About cut elimination for logics of common knowledge

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Dedicated to H. Schwichtenberg for his 60th birthday

Abstract

The notions of *common knowledge* or *common belief* play an important role in several areas of computer science (e.g. distributed systems, communication), in philosophy, game theory, artificial intelligence, psychology and many other fields which deal with the interaction within a group of "agents", agreement or coordinated actions. In the following we will present several deductive systems for common knowledge above epistemic logics – such as **K**, **T**, **S4** and **S5** – with a fixed number of agents. We focus on structural and proof-theoretic properties of these calculi.

1 Introduction

The notions of *common knowledge* or *common belief* play an important role in several areas of computer science (e.g. distributed systems, communication), in philosophy, game theory, artificial intelligence, psychology and many other fields which deal with the interaction within a group of "agents", agreement or coordinated actions. Everybody has a vague intuitive understanding of what common knowledge (belief) should be, and for a lot of applications such informal approaches may suffice. On the other hand, in many cases a formal mathematical treatment of common knowledge (belief) is required.

There are two main directions in developing formalizations of reasoning with and about common knowledge:

- Barwise (cf. e.g. [3, 4]) discusses common knowledge within his *Situation Semantics* and his general treatment *Situation in Logic.* Basic ingredients are the sets SIT of situations and FACTS of facts.

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- Alternatively, common knowledge may be studied by starting off from epistemic logics, i.e. in the context of multi-modal logics; see the textbooks Fagin, Halpern, Moses and Vardi [5] and Meyer and van der Hoek [12] for a good introduction.

Although being built up from different "atoms", there exist interesting connections between these two formal frameworks for common knowledge. For example, largest fixed points of suitable operators are used in a crucial way in both cases. In this article, however, we will confine ourselves to common knowledge in its multi-modal version. More about its relationship to common knowledge à la Barwise can be found in Graf [7] and Lismont [11].

In the following we will present several deductive systems for common knowledge above epistemic logics – such as **K**, **T**, **S4** and **S5** – with a fixed number of agents. We focus on structural and proof-theoretic properties of these calculi, in particular in connection with cuts and cut elimination.

For completeness we recall the basic syntactic and semantic notions of our logics of common knowledge and introduce their standard Hilbert-style formulations. In the later sections we turn to finitary and infinitary Tait-calculi, present results about partial cut elimination for the finitary system and total cut elimination for the infinitary one. In addition we study two interesting finite and cut-free fragments of the infinitary calculus.

2 Syntax and semantics of logics of common knowledge

Let $L_n(\mathsf{C})$ be our standard language for multi-modal logic which comprises a set PROP of *atomic propositions*, typically indicated by $P, Q, \ldots$ (possibly with subscripts), the *propositional connectives* $\vee$ and $\wedge$, the *epistemic operators* $\mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_n$ and the *common knowledge operator* C; in addition we assume that there is an auxiliary symbol $\sim$ for forming the complements of atomic propositions and dual epistemic operators. The *formulas* $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts) of $L_n(\mathsf{C})$ and the *depth* $\mathsf{dpt}(\alpha)$ for each $L_n(\mathsf{C})$ formula α are inductively generated as follows:

1. All atomic propositions P and their complements $\sim P$ are $L_n(\mathsf{C})$ formulas;
$$\mathsf{dpt}(P) \;:=\; \mathsf{dpt}(\sim P) \;:=\; 0.$$
2. If α and β are $L_n(\mathsf{C})$ formulas, so are $(\alpha \vee \beta)$ and $(\alpha \wedge \beta)$;
$$\mathsf{dpt}((\alpha \vee \beta)) \;:=\; \mathsf{dpt}((\alpha \wedge \beta)) \;:=\; \max(\mathsf{dpt}(\alpha), \mathsf{dpt}(\beta)) + 1.$$

3. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{K}_i(\alpha)$ and $\sim\mathsf{K}_i(\alpha)$;

$$\mathsf{dpt}(\mathsf{K}_i(\alpha)) \;:=\; \mathsf{dpt}(\sim\mathsf{K}_i(\alpha)) \;:=\; \mathsf{dpt}(\alpha) + 1.$$

4. If α is an $L_n(\mathsf{C})$ formula, so are $\mathsf{C}(\alpha)$ and $\sim\mathsf{C}(\alpha)$;

$$\mathsf{dpt}(\mathsf{C}(\alpha)) \;:=\; \mathsf{dpt}(\sim\mathsf{C}(\alpha)) \;:=\; \mathsf{dpt}(\alpha) + n + 1.$$

See below for an explanation why the number n, i.e. the number of agents, has to be added in the last clause. The $L_n(\mathsf{C})$ formulas $\sim P$ act as negations of the atomic proposition P; the duals $\sim\mathsf{K}_i$ and $\sim\mathsf{C}$ of the modal operators K_i and C, respectively, are needed in forming the negations $\neg\alpha$ of general $L_n(\mathsf{C})$ formulas α (by making use of de Morgan's laws and the law of double negation):

1. If α is the atomic proposition P, then $\neg\alpha$ is $\sim P$; if α is the formula $\sim P$, then $\neg\alpha$ is P.

2. If α is the formula $(\beta \vee \gamma)$, then $\neg\alpha$ is $(\neg\beta \wedge \neg\gamma)$; if α is the formula $(\beta \wedge \gamma)$, then $\neg\alpha$ is $(\neg\beta \vee \neg\gamma)$.

3. If α is the formula $\mathsf{K}_i(\beta)$, then $\neg\alpha$ is $\sim\mathsf{K}_i(\neg\beta)$; if α is the formula $\sim\mathsf{K}_i(\beta)$, then $\neg\alpha$ is $\mathsf{K}_i(\neg\beta)$.

4. If α is the formula $\mathsf{C}(\beta)$, then $\neg\alpha$ is $\sim\mathsf{C}(\neg\beta)$; if α is the formula $\sim\mathsf{C}(\beta)$, then $\neg\alpha$ is $\mathsf{C}(\neg\beta)$.

Often we omit parentheses if there is no danger of confusion and abbreviate the remaining logical connectives as usual; in addition we set

$$\mathsf{E}(\alpha) \;:=\; \mathsf{K}_1(\alpha) \wedge \ldots \wedge \mathsf{K}_n(\alpha).$$

The definition of the depth of the formulas $\mathsf{C}(\alpha)$ and $\sim\mathsf{C}(\alpha)$ has been tailored so that we always have

$$\mathsf{dpt}(\mathsf{E}(\alpha)) \;=\; \mathsf{dpt}(\sim\mathsf{E}(\alpha)) \;<\; \mathsf{dpt}(\mathsf{C}(\alpha)) \;=\; \mathsf{dpt}(\sim\mathsf{C}(\alpha)).$$

A possible intuitive interpretation of $\mathsf{K}_i(\alpha)$ is "agent i knows (believes) that α", and thus $\mathsf{E}(\alpha)$ can be understood as "everybody knows (believes) that α". The latter formula has to be strictly distinguished from $\mathsf{C}(\alpha)$, which expresses common knowledge of α among the agents 1 to n (see below). We also need the iterations $\mathsf{E}^m(\alpha)$ for all natural numbers m, inductively introduced as

$$\mathsf{E}^0(\alpha) \;:=\; \alpha \quad \text{and} \quad \mathsf{E}^{m+1}(\alpha) \;:=\; \mathsf{E}(\mathsf{E}^m(\alpha)).$$

Turning to the semantics of $L_n(\mathsf{C})$, we define a *Kripke-frame* (for $L_n(\mathsf{C})$) to be an $(n+1)$-tuple

$$\mathfrak{M} \;=\; (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$$

for a non-empty set W of *worlds* and binary relations $\mathbb{K}_1, \ldots, \mathbb{K}_n$ on W; the set of worlds of a Kripke-frame $\mathfrak{M}$ is often denoted by $|\mathfrak{M}|$. A *valuation* in $\mathfrak{M}$ then is a function $\mathcal{V}$ from the atomic propositions PROP to the power set $Pow(|\mathfrak{M}|)$ of $|\mathfrak{M}|$,

$$\mathcal{V} \;:\; \mathsf{PROP} \;\rightarrow\; Pow(|\mathfrak{M}|).$$

The *truth-set* $\|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$ of an $L_n(\mathsf{C})$ formula α with respect to the Kripke-frame $\mathfrak{M} = (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$ and a valuation $\mathcal{V}$ is defined, as usual in multi-modal logics, by induction an the complexity of α with an additional clause for treating the operator C:

$$\begin{aligned}
\|P\|^{\mathfrak{M}}_{\mathcal{V}} &:= \mathcal{V}(P), \\
\|{\sim}P\|^{\mathfrak{M}}_{\mathcal{V}} &:= W \setminus \|P\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\alpha \vee \beta\|^{\mathfrak{M}}_{\mathcal{V}} &:= \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cup \|\beta\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\alpha \wedge \beta\|^{\mathfrak{M}}_{\mathcal{V}} &:= \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \cap \|\beta\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \{\, v \in W : w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}} \text{ for all } w \text{ so that } (v, w) \in \mathbb{K}_i \,\}, \\
\|{\sim}\mathsf{K}_i(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= W \setminus \|\mathsf{K}_i(\neg\alpha)\|^{\mathfrak{M}}_{\mathcal{V}}, \\
\|\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= \bigcap \{\, \|\mathsf{E}^m(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} : m \geq 1 \,\}, \\
\|{\sim}\mathsf{C}(\alpha)\|^{\mathfrak{M}}_{\mathcal{V}} &:= W \setminus \|\mathsf{C}(\neg\alpha)\|^{\mathfrak{M}}_{\mathcal{V}}.
\end{aligned}$$

By means of these truth-sets we can easily express that the $L_n(\mathsf{C})$ formula α is valid in the Kripke-frame $\mathfrak{M}$ with respect to valuation $\mathcal{V}$ and world w; this is the case if $w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}$. The following notation is convenient for expressing this situation:

$$(\mathfrak{M}, \mathcal{V}, w) \models \alpha \quad :\Longleftrightarrow \quad w \in \|\alpha\|^{\mathfrak{M}}_{\mathcal{V}}.$$

Observe that these semantics do not imply that α is true in all worlds which satisfy $\mathsf{C}(\alpha)$. In the literature sometimes a distinction is made between knowledge and belief: knowledge of a fact implies the truth of this fact, whereas the belief of something may be compatible with its falsity. But since the intuitive meaning of knowledge or belief can only be approximated and can never be completely grasped by formal semantics, we will not pay attention to this subtlety.

If we have $(\mathfrak{M}, \mathcal{V}, w) \models \alpha$ for all valuations $\mathcal{V}$ in $\mathfrak{M}$ and all worlds $w \in |\mathfrak{M}|$ of a Kripke-frame $\mathfrak{M}$, then α is *valid* in $\mathfrak{M}$,

$$\mathfrak{M} \models \alpha.$$

Our semantics reflects the so-called *iterative interpretation* of common knowledge:

$$(\mathfrak{M}, \mathcal{V}, w) \models \mathsf{C}(\alpha) \quad\Longleftrightarrow\quad (\mathfrak{M}, \mathcal{V}, w) \models \bigwedge_{m \geq 1} \mathsf{E}^m(\alpha).$$

Thus α is common knowledge if everybody knows α and everybody knows that everybody knows α and everybody knows that everybody knows that everybody knows α and so on. Alternatively, we could also treat common knowledge in the sense of the *greatest fixed point interpretation* since

$$(\star) \qquad \|\mathsf{C}(\alpha)\|_{\mathcal{V}}^{\mathfrak{M}} \;=\; \bigcup \{\, X \subset |\mathfrak{M}| : X = \|\mathsf{E}(\alpha) \wedge \mathsf{E}(Q)\|_{\mathcal{V}[Q:=X]}^{\mathfrak{M}} \,\}$$

where Q is chosen to be an atomic proposition which does not occur in α and $\mathcal{V}[Q := X]$ is the valuation which maps Q to X and otherwise agrees with $\mathcal{V}$. A proof of equation $(\star)$ can be found, for example, in Fagin, Halpern, Moses and Vardi [5].

