On Modal μ-Calculus And Non-Well-Founded Set Theory

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Abstract. A *finitary* characterization for non-well-founded sets with finite transitive closure is established in terms of a greatest fixpoint formula of the modal μ-calculus. This generalizes the standard result in the literature where a finitary modal characterization is provided only for wellfounded sets with finite transitive closure. The proof relies on the concept of automaton, leading then to new interlinks between automata theory and non-well-founded sets.

Keywords: modal μ-calculus, alternating tree automata, bisimulation, transition systems, non-well-founded sets

1. Introduction

We present a research-line connecting the modal μ-calculus to the theory of non-well-founded sets. A *finitary* characterization for non-well-founded sets with finite transitive closure is established in terms of modal μ-formulae. This generalizes the standard result of Baltag [4] where a finitary characterization is provided only for wellfounded sets with finite transitive closure. The proof proceeds in two steps: We first construct a *characteristic automaton*; then, relying on the equivalence between automata and the modal μ-calculus, we get the corresponding *characteristic μ-formula*. Thus, as an important additional issue, we also establish a characterization for non-well-founded sets in terms of automata.

The modal μ-calculus is an extension of modal logic, with least and greatest fixpoint constructors. The term "μ-calculus" and the idea of extending modal logic with fixpoints appeared for the first time in the paper of Scott and De Bakker [12] and was further developed by others. Nowadays, the term "modal μ-calculus" stands for the formal system introduced by Kozen [8]. It is a powerful logic of programs subsuming dynamic and temporal logics like $PDL, PLTL, CTL$ and CTL^*. Hence, it provides us with the capability of expressing and reasoning about assertions concerning "temporal" properties of dynamic (reactive

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and parallel) systems with potentially infinite behaviour. We refer to Bradfield and Stirling's tutorial article [6] for a thorough introduction to the modal μ-calculus.

Semantics of the modal μ-calculus is usually based on the concept of *transition systems*. Transition systems are of well-established importance in theoretical computer science chiefly as vehicles for operational semantics. Indeed, inherently dynamic structures, occurring in computation, usually involve a notion of state, which can be transformed in various ways. Abstract mathematical description of the *observable behaviour* of such state-based dynamic structures relies on the notion of transition system. Clearly, it can happen that two transition systems represent *observationally indistinguishable* behaviours. Hence, a central question arises: When should two transition systems be identified? Obviously, an equivalence relation between two transition systems must depend on an appropriate space of observations. The intuitive concept of observational indistinguishability is captured by the formal notion of bisimilarity. Thus, bisimilarity between two transition systems may be regarded as the formal equivalent of the fact that the corresponding observable behaviours are equal as far as we can see. We turn to a formal development of our theoretic framework in Section 2, where all these aspects will be made precise by defining appropriate notions of syntax and semantics.

Since the seminal work of Aczel [1], non-well-founded sets have been regarded as an alternative and uniform treatment of dynamic systems. Nowadays, the application area of non-well-founded sets ranges from knowledge representation and theoretical economics to semantics of natural and programming languages. On the other hand, it is worth mentioning that work in non-well-founded set theory had begun long before Aczel [1]. Forster in [7] claims:

> ... set theory was *born* ill-founded (had it not been, Russell's and Cantor's paradoxes would not have been discovered when they were, or indeed, at all!) and wellfounded set theory is merely a pampered part of it.

In 1917 it was Mirimianoff [9, 10] who first distinguished between well-founded and non-well-founded sets. In 1954 the relative independence of the axiom of foundation was established by Bernays [5], and since then several existence axioms for non-well-founded sets have been proposed. In the sequel we adopt Aczel's formulation of the anti-foundation axiom, AFA, asserting that

every graph has a unique decoration.

A strong connection between modal logic and non-well-founded sets has been established by Baltag, Barwise and Moss in [3, 4] where the following characterization of sets in terms of an *infinitary modal logic* is proved:

Every non-well-founded set is characterizable by some formula of an infinitary modal language.

The above-mentioned characterization would not hold if we restrict ourselves to a finitary language (see [4], Proposition 11.6, p.135). Thus, we may ask: When is a non-well-founded set characterizable by some finitary modal formula? Baltag in [4] provides the answer:

A non-well-founded set a is characterizable by some formula of a finitary modal language if and only if a is wellfounded and $TC(\{a\})$ *is finite.*

Hence, it turns out that characterizability of sets by modal formulae depends, in a sensitive way, on whether we are using the full infinitary language or just the finitary one. In particular, if a set - with finite transitive closure - is *not* wellfounded then it is only characterizable by an infinitary modal formula. It is the main purpose of the present contribution to provide a *finitary characterization* for such sets by means of the modal μ-calculus. Our main result is:

A non-well-founded set a is characterizable by some formula of the modal μ*-calculus if and only if* $TC(\{a\})$ *is finite.*

The rest of the paper is organized as follows. In Section 2, basic notions and results, including Wilke's alternating tree automata and their equivalence to the modal μ-calculus, are formally introduced. In Section 3, we provide a new proof of the existence of the, so-called, *characteristic formulae* for finite transition systems, by means of the above-mentioned equivalence. In Section 4, a finitary characterization for non-well-founded sets is established.

2. Preliminaries

In the language of the modal μ-calculus all the primitive symbols are among the set $\mathsf{P} = \{p, q, ..., X, Y, ...\}$ of *propositional variables* and the symbols $\top, \bot, \wedge, \vee, \neg, \Box, \Diamond, \mu, \nu$.

The class of μ-formulae, denoted by $\varphi, \psi, \alpha, \beta, \gamma, ...$, is defined as follows:

$$\varphi \quad ::= \quad \bot \quad | \quad \top \quad | \quad p \quad | \quad (\alpha \wedge \beta) \quad | \quad (\alpha \vee \beta) \quad | \quad \neg\alpha \quad | \quad \Box\alpha \quad | \quad \Diamond\alpha \quad | \quad \mu X.\alpha \quad | \quad \nu X.\alpha$$

As usual for formulae of the form $\mu X.\alpha$, $\nu X.\alpha$ we require the *syntactic monotonicity* of α with respect to X: Every occurrence of the variable X in α must be within the scope of an even number of negations.

