Iterating Σ operations in admissible set theory without foundation: a further aspect of metapredicative Mahlo

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Abstract

In this article we study the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ which (i) describes a recursively inaccessible universe, (ii) permits the iteration of Σ operations along the ordinals, (iii) does not comprise $\in$ induction, and (iv) restricts complete induction on the natural numbers to sets. It is shown that the proof-theoretic ordinal of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is the metapredicative Mahlo ordinal $\varphi\omega 00$. Our system $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is closely related to the system of second order arithmetic for Σ^1_1 transfinite dependent choice introduced in Rüede [8].

1 Introduction

The theory KPi^0, introduced in Jäger [3], is a natural subsystem of set theory whose proof-theoretic strength is characterized by the well-known ordinal Γ_0. It describes a set-theoretic universe above the natural numbers as urelements which is admissible and a limit of admissibles, i.e. recursively inaccessible. However, KPi^0 is very weak with respect to induction principles: $\in$ induction is not available at all, and complete induction on the natural numbers is restricted to sets.

In this article we study the effect of extending KPi^0 by the possibility of iterating Σ operations (on the universe) along ordinals. It will be shown that the resulting system, we call it $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$, has the proof-theoretic ordinal $\varphi\omega 00$ and thus the same proof-theoretic strength as the theory KPm^0 of Jäger and Strahm [6], which plays an important role in connection with the concept of metapredicative Mahlo; see Jäger [2] for a discussion of this topic in a wider context.

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There exists a close relationship between our $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and the system of second order arithmetic for Σ^1_1 transfinite dependent choice which Rüede describes in [8, 9]. Actually, we will adapt Rüede's well-ordering proof in [9] to the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ in order to establish $\varphi\omega 00$ as its lower proof-theoretic bound. A straightforward argument yields that $\varphi\omega 00$ is also the upper proof-theoretic bound of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$.

The plan of this paper is as follows: In the next section we introduce the basic definitions and describe the theories KPi^0, $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and KPm^0. Section 3 is dedicated to delimiting the system $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ from above by embedding it into KPm^0. Section 4 deals with some important properties of KPi^0 and $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$, whereas in Section 5 we carry through a well-ordering proof in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$.

2 The theories KPi^0, $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and KPm^0

Let $\mathcal{L}_1$ be some of the standard languages of first order arithmetic with variables $a, b, c, d, e, f, g, h, u, v, w, x, y, z, \ldots$ (possibly with subscripts), a constant 0 as well as function and relation symbols for all primitive recursive functions and relations. The theory KPi^0 is formulated in the extension $\mathcal{L}^\star = \mathcal{L}_1(\in, \mathsf{N}, \mathsf{S}, \mathsf{Ad})$ of $\mathcal{L}_1$ by the membership relation symbol $\in$, the set constant N for the set of natural numbers and the unary relation symbols S and Ad for sets and admissible sets, respectively.

The *number terms* of $\mathcal{L}^\star$ are inductively generated from the variables, the constant 0 and the symbols for the primitive recursive functions; the *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\star$ are the number terms of $\mathcal{L}_1$ plus the set constant N. The formulas $(A, B, C, A_1, B_1, C_1, \ldots)$ of $\mathcal{L}^\star$ as well as the Δ_0, Σ, Π, Σ_n and Π_n formulas of $\mathcal{L}^\star$ are defined as usual. Equality between objects is not represented by a primitive symbol but defined by

$$(s = t) \;:=\; \begin{cases} (s \in \mathsf{N} \,\wedge\, t \in \mathsf{N} \,\wedge\, (s =_{\mathsf{N}} t)) \,\vee \\ (\mathsf{S}(s) \,\wedge\, \mathsf{S}(t) \,\wedge\, (\forall x \in s)(x \in t) \,\wedge\, (\forall x \in t)(x \in s)) \end{cases}$$

where $=_{\mathsf{N}}$ is the symbol for the primitive recursive equality on the natural numbers. The formula A^s is the result of replacing each unrestricted quantifier $(\exists x)(\ldots)$ and $(\forall x)(\ldots)$ in A by $(\exists x \in s)(\ldots)$ and $(\forall x \in s)(\ldots)$, respectively. In addition, we freely make use of all standard set-theoretic notations and write, for example, $\mathsf{Tran}(s)$ for the Δ_0 formula saying that s is a transitive set.

Since the axioms of KPi^0 do not comprise $\in$ induction, i.e. the principle of foundation with respect to $\in$, we build it directly into the notion of *ordinal*,

$$\mathsf{Wf}(a, \in) \;:=\; \forall x(x \subset a \wedge x \neq \emptyset \;\rightarrow\; (\exists y \in x)(\forall z \in y)(z \notin x)),$$

$$\mathsf{Ord}(a) \;:=\; \mathsf{Tran}(a) \wedge (\forall x \in a)\mathsf{Tran}(x) \wedge \mathsf{Wf}(a, \in).$$

Thus $\mathsf{Ord}(a)$ is a Π formula of $\mathcal{L}^\star$; in the following small Greek letters range over ordinals.

The theory KPi^0 is formulated in the language $\mathcal{L}^\star$; its logical axioms comprise the usual axioms of classical first order logic with equality. The non-logical axioms of KPi^0 can be divided into the following five groups.

