$\mathcal{PALC}$: Extending $\mathcal{ALC}$ ABoxes with Probabilities

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Abstract

Most probabilistic extensions of description logics focus on the terminological apparatus. While some allow for expressing probabilistic knowledge about concept assertions, systems which can express probabilistic knowledge about role assertions have received very little attention as yet. We present a system $\mathcal{PALC}$ which allows us to express degrees of belief in concept and role assertions for individuals. We introduce syntax and semantics for $\mathcal{PALC}$ and we define the corresponding reasoning problem. An independence assumption regarding the assertions for different individuals yields additional constraints on the possible interpretations.

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Introduction

Motivation

Nowadays, description logics are a standard formalism for knowledge representation. An overview of the existing work on description logics can be found in [BCM+03] and [BKW03]. For many applications it is important to extend description logics such that they can handle probabilistic knowledge. Halpern [Hal90] and Bacchus [Bac90] define logics to reason with and about probabilities. Since they work with full first-order logic, their systems are undecidable. Moreover the independence assumption fundamental to our approach does not hold for full first-order logic. Most of the probabilistic extensions of description logics focus on probabilities on terminological axioms, see for example [BKW03, Hei94, KLP97]. Notable exceptions are [Jae94, GL02]. Jaeger [Jae94] presents a system which allows for probabilistic knowledge about concept instances. He concentrates mainly on combining probabilistic terminological knowledge with probabilistic assertional knowledge by means of cross-entropy minimization. Giugno and Lukasiewicz [GL02] examine probabilistic ontologies for the semantic web. There, assertions of the form $C(a)$ are expressed as inclusion axioms $\{a\} \sqsubseteq C$ and probabilistic assertions are simply probabilistic inclusion axioms. None of the approaches however allows for expressing probabilistic knowledge on role instances.

In our work, we propose the language $\mathcal{PALC}$ as a probabilistic extension to the description logic $\mathcal{ALC}$. It allows us to express degrees of belief in both, concept and role assertions for individuals. We introduce syntax and semantics of $\mathcal{PALC}$. A reasoning problem for this system is to find the possible valuations for the probabilistic constants. The expressive power of $\mathcal{ALC}$ is relatively low when compared to first-order logic. This results in a high degree of independence regarding the assertions for different individuals. The identification of some specific sets of independent assertions yields additional constraints on the interpretation of the probabilistic constants. In particular, we have less freedom in the interpretation of assertions about value restrictions. Straccia [Str04] introduces similar constraints in a lattice based approach to uncertainty in description logics. However, because his approach is not based on probabilities, there is no notion of independence.

Investigating the structure of the solution space of a $\mathcal{PALC}$ reasoning problem, we will first identify linear constraints similar to those in [Nil86]. The origin of these constraints are the probability axioms. We identify further, non linear constraints which are imposed by our independence assumption. Whether the constraints identified in this work fully characterize the solutions of a $\mathcal{PALC}$ reasoning problem remains an open question as yet.

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Chapter 1

Preliminaries

1.1 Probability

In this section we will briefly review some topics of probability theory. See [Kol33] for Kolmogorov's original work on the mathematical foundations of probability. A more exhaustive treatment especially regarding the Borel-Cantelli Lemmas can be found in [Nev69].

A triple (Ω, F, P) on the domain Ω is called *probability space* for

i) Ω is a non-empty set called *sample space*.

ii) The set of *events* F is a σ-algebra over Ω.

iii) The *probability distribution* $\mathrm{P} : F \to \mathbb{R}$ on F is a mapping from the events to the reals such that the *probability axioms* hold:

   - $\mathrm{P}(A) \geq 0$ for all $A \in F$,
   - $\mathrm{P}(\Omega) = 1$,
   - $\mathrm{P}(\bigcup_{j \in J} A_j) = \sum_{j \in J} \mathrm{P}(A_j)$ for any countable sequence $(A_j)_{j \in J}$ of pairwise disjoint events.