The standard semantics of the modal μ-calculus is given by transition systems. A transition system $\mathcal{S}$ is a triple $(S, \to_{\mathcal{S}}, \lambda)$ consisting of

- a set S of *states*,
- a binary relation $\to_{\mathcal{S}} \subseteq S \times S$ known as *transition relation*,
- the *valuation* $\lambda : \mathsf{P} \to \mathcal{P}(S)$ assigning to each propositional variable p a subset $\lambda(p)$ of S.

We write $s \to_{\mathcal{S}} t$ for $(s,t) \in \to_{\mathcal{S}}$. Let λ be a valuation on $\mathcal{P}(S)$, p a propositional variable and S' an element of $\mathcal{P}(S)$; we set for all propositional variables p'

$$\lambda[p \mapsto S'](p') = \begin{cases} S' & \text{if } p' = p \\ \lambda(p') & \text{otherwise.} \end{cases}$$

Given a transition system $\mathcal{S} = (S, \to_{\mathcal{S}}, \lambda)$, then $\mathcal{S}[p \mapsto S']$ denotes the transition system $(S, \to_{\mathcal{S}}, \lambda[p \mapsto S'])$. Let φ be a μ-formula and $\mathcal{S}$ a transition system, the set of states where φ holds, denoted by $\|\varphi\|_{\mathcal{S}}$, is called the *denotation of* φ *in* $\mathcal{S}$. The definition of $\|\varphi\|_{\mathcal{S}}$ proceeds by induction on the complexity of φ . Simultaneously for all transition systems $\mathcal{S}$, we set

- $\|p\|_{\mathcal{S}} = \lambda(p)$ for all $p \in \mathsf{P}$,
- $\|\neg\alpha\|_{\mathcal{S}} = S - \|\alpha\|_{\mathcal{S}}$,
- $\|\alpha \wedge \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cap \|\beta\|_{\mathcal{S}}$,
- $\|\alpha \vee \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cup \|\beta\|_{\mathcal{S}}$,
- $\|\Box\alpha\|_{\mathcal{S}} = \{s \in S \mid \forall t((s \to_{\mathcal{S}} t) \to t \in \|\alpha\|_{\mathcal{S}})\}$,
- $\|\Diamond\alpha\|_{\mathcal{S}} = \{s \in S \mid \exists t((s \to_{\mathcal{S}} t) \wedge t \in \|\alpha\|_{\mathcal{S}})\}$,
- $\|\nu X.\alpha\|_{\mathcal{S}} = \bigcup\{S' \subseteq S \mid S' \subseteq \|\alpha(X)\|_{\mathcal{S}[X \mapsto S']}\}$,
- $\|\mu X.\alpha\|_{\mathcal{S}} = \bigcap\{S' \subseteq S \mid \|\alpha(X)\|_{\mathcal{S}[X \mapsto S']} \subseteq S'\}$.

By Tarski-Knaster Theorem, cf. [15], $\|\nu X.\alpha\|_{\mathcal{S}}$ ($\|\mu X.\alpha\|_{\mathcal{S}}$) is the greatest (least) fixpoint of the operator

$$S' \mapsto \|\alpha(X)\|_{\mathcal{S}[X \mapsto S']}.$$

By $s \models \varphi$ we mean that φ is *valid in* s (i.e. $s \in \|\varphi\|_{\mathcal{S}}$). If φ is valid in all states s of $\mathcal{S}$, then φ is said to be *valid in* $\mathcal{S}$ and we write $\mathcal{S} \models \varphi$. If φ is valid in all transition systems $\mathcal{S}$, we say φ is *valid* and we write $\models \varphi$.

A *pointed transition system* is a pair $(\mathcal{S}, s_I)$ consisting of a transition system $\mathcal{S}$ and a state $s_I \in S$. Sometimes, we denote a pointed transition system $(\mathcal{S}, s_I)$ by $(S, \rightarrow_{\mathcal{S}}, \lambda, s_I)$ for $\mathcal{S}$ being of the form $(S, \rightarrow_{\mathcal{S}}, \lambda)$. The *extension of* φ, denoted by $\|\varphi\|$, is defined to be the class of all pointed transition systems $(\mathcal{S}, s_I)$ such that $s_I \in \|\varphi\|_{\mathcal{S}}$.

In our setting, the definition of bisimilarity is formulated for pointed transition systems. Let L be a subset of P. A relation $R \subset S \times S'$ is a L-*bisimulation* between two pointed transition systems $(S, \rightarrow_{\mathcal{S}}, \lambda, s_I)$ and $(S', \rightarrow_{\mathcal{S}'}, \lambda', s'_I)$ if the following conditions hold:

(i) $(s_I, s'_I) \in R$,

(ii) if $(s, s') \in R$ and $s \rightarrow_{\mathcal{S}} t$ then there is a $t' \in S'$ such that $s' \rightarrow_{\mathcal{S}'} t'$ and $(t, t') \in R$,

(iii) if $(s, s') \in R$ and $s' \rightarrow_{\mathcal{S}'} t'$ then there is a $t \in S$ such that $s \rightarrow_{\mathcal{S}} t$ and $(t, t') \in R$,

(iv) if $(s, s') \in R$ then $\{p \in \mathsf{L} \mid s \in \lambda(p)\} = \{p \in \mathsf{L} \mid s' \in \lambda'(p)\}$.

Two pointed transition systems $(\mathcal{S}, s_I)$ and $(\mathcal{S}', s'_I)$ are said to be L-*bisimilar* if and only if there is a L-bisimulation between them. We then write

$$(\mathcal{S}, s_I) \sim_{\mathsf{L}} (\mathcal{S}', s'_I).$$

For any pointed transition system $(\mathcal{S}, s_I)$ we define $\|(\mathcal{S}, s_I)\|_{\sim_{\mathsf{L}}}$ to be the class of all pointed transition systems L-bisimilar to $(\mathcal{S}, s_I)$, that is,

$$\|(\mathcal{S}, s_I)\|_{\sim_{\mathsf{L}}} = \{(\mathcal{S}', s'_I) \mid (\mathcal{S}, s_I) \sim_{\mathsf{L}} (\mathcal{S}', s'_I)\}.$$

Lemma 1
Let φ be a μ-formula whose propositional variables range over L and let $(\mathcal{S}, s_I)$ and $(\mathcal{S}', s'_I)$ be L-bisimilar, then we have

$$(\mathcal{S}, s_I) \in \|\varphi\| \quad \Leftrightarrow \quad (\mathcal{S}', s'_I) \in \|\varphi\|.$$

The proof follows from the definitions of denotation and extension; it goes by induction on the complexity of φ.