I. Ontological axioms. We have for all function symbols $\mathcal{H}$ and relation symbols $\mathcal{R}$ of $\mathcal{L}_1$ and all axioms $A(\vec{u})$ of group III whose free variables belong to the list $\vec{u}$:

(1) $$a \in \mathsf{N} \;\leftrightarrow\; \neg\mathsf{S}(a),$$

(2) $$\vec{a} \in \mathsf{N} \;\rightarrow\; \mathcal{H}(\vec{a}) \in \mathsf{N},$$

(3) $$\mathcal{R}(\vec{a}) \;\rightarrow\; \vec{a} \in \mathsf{N},$$

(4) $$a \in b \;\rightarrow\; \mathsf{S}(b),$$

(5) $$\mathsf{Ad}(a) \;\rightarrow\; (\mathsf{N} \in a \wedge \mathsf{Tran}(a)),$$

(6) $$\mathsf{Ad}(a) \;\rightarrow\; (\forall \vec{x} \in a)A^a(\vec{x}),$$

(7) $$\mathsf{Ad}(a) \wedge \mathsf{Ad}(b) \;\rightarrow\; a \in b \vee a = b \vee b \in a.$$

II. Number-theoretic axioms. We have for all axioms $A(\vec{u})$ of Peano arithmetic PA which are not instances of the schema of complete induction and whose free variables belong to the list $\vec{u}$:

(Number theory) $$\vec{u} \in \mathsf{N} \;\rightarrow\; A^{\mathsf{N}}(\vec{u}).$$

III. Kripke Platek axioms. We have for all Δ_0 formulas $A(u)$ and $B(u, v)$ of $\mathcal{L}^\star$:

(Pair) $$\exists x(a \in x \wedge b \in x),$$

(Tran) $$\exists x(a \subset x \wedge \mathsf{Tran}(x)),$$

(Δ_0-Sep) $$\exists y(\mathsf{S}(y) \wedge y = \{x \in a : A(x)\}),$$

(Δ_0-Col) $$(\forall x \in a)\exists y B(x, y) \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)B(x, y).$$

IV. Limit axiom. It is used to formalize that each set is element of an admissible set, hence we claim:

$$\text{(Lim)} \qquad \exists x(a \in x \wedge \mathsf{Ad}(x)).$$

V. Complete induction on N. The only induction principle included in the axioms of KPi^0 is the following axiom of complete induction on the natural numbers for sets:

$$(\mathsf{S}\text{-}\mathsf{I}_{\mathsf{N}}) \qquad 0 \in a \wedge (\forall x \in \mathsf{N})(x \in a \rightarrow x+1 \in a) \rightarrow \mathsf{N} \subset a.$$

The monograph Barwise [1] provides an excellent introduction into general admissible set theory. Theories of admissible sets without foundation, on the other hand, have been studied, in particular, in Jäger [3, 4]. It is shown there, among other things, that the proof-theoretic ordinal of KPi^0 is Γ_0.

In Section 5 we will also mention an auxiliary *basic set theory* BS^0. It is obtained from KPi^0 by simply dropping the schema of Δ_0 collection.

In this article, however, we are primarily interested in the axiom schema (Σ-TR) about the iteration of Σ operations, added to KPi^0. To formulate this principle, we introduce for each Σ formula $D(\vec{u}, x, y, z)$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}, x, y, z$ free the formula

$$\mathsf{Hier}_D(\vec{a}, b, f) := \begin{cases} \mathsf{Ord}(b) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall \xi \in b) D(\vec{a}, \xi, f{\upharpoonright}\xi, f(\xi)). \end{cases}$$

$\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is now defined to be the theory obtained from KPi^0 by adding the axiom

$$(\Sigma\text{-}\mathsf{TR}) \qquad \mathsf{Ord}(\alpha) \wedge (\forall \xi < \alpha)\forall x \exists! y D(\vec{a}, \xi, x, y) \rightarrow \exists f \mathsf{Hier}_D(\vec{a}, \alpha, f)$$

for all Σ formulas $D(\vec{u}, x, y, z)$ of $\mathcal{L}^\star$. This axiom says that the Σ operation defined by D (depending on the parameters $\vec{a}$) can be iterated along the ordinal α.

$\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is an interesting theory which reveals a further aspect of metapredicative Mahlo. In the sequel we will show that this system is proof-theoretically equivalent to the theory KPm^0 and that it has proof-theoretic ordinal $\varphi\omega00$.

The theory KPm^0, introduced in Jäger and Strahm [6], is also formulated in the language $\mathcal{L}^\star$ and extends KPi^0 by the schema of Π_2 reflection on the admissibles,

$$(\Pi_2\text{-}\mathsf{Ref}^{\mathsf{Ad}}) \qquad A(\vec{a}) \rightarrow \exists x(\vec{a} \in x \wedge \mathsf{Ad}(x) \wedge A^x(\vec{a}))$$

for all Π_2 formulas $A(\vec{u})$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}$ free. In [6] it is also shown that KPm^0, i.e. $\mathsf{KPi}^0 + (\Pi_2\text{-}\mathsf{Ref}^{\mathsf{Ad}})$, is of the same proof-theoretic strength as a natural system of explicit mathematics.

3 Embedding $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ into KPm^0

The only purpose of this very short section is to prove that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ can be directly embedded into KPm^0. Given a Σ formula $D(\vec{u}, x, y, z)$ of $\mathcal{L}^\star$, parameters $\vec{a}$ for which D defines a Σ operation and an ordinal α, the principle of Π_2 reflection on admissibles tells us that there must exist an admissible set d, containing the parameters $\vec{a}$ and α, so that this operation maps d into d; working inside d allows us to iterate this operation along α.

Lemma 1 *For all Σ formulas $D(\vec{u}, x, y, z)$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}, x, y, z$ free we have that*

$$\mathsf{KPm}^0 \vdash \mathsf{Ord}(\alpha) \wedge (\forall \xi < \alpha)\forall x \exists! y D(\vec{a}, \xi, x, y) \rightarrow \exists f \mathsf{Hier}_D(\vec{a}, \alpha, f).$$

Proof We work informally in KPm^0 and choose an ordinal α and parameters $\vec{a}$ so that

$$(1) \qquad (\forall \xi < \alpha)\forall x \exists! y D(\vec{a}, \xi, x, y).$$

By Π_2 reflection on the admissibles applied to (1) it follows that there exists an admissible set d which contains α and $\vec{a}$ as elements and satisfies

$$(2) \qquad (\forall \xi < \alpha)(\forall x \in d)(\exists y \in d) D^d(\vec{a}, \xi, x, y).$$

Furthermore, because of Σ persistence and (1), we can now conclude from assertion (2) that

$$(3) \qquad (\forall \xi < \alpha)(\forall x \in d)(\exists y \in d) D(\vec{a}, \xi, x, y),$$

$$(4) \qquad (\forall \xi < \alpha)(\forall x, y \in d)(D(\vec{a}, \xi, x, y) \leftrightarrow D^d(\vec{a}, \xi, x, y)).$$

Working within the admissible set d, a straightforward adaptation of the usual proof of Σ recursion yields

$$(5) \qquad (\exists f \in d)(\mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \beta \wedge (\forall \xi < \beta) D^d(\vec{a}, \xi, f{\restriction}\xi, f(\xi)))$$

by transfinite induction for all ordinals $\beta \leq \alpha$. Because of (4), the assertion of our lemma follows immediately from (5). □

This lemma states that $(\Sigma\text{-}\mathsf{TR})$ is provable in KPm^0. Consequently, our system $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ is a subtheory of KPm^0.