Property $(\star)$ follows from the continuity of the operator defined by the formula $(\mathsf{E}(\alpha) \wedge \mathsf{E}(Q))$. There are variants of common knowledge like ϵ-common knowledge or $\diamond$-common knowledge so that $\mathsf{C}^{\epsilon}(\alpha)$ and $\mathsf{C}^{\diamond}(\alpha)$ cannot be characterized by the union of the *finite* iterations of the corresponding operators; then only the greatest fixed point approach makes sense (cf. e.g. [8, 5]).

Now we recall the Hilbert-style formulations of a few multi-modal logics of common knowledge. We begin with the usual logic $\mathbf{K}$, extended to n agents plus C, and denote it by $\mathbf{K}_n(\mathsf{C})$.

Basic axioms of $\mathbf{K}_n(\mathsf{C})$

(TAUT) All propositional tautologies

(K) $\mathsf{K}_i(\alpha) \wedge \mathsf{K}_i(\alpha \to \beta) \;\to\; \mathsf{K}_i(\beta)$

Basic rules of inference of $\mathbf{K}_n(\mathsf{C})$

(MP) $$\frac{\alpha \qquad \alpha \to \beta}{\beta}$$

(NEC) $$\frac{\alpha}{\mathsf{K}_i(\alpha)}$$

Co-closure axioms of $\mathbf{K}_n(\mathsf{C})$

(CCL) $$\mathsf{C}(\alpha) \ \rightarrow \ (\mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha)))$$

Induction rules of $\mathbf{K}_n(\mathsf{C})$

(IND) $$\frac{\beta \ \rightarrow \ \mathsf{E}(\alpha) \wedge E(\beta)}{\beta \ \rightarrow \ \mathsf{C}(\alpha)}$$

In these axioms and rules and in the ones which will be formulated below, α and β may be arbitrary $L_n(\mathsf{C})$ formulas. The system $\mathbf{T}_n(\mathsf{C})$ is obtained from $\mathbf{K}_n(\mathsf{C})$ by adding all axioms

(T) $$\mathsf{K}_i(\alpha) \ \rightarrow \ \alpha.$$

$\mathbf{S4}_n(\mathsf{C})$ is the multi-modal version of **S4** with common knowledge and extends $\mathbf{T}_n(\mathsf{C})$ by all axioms (4) for positive introspection

(4) $$\mathsf{K}_i(\alpha) \ \rightarrow \ \mathsf{K}_i(\mathsf{K}_i(\alpha)).$$

Finally, adding the corresponding axioms (5) of negative introspection to the theory $\mathbf{S4}_n(\mathsf{C})$ gives the system $\mathbf{S5}_n(\mathsf{C})$,

(5) $$\neg\mathsf{K}_i(\alpha) \ \rightarrow \ \mathsf{K}_i(\neg\mathsf{K}_i(\alpha)).$$

Now let $\mathbf{F}$ be one of the theories $\mathbf{K}_n(\mathsf{C})$, $\mathbf{T}_n(\mathsf{C})$, $\mathbf{S4}_n(\mathsf{C})$ or $\mathbf{S5}_n(\mathsf{C})$. We employ the standard notion of provability of an $L_n(\mathsf{C})$ formula α in the theory $\mathbf{F}$ and write this fact as

$$\mathbf{F} \vdash \alpha.$$

A Kripke-frame $\mathfrak{M}$ is a *model* of $\mathbf{F}$ if all axioms of $\mathbf{F}$ are valid in $\mathfrak{M}$ and if $\mathfrak{M}$ is closed under the rules of inference of $\mathbf{F}$ with respect to validity. A standard result of modal logic characterizes the Kripke-frames $\mathfrak{M} = (W, \mathbb{K}_1, \ldots, \mathbb{K}_n)$ which are models of these theories:

(1) $\mathfrak{M}$ is a model of $\mathbf{K}_n(\mathsf{C})$ for arbitrary (binary) $\mathbb{K}_1, \ldots, \mathbb{K}_n$.

(2) $\mathfrak{M}$ is a model of $\mathbf{T}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are reflexive.

(3) $\mathfrak{M}$ is a model of $\mathbf{S4}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are reflexive and transitive.

(4) $\mathfrak{M}$ is a model of $\mathbf{S5}_n(\mathsf{C})$ if and only if the $\mathbb{K}_1, \ldots, \mathbb{K}_n$ are equivalence relations.

Following the standard patterns, we call the $L_n(\mathsf{C})$ formula α a *semantic consequence* of $\mathbf{F}$,

$$\mathbf{F} \models \alpha,$$

if α is valid in all models of $\mathbf{F}$. The subsequent theorem states that syntactic derivability is adequate for semantic consequence in all our logics.

Theorem 1 (Soundness and completeness) *Let* $\mathbf{F}$ *be one of the logics* $\mathbf{K}_n(\mathsf{C})$*,* $\mathbf{T}_n(\mathsf{C})$*,* $\mathbf{S4}_n(\mathsf{C})$ *or* $\mathbf{S5}_n(\mathsf{C})$*. Then we have*

$$\mathbf{F} \vdash \alpha \quad \Longleftrightarrow \quad \mathbf{F} \models \alpha.$$

Let us now come back to the co-closure axioms and induction rules of, say, $\mathbf{K}_n(\mathsf{C})$. The axiom (CCL) states that each formula $\mathsf{C}(\alpha)$ describes a set of states co-closed under the operator

$$\mathrm{Op}_\alpha(X) \;:=\; \mathsf{E}(\alpha) \wedge \mathsf{E}(X)$$

mapping sets of states to sets of states, with respect to a given frame and valuation. The rules (IND), on the other hand, formulate, that $\mathsf{C}(\alpha)$ is the greatest (definable) set co-closed under Op_α. So we immediately obtain that $\mathsf{C}(\alpha)$ is the largest fixed point of Op_α, i.e.

$$\mathbf{K}_n(\mathsf{C}) \;\vdash\; \mathsf{C}(\alpha) \;\leftrightarrow\; \mathsf{E}(\alpha) \wedge \mathsf{E}(\mathsf{C}(\alpha)).$$

Proof-theoretic experience should provide a clear indication that the interplay of (CCL) and (IND) may cause serious difficulties in finding good deductive systems for $\mathbf{K}_n(\mathsf{C})$ and the other multi-modal logics mentioned before.

3 A Tait-style reformulation of $\mathbf{K}_n(\mathsf{C})$

In this and the following sections we will look more carefully at the deductive and procedural aspects of our logics of common knowledge. For simplicity we restrict ourselves to the theory $\mathbf{K}_n(\mathsf{C})$; other logics are treated in Alberucci [2, 1].

Obviously, inference rules like modus ponens (MP), which violate the subformula property, make reasonable backward proof search impossible. The first steps thus are a reformulation of $\mathbf{K}_n(\mathsf{C})$ as a Tait-style system with cuts and an attempt to "tame" general cuts in a suitable way.

The Tait-calculus $\overline{\mathbf{K}}_n(\mathsf{C})$ derives finite sets of $L_n(\mathsf{C})$ formulas which are denoted by the capital Greek letters $\Gamma, \Delta, \Pi, \Sigma \ldots$ (possibly with subscripts) and have to be interpreted disjunctively. We often write (for example) $\alpha, \beta, \Gamma, \Delta$

for the union $\{\alpha, \beta\} \cup \Gamma \cup \Delta$. In addition, if Γ is the set $\{\alpha_1, \ldots, \alpha_m\}$, we often use the following convenient abbreviations:

$$\begin{aligned}
\Gamma^\vee &:= \alpha_1 \vee \ldots \vee \alpha_m, \\
\neg\Gamma &:= \{\neg\alpha_1, \ldots, \neg\alpha_m\}, \\
\neg\mathsf{K}_i(\Gamma) &:= \{\neg\mathsf{K}_i(\alpha_1), \ldots, \neg\mathsf{K}_i(\alpha_m)\}, \\
\neg\mathsf{C}(\Gamma) &:= \{\neg\mathsf{C}(\alpha_1), \ldots, \neg\mathsf{C}(\alpha_m)\}.
\end{aligned}$$

The axioms and rules of $\overline{\mathbf{K}}_n(\mathsf{C})$ consist of the usual propositional axioms and rules of Tait-calculi, of rules for the epistemic operators K_i with incorporated formulas $\neg\mathsf{C}(\Delta)$ plus specific C-rules and induction rules.

Axioms of $\overline{\mathbf{K}}_n(\mathsf{C})$

(ID) $$P,\ \neg P,\ \Gamma$$

Basic rules of inference of $\overline{\mathbf{K}}_n(\mathsf{C})$

($\vee$) $$\frac{\alpha,\ \beta,\ \Gamma}{\alpha \vee \beta,\ \Gamma}$$

($\wedge$) $$\frac{\alpha,\ \Gamma \qquad \beta,\ \Gamma}{\alpha \wedge \beta,\ \Gamma}$$

(K_i) $$\frac{\alpha,\ \neg\Gamma,\ \neg\mathsf{C}(\Delta)}{\mathsf{K}_i(\alpha),\ \neg\mathsf{K}_i(\Gamma),\ \neg\mathsf{C}(\Delta),\ \Pi}$$

C-rules of $\overline{\mathbf{K}}_n(\mathsf{C})$

($\neg\mathsf{C}$) $$\frac{\neg\mathsf{E}(\alpha),\ \Gamma}{\neg\mathsf{C}(\alpha),\ \Gamma}$$

(C) $$\frac{\mathsf{E}(\alpha),\ \neg\mathsf{C}(\Delta)}{\mathsf{C}(\alpha),\ \neg\mathsf{C}(\Delta),\ \Pi}$$

Induction rules of $\overline{\mathbf{K}}_n(\mathsf{C})$

(Ind) $$\frac{\neg\beta,\ \mathsf{E}(\alpha),\ \neg\mathsf{C}(\Delta) \qquad \neg\beta,\ \mathsf{E}(\beta),\ \neg\mathsf{C}(\Delta)}{\neg\beta,\ \mathsf{C}(\alpha),\ \neg\mathsf{C}(\Delta),\ \Pi}$$

The axioms and rules of our Tait-style formalization of $\mathbf{K}_n(\mathsf{C})$ do not comprise cuts; since we want to distinguish between various cut rules, we always mention explicitly what sort of cuts we use.

Now we introduce the usual cuts, called general cuts in our present context; restrictions of the cut rule will be discussed later.

General cuts

(G-Cut)
$$\frac{\alpha, \Gamma \qquad \neg\alpha, \Gamma}{\Gamma}$$

The designated formulas α and $\neg\alpha$ are called the *cut formulas* of this general cut.

Derivability of a finite set Γ of $L_n(\mathsf{C})$ formulas in $\overline{\mathbf{K}}_n(\mathsf{C})$ with possible additional cuts from ($*$-Cut) is introduced as usual and written as

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (*\text{-Cut}) \vdash \Gamma.$$

Before saying more about general and special cuts, we have to make sure that $\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut})$ is a reformulation of $\mathbf{K}_n(\mathsf{C})$. One direction is straightforward and formulated below.

Lemma 2 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \Gamma \quad \Longrightarrow \quad \mathbf{K}_n(\mathsf{C}) \vdash \Gamma^{\vee}.$$

The proof of this lemma is unproblematic but requires some tedious work within the theory $\mathbf{K}_n(\mathsf{C})$ which we omit. For establishing the reduction of $\mathbf{K}_n(\mathsf{C})$ to $\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut})$, it is convenient to begin with some auxiliary considerations. A first remark refers to the propositional completeness and the co-closure properties of $\overline{\mathbf{K}}_n(\mathsf{C})$.