Let us define alternating tree automata as introduced by Wilke in [16]. An *alternating tree automaton* $\mathcal{A}$ is a tuple $\mathcal{A} = (Q, \mathsf{P}, q_I, \delta, \Omega)$ consisting of

- a finite set Q of *states*,
- a set P of *propositional variables*,
- an *initial state* $q_I \in Q$,
- a *priority function* $\Omega : Q \to \omega$,
- the *transition function* $\delta : Q \to \mathsf{TC}^{Q \cup \mathsf{P}}$ where the set $\mathsf{TC}^{Q \cup \mathsf{P}}$, consisting of all *transition conditions over* $Q \cup \mathsf{P}$, is inductively defined as follows
  - $\bot, \top \in \mathsf{TC}^{Q \cup \mathsf{P}}$,
  - $p, \neg p \in \mathsf{TC}^{Q \cup \mathsf{P}}$ for all $p \in \mathsf{P}$,
  - $q \in \mathsf{TC}^{Q \cup \mathsf{P}}$ for all $q \in Q$,
  - if $t \in \mathsf{TC}^{Q \cup \mathsf{P}}$ then $\Box t, \Diamond t \in \mathsf{TC}^{Q \cup \mathsf{P}}$,
  - if $t, t' \in \mathsf{TC}^{Q \cup \mathsf{P}}$ then $(t \wedge t'), (t \vee t') \in \mathsf{TC}^{Q \cup \mathsf{P}}$.

Remark 2
Notice that a transition condition $\delta(q)$ is a μ-formula over propositional variables in $Q \cup P$. We sometimes write $\delta_q(q_1, \ldots, q_n)$ if $\delta(q)$ can be interpreted as a μ-formula whose variables are among $\{q_1, \ldots, q_n\} \cup \mathsf{P}$.

Let $\mathcal{A}$ be an alternating tree automaton containing a state q_0 and let $\mathcal{S}$ be a transition system containing a state s_0. We define ϱ to be a *q_0-run on s_0 of $\mathcal{A}$ on $\mathcal{S}$* if ϱ is a $(S \times Q)$-vertex-labeled tree of the form (V, E, ℓ) for V being a set of *vertices*, E being a binary relation on V, and $\ell : V \to (S \times Q)$ being the *labeling function*. If v_0 is the root of V then $\ell(v_0)$ must be (s_0, q_0). Moreover, for all vertices $v \in V$ with $\ell(v) = (s, q)$, the following conditions must be fulfilled

- $\delta(q) \neq \bot$,
- if $\delta(q) = p$ then $s \in \|p\|_{\mathcal{S}}$ and if $\delta(q) = \neg p$ then $s \notin \|p\|_{\mathcal{S}}$,
- if $\delta(q) = q'$ then there exists a $v' \in E(v)$ such that $\ell(v') = (s, q')$,
- if $\delta(q) = \Diamond q'$ then there is a $v' \in E(v)$ such that $\ell(v') = (s', q')$ for $s \to_{\mathcal{S}} s'$,
- if $\delta(q) = \Box q'$ then for all s' such that $s \to_{\mathcal{S}} s'$ there is a $v' \in E(v)$ such that $\ell(v') = (s', q')$,
- if $\delta(q) = q' \vee q''$ then there is a $v' \in E(v)$ such that either $\ell(v') = (s, q')$ or $\ell(v') = (s, q'')$,

- if $\delta(q) = q' \wedge q''$ then there are $v', v'' \in E(v)$ such that $\ell(v') = (s, q')$ and $\ell(v'') = (s, q'')$.

The following lemma provides us with an alternative definition of q_0-run on s_0 of $\mathcal{A}$ on $\mathcal{S}$. This definition allows us to extend the previous notion of automaton in such a way that arbitrary fixpoint-free modal μ-formulae can be used as transition conditions.

Let $\mathcal{A} = (Q, P, q_0, \delta, \Omega)$ be an automaton, $\mathcal{S} = (S, R, \lambda)$ a transition system and $\varrho = (V, E, \ell)$ a $(S \times Q)$-vertex-labeled tree. For all $v \in V$ and $q \in Q$ we define

$$S_{E(v)|q} := \{s \in S \mid (\exists v' \in E(v))\ (\ell(v') = (s, q))\}.$$

The following lemma can be proven by unwinding the definitions.

Lemma 3
Let $\mathcal{A} = (\{q_0, \ldots, q_n\}, P, q_0, \delta, \Omega)$ be an automaton, $\mathcal{S} = (S, R, \lambda)$ a transition system and $\varrho = (V, E, \ell)$ a $(S \times Q)$-vertex-labeled tree with root v_0. For all $s_0 \in S$ the following two sentences are equivalent:

– *$\varrho = (V, E, \ell)$ is a q_0-run on s_0 of $\mathcal{A}$ on $\mathcal{S}$,*

– *$\ell(v_0) = (s_0, q_0)$ and for all vertices v which are labeled by (s, q) we have*

$$s \in \|\delta(q)\|_{\mathcal{S}[q_0 \mapsto S_{E(v)|q_0}, \ldots, q_n \mapsto S_{E(v)|q_n}]}.$$

An *infinite branch* of a run is *accepting* if the highest priority appearing infinitely often is even. A *run is accepting* if so are all infinite branches. An automaton $\mathcal{A}$ *accepts a pointed transition system* $(\mathcal{S}, s_I)$ if and only if there exists an accepting q_I-run on s_I of $\mathcal{A}$ on $\mathcal{S}$ (for q_I being $\mathcal{A}$'s initial state). The following two abbreviations are adopted. Let $\mathcal{A}$ be an automaton and $\mathcal{S}$ a transition system. $\|\mathcal{A}\|_{\mathcal{S}}$ denotes the set of all states s of $\mathcal{S}$ such that $\mathcal{A}$ accepts $(\mathcal{S}, s)$. And $\|\mathcal{A}\|$ denotes the class of all pointed transition systems accepted by $\mathcal{A}$.