Let (Ω, F, P) be a probability space, Ω' a set, and F' a σ-algebra on Ω'. A function $X : \Omega \to \Omega'$ is called a *random variable* (for F') if $X^{-1}(A) \in F$ for every $A \in F'$. For brevity of notation we write

$$\{X \in A\} := \{s \in \Omega | X(s) \in A\}$$

for the inverse image of $A \in F'$ under X. We can obtain a new probability space $(\Omega', F', \mathrm{P}')$ with the probability distribution

$$\mathrm{P}' : \Omega' \to \mathbb{R}, \mathrm{P}'(A) = \mathrm{P}(X \in A).$$

P' is called the *distribution of the random variable* X.

Let Ω_j be a set with the σ-algebra F_j for each j in a non-empty index set J, and let $X_j : \Omega \to \Omega_j$ be a random variable for F_j for each j. The σ-algebra generated by $\{X_j | j \in J\}$ is

$$\sigma(\{X_j | j \in J\}) = \sigma(\bigcup_{j \in J} \{X_j^{-1}(A_j) \in F | A_j \in F_j\}).$$

Two events A and B are *independent* if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B).$$

Note that Ω and $\emptyset$ are always independent from any other event. For any event A with $\mathrm{P}(A) \neq 0$ we have $\mathrm{P}(\emptyset \cap A) = \mathrm{P}(\emptyset) = 0 = \mathrm{P}(\emptyset)\,\mathrm{P}(A)$ and $\mathrm{P}(\Omega \cap A) = \mathrm{P}(A) = \mathrm{P}(\Omega)\,\mathrm{P}(A)$. In general a set of events $\{A_j \in F | j \in J\}$ is independent if for any finite, non-empty subset $K \subseteq J$

$$\mathrm{P}(\bigcap_{k \in K} A_k) = \prod_{k \in K} \mathrm{P}(A_k).$$

Independence of random variables reduces to independence of sets of events: $\{X_j | j \in J\}$ is independent if for any finite, non-empty subset $K \subseteq J$

$$\mathrm{P}(\bigcap_{k \in K} \{X_k \in A_k\}) = \prod_{k \in K} \mathrm{P}(X_k \in A_k) \text{ for all } A_k \in F_k.$$

1.1.1 Convergence

We now provide some important results about infinite sequences of events. We will draw on these results later when we define the semantics of probabilistic value restrictions and probabilistic existential restrictions. Further we give a definition for infinite products. Since we work with probabilities we restrict our interest in products where all factors are restraint to the interval $[0,1]$.

For the remainder of this section J will denote an arbitrary, non-empty, and countable index set. $(A_j)_{j\in J}$ will denote a sequence of events in a probability space (Ω, F, P).

Definition 1. We define the events

$$\lim_{j\to\infty} \sup A_j = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k,$$
$$\lim_{j\to\infty} \inf A_j = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} A_k.$$

Remark 2.

$$\lim_{j\to\infty} \inf A_j \subseteq \lim_{j\to\infty} \sup A_j.$$

Definition 3. In the case of equality we define the limit of the sequence $(A_j)_{j\in J}$ as

$$\lim_{j\to\infty} A_j = \lim_{j\to\infty} \inf A_j = \lim_{j\to\infty} \sup A_j.$$

Lemma 4.

i) Let $(A_j)_{j\in J}$ be an increasing sequence, that is $A_j \subseteq A_{j+1}$ for all $j \in J$. Then

$$\lim_{j\to\infty} A_j = \bigcup_{j=1}^{\infty} A_j.$$

ii) Let $(A_j)_{j\in J}$ be a decreasing sequence, that is $A_j \supseteq A_{j+1}$ for all $j \in J$. Then

$$\lim_{j\to\infty} A_j = \bigcap_{j=1}^{\infty} A_j.$$

Proof.

i)
- $\lim_{j\to\infty} \inf A_j = \bigcup_{j=1}^{\infty} \underbrace{\bigcap_{k=j}^{\infty} A_k}_{=A_j} = \bigcup_{j=1}^{\infty} A_j.$
- Since $\lim_{j\to\infty} \sup A_j \supseteq \lim_{j\to\infty} \inf A_j$ it is sufficient to show that

$$\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k \subseteq \bigcup_{j=1}^{\infty} A_j.$$

  Let $a \in \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$. Then $a \in \bigcup_{k=j}^{\infty} A_k$ for all $j \in J$ and thus $a \in \bigcup_{k=1}^{\infty} A_k$.