Theorem 2 (Embedding) *For all $\mathcal{L}^\star$ formulas A we have that*

$$\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR}) \vdash A \quad \Longrightarrow \quad \mathsf{KPm}^0 \vdash A.$$

In view of this theorem and the results of Jäger and Strahm [6] we know that the ordinal $\varphi\omega 00$ is an upper bound for the proof-theoretic strength of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$.

4 Some basic properties of the theories KPi^0 and $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$

The foundation axiom is not available in KPi^0, and thus it is not ruled out in general that there are sets which contain themselves. Nevertheless it can be shown that this is not possible for admissibles.

Lemma 3 *In KPi^0 it can be proved that*

$$\mathsf{Ad}(a) \;\rightarrow\; a \notin a.$$

Proof Suppose, on the contrary, that a is an admissible set which contains itself as an element. By Δ_0 separation within a this a then also contains the "Russell" set r,

$$r \;:=\; \{x \in a : x \notin x\}.$$

Hence we have that $r \in r$ if and only if $r \notin r$. This is a contradiction, and our lemma is proved. □

Now we turn to two further properties of KPi^0 which – or better: generalizations thereof – will be crucial for the well-ordering proof in the next section: (i) every set s is provably contained in a least admissible set s^+, (ii) every set s is provably contained in a least set which is admissible or limit of admissibles.

If $\in$ induction (foundation) were available we could simply carry over the standard recursion-theoretic proof. In the present context, however, a different strategy has to be chosen.

Lemma 4 *Let $D(\vec{u}, v)$ be a Δ_0 formula of $\mathcal{L}^\star$ with at most the variables $\vec{u}, v$ free. Then KPi^0 proves that*

$$\exists x(D(\vec{a}, x) \,\wedge\, \vec{a} \in x \,\wedge\, \mathsf{Ad}(x)) \;\rightarrow$$
$$\exists y(y = \bigcap\{x : D(\vec{a}, x) \,\wedge\, \vec{a} \in x \,\wedge\, \mathsf{Ad}(x)\} \,\wedge\, D(\vec{a}, y) \,\wedge\, \vec{a} \in y \,\wedge\, \mathsf{Ad}(y)).$$

Proof We work informally in the theory KPi^0 and fix some arbitrary parameters $\vec{a}$. The assertion is obvious by the linearity of the admissibles if the class $\{x : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}$ contains only finitely many elements. Hence we may assume that there exist elements b, c, d, e with the properties

$$b \in c \in d \in e, \tag{1}$$

$$b, c, d, e \in \{x : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}. \tag{2}$$

Now we use Δ_0 separation in order to define the set

$$s_0 := \bigcap\{x \in e : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}.$$

Because of the linearity of Ad we obtain the following further properties of this set s_0:

$$s_0 = \bigcap\{x : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}, \tag{3}$$

$$s_0 = \bigcap\{x \in c : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}, \tag{4}$$

$$s_0 \in d. \tag{5}$$

Assertion (5) follows from (4) by Δ_0 separation in d. In a next step the further set

$$s_1 := \bigcap\{x \in e : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x) \wedge s_0 \in x\}$$

is introduced. Given these sets s_0 and s_1, we convince ourselves that they are really different,

$$s_0 \neq s_1. \tag{6}$$

Assume, for the contrary, that $s_0 = s_1$ and employ Δ_0 separation once more for defining the Russell set

$$r := \{x \in s_0 : x \notin x\}.$$

Then $r \in u$ for each set u satisfying $\mathsf{Ad}(u)$ and $s_0 \in u$. This implies $r \in s_1$ and therefore, because of our assumption, also $r \in s_0$. Therefore we have

$$r \in r \;\leftrightarrow\; r \in s_0 \wedge r \notin r \;\leftrightarrow\; r \notin r.$$

This is a contradiction, and so (6) is proved. This assertion (6), however, implies that there exists a set t with the property

$$t \in e \wedge D(\vec{a}, t) \wedge \vec{a} \in t \wedge \mathsf{Ad}(t) \wedge s_0 \notin t. \tag{7}$$

It remains to show that $s_0 = t$. The inclusion $s_0 \subset t$ is obvious. In order to prove $t \subset s_0$, we pick an arbitrary $u \in d$ with $D(\vec{a}, u) \wedge \vec{a} \in u \wedge \mathsf{Ad}(u)$ and establish $t \subset u$. The linearity of Ad gives

$$t \in u \vee t = u \vee u \in t. \tag{8}$$

In the cases of $t \in u$ and $t = u$, the property $t \subset u$ is clear. On the other hand, $u \in t$ would imply that

$$s_0 = \bigcap\{x \in u \cup \{u\} : D(\vec{a}, x) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\} \in t,$$

contradicting the choice of t. Therefore, putting everything together, we know that $s_0 = t$, hence $D(\vec{a}, s_0) \wedge \vec{a} \in s_0 \wedge \mathsf{Ad}(s_0)$. Remembering (3), this concludes the proof of our lemma. $\square$

This lemma implies, for example, that in KPi^0 for any set a the intersection a^+ of all admissibles containing a is an admissible itself,

$$a^+ := \bigcap\{x : a \in x \wedge \mathsf{Ad}(x)\}.$$

Given a set a and a binary relation $b \subset a \times a$, we write $\mathsf{Lin}(a, b)$ if b is a strict linear ordering on a. A linear ordering is a well-ordering if any non-empty subset of its domain has a least element with respect to this ordering,

$$\mathsf{Wo}(a, b) := \mathsf{Lin}(a, b) \wedge \forall x(x \subset a \wedge x \neq \emptyset \to (\exists y \in x)(\forall z \in x)(\langle z, y\rangle \notin b)).$$