The following two theorems state the equivalence between alternating tree automata and the modal μ-calculus. The first one translates the modal μ-calculus into alternating tree automata; it is due to Wilke [16].

Theorem 4
For all modal μ-formulae φ we can construct an automaton $\mathcal{A}_\varphi$ such that

$$\|\varphi\| = \|\mathcal{A}_\varphi\|.$$

The next theorem translates alternating tree automata into the modal μ-calculus; its proof is due to Wilke and Alberucci and can be found in Alberucci [2]. It is a generalization of an analogous result by Niwinski in [11] where a translation for binary transition system is done.

Theorem 5
For any alternating tree automaton $\mathcal{A} = (Q, \mathsf{P}, q_I, \delta, \Omega)$ we can construct a a μ-formula $\varphi_{\mathcal{A}}$ over propositional variables $\mathsf{P} \cup Q$ such that

$$\|\mathcal{A}\| = \|\varphi_{\mathcal{A}}\|.$$

Let us conclude this section with an example illustrating the translation of automata into the modal μ-calculus.

Example 6
Let $\mathcal{A}$ be the automaton of the form

$$\mathcal{A} = (\{q_0, q_1\}, \{p_0, p_1\}, q_0, \delta, \Omega),$$

such that $\Omega(q_0) = \Omega(q_1) = 0$ and $\delta(q_0) = \alpha_0$ and $\delta(q_1) = \alpha_1$ where

$$\begin{aligned} \alpha_0 &:= \Box(q_0 \vee q_1) \wedge \Diamond q_0 \wedge \Diamond q_1 \wedge p_0 \wedge \neg p_1 \\ \alpha_1 &:= \Box\bot \wedge \neg p_0 \wedge p_1. \end{aligned}$$

The μ-formula $\varphi_{\mathcal{A}}$, such that $\|\mathcal{A}\| = \|\varphi_{\mathcal{A}}\|$, is defined as

$$\varphi_{\mathcal{A}} \equiv \alpha_0[q_0/(\nu X.\Box(X \vee \alpha_1) \wedge \Diamond X \wedge \Diamond\alpha_1 \wedge p_0 \wedge \neg p_1);\;\; q_1/\alpha_1].$$

3. Characteristic Formulae

The existence of *characteristic formulae* for finite transition systems has been proved by Steffen in [13]. In the sequel, we give a new proof by making use of the equivalence between the modal μ-calculus and automata.

The theorem below states that for each pointed transition system there exists a *characteristic formula*, that is, a modal μ-formula representing it modulo bisimulation.

Theorem 7
For each finite pointed transition system $(\mathcal{S}, s_I)$ and each finite set of propositional variables $\mathsf{L} \subset \mathsf{P}$ there is a modal μ-formula $\varphi^{\mathsf{L}}_{(\mathcal{S},s_I)}$ such that

$$\|(\mathcal{S}, s_I)\|_{\sim_{\mathsf{L}}} = \|\varphi^{\mathsf{L}}_{(\mathcal{S},s_I)}\|.$$

Our proof strategy is as follows: We first construct a *characteristic automaton* $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ representing the pointed transition system modulo bisimulation and then apply Theorem 5 to get the desired μ-formula. Let us define $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$:

Let $(S, \rightarrow_{\mathcal{S}}, \lambda, s_I)$ be a finite pointed transition system and $\mathsf{L} \subset \mathsf{P}$ a finite subset of propositional variables. $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ is the automaton of the form

$$(Q_S, \mathsf{L}, q_{s_I}, \delta, \Omega)$$

such that $Q_S = \{q_s \mid s \in S\}$, $\Omega(q_s) = 0$ for all $q_s \in Q_S$, and

$$\delta(q_s) = \mathsf{suc}(\{q_{s'} \mid s \rightarrow_{\mathcal{S}} s'\}) \wedge \bigwedge_{\substack{p \in \mathsf{L},\\ s \in \lambda(p)}} p \wedge \bigwedge_{\substack{p \in \mathsf{L},\\ s \notin \lambda(p)}} \neg p$$

whereby

– $\mathsf{suc}(\emptyset) = \Box\bot$ and

– $\mathsf{suc}(\{q_1, \ldots, q_n\}) = \Box(q_1 \vee \ldots \vee q_n) \wedge \Diamond q_1 \wedge \ldots \wedge \Diamond q_n$.

The following lemma gives an ad hoc definition of accepting runs of automata of the form $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$.

Lemma 8
Let $\mathcal{S} = (S, \rightarrow_{\mathcal{S}}, \lambda)$ and $\mathcal{S}' = (S', \rightarrow_{\mathcal{S}'}, \lambda')$ be finite transition systems. $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ accepts $(\mathcal{S}', s'_I)$ if and only if there is a function ℓ^T from a rooted tree (T, t_0) to $S' \times Q_S$ such that $\ell^T(t_0) = (s'_I, q_{s_I})$ and such that if $\ell^T(t) = (s', q_s)$ then:

1. *$\{p \in \mathsf{L} \mid s' \in \lambda'(p)\} = \{p \in \mathsf{L} \mid s \in \lambda(p)\}$.*
2. *For each $\bar{s}$ with $s \rightarrow_{\mathcal{S}} \bar{s}$ there is a child t' of t and a $\bar{s}'$ such that $s' \rightarrow_{\mathcal{S}'} \bar{s}'$ with*
$$\ell^T(t') = (\bar{s}', q_{\bar{s}}).$$
3. *For each $\bar{s}'$ with $s' \rightarrow_{\mathcal{S}'} \bar{s}'$ there is a child t' of t and a $\bar{s}$ such that $s \rightarrow_{\mathcal{S}} \bar{s}$ with*
$$\ell^T(t') = (\bar{s}', q_{\bar{s}}).$$

Proof Let us show that there is a function ℓ^T fulfilling the requirements of the lemma if and only if there is an accepting run. First we observe that, since Ω maps everything to 0, we accept any infinite branch and that the initial node must be labeled by (s'_I, q_{s_I}) for both ℓ^T and an accepting run. Hence, in order to prove the "only if" implication we only have to show that the conditions formulated in this lemma imply those formulated in Lemma 3. So, we assume that we have a function ℓ^T fulfilling the requirements of this lemma and that $\ell^T(t) = (s', q_s)$. Further, assume that for all internal states q_{s_i}, S_i is the set containing all $s \in S'$ such that there is a child t' of t with $\ell^T(t') = (s, q_i)$. It is enough to show that

$$s' \in \|\delta(q_s)\|_{\mathcal{S}'[q_0 \mapsto S_0, \ldots, q_n \mapsto S_n]}.$$

