Variation on a theme of Schütte

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Abstract

Let $\prec$ be a primitive recursive well-ordering on the natural numbers and assume that its order-type is greater than or equal to the proof-theoretic ordinal of the theory T. We show that the proof-theoretic strength of T is not increased if we add the negation of the statement which formalizes transfinite induction along $\prec$.

Key words Proof theory, proof-theoretic strength, transfinite induction.

MSC (2000) 03F03, 03F05, 03F15, 03F25

1 Introduction

The technique of *pseudo-hierarchies* has become a powerful tool in several areas of mathematical logic and goes back to Spector [13], Gandy [3] and – in its full form – to Feferman and Spector [2]. The method of pseudo-hierarchies in connection with subsystems of second order arithmetic is described in Simpson [12] in extenso; a typical application for specific fixed point definitions is given in Avigad [1].

Recent work on metapredicative theories for iterated admissible sets and explicit mathematics with comparatively weak induction principles – see Probst [9] and some papers in preparation [5, 8] for all relevant details – makes it desirable to apply similar strategies, but the use of pseudo-hierarchies in subsystems of set theory and explicit mathematics seems to be a different matter.

For the proof-theoretic applications we have in mind, pseudo-hierarchy constructions cannot directly be employed. However, there is a way around this:

(i) Given a suitable set theory T, extend it to a system $\mathsf{T}^\dagger$ by adding the negations of certain instances of transfinite induction which are not derivable in T.

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(ii) Work now within $\mathsf{T}^\dagger$ where pseudo-hierarchies can be employed, similar to as it is done in subsystems of second order arithmetic.

(iii) Finally by showing that $\mathsf{T}^\dagger$ has the same proof-theoretic strength as T, we achieve what we desire.

In this short technical note we are concerned with step (iii) which is interesting in its own right. Let T be any theory with proof-theoretic ordinal α, and let $\prec$ be a primitive recursive well-ordering of the natural numbers of order-type greater than or equal to α. Then it is clear that T cannot prove the statement $\mathcal{I}_\prec$,

$$\mathcal{I}_\prec \;:=\; (\forall X \subset \mathbb{N})[(\forall u \in \mathbb{N})((\forall v \prec u)(v \in X) \to (u \in X)) \;\to\; \mathbb{N} \subset X],$$

formalizing transfinite induction along the well-ordering $\prec$. This implies that the theory $\mathsf{T} + \neg\mathcal{I}_\prec$, which results from T by adding the negation of the true statement $\mathcal{I}_\prec$, is consistent. It is a natural question to ask whether such extensions lead to an increase of proof-theoretic strength.

This question is answered negatively below. To achieve this result we only have to modify the proof of Schütte's famous boundedness theorem for infinitary number theory PA^∞.

2 The semiformal system PA^∞

Let $\mathcal{L}_2$ be any standard language of second order arithmetic with number variables $a, b, c, u, v, w, x, y, z, \ldots$, set variables $X, Y, Z, \ldots$ (both possibly with subscripts), the symbol $\in$ for membership, the constant 0, the unary function symbol S for the successor function as well as function and relation symbols for all other primitive recursive functions and relations. Starting off from this alphabet, the *number terms* $r, s, t, \ldots$ (possibly with subscripts) of $\mathcal{L}_2$ are defined as usual; the set terms of $\mathcal{L}_2$ are just its set variables.

The formulas of $\mathcal{L}_2$ and the axioms and rules of the system PA^∞ will be formulated in a Tait-style. This is only in order to fix a framework which is convenient for our proof-theoretic considerations below. Gentzen or Schütte calculi, for example, would work just as well.

The *positive literals* of $\mathcal{L}_2$ are all expressions of the form $\mathsf{R}(t_1, \ldots, t_n)$ and $(t \in X)$ where R is a relation symbol for an n-ary primitive recursive relation, X is a set variable and $t, t_1, \ldots, t_n$ are number terms of $\mathcal{L}_2$. The *negative literals* of $\mathcal{L}_2$ have the form $\sim\!E$ so that E is a positive literal of $\mathcal{L}_2$; for $\sim\!(t \in X)$ we write $(t \notin X)$.

