\mathsf{T}|$, is the least ordinal which is not provable in T.

The classes of formulas rel-$\Sigma^1_k(\mathsf{U})$, rel-$\Sigma^1_k(\mathsf{U},\mathcal{U})$, rel-$\Pi^1_k(\mathsf{U})$ and rel-$\Pi^1_k(\mathsf{U},\mathcal{U})$

We introduce new classes of formulas. First, we define the class of formulas rel-$\Pi^1_0(\mathsf{U})$ (relative arithmetic $\mathcal{L}_2(\mathsf{U})$-formulas).

1. Each arithmetic $\mathcal{L}_2(\mathsf{U})$ formula is a rel-$\Pi^1_0(\mathsf{U})$ formula.
2. If φ and ψ are rel-$\Pi^1_0(\mathsf{U})$ formulas, so also are $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$.
3. If φ is a rel-$\Pi^1_0(\mathsf{U})$ formula, so also are $\exists x\varphi$ and $\forall x\varphi$.
4. If φ is a rel-$\Pi^1_0(\mathsf{U})$ formula, so also are $(\exists X \mathrel{\dot\in} S)\varphi$ and $(\forall X \mathrel{\dot\in} S)\varphi$.

rel-$\Sigma^1_1(\mathsf{U})$ is the collection of all rel-$\Pi^1_0(\mathsf{U})$ formulas and of all formulas $\exists X\varphi(X)$ with $\varphi(X)$ a rel-$\Pi^1_0(\mathsf{U})$ formula. rel-$\Pi^1_k(\mathsf{U})$ and rel-$\Sigma^1_k(\mathsf{U})$ are defined as usual. rel-$\Pi^1_k(\mathsf{U},\mathcal{U})$ and rel-$\Sigma^1_k(\mathsf{U},\mathcal{U})$ are similarly defined.

Let φ be an $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula. Then we mean by $\mathsf{U}(\{x : \varphi(x)\})$ the expression $(\exists X)[(\forall x)(x \in X \leftrightarrow \varphi(x)) \wedge \mathsf{U}(X)]$, and by $t \in \mathcal{U}(\{x : \varphi(x)\})$ we mean the expression $(\exists X)[(\forall x)(x \in X \leftrightarrow \varphi(x)) \wedge t \in \mathcal{U}(X)]$.

3 Definition of the theories

First we define the theory of universes NUT (Non-uniform Universes Theory). It is formulated in $\mathcal{L}_2(\mathsf{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. The non-logical axioms are:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) *equality axioms*
$X = Y \to (\mathsf{U}(X) \to \mathsf{U}(Y))$.

(3) *set operations*

(rel-$\Pi^1_0(\mathsf{U})$-CA): For all rel-$\Pi^1_0(\mathsf{U})$ formulas $\varphi(x)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x))$.

(rel-$\Sigma^1_1(\mathsf{U})$-AC): For all rel-$\Sigma^1_1(\mathsf{U})$ formulas $\varphi(x, X)$:
$(\forall x)(\exists X)\varphi(x, X) \to (\exists X)(\forall x)\varphi(x, (X)_x)$.

(4) *closure conditions for universes*

(4.1) For all $rel\text{-}\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, \vec{Z}]$:
$\mathsf{U}(D) \wedge \vec{Z} \dot{\in} D \rightarrow (\exists Y \dot{\in} D)(\forall x)(x \in Y \leftrightarrow \varphi[x, \vec{z}, \vec{Z}])$.

(4.2) For all $rel\text{-}\Pi^1_0(\mathsf{U})$ formulas $\varphi[x, \vec{z}, X, Y, \vec{Z}]$:
$\mathsf{U}(D) \wedge \vec{Z} \dot{\in} D \rightarrow (\forall x)(\exists Y \dot{\in} D)(\exists X \dot{\in} D)\varphi[x, \vec{z}, X, Y, \vec{Z}]$
$\rightarrow (\exists Y \dot{\in} D)(\forall x)(\exists X \dot{\in} D)\varphi[x, \vec{z}, X, (Y)_x, \vec{Z}]$.

(5) *non-uniform limit axioms* $(\exists D)(X \dot{\in} D \wedge \mathsf{U}(D))$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U})$.

The theory MUT (Minimal Universes Theory) is also formulated in $\mathcal{L}_2(\mathsf{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. It is a strengthening of NUT. The non-logical axioms are:

(1)-(4) same as for NUT.

(5) (5.1) *non-uniform limit axioms*
$(\exists D)(X \dot{\in} D \wedge \mathsf{U}(D))$.

(5.2) *linearity*
$\mathsf{U}(D) \wedge \mathsf{U}(E) \rightarrow D \dot{\in} E \vee D \doteq E \vee E \dot{\in} D$.

(5.3) *minimal universe axioms*
For all $\varphi(X) \in rel\text{-}\Sigma^1_1(\mathsf{U})$ and for all $\psi(X) \in rel\text{-}\Pi^1_1(\mathsf{U})$:
$(\forall X)(\psi(X) \leftrightarrow \varphi(X)) \wedge (\exists D)(\varphi(D) \wedge \mathsf{U}(D))$
$\rightarrow (\exists D)[\varphi(D) \wedge \mathsf{U}(D) \wedge (\forall X \dot{\in} D)(\mathsf{U}(X) \rightarrow \neg\varphi(X))]$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U})$.

Finally, we introduce a uniform variant of NUT, the theory UUT (Uniform Universes Theory). It is formulated in $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ and is based on the usual axioms and rules for the two-sorted predicate calculus. The non-logical axioms are:

(1) *defining axioms for all primitive recursive functions and relations.*

(2) *equality axioms*

(2.1) $S = R \rightarrow (\mathsf{U}(S) \rightarrow \mathsf{U}(R))$.
(2.2) $S = R \rightarrow (\mathcal{U}(S) = \mathcal{U}(R))$.

(3) *set operations*
As in NUT but extended to all $rel\text{-}\Pi^1_0(\mathsf{U},\mathcal{U})$ ($rel\text{-}\Sigma^1_1(\mathsf{U},\mathcal{U})$, resp.) formulas φ.

(4) *closure conditions for universes*
Exactly as for NUT.

(5) *uniform limit axioms*
$X \dot{\in} \mathcal{U}(X) \wedge \mathsf{U}(\mathcal{U}(X))$.

(6) *induction scheme for arbitrary formulas of* $\mathcal{L}_2(\mathsf{U},\mathcal{U})$.

Notice that in UUT we can prove $(\forall X)\varphi(X) \to \varphi(S)$ for each formula φ and set term S of $\mathcal{L}_2(\mathsf{U},\mathcal{U})$. NUT_0, MUT_0 and UUT_0 are taken to be the theories NUT, MUT, UUT with set-induction

$$(0 \in S \wedge (\forall x)(x \in S \to x+1 \in S)) \to (\forall x)(x \in S)$$

instead of full induction (6). We end this section with some remarks.

NUT_0 is included in UUT_0 and MUT_0

A trivial induction on the length of the derivation $\mathsf{NUT}_0 \vdash \varphi$ shows

$$\mathsf{NUT}_0 \vdash \varphi \quad \Longrightarrow \quad \mathsf{UUT}_0 \vdash \varphi \text{ and } \mathsf{MUT}_0 \vdash \varphi.$$

Therefore, NUT_0 is included in UUT_0 and MUT_0.

Closure conditions of universes in UUT_0

Notice that the closure conditions for universes in UUT_0 are formulated for $rel\text{-}\Pi^1_0(\mathsf{U})$ and not for $rel\text{-}\Pi^1_0(\mathsf{U},\mathcal{U})$ formulas. If we took, for instance,

for all $rel\text{-}\Pi^1_0(\mathsf{U},\mathcal{U})$ formulas $\varphi[x,\vec{z},\vec{S}]$:
$\mathsf{U}(R) \wedge \vec{S} \dot{\in} R \to (\exists Z \dot{\in} R)(\forall x)(x \in Z \leftrightarrow \varphi[x,\vec{z},\vec{S}])$,

then the corresponding theory would be inconsistent. To see this, set $\varphi := x \in \mathcal{U}(X)$. Then the axiom yields

$$\mathsf{U}(\mathcal{U}(X)) \wedge X \dot{\in} \mathcal{U}(X) \to (\exists Z \dot{\in} \mathcal{U}(X))(Z = \mathcal{U}(X)).$$

We conclude that $\mathcal{U}(X) \dot{\in} \mathcal{U}(X)$ holds. This contradicts lemma 1b).

Motivation of the axioms

The use of the axiom scheme (1) makes working in the above theories more convenient. (2) assures the compatibility of the introduced symbols U, $\mathcal{U}$ with the extensional equality of the sets.

With our theories we intend to describe countable coded ω-models of Σ^1_1-AC. It is natural to demand at least the same set principles for dealing with these models. Therefore we have imposed the axiom scheme (3). The closure conditions of these models are listed in (4). We have closure under arithmetical comprehension (4.1) and closure under Σ^1_1-choice (4.2).

In (5) the existence of universes is ensured by a limit axiom. In MUT we can choose these universes minimal with respect to rel-$\Delta^1_1(\mathsf{U})$ formulas and the given notion of linearity. In UUT we can choose universes uniformly.