If

$$\delta(q_s) = \Box\bot \wedge \bigwedge_{\substack{p \in \mathsf{L}, \\ s \in \lambda(p)}} p \wedge \bigwedge_{\substack{p \in \mathsf{L}, \\ s \notin \lambda(p)}} \neg p$$

then $\{\bar{s} \mid s \rightarrow_{\mathcal{S}} \bar{s}\} = \emptyset$ and by condition 3 of the lemma there is no $\bar{s}'$ such that $s' \rightarrow_{\mathcal{S}'} \bar{s}'$. Further, by condition 1 we have

$$\{p \in \mathsf{L} \mid s' \in \lambda'(p)\} = \{p \in \mathsf{L} \mid s \in \lambda(p)\}.$$

Hence, we get

$$s' \in \|\delta(q_s)\|_{\mathcal{S}'[q_0 \mapsto S_0, \ldots, q_n \mapsto S_n]}.$$

If

$$\delta(q_s) = \Box(q_{s_1} \vee \ldots \vee q_{s_n}) \wedge \Diamond q_{s_1} \wedge \ldots \wedge \Diamond q_{s_n} \wedge \bigwedge_{\substack{p \in \mathsf{L}, \\ s \in \lambda(p)}} p \wedge \bigwedge_{\substack{p \in \mathsf{L}, \\ s \notin \lambda(p)}} \neg p$$

then $\{\bar{s} \mid s \rightarrow_{\mathcal{S}} \bar{s}\} = \{s_1, \ldots, s_n\}$ and by condition 2 of the lemma for each q_{s_i} there is a $\bar{s}'$ such that $s' \rightarrow_{\mathcal{S}'} \bar{s}'$ and there is a t', child of t, such that $\ell^T(t') = (\bar{s}', q_{s_i})$. Thus, we get

$$s' \in \|\Diamond q_{s_i}\|_{\mathcal{S}'[q_0 \mapsto S_0, \ldots, q_n \mapsto S_n]}. \tag{1}$$

By condition 3 for each $\bar{s}'$ such that $s' \rightarrow_{\mathcal{S}'} \bar{s}'$ there is a child t' of t and a $\bar{s}$ such that $s \rightarrow_{\mathcal{S}} \bar{s}$ with $\ell(t') = (\bar{s}', q_{\bar{s}})$. And thus we get

$$s' \in \|\Box(q_{s_1} \vee \ldots \vee q_{s_n})\|_{\mathcal{S}'[q_0 \mapsto S_0, \ldots, q_n \mapsto S_n]}. \tag{2}$$

By condition 1, we have

$$\{p \in \mathsf{L} \mid s' \in \lambda'(p)\} = \{p \in \mathsf{L} \mid s \in \lambda(p)\}$$

and this, together with (1) and (2), implies

$$s' \in \|\delta(q_s)\|_{\mathcal{S}'[q_0 \mapsto S_0, \ldots, q_n \mapsto S_n]}.$$

This proves the “only if” direction. The “if” direction, that is, the fact the local conditions of Lemma 3 imply the conditions of this lemma, can be shown with similar arguments. □

Before formally proving that $\|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\| = \|(\mathcal{S},s_I)\|_{\sim_{\mathsf{L}}}$ for all pointed transition systems $(\mathcal{S},s_I)$ let us elaborate an example to get a feeling for what is going on.

Example 9
Suppose we are given two non-bisimilar *pointed transition systems* $(\mathcal{S},s_0)$ *and* $(\mathcal{S}',s'_0)$ *as depicted in* FIGURE 1 *and* FIGURE 2 *respectively.* $(\mathcal{S},s_0)$ *is defined by*

$$\mathcal{S} = (\{s_0,s_1\},\rightarrow_{\mathcal{S}},\lambda)$$

with $\lambda(p_0) = \{s_0\}$, $\lambda(p_1) = \{s_1\}$, *and* $\rightarrow_{\mathcal{S}} = \{(s_0,s_1),(s_0,s_0)\}$. $(\mathcal{S}',s'_0)$ *is given by*

$$\mathcal{S}' = (\{s'_0\},\rightarrow_{\mathcal{S}'},\lambda')$$

with $\lambda'(p_0) = \{s'_0\}$, $\lambda(p_1) = \emptyset$, *and* $\rightarrow_{\mathcal{S}'} = \{(s'_0,s'_0)\}$.

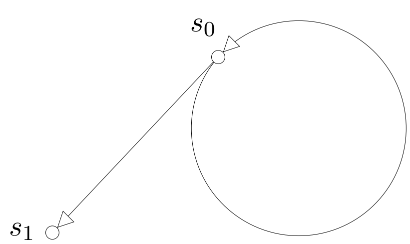

FIGURE 1

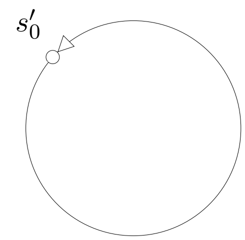

FIGURE 2

Let us first introduce the automaton $\mathcal{A}^{\{p_0,p_1\}}_{(\mathcal{S},s_0)}$ *representing the pointed transition system* $(\mathcal{S},s_0)$. *It is of the form*

$$\mathcal{A}^{\{p_0,p_1\}}_{(\mathcal{S},s_0)} = (\{q_{s_0},q_{s_1}\},\{p_0,p_1\},q_{s_0},\delta,\Omega)$$

such that $\Omega(q_{s_0}) = \Omega(q_{s_1}) = 0$ *and*

$$\begin{aligned}\delta(q_{s_0}) &= \Box(q_{s_0} \vee q_{s_1}) \wedge \Diamond q_{s_0} \wedge \Diamond q_{s_1} \wedge p_0 \wedge \neg p_1,\\ \delta(q_{s_1}) &= \Box\bot \wedge \neg p_0 \wedge p_1.\end{aligned}$$

By Lemma 8, an accepting run (among several others) of $\mathcal{A}^{\{p_0,p_1\}}_{(\mathcal{S},s_0)}$ *on* $(\mathcal{S},s_0)$ *might be as depicted in* FIGURE 3.