The *formulas* $A, B, C, D, \dots$ (possibly with subscripts) of $\mathcal{L}_2$ and their ranks are inductively generated from the positive and negative literals by closing under disjunctions, conjunctions and existential and universal quantifications; i.e.

1. Each positive and negative literal E is a formula; $\mathrm{rn}(E) := 0$.
2. If A and B are formulas, then so also are $(A \vee B)$ and $(A \wedge B)$;
$$\mathrm{rn}(A \vee B) \;:=\; \mathrm{rn}(A \wedge B) \;:=\; \max(\mathrm{rn}(A), \mathrm{rn}(B)) + 1.$$
3. If A is a formula, then so also are $\exists u A$ and $\forall u A$;
$$\mathrm{rn}(\exists u A) \;:=\; \mathrm{rn}(\forall u A) \;:=\; \mathrm{rn}(A) + 1.$$
4. If A is a formula, then so also are $\exists X A$ and $\forall X A$;
$$\mathrm{rn}(\exists X A) \;:=\; \mathrm{rn}(\forall X A) \;:=\; \mathrm{rn}(A) + 1.$$

The negation $\neg A$ of an $\mathcal{L}_2$ formula A is defined by making use of the law of double negation and de Morgan's laws, and all other logical connectives are abbreviated in the standard way.

The *arithmetic* formulas of $\mathcal{L}_2$ are the formulas of $\mathcal{L}_2$ which do not contain set quantifiers. The terms of $\mathcal{L}_2$ which do not contain free occurrences of number variables are the *closed* terms of $\mathcal{L}_2$; the formulas of $\mathcal{L}_2$ which do not contain free occurrences of number variables are the *semi-closed* formulas of $\mathcal{L}_2$. Semi-closed formulas of $\mathcal{L}_2$ may contain set variables. Based on the standard interpretation of all function symbols, a uniquely determined natural number $\mathrm{Val}(t)$ – the value of t – is associated to any closed number term t of $\mathcal{L}_2$. We call two semi-closed literals of $\mathcal{L}_2$ *numerically equivalent* if they are syntactically identical modulo subterms which have the same value.

The *true [false] positive literals* of $\mathcal{L}_2$ are the literals $\mathsf{R}(t_1, \dots, t_n)$ where R is the relation symbol for an n-ary primitive recursive relation $\mathcal{R}$, the number terms $t_1, \dots, t_n$ are closed and $\mathcal{R}(\mathrm{Val}(t_1), \dots, \mathrm{Val}(t_n))$ is true [false]. Accordingly, the *true [false] negative literals* of $\mathcal{L}_2$ are the literals $\sim\! E$ so that E is a false [true] positive literal. Let $\mathbb{T}$ be the collection of all true positive and true negative literals of $\mathcal{L}_2$ and $\mathbb{F}$ the collection of all false positive and false negative literals of $\mathcal{L}_2$.

The semiformal system PA^∞ works with finite sets $\Gamma, \Delta, \Phi, \Psi, \dots$ (possibly with subscripts) of semi-closed arithmetic formulas of $\mathcal{L}_2$, which have to be interpreted disjunctively. If A is a semi-closed arithmetic $\mathcal{L}_2$ formula, then

Γ, A is shorthand for $\Gamma \cup \{A\}$, and similar for expressions, for example, of the form Γ, A, B.

The axioms and rules of inference of the system PA^∞ can now be divided into the following four groups:

I. Axioms of PA^∞. For all finite sets Γ of semi-closed arithmetic $\mathcal{L}_2$ formulas, all elements A of $\mathbb{T}$ and all numerically equivalent literals B and C:

$$\Gamma,\, A \qquad \text{and} \qquad \Gamma,\, \neg B,\, C.$$

II. Propositional rules of PA^∞. For all finite sets Γ of semi-closed arithmetic $\mathcal{L}_2$ formulas and all semi-closed arithmetic $\mathcal{L}_2$ formulas A and B:

$$\frac{\Gamma,\, A}{\Gamma,\, A \vee B}, \qquad \frac{\Gamma,\, B}{\Gamma,\, A \vee B}, \qquad \frac{\Gamma,\, A \quad \Gamma,\, B}{\Gamma,\, A \wedge B}.$$