  Together we thus find $\lim_{j\to\infty} \inf A_j = \lim_{j\to\infty} \sup A_j = \bigcup_{j=1}^{\infty} A_j$.

ii) Similar to the proof of the first part.

□

A fundamental property of probability distributions is their continuity. That is, probability distributions and limits commute.

Theorem 5. *Let* $A = \lim_{j\to\infty} A_j$. *Then*

$$\lim_{j\to\infty} \mathrm{P}(A_j) = \mathrm{P}(\lim_{j\to\infty} A_j) = \mathrm{P}(A).$$

Proof.

$$\lim_{j\to\infty} \sup \mathrm{P}(A_j) = \lim_{j\to\infty} \mathrm{P}(\bigcup_{k=j}^{\infty} A_k)$$
$$= \mathrm{P}(\lim_{j\to\infty} \sup A_j) = \mathrm{P}(A) = \mathrm{P}(\lim_{j\to\infty} \inf A_j)$$
$$= \lim_{j\to\infty} \mathrm{P}(\bigcap_{k=j}^{\infty} A_k) = \lim_{j\to\infty} \inf \mathrm{P}(A_j).$$

□

Lemma 6.

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mathrm{P}(\bigcap_{k=1}^{j} A_k).$$

Proof.

i) $(B_j)_{j\in J}$ with $B_j = \bigcap_{k=1}^{j} A_k$ is a decreasing sequence,

ii) $\lim\limits_{j\to\infty} B_j = \bigcap_{j=1}^{\infty} B_j = \bigcap_{j=1}^{\infty} A_j$,

iii) $\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = \mathrm{P}(\lim\limits_{j\to\infty} B_j) = \lim\limits_{j\to\infty} \mathrm{P}(B_j) = \lim\limits_{j\to\infty} \mathrm{P}(\bigcap_{k=1}^{j} A_k)$.

□

Lemma 7 (Borel-Cantelli Part 1)**.** *Let* $\sum_{j=1}^{\infty} \mathrm{P}(A_j) < \infty$. *Then*

$$\mathrm{P}(\lim_{j\to\infty} \sup A_j) = 0.$$

Proof.

$$\mathrm{P}(\lim_{j\to\infty} \sup A_j) = \lim_{j\to\infty} \mathrm{P}(\bigcup_{k=j}^{\infty} A_k) \le \lim_{j\to\infty} \sum_{k=j}^{\infty} \mathrm{P}(A_k) = 0.$$

□

Lemma 8 (Borel-Cantelli Part 2)**.** *Let* $\{A_j | j \in J\}$ *be independent and* $\sum_{j=1}^{\infty} \mathrm{P}(A_j) = \infty$. *Then*

$$\mathrm{P}(\lim_{j\to\infty} \sup A_j) = 1.$$

Proof. First note that

$$1 - \mathrm{P}(\lim_{j\to\infty} \sup A_j) = \mathrm{P}(\lim_{j\to\infty} \inf \Omega \setminus A_j) = \lim_{j\to\infty} \mathrm{P}(\bigcap_{k=j}^{\infty} \Omega \setminus A_k).$$

Because of independence, we find for every $j \in J$

$$\begin{aligned}
\mathrm{P}(\bigcap_{k=j}^{\infty} \Omega \setminus A_k) &= \lim_{n\to\infty} \mathrm{P}(\bigcap_{k=j}^{n} \Omega \setminus A_k) \\
&= \lim_{n\to\infty} \prod_{k=j}^{n} \mathrm{P}(\Omega \setminus A_k) \\
&= \lim_{n\to\infty} \prod_{k=j}^{n} 1 - \mathrm{P}(A_k) \\
&\leq \lim_{n\to\infty} \prod_{k=j}^{n} \exp(-\mathrm{P}(A_k)) \\
&= \lim_{n\to\infty} \exp(-\sum_{k=j}^{n} \mathrm{P}(A_k)) \\
&= 0
\end{aligned}$$

since the sum diverges. □

Definition 9 (Infinite Product)**.** Let $(x_j)_{j\in J}$ be a sequence in $(0,1]$. The infinite product of this sequence converges if the limit of its partial products exists and is then defined as

$$\prod_{j=1}^{\infty} x_j = \lim_{j\to\infty} \prod_{k=1}^{j} x_k.$$

Remark 10. $\prod_{j=1}^{\infty} x_j = 0$ if $x_k = 0$ for some $k \in J$.