We first observe that in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ all Σ operations can be iterated along arbitrary well-orderings, not only along ordinals as stated by $(\Sigma\text{-}\mathsf{TR})$. To do so, we write for each Σ formula $D(\vec{u}, x, y, z)$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}, x, y, z$ free in analogy to Hier_D

$$\mathsf{Hier}^+_D(\vec{a}, b, c, f) := \begin{cases} \mathsf{Wo}(b, c) \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = b \wedge \\ (\forall x \in b) D(\vec{a}, x, \{\langle y, f(y)\rangle : \langle y, x\rangle \in c\}, f(x)). \end{cases}$$

For the proof of Theorem 6 below it is convenient to convince ourselves that for every well-ordering b on a set a there exists a function f – provable in KPi^0 – so that the range $\mathsf{Rng}(f)$ of f is an ordinal and f is 1-1 mapping from a to $\mathsf{Rng}(f)$ translating the order relation b on a into the $<$-relation on $\mathsf{Rng}(f)$. This function f is called the *collapse* of b on a;

$$\mathsf{Clp}(a, b, f) := \begin{cases} b \subset a \times a \wedge \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = a \wedge \\ (\forall x \in a)(f(x) = \{f(y) : \langle y, x\rangle \in b\}). \end{cases}$$

The following lemma states that any well-ordering b on any set a has a collapse which is uniquely determined by a and b.

Lemma 5 *The following two assertions can be proved in* KPi^0*:*

1. $\mathsf{Wo}(a,b) \land \mathsf{Clp}(a,b,f) \land \mathsf{Clp}(a,b,g) \to f = g.$
2. $\mathsf{Wo}(a,b) \land a,b \in d \land \mathsf{Ad}(d) \to (\exists f \in d)\mathsf{Clp}(a,b,f).$

PROOF The uniqueness property is obtained by straightforward induction along the well-ordering b on a. For the proof of the existence of a collapse we work informally in KPi^0 and choose any a, b, d so that

$$\mathsf{Wo}(a,b) \land a,b \in d \land \mathsf{Ad}(d).$$

Writing $a|u$ and $b||u$ for the restrictions of a and b to the predecessors of u,

$$a|u := \{x \in a : \langle x,u\rangle \in b\} \quad \text{and} \quad b||u := \{\langle x,y\rangle \in b : \langle y,u\rangle \in b\},$$

we can easily establish by induction along b on a that

$$(\exists! g \in d)\mathsf{Clp}(a|u, b||u, g)$$

for all elements u of a. With Δ_0 collection we can now immediately derive what we wish. □

Theorem 6 *For all* Σ *formulas* $D(\vec{u},x,y,z)$ *of* $\mathcal{L}^\star$ *with at most the variables* $\vec{u},x,y,z$ *free we have that* $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ *proves*

$$\mathsf{Wo}(b,c) \land (\forall x \in b)\forall y \exists! z D(\vec{a},x,y,z) \to \exists f \mathsf{Hier}^+_D(\vec{a},b,c,f)).$$

PROOF We work informally in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and choose arbitrary parameters $\vec{a}$ and sets b, c so that

$$(1) \qquad \mathsf{Wo}(b,c) \land (\forall x \in b)\forall y \exists! z D(\vec{a},x,y,z).$$

By the previous lemma there exists a collapse h of c on b, i.e. we may assume that

$$(2) \qquad \mathsf{Clp}(b,c,h)$$

for a suitable function h. Now set $\alpha := \mathsf{Rng}(h)$ and apply the principle $(\Sigma\text{-}\mathsf{TR})$ to a properly tailored modification D' of the formula D. This gives us a function g which can then be modified to the desired witness for Hier_D. More precisely, let $B(u,v)$ be the Σ formula of $\mathcal{L}^\star$ which is the disjunction of the following three (mutually exclusive) $\mathcal{L}^\star$ formulas:

(i) $\mathsf{Ord}(u) \land v \in b \land h(v) = u,$

(ii) $u = \emptyset \ \wedge \ v = \emptyset$,

(iii) $\neg \mathsf{Ord}(u) \ \wedge \ (\forall x \in u)(\exists y \in b)\exists z(x = \langle h(y), z \rangle \ \wedge \ \langle y, z \rangle \in v) \ \wedge$
$(\forall x \in v)(\exists y \in b)\exists z(x = \langle y, z \rangle \ \wedge \ \langle h(y), z \rangle \in u)$.

In addition we set

$$D'(\vec{a}, u, v, w) \ := \ \exists x \exists y (B(u, x) \wedge B(v, y) \wedge D(\vec{a}, x, y, w))$$

and observe that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ proves

$$(\forall \xi < \alpha)\forall x \exists ! y D'(\vec{a}, \xi, x, y). \tag{3}$$

Because of Σ reflection, the formula $D'(\vec{a}, u, v, w)$ is provably equivalent in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ to a Σ formula. In view of (3) we can apply $(\Sigma\text{-}\mathsf{TR})$, and thus there exists a function g for which we have

$$\mathsf{Hier}_{D'}(\vec{a}, \alpha, g). \tag{4}$$

To finish our proof, let f be the function with domain b which is defined for all elements u of b by

$$f(u) \ := \ g(h(u)).$$

¿From (4), the definition of the formula $D'(\vec{a}, u, v, w)$ and this definition of f we obtain

$$(\forall \xi < \alpha)\exists y \exists z (B(\xi, y) \wedge B(g{\restriction}\xi, z) \wedge D(\vec{a}, y, z, g(\xi))). \tag{5}$$

Because of (2) this immediately implies that

$$(\forall x \in b)\exists y \exists z (B(h(x), y) \wedge B(g{\restriction}h(x), z) \wedge D(\vec{a}, y, z, g(h(x)))). \tag{6}$$