It is very important to remark that in our theories universes can only introduced by the limit axioms (and the minimal universe axioms). All these axioms are existence axioms only. In a certain sense the universes are given implicitly. We have not *defined* the universes, in this sense the universes are not given explicitly.

Inconsistencies

In [15] some inconsistencies are proved. For instance, ATR_0 plus

$$(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y \rightarrow X \dot{\in} Y \vee X \doteq Y \vee Y \dot{\in} X$$

is inconsistent. Here, we have written $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ for a finite axiomatization of $(\mathsf{ACA}) + (\Sigma^1_1$-$\mathsf{AC})$. (Later on, we prove that ATR_0 is included in NUT_0 and has the same proof-theoretic strength.) A further result is the following. NUT_0 plus (*linearity of universes*) is consistent, since MUT_0 is consistent. But NUT_0 plus (*linearity of universes*) plus

$$\mathsf{U}(X) \wedge X \doteq Y \rightarrow \mathsf{U}(Y)$$

is inconsistent.

Universes and countable coded ω-models of Σ^1_1-AC

We mention the following fact: If there is a set X such that $\mathsf{U}(X)$holds, then we can define for example

$$Y := \{\langle x, 2k+1\rangle : \langle x, k\rangle \in X\}.$$

We see immediately that Y is also a countable coded ω-model of Σ^1_1-AC, but we cannot prove that Y is a universe. In this sense we use the notation "universe" only for sets X with $\mathsf{U}(X)$. On the other hand we use the notation "countable coded ω-model of Σ^1_1-AC" for sets which satisfy the closure conditions (4.1) and (4.2) for universes. Each universe is a countable coded ω-model of Σ^1_1-AC but not vice versa.

In our theories there are much more countable coded ω-models of Σ^1_1-AC than universes. Since we can embed ATR_0 into these theories (cf. lemma 4) we can even construct in our theories countable coded ω-models of Σ^1_1-AC (cf. theorem 6), because these models are defined explicitly (and of course because ATR_0 is strong enough). But we cannot prove that these so constructed models are universes. That is, we can choose for example in MUT_0 a minimal universe but not a minimal countable coded ω-model of Σ^1_1-AC.

What about a uniform variant of MUT?

We can create a lot of further theories by mixing the stated axioms (and adding further axioms). For instance, we can replace the non-uniform limit axiom in MUT by a uniform limit axiom for minimal universes and adapt the other axioms of MUT. Later on we show that this extension has the same proof-theoretic ordinal. On the other hand it is an open question whether the stated linearity axiom of MUT is strong enough to define in MUT a universe operator. In this context we will prove that by the (in a certain sense stronger) linearity axiom

$$\mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X = Y \vee Y \mathrel{\dot\in} X$$

we can define in MUT a universe operator. This universe operator will be a minimal universe operator.

Our theories of universes in comparison with theories of universes in other contexts

Our theories are built in a similar way as the theories of universes in explicit mathematics, or theories about admissibles without foundation in the framework of set theory (cf. for example KPi^0 [3]). We find always the same structure: some ontological axioms and ground structures (here (1) and (2)), some set operations (here (3)), axioms about the properties of universes (here (4)), then universes with the aid of limit axioms are introduced (here (5)) and finally there is some kind of induction (here (6)). The purpose of our theories of universes is not to give another possibility to deal with universes, but rather to show that we can build similar theories (as for example KPi^0) in second order arithmetic and that these theories have the same proof-theoretic strength.

Notice that our universes correspond to admissibles *without* foundation. The reason is that the properties of our universes are not strong. We have only closure under arithmetical comprehension and under the Σ^1_1-choice axiom. But, for example, we cannot prove that our universes are equivalent (with respect to $\doteq$) to sets of the form $\{X \subseteq \omega : X$ is hyperarithmetical in $Z\}$. (That is, we cannot prove that our universes are least (with respect to $\dot\in$) countable coded ω-models of Σ^1_1-AC.)

Universes as countable coded ω-models of Σ^1_1-DC

Our universes satisfy the axiom of Σ^1_1-choice. Assume that we had "$\mathsf{U}(X)$ implies that X is a countable coded ω-model of Σ^1_1-DC" instead of "$\mathsf{U}(X)$ implies that X is a countable coded ω-model of Σ^1_1-AC". Is the corresponding theory of such universes proof-theoretically stronger than the theory NUT (or UUT, MUT)? We do not give a proof here but only mention that the proof-theoretic strength does not change. There is the following reason for this fact: In the sequel we use that in ATR_0 we can prove the existence of countable coded ω-models of Σ^1_1-AC (theorem VIII.4.20 [21]). But the same theorem states also that ATR_0 proves the existence of countable coded ω-models of Σ^1_1-DC. This fact leads to the proof-theoretic equivalence of the mentioned theories.

But notice that the situation is different if we add rel-$\Sigma^1_1(\mathsf{U})$-DC to these theories. Then, e.g., the adapted theory NUT will be proof-theoretically stronger than the original NUT.

4 Properties of $\mathsf{NUT}_0, \mathsf{UUT}_0, \mathsf{MUT}_0$

The purpose of this section is to present ontological properties of our theories, especially the closure properties of our classes of formulas. We often use these properties tacitly in the following. First we collect two properties of universes in lemma 1. Assertion a) is a kind of transitivity and assertion b) says that "a universe cannot speak about itself".

Lemma 1 *In* NUT_0, UUT_0 *and* MUT_0 *we have*

a) $\mathsf{U}(T) \wedge R \dot{\in} S \wedge S \dot{\in} T \rightarrow R \dot{\in} T$,

b) $\mathsf{U}(T) \rightarrow T \dot{\notin} T$.

Proof. Here and in the following we work informally in the theories. Assertion a) follows easily by arithmetical comprehension in the universe T. There remains assertion b). Let us assume $\mathsf{U}(T)$ and $T \dot{\in} T$. We show by a diagonalization argument that this leads to a contradiction. By $T \dot{\in} T$ and closure of the universe T under arithmetical comprehension there exists a set Z in T such that

$$(\forall x)[x \in Z \leftrightarrow (Seq_2 x \wedge (T)_{(x)_1} \dot{\notin} (T)_{(x)_1} \wedge (x)_0 \in (T)_{(x)_1})].$$

First, we prove

$$(\forall X \dot{\in} T)[X \neq \emptyset \rightarrow (X \dot{\in} Z \leftrightarrow X \dot{\notin} X)]. \tag{1}$$

Choose X in T such that $X \neq \emptyset$. We have to show $X \dot{\in} Z \leftrightarrow X \dot{\notin} X$.
$\rightarrow$: Since X is in Z there is an index l with $X = (Z)_l$. The definition of Z yields

$$(\forall x)[x \in X \leftrightarrow ((T)_l \dot{\notin} (T)_l \wedge x \in (T)_l)].$$

Since X is not empty we can choose an x in X and conclude $(T)_l \dot{\notin} (T)_l$. Then we have $(\forall x)(x \in X \leftrightarrow x \in (T)_l)$. This is just $X = (T)_l$ and therefore $X \dot{\notin} X$.
$\leftarrow$: We have $X \dot{\notin} X$. Furthermore we know $X \dot{\in} T$. Therefore we can choose an index l with $X = (T)_l$. Since we have $X \dot{\notin} X$ we conclude

$$(\forall x)[x \in X \leftrightarrow ((T)_l \dot{\notin} (T)_l \wedge x \in (T)_l)].$$

By definition of Z we immediately get $X = (Z)_l$ and therefore $X \dot{\in} Z$. Hence (1). In a next step we show $Z \neq \emptyset$. The injectivity of the coding function yields that there exists a z such that $(\forall l)\langle z, l\rangle \neq z$. Then $\{z\} \dot{\notin} \{z\}$. Finally, we know $\{z\} \dot{\in} T$ and we conclude $\{z\} \dot{\in} Z$. This together with (1) and $Z \dot{\in} T$ yields the desired contradiction $Z \dot{\in} Z \leftrightarrow Z \dot{\notin} Z$. □

Notice that the proof of lemma 1b) does not use the closure property (4.2) of universes. This means: For each countable coded ω-model T of ACA we have $T \dot{\notin} T$. In a next step we prove that in NUT_0 (UUT_0, resp.) we have (rel-$\Delta^1_1(\mathsf{U})$-CA) ((rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$), resp.). Since the proof of this statement is an imitation of the proof of "Π^1_0-CA and Σ^1_1-AC imply Δ^1_1-CA" (cf. lemma VII.6.6 in [21]), we omit it.

Lemma 2 *For all $\varphi_1 \in rel$-$\Sigma^1_1(\mathsf{U})$, $\varphi_2 \in rel$-$\Pi^1_1(\mathsf{U})$, $\psi_1 \in rel$-$\Sigma^1_1(\mathsf{U},\mathcal{U})$, $\psi_2 \in rel$-$\Pi^1_1(\mathsf{U},\mathcal{U})$ the following hold.*

a) NUT_0 *proves (rel-$\Delta^1_1(\mathsf{U})$-CA), i.e., for $i \in \{1,2\}$ we have*

$$\mathsf{NUT}_0 \vdash (\varphi_1(x) \leftrightarrow \varphi_2(x)) \to (\exists X)(\forall x)(x \in X \leftrightarrow \varphi_i(x)).$$

b) UUT_0 *proves (rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$), i.e., for $i \in \{1,2\}$ we have*

$$\mathsf{UUT}_0 \vdash (\psi_1(x) \leftrightarrow \psi_2(x)) \to (\exists X)(\forall x)(x \in X \leftrightarrow \psi_i(x)).$$

In the next lemma we formulate properties which correspond to the usual closure conditions of the class of Σ^1_1-formulas (Π^1_1-formulas, resp.). For that purpose we define: If Th is a theory and $\mathcal{F}$ is a class of $\mathcal{L}(\mathsf{Th})$ formulas, then $\mathcal{F}^{\mathsf{Th}}$ denotes the class of all $\mathcal{L}(\mathsf{Th})$ formulas which in Th are equivalent to some $\varphi \in \mathcal{F}$. Since the proof of lemma 3 uses only standard arguments, we omit it.