Theorem 11. *Let $\{A_j | j \in J\}$ be independent. Then*

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) \in \{0,1\}$$

and further

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = 1 \textit{ iff } \mathrm{P}(A_j) = 1 \textit{ for all } j \in J.$$

Proof. Let $B_j = \bigcap_{k=1}^{j} A_k$. With Lemma 6, Definition 3, Lemma 7 and

Lemma 8 we find

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = \mathrm{P}(\lim_{j\to\infty} B_j) = \mathrm{P}(\lim_{j\to\infty} \sup B_j) = \begin{cases} 0 \text{ if } \sum_{j=1}^{\infty} \mathrm{P}(B_j) < \infty, \\ 1 \text{ if } \sum_{j=1}^{\infty} \mathrm{P}(B_j) = \infty. \end{cases}$$

For the proof of the second part assume $\mathrm{P}(A_j) = 1$ for all $j \in J$. Then

$$\sum_{j=1}^{\infty} \mathrm{P}(B_j) = \sum_{j=1}^{\infty} \prod_{k=1}^{j} \mathrm{P}(A_k) = \infty$$

and thus

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = 1$$

according to the first part. Finally assume $\mathrm{P}(A_k) < 1$ for some $k \in J$. Then from Lemma 6 and Definition 9 we find

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = \prod_{j=1}^{\infty} \mathrm{P}(A_j) < 1$$

which leaves us with

$$\mathrm{P}(\bigcap_{j=1}^{\infty} A_j) = 0$$

according to the first part. □

1.2 The Description Logic $\mathcal{ALC}$

This section gives a short introduction on the description logic $\mathcal{ALC}$ as far as it is of concern for our aim. For a more in depth treatment see [BCM+03].

In description logics, a description takes the form of a concept. *Complex concepts* are built from *atomic concepts* and *roles* with *concept constructors*. We use the letters A and B for atomic concepts, C and D for complex concepts and the letters R and S for roles. Concepts in $\mathcal{ALC}$ are formed according to the following syntax:

$$\begin{array}{rcll} C, D & \longrightarrow & A \mid & \text{(atomic concept)} \\ & & \top \mid & \text{(universal concept)} \\ & & \bot \mid & \text{(bottom concept)} \\ & & \neg C \mid & \text{(negation)} \\ & & C \sqcap D \mid & \text{(conjunction)} \\ & & C \sqcup D \mid & \text{(disjunction)} \\ & & \forall R.C \mid & \text{(value restriction)} \\ & & \exists R.C & \text{(existential restriction).} \end{array}$$

The semantics of concepts and roles is given by an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where the *domain* $\Delta^{\mathcal{I}}$ consists of a non-empty set of individuals and the *interpretation function* $\cdot^{\mathcal{I}}$ assigns to every atomic concept A a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and to every role R a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function is extended to concepts by the following inductive definitions:

$$\begin{aligned} \top^{\mathcal{I}} &:= \Delta^{\mathcal{I}} \\ \bot^{\mathcal{I}} &:= \emptyset \\ (\neg C)^{\mathcal{I}} &:= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\ (C \sqcap D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\ (C \sqcup D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\ (\forall R.C)^{\mathcal{I}} &:= \{r \in \Delta^{\mathcal{I}} \mid \forall s((r,s) \in R^{\mathcal{I}} \rightarrow s \in C^{\mathcal{I}})\} \\ (\exists R.C)^{\mathcal{I}} &:= \{r \in \Delta^{\mathcal{I}} \mid \exists s((r,s) \in R^{\mathcal{I}} \wedge s \in C^{\mathcal{I}})\}. \end{aligned}$$

A concept C is *satisfiable* if there is an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$. Further two concepts C and D are *equivalent* - in symbols $C \equiv D$ - if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every interpretation $\mathcal{I}$.

We introduce a set of individual constants $a, b, c, \ldots$ and we assert properties of these by stating *concept assertions* $C(a)$ and *role assertions* $R(b, c)$. Informally, $C(a)$ states that a belongs to the concept C and $R(b, c)$ states that

b and c are related by the role R. A *world description* or *ABox* $\mathcal{A}$ is a finite set of (concept and role) assertions.