In view of of the definitions of the function f and the formula $B(u, v)$ we can transform this assertion into

$$(\forall x \in b)\exists z (B(\{\langle h(y), f(y) \rangle : \langle y, x \rangle \in c\}, z) \wedge D(\vec{a}, x, z, f(x))). \tag{7}$$

Looking at the definition of $B(u, v)$ once more, we see that (7) can be simplified to

$$(\forall x \in b) D(\vec{a}, x, \{\langle y, f(y) \rangle : \langle y, x \rangle \in c\}, f(x)). \tag{8}$$

This means, however, that we have $\mathsf{Hier}^+_D(\vec{a}, b, c, f)$, and therefore the proof of our lemma is completed. $\square$

In his well-ordering proof for second order arithmetic with Σ^1_1 transfinite dependent choice, Rüede often makes use of Π^1_2 reflection on ω-models of ACA_0. In our present context, this part is taken over by Π_2 reflection on $\overline{\mathsf{Ad}}$. The "topological closure" $\overline{\mathsf{Ad}}$ of the predicate Ad is obtained by adding to the admissibles also the limits of admissibles,

$$\overline{\mathsf{Ad}}(d) \;:=\; \mathsf{Ad}(d) \,\vee\, (d \neq \emptyset \,\wedge\, d = \bigcup\{x \in d : \mathsf{Ad}(x)\}).$$

Clearly each element of $\overline{\mathsf{Ad}}$ satisfies Δ_0 separation and models the theory BS^0. We prove a uniform version of Π_2 reflection on $\overline{\mathsf{Ad}}$ in a form tailored for our later purposes.

Lemma 7 (Π_2 reflection on $\overline{\mathsf{Ad}}$) *For any Σ formula $A(\vec{u}, v, w)$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}, v, w$ free there exists a Σ formula $A^\sharp(\vec{u}, v)$ of $\mathcal{L}^\star$ with at most the variables $\vec{u}, v$ free so that the following two assertions can be proved in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$:*

1. $\forall x \exists y A(\vec{a}, x, y) \;\rightarrow\; \exists! z A^\sharp(\vec{a}, z)$.
2. $\forall x \exists y A(\vec{a}, x, y) \;\rightarrow$
   $\forall z (A^\sharp(\vec{a}, z) \;\rightarrow\; \vec{a} \in z \,\wedge\, \overline{\mathsf{Ad}}(z) \,\wedge\, (\forall x \in z)(\exists y \in z) A^z(\vec{a}, x, y))$.

Proof We work informally in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and begin with introducing the following abbreviations:

$$B_A(\vec{u}, v, w) \;:=\; \vec{u}, v \in w \,\wedge\, \mathsf{Ad}(w) \,\wedge\, (\forall x \in v)(\exists y \in w) A^w(\vec{u}, x, y)$$

$$C_A(\vec{u}, v, w) \;:=\; B_A(\vec{u}, v, w) \,\wedge\, (\forall z \in w) \neg B_A(\vec{u}, v, z)$$

Obviously, $B_A(\vec{u}, v, w)$ and $C_A(\vec{u}, v, w)$ are Δ_0 formulas, and with Σ reflection and the limit axiom (Lim) we obtain for arbitrary parameters $\vec{a}$ that

$$\forall x \exists y A(\vec{a}, x, y) \;\rightarrow\; \forall v \exists w B_A(\vec{a}, v, w). \tag{1}$$

Hence Lemma 3, i.e. the fact that an admissible cannot contain itself, and Lemma 4 imply in view of (1) that

$$\forall x \exists y A(\vec{a}, x, y) \;\rightarrow\; \forall v \exists! w C_A(\vec{a}, v, w). \tag{2}$$

The operation described by this fact will now be iterated along the standard less relation $<_{\mathsf{N}}$ on the natural numbers N. To adjust everything to the formulation of Theorem 6 we define

$$D_A(\vec{u}, x, y, z) \;:=\; C_A(\vec{u}, \bigcup\{v : (\exists w \in \mathsf{N})(w <_{\mathsf{N}} x \,\wedge\, \langle w, v\rangle \in y)\}, z).$$

Trivially, we have $\mathsf{Wo}(\mathsf{N}, <_{\mathsf{N}})$. Furthermore, (2) implies that

$$\forall x \exists y A(\vec{a}, x, y) \;\rightarrow\; (\forall x \in \mathsf{N}) \forall y \exists! z D_A(\vec{a}, x, y, z). \tag{3}$$

Because of Theorem 6 we consequently know that there exists a function f whose domain is the set N and which satisfies $\mathsf{Hier}^+_{D_A}(\vec{a}, \mathsf{N}, <_{\mathsf{N}}, f)$, i.e.

$$\forall x \exists y A(\vec{a}, x, y) \;\rightarrow\; \exists f \mathsf{Hier}^+_{D_A}(\vec{a}, \mathsf{N}, <_{\mathsf{N}}, f). \tag{4}$$

The function f which is claimed to exist in (4) has to be unique. Thus also the set d,

$$d \;:=\; \bigcup \{f(x) : x \in \mathsf{N}\},$$

is uniquely determined. Under the assumption $\forall x \exists y A(\vec{a}, x, y)$ it is now easily verified that we have for this set d the desired properties $\vec{a} \in d$, $\overline{\mathsf{Ad}}(d)$ and $(\forall x \in d)(\exists y \in d) A^d(\vec{a}, x, y))$.

To finish the proof of our lemma, we set

$$A^\sharp(\vec{u}, v) \;:=\; \exists f (\mathsf{Hier}^+_{D_A}(\vec{u}, \mathsf{N}, <_{\mathsf{N}}, f) \;\wedge\; v = \bigcup \{f(x) : x \in \mathsf{N}\}).$$

Our previous considerations make it clear that for this Σ formula $A^\sharp(\vec{u}, v)$ both assertions of our lemma are satisfied. $\square$

5 The well-ordering proof in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$

Now the stage is set for extending the well-ordering proof in Jäger, Setzer, Kahle and Strahm [5] to our set theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$, similar to how Rüede [8] adopts it for the treatment of Σ^1_1 transfinite dependent choice. We will show that all ordinals less than $\varphi\omega00$ are provable in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$.