Let us show that $\mathcal{A}^{\{p_0.p_1\}}_{(\mathcal{S},s_0)}$ *does not accept* $(\mathcal{S}',s'_0)$. *For, if this is not the case then by Lemma 8 there would be a tree as depicted in* FIGURE 4 *with root* $l^T(t_0) = (s'_0,q_{s_0})$ *and children* (s'_0,q_{s_1}), (s'_0,q_{s_0}) *(whose occurrences being required by* $\Diamond q_{s_0}$ *and* $\Diamond q_{s_1}$ *in* $\delta(q_{s_0})$*). Applying the condition* $\delta(q_{s_1}) = \Box\bot\wedge\neg p_0\wedge p_1$ *to* (s'_0,q_{s_1}), *by Lemma 8, it follows that* s'_0 *must be a terminal state. This is not the case, so,* $\mathcal{A}^{\{p_0.p_1\}}_{(\mathcal{S},s_0)}$ *cannot accept* $(\mathcal{S}',s'_0)$.

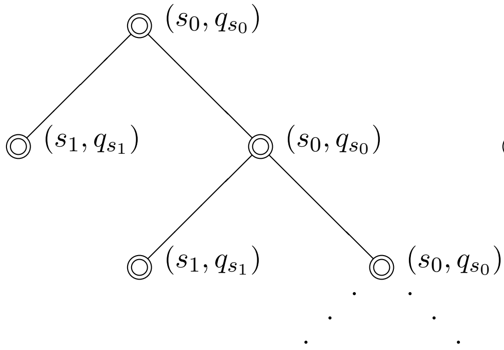

FIGURE 3

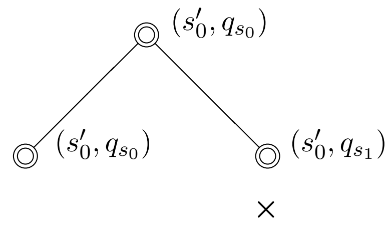

FIGURE 4

The next lemma shows us that the automaton $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ indeed characterizes $(\mathcal{S}, s_I)$ modulo L-bisimulation.

Lemma 10
For all pointed transition systems $(\mathcal{S}, s_I)$ and finite $\mathsf{L} \subset \mathsf{P}$ we have

$$\|(\mathcal{S}, s_I)\|_{\sim_{\mathsf{L}}} = \|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\|.$$

PROOF "$\subseteq$": It is enough to show that

$$(\mathcal{S}, s_I) \in \|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\| \tag{1}$$

since the acceptance of all L-bisimilar ones follows from Lemma 1. To prove (1) we inductively construct a labeled tree $\ell^T : (T, t_0) \to S \times Q_S$ satisfying the conditions of Lemma 8: We set $\ell^T(t_0) = (s_I, q_{s_I})$. For the induction step, if $\ell^T(t) = (s, q_s)$ then for each s' such that $s \to_{\mathcal{S}} s'$ we take a successor $t_{s'}$ of t and set $\ell^T(t_{s'}) = (s', q_{s'})$. Clearly, ℓ^T satisfies the conditions of Lemma 8.

"$\supseteq$": Let $(\mathcal{S}', s'_I)$ be a pointed transition system where $\mathcal{S}' = (S', \to_{\mathcal{S}'}, \lambda')$. It is enough to show that if $(\mathcal{S}', s'_I) \in \|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\|$ then there is a L-bisimulation R between $(\mathcal{S}, s_I)$ and $(\mathcal{S}', s'_I)$. So, assume that there is an accepting run of $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ on $(\mathcal{S}', s'_I)$. By Lemma 8 there is a rooted tree (T, t_0) and a function $\ell^T : T \to S' \times Q_S$ fulfilling the requirements of Lemma 8. We define the relation $R \subseteq S' \times S$ as

$$R = \{(s, s') \in S \times S' \mid \text{ there is a } t \in T \text{ such that } \ell^T(t) = (s', q_s)\}.$$

Let us prove that R is a L-bisimulation.

(i) We have that $(s'_I, s_I) \in R$.

(ii) Assume that $(s', s) \in R$ and $s' \to_{\mathcal{S}'} m'$. Then, there is a $t \in T$ such that $\ell^T(t) = (s', q_s)$. By condition 3 of Lemma 8, there is a $t' \in T$ and a $m \in S$ such that $s \to_{\mathcal{S}} m$ and such that $\ell^T(t') = (m', q_m)$. Since by construction of R we have $(m', m) \in R$ the second assertion is shown.

(iii) Assume that $(s', s) \in R$ and $s \to_{\mathcal{S}} m$. Hence, there is a $t \in T$ such that $\ell^T(t) = (s', q_s)$. By condition 2 of Lemma 8, there is a $t' \in T$ and a $m' \in S'$ such that $s' \to_{\mathcal{S}'} t'$ and such that $\ell^T = (m', q_m)$. Since by construction of R we have $(m', m) \in R$ the third assertion is shown.

(iv) Let $(s', s) \in R$, then, there is $t \in T$ such that $\ell^T(t) = (s', q_s)$. By condition 1 of Lemma 8, the same propositional variables $p \in \mathsf{L}$ are satisfied in s and s'. This proves the final assertion and the lemma.

□

We are now able to complete the proof of Theorem 7.

PROOF Let $(\mathcal{S}, s_I)$ be a finite pointed transition system and $\mathsf{L} \subset \mathsf{P}$ a finite set of propositional variables. By Lemma 10, for the automaton $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ we have

$$\|(\mathcal{S}, s_I)\|_{\sim_{\mathsf{L}}} = \|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\|.$$

By Theorem 5 there is a μ-formula $\varphi^{\mathsf{L}}_{(\mathcal{S},s_I)}$ such that

$$\|\varphi^{\mathsf{L}}_{(\mathcal{S},s_I)}\| = \|\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}\|.$$

This proves the theorem. □

Remark 11
By looking more closely at the proof of Theorem 5 and at the automaton $\mathcal{A}^{\mathsf{L}}_{(\mathcal{S},s_I)}$ we could notice that the formula $\varphi^{\mathsf{L}}_{(\mathcal{S},s_I)}$ only contains greatest fixpoints; that is, it belongs to the first level of the modal μ-calculus hierarchy.