Lemma 3 *The class of rel-$\Sigma^1_1(\mathsf{U})^{\mathsf{NUT}_0}$ (rel-$\Pi^1_1(\mathsf{U})^{\mathsf{NUT}_0}$, resp.) formulas is closed under $\wedge$, $\vee$, $\exists x$, $\forall x$, $\exists X \dot{\in} Y$, $\forall X \dot{\in} Y$, $\exists X$ ($\forall X$, resp.). The same holds for rel-$\Sigma^1_1(\mathsf{U},\mathcal{U})^{\mathsf{NUT}_0}$ (rel-$\Pi^1_1(\mathsf{U},\mathcal{U})^{\mathsf{NUT}_0}$, resp.).*

In the following we often use the notion of a rel-$\Delta^1_1(\mathsf{U})$ (rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$, resp.) formula which is defined with respect to a theory as usual. It will always be clear from the context which theory we mean. The lemmas 2 and 3 show that for theories which contain NUT_0 (UUT_0, resp.) we have formula comprehension and (usual) closure conditions for rel-$\Delta^1_1(\mathsf{U})$ (rel-$\Delta^1_1(\mathsf{U},\mathcal{U})$, resp.).

5 ATR and NUT

We show that there is an embedding of ATR into NUT and of NUT into ATR.

The embedding of ATR_0 into NUT_0 corresponds exactly to the embedding of ATR_0 into KPi_0 (cf. [3]). Therefore we omit the proof of the following lemma.

Lemma 4 *For each $\mathcal{L}_2$ formula φ we have*

a) $\mathsf{ATR}_0 \vdash \varphi \quad \Longrightarrow \quad \mathsf{NUT}_0 \vdash \varphi$,

b) $\mathsf{ATR} \vdash \varphi \quad \Longrightarrow \quad \mathsf{NUT} \vdash \varphi$.

Now we use results of Simpson [21] to embed NUT_0 into ATR_0. In [21] it is shown that ATR_0 proves the existence of countable coded ω-models of Σ^1_1-AC. Simpsons definition of countable coded ω-models makes use of the notion of valuation functions (cf. definition VII.2.1 in [21]). Our countable coded ω-models however are sets which reflect (not satisfy) appropriate properties. In order to apply the results of Simpson we proceed as follows. First we give a finite axiomatization $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ of $(\Sigma^1_1\text{-}\mathsf{AC}) + (\mathsf{ACA})$. Next we investigate Simpsons proof which leads to lemma VIII.4.19 in [21]. This investigation shows that more or less the same proof leads to the proposition: "ATR_0 proves the existence of a set D with $X \dot{\in} D$ and $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D$". Then we can translate the predicate $\mathsf{U}(D)$ as "D is a countable coded ω-model of Σ^1_1-AC" and the embedding goes through. We need universal relations for the exact formulation. For each n and m let $\pi^0_{1,n,m}[e, x_1, \ldots, x_n, X_1, \ldots, X_m]$ be a universal Π^0_1 formula (of $\mathcal{L}_2$). Now the finite axiomatization is given by the formula $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$.

$$\begin{aligned} Ax_{\Sigma^1_1\text{-}\mathsf{AC}} \;:=\; & (\forall X, Y)(\exists Z)(Z = X \oplus Y) \land \\ & (\forall e, z)(\forall Z)(\exists Y)(\forall x)(x \in Y \leftrightarrow \pi^0_{1,2,1}[e, x, z, Z]) \land \\ & [(\forall e, z)(\forall Z)[(\forall x)(\exists Y)\pi^0_{1,2,2}[e, x, z, Y, Z] \\ & \qquad \to (\exists Y)(\forall x)\pi^0_{1,2,2}[e, x, z, (Y)_x, Z]]]. \end{aligned}$$

Again we adopt the standard notation φ^D for the relativization of the $\mathcal{L}_2$ formula φ to D (for example $(\forall X \varphi(X))^D := (\forall X \dot{\in} D)\varphi^D(X)$). The following lemma shows that the formula $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ serves the right role. Its proof is standard and therefore omitted.

Lemma 5 *Let φ be an instance of $(\Sigma^1_1\text{-}\mathsf{AC}) + (\mathsf{ACA})$. Then ACA_0 proves*

$$(\forall \vec{z})(\forall \vec{Z})((Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D \land \vec{Z} \dot{\in} D \to \varphi^D[\vec{z}, \vec{Z}]).$$

Now, Simpsons theorem VIII.3.15 [21] and more or less the same proof which leads to lemma VIII.4.19 [21] yields the following theorem.

Theorem 6 $\mathsf{ATR}_0 \vdash (\exists D)(X \dot{\in} D \land (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^D)$.

This theorem is the crucial point in the embedding of NUT_0 into ATR_0. We now introduce the translation. For every $\mathcal{L}_2(\mathsf{U})$ formula we write φ^{Ax} for the $\mathcal{L}_2$ formula which is obtained by replacing each instance $\mathsf{U}(X)$ in φ by $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$. Then we have the following embedding theorem.

Theorem 7 *For all $\mathcal{L}_2(\mathsf{U})$ formulas φ the following holds.*

a) $\mathsf{NUT}_0 \vdash \varphi \quad \Longrightarrow \quad \mathsf{ATR}_0 \vdash \varphi^{Ax}$.

b) $\mathsf{NUT} \vdash \varphi \quad \Longrightarrow \quad \mathsf{ATR} \vdash \varphi^{Ax}$.

Proof. We show b) by induction on the length of derivation $\mathsf{NUT} \vdash \varphi$ (the proof of the assertion a) is identical). We consider only the mathematical axioms (3) of NUT, the other mathematical axioms (1), (2), (4) – (6) and the logical rules and logical axioms are easily verified.

Discussing (3), we prove only (rel-$\Sigma^1_1(\mathsf{U})$-AC), since the proof of (rel-$\Pi^1_0(\mathsf{U})$-CA) is similar. Let us assume $((\forall x)(\exists X)\varphi(x,X))^{Ax}$ and $\varphi \in rel$-$\Sigma^1_1(\mathsf{U})$. We have to show (within ATR) $((\exists X)(\forall x)\varphi(x,(X)_x))^{Ax}$. First we notice

$$\begin{aligned} ((\forall x)(\exists X)\varphi(x,X))^{Ax} &\leftrightarrow (\forall x)(\exists X)\varphi^{Ax}(x,X), \\ ((\exists X)(\forall x)\varphi(x,(X)_x))^{Ax} &\leftrightarrow (\exists X)(\forall x)\varphi^{Ax}(x,(X)_x). \end{aligned}$$

Since $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^X$ is equivalent to an arithmetic formula, the formula φ^{Ax} is equivalent to a Σ^1_1 formula θ, and we have $(\forall x)(\exists X)\theta(x,X)$. Note that we have ($\Sigma^1_1$-$\mathsf{AC}$) in ATR. Hence

$$(\exists X)(\forall x)\theta(x,(X)_x) \qquad \text{and} \qquad (\exists X)(\forall x)\varphi^{Ax}(x,(X)_x).$$

□

By lemma 4 and theorem 7 NUT_0 (NUT, resp.) is conservative over ATR_0 (ATR, resp.) for arithmetic formulas. This yields the following corollary (cf. e.g. [1, 7]).

Corollary 8 $|\mathsf{NUT}_0| = |\mathsf{ATR}_0| = \Gamma_0$ *and* $|\mathsf{NUT}| = |\mathsf{ATR}| = \Gamma_{\varepsilon_0}$.

6 An embedding of UUT_0 into $\mathsf{MUT}_0^=$

In this section we show that in a strengthening of MUT_0 we can define unique universes using an appropriate rel-$\Delta^1_1(\mathsf{U})$ formula. This yields an embedding of UUT_0 into this strengthened theory. We do not know whether an embedding of UUT into MUT is possible, since we do not know how to define unique minimal universes with respect to the linear ordering of universes in MUT. Therefore, we strengthen the linearity axiom in such a way that we are able to show the existence of (unique) minimal universes. Then we can define a universe operator and the embedding goes through.

First we describe the strengthening of MUT_0. We add to the theory MUT_0 the linearity axioms

$$(\mathrm{Lin}^=) \quad \mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X = Y \vee Y \mathrel{\dot\in} X.$$

The difference between $(\mathrm{Lin}^=)$ and (Lin) is only by a small dot " $\dot{}$ ". (Lin) are the axioms

$$(\mathrm{Lin}) \quad \mathsf{U}(X) \wedge \mathsf{U}(Y) \to X \mathrel{\dot\in} Y \vee X \doteq Y \vee Y \mathrel{\dot\in} X.$$

Note that $X = Y$ means that X and Y are the same sets. On the other hand, $X \doteq Y$ only implies that X and Y have the same projections. $X = Y$ implies $X \doteq Y$ but not vice versa.