As in, for example, Jäger and Strahm [6] we work with the *ternary Veblen functions* for coping with an sufficiently long initial segment of the ordinals. The usual Veblen hierarchy is generated by the binary function φ, starting off with $\varphi 0 \beta = \omega^\beta$, and often discussed in the literature, cf. e.g. Pohlers [7] or Schütte [10]. The ternary Veblen function φ is easily obtained from the binary φ as follows:

1. $\varphi 0 \beta \gamma$ is $\varphi \beta \gamma$.
2. If $\alpha > 0$, then $\varphi \alpha 0 \gamma$ denotes the γth ordinal which is strongly critical with respect to all functions $\lambda \xi . \lambda \eta . \varphi \delta \xi \eta$ for $\delta < \alpha$.

3. If $\alpha > 0$ and $\beta > 0$, then $\varphi\alpha\beta\gamma$ denotes the γth common fixed point of the functions $\lambda\eta.\varphi\alpha\delta\eta$ for $\delta < \beta$.

Let Ξ_0 be the least ordinal greater than 0 which is closed under addition and the ternary φ. In the following we will work with a standard primitive recursive notation system $(\mathsf{OT}, \prec)$ for all ordinals less than Ξ_0. All required definitions are straightforward generalizations of those used for building a notation system for Γ_0 (cf. [7, 10]) and can be omitted.

In this section we let $\mathfrak{a}\,\mathfrak{b}, \mathfrak{c}, \dots$ (possibly with subscripts) range over the set OT; in addition, ℓ is used for codes of limit ordinals; the terms $\hat{0}, \hat{1}, \hat{2}, \dots$ act as codes for the finite ordinals. To simplify the notation we often write the ordinal constants and ordinal functions such as, for example,

$$0, \quad 1, \quad \omega, \quad \lambda\xi.\lambda\eta.(\xi + \eta), \quad \lambda\xi.\omega^\xi, \quad \lambda\zeta.\lambda\xi.\lambda\eta.\varphi\zeta\xi\eta$$

instead of the corresponding codes and primitive recursive functions. Another useful binary operation on ordinal notations, introduced in Jäger, Setzer, Kahle and Strahm [5], is given by

$$\mathfrak{a} \upharpoonright \mathfrak{b} \;:=\; \exists\mathfrak{c}\exists\ell(\mathfrak{b} \;=\; \mathfrak{c} + \mathfrak{a} \cdot \ell).$$

For completeness we also recall how it is expressed that our specific primitive recursive relation $\prec$ is a *well-ordering*, that a formula is *progressive* with respect to $\prec$ and how *transfinite induction along* $\prec$ is defined for arbitrary formulas:

$$\begin{aligned}
\mathsf{Wo}(\mathfrak{a}) &:= \mathsf{Wo}(\{\mathfrak{b} : \mathfrak{b} \prec \mathfrak{a}\}, \{\langle \mathfrak{c}, \mathfrak{b}\rangle : \mathfrak{c} \prec \mathfrak{b} \prec \mathfrak{a}\}),\\
\mathsf{Prog}(A) &:= \forall\mathfrak{a}((\forall\mathfrak{b} \prec \mathfrak{a})A(\mathfrak{b}) \to A(\mathfrak{a})),\\
\mathsf{TI}(A, \mathfrak{a}) &:= \mathsf{Prog}(A) \to (\forall\mathfrak{b} \prec \mathfrak{a})A(\mathfrak{b}).
\end{aligned}$$

Maybe apart from $\mathfrak{a} \upharpoonright \mathfrak{b}$, all these notions are standard in the context of well-ordering proofs. For dealing with $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ we need further predicates $\mathcal{K}_n(u)$ and $\mathcal{H}_n(\mathfrak{a}, u, f)$ which are defined simultaneously by induction on the natural number n as well as the predicates $\mathcal{I}(\mathfrak{b}, f, \mathfrak{a})$ and $\mathcal{M}_n(\mathfrak{b}, f, \mathfrak{a})$:

$$\begin{aligned}
\mathcal{T}(f) &:= \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \mathsf{OT},\\
\mathcal{K}_1(a) &:= \mathsf{Ad}(a),\\
\mathcal{K}_{n+1}(a) &:= \overline{\mathsf{Ad}}(a) \wedge [\forall x \exists f(\mathcal{T}(f) \wedge \forall\mathfrak{a}(\mathsf{Wo}(\mathfrak{a}) \to \mathcal{H}_n(\mathfrak{a}, x, f)))]^a,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_n(\mathfrak{a}, u, f) &:= \mathcal{T}(f) \wedge (\forall \mathfrak{b} \prec \mathfrak{a})(f{\restriction}\mathfrak{b} \in f(\mathfrak{b}) \wedge u \in f(\mathfrak{b}) \wedge \mathcal{K}_n(f(\mathfrak{b}))),\\
\mathcal{I}(\mathfrak{b}, f, \mathfrak{a}) &:= (\forall \mathfrak{c} \prec \mathfrak{b})(\forall x \in f(\mathfrak{c}))\mathsf{TI}(x, \mathfrak{a}),\\
\mathcal{M}_n(\mathfrak{b}, f, \mathfrak{a}) &:= \forall \mathfrak{c}(\forall \mathfrak{d} \preceq \mathfrak{b})(\omega^{1+\mathfrak{a}} \uparrow \mathfrak{d} \wedge \mathcal{I}(\mathfrak{d}, f, \mathfrak{c}) \rightarrow \mathcal{I}(\mathfrak{d}, f, \varphi\hat{n}\mathfrak{a}\mathfrak{c})).
\end{aligned}$$

The first lemma concerning this machinery states that each set a is provably element of a set b which satisfies the property $\mathcal{K}_n$. It plays a key role in our well-ordering proof.