Lemma 3 *For all $L_n(\mathsf{C})$ formulas α the following two assertions can be proved in $\overline{\mathbf{K}}_n(\mathsf{C})$:*

1. $\neg\alpha, \alpha$.
2. $\neg\mathsf{C}(\alpha), \mathsf{E}(\alpha) \land \mathsf{E}(\mathsf{C}(\alpha))$.

Proof The first assertion can be easily established by induction on the depth $\mathsf{dpt}(\alpha)$ of α; details are left to the reader. Thus we have

(1) $$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{E}(\alpha), \mathsf{E}(\alpha),$$

(2) $$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha), \mathsf{C}(\alpha).$$

From (1) we can immediately deduce by rule $(\neg\mathsf{C})$ that

(3) $$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha),\, \mathsf{E}(\alpha),$$

Moreover, (2) and applications of the rules $(\mathsf{K}_1), \ldots, (\mathsf{K}_n)$ and $(\wedge)$ yield

(4) $$\overline{\mathbf{K}}_n(\mathsf{C}) \vdash \neg\mathsf{C}(\alpha),\, \mathsf{E}(\mathsf{C}(\alpha)).$$

Altogether, statements (3) and (4) plus once more the rule $(\wedge)$ give us what we want. □

Lemma 4 *Let α and β be two $L_n(\mathsf{C})$ formulas so that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \beta \to \mathsf{E}(\alpha) \wedge \mathsf{E}(\beta).$$

Then we also have that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \beta \to \mathsf{C}(\alpha).$$

This lemma is a direct consequence of the induction rule (Ind) of our calculus and some trivial formula manipulations within $\overline{\mathbf{K}}_n(\mathsf{C})$. Thus, recapitulating what we have obtained so far, we see that $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) is a Tait-style reformulation of $\mathbf{K}_n(\mathsf{C})$.

Theorem 5 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \vdash \Gamma \quad\Longleftrightarrow\quad \mathbf{K}_n(\mathsf{C}) \vdash \Gamma^{\vee}.$$

Proof The direction from left to right is a direct consequence of Lemma 2. In order to prove the converse direction, we first observe that the basic axioms of $\mathbf{K}_n(\mathsf{C})$ are trivially derivable in $\overline{\mathbf{K}}_n(\mathsf{C})$ and that the co-closure axioms are proved in Lemma 3(2). Hence all axioms of $\mathbf{K}_n(\mathsf{C})$ are provable in $\overline{\mathbf{K}}_n(\mathsf{C})$. Since Lemma 4 states that $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) is closed under the induction rule of $\mathbf{K}_n(\mathsf{C})$ and since all other derivation rules of $\mathbf{K}_n(\mathsf{C})$ have obvious counterparts in $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut), the direction from right to left of our theorem follows by induction on the derivations in $\mathbf{K}_n(\mathsf{C})$. □

The rule (G-Cut) is a stumbling block to using $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) as a meaningful procedural framework for common knowledge. However, total cut elimination for this calculus is not possible: Let us work with two agents only, choose two different atomic propositions P and Q and consider the formula α given by

$$\neg\mathsf{K}_1(P \wedge \mathsf{C}(Q)) \;\vee\; \neg\mathsf{K}_2(Q \wedge \mathsf{C}(P)) \;\vee\; \mathsf{C}(P \vee Q).$$

Then it can be easily checked that $\mathbf{K}_n(\mathsf{C}) \models \alpha$, hence $\overline{\mathbf{K}}_n(\mathsf{C}) + \text{(G-Cut)} \vdash \alpha$ in view of Theorem 1 and Theorem 5. But it is also not too complicated to show that α cannot be proved in (the cut-free system) $\overline{\mathbf{K}}_n(\mathsf{C})$.

Because of this and related examples we doubt that there is a natural and perspicuous (more sophisticated) cut-free Tait- or Gentzen-calculus which is equivalent to $\mathbf{K}_n(\mathsf{C})$ and enjoys the subformula property.

4 Fischer-Ladner cuts

An interesting *partial* cut elimination result for a Tait-style version of $\mathbf{K}_n(\mathsf{C})$ is obtained by restricting cuts to specific formulas generated from the so-called *Fischer-Ladner closure* of provable formulas. The exact details will be described below; first we introduce some auxiliary notions.

Let Ω be a set of $L_n(\mathsf{C})$ formulas which is closed under negation; i.e. Ω has the property that $\Omega = \neg\Omega$. Then the Ω-*cuts* are all cuts

$$(\Omega\text{-Cut}) \qquad \frac{\alpha,\, \Gamma \qquad \neg\alpha,\, \Gamma}{\Gamma}$$

so that their cut formulas α and $\neg\alpha$ belong to the set Ω. Such Ω-cuts, for very specific sets of formulas Ω, will play an important role later.

Lemma 6 *Let Ω be a set of $L_n(\mathsf{C})$ formulas which is closed under negation. Then we have for all finite sets Γ of $L_n(\mathsf{C})$ formulas and all formulas $(\alpha \vee \beta)$ and $(\alpha_0 \wedge \alpha_1)$ which belong to Ω:*

1. $\overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \vdash (\alpha \vee \beta),\, \Gamma \quad \Longrightarrow \quad \overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \vdash \alpha,\, \beta,\, \Gamma.$
2. $\overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \vdash (\alpha_0 \wedge \alpha_1),\, \Gamma \quad \Longrightarrow \quad \overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \vdash \alpha_i,\, \Gamma.$

Proof Obvious derivations in $\overline{\mathbf{K}}_n(\mathsf{C})$ yield that

$$(1) \qquad \overline{\mathbf{K}}_n(\mathsf{C}) \vdash (\neg\alpha \wedge \neg\beta),\, \alpha,\, \beta,$$

$$(2) \qquad \overline{\mathbf{K}}_n(\mathsf{C}) \vdash (\neg\alpha_0 \vee \neg\alpha_1),\, \alpha_i$$

for $i = 0, 1$. Since the formulas $(\neg\alpha \wedge \neg\beta)$ and $(\neg\alpha_0 \vee \neg\alpha_1)$ belong to Ω, the assertions of our lemma follow from (1) and (2) by simple Ω-cuts. $\Box$

For the next considerations let Ω and Σ be two sets of $L_n(\mathsf{C})$ formulas which are closed under negation and assume that Σ is a finite subset of Ω. For those Ω and Σ we introduce as auxiliary notions:

- A finite subset Γ of $L_n(\mathsf{C})$ formulas is called Ω-*consistent* in case that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \nvdash \neg\Gamma.$$

- A subset Γ of Σ is called *maximal* Ω-*consistent with respect to* Σ if Γ is Ω-consistent and if there exists no Ω-consistent subset of Σ which is a proper superset of Γ.

Some important properties of maximal Ω-consistent sets with respect to Σ are summarized in the subsequent lemma. Its proof is standard and can be omitted.

Lemma 7 *Let Ω and Σ be sets of $L_n(\mathsf{C})$ formulas as above. Then we have for all subsets Γ of Σ which are maximal Ω-consistent with respect to Σ and all $L_n(\mathsf{C})$ formulas α, β:*

1. $\alpha \in \Sigma \quad \Longrightarrow \quad \alpha \in \Gamma$ *or* $\neg\alpha \in \Gamma$.
2. $\alpha \in \Sigma$ *and* $\overline{\mathbf{K}}_n(\mathsf{C}) + (\Omega\text{-Cut}) \vdash \alpha, \neg\Gamma \quad \Longrightarrow \quad \alpha \in \Gamma$.
3. $\alpha, \beta \in \Sigma$ *and* $(\alpha \vee \beta) \in \Gamma \quad \Longrightarrow \quad \alpha \in \Gamma$ *or* $\beta \in \Gamma$.
4. $\alpha, \beta \in \Sigma$ *and* $(\alpha \wedge \beta) \in \Gamma \quad \Longrightarrow \quad \alpha \in \Gamma$ *and* $\beta \in \Gamma$.

Again, let Ω and Σ be sets of $L_n(\mathsf{C})$ formulas as above. Each subset Γ of Σ which is Ω-consistent can be easily extended to a maximal Ω-consistent set with respect to Σ. To see why, simply fix an enumeration $\gamma_0, \ldots, \gamma_k$ of Σ and define $\Gamma_0 := \Gamma$ as well as

$$\Gamma_{i+1} \;:=\; \begin{cases} \Gamma_i \cup \{\gamma_i\} & \text{if } \Gamma_i \cup \{\gamma_i\} \text{ is } \Omega\text{-consistent with respect to } \Sigma, \\ \Gamma_i \cup \{\neg\gamma_i\} & \text{otherwise} \end{cases}$$

for all natural numbers $i \leq k$. Then simple induction on $i \leq k$ shows that each Γ_i is Ω-consistent and contained in Σ. Hence the union of all sets Γ_i, $0 \leq i \leq k$, is a possible candidate for the set Δ which is claimed to exist in the following lemma.

Lemma 8 *Let Ω and Σ be sets of $L_n(\mathsf{C})$ formulas as above and assume that Γ is Ω-consistent subset of Σ. Then there exists a subset Δ of Σ which is maximal Ω-consistent with respect to Σ and contains Γ.*

Before formulating and proving the main results of this section, we have to fix those sets of $L_n(\mathsf{C})$ formulas which we have to substitute for Σ and Ω.

The so-called *Fischer-Ladner closure* $\mathbb{FL}(\alpha)$ of an $L_n(\mathsf{C})$ formula α (see Fischer and Ladner [6]) is the set of $L_n(\mathsf{C})$ formulas which is inductively generated as follows:

1. α belongs to $\mathbb{FL}(\alpha)$.
2. If β belongs to $\mathbb{FL}(\alpha)$, then $\neg\beta$ belongs to $\mathbb{FL}(\alpha)$.
3. If $(\beta \vee \gamma)$ belongs to $\mathbb{FL}(\alpha)$, then β and γ belong to $\mathbb{FL}(\alpha)$.
4. If $(\beta \wedge \gamma)$ belongs to $\mathbb{FL}(\alpha)$, then β and γ belong to $\mathbb{FL}(\alpha)$.
5. If $\mathsf{K}_i(\beta)$ belongs to $\mathbb{FL}(\alpha)$, then β belongs to $\mathbb{FL}(\alpha)$.
6. If $\mathsf{C}(\beta)$ belongs to $\mathbb{FL}(\alpha)$, then β, $\mathsf{E}(\beta)$ and $\mathsf{E}(\mathsf{C}(\beta))$ belong to $\mathbb{FL}(\alpha)$.

Moreover, for any finite set Γ of $L_n(\mathsf{C})$ formulas, its Fisher-Ladner closure $\mathbb{FL}(\Gamma)$ is introduced by

$$\mathbb{FL}(\Gamma) \ := \ \mathbb{FL}(\Gamma^{\vee}).$$

The Fischer-Ladner closure $\mathbb{FL}(\alpha)$ of an $L_n(\mathsf{C})$ formula α is obviously finite and, according to [6], the number of elements of $\mathbb{FL}(\alpha)$ is of order $\mathcal{O}(|\alpha|)$ where $|\alpha|$ denotes the length of the formula α.

Sets $\mathbb{FL}(\Gamma)$ will take over the role of the set Σ in the previous considerations; the counterpart of the set Ω will be the *disjunctive-conjunctive closure* $\mathbb{DC}(\Gamma)$ of $\mathbb{FL}(\Gamma)$ which is carefully introduced now.