$\mathsf{MUT}^=$ denotes the theory $\mathsf{MUT} + (\mathrm{Lin}^=)$. Later on, we will show that $\mathsf{MUT}^=$ and MUT have the same proof-theoretic strength.

In the theory MUT the universes are stratified in the following sense: All minimal universes over the empty set contain the same projections and all these universes make up the first, lowest stratum. If, for example, the universes A and B are in the first stratum, then they have the same projections ($A \doteq B$), but they may have different indices for the same projections (i.e., we may have $(A)_k \neq (B)_k$). Now choose a universe D in this first stratum. Then the next stratum contains all minimal universes over D. That this second stratum does not depend on the choice of D is stated in lemma 16 in [15]. There, we proved

$$\mathsf{MUT}_0 \vdash \mathsf{U}(X) \wedge \mathsf{U}(Y) \wedge \mathsf{U}(Z) \wedge X \doteq Y \wedge Y \mathrel{\dot\in} Z \to X \mathrel{\dot\in} Z.$$

That is, each universe C in the first stratum is contained in each universe of the second stratum; and so on. In the stratification of $\mathsf{MUT}^=$ each stratum contains only one universe. It is an open question whether $\mathsf{NUT} + (\mathrm{Lin}^=)$ is proof-theoretically stronger than NUT.

The uniqueness in $\mathsf{MUT}^=$ of the universes in a stratum implies that the following abbreviation is in fact a $rel\text{-}\Delta^1_1(\mathsf{U})$ formula.

$$minU(x,X) := (\exists Z)[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y) \wedge x \in Z].$$

In $\mathsf{MUT}_0^=$ the meaning of the formula $minU(x,X)$ is: x is in the (unique!) minimal universe which contains X. The following lemma is the formalization of this idea.

Lemma 9 *The following are theorems of* $\mathsf{MUT}_0^=$.

a) $[\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y)(\mathsf{U}(Y) \wedge X \dot{\in} Y \rightarrow Y = D \vee D \dot{\in} Y)] \leftrightarrow$
$[\mathsf{U}(D) \wedge X \dot{\in} D \wedge (\forall Y \dot{\in} D)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)]$.

b) $(\exists! Z)[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)]$.

c) $minU(x,X) \leftrightarrow$
$(\forall Z)[[X \dot{\in} Z \wedge \mathsf{U}(Z) \wedge (\forall Y \dot{\in} Z)(\mathsf{U}(Y) \rightarrow X \dot{\notin} Y)] \rightarrow x \in Z]$.

Proof. a) follows from lemma 1 and (Lin$^=$). The existence of Z in b) is assured by the limit axiom and the minimal universe axiom. Uniqueness follows from (Lin$^=$). c) follows from b). □

We now give an embedding of UUT into $\mathsf{MUT}^=$. The idea is to interpret $x \in \mathcal{U}(S)$ as "x is in the minimal universe which contains S". $\mathsf{U}(S)$ will be interpreted essentially as $\mathsf{U}(S)$ (more precisely: U(S) will be interpreted as $\mathsf{U}(\{x : (x \in S)^{min}\})$). We define for each $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formula φ an $\mathcal{L}_2(\mathsf{U})$ formula φ^{min}. It is inductively defined. If φ is an $\mathcal{L}_2$ literal, then $\varphi^{min} := \varphi$. Otherwise we set

$$
\begin{array}{lll}
1. & (x \in \mathcal{U}(S))^{min} & := (\exists Z)[(\exists k)(\forall z)[(z \in S)^{min} \leftrightarrow \langle z,k\rangle \in Z] \wedge \\
 & & \quad \mathsf{U}(Z) \wedge x \in Z \wedge \\
 & & \quad (\forall Y \dot{\in} Z)[\mathsf{U}(Y) \rightarrow \\
 & & \quad\quad \neg(\exists k)(\forall z)[(z \in S)^{min} \leftrightarrow \langle z,k\rangle \in Y]]], \\
2. & (x \notin \mathcal{U}(S))^{min} & := \neg(x \in \mathcal{U}(S))^{min}, \\
3. & (\mathsf{U}(S))^{min} & := (\exists Z)[(\forall x)(x \in Z \leftrightarrow (x \in S)^{min}) \wedge \mathsf{U}(Z)], \\
4. & (\neg\mathsf{U}(S))^{min} & := \neg(\mathsf{U}(S))^{min}, \\
5. & (\varphi \circ \psi)^{min} & := \varphi^{min} \circ \psi^{min} \qquad \circ \in \{\wedge, \vee\}, \\
6. & (\mathrm{Q}x\varphi)^{min} & := \mathrm{Q}x\varphi^{min} \qquad \mathrm{Q} \in \{\exists, \forall\}, \\
7. & (\mathrm{Q}X\varphi)^{min} & := \mathrm{Q}X\varphi^{min} \qquad \mathrm{Q} \in \{\exists, \forall\}.
\end{array}
$$

Note that NUT_0 proves $(x \in \mathcal{U}(S))^{min} \leftrightarrow (minU(x,S))^{min}$.

Theorem 10 *For all $\mathcal{L}_2(\mathsf{U},\mathcal{U})$ formulas φ we have*

a) $\mathsf{UUT}_0 \vdash \varphi \quad \Longrightarrow \quad \mathsf{MUT}_0^= \vdash \varphi^{min}$,

b) $\mathsf{UUT} \vdash \varphi \quad \Longrightarrow \quad \mathsf{MUT}^= \vdash \varphi^{min}$.

Proof. We show a) by induction on the length of the derivation $\mathsf{UUT}_0 \vdash \varphi$ (an analogous argument shows b)). The logical rules and logical axioms are easily dealt with. Let us consider the mathematical axioms (1)-(6) of UUT_0.

(1) We have these axioms in $\mathsf{MUT}_0^=$ too.

(2) An easy induction on the build-up of set terms implies the claim.

(3) If φ is a rel-$\Pi_0^1(\mathsf{U},\mathcal{U})$ formula, then we can prove by induction on the build-up of φ, using lemma 9c) and the closure properties of rel-$\Delta_1^1(\mathsf{U})$ formulas (cf. lemma 3), that φ^{min} is a rel-$\Delta_1^1(\mathsf{U})$ formula. In $\mathsf{MUT}_0^=$ we have (rel-$\Delta_1^1(\mathsf{U})$-CA) (lemma 2). This immediately proves the translation of (rel-$\Pi_0^1(\mathsf{U},\mathcal{U})$-$\mathsf{CA}$). For the proof of the translation of (rel-$\Sigma_1^1(\mathsf{U},\mathcal{U})$-$\mathsf{AC}$) we notice that in $\mathsf{MUT}_0^=$ we have (rel-$\Sigma_1^1(\mathsf{U})$-AC) and that for $\varphi \in rel$-$\Sigma_1^1(\mathsf{U},\mathcal{U})$ the formula φ^{min} is equivalent to a rel-$\Sigma_1^1(\mathsf{U})$ formula (again by induction on the build-up of φ).

(4) Since $(\mathsf{U}(X))^{min}$ is equivalent to $\mathsf{U}(X)$ the claim is immediately evident.

(5) Follows from lemma 9b) and the definition of the $(\ldots)^{min}$ translation.

(6) We have set induction in $\mathsf{MUT}_0^=$ too. □

We obtain the following corollary.

Corollary 11 $|\mathsf{UUT}_0| \leq |\mathsf{MUT}_0^=|$ *and* $|\mathsf{UUT}| \leq |\mathsf{MUT}^=|$.

7 A well-ordering proof for UUT

In this section we show that UUT proves transfinite induction for each initial segment of the ordinal $\varphi 1 \varepsilon_0 0$. We follow the presentation in [22]. (Here we give well-ordering proofs although it is also possible to embed other theories, for instance $\widehat{\mathsf{ID}}_{<\varepsilon_0}$.)

In what follows we presuppose the same ordinal-theoretic facts as given in section 2 of [6]. That is, we let Φ_0 denote the least ordinal greater than 0 which is closed under all n-ary φ functions, and we assume that a standard notation system of order type Φ_0 is given in a straightforward manner. We write $\prec$ for the corresponding primitive recursive well-ordering. We assume without loss of generality that the field of $\prec$ is the set of all natural numbers and that 0 is the least element with respect to $\prec$. Hence, each natural number codes an ordinal less than Φ_0. When working in UUT in this section, we let $a, b, c, \ldots$ range over the field of $\prec$, and ℓ denotes limit notations. There exist primitive recursive functions acting on the codes of this notation system which correspond to the usual operations on ordinals. In what follows it is often convenient in order to simplify notation to use ordinals and ordinal operations instead of their codes and primitive recursive analogues. Then (for example) ω and $\omega + \omega$ stand for the natural numbers whose order type with respect to $\prec$ are ω and $\omega+\omega$. Finally, we write $Prog(\varphi)$ for $Prog(\prec, \varphi)$ and $TI(a, \varphi)$ for $TI(\prec\restriction a, \varphi)$.