III. Quantifier rules of PA^∞. For all finite sets Γ of semi-closed arithmetic $\mathcal{L}_2$ formulas and all semi-closed arithmetic $\mathcal{L}_2$ formulas $A(a)$:

$$\frac{\Gamma,\, A(s)}{\Gamma,\, \exists u A(u)}, \qquad \frac{\Gamma,\, A(t) \quad \text{for all closed terms } t}{\Gamma,\, \forall u A(u)} \quad (\omega\text{-rule}).$$

IV. Cut rules of PA^∞. For all finite sets Γ of semi-closed arithmetic $\mathcal{L}_2$ formulas and all semi-closed arithmetic $\mathcal{L}_2$ formulas A:

$$\frac{\Gamma,\, A \qquad \Gamma,\, \neg A}{\Gamma}.$$

The formulas A and $\neg A$ are the cut formulas of this cut; the rank of this cut is the rank $\mathrm{rn}(A) = \mathrm{rn}(\neg A)$ of its cut formulas.

The (ω-rule) is an inference rule with infinitely many premises, and as a consequence, derivations in PA^∞ may be of infinite depth. The exact definition of derivability in PA^∞ is as follows.

Definition 1 *Let Γ be a finite set of semi-closed arithmetic $\mathcal{L}_2$ formulas. Then $\mathsf{PA}^\infty \vdash_n^\alpha \Gamma$ is defined for all ordinals α and natural numbers n by induction on α.*

1. *If Γ is an axiom of PA^∞, then we have $\mathsf{PA}^\infty \vdash_n^\alpha$ for all ordinals α and all natural numbers n.*

2. *If $\mathsf{PA}^\infty \vdash_n^{\alpha_\iota} \Gamma_\iota$ and $\alpha_\iota < \alpha$ for all premises Γ_ι of a propositional rule, a quantifier rule or a cut whose rank is less than n, then we have $\mathsf{PA}^\infty \vdash_n^\alpha \Gamma$ for the conclusion Γ of this rule.*

Thus $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma$ means that Γ is provable in PA^∞ by a proof of depth bounded by α which does not make use of any cut rule. Furthermore, we write $\mathsf{PA}^\infty \vdash_0^{<\alpha} \Gamma$ if there exists an ordinal $\beta < \alpha$ so that $\mathsf{PA}^\infty \vdash_0^\beta \Gamma$.

Our system PA^∞ simply is one specific formulation of infinitary first order arithmetic. One of the main features of such systems is their cut elimination property.

Theorem 2 (Gentzen, Schütte) *For all finite sets Γ of semi-closed arithmetic formulas of $\mathcal{L}_2$, all ordinals α and all natural numbers n we have that*

$$\mathsf{PA}^\infty \vdash_n^\alpha \Gamma \quad \Longrightarrow \quad \mathsf{PA}^\infty \vdash_0^{2(n,\alpha)} \Gamma.$$

For a proof of this theorem and further details we refer, for example, to Schütte [11]. The ordinal $2(n,\alpha)$ is inductively defined by setting $2(0,\alpha) := \alpha$ and $2(n{+}1,\alpha) := 2^{2(n,\alpha)}$.

3 An extension of Schütte's boundedness theorem

One of the applications of cut-free proofs in infinitary first order arithmetic is Schütte's celebrated boundedness theorem. It plays a crucial rôle in determining upper proof-theoretic bounds by means of infinitary proof theory.

We have to fix some notation first. If $\mathsf{R}_\prec$ is the binary relation symbol of $\mathcal{L}_2$ denoting the primitive recursive well-ordering $\prec$, we usually write $(s \prec t)$ or simply $s \prec t$ instead of $\mathsf{R}_\prec(s,t)$. Further convenient abbreviations are:

$$\begin{aligned}
(\exists u \prec t)A(u) &:= \exists u(u \prec t \,\land\, A(u)),\\
(\forall u \prec t)A(u) &:= \forall u(u \prec t \,\to\, A(u)),\\
\mathsf{Prog}_\prec(Z) &:= \forall u((\forall v \prec u)(v \in Z) \,\to\, (u \in Z)),\\
\mathsf{TI}_\prec(t,Z) &:= \mathsf{Prog}_\prec(Z) \,\to\, (\forall u \prec t)(u \in Z),\\
\mathsf{I}_\prec(t) &:= \forall Z \mathsf{TI}_\prec(t,Z).
\end{aligned}$$

The order-type of a primitive recursive well-ordering $\prec$ is denoted by $|\prec|$, and, for any closed number term t of $\mathcal{L}_2$, we let $|t|_\prec$ be the order-type of the natural number $\mathrm{Val}(t)$ with respect to this well-ordering.