Until the end of this section we fix an arbitrary finite set Γ of $L_n(\mathsf{C})$ formulas and associate to this Γ (arbitrary but fixed) enumerations

$$(\star) \qquad \delta_1, \delta_2, \ldots, \delta_p \quad \text{and} \quad \Delta_1, \Delta_2, \ldots, \Delta_q$$

of the elements of $\mathbb{FL}(\Gamma)$ and the subsets of $\mathbb{FL}(\Gamma)$, respectively. Each set $\Delta \subset \mathbb{FL}(\Gamma)$ can then be written as

$$\{\, \delta_{s(1)}, \delta_{s(2)}, \ldots, \delta_{s(m_\Delta)} \,\}$$

so that $1 \leq s(1) < s(2) < \ldots < s(m_\Delta) \leq p$, and we define the $L_n(\mathsf{C})$ formula

$$(\star\star) \qquad \varphi_\Delta \ := \ (\ldots(\delta_{s(1)} \wedge \delta_{s(2)}) \wedge \ldots) \wedge \delta_{s(m_\Delta)}).$$

In addition, each $D \subset \mathit{Pow}(\mathbb{FL}(\Gamma))$ can be brought into the form

$$\{\, \Delta_{t(1)}, \Delta_{t(2)}, \ldots, \Delta_{t(m_D)} \,\}$$

so that $1 \leq t(1) < t(2) < \ldots < t(m_D) \leq q$, and now we define

$$\varphi_D \ := \ (\ldots(\varphi_{\Delta_{t(1)}} \vee \varphi_{\Delta_{t(2)}}) \vee \ldots) \vee \varphi_{\Delta_{t(m_D)}}).$$

Finally, we let $\mathbb{DC}(\Gamma)$ be the set of all formulas φ_D, for $D \subset Pow(\mathbb{FL}(\Gamma))$, and their negations,

$$\mathbb{DC}(\Gamma) \ := \ \{\, \varphi_D : D \subset Pow(\mathbb{FL}(\Gamma))\,\} \cup \{\, \neg\varphi_D : D \subset Pow(\mathbb{FL}(\Gamma))\,\}.$$

According to these definitions, $\mathbb{FL}(\Gamma)$ is contained in the set $\mathbb{DC}(\Gamma)$. Futhermore, since $\mathbb{DC}(\Gamma)$ contains a representative (modulo logical equivalence) of each formula which is built up from the elements of $\mathbb{FL}(\Gamma)$ by disjunctions and conjunctions, it is justified to regard it as the disjunctive-conjunctive closure of $\mathbb{FL}(\Gamma)$.

Depending on Γ, we can introduce the *canonical Kripke-frame*

$$\mathfrak{M}^\Gamma \ := \ (W^\Gamma, \mathbb{K}_1^\Gamma, \ldots, \mathbb{K}_n^\Gamma)$$

whose set of worlds W^Γ is the collection of all maximal $\mathbb{DC}(\Gamma)$-consistent sets with respect to $\mathbb{FL}(\Gamma)$; the accessibility relations $\mathbb{K}_i^\Gamma$ consist of all pairs (Δ, Σ) of elements of W^Γ so that

$$\Delta/\mathsf{K}_i \ := \ \{\alpha : \mathsf{K}_i(\alpha) \in \Delta\}$$

is contained in Σ, i.e.

$$\mathbb{K}_i^\Gamma \ := \ \{\, (\Delta, \Sigma) \in W^\Gamma \times W^\Gamma : \Delta/\mathsf{K}_i \subset \Sigma \,\}.$$

The following lemma takes care of one specific case in the proof of Lemma 10 and is treated separately in order to "disburden" this rather lengthy proof.

Lemma 9 *Assume that* $\Delta \in W^\Gamma$, $(\Delta, \Sigma) \in \mathbb{K}_i^\Gamma$ *and* $\mathsf{C}(\alpha) \in \Delta$. *Then we have* $\mathsf{C}(\alpha) \in \Sigma$ *and* $\alpha \in \Sigma$.

Proof Recall from Lemma 3(2) that

(1) $$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \ \vdash \ \neg\mathsf{C}(\alpha),\, \mathsf{E}(\mathsf{C}(\alpha)),$$

(2) $$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \ \vdash \ \neg\mathsf{C}(\alpha),\, \mathsf{E}(\alpha).$$

Since $\mathsf{E}(\mathsf{C}(\alpha))$ and $\mathsf{E}(\alpha)$ belong to $\mathbb{FL}(\Gamma)$, we are in the position of applying Lemma 7(2) to (1) and (2) and know that

(3) $$\mathsf{E}(\mathsf{C}(\alpha)) \in \Delta,$$

(4) $$\mathsf{E}(\alpha) \in \Delta.$$

Because of Lemma 7(4) we thus have $\mathsf{K}_i(\mathsf{C}(\alpha)) \in \Delta$ and $\mathsf{K}_i(\alpha) \in \Delta$. The definition of $\mathbb{K}_i^\Gamma$ therefore implies the assertion of our lemma. $\Box$

As *canonical valuation* (with respect to Γ) we fix the mapping $\mathcal{V}^\Gamma$ from the atomic propositions to $Pow(W^\Gamma)$ given by

$$\mathcal{V}^\Gamma(P) \;:=\; \{\Delta \in W^\Gamma : P \in \Delta\}$$

for all elements P of PROP. With $\mathfrak{M}^\Gamma$ and $\mathcal{V}^\Gamma$ being provided, we are ready for establishing the main lemma for the proof of Theorem 11.

Lemma 10 *Let Γ be our finite set of $L_n(\mathsf{C})$ formulas. Then we have for all $\Sigma \in W^\Gamma$ and all $\alpha \in \mathbb{FL}(\Gamma)$ that*

$$\alpha \in \Sigma \quad\Longleftrightarrow\quad (\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Sigma) \models \alpha.$$

Proof We show this equivalence by induction on the structure of the formula α and carry through the following distinction by cases.

1. α is an atomic proposition or the negation of an atomic proposition. Then the assertion follows from the definition of $\mathcal{V}^\alpha$.

2. α is of the form $(\beta_0 \vee \beta_1)$ or $(\beta_0 \wedge \beta_1)$. Then the assertion follows from the induction hypothesis by means of Lemma 7.

3. α is of the form $\mathsf{K}_i(\beta)$. The direction from left to right is immediate from the definition of $\mathbb{K}_i^\Gamma$ and the induction hypothesis. For the converse direction, assume that $\mathsf{K}_i(\beta) \notin \Sigma$. Then $\neg\mathsf{K}_i(\beta) \in \Sigma$ by Lemma 7(1) and

$$(1) \qquad \overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\nvdash\; \mathsf{K}_i(\beta), \{\neg\mathsf{K}_i(\gamma) : \mathsf{K}_i(\gamma) \in \Sigma\}.$$

Because of the rule (K_i) we therefore also have

$$(2) \qquad \overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\nvdash\; \beta, \{\neg\gamma : \mathsf{K}_i(\gamma) \in \Sigma\}.$$

This means that the set $\{\neg\beta\} \cup \{\gamma : \mathsf{K}_i(\gamma) \in \Sigma\}$ is $\mathbb{DC}(\Gamma)$-consistent. Since it is also contained in $\mathbb{FL}(\Gamma)$, Lemma 8 claims the existence of an element Δ of W^Γ with

$$(3) \qquad \neg\beta \in \Delta,$$

$$(4) \qquad \{\gamma : \mathsf{K}_i(\gamma) \in \Sigma\} \;\subset\; \Delta.$$

From (3) we conclude with the induction hypothesis that $(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Delta) \not\models \beta$. Further, (4) yields that $(\Sigma, \Delta) \in \mathbb{K}_i^\Gamma$. Hence $(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Sigma) \not\models \mathsf{K}_i(\beta)$, and the direction from right to left is proved.

4. α is of the form $\sim\!\mathsf{K}_i(\beta)$. The treatment of this case is analogous to the previous one.

5. α is of the form $\mathsf{C}(\beta)$. For showing the direction from left to right we assume $\mathsf{C}(\beta) \in \Sigma$. Lemma 9 and a simple proof by induction on m entails that

$$\mathsf{C}(\beta) \in \Delta \quad \text{and} \quad \beta \in \Delta \tag{5}$$

for all elements $\Delta \in W^\Gamma$ which are accessible from Σ in m steps. But then the induction hypothesis implies

$$(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Delta) \models \beta \tag{6}$$

for such Δ. Given the definition of the validity of the formula $\mathsf{C}(\beta)$, we have herewith shown that $(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Sigma) \models \mathsf{C}(\beta)$.

For dealing with the converse direction, we first recall the enumerations $(\star)$ and let

$$\Delta_{w(1)},\, \Delta_{w(2)},\, \ldots,\, \Delta_{w(u)}$$

with $1 \leq w(1) < w(2) < \ldots < w(u) \leq q$ be the list of all sets Δ_j so that $(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Delta_j) \models \mathsf{C}(\beta)$. Now introduce the formula $\psi_{\mathsf{C}(\beta)}$,

$$\psi_{\mathsf{C}(\beta)} \;:=\; (\ldots(\varphi_{\Delta_{w(1)}} \vee \varphi_{\Delta_{w(2)}}) \vee \ldots) \vee \varphi_{\Delta_{w(u)}})$$

for each $\varphi_{\Delta_{w(j)}}$ being defined as in $(\star\star)$. From the definition of $\mathbb{DC}(\Gamma)$ above we learn that $\psi_{\mathsf{C}(\beta)} \in \mathbb{DC}(\Gamma)$. For this formula $\psi_{\mathsf{C}(\beta)}$ we want to show:

$$\overline{\mathbf{K}}_n(\mathsf{C}) \;\vdash\; \neg\psi_{\mathsf{C}(\beta)},\, \mathsf{E}(\beta), \tag{7}$$

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\psi_{\mathsf{C}(\beta)},\, \mathsf{E}(\psi_{\mathsf{C}(\beta)}). \tag{8}$$

To prove (7), observe that

$$(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Delta) \models \mathsf{C}(\beta) \quad\Longrightarrow\quad (\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Delta) \models \mathsf{E}(\beta)$$

for all $\Delta \in W^\Gamma$. Hence the induction hypothesis tells us that $\mathsf{E}(\beta) \in \Delta_{w(j)}$ for $j = 1, \ldots, u$. Consequently, we have

$$\overline{\mathbf{K}}_n(\mathsf{C}) \;\vdash\; \neg\varphi_{\Delta_{w(j)}},\, \mathsf{E}(\beta) \tag{9}$$

for $j = 1, \ldots, u$. From (9) and the definition of $\neg\psi_{\mathsf{C}(\beta)}$ we obtain assertion (7) by some obvious basic inferences.

The proof of (8) is more complicated: We first observe that for all $\Delta \in W^\Gamma$

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg(\Delta/\mathsf{K}_i),\, \{\varphi_{\Delta_j} : j \in N_\Delta\} \tag{10}$$

where N_Δ is the set of all natural numbers given by

$$N_\Delta \;:=\; \{\, j : 1 \leq j \leq q \;\text{ and }\; (\Delta, \Delta_j) \in \mathbb{K}_i^\Gamma \,\},$$

again referring to the enumerations ($\star$). If this were not the case, then we could pick for each $j \in N_\Delta$ a formula $\chi_j \in \Delta_j$ satisfying

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\not\vdash\; \neg(\Delta/\mathsf{K}_i),\, \{\chi_j : j \in N_\Delta\}.$$

However, this would imply that the set

$$(\Delta/\mathsf{K}_i) \,\cup\, \{\neg\chi_j : j \in N_\Delta\}$$

is $\mathbb{DC}(\Gamma)$-consistent and therefore, by Lemma 8, contained in a set Π which is maximal $\mathbb{DC}(\Gamma)$-consistent with respect to $\mathbb{FL}(\Gamma)$. But then we had $(\Delta/\mathsf{K}_i) \subset \Pi$, hence $(\Delta, \Pi) \in \mathbb{K}_i^\Gamma$, and $\Pi \neq \Delta_j$ for all $j \in N_\Delta$ because of the choice of the formulas χ_j. This is a contradiction, and (10) has been established.