If we want to stress the relevant induction variable of a formula φ, we sometimes write $Prog(\lambda a.\varphi(a))$ instead of $Prog(\varphi)$. If S is a set term, then $Prog(S)$ and $TI(a, S)$ have their obvious meanings.

7.1 Hierarchies of universes

It is our aim to derive $(\forall X)TI(\alpha, X)$ in UUT for each ordinal α less than $\varphi 1\varepsilon_0 0$. A crucial step towards this aim is the construction of a transfinite hierarchy H of universes along $\prec$ above a given S. We choose $\mathcal{U}(S)$ for the universe containing S.

We let $Hier(S, H, a)$ denote the formula which formalizes the property "H is a hierarchy of universes along $\prec$ up to a above S".

$$\begin{aligned} Hier(S, H, a) \;:=\; & (\forall x)[x \in (H)_0 \leftrightarrow x \in \mathcal{U}(S)] \wedge \\ & (\forall b)[0 \prec b \preceq a \to (\forall x)(x \in (H)_b \leftrightarrow \mathcal{U}((H)_{\prec b}))]. \end{aligned}$$

We recall that $(H)_{\prec b}$ is the disjoint union of all $(H)_c$ with $c \prec b$. The uniqueness of such hierarchies is proved by transfinite induction up to ordinals α less than ε_0, which is available in UUT.

Lemma 12 *For all ordinals α less than ε_0 we have*

$$\mathsf{UUT} \vdash (\forall a \prec \alpha)[Hier(S, H, a) \wedge Hier(S, G, a) \to (\forall b \prec a)((H)_b = (G)_b)].$$

We mention two ontological properties of such hierarchies of universes.

Lemma 13 *The following hold in* UUT.

a) $Hier(S, H, a) \to (\forall b \preceq a)\mathsf{U}((H)_b)$.

b) $Hier(S, H, a) \to (\forall b, c)(c \prec b \preceq a \to (H)_c \dot{\in} (H)_b)$.

Proof. Assume $Hier(S, H, a)$ and $b \preceq a$. Each step $(H)_b$ of the hierarchy H is of the form $\mathcal{U}(S)$ or $\mathcal{U}((H)_{\prec b})$. We know $\mathsf{U}(\mathcal{U}(S))$ for all set terms S. This gives a). In order to prove assertion b) we assume $Hier(S, H, a)$ and $c \prec b \preceq a$. We know $(H)_c \dot{\in} (H)_{\prec b}$, $(H)_{\prec b} \dot{\in} \mathcal{U}((H)_{\prec b})$ and $(H)_b = \mathcal{U}((H)_{\prec b})$. $(H)_b$ is a universe and lemma 1a) yields $(H)_c \dot{\in} (H)_b$. □

The next lemma states the existence of such hierarchies up to ordinals less than ε_0. The prove is by induction up to $\alpha < \varepsilon_0$ which is available in UUT. Since this proof uses only standard arguments, we omit it.

Lemma 14 *For all ordinals* α *less than* ε_0 *we have*

$$\mathsf{UUT} \vdash (\forall a \prec \alpha)(\exists Y)Hier(S, Y, a).$$

7.2 Well-ordering proof

Crucial for carrying out the well-ordering proof in UUT is the very natural notion $I_H^c(a)$ of *transfinite induction up to a for all sets belonging to a universe* $(H)_b$ *such that* $b \prec c$ *(and* $Hier(R, H, c)$*) holds*, which is given as follows:

$$I_H^c(a) := (\forall b \prec c)(\forall Y \dot{\in} (H)_b)TI(a, Y).$$

The next lemma tells us that I_H^c can be represented by a set in $(H)_c$, if $Hier(R, H, c)$ holds.

Lemma 15 *For each ordinal* α *less than* ε_0 *the following is a theorem of* UUT.

$$(\forall c \prec \alpha)[Hier(R, H, \alpha) \to (\exists Z \dot{\in} (H)_c)(\forall x)(x \in Z \leftrightarrow I_H^c(x))].$$

Proof. Assuming $c \prec \alpha$ and $Hier(R, H, \alpha)$ we know by definition

$$b \prec c \to ((H)_{\prec c})_b = (H)_b \qquad \text{and} \qquad (H)_{\prec c} \dot{\in} (H)_c.$$

Hence $(\forall b \prec c)(\forall Y \dot{\in} (H)_b)TI(a, Y)$ is equivalent to a rel-$\Pi_0^1(\mathsf{U})$ formula with set parameter $H_{\prec c}$ in $(H)_c$. Hence closure of $(H)_c$ under rel-$\Pi_0^1(\mathsf{U})$

comprehension implies the existence of a set Z in $(H)_c$ such that $Z = I^c_H$. $\square$

In the next theorem we use the binary relation $\uparrow$

$$a \uparrow b := (\exists c, \ell)(b = c + a \cdot \ell)$$

and the abbreviation

$$
\begin{aligned}
&Main_\alpha(a) := \\
&\quad (\forall X, Y)(\forall b, c)[c \preceq \alpha \wedge \omega^{1+a} \uparrow c \wedge Hier(X, Y, c) \wedge I^c_Y(b) \rightarrow I^c_Y(\varphi 1 a b)].
\end{aligned}
$$

We omit the proof of the following theorem, because the statements correspond to analogous results in [22] and [6].

Theorem 16 *For each ordinal α less than ε_0 we can prove in* UUT

a) $(\forall X, Y)(\forall \ell, a)[\ell \prec \alpha \wedge Hier(X, Y, \alpha) \wedge I^\ell_Y(a) \rightarrow I^\ell_Y(\varphi a 0)]$,

b) $(\forall X, Y)(\forall \ell)[\ell \prec \alpha \wedge Hier(X, Y, \alpha) \rightarrow Prog(\lambda a. I^\ell_Y(\Gamma_a))]$,

c) $Prog(\lambda a. Main_\alpha(a))$.

And for each ordinal α less than $\varphi 1 \varepsilon_0 0$ the following is a theorem of UUT.

$$(\forall X) TI(\alpha, X).$$

The methods of this section can also be applied to the theory MUT in order to obtain the lower bound of MUT (cf. [15]). We collect these lower bounds in a corollary.

Corollary 17 *We have* $\varphi 1 \varepsilon_0 0 \leq |\mathsf{UUT}|$ *and* $\varphi 1 \varepsilon_0 0 \leq |\mathsf{MUT}| \leq |\mathsf{MUT}^=|$.

8 Upper bounds of $\mathsf{MUT}^=_0$ and $\mathsf{MUT}^=$

In this section we give an asymmetric interpretation of $\mathsf{MUT}^=$ into the semiformal system T_α. In T_α we have constants $\mathsf{D}_\beta, \mathsf{D}_{<\gamma}$ $(\beta < \alpha, \gamma \leq \alpha)$. Each D_β satisfies the closure conditions for universes. Moreover we have $\mathsf{D}_{<\beta} \mathrel{\dot\in} \mathsf{D}_\beta$. We now sum up the proceeding: We first show that without loss of generality we can take the minimality condition (5.3) in $\mathsf{MUT}^=$ only for the rel-$\Pi^1_0(\mathsf{U})$ formulas instead for the whole class of rel-$\Delta^1_1(\mathsf{U})$ formulas. Considering this we will introduce the corresponding Tait-style reformulation $(\mathsf{MUT}^=)^T$ of MUT. Then we prove an asymmetric interpretation of $(\mathsf{MUT}^=)^T$ into T_α. This leads finally to the interpretation of $\mathsf{MUT}^=$ into T_α $(\alpha < \varepsilon_0)$. The proof-theoretic analysis of T_α is given in [15].

8.1 The semi-formal system T_α

The semi-formal system T_α is formulated with bounded second order quantifiers $\exists X \dot{\in} \mathsf{D}_\beta$ and $\forall X \dot{\in} \mathsf{D}_\beta$ for $\beta < \alpha$. Note that in $\mathsf{MUT}^=$ we have used $(\exists X \dot{\in} Y)\varphi(X)$ as an abbreviation for $(\exists X)(X \dot{\in} Y \wedge \varphi(X))$. In T_α, $(\exists X \dot{\in} \mathsf{D}_\beta)\varphi(X)$ is in fact a formula and not an abbreviation.

T_α is based on the language $\mathcal{L}_\alpha$. $\mathcal{L}_\alpha$ is the extension of $\mathcal{L}_2$ by new unary relation symbols D_β, $\mathsf{D}_{<\gamma}$ ($\beta < \alpha$, $\gamma \leq \alpha$). The $\mathcal{L}_\alpha$ *literals* are the $\mathcal{L}_2$ literals and all formulas $[\neg]\mathsf{D}_\beta(t)$, $[\neg]\mathsf{D}_{<\gamma}(t)$ ($\beta < \alpha$, $\gamma \leq \alpha$). Furthermore, the class of $\mathcal{L}_\alpha$ formulas is closed under $\wedge, \vee, \forall x, \exists x, \exists X \dot{\in} \mathsf{D}_\beta, \forall X \dot{\in} \mathsf{D}_\beta, \exists X, \forall X$ for each $\beta < \alpha$. The exact meaning of the bounded second order quantifiers will be given in the definition of T_α. We shall write for instance $t \in \mathsf{D}_\beta$ for $\mathsf{D}_\beta(t)$, $t \in \mathsf{D}_{<\beta}$ for $\mathsf{D}_{<\beta}(t)$ etc. We take as $\mathcal{L}_\alpha$ *formulas of* T_α the $\mathcal{L}_\alpha$ formulas without free number variables.