Schütte's boundedness theorem states that there is a close relationship between the cut-free provability of the assertion $\mathsf{TI}_\prec(t,Z)$ within PA^∞ and the ordinal $|t|_\prec$: a cut-free PA^∞ proof of depth, more or less, $|t|_\prec$ is required in order to establish within PA^∞ that the initial part of $\prec$ up to t is well-ordered.

Theorem 3 (Boundedness Theorem) *Let $\prec$ be some primitive recursive well-ordering. For any closed number term t of $\mathcal{L}_2$ and any ordinal α we have that*

$$\mathsf{PA}^\infty \vdash_0^\alpha \mathsf{TI}_\prec(t, Z) \quad \Longrightarrow \quad |t|_\prec \leq \omega\alpha.$$

The proof of this theorem is given in Schütte [11] in all details; alternatively, it can also be found in Pohlers [7].

Now we turn to the variation or extension of Schütte's theorem which is the main topic of this note. The crucial step is the following main lemma whose proof is tailored according to a corresponding lemma in Schütte [11].

Lemma 4 (Main Lemma) *Let $\prec$ be a primitive recursive well-ordering and suppose that we are given two ordinals $\alpha, \beta < |\prec|$, two sets Γ and Δ of semi-closed arithmetic $\mathcal{L}_2$ formulas and two finite sets M_+ and M_- of closed number terms of $\mathcal{L}_2$ so that the following assumptions are satisfied:*

(1) $M_+ \neq \emptyset$ *and* $\beta = \min\{|r|_\prec : r \in M_+\}$,

(2) $\{|r|_\prec : r \in M_+\} \cap \{|r|_\prec : r \in M_-\} = \emptyset$,

(3) $\Delta \subset \{\neg \mathsf{Prog}_\prec(Z)\} \cup \{(r \in Z) : r \in M_+\} \cup \{(r \notin Z) : r \in M_-\} \cup \mathbb{F}$,

(4) the relation variable Z does not occur in Γ,

(5) $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma, \Delta$ *and* $\omega\alpha \leq \beta$.

Then we even have that $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma$.

Proof We prove this assertion by induction on α and distinguish the following cases.

1. The set $\Gamma \cup \Delta$ is an axiom of PA^∞. Then already Γ has to be an axiom, hence $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma$.

2. The main formula of the last inference (S) belongs to Γ. Then we apply the induction hypothesis to the premises of (S) and derive Γ afterwards by applying (S) again.

3. The main formula of the last inference (S) belongs to Δ. Then this formula has to be the formula $\neg \mathsf{Prog}_\prec(Z)$, and there exist a $\gamma < \alpha$ and a closed number term t of $\mathcal{L}_2$ so that

$$(1) \qquad \mathsf{PA}^\infty \vdash_0^\gamma \Gamma, \Delta, (\forall u \prec t)(u \in Z) \wedge (t \notin Z).$$

By simple inversion we deduce from (1) that

$$\text{(2)} \qquad \mathsf{PA}^\infty \vdash_0^\gamma \Gamma,\, \Delta,\, (\forall u \prec t)(u \in Z),$$

$$\text{(3)} \qquad \mathsf{PA}^\infty \vdash_0^\gamma \Gamma,\, \Delta,\, (t \notin Z).$$

We continue by considering the following two subcases:

3.1. $|t|_\prec \notin \{|r|_\prec : r \in M_+\}$. Considering (3), we now immediately realize that the assumptions (1)–(5) of our lemma are satisfied if we replace the ordinal α by γ, the set of formulas Δ by $\Delta, (t \notin Z)$ and the set of terms M_- by $M_- \cup \{t\}$. Thus the induction hypothesis implies $\mathsf{PA}^\infty \vdash_0^\gamma \Gamma$, and, consequently, we have $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma$.