The next step is to choose an arbitrary $\Delta_{w(k)}$ with $1 \leq k \leq u$. By (10) we have

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg(\Delta_{w(k)}/\mathsf{K}_i),\, \psi_{\mathsf{C}(\beta)}, \tag{11}$$

simply because $N_{\Delta_{w(k)}} \subset \{w(1), w(2), \ldots, w(u)\}$. By applying the rule (K_i) to (11) we gain

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\Delta_{w(k)},\, \mathsf{K}_i(\psi_{\mathsf{C}(\beta)}), \tag{12}$$

hence also

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\Delta_{w(k)},\, \mathsf{E}(\psi_{\mathsf{C}(\beta)}), \tag{13}$$

since (12) holds for all operators $\mathsf{K}_1, \ldots, \mathsf{K}_n$. Assertion (13) is immediately transformed into

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\varphi_{\Delta_{w(k)}},\, \mathsf{E}(\psi_{\mathsf{C}(\beta)}) \tag{14}$$

and available for all $1 \leq k \leq u$. Therefore assertion (8) follows from (14) by several applications of the rule $(\wedge)$.

Having proved assertions (7) and (8), the induction rule (Ind) comes into play and yields

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\psi_{\mathsf{C}(\beta)},\, \mathsf{C}(\beta). \tag{15}$$

Since $\psi_{\mathsf{C}(\beta)}$ belongs to $\mathbb{DC}(\Gamma)$, assertion (15) gives us in view of Lemma 6(2) that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\varphi_{\Delta_{w(k)}},\, \mathsf{C}(\beta) \tag{16}$$

for all $1 \leq k \leq u$. The formulas $\varphi_{\Delta_{w(k)}}$ are elements of $\mathbb{DC}(\Gamma)$ as well, and now we apply Lemma 6(1) to (16) in order to obtain

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\Delta_{w(k)},\, \mathsf{C}(\beta) \tag{17}$$

for all $1 \leq k \leq u$. To conclude the proof of the direction from right to left, assume that $\Sigma \in W^\Gamma$ and

$$(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Sigma) \models \mathsf{C}(\beta).$$

Then the set Σ is identical to some $\Delta_{w(k)}$, $1 \leq k \leq u$, and thus (17) entails

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\vdash\; \neg\Sigma,\, \mathsf{C}(\beta). \tag{18}$$

Finally we make use of Lemma 7(2) and gain $\mathsf{C}(\beta) \in \Sigma$, as desired.

6. α is of the form $\sim\mathsf{C}(\beta)$. The treatment of this case is analogous to the previous one. □

Theorem 11 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \vdash \Gamma \quad\Longleftrightarrow\quad \mathbf{K}_n(\mathsf{C}) \models \Gamma^\vee.$$

Proof The direction from left to right of this equivalence is implied by Theorem 5 and Theorem 1. Conversely, fix a finite set Γ of $L_n(\mathsf{C})$ formulas and assume that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\nvdash\; \Gamma.$$

Then the formula $\Gamma^\vee$ is an element of $\mathbb{FL}(\Gamma) \subset \mathbb{DC}(\Gamma)$, and Lemma 6(1) implies that

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \;\nvdash\; \Gamma^\vee.$$

Hence $\{\neg(\Gamma^\vee)\}$ is $\mathbb{DC}(\Gamma)$-consistent and, because of Lemma 8, there must be a set Σ which contains $\neg(\Gamma^\vee)$ and is maximal $\mathbb{DC}(\Gamma)$-consistent with respect to $\mathbb{FL}(\Gamma)$, i.e.

$$\Sigma \in W^\Gamma \quad\text{and}\quad \neg(\Gamma^\vee) \in \Sigma.$$

Now we can apply the previous lemma in order to obtain

$$(\mathfrak{M}^\Gamma, \mathcal{V}^\Gamma, \Gamma) \models \neg(\Gamma^\vee).$$

So we know that $\Gamma^\vee$ is not valid in the canonical $\mathfrak{M}^\Gamma$, and, consequently, $\mathbf{K}_n(\mathsf{C}) \not\models \Gamma^\vee$. This completes the proof of our theorem. □

Corollary 12 (Partial cut elimination for $\overline{\mathbf{K}}_n(\mathsf{C})$) *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\overline{\mathbf{K}}_n(\mathsf{C}) + (\text{G-Cut}) \ \vdash \ \Gamma \quad \Longleftrightarrow \quad \overline{\mathbf{K}}_n(\mathsf{C}) + (\mathbb{DC}(\Gamma)\text{-Cut}) \ \vdash \ \Gamma.$$

The last assertion is a trivial consequence of Theorem 1, Theorem 5 and Theorem 11 just above. It says that for each proof of a finite set Γ of $L_n(\mathsf{C})$ formulas in the calculus $\overline{\mathbf{K}}_n(\mathsf{C})$ + (G-Cut) there exists a proof with cuts so that all their cut formulas belong to the representation system $\mathbb{DC}(\Gamma)$ of the disjunctive-conjunctive closure of the Fischer-Ladner closure of Γ.

This corollary allows us to replace the infinite number of all possible cuts in a derivation of a set Γ by cuts whose cut formulas belong to the finite set $\mathbb{DC}(\Gamma)$. However, from the point of view of efficient proof search, the cardinality of $\mathbb{DC}(\Gamma)$ is still infeasible.

In Alberucci [2, 1] our partial cut elimination technique has been refined by showing that cuts with cut formulas from the conjunctive closure of the Fischer-Ladner closure are sufficient. It is an interesting question whether the cuts can be further restricted.

5 The infinitary system $\mathbf{K}_n^\omega(\mathsf{C})$

The iterative approach to common knowledge can most easily be reflected in a deductive system by working with an analogue of the ω-rule which permits the derivation of the formula $\mathsf{C}(\alpha)$ from the infinitely many premises

$$\mathsf{E}^1(\alpha),\ \mathsf{E}^2(\alpha),\ \ldots,\ \mathsf{E}^m(\alpha),\ \ldots$$

for all natural numbers $m \geq 1$, just as in the semantic interpretation of $\mathsf{C}(\alpha)$, introduced in Section 2 above.

Our infinitary system $\mathbf{K}_n^\omega(\mathsf{C})$ is formulated in the finitary language $L_n(\mathsf{C})$ and derives finite sets of $L_n(\mathsf{C})$ formulas. It is infinitary only because of the rule ($\omega\mathsf{C}$) for introducing common knowledge; ($\omega\mathsf{C}$) has infinitely many premises and thus may give rise to infinite proof trees.

The axioms and basic rules of $\mathbf{K}_n^\omega(\mathsf{C})$ are those of $\overline{\mathbf{K}}_n(\mathsf{C})$, in particular we have the rules (K_i for introducing the epistemic operators K_i and their negations,

$$(\mathsf{K}_i) \qquad \frac{\alpha,\ \neg\Gamma,\ \neg\mathsf{C}(\Delta)}{\mathsf{K}_i(\alpha),\ \neg\mathsf{K}_i(\Gamma),\ \neg\mathsf{C}(\Delta),\ \Pi}$$

with the formulas $\mathsf{K}_i(\alpha)$ and $\neg\mathsf{K}_i(\Gamma)$ as *main formulas* and the negated formulas about common knowledge as *side formulas.* As further rule for introducing positive knowledge we add

$$(\mathsf{K}^\star) \qquad \frac{\alpha}{\mathsf{K}_i(\mathsf{E}^m(\alpha)),\, \Pi}$$

for any natural number m. The rule $(\mathsf{K}^\star)$ might appear to be superfluous for the following reason: Suppose that α is provable. Then a series of applications of the rules $(\mathsf{K}_1), \dots, (\mathsf{K}_n)$ and $(\wedge)$ allows us to derive $\mathsf{K}_i(\mathsf{E}^m(\alpha)),\, \Pi$ for all m. However, these derivations depend on m, whereas an application of $(\mathsf{K}^\star)$ enables us to accomplish the same in one step. Together with the rule $(\omega\mathsf{C})$ from below, we only need n additional steps to derive $\mathsf{C}(\alpha)$ in case that α has been proved already. Without $(\mathsf{K}^\star)$ infinitely many additional steps would be required.

Negated common knowledge is introduced by the rule $(\neg\mathsf{C})$ as before, for positive common knowledge we now have the infinitary rule $(\omega\mathsf{C})$.

C-rules of $\mathbf{K}_n^\omega(\mathsf{C})$

$$(\neg\mathsf{C}) \qquad \frac{\neg\mathsf{E}(\alpha),\, \Gamma}{\neg\mathsf{C}(\alpha),\, \Gamma}$$

$$(\omega\mathsf{C}) \qquad \frac{\mathsf{E}^m(\alpha),\, \Gamma \quad (\text{for all } m \geq 1)}{\mathsf{C}(\alpha),\, \Gamma}$$

Although all formulas of the language $L_n(\mathsf{C})$ are finite strings of symbols, the rule $(\omega\mathsf{C})$ has the effect of treating the formulas $\mathsf{C}(\alpha)$ as the infinite conjunctions $\bigwedge\{\mathsf{E}^m(\alpha) : m \geq 1\}$. Accordingly, the rank $\mathsf{me}(\alpha)$ of each $L_n(\mathsf{C})$ formula α is an ordinal which is inductively generated as follows:

1. $\mathsf{me}(P) \;:=\; \mathsf{me}(\sim P) \;:=\; 0.$
2. $\mathsf{me}(\alpha \vee \beta) \;:=\; \mathsf{me}(\alpha \wedge \beta) \;:=\; \max(\mathsf{me}(\alpha), \mathsf{me}(\beta)) + 1.$
3. $\mathsf{me}(\mathsf{K}_i(\alpha)) \;:=\; \mathsf{me}(\sim\mathsf{K}_i(\alpha)) \;:=\; \mathsf{me}(\alpha) + 1.$
4. $\mathsf{me}(\mathsf{C}(\alpha)) \;:=\; \mathsf{me}(\sim\mathsf{C}(\alpha)) \;:=\; \sup(\mathsf{me}(\mathsf{E}^m(\alpha) : m \geq 1).$

Because of the rule $(\omega\mathsf{C})$, our system $\mathbf{K}_n^\omega(\mathsf{C})$ allows proof trees which consist of infinitely many nodes, and thus ordinals, which are denoted by the small Greek letters $\sigma, \tau, \eta, \xi, \dots$ (possibly with subscripts) come into the picture.

Starting from these axioms and rules of inference, derivability in $\mathbf{K}_n^\omega(\mathsf{C})$ is introduced as usual. For arbitrary ordinals σ and finite sets Γ of $L_n(\mathsf{C})$ formulas the notion $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ is defined by induction on σ as follows:

1. If Γ is an axiom of $\mathbf{K}_n^\omega(\mathsf{C})$, then we have $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ for all σ.

2. If $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{\sigma_i} \Gamma_i$ and $\sigma_i < \sigma$ for all premises Γ_i of a rule of $\mathbf{K}_n^\omega(\mathsf{C})$, then we have $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma$ for the conclusion Γ of this rule.

$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{<\sigma} \Gamma$ means $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\tau \Gamma$ for some ordinal $\tau < \sigma$, and $\mathbf{K}_n^\omega(\mathsf{C}) \vdash \Gamma$ means $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\tau \Gamma$ for some ordinal τ.

If general cuts (G-Cut) are added to our legitimate rules of inference, we define the corresponding notions

$$\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)} \vdash^\sigma \Gamma \quad \text{and} \quad \mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)} \vdash \Gamma$$

of derivability of the finite set Γ of $L_n(\mathsf{C})$ formulas in the extended system $\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)}$ accordingly.

Two structural properties of $\mathbf{K}_n^\omega(\mathsf{C})$ – weakening and inversion – will play a certain role in the next section. Both can be established trivially by induction on the derivations involved.