We now introduce the Tait-calculus T_α. It is an extension of the classical Tait-calculus [20]. In the formulation below we simply write $\pi_1^0[e, \vec{x}, \vec{X}]$ for the universal Π_1^0 predicate $\pi_{1,n,m}^0[e, \vec{x}, \vec{X}]$; φ, ψ range over $\mathcal{L}_\alpha$ formulas and Γ, Λ range over finite sets of such formulas. We often write (for instance) Γ, φ for the union of Γ and $\{\varphi\}$

1. Ontological axioms I.

Γ, φ for each true $\mathcal{L}_1$ literal φ and $\Gamma, \varphi, \neg\varphi$ for each $\mathcal{L}_\alpha$ literal φ of T_α.

2. Propositional rules.

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}.$$

3. Quantifier rules. For all closed number terms s and all set variables Y:

$$\frac{\Gamma, \varphi(s)}{\Gamma, (\exists x)\varphi(x)}, \qquad \frac{\Gamma, \varphi(t) \quad \text{for all closed terms } t}{\Gamma, (\forall x)\varphi(x)},$$

$$\frac{\Gamma, \psi(Y)}{\Gamma, (\exists X)\psi(X)}, \qquad \frac{\Gamma, \psi(Y)}{\Gamma, (\forall X)\psi(X)} \quad (vc),$$

$$\frac{\Gamma, Y \dot{\in} \mathsf{D}_\beta \wedge \psi(Y)}{\Gamma, (\exists X \dot{\in} \mathsf{D}_\beta)\psi(X)}, \qquad \frac{\Gamma, Y \dot{\in} \mathsf{D}_\beta \to \psi(Y)}{\Gamma, (\forall X \dot{\in} \mathsf{D}_\beta)\psi(X)} \quad (vc),$$

By (vc) we indicate that the rule has to respect the usual variable conditions. That is, Y must not occur in the conclusion.

4. Ontological axioms II. For all closed terms s such that $Seq_2 s$ is false, all closed terms t such that $Seq_2 t$, $Seq_2(t)_0$ and $\beta \preceq (t)_1$ is true:

$$\Gamma, s \notin \mathsf{D}_{<\beta} \qquad \text{and} \qquad \Gamma, t \notin \mathsf{D}_{<\beta}.$$

5. Ontological rules III. For all closed terms t so that $Seq_2 t$ and $(t)_1 = \gamma$ is true:

$$\frac{\Gamma, (t)_0 \in \mathsf{D}_\gamma}{\Gamma, t \in \mathsf{D}_{<\beta}}, \qquad \frac{\Gamma, (t)_0 \notin \mathsf{D}_\gamma}{\Gamma, t \notin \mathsf{D}_{<\beta}}.$$

6. Closure axioms. For all closed number terms s, r:

$$\Gamma, (U, V \mathrel{\dot{\notin}} \mathsf{D}_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}_\beta)(X = U \oplus V),$$
$$\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}_\beta)(\forall x)(x \in X \leftrightarrow \pi^0_1[s, x, r, U, \mathsf{D}_{<\beta}]).$$

7. Closure rules. For all closed number terms s, r:

$$\frac{\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}_\beta), (\forall x)(\exists X \mathrel{\dot{\in}} \mathsf{D}_\beta)\pi^0_1[s, x, r, X, U, \mathsf{D}_{<\beta}]}{\Gamma, (U \mathrel{\dot{\notin}} \mathsf{D}_\beta), (\exists X \mathrel{\dot{\in}} \mathsf{D}_\beta)(\forall x)\pi^0_1[s, x, r, (X)_x, U, \mathsf{D}_{<\beta}]},$$

8. Cut rules.

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}.$$

8.2 Asymmetric interpretation of $\mathsf{MUT}^=$ into T_α

As mentioned we can reduce the minimality condition (5.3) for rel-$\Delta^1_1(\mathsf{U})$ to a minimality condition in $\mathsf{MUT}^=$ for rel-$\Pi^1_0(\mathsf{U})$.

Lemma 18 *Let T denote the theory $\mathsf{MUT}^=$ where the minimal universe axiom (5.3) is formulated only for rel-$\Pi^1_0(\mathsf{U})$ formulas. Then T proves the (full) minimal universe axiom (5.3).*

Proof. We argue in T. Choose rel-$\Pi^1_0(\mathsf{U})$ formulas φ, ψ such that

$$(\forall E)((\exists Z)\varphi(Z, E) \leftrightarrow (\forall Z)\psi(Z, E)) \wedge (\exists D)(\mathsf{U}(D) \wedge (\exists Z)\varphi(Z, D)).$$

We have to show that there is a minimal universe F such that $(\exists Z)\varphi(Z, F)$. We can choose a universe E such that there is a universe D in E and such that $(\exists Z)\varphi(Z, D)$ holds. Now set

$$H := \{\langle x, k\rangle : x \in (E)_k \wedge (\exists Z)\varphi(Z, (E)_k) \wedge \mathsf{U}((E)_k)\}.$$

H is a rel-$\Delta^1_1(\mathsf{U})$ set, since we have

$$\begin{aligned} \langle x, k\rangle \in H &\leftrightarrow (\forall Z)(x \in (E)_k \wedge \psi(Z, (E)_k) \wedge \mathsf{U}((E)_k)) \\ &\leftrightarrow (\exists Z)(x \in (E)_k \wedge \varphi(Z, (E)_k) \wedge \mathsf{U}((E)_k)). \end{aligned}$$

We also know

$$\mathsf{U}(X) \rightarrow (X \dot{\in} H \leftrightarrow (X \dot{\in} E \wedge (\exists Z)\varphi(Z, X))).$$

Hence the universe D is in H. An application of the minimal universe axiom (of T) to the formula $D \dot{\in} H$ yields a universe F such that

$$F \dot{\in} H \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \rightarrow X \dot{\notin} H).$$

Hence, we conclude

$$F \dot{\in} E \wedge (\exists Z)\varphi(Z, F) \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \rightarrow (X \dot{\notin} E \vee \neg(\exists Z)\varphi(Z, X))).$$

We have for all universes X in F that X is in E. Thus

$$(\exists Z)\varphi(Z, F) \wedge (\forall X \dot{\in} F)(\mathsf{U}(X) \rightarrow \neg(\exists Z)\varphi(Z, X)).$$

This is the claim. □

Next we give an infinitary Tait-style version $(\mathsf{MUT}^=)^T$ of $\mathsf{MUT}^=$. Now $\Gamma, \Lambda, \ldots$ denote finite sets of $\mathcal{L}_2(\mathsf{U})$ formulas and Γ, φ is a shorthand for $\Gamma \cup \{\varphi\}$. The system $(\mathsf{MUT}^=)^T$ contains the following axioms and rules of inference.

1. Ontological axioms I. For all closed number terms s, t with identical value, all true literals φ of $\mathcal{L}_1$ and all set variables X:

$$\Gamma, \varphi \qquad \text{and} \qquad \Gamma, t \in X, s \notin X \qquad \text{and} \qquad \Gamma, \mathsf{U}(X), \neg\mathsf{U}(X).$$

2. Propositional and quantifier rules. These include the usual Tait-style inference rules for the propositional connectives and all sorts of quantifiers (especially the ω-rule).

3. Ontological axioms II.

$$\Gamma, \neg \mathsf{U}(X), X \neq Y, \mathsf{U}(Y).$$

4. Set axioms and rules. For all rel-$\Pi^1_0(\mathsf{U})$ formulas φ:

$$\Gamma, (\exists X)(x \in X \leftrightarrow \varphi(x)), \qquad \frac{\Gamma, (\forall x)(\exists X)\varphi(x, X)}{\Gamma, (\exists X)(\forall x)\varphi(x, (X)_x)}.$$

5. Closure axioms. For all rel-$\Pi^1_0(\mathsf{U})$ formulas φ:

$$\Gamma, \neg\mathsf{U}(D), X \mathrel{\dot{\notin}} D, Z \mathrel{\dot{\notin}} D, X \oplus Z \mathrel{\dot{\in}} D,$$
$$\Gamma, \neg\mathsf{U}(D), Z \mathrel{\dot{\notin}} D, (\exists Y \mathrel{\dot{\in}} D)(\forall x)(x \in Y \leftrightarrow \varphi[x, z, Z]),$$
$$\Gamma, \neg\mathsf{U}(D), Z \mathrel{\dot{\notin}} D, \neg(\forall x)(\exists Y \mathrel{\dot{\in}} D)\varphi[x, z, Y, Z], (\exists Y \mathrel{\dot{\in}} D)(\forall x)\varphi[x, z, (Y)_x, Z].$$

6. Universe axioms. For all rel-$\Pi^1_0(\mathsf{U})$ formulas φ:

$$\Gamma, (\exists Z)(X \mathrel{\dot{\in}} Z \wedge \mathsf{U}(Z)),$$
$$\Gamma, \neg\mathsf{U}(D), \neg\mathsf{U}(E), D \mathrel{\dot{\in}} E, D = E, E \mathrel{\dot{\in}} D,$$
$$\Gamma, (\forall Z)(\neg\mathsf{U}(Z) \vee \neg\varphi(Z)), (\exists Z)[\mathsf{U}(Z) \wedge \varphi(Z) \wedge (\forall F \mathrel{\dot{\in}} Z)(\mathsf{U}(F) \to \neg\varphi(F))].$$

7. Cut rules. These include the usual cut rules.

In a next step we define the classes of $\mathcal{L}_2(\mathsf{U})$ formulas $essrel$-$\Sigma^1_1(\mathsf{U})$ and $essrel$-$\Pi^1_1(\mathsf{U})$. They correspond to ess-Σ^1_1 and ess-Π^1_1 (cf. for example [2]).