Lemma 8 (Main lemma) *For any natural number n greater than 0 there exists a Σ formula $F_n(u, v)$ of $\mathcal{L}^\star$ so that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ proves*

$$\forall x \exists! y F_n(x, y) \wedge \forall x \forall y(F_n(x, y) \rightarrow x \in y \wedge \mathcal{K}_n(y)).$$

Proof We prove this assertions by complete induction on n. For $n = 1$ we simply have to set

$$F_1(u, v) := (v = u^+).$$

Because of Lemma 4 and the discussion following this lemma we know that this formula $F_1(u, v)$ satisfies our requirements. Now we assume $n > 1$ and apply the induction hypothesis to provide a Σ formula $F_{n-1}(u, v)$ so that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ proves

$$\forall x \exists! y F_{n-1}(x, y) \wedge \forall x \forall y(F_{n-1}(x, y) \rightarrow x \in y \wedge \mathcal{K}_n(y)). \tag{1}$$

Based on this Σ formula $F_{n-1}(u, v)$ we introduce the auxiliary Σ formula $B_n(u, v, w)$,

$$B_n(u, v, w) := F_{n-1}(\{u, v\}, w).$$

¿From (1) we immediately conclude that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ proves

$$\forall x \forall y \exists! z B_n(x, y, z). \tag{2}$$

Hence $B_n(u, v, w)$ defines a Σ operation to which we want to apply Theorem 6 in a next step. This theorem 6 implies for any parameter u and any element $\mathfrak{c}$ of OT that, provably in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$,

$$\mathsf{Wo}(\mathfrak{c}) \rightarrow \begin{cases} \exists g[\mathsf{Fun}(g) \wedge \mathsf{Dom}(g) = \{\mathfrak{d} : \mathfrak{d} \prec \mathfrak{c}\} \wedge \\ \quad (\forall \mathfrak{d} \prec \mathfrak{c}) B_n(u, g{\restriction}\mathfrak{d}, g(\mathfrak{d}))]. \end{cases} \tag{3}$$

Now define $E_n(u, \mathfrak{c}, g)$ to be the Σ formula

$$\mathsf{Fun}(g) \wedge \mathsf{Dom}(g) = \{\mathfrak{d} : \mathfrak{d} \prec \mathfrak{c}\} \wedge (\forall \mathfrak{d} \prec \mathfrak{c}) B_n(u, g{\restriction}\mathfrak{d}, g(\mathfrak{d})).$$

Until the end of this section we work informally in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and rewrite statement (3) as

$$\forall \mathfrak{c} \exists g (\mathsf{Wo}(\mathfrak{c}) \to E_n(u, \mathfrak{c}, g)). \tag{4}$$

Σ reflection, the limit axiom (Lim) and Σ persistence in connection with (4) guarantee the existence of an admissible set d such that

$$\forall \mathfrak{c} (\exists g \in d)(\mathsf{Wo}^d(\mathfrak{c}) \to E_n^d(u, \mathfrak{c}, g)). \tag{5}$$

We further claim that for all elements g and g' of d

$$\mathsf{Wo}^d(\mathfrak{c}) \wedge \mathfrak{a} \prec \mathfrak{b} \prec \mathfrak{c} \wedge E_n^d(u, \mathfrak{b}, g) \wedge E_n^d(u, \mathfrak{c}, g') \ \to \ g(\mathfrak{a}) = g'(\mathfrak{a}), \tag{6}$$

a fact that can be easily checked by inspecting the definitions of $E_n(u, \mathfrak{b}, g)$ and $E_n(u, \mathfrak{c}, g')$. A next step in the proof of our lemma is to set

$$f \ := \ \bigcup \{ g \in d : \exists \mathfrak{c} (\mathsf{Wo}^d(\mathfrak{c}) \wedge E_n^d(u, \mathfrak{c}, g)\} \ \cup \ \{ \langle \mathfrak{c}, \emptyset \rangle : \neg \mathsf{Wo}^d(\mathfrak{c}) \}.$$

f is an element of d^+; by (6) we know that it is a function, and $\mathsf{Dom}(f) = \mathsf{OT}$ is immediate from its definition. Assertion (5) and the definition of f yield, in addition, that

$$\mathsf{Wo}^d(\mathfrak{c}) \ \to \ (\forall \mathfrak{d} \prec \mathfrak{c})(u \in f(\mathfrak{d}) \wedge f{\restriction}\mathfrak{d} \in f(\mathfrak{d}) \wedge \mathcal{K}_{n-1}(f(\mathfrak{d}))) \tag{7}$$

and this, in turn, implies because of Σ persistence and our previous remarks about f that

$$\forall x \exists f (\mathcal{T}(f) \wedge \forall \mathfrak{c} (\mathsf{Wo}(\mathfrak{c}) \to \mathcal{H}_{n-1}(\mathfrak{c}, x, f))). \tag{8}$$

The last step of our proof consists in applying Π_2 reflection on $\overline{\mathsf{Ad}}$, to the Σ formula $A_n(u, v, w)$,

$$A_n(u, v, w) \ := \ (u = u) \wedge \mathcal{T}(w) \wedge \forall \mathfrak{c} (\mathsf{Wo}(\mathfrak{c}) \to \mathcal{H}_{n-1}(\mathfrak{c}, v, w)).$$

Lemma 7 implies the existence of a Σ formula $A_n^\sharp(u, v)$ so that from (8) we may deduce for any parameter a that

$$\exists ! y A_n^\sharp(a, y), \tag{9}$$

$$\forall y (A_n^\sharp(a, y) \ \to \ a \in y \wedge \overline{\mathsf{Ad}}(y) \wedge (\forall x \in y)(\exists f \in y) A_n^y(a, x, f)). \tag{10}$$

By choosing $F_n(u, v)$ to be the formula $A_n^\sharp(u, v)$, the assertions (9) and (10) together with the definition of the formula $\mathcal{K}_n(v)$ immediately imply

$$\forall x \exists ! y F_n(x, y) \ \wedge \ \forall x \forall y (F_n(x, y) \ \to \ x \in y \wedge \mathcal{K}_n(y)).$$

This finishes the induction step, and therefore also the proof of our lemma is completed. □

The proofs of the following three lemmas can be easily recaptured by (more or less notational) adaptations of the corresponding proofs in Jäger, Kahle Setzer and Strahm [5] and Rüede [8]. Therefore we omit all details and confine ourselves to providing exact references.