Lemma 13 (Weakening) *For all finite sets Γ, Δ of $L_n(\mathsf{C})$ formulas and all ordinals σ, τ we have that*

$$\Gamma \subset \Delta, \quad \sigma \leq \tau \quad \textit{and} \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash^\tau \Delta.$$

Lemma 14 (Inversion) *For all finite sets Γ of $L_n(\mathsf{C})$ formulas, all $L_n(\mathsf{C})$ formulas α, β and all ordinals σ we have:*

1. $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha \vee \beta, \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha, \beta, \Gamma.$

2. $\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha \wedge \beta, \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \alpha, \Gamma \quad \textit{and} \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \beta, \Gamma.$

Likewise, weakening and inversion can also be accomplished for the extended system $\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)}$. However, since we do not need them in this form, we omit formulating them explicitly.

Straightforward – in general transfinite – induction on the lengths of derivations yields the correctness of $\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)}$ with respect to the semantics introduced in Section 2. This means that we have the following theorem.

Theorem 15 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)} \vdash \Gamma \quad \Longrightarrow \quad \mathbf{K}_n(\mathsf{C}) \models \Gamma^\vee.$$

The converse of this theorem is also true; we will even prove a much stronger form of completeness later. The next two lemmas deal with the co-closure and induction properties of $\mathbf{K}_n(\mathsf{C})$ and illustrate the use of the infinitary rule $(\omega\mathsf{C})$.

As for the system $\overline{\mathbf{K}}_n(\mathsf{C})$, we first observe that $\mathbf{K}_n^\omega(\mathsf{C})$ is propositionally complete and proves the co-closure axioms of $\mathbf{K}_n(\mathsf{C})$. Because of the infinitary rule $(\omega\mathsf{C})$, however, infinitary derivations may arise.

Lemma 16 *For all $L_n(\mathsf{C})$ formulas α the following two assertions can be proved in $\mathbf{K}_n^\omega(\mathsf{C})$:*

1. $\neg\alpha, \alpha$.
2. $\neg\mathsf{C}(\alpha), \mathsf{E}(\alpha) \land \mathsf{E}(\mathsf{C}(\alpha))$.

Proof The first part of this lemma can be easily established by induction on the structure of α; since the rule $(\omega\mathsf{C})$ has to be used for showing assertions of the form

$$\mathbf{K}_n^\omega(\mathsf{C}) \ \vdash \ \neg\mathsf{C}(\beta), \mathsf{C}(\beta)$$

proofs of infinite depth are needed in general. The second part of this lemma follows from the first exactly as in the proof of Lemma 3. □

The rule $(\omega\mathsf{C})$ also enables us to deal with the induction rules of $\mathbf{K}_n(\mathsf{C})$ within $\mathbf{K}_n^\omega(\mathsf{C})$ + (G-Cut); the price being again the use of infinite derivations.

Lemma 17 *Let α and β be $L_n(\mathsf{C})$ formulas and suppose that*

$$\mathbf{K}_n^\omega(\mathsf{C}) + (\text{G-Cut}) \ \vdash \ \beta \ \to \ \mathsf{E}(\alpha) \land \mathsf{E}(\beta).$$

Then we also have that

$$\mathbf{K}_n^\omega(\mathsf{C}) + (\text{G-Cut}) \ \vdash \ \beta \ \to \ \mathsf{C}(\alpha).$$

Proof We work informally in $\mathbf{K}_n^\omega(\mathsf{C})$ + (G-Cut) and obtain from the assumption that

$$\neg\beta, \mathsf{E}(\alpha), \tag{1}$$

$$\neg\beta, \mathsf{E}(\beta). \tag{2}$$

From (1) and (2) we deduce by several applications of the rules $(\mathsf{K}_1), \ldots, (\mathsf{K}_n)$ and some intermediate steps that

$$\neg\mathsf{E}^m(\beta), \mathsf{E}^{m+1}(\alpha), \tag{3}$$

$$\neg\mathsf{E}^m(\beta), \mathsf{E}^{m+1}(\beta) \tag{4}$$

for all natural numbers m. Therefore, a series of cuts yields

(5) $$\neg\beta,\ \mathsf{E}^{m+1}(\alpha)$$

for all natural numbers m. We can thus apply the rule ($\omega\mathsf{C}$) in order to conclude that

(6) $$\neg\beta,\ \mathsf{C}(\alpha).$$

Since the length of the derivation of (5) is m or more, this derivation of (6) is infinite. Some trivial modifications of (6) finish our proof. $\Box$

Lemma 16 and Lemma 17 make it clear that the Hilbert system $\mathbf{K}_n(\mathsf{C})$ can be embedded into $\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)}$. Recalling Theorem 1 and Theorem 15, we can state the following intermediate result.

Theorem 18 *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)} \vdash \Gamma \quad\Longleftrightarrow\quad \mathbf{K}_n(\mathsf{C}) \vdash \Gamma^\vee.$$

The proof of the inclusion of $\mathbf{K}_n(\mathsf{C})$ in $\mathbf{K}_n^\omega(\mathsf{C})+\text{(G-Cut)}$ given above has only been included in order to illustrate the use of the rules (K_i), ($\neg\mathsf{C}$) and ($\omega\mathsf{C}$) in this infinitary system. It also follows from the fact that $\mathbf{K}_n^\omega(\mathsf{C})$, i.e. the system without any cuts, is complete. This completeness result for $\mathbf{K}_n^\omega(\mathsf{C})$ will be proved now by semantic methods.

A finite set Γ of $L_n(\mathsf{C})$ formulas is called $\mathbf{K}_n^\omega(\mathsf{C})$ *saturated* if the following conditions are satisfied:

(ωS.1) $\mathbf{K}_n^\omega(\mathsf{C}) \nvdash \Gamma$.

(ωS.2) For all $L_n(\mathsf{C})$ formulas $(\alpha \vee \beta)$ we have

$$(\alpha \vee \beta) \in \Gamma \quad\Longrightarrow\quad \alpha \in \Gamma \ \text{ and } \ \beta \in \Gamma.$$

(ωS.3) For all $L_n(\mathsf{C})$ formulas $(\alpha \wedge \beta)$ we have

$$(\alpha \wedge \beta) \in \Gamma \quad\Longrightarrow\quad \alpha \in \Gamma \ \text{ or } \ \beta \in \Gamma.$$

(ωS.4) For all $L_n(\mathsf{C})$ formulas $\neg\mathsf{C}(\alpha)$ we have

$$\neg\mathsf{C}(\alpha) \in \Gamma \quad\Longrightarrow\quad \neg\mathsf{E}(\alpha) \in \Gamma.$$

(ωS.5) For all $L_n(\mathsf{C})$ formulas $\mathsf{C}(\alpha)$ we have

$$\mathsf{C}(\alpha) \in \Gamma \quad\Longrightarrow\quad \mathsf{E}^m(\alpha) \in \Gamma \ \text{ for some } m \geq 1.$$

If Γ is a finite set of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ and which, in addition, is not $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, then one of the conditions (ωS.2) to (ωS.5) is violated for Γ. By systematically correcting such deficiencies, we can extend any finite set Γ of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ to a $\mathbf{K}_n^\omega(\mathsf{C})$ saturated set.

Lemma 19 *For every finite set Γ of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ there exists a $\mathbf{K}_n^\omega(\mathsf{C})$ saturated set Δ which contains Γ.*

Proof We assume that we have fixed an enumeration $\delta_0, \delta_1, \dots$ of all $L_n(\mathsf{C})$ formulas. If the formula α is the formula δ_i in this enumeration, we call i the *index* of α.

Depending on this enumeration we now define for each finite set Π of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ a new set Π':

1. If Π is $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, then $\Pi' := \Pi$.

2. If Π is not $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, then we choose the formula α with the smallest index for which one of the conditions (ωS.1) $-$ (ωS.5) is violated and determine Π' by distinguishing between the possible forms of α.

2.1. α is of the form $(\beta \vee \gamma)$. Then we set

$$\Pi' \ := \ \Pi \cup \{\beta, \gamma\}.$$

2.2. α is of the form $(\beta \wedge \gamma)$. Since Π is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ we know that

$$\mathbf{K}_n^\omega(\mathsf{C}) \nvdash \beta, \Pi \qquad \text{or} \qquad \mathbf{K}_n^\omega(\mathsf{C}) \nvdash \gamma, \Pi.$$

Then we set

$$\Pi' \ := \ \begin{cases} \Pi \cup \{\beta\} & \text{if } \mathbf{K}_n^\omega(\mathsf{C}) \nvdash \beta, \Pi\,, \\ \Pi \cup \{\gamma\} & \text{otherwise.} \end{cases}$$

2.3. α is of the form $\mathsf{C}(\beta)$. Since Π is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ we know that

$$\mathbf{K}_n^\omega(\mathsf{C}) \nvdash \mathsf{E}^m(\beta), \Pi$$

for some natural number $m \geq 1$. We choose the least such k and set

$$\Pi' \ := \ \Pi \cup \{\mathsf{E}^k(\beta)\}.$$

2.4. α is of the form $\neg\mathsf{C}(\beta)$. Then we set

$$\Pi' \ := \ \Pi \cup \{\neg\mathsf{E}(\beta)\}.$$

Observe that this construction implies that the so defined Π' is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$.

In the next step we assign to each finite set Π of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ its *deficiency-number* $\mathsf{dn}(\Pi)$. If Π is $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, then $\mathsf{dn}(\Pi) := 0$. Otherwise fix the set $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of all elements of Π for which one of the conditions (ωS.1) – (ωS.5) is violated and set

$$\mathsf{dn}(\Pi) \;:=\; \omega^{\mathsf{me}(\alpha_1)} \,\#\, \omega^{\mathsf{me}(\alpha_2)} \,\#\, \ldots \,\#\, \omega^{\mathsf{me}(\alpha_m)}$$

where we make use of the natural sum of ordinals as introduced, for example, in Schütte [13].

Now we take the given finite set Γ of $L_n(\mathsf{C})$ formulas which is not provable in $\mathbf{K}_n^\omega(\mathsf{C})$ and define the sequence $\Gamma_0, \Gamma_1, \ldots$ of finite sets by

$$\Gamma_0 \;:=\; \Gamma \qquad \text{and} \qquad \Gamma_{m+1} \;:=\; \Gamma'_m$$

for all natural numbers m. What we have done so far guarantees that

$$\Gamma \subset \Gamma_m, \tag{1}$$

$$\mathbf{K}_n^\omega(\mathsf{C}) \nvdash \Gamma_m, \tag{2}$$

$$\mathsf{dn}(\Gamma_m) \neq 0 \quad \Longrightarrow \quad \mathsf{dn}(\Gamma_{m+1}) < \mathsf{dn}(\Gamma_m) \tag{3}$$

for all natural numbers m. Since there are no infinite decreasing sequences of ordinals, one of the sets Γ_m has to be $\mathbf{K}_n^\omega(\mathsf{C})$ saturated and is a possible candidate for the choice of Δ. $\square$

Before turning to the intended Kripke-frame $\mathfrak{S}^\omega$, we introduce for all finite sets Γ of $L_n(\mathsf{C})$ formulas the following shorthand notations:

$$\begin{aligned}
(\Gamma \cap \neg\mathsf{K}_i) &:= \{\neg\mathsf{K}_i(\alpha) : \neg\mathsf{K}_i(\alpha) \in \Gamma\}, \\
(\Gamma \cap \neg\mathsf{C}) &:= \{\neg\mathsf{C}(\alpha) : \neg\mathsf{C}(\alpha) \in \Gamma\}, \\
(\Gamma/\neg\mathsf{K}_i) &:= \{\neg\alpha : \neg\mathsf{K}_i(\alpha) \in \Gamma\}.
\end{aligned}$$

Then the Kripke-frame $\mathfrak{S}^\omega$, which is built up by the $\mathbf{K}_n^\omega(\mathsf{C})$ saturated sets, is the Kripke-frame

$$\mathfrak{S}^\omega \;:=\; (\mathit{Sat}^\omega, \mathbb{K}_1^\omega, \mathbb{K}_2^\omega, \ldots, \mathbb{K}_n^\omega)$$

whose universe Sat^ω is the set of all $\mathbf{K}_n^\omega(\mathsf{C})$ saturated sets and whose accessibility relations $\mathbb{K}_i^\omega$ comprise exactly those pairs $(\Gamma, \Delta) \in \mathit{Sat}^\omega \times \mathit{Sat}^\omega$ such that

- $\alpha \in \Delta$ and $\mathsf{K}_i(\alpha) \in \Gamma$ for some $L_n(\mathsf{C})$ formula α,
- $(\Gamma/\neg\mathsf{K}_i) \cup (\Gamma \cap \neg\mathsf{C}) \subset \Delta$.