Definition 19 The $essrel$-$\Sigma^1_1(\mathsf{U})$ ($essrel$-$\Pi^1_1(\mathsf{U})$) formulas are inductively defined as follows:

1. Each rel-$\Pi^1_0(\mathsf{U})$ formula is an $essrel$-$\Sigma^1_1(\mathsf{U})$ and an $essrel$-$\Pi^1_1(\mathsf{U})$ formula.
2. If φ, ψ are $essrel$-$\Sigma^1_1(\mathsf{U})$ ($essrel$-$\Pi^1_1(\mathsf{U})$, resp.) formulas, then so also are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\forall x\varphi$, $\exists x\varphi$, $(\forall X \mathrel{\dot{\in}} Y)\varphi$, $(\exists X \mathrel{\dot{\in}} Y)\varphi$, $\exists X\varphi$ ($\forall X\varphi$, resp.).

Definition 20 The rank $rk(\varphi)$ of an $\mathcal{L}_2(\mathsf{U})$ formula φ is inductively defined as follows:

If φ is an $essrel$-$\Sigma^1_1(\mathsf{U})$ or an $essrel$-$\Pi^1_1(\mathsf{U})$ formula, then $rk(\varphi) := 0$. Otherwise:

1. If φ is a formula $\psi \vee \theta$ or $\psi \wedge \theta$, then $rk(\varphi) := max(rk(\psi), rk(\theta)) + 1$.

2. If φ is a formula $\exists x\psi$, $\forall x\psi$, $\exists X\psi$, $\forall X\psi$, then $rk(\varphi) := rk(\psi) + 1$.

Corresponding to this rank we have partial cut elimination. Furthermore, we can embed $\mathsf{MUT}^=$ into $(\mathsf{MUT}^=)^T$. Again the proof is standard and we omit it.

Lemma 21 *We have*

a) $(\mathsf{MUT}^=)^T \vdash^{\alpha}_{k+1} \Gamma \quad \Longrightarrow \quad (\mathsf{MUT}^=)^T \vdash^{\omega_k(\alpha)}_{1} \Gamma$,

b) $\mathsf{MUT}^= \vdash \varphi[\vec{x}, \vec{X}] \quad \Longrightarrow \quad (\mathsf{MUT}^=)^T \vdash^{<\omega+\omega}_{<\omega} \varphi[\vec{t}, \vec{X}]$ *for all closed number terms* $\vec{t}$.

Now we define the translation which is used in the asymmetric interpretation.

Definition 22 For all $\mathcal{L}_2(\mathsf{U})$ formulas φ and ordinals $\alpha, \beta < \gamma$ let $\varphi^{\alpha,\beta,\gamma}$ denote the $\mathcal{L}_\gamma$ formula of T_γ which results from φ when each subformula $[\neg]\mathsf{U}(X)$ is replaced by $[\neg](\exists d \prec \gamma)(X = (\mathsf{D}_{<\gamma})_d)$, and each unbounded quantifier $\exists X$ ($\forall X$, resp.) is replaced by $(\exists X \dot{\in} \mathsf{D}_\beta)$ $((\forall X \dot{\in} \mathsf{D}_\beta)$, resp.). A quantifier $\exists X$ ($\forall X$, resp.) is called unbounded if its range is not of the form $X \dot{\in} Y \wedge \ldots$ ($X \dot{\in} Y \rightarrow \ldots$, resp.).

In the next lemma we formulate the persistency of our translation. The proof is by induction on the length of derivation, we omit it.

Lemma 23 *If* $\beta' < \beta < \delta$, $\gamma < \gamma' < \delta \leq \alpha$ *and* $\mathsf{T}_\alpha \vdash^{\rho}_{<\omega} \Gamma, \varphi^{\beta,\gamma,\delta}$ *then* $\mathsf{T}_\alpha \vdash^{<\rho+\omega}_{<\omega} \Gamma, \varphi^{\beta',\gamma',\delta}$.

Now we are ready to state the asymmetric interpretation.

Theorem 24 *For all finite sets* Γ *of* $\mathcal{L}_2(\mathsf{U})$ *formulas and all ordinals* α, β, γ *with* $\beta + \omega^\gamma < \alpha < \varepsilon_0$ *we have:*

$$(\mathsf{MUT}^=)^T \vdash^{\gamma}_{1} \Gamma[\vec{X}] \quad \Longrightarrow \quad \mathsf{T}_\alpha \vdash^{\omega^{\beta+\omega^\gamma}}_{<\omega} \vec{X} \dot{\notin} \mathsf{D}_\beta, \Gamma^{\beta,\beta+\omega^\gamma,\alpha}[\vec{X}].$$

Proof. This theorem is proved by induction on γ. As an example we discuss the minimal universe axioms. The other axioms and rules are dealt with as in similar asymmetric interpretations, cf. e.g. [2, 15, 19]. We write in this proof only $\varphi^{\delta,\lambda}$ short for $\varphi^{\delta,\lambda,\alpha}$.

For technical reasons we introduce a formal system $\bar{\mathsf{T}}_\alpha$. The semi-formal system T_α is a Tait-style version of $\bar{\mathsf{T}}_\alpha$. $\bar{\mathsf{T}}_\alpha$ is formulated in $\mathcal{L}_\alpha$ and is based on the usual axioms and rules for the two-sorted predicate calculus extended by rules for the $\exists X \dot{\in} \mathsf{D}_\beta$ and $\forall X \dot{\in} \mathsf{D}_\beta$ quantifiers ($\beta < \alpha$). We have defining axioms for all primitive recursive functions and relations and

(1) *ontological properties for* $\gamma < \beta < \alpha$

$$(\forall x)(x = \gamma \rightarrow (\mathsf{D}_{<\beta})_x = \mathsf{D}_\gamma),$$

(2) *closure conditions for all* D_β *(*$\beta < \alpha$*)*

$$\begin{array}{ll}(2.1) & Y, Z \dot{\in} \mathsf{D}_\beta \rightarrow (\exists X \dot{\in} \mathsf{D}_\beta)(X = Y \oplus Z),\\ (2.2) & Z \dot{\in} \mathsf{D}_\beta \rightarrow (\exists X \dot{\in} \mathsf{D}_\beta)(\forall x)(x \in X \leftrightarrow \pi_1^0[e, x, z, Z, \mathsf{D}_{<\beta}]).\\ (2.3) & Z \dot{\in} \mathsf{D}_\beta \wedge (\forall x)(\exists X \dot{\in} \mathsf{D}_\beta)\pi_1^0[e, x, z, X, Z, \mathsf{D}_{<\beta}]\\ & \quad \rightarrow (\exists X \dot{\in} \mathsf{D}_\beta)\pi_1^0[e, x, z, (X)_x, Z, \mathsf{D}_{<\beta}].\end{array}$$

We now assume that an instance of the minimal universe axiom occurs in Γ. Let φ be a rel-$\Pi_0^1(\mathsf{U})$ formula where all free set parameters are among $\vec{X}$. Then we have to show

$$\begin{aligned}\mathsf{T}_\alpha \; \frac{\omega\beta+\omega^\gamma}{<\omega} \quad & \vec{X} \dot{\notin} \mathsf{D}_\beta, (\forall Z \dot{\in} \mathsf{D}_\beta)((\forall d \prec \alpha)(Z \neq (\mathsf{D}_{<\alpha})_d) \vee (\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z)),\\ & (\exists Z \dot{\in} \mathsf{D}_{\beta+\omega^\gamma})[(\exists d \prec \alpha)(Z = (\mathsf{D}_{<\alpha})_d) \wedge \varphi^{\beta,\beta+\omega^\gamma}(Z) \wedge \\ & \qquad (\forall F \dot{\in} Z)((\exists d \prec \alpha)(F = (\mathsf{D}_{<\alpha})_d) \rightarrow (\neg\varphi)^{\beta,\beta+\omega^\gamma}(F))].\end{aligned} \tag{2}$$

First we show within $\bar{\mathsf{T}}_\alpha$ that $(\forall X)TI(\beta, X)$ implies

$$\begin{aligned}&\vec{X} \dot{\in} \mathsf{D}_\beta \wedge (\exists Z \dot{\in} \mathsf{D}_\beta)((\exists d \prec \alpha)(Z = (\mathsf{D}_{<\alpha})_d) \wedge \neg(\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z))\\ &\quad \rightarrow (\exists Z \dot{\in} \mathsf{D}_{\beta+\omega^\gamma})[(\exists d \prec \alpha)(Z = (\mathsf{D}_{<\alpha})_d) \wedge \varphi^{\beta,\beta+\omega^\gamma}(Z) \wedge \\ &\qquad (\forall F \dot{\in} Z)((\exists d \prec \alpha)(F = (\mathsf{D}_{<\alpha})_d) \rightarrow (\neg\varphi)^{\beta,\beta+\omega^\gamma}(F))].\end{aligned} \tag{3}$$