3.2. $|t|_\prec \in \{|r|_\prec : r \in M_+\}$. In this case we have

$$\text{(4)} \qquad \omega\gamma \;<\; \omega\alpha \;\leq\; \beta \;\leq\; |t|_\prec.$$

Since M_- is finite, there exists a closed number term s of $\mathcal{L}_2$ with the properties

$$\text{(5)} \qquad |s|_\prec \notin \{|r|_\prec : r \in M_-\},$$

$$\text{(6)} \qquad \omega\gamma \;<\; |s|_\prec \;<\; \beta.$$

From assertions (4) and (6) we conclude s is smaller than t in the sense of $\prec$ so that the formula $s \not\prec t$ belongs to $\mathbb{F}$. In addition, an inversion applied to (2) yields

$$\text{(7)} \qquad \mathsf{PA}^\infty \vdash_0^\gamma \Gamma,\, \Delta,\, s \not\prec t,\, (s \in Z).$$

The next step is to set $M'_+ := M_+ \cup \{s\}$ and to note that

$$\text{(8)} \qquad \omega\gamma \;<\; |s|_\prec \;=\; \min\{|r|_\prec : r \in M'_+\}.$$

In view of (7) we easily check, as in the previous subcase, that the assumptions (1)–(5) of our lemma are satisfied if we replace, this time, the ordinals α and β by γ and $|s|_\prec$, respectively, the set of formulas Δ by $\Delta, s \not\prec t, (s \in Z)$ and the set of terms M_+ by M'_+. Hence $\mathsf{PA}^\infty \vdash_0^\gamma \Gamma$ by induction hypothesis and therefore $\mathsf{PA}^\infty \vdash_0^\alpha \Gamma$, as required. This completes the proof of our lemma. $\square$

In the formulations below we confine ourselves to theories T whose languages $\mathcal{L}(\mathsf{T})$ comprise our language $\mathcal{L}_2$. However, everything works as well provided that there is a natural and canonical embedding of $\mathcal{L}_2$ into $\mathcal{L}(\mathsf{T})$.

Definition 5 *Let* T *be any theory which is formulated in a language* $\mathcal{L}(\mathsf{T})$ *comprising* $\mathcal{L}_2$. *Then* T *is called* α-equivalent *to* PA^∞ *if* α *is the least ordinal so that we have*

$$\mathsf{T} \vdash A \quad \Longrightarrow \quad \mathsf{PA}^\infty \vdash_0^{<\alpha} A$$

for all semi-closed arithmetic formulas A *of* $\mathcal{L}_2$. *In this situation we write* $\mathsf{T} \simeq_\alpha \mathsf{PA}^\infty$.

The notion of α-equivalence to PA^∞ is technically well-suited for our present purpose. It is closely related to the proof-theoretic strength of a theory measured in terms of its proof-theoretic ordinal which can be defined, in our present context, as follows.

Definition 6 *Let* T *be a theory which is formulated in a language* $\mathcal{L}(\mathsf{T})$ *comprising* $\mathcal{L}_2$.

1. *The ordinal* α *is* provable *in* T *if there exists a primitive recursive well-ordering* $\prec$ *of order-type* α *so that*

$$\mathsf{T} \vdash \forall u \mathsf{I}_\prec(u).$$

2. *The* proof-theoretic ordinal *of* T, *denoted by* $|\mathsf{T}|$, *is the least ordinal which is not provable in* T.

Although we do not formulate an abstract theorem stating that a theory T is α-equivalent to PA^∞ if and only if its proof-theoretic ordinal is α, we nevertheless want to point out that this is the case in natural situations. In general, a non-artificial theory which is α-equivalent to PA^∞ has proof-theoretic ordinal α, and vice versa.

To verify this observation, let us have a look at, for example, the proof-theoretic machinery and the subsystems of first order arithmetic, second order arithmetic and set theory studied in Schütte [11], Pohlers [7] and Jäger [4]. For all these theories T it can be checked immediately that T is α-equivalent to PA^∞ if and only if α is its proof-theoretic ordinal. More precisely: let T be any theory whose proof-theoretic ordinal has been determined to be the ordinal α via traditional ordinal analysis; then we may expect that it is proof-theoretic routine to read of from this ordinal analysis that T is α-equivalent to PA^∞.