Based on these relations $\mathbb{K}_1^\omega, \ldots, \mathbb{K}_n^\omega$ we define the *reachability in m steps* of a set $\Delta \in Sat^\omega$ from a set $\Gamma \in Sat^\omega$ by induction on m as follows:

(i) Γ is reachable from Γ in 0 steps.

(ii) Δ is reachable from Γ in $m+1$ step if and only if there exist an i, $1 \leq i \leq n$, and a set $\Pi \in Sat^\omega$ so that Π is reachable from Γ in m steps and $(\Pi, \Delta) \in \mathbb{K}_i^\omega$.

Δ is called *reachable* from Γ if Δ is reachable from Γ in m steps for some natural number m. Observe that the definition of the relations $\mathbb{K}_1^\omega, \ldots, \mathbb{K}_n^\omega$ immediately implies the following property.

Lemma 20 *Assume $\Gamma, \Delta \in Sat^\omega$. If Δ is reachable from Γ, then we have for all $L_n(\mathsf{C})$ formulas α that*

$$\neg\mathsf{C}(\alpha) \in \Gamma \quad \Longrightarrow \quad \neg\mathsf{C}(\alpha) \in \Delta.$$

For the Kripke-frame $\mathfrak{S}^\omega$ we also fix the valuation $\mathcal{V}^\omega$ from the atomic propositions to $Pow(Sat^\omega)$ defined by

$$\mathcal{V}^\omega(P) \;:=\; \{\Gamma \in Sat^\omega : P \notin \Gamma\}.$$

This finishes this construction and allows us to prove the following lemma, which is the core in proving the completeness of $\mathbf{K}_n^\omega(\mathsf{C})$.

Lemma 21 *For all $L_n(\mathsf{C})$ formulas α and all $\Gamma \in Sat^\omega$ we have that*

$$\alpha \in \Gamma \quad \Longrightarrow \quad (\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \not\models \alpha.$$

Proof We obtain this implication by induction on the structure of α and distinguish the following cases.

1. α is an atomic proposition or the negation of an atomic proposition. Then the assertion follows from the definition of $\mathcal{V}^\omega$.

2. α is of the form $(\beta \vee \gamma)$, $(\beta \wedge \gamma)$ or $\mathsf{C}(\beta)$. Then the assertion follows immediately from the induction hypothesis since Γ is $\mathbf{K}_n^\omega(\mathsf{C})$ saturated.

3. α is of the form $\mathsf{K}_i(\beta)$. We know that Γ is $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, and thus $\mathbf{K}_n^\omega(\mathsf{C}) \not\vdash \Gamma$. Hence we also have

$$\mathbf{K}_n^\omega(\mathsf{C}) \not\vdash \alpha, (\Gamma \cap \neg\mathsf{K}_i), (\Gamma \cap \neg\mathsf{C}). \tag{1}$$

Because of the form of the rule (K_i) this implies

$$\mathbf{K}_n^\omega(\mathsf{C}) \not\vdash \beta, (\Gamma/\neg\mathsf{K}_i), (\Gamma \cap \neg\mathsf{C}). \tag{2}$$

Moreover, in view of Lemma 19, there exists a set $\Delta \in \mathit{Sat}^\omega$ so that $\{\beta\} \cup (\Gamma/\neg\mathsf{K}_i) \cup (\Gamma \cap \neg\mathsf{C})$ is contained in Δ. This implies, in particular, that

$$\beta \in \Delta, \tag{3}$$

$$(\Gamma, \Delta) \in \mathbb{K}_i^\omega. \tag{4}$$

From the induction hypothesis and (3) we obtain

$$(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Delta) \not\models \beta.$$

This implies the assertion since, by (4), Δ is reachable in one step from Γ via the accessibility relation $\mathbb{K}_i^\omega$.

4. α is of the form $\sim\mathsf{K}_i(\beta)$. In this case we carry though the following further distinction.

4.1. There exists no Δ so that $(\Gamma, \Delta) \in \mathbb{K}_i^\omega$. Then we obviously must have that $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \models \mathsf{K}_i(\neg\beta)$, hence also $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \not\models \alpha$.

4.2. There exists a Δ so that $(\Gamma, \Delta) \in \mathbb{K}_i^\omega$. By the definition of $\mathbb{K}_i^\omega$ we have $\beta \in \Delta$ for all Δ so that $(\Gamma, \Delta) \in \mathbb{K}_i^\omega$. Therefore, the induction hypothesis implies

$$(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Delta) \not\models \beta \tag{5}$$

for all Δ so that $(\Gamma, \Delta) \in \mathbb{K}_i^\omega$. From this observation we can immediately deduce that

$$(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \models \mathsf{K}_i(\neg\beta). \tag{6}$$

However, this is exactly our assertion that $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \not\models \alpha$ since α is the formula $\sim\mathsf{K}_i(\beta)$ in this case.

5. α is of the form $\sim\mathsf{C}(\beta)$. Consequently, because of Lemma 20, $\neg\mathsf{C}(\neg\beta) \in \Delta$ for all sets Δ which are reachable from Γ. All these Δ are $\mathbf{K}_n^\omega(\mathsf{C})$ saturated, and so we have $\neg\mathsf{E}(\neg\beta) \in \Delta$. We apply the induction hypothesis and obtain

$$(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Delta) \not\models \neg\mathsf{E}(\neg\beta) \quad \text{i.e.} \quad (\mathfrak{S}^\omega, \mathcal{V}^\omega, \Delta) \models \mathsf{E}(\neg\beta) \tag{7}$$

for all Δ which are reachable from Γ. But since $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \models \mathsf{E}^{k+1}(\neg\beta)$ is equivalent to the fact that $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Delta) \models \mathsf{E}(\neg\beta)$ for all Δ which are reachable from Γ in k steps, assertion (7) gives us

$$(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \models \mathsf{E}^m(\neg\beta) \tag{8}$$

for all natural numbers $m \geq 1$. This implies $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \models \mathsf{C}(\neg\beta)$, hence $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \not\models \alpha$ and finishes the proof of our lemma. $\square$

Theorem 22 (Completeness of $\mathbf{K}_n^\omega(\mathsf{C})$) *If the $L_n(\mathsf{C})$ formula α is valid in all Kripke-frames for $L_n(\mathsf{C})$, then we have $\mathbf{K}_n^\omega(\mathsf{C}) \vdash \alpha$.*

PROOF Assume that the $L_n(\mathsf{C})$ formula α cannot be proved in $\mathbf{K}_n^\omega(\mathsf{C})$. Then by Lemma 19 there has to exist an $\mathbf{K}_n^\omega(\mathsf{C})$ saturated set Γ which contains α. Thus the previous lemma implies $(\mathfrak{S}^\omega, \mathcal{V}^\omega, \Gamma) \not\models \alpha$, and hence α is not valid in all Kripke-frames for $L_n(\mathsf{C})$. $\square$

Since the system $\mathbf{K}_n^\omega(\mathsf{C})$+(G-Cut) is sound and the system $\mathbf{K}_n^\omega(\mathsf{C})$ is complete, all cuts are superfluous. Hence we have a semantic proof of cut elimination for $\mathbf{K}_n^\omega(\mathsf{C})$ + (G-Cut).

Corollary 23 (Total cut elimination for $\mathbf{K}_n^\omega(\mathsf{C})$) *For all finite sets Γ of $L_n(\mathsf{C})$ formulas we have that*

$$\mathbf{K}_n^\omega(\mathsf{C}) + \text{(G-Cut)} \vdash \Gamma \quad \Longleftrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}) \vdash \Gamma.$$

PROOF Let Γ be a set of $L_n(\mathsf{C})$ formulas which is provable in $\mathbf{K}_n^\omega(\mathsf{C})$+(G-Cut). Then Theorem 15 and Theorem 22 imply that $\mathbf{K}_n^\omega(\mathsf{C}) \vdash \Gamma^\vee$. Thus Γ has to be derivable in $\mathbf{K}_n^\omega(\mathsf{C})$, as can be seen by applying some inversions. The other direction of the claimed equivalence is obvious. $\square$

This result also interesting in connection with work by Kaneko and Nagashima (cf. e.g. [10, 9]). They introduce an infintary system $\mathsf{GL(G)}$ for common knowledge and obtain cut elimination for $\mathsf{GL(G)}$. However, after cut elimination their "cut-free" proofs do not have the subformula property; something like cuts may creep in in the context of another rule. In contrast to that situation, proofs in $\mathbf{K}_n^\omega(\mathsf{C})$ enjoy the subformula property, provided, of course, that formulas $\mathsf{E}^m(\alpha)$ are regarded as subformulas of $\mathsf{C}(\alpha)$.

6 The positive and the negative fragment of the system $\mathbf{K}_n^\omega(\mathsf{C})$

In the previous section we have seen that $\mathbf{K}_n^\omega(\mathsf{C})$ is a cut-free and complete deductive system for common knowledge – the price for this cut-freeness being the allowance of infinitary derivations. However, although $\mathbf{K}_n^\omega(\mathsf{C})$ is a system permitting infinitary derivations, it contains (at least) two interesting finite subsystems: its positive and its negative fragment. They are both finite in the sense that each positive (negative) assertion of $L_n(\mathsf{C})$ which is provable in $\mathbf{K}_n^\omega(\mathsf{C})$ has a finite proof in the positive (negative) fragment of $\mathbf{K}_n^\omega(\mathsf{C})$.

An $L_n(\mathsf{C})$ formula α is called *positive* if it does not contain occurrences of the dual $\sim\mathsf{C}$ of the common knowledge operator; α is called *negative* if it does not contain the common knowledge operator C (this means that a negative formula α may contain occurrences of C only in the form $\sim\mathsf{C}$). Recall that all our formulas are in negation normal form and that the negation of complex formulas is defined; thus "positive" and "negative" refer to the occurrences of the subformulas about common knowledge.

The *positive fragment* $\mathbf{K}_n^\omega(\mathsf{C}^+)$ of $\mathbf{K}_n^\omega(\mathsf{C})$ is obtained from $\mathbf{K}_n^\omega(\mathsf{C})$ by dropping the rule $(\neg\mathsf{C})$ and restricting all other rules to finite sets of positive $L_n(\mathsf{C})$ formulas; accordingly, the *negative fragment* $\mathbf{K}_n^\omega(\mathsf{C}^-)$ of $\mathbf{K}_n^\omega(\mathsf{C})$ is obtained from $\mathbf{K}_n^\omega(\mathsf{C})$ by dropping the rule $(\omega\mathsf{C})$ and restricting all other rules to finite sets of negative $L_n(\mathsf{C})$ formulas. Thus $\mathbf{K}_n^\omega(\mathsf{C}^+)$ contains the infinitary rule $(\omega\mathsf{C})$, whereas all rules of $\mathbf{K}_n^\omega(\mathsf{C}^-)$ are finite.