4. A Finitary Modal Characterization for Non-Well-Founded Sets

We provide a new characterization result by using Theorem 7. Let us first review some well-known preliminary definitions and facts. We refer to Barwise and Moss [4] and Aczel [1] for a thorough introduction to the theory of non-well-founded sets.

We assume a class $\mathcal{U}$ of *urelements* which will be regarded as propositional variables. Since $\mathcal{U}$ is a proper class we always consider only an arbitrary set A of urelements. Let $V_{afa}[A]$ be the class of sets over A; note that no urelements belong to $V_{afa}[A]$.

A *(A-)labeled graph* $\mathcal{G} = (N_{\mathcal{G}}, \rightarrow_{\mathcal{G}}, \lambda^{\mathcal{G}})$ is a transition system labeled with subsets of a set $A \subset \mathcal{U}$, that is,

$$\lambda^{\mathcal{G}} : N_{\mathcal{G}} \rightarrow \mathcal{P}(A).$$

When no ambiguity arises, we simply write labeled graph without specifying the set A. Note that graphs and transition systems are equivalent notions. A *pointed labeled graph* $(\mathcal{G}, g_0)$ is a labeled graph together with a distinguished node g_0 which is called its *point*. A *(labeled) decoration of a labeled graph* $\mathcal{G}$ is a function

$$d_{\mathcal{G}} : N_{\mathcal{G}} \rightarrow V_{afa}[A],$$

such that

$$d_{\mathcal{G}}(a) = \{d_{\mathcal{G}}(b) \mid a \rightarrow_{\mathcal{G}} b\} \cup \lambda^{\mathcal{G}}(a).$$

Aczel's formulation of the (labeled) anti-foundation axiom, AFA, asserts that *every (labeled) graph has a unique (labeled) decoration.* Let $\mathsf{ZFC}^{-}{+}\mathsf{AFA}$ denote the axiom system obtained by replacing the axiom of foundation in ZFC (including urelements cf.[4]) by AFA. For the rest of this section $\mathsf{ZFC}^{-}{+}\mathsf{AFA}$ is our underlying theory.

Given any set a it is possible to associate to it a unique (labeled) pointed graph $(\mathcal{TC}(a), a) = (TC(\{a\}), \in^{-1}, a, l)$ with the set of nodes given by the transitive closure of $\{a\}$, the edge relation given by the *converse membership* condition on $TC(\{a\})$, the *point* given by the set a itself and for any set b belonging to the node set $TC(\{a\})$, $l(b) = b \cap \mathcal{U}$. It is easily seen that the identity function $d_{\mathcal{TC}(a)}(x) = x$ is a decoration of $(TC(\{a\}), \in^{-1}, a, l)$, $a \in TC(\{a\})$ and $d_{\mathcal{TC}(a)}(a) = a$.

In [3, 4] Baltag, Barwise and Moss have established a characterization of sets in terms of an *infinitary modal logic*. The class of formulae of the *infinitary modal language* $\mathcal{L}_{\infty}(A)$ is defined to be the smallest class containing $\top$, the set A of urelements (regarded as propositional variables), and closed under negation $\neg$, conjunction $\wedge$, the modal operator $\Diamond$ and infinitary conjunction $\bigwedge$. The *finitary modal language* $\mathcal{L}(A)$ is obtained from $\mathcal{L}_{\infty}(A)$ by omitting the infinitary conjunction $\bigwedge$. The *satisfaction relation* $\models$ between pointed graphs and formulae of $\mathcal{L}(A)$ or $\mathcal{L}_{\infty}(A)$ is the standard Kripke definition previously adopted. This definition can be easily transferred from pointed graphs to sets via the above-mentioned "identification" of sets a with their associated converse-membership graphs $(\mathcal{TC}(a), a)$, leading then to the following definitions of *set-theoretical semantics* for modal logic and the corresponding notion of *characterizability*.

According to [3, 4], let A be a set of urelements and a any set. The definition of $a \models_{V_{afa}[A]} \varphi$ proceeds by induction on the complexity of φ:

$$
\begin{array}{ll}
a \models_{V_{afa}[A]} p & \text{if } p \in a, \text{ for all } p \in A \\
a \models_{V_{afa}[A]} \top & \text{for all } a \\
a \models_{V_{afa}[A]} \neg\varphi & \text{iff } a \not\models_{V_{afa}[A]} \varphi \\
a \models_{V_{afa}[A]} \Diamond\varphi & \text{iff for some set } b \in a, b \models_{V_{afa}[A]} \varphi \\
a \models_{V_{afa}[A]} \varphi \wedge \psi & \text{iff } a \models_{V_{afa}[A]} \varphi \text{ and } a \models_{V_{afa}[A]} \psi \\
a \models_{V_{afa}[A]} \bigwedge \Phi & \text{iff } a \models_{V_{afa}[A]} \varphi \text{ for all } \varphi \in \Phi
\end{array}
$$

Obviously, for every formula φ in $\mathcal{L}(A)$ or $\mathcal{L}_\infty(A)$ and a set $a \in V_{afa}[A]$ we have

$$a \models_{V_{afa}[A]} \varphi \quad \text{iff} \quad (\mathcal{TC}(a), a) \models \varphi.$$

Let $a \in V_{afa}[A]$, and let φ be a formula in $\mathcal{L}_\infty(A)$ or $\mathcal{L}(A)$. We say that φ *characterizes* a *in* $V_{afa}[A]$ provided that $b \models_{V_{afa}[A]} \varphi$ iff $b = a$, for all $b \in V_{afa}[A]$.

Theorem 12 (Baltag,Barwise,Moss)
Every set $a \in V_{afa}[A]$ is characterizable in $V_{afa}[A]$ by some formula of the infinitary modal language $\mathcal{L}_\infty(A)$.

Note that this theorem would not hold if we restrict ourselves to a finitary language (see for example [4], Proposition 11.6, p.135). Then a natural question arises: When is a set characterizable by some finitary modal formula? The following result provides the answer.