The following definition introduces extensions of a given theory by negations of instances of certain transfinite inductions. Afterwards, it is shown that no proof-theoretic strength is gained by moving to these extensions.

Definition 7 *Let* T *be a theory which is formulated in a language* $\mathcal{L}(\mathsf{T})$ *comprising* $\mathcal{L}_2$*, and let* $\prec$ *be a primitive recursive well-ordering. For any ordinal* α *less than* $|\prec|$ *we define the* α-extension $\mathcal{E}_\prec(\mathsf{T},\alpha)$ *of* T *to be the theory which consists of* T *plus all formulas of the form* $\neg\mathsf{I}_\prec(t)$ *where* t *is a closed number term of* $\mathcal{L}_2$ *so that* $\alpha \leq |t|_\prec$*.*

To simplify the formulation of the following theorem, an ordinal α is denoted *ω-closed* if and only if $\omega\alpha = \alpha$. We will see now that for any ω-closed α and any theory T which is α-equivalent to PA^∞, its α-extension is α-equivalent to PA^∞ as well.

Theorem 8 *Suppose that* T *is a theory which is formulated in a language* $\mathcal{L}(\mathsf{T})$ *comprising* $\mathcal{L}_2$*, that* $\prec$ *is a primitive recursive well-ordering and that* α *is an* ω*-closed ordinal less than* $|\prec|$*. Then we have that*

$$\mathsf{T} \simeq_\alpha \mathsf{PA}^\infty \quad \Longrightarrow \quad \mathcal{E}_\prec(\mathsf{T},\alpha) \simeq_\alpha \mathsf{PA}^\infty.$$

Proof We assume that T is α-equivalent to PA^∞ and have to show that $\mathcal{E}_\prec(\mathsf{T},\alpha)$ is also α-equivalent to PA^∞. If $\mathcal{E}_\prec(\mathsf{T},\alpha)$ is β-equivalent to PA^∞, then $\alpha \leq \beta$ since $\mathcal{E}_\prec(\mathsf{T},\alpha)$ is an extension of T. Now let A be a semi-closed arithmetic formula of $\mathcal{L}_2$ which is provable in $\mathcal{E}_\prec(\mathsf{T},\alpha)$. Then there exist finitely many closed number terms $t_1,\ldots,t_n$ of $\mathcal{L}_2$ with the following properties:

$$\alpha \;\leq\; |t_1|_\prec,\ldots,|t_n|_\prec, \tag{1}$$

$$\mathsf{T} \;\vdash\; \mathsf{I}_\prec(t_1) \,\vee\, \ldots \,\vee\, \mathsf{I}_\prec(t_n) \,\vee\, A. \tag{2}$$

From (2) we immediately obtain

$$\mathsf{T} \;\vdash\; \mathsf{TI}_\prec(t_1,Z) \,\vee\, \ldots \,\vee\, \mathsf{TI}_\prec(t_n,Z) \,\vee\, A \tag{3}$$

for some set variable Z which does not occur in A. Thus some simple manipulations within T yield

$$\mathsf{T} \;\vdash\; \neg\mathsf{Prog}_\prec(Z) \,\vee\, (t_1 \in Z) \,\vee\, \ldots \,\vee\, (t_n \in Z) \,\vee\, A. \tag{4}$$

Exploiting the fact that T is α-equivalent to PA^∞ we can deduce from (4) that

$$\mathsf{PA}^\infty \;\vdash_0^{<\alpha}\; \neg\mathsf{Prog}_\prec(Z) \,\vee\, (t_1 \in Z) \,\vee\, \ldots \,\vee\, (t_n \in Z) \,\vee\, A \tag{5}$$

and therefore also

$$\mathsf{PA}^\infty \;\vdash_0^{<\alpha}\; \neg\mathsf{Prog}_\prec(Z),\, (t_1 \in Z),\, \ldots,\, (t_n \in Z),\, A. \tag{6}$$

In view of our assumption that α is ω-closed and in view of Lemma 4 we obtain from assertion (6) that $\mathsf{PA}^\infty \vdash_0^{<\alpha} A$. This completes the proof of our theorem. $\square$

Ordinal notation systems – for representing initial segments of the ordinal numbers – can be based in a very natural and perspicuous way on ordinal addition and the so-called Veblen functions. For the binary Veblen function, providing a notation system for all ordinals less than the famous Feferman-Schütte ordinal Γ_0, this is done, for example, in Schütte [11] and Pohlers [7] in full detail. Jäger and Strahm [6] and Strahm [14] employ the ternary Veblen function φ in connetion with their proof-theoretic analysis of metapredicative subsystems of explicit mathematics and set theory.