Theorem 24 (Negative fragment) *Let Γ be a finite set of negative $L_n(\mathsf{C})$ formulas. For all ordinals σ we then have that*

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^\sigma \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}^-) \vdash^{<\omega} \Gamma.$$

PROOF We show this assertion by induction of σ and observe that Γ, which is a set of negative $L_n(\mathsf{C})$ formulas, cannot have been derived by the rule $(\omega\mathsf{C})$. All other rules have only finitely many premises which are sets of negative $L_n(\mathsf{C})$ formulas since no cuts are permitted. With the help of the induction hypothesis we therefore immediately obtain what we want. $\Box$

The situation is much more complicated in the case of the positive fragment $\mathbf{K}_n^\omega(\mathsf{C}^+)$ of $\mathbf{K}_n^\omega(\mathsf{C})$: we have to finitize each application of the infinitary rule $(\omega\mathsf{C})$. To achieve this, we count the nestings of the $\sim\mathsf{K}_i$ and prove Lemma 25 below. For all positive $L_n(\mathsf{C})$ formulas α the number $\partial(\alpha)$ is inductively defined as follows:

1. $\partial(P) \;:=\; \partial(\sim P) \;:=\; 0.$
2. $\partial(\alpha \vee \beta) \;:=\; \partial(\alpha \wedge \beta) \;:=\; \max(\partial(\alpha), \partial(\beta)).$
3. $\partial(\mathsf{K}_i(\alpha)) \;:=\; \partial(\mathsf{C}(\alpha)) \;:=\; \partial(\alpha).$
4. $\partial(\sim\mathsf{K}_i(\alpha)) \;:=\; \partial(\alpha) + 1.$

Besides that, we extend this definition to all finite sets Γ of positive $L_n(\mathsf{C})$ formulas by setting

$$\partial(\Gamma) \;:=\; \sup(\partial(\alpha) : \alpha \in \Gamma).$$

It is easy to verify that for all positive $L_n(\mathsf{C})$ formulas α we have the equality $\partial(\neg\mathsf{K}_i(\alpha)) = \partial(\neg\alpha) + 1$ and, as a consequence thereof, for all finite sets Γ and $\neg\Delta$ of positive $L_n(\mathsf{C})$ formulas

$$\neg\mathsf{K}_i(\Delta) \subset \Gamma \quad \text{and} \quad \Delta \neq \emptyset \quad \Longrightarrow \quad \partial(\neg\Delta) < \partial(\Gamma).$$

Lemma 25 *Let Γ be a finite set of positive $L_n(\mathsf{C})$ formulas and α a positive $L_n(\mathsf{C})$ formula. Then we have for all ordinals σ, all natural numbers ℓ, m with $\partial(\Gamma) \leq \ell \leq m$ and all $i \in \{1, \ldots, n\}$ that*

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^\sigma \mathsf{K}_i(\mathsf{E}^\ell(\alpha)), \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{n\sigma} \mathsf{K}_i(\mathsf{E}^m(\alpha)), \Gamma.$$

Proof We show this assertion by induction on σ and have to distinguish between the following cases.

1. $\{\mathsf{K}_i(\mathsf{E}^\ell(\alpha))\} \cup \Gamma$ is an axiom of $\mathbf{K}_n^\omega(\mathsf{C}^+)$. Then $\{\mathsf{K}_i(\mathsf{E}^m(\alpha))\} \cup \Gamma$ is an axiom of $\mathbf{K}_n^\omega(\mathsf{C}^+)$ as well.

2. $\{\mathsf{K}_i(\mathsf{E}^\ell(\alpha))\} \cup \Gamma$ is the conclusion of a rule $(\vee)$, $(\wedge)$ or $(\omega\mathsf{C})$. Then we apply the induction hypothesis to the premise(s) of this rule and carry it through again afterwards in order to derive what we want.

3. $\{\mathsf{K}_i(\mathsf{E}^\ell(\alpha))\} \cup \Gamma$ is the conclusion of a rule (K_j), and the formula $\mathsf{K}_i(\mathsf{E}^\ell(\alpha))$ is not its main formula of this inference. Then there exist an ordinal $\tau < \sigma$, a positive $L_n(\mathsf{C})$ formula β and a finite set $\neg\Delta$ of positive $L_n(\mathsf{C})$ formulas so that

$$\text{(1)} \qquad \mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^\tau \beta, \neg\Delta,$$

$$\text{(2)} \qquad \{\mathsf{K}_j(\beta)\} \cup \neg\mathsf{K}_j(\Delta) \subset \{\mathsf{K}_i(\mathsf{E}^\ell(\alpha))\} \cup \Gamma.$$

Keeping (2) in mind, we can simply apply the rule (K_j) to (1) in order to obtain our assertion. The induction hypothesis is not needed in this case.

4. $\{\mathsf{K}_i(\mathsf{E}^\ell(\alpha))\} \cup \Gamma$ is the conclusion of a rule (K_i) with the main formula $\mathsf{K}_i(\mathsf{E}^\ell(\alpha))$. Then there exist an ordinal $\tau < \sigma$ and a finite set $\neg\Delta$ of positive $L_n(\mathsf{C})$ formulas so that

$$\text{(3)} \qquad \mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^\tau \mathsf{E}^\ell(\alpha), \neg\Delta,$$

$$\text{(4)} \qquad \neg\mathsf{K}_i(\Delta) \subset \Gamma.$$

It simplifies matters to proceed with a further distinction of cases: whether Δ is the empty set or not.

4.1. $\Delta = \emptyset$. By Lemma 14 we obtain from (3)

$$\text{(5)} \qquad \mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^\tau \alpha,$$

and from that together with the rule $(\mathsf{K}^\star)$ that

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{\tau+1} \mathsf{K}_i(\mathsf{E}^k(\alpha)),\, \Gamma \tag{6}$$

for all natural numbers k and all $i \in \{1, \dots, n\}$. Since $\tau + 1 \leq n\sigma$, Lemma 13 yields our claim.

4.2. $\Delta \neq \emptyset$. In view of a previous consideration we know that in this case $\partial(\neg\Delta) < \partial(\Gamma)$, thus $\ell = k + 1$ for some natural number k. Moreover, Lemma 14 applied to (3) gives us

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{\tau} \mathsf{K}_j(\mathsf{E}^k(\alpha)),\, \neg\Delta \tag{7}$$

for all $j \in \{1, \dots, n\}$. But $k \geq \partial(\neg\Delta)$, and so the induction hypothesis applied to (7) yields

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{n\tau} \mathsf{K}_j(\mathsf{E}^r(\alpha)),\, \neg\Delta \tag{8}$$

for all natural numbers $r \geq k$ and all $j \in \{1, \dots, n\}$. We continue with $n - 1$ applications of the rule $(\wedge)$ and conclude

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{n\tau+(n-1)} \mathsf{E}^m(\alpha),\, \neg\Delta \tag{9}$$

for all natural numbers $m \geq k + 1 = \ell$. Finally, by rule (K_i) we can go over from (9) to

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{n\tau+n} \mathsf{K}_i(\mathsf{E}^m(\alpha)),\, \neg\mathsf{K}_i(\Delta) \tag{10}$$

for all natural numbers $m \geq \ell$. Since $n\tau + n \leq n\sigma$, our assertion follows immediately from (10), (4) and Lemma 13. This completes the treatment of the last case and finishes our proof. $\Box$

Theorem 26 (Positive fragment) *Let Γ be a finite set of positive $L_n(\mathsf{C})$ formulas. For all ordinals σ we then have that*

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{\sigma} \Gamma \quad \Longrightarrow \quad \mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{<\omega} \Gamma.$$

Proof We prove this assertion by induction on σ and distinguish the following cases.

1. Γ is an axiom $\mathbf{K}_n^\omega(\mathsf{C})$. Then we trivially have $\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^0 \Gamma$.

2. Γ is the conclusion of a rule $(\vee)$, $(\wedge)$, (K_i) or $(\sim\mathsf{K}_i)$. Each of these rules has only finitely many premises to which the induction hypothesis is applied.

Hence these premises are finitely derivable, and in one further derivation step we thus have our assertion.

3. Γ is the conclusion of a rule ($\omega\mathsf{C}$). Then Γ contains an $L_n(\mathsf{C})$ formula $\mathsf{C}(\alpha)$ and there exist ordinals $\sigma_1, \sigma_2, \ldots$, so that for all natural numbers $k \geq 1$

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{\sigma_k} \mathsf{E}^k(\alpha), \Gamma, \tag{1}$$

$$\sigma_k < \sigma. \tag{2}$$

Set $\ell := \partial(\Gamma)$ and apply the previous lemma to the "branches" 1 to $\ell + 1$ of (1) in order to obtain natural numbers $r_1, \ldots, r_{\ell+1}$ satisfying

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{r_k} \mathsf{E}^k(\alpha), \Gamma \tag{3}$$

for all $k \leq \ell + 1$. Now we make use of Lemma 14 to obtain from (3)

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{r_{\ell+1}} \mathsf{K}_i(\mathsf{E}^\ell(\alpha)), \Gamma \tag{4}$$

for all $i \in \{1, \ldots, n\}$. Thus the previous lemma can be applied to (4), and we accomplish

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{nr_{\ell+1}} \mathsf{K}_i(\mathsf{E}^m(\alpha)), \Gamma \tag{5}$$

for all natural numbers $m \geq \ell$ and for all $i \in \{1, \ldots, n\}$. Consequently, by $(n-1)$ applications of the rule ($\wedge$) and Lemma 13 we derive

$$\mathbf{K}_n^\omega(\mathsf{C}) \vdash^{(n+1)r_{\ell+1}} \mathsf{E}^{m+1}(\alpha), \Gamma \tag{6}$$

for all $m \geq \ell$. Summing up, we have for all natural numbers k with $1 \leq k \leq \ell$ that

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{r_k} \mathsf{E}^k(\alpha), \Gamma \tag{7}$$

and for all natural numbers $s > \ell$ that

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{(n+1)r_{\ell+1}} \mathsf{E}^s(\alpha), \Gamma. \tag{8}$$

For any natural number t greater than $r_1, \ldots, r_\ell$ and $(n+1)r_{\ell+1}$ we obtain from (7) and (8) with the help of the rule ($\omega\mathsf{C}$) that

$$\mathbf{K}_n^\omega(\mathsf{C}^+) \vdash^{t} \mathsf{C}(\alpha), \Gamma. \tag{9}$$

Since Γ is a set of positive $L_n(\mathsf{C})$ formulas, we have covered all possibilities of its derivation, and our theorem is proved. $\square$

Notice that our positive fragment does not guarantee that all proofs are finite since infinite branchings are still permitted. However, also this problem can be solved by introducing a new calculus $\mathbf{K}_n^{<\omega}(\mathsf{C}^+)$ which is obtained from $\mathbf{K}_n^{\omega}(\mathsf{C}^+)$ by replacing the infinitary rule

$$(\omega\mathsf{C}) \qquad \frac{\mathsf{E}^m(\alpha),\, \Gamma \quad (\text{for all } m \geq 1)}{\mathsf{C}(\alpha),\, \Gamma}$$

by the finitary rule

$$(<\omega\mathsf{C}) \qquad \frac{\mathsf{E}^m(\alpha),\, \Gamma \quad (\text{for all } m \text{ so that } 1 \leq m \leq \partial(\Gamma)+1)}{\mathsf{C}(\alpha),\, \Gamma}$$

Working within the positive fragment, we can now show that the infinitary rule $(\omega\mathsf{C})$ can be restricted to the finitary $(<\omega\mathsf{C})$.

Theorem 27 *For any set Γ of positive $L_n(\mathsf{C})$ formulas we have that*

$$\mathbf{K}_n^{\omega}(\mathsf{C}^+) \vdash \Gamma \quad \Longleftrightarrow \quad \mathbf{K}_n^{<\omega}(\mathsf{C}^+) \vdash \Gamma.$$

Proof The direction from left to right follows by induction on the proofs in $\mathbf{K}_n^{\omega}(\mathsf{C}^+)$. The reverse direction is proved by induction on the lengths of the proofs in $\mathbf{K}_n^{<\omega}(\mathsf{C}^+)$, using Lemma 25 in the induction step whenever the last rule applied in $\mathbf{K}_n^{<\omega}(\mathsf{C}^+)$ is an instance of $(<\omega\mathsf{C})$. □

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