Definition 12. Let $\mathcal{A}$ be an ABox. We define the set of concepts $\mathrm{cns}(\mathcal{A})$, roles $\mathrm{rls}(\mathcal{A})$, and individual constants $\mathrm{ind}(\mathcal{A})$ occurring in assertions in $\mathcal{A}$ as

$$\begin{aligned}
\mathrm{cns}(\mathcal{A}) &= \{C | C(a) \in \mathcal{A} \text{ for some individual constant } a\}, \\
\mathrm{rls}(\mathcal{A}) &= \{R | R(a,b) \in \mathcal{A} \text{ for some individual constants } a,\ b\}, \\
\mathrm{ind}(\mathcal{A}) &= \{a | R(a,b) \in \mathcal{A} \text{ or } R(b,a) \in \mathcal{A} \\
&\qquad \text{for some role } R \text{ and some individual constant } b\} \cup \\
&\qquad \{a | C(a) \in \mathcal{A} \text{ for some concept } C\}.
\end{aligned}$$

We give semantics to the ABox by extending interpretations to individual constants. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. The interpretation function $\cdot^{\mathcal{I}}$ not only maps atomic concepts and roles to sets and relations, but in addition maps each individual constant a to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. We assume that distinct individual constants denote distinct objects in $\Delta^{\mathcal{I}}$. Therefore, the mapping $\cdot^{\mathcal{I}}$ respects the *unique name assumption*, that is if a and b are different individual constants, then $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. The interpretation $\mathcal{I}$ satisfies the concept assertion $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and it satisfies the role assertion $R(b,c)$ if $(b^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}}$. An interpretation satisfies an ABox $\mathcal{A}$ if it satisfies each assertion in it. We then say $\mathcal{A}$ is *satisfiable* and call $\mathcal{I}$ a *model* of $\mathcal{A}$. We will use the following notation:

$\mathcal{I} \models \alpha$ means that $\mathcal{I}$ satisfies the (role or concept) assertion α and $\mathcal{I} \models \mathcal{A}$ means that $\mathcal{I}$ is a model of the ABox $\mathcal{A}$.

$\mathcal{A} \models \alpha$ states that $\mathcal{A}$ *entails* the assertion α. That is, every model $\mathcal{I}$ of $\mathcal{A}$ also satisfies α.

$\mathcal{A} \not\models \alpha$ states that there exists a model $\mathcal{I}$ of $\mathcal{A}$ which does not satisfy the assertion α.

The next Lemma proves some equivalencies between concepts which will be useful in subsequent sections.

Lemma 13. *Let R be a role and C, D concepts. Then*

i) $\neg\neg C \equiv C$.

ii) $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$.

iii) $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$.

iv) $\forall R.C \equiv \neg\exists R.\neg C$.

v) $\exists R.C \equiv \neg\forall R.\neg C$.

Proof. Let $\mathcal{I}$ be an interpretation. Then

i) $(\neg\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus (\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}) = C^{\mathcal{I}}$.

ii) $(\neg(C \sqcap D))^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus (C^{\mathcal{I}} \cap D^{\mathcal{I}}) = (\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}) \cup (\Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}) = (\neg C \sqcup \neg D)^{\mathcal{I}}$.

iii) Similar to ii).

iv) $$\begin{aligned}(\neg\exists R.\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus \{r \in \Delta^{\mathcal{I}} \mid \exists s((r,s) \in R^{\mathcal{I}} \wedge s \notin C^{\mathcal{I}})\} \\ &= \{r \in \Delta^{\mathcal{I}} \mid \forall s((r,s) \in R^{\mathcal{I}} \rightarrow s \in C^{\mathcal{I}})\} \\ &= (\forall R.C)^{\mathcal{I}}.\end{aligned}$$

v) Similar to iv).

□

An ABox contains implicit knowledge that can be made explicit through inferences. The main ABox reasoning tasks are satisfiability checking and entailment checking. By satisfiability checking we want to determine if a given ABox has a model while entailment checking determines whether a given ABox entails a certain assertion.