In the following example we write $\sqsubset$ for the primitive recursive well-ordering whose order-type is the least ordinal Υ so that $\Upsilon = \varphi\Upsilon00$. The ordinal $\varphi\omega00$ mentioned below is the metapredicative Mahlo number; see Jäger and Strahm [6] for details.

Example 9 We consider two subsystems of second order arithmetic and one theory of (iterated) admissible sets.

1. It is proof-theoretic folklore that $\varphi\varepsilon_0 0$ is the proof-theoretic ordinal of the theory Δ^1_1-CA and that Δ^1_1-CA is $\varphi\varepsilon_0 0$-equivalent to PA^∞. So if we pick a closed number term r of $\mathcal{L}_2$ whose order type with respect to $\sqsubset$ is greater than or equal to $\varphi\varepsilon_0 0$, then the assertion $\mathsf{I}_\sqsubset(r)$ – although true in the standard model – cannot be proved in Δ^1_1-CA. In view of Theorem 8 its negation can be added to Δ^1_1-CA without providing any additional strength;
$$\Delta^1_1\text{-}\mathsf{CA} + \neg\mathsf{I}_\sqsubset(r) \ \simeq_{\varphi\varepsilon_0 0}\ \mathsf{PA}^\infty \quad \text{and} \quad |\Delta^1_1\text{-}\mathsf{CA} + \neg\mathsf{I}_\sqsubset(r)| = \varphi\varepsilon_0 0.$$

2. The same considerations can be applied to the system Σ^1_1-TDC of Σ^1_1 transfinite dependent choice which has been introduced by Rüede and is studied in [10]. It follows from the work there that $|\Sigma^1_1\text{-}\mathsf{TDC}| = \varphi\omega00$ and that Σ^1_1-TDC is $\varphi\omega00$-equivalent to PA^∞. Hence for any closed number term s of $\mathcal{L}_2$ so that $\varphi\omega00 \leq |s|_\sqsubset$ Theorem 8 yields that
$$\Sigma^1_1\text{-}\mathsf{TDC} + \neg\mathsf{I}_\sqsubset(s) \ \simeq_{\varphi\omega00}\ \mathsf{PA}^\infty \quad \text{and} \quad |\Sigma^1_1\text{-}\mathsf{TDC} + \neg\mathsf{I}_\sqsubset(s)| = \varphi\omega00.$$

3. Now we turn to a subsystem of set theory and consider the theory KPm^0 of Jäger and Strahm [6]. It formalizes that we have a Mahlo universe, i.e. Π_2 reflection on the admissibles; $\in$-induction, however, is not available. The language $\mathcal{L}(\mathsf{KPm}^0)$ of KPm^0 contains the language

$\mathcal{L}_2$ modulo an obvious translation A^* of any $\mathcal{L}_2$ formula A. According to [6], KPm^0 is $\varphi\omega 00$-equivalent to PA^∞ and $\varphi\omega 00$ is the proof-theoretic ordinal of KPm^0. As in the case above we can therefore deduce with the help of Theorem 8 that

$$\mathsf{KPm}^0 + \neg \mathsf{I}^*_\sqsubset(s) \ \simeq_{\varphi\omega 00} \ \mathsf{PA}^\infty \quad \text{and} \quad |\mathsf{KPm}^0 + \neg \mathsf{I}^*_\sqsubset(s)| = \varphi\omega 00,$$

provided that, as before, s is a closed number term whose order type with respect to the well-ordering $\sqsubset$ is greater than or equal to $\varphi\omega 00$.

Obviously, Theorem 8 can also be applied in the context of theories of explicit mathematics, and corresponding examples can be found. However, in this short note we do not want to go into details of the application of this theorem, instead we refer the reader to forthcoming publications.

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