Theorem 13 (Baltag)
Assume that A is finite. A set $a \in V_{afa}[A]$ is characterizable by some formula of the finitary modal language $\mathcal{L}(A)$ if and only if a is well-founded and $TC(\{a\})$ is finite.

Hence, it turns out that the above-mentioned characterizability of sets by modal formulae depends, in a sensitive way, on whether we are using the full infinitary language $\mathcal{L}_\infty(A)$ or just the finitary one $\mathcal{L}(A)$. In particular, if a set is *not* wellfounded then it is only characterizable by an infinitary modal formula.

In order to state a *finitary* characterization for such sets - with finite transitive closure - in terms of the modal μ-calculus we proceed as follows.

Theorem 14 (Aczel)
For all pointed graphs $(\mathcal{G}, g_0)$ and $(\mathcal{G}', g_0')$ with labels in A and associated decorations $d_{\mathcal{G}}$, $d_{\mathcal{G}'}$ we have

$$d_{\mathcal{G}}(g_0) = d_{\mathcal{G}'}(g_0') \quad \Longleftrightarrow \quad (\mathcal{G}, g_0) \sim_A (\mathcal{G}', g_0').$$

This theorem combined with the fact that $d_{\mathcal{TC}(a)}(a) = a$ yields the following:

every set can be understood as a pointed graph modulo bisimilarity.

The following lemma is proven by combining Theorem 14 with the above-defined notion of characterizability.

Lemma 15
Let $a \in V_{afa}[A]$. A formula φ in $\mathcal{L}(A)$ or $\mathcal{L}_\infty(A)$ characterizes a in $V_{afa}[A]$ iff

$$\|\varphi\| = \|(\mathcal{TC}(a), a)\|_{\sim_A}.$$

Proof Let φ characterize a in $V_{afa}[A]$. By definition this is equivalent to the fact that for any set $b \in V_{afa}[A]$ we have

$$b \models_{V_{afa}[A]} \varphi \quad \text{iff} \quad b = a.$$

Since $b \models_{V_{afa}[A]} \varphi$ if and only if $(\mathcal{TC}(b), b) \models \varphi$ the equivalence above can be restated as

$$(\mathcal{TC}(b), b) \models \varphi \quad \text{iff} \quad b = a.$$

Since $d_{\mathcal{TC}(a)}(a) = a$ and $d_{\mathcal{TC}(b)}(b) = b$, by Theorem 14, this is equivalent to

$$(\mathcal{TC}(b), b) \models \varphi \quad \text{iff} \quad (\mathcal{TC}(b), b) \sim_A (\mathcal{TC}(a), a).$$

Hence, by definition of $\|\varphi\|$ and $\|(\mathcal{TC}(a), a)\|_{\sim_A}$, it can be restated as

$$\|\varphi\| = \|(\mathcal{TC}(a), a)\|_{\sim_A}.$$

□

Hence, by Lemma 15, the notion of characterizability can be uniformly extended to the modal μ-calculus so that, a μ-formula φ *characterizes a set a in $V_{afa}[A]$* if and only if

$$\|\varphi\| = \|(\mathcal{TC}(a), a)\|_{\sim_A}.$$

The existence of characteristic formulae for finite transition systems, proved in Theorem 7, yields a *finitary* characterization for any non-well-founded set with finite transitive closure.

Theorem 16
Every set $a \in V_{afa}[A]$ with finite $TC(\{a\})$ is characterizable by some formula φ_a of the modal μ-calculus.

Note that Theorem 16 can also be restated in terms of *characteristic automata*.

Theorem 17
Every set $a \in V_{afa}[A]$ with finite $TC(\{a\})$ is characterizable by some automaton $\mathcal{A}_a$.

In order to establish our main result we proceed as follows.

Theorem 18 (Finite Model Property)
For all formulae φ of the modal μ-calculus there exists a finite pointed graph $(\mathcal{G}, g_0)$ such that

$$(\mathcal{G}, g_0) \in \|\varphi\|.$$

For a proof the reader is referred to Streett and Emerson [14].

Lemma 19
Let $(\mathcal{G}, g_0)$ be a finite pointed graph and let $a \in V_{afa}[A]$ be such that $(\mathcal{G}, g_0) \sim_A (\mathcal{TC}(a), a)$. Then $(\mathcal{TC}(a), a)$ is finite.

Proof Assume that $(\mathcal{TC}(a), a)$ is infinite. Since any node of $(\mathcal{TC}(a), a)$ is reachable from the point a we have that either there is an infinite path which does not loop or there is a set $b \in TC(\{a\})$ which is infinite. In the second case, by construction of $(\mathcal{TC}(a), a)$, for all distinct $b_i, b_j \in b$ we have that $(\mathcal{TC}(a), b_i)$ is not bisimilar to $(\mathcal{TC}(a), b_j)$. By unwinding the definition of the bisimulation relation it can easily be seen that the path from a to b in $(\mathcal{TC}(a), a)$ is reproduced in $(\mathcal{G}, g_0)$, that is, there is a path (of the same length) starting from g_0 and reaching a point g bisimilar to b. Since b has infinitely many non-bisimilar successor nodes the same must hold for g. Thus, $(\mathcal{G}, g_0)$ must be infinite. This is a contradiction. A similar argument holds for the first case. $\square$

Theorem 20
No set $a \in V_{afa}[A]$ with infinite $TC(\{a\})$ is characterizable by a formula of the modal μ-calculus.

Proof Suppose that there exists a set $a \in V_{afa}[A]$ with infinite $TC(\{a\})$ such that

$$\|\varphi\| = \|(\mathcal{TC}(a), a)\|_{\sim_A},$$

for some formula φ of the modal μ-calculus. By Theorem 18 there is a finite pointed graph $(\mathcal{G}, g_0)$ such that

$$(\mathcal{G}, g_0) \in \|\varphi\|.$$

Thus by Lemma 19, $TC(\{a\})$ must be finite; a contradiction. □

This theorem in combination with Theorem 16 gives our main result.

Theorem 21
A set $a \in V_{afa}[A]$ is characterizable by some formula φ_a of the modal μ-calculus if and only if $TC(\{a\})$ is finite.

Also this result can be restated in terms of *characteristic automata*.

Theorem 22
A set $a \in V_{afa}[A]$ is characterizable by some automaton $\mathcal{A}_a$ if and only if $TC(\{a\})$ is finite.

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