Lemma 14. *Let $\mathcal{A}$ be an ABox and $C(a)$ an assertion. Then*

$$\mathcal{A} \models C(a) \text{ iff } \mathcal{A} \cup \{\neg C(a)\} \text{ is not satisfiable.}$$

Proof. Let $\mathcal{A} \models C(a)$. Assume $\mathcal{I} \models \mathcal{A} \cup \{\neg C(a)\}$. Then $a^{\mathcal{I}} \notin C^{\mathcal{I}}$ and $\mathcal{I} \models \mathcal{A}$. But the latter implies $a^{\mathcal{I}} \in C^{\mathcal{I}}$ which contradicts the former. The assumption was thus wrong and $\mathcal{A} \cup \{\neg C(a)\}$ is not satisfiable.

Conversely let $\mathcal{I} \not\models \mathcal{A} \cup \{\neg C(a)\}$ for every interpretation $\mathcal{I}$. If $\mathcal{I} \not\models \mathcal{A}$ for every interpretation $\mathcal{I}$, then $C(a)$ holds in every model of $\mathcal{A}$. Otherwise $\mathcal{J} \models \mathcal{A}$ for some interpretation $\mathcal{J}$. But then $\mathcal{J} \not\models \neg C(a)$ and thus $\mathcal{J} \models C(a)$. □

An important property of $\mathcal{ALC}$ is its decidability. In particular:

Theorem 15. *Let $\mathcal{A}$ be an ABox. Then it is decidable whether or not $\mathcal{A}$ is satisfiable.*

Proof. See [BCM+03] for a proof sketch and [BS99] for the full proof although done for the more expressive description logic $\mathcal{ALCN}(\circ)$. □

1.2.1 Independence

In this section we provide some rather specific results about $\mathcal{ALC}$ ABoxes. In particular we show that assertions occurring in these ABoxes are strongly independent. We will make use of these results later when we define probabilistic ABoxes and their semantics.

The proofs in this section make use of the tableau algorithm for $\mathcal{ALC}$. See [BCM+03] for an introduction or [BS99] and [BS01] for a more exhaustive treatment.

Definition 16. Let $\mathcal{A}$ be an ABox. We define the binary relation $<^{\mathcal{A}}$ on the individual constants $a, b \in \mathrm{ind}(\mathcal{A})$ as

i) $a <^{\mathcal{A}} b$ if $R(a,b) \in \mathcal{A}$ for some role R,

ii) $a <^{\mathcal{A}} b$ if $a <^{\mathcal{A}} c$ and $c <^{\mathcal{A}} b$ for some $c \in \mathrm{ind}(\mathcal{A})$,

iii) $a \not<^{\mathcal{A}} b$ otherwise.

Definition 17. An ABox $\mathcal{A}$ is *separable* if $<^{\mathcal{A}}$ is irreflexive (i.e. $a \not<^{\mathcal{A}} a$ for all individual constants a) and further

$$a <^{\mathcal{A}} b \wedge a' <^{\mathcal{A}} b \implies a <^{\mathcal{A}} a' \vee a' <^{\mathcal{A}} a \vee a = a'$$

for all individual constants $a, a', b \in \mathrm{ind}(\mathcal{A})$.

Lemma 18. *Let $\mathcal{A}$ be a separable ABox and $\mathcal{A}_n$ an ABox occurring in a run of the tableau algorithm after n completion rules have been applied to $\mathcal{A}$. Then $\mathcal{A}_n$ is separable and further for every individual constant $b \in \mathrm{ind}(\mathcal{A}_n)$ there is an individual constant $a \in \mathrm{ind}(\mathcal{A})$ such that either $a = b$ or $a <^{\mathcal{A}_n} b$. We call a the* root *of b in $\mathcal{A}$.*

Proof. First note that since $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ also $<^{\mathcal{A}_n} \subseteq <^{\mathcal{A}_{n+1}}$. We proceed by induction on the number of completion rule applications n of the tableau algorithm applied to $\mathcal{A}$.