Lemma 9 *The following three assertions can be proved in* BS^0*:*

1. $\mathcal{H}_1(\ell, u, f) \wedge \mathcal{I}(\ell, f, \mathfrak{a}) \rightarrow \mathcal{I}(\ell, f, \varphi\mathfrak{a}0)$.
2. $\mathcal{H}_1(\ell, u, f) \rightarrow \mathsf{Prog}(\{\mathfrak{a} : \mathcal{I}(\ell, f, \varphi 10\mathfrak{a})\})$.
3. $\mathcal{H}_1(\mathfrak{b}, u, f) \rightarrow \mathsf{Prog}(\{\mathfrak{a} : \mathcal{M}_1(\mathfrak{b}, f, \mathfrak{a})\}$.

Proof For all details concerning the proof of these three assertions see Lemma 5, Lemma 6 and Lemma 7 of [5]. □

Lemma 10 *For any natural number n greater than* 0 *the following three assertions can be proved in* BS^0*:*

1. $\mathcal{K}_{n+1}(a) \wedge [\forall x \forall f \forall \mathfrak{b}(\mathcal{H}_n(\mathfrak{b}, x, f) \rightarrow \mathsf{Prog}(\{\mathfrak{c} : \mathcal{M}_n(\mathfrak{b}, f, \mathfrak{c})\}))]^a$
   $\rightarrow \forall \mathfrak{c}[(\forall x \in a)\mathsf{TI}(x, \mathfrak{c}) \rightarrow (\forall x \in a)\mathsf{TI}(x, \varphi\hat{n}\mathfrak{c}0)]$.
2. $\mathcal{K}_{n+1}(a) \wedge \forall \mathfrak{c}[(\forall x \in a)\mathsf{TI}(x, \mathfrak{c}) \rightarrow (\forall x \in a)\mathsf{TI}(x, \varphi\hat{n}\mathfrak{c}0)]$
   $\rightarrow \mathsf{Prog}(\{\mathfrak{c} : (\forall x \in a)\mathsf{TI}(x, \varphi(\hat{n}{+}1)0\mathfrak{c})\})$.
3. $\mathcal{H}_n(\mathfrak{b}, u, f) \wedge \forall a[\mathcal{K}_n(a) \rightarrow \mathsf{Prog}(\{\mathfrak{c} : (\forall x \in a)\mathsf{TI}(x, \varphi\hat{n}0\mathfrak{c})\})]$
   $\rightarrow \mathsf{Prog}(\{\mathfrak{c} : \mathcal{M}_n(\mathfrak{b}, f, \mathfrak{c}\})$.

Proof For all details concerning the proof of these three assertions see Lemma 4, Lemma 5 and Lemma 6 of [8]. □

Lemma 11 *For any natural number n greater than* 0 *the following three assertions can be proved in* BS^0*:*

1. $\overline{\mathsf{Ad}}(a) \rightarrow [\forall x \forall f \forall \mathfrak{b} \mathcal{H}_n(\mathfrak{b}, x, f) \rightarrow \mathsf{Prog}(\{\mathfrak{c} : \mathcal{M}_n(\mathfrak{b}, f, \mathfrak{c})\})]^a$.
2. $\mathcal{K}_{n+1}(a) \rightarrow \forall \mathfrak{c}[(\forall x \in a)\mathsf{TI}(x, \mathfrak{c}) \rightarrow (\forall x \in a)\mathsf{TI}(x, \varphi\hat{n}\mathfrak{c}0)]$.
3. $\mathcal{K}_{n+1}(a) \rightarrow \mathsf{Prog}(\{\mathfrak{c} : (\forall x \in a)\mathsf{TI}(x, \varphi(\hat{n}{+}1)0\mathfrak{c})\})$.

Proof Start with showing that the first assertion implies the second and the second the third. Then prove the first assertion by induction on n. For details see Theorem 6 of [8]. □

Theorem 12 (Lower bound) *For any natural number n we have that*

$$\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR}) \vdash \forall x \mathsf{TI}(x, \varphi\hat{n}00).$$

Proof Fix any natural number n greater than 0. Arguing informally in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$, let a be an arbitrary set. In view of Lemma 8 we know that there exist a set b satisfying

$$a \in b \wedge \mathcal{K}_{n+1}(b). \tag{1}$$

The second part of Lemma 11 yields, in addition, that

$$\mathcal{K}_{n+1}(b) \rightarrow \forall \mathfrak{c}[(\forall x \in b)\mathsf{TI}(x, \mathfrak{c}) \rightarrow (\forall x \in b)\mathsf{TI}(x, \varphi\hat{n}\mathfrak{c}0)], \tag{2}$$

and, consequently, if we set $\mathfrak{c} = 0$,

$$\mathcal{K}_{n+1}(b) \rightarrow (\forall x \in b)\mathsf{TI}(x, \varphi\hat{n}00)]. \tag{3}$$

¿From (1) and (3) we conclude $\mathsf{TI}(a, \varphi\hat{n}00)$, which is exactly what we had to show. □

An ordinal α is called provable in the theory T – formulated in $\mathcal{L}^\star$ or a similar language – if there exists a primitive recursive well-ordering $\lhd$ on the natural numbers of order-type α so that

$$T \vdash \mathsf{Wo}(\mathsf{N}, \lhd).$$

The least ordinal which is not provable in T is called the *proof-theoretic ordinal of* T and denoted by $|\mathsf{T}|$.

¿From Theorem 2, Theorem 12 above and the results of Jäger and Strahm [6], which tell us that $|\mathsf{KPm}^0| \leq \varphi\omega00$, we derive the following characterization of the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ in terms of their proof-theoretic ordinal.

Corollary 13 *The set theories $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})$ and KPm^0 have the same proof-theoretic strength, namely*

$$|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{TR})| = |\mathsf{KPm}^0| = \varphi\omega00.$$

Of course, Theorem 2 and Theorem 12 also show that any ordinal less than $\varphi\omega00$ is provable in KPm^0. A direct proof of this result can be found in Strahm [11].

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