By induction on the build-up of φ it can be proved that there is a Π_0^1 formula ψ such that $\bar{\mathsf{T}}_\alpha$ proves

$$\vec{X} \dot{\in} (\mathsf{D}_{<\alpha})_c \wedge c \prec \alpha \rightarrow (\varphi^{\beta,\beta+\omega^\gamma}(Z) \leftrightarrow \psi(y, Z, Y)[y \backslash c, Y \backslash \mathsf{D}_{<\alpha}]).$$

Since φ is in rel-$\Pi_0^1(\mathsf{U})$ we know $(\neg\varphi)^{\beta,\beta+\omega^\gamma} \equiv \neg\varphi^{\beta,\beta+\omega^\gamma}$ and $\neg(\neg\varphi)^{\beta,\beta+\omega^\gamma} \equiv \varphi^{\beta,\beta+\omega^\gamma}$. By assumption there is a Z in D_β such that $Z = (\mathsf{D}_{<\alpha})_d$, $d \prec \alpha$

and $\neg(\neg\varphi)^{\beta,\beta+\omega^\gamma}(Z)$. Hence we can choose $Z \dot\in (\mathsf{D}_{<\alpha})_\beta$, $d \prec \alpha$ such that $Z = (\mathsf{D}_{<\alpha})_d \wedge \psi(\beta, Z, (\mathsf{D}_{<\alpha}))$. We have to prove

$$(\exists G \dot\in \mathsf{D}_{\beta+\omega^\gamma})[(\exists e \prec \alpha)(G = (\mathsf{D}_{<\alpha})_e) \wedge \psi(\beta, G, \mathsf{D}_{<\alpha}) \wedge \\ (\forall F \dot\in G)((\exists e \prec \alpha)(F = (\mathsf{D}_{<\alpha})_e) \to \neg\psi(\beta, F, \mathsf{D}_{<\alpha}))]. \tag{4}$$

We define $H := \{c : c \prec \beta \wedge \psi(\beta, (\mathsf{D}_{<\alpha})_c, \mathsf{D}_{<\alpha})\}$. We have $(\mathsf{D}_{<\alpha})_d \dot\in \mathsf{D}_\beta$, hence $d \prec \beta, d \in H, H \neq \emptyset$. Therefore, we can choose a least c with $c \in H$, since we have assumed $(\forall X)TI(\beta, X)$. This immediately proves (4).

There is an embedding of $\bar{\mathsf{T}}_\alpha$ into T_α. For each formula $\vartheta[\vec{x}, \vec{X}]$ and all number terms $\vec{t}$ we have

$$\bar{\mathsf{T}}_\alpha \vdash \vartheta[\vec{x}, \vec{X}] \quad \Longrightarrow \quad \mathsf{T}_\alpha \mathrel{\vdash\!\!\frac{<\omega}{<\omega}} \vartheta[\vec{t}, \vec{X}].$$

Let θ denote the formula (3). Then we can prove in T_α with finite deduction length $\neg TI(\beta, Y), \theta$. Furthermore, standard arguments show $\mathsf{T}_\alpha \mathrel{\vdash\!\!\frac{\omega\beta}{<\omega}} TI(\beta, Y)$ and a cut implies $\mathsf{T}^0_\alpha \mathrel{\vdash\!\!\frac{<\omega^{\beta+\omega^\gamma}}{<\omega}} \theta$. $\wedge$-inversion and $\vee$-exportation now imply the claim (2). □

We can carry-out an analogous analysis of the theory $\mathsf{MUT}^=_0$, with the difference that here only finitely many D_n $(n \in \mathbb{N})$ are necessary. Instead of a rigorous proof, we give a short sketch of how one proceeds:

1. We fix a Tait-style reformulation $(\mathsf{MUT}^=_0)^T$ of $\mathsf{MUT}^=_0$. It looks like $(\mathsf{MUT}^=)^T$ but instead of the ω-rule we take the $(\forall x)$-rule; we also have to add set-induction.

2. As for $(\mathsf{MUT}^=)^T$ we prove partial cut elimination for $(\mathsf{MUT}^=_0)^T$ and embedding of $\mathsf{MUT}^=_0$ into $(\mathsf{MUT}^=_0)^T$. Notice that all lengths are finite.

3. We introduce the corresponding translation $\varphi^{m,n,k}$ $(m, n, k \in \mathbb{N})$ and prove a corresponding asymmetric interpretation theorem where we need only finitely many universes.

We collect all results in the following corollary.

Corollary 25 *We have for all arithmetic sentences φ the following reductions.*

a) $\mathsf{MUT}^=_0 \vdash \varphi \quad \Longrightarrow \quad$ *There is a $k \in \mathbb{N}$ and a $\gamma < \varepsilon_0$ such that* $\mathsf{T}_k \mathrel{\vdash\!\!\frac{\gamma}{1}} \varphi$.

b) $\mathsf{MUT}^= \vdash \varphi \quad \Longrightarrow \quad$ *There are* $\alpha, \gamma < \varepsilon_0$ *such that* $\mathsf{T}_\alpha \,|\!\frac{\gamma}{1}\, \varphi$.

Since the proof-theoretic analysis of the semi-formal systems T_α is given in [15, 17], we only sketch the computation of the upper bound of $(\mathsf{T}_\alpha)_{\alpha<\varepsilon_0}$ and $(\mathsf{T}_n)_{n<\omega}$. Very briefly, this computation mimics the proof-theoretic analysis of e.g. $\widehat{\mathsf{ID}}_\alpha$ [6].

First, we notice that there is a partial cut elimination theorem for T_α. For technical reasons we embed $\mathsf{T}_{\alpha+1}$ into $\mathsf{E}_{\alpha+1}$, a first order reformulation of $\mathsf{T}_{\alpha+1}$. The formulas of $\mathsf{E}_{\alpha+1}$ are the formulas of $\mathsf{T}_{\alpha+1}$ in which no set variables occur. Establishing a partial cut elimination theorem for $\mathsf{E}_{\alpha+1}$ too, we get for all first order sentences φ

$$\mathsf{T}_{\alpha+1} \,|\!\frac{\gamma}{<\omega}\, \varphi \quad \Longrightarrow \quad \mathsf{E}_{\alpha+1} \,|\!\frac{<\varepsilon(\gamma)}{1}\, \varphi.$$

The proof-theoretic analysis of the semi-formal systems E_α consists of two parts: the finite reduction and the transfinite reduction. For the finite reduction we introduce semi-formal systems $H_\nu\mathsf{E}_\alpha$ in which we have in addition iterated arithmetical comprehension up to ν. Then we prove an asymmetric interpretation of $\mathsf{E}_{\alpha+1}$ into $H_\nu\mathsf{E}_\alpha$ (cf. [2] for a similar argument in the context of choice axioms and comprehension principles). The next step is the elimination of "H_ν" in $H_\nu\mathsf{E}_\alpha$. To achieve this we introduce a system RA_α of ramified analysis. The first order part of RA_α essentially corresponds to E_α. We can embed $H_\nu\mathsf{E}_\alpha$ into RA_α. There is also a partial (second) cut elimination theorem for RA_α. Finally, we embed the first order fragment of RA_α into E_α and obtain for all first order sentences φ

$$\mathsf{E}_{\alpha+1} \,|\!\frac{\gamma}{1}\, \varphi \quad \Longrightarrow \quad \mathsf{E}_\alpha \,|\!\frac{<\varphi\varepsilon(\gamma)0}{1}\, \varphi.$$

The transfinite reduction of E_α is an iteration of this finite reduction and very similar to the reduction of transfinitely many fixed points (cf. [6] Main Lemma II). In particular we can prove for all first order sentences φ

$$\mathsf{E}_{\beta+\omega^{1+\rho}} \,|\!\frac{\gamma}{1}\, \varphi \quad \Longrightarrow \quad \mathsf{E}_\beta \,|\!\frac{\varphi 1\rho\gamma}{1}\, \varphi.$$

Carrying through everything in detail (cf. [15, 17]) finally gives the upper bound of $(\mathsf{T}_\alpha)_{\alpha<\varepsilon_0}$ ($(\mathsf{T}_n)_{n<\omega}$, resp.): $\varphi 1\varepsilon_0 0$ (Γ_0, resp.). Together with corollary 8, 17 and 25 we obtain $|\mathsf{MUT}^=_0| = \Gamma_0$ and $|\mathsf{MUT}^=| = \varphi 1\varepsilon_0 0$. Let us collect the proof-theoretic strengths of the theories of universes in the following corollary.

Corollary 26 *We have*

a) $|\mathsf{NUT}_0| = |\mathsf{UUT}_0| = |\mathsf{MUT}_0| = |\mathsf{MUT}_0^=| = \Gamma_0$,

b) $|\mathsf{NUT}| = \Gamma_{\varepsilon_0}$,

c) $|\mathsf{UUT}| = |\mathsf{MUT}| = |\mathsf{MUT}^=| = \varphi 1 \varepsilon_0 0$.

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