- For $n = 0$ the claim holds by precondition.
- Assume the claim holds for n. If $\mathcal{A}_n$ is complete we are done. So assume $\mathcal{A}_n$ is not complete. Then there exists an assertion $C(b) \in \mathcal{A}_n$ such that a completion rule is applicable to $C(b)$. If the applicable rule is one of $\rightarrow_{\sqcup}$, $\rightarrow_{\sqcap}$, or $\rightarrow_{\forall}$ then the ABoxes $\mathcal{A}_{n+1}$ (and $\mathcal{A}'_{n+1}$) produced by the completion rule do neither contain new role assertions nor new individual constants. Thus $<^{\mathcal{A}_{n+1}} = <^{\mathcal{A}_n}$ (and $<^{\mathcal{A}'_{n+1}} = <^{\mathcal{A}_n}$) and the claim holds by hypothesis. Otherwise the applicable rule is $\rightarrow_{\exists}$. Thus $R(b, z) \in \mathcal{A}_{n+1}$ where $z \notin \mathrm{ind}(\mathcal{A}_n)$ and thus $b <^{\mathcal{A}_{n+1}} z$. By hypothesis $a <^{\mathcal{A}_n} b$ for some $a \in \mathrm{ind}(\mathcal{A})$. Using $<^{\mathcal{A}_n} \subseteq <^{\mathcal{A}_{n+1}}$ and transitivity we find $a <^{\mathcal{A}_{n+1}} z$. Further $<^{\mathcal{A}_{n+1}}$ is irreflexive since by hypothesis $<^{\mathcal{A}_n}$ is irreflexive, $z \notin \mathrm{ind}(\mathcal{A}_n)$ and $z \not<^{\mathcal{A}_{n+1}} z$. Finally assume $a <^{\mathcal{A}_{n+1}} z$ and $a' <^{\mathcal{A}_{n+1}} z$ for $a \neq a'$. If $a <^{\mathcal{A}_{n+1}} b$ and $a' <^{\mathcal{A}_{n+1}} b$ then either $a <^{\mathcal{A}_{n+1}} a'$ or $a' <^{\mathcal{A}_{n+1}} a$ by hypothesis. Otherwise without loss of generality assume $a <^{\mathcal{A}_{n+1}} z$ and $b <^{\mathcal{A}_{n+1}} z$. But then $a <^{\mathcal{A}_{n+1}} b$ because $a <^{\mathcal{A}_{n+1}} z$ implies either $a = b$ or $a <^{\mathcal{A}_{n+1}} b$.

□

Remark 19. In a separable ABox $\mathcal{A}$ the interaction of individuals of different assertions is of a limited kind. That is, when running the tableau algorithm on $\mathcal{A}$ and a completion rule is applicable at a certain point then the individuals occurring in the assertions involved all have the same root a in $\mathcal{A}$. Similarly if a completion rule is not applicable at a certain point then the individuals occurring in the assertions blocking the completion rule from being applied all have the same root a' in $\mathcal{A}$.

Lemma 20. *For every $1 \leq i \leq n$ let C_i be a concept and b_i an individual constant. Then the ABox*

$$\mathcal{A} = \{C_i(b_i) | 1 \leq i \leq n\}$$

is satisfiable iff each of its elements is satisfiable (i.e. each $C_i(b_i)$ for $1 \leq i \leq n$ is satisfiable).

Proof. The only if direction holds by definition.
For the other direction assume each element of $\mathcal{A}$ is satisfiable. Then for every $1 \leq i \leq n$ applying the tableau algorithm to $\mathcal{A}_i = \{C_i(b_i)\}$ there is a sequence S_i with length l_i of completion rules which leads to a complete and clash free ABox $\mathcal{A}_i^c$. We show by induction on n that then the sequence of completion rules $S_1, \ldots, S_n$ obtained by concatenating the individual sequences can be applied to $\mathcal{A}$ and leads to an ABox $\mathcal{A}^c = \bigcup_{1 \leq i \leq n} \mathcal{A}_i^c$ which is complete and clash free.

- For $n = 1$ the claim holds since $\mathcal{A} = \mathcal{A}_1$.
- Assume the claim holds for n. Let S_i^l denote the starting segment consisting of the first l members of S_i. For $l \geq l_i$ we set $S_i^l = S_i$. We show by induction on l that $S_1, \ldots, S_{n+1}$ can be applied to $\mathcal{A}$.
  - For $l = 0$ we have $S_1, \ldots, S_{n+1}^l = S_1, \ldots, S_n$ which by hypothesis can be applied to $\mathcal{A}$.

– Assume $S_1, \dots, S^l_{n+1}$ can be applied to $\mathcal{A}$. If $l \geq l_{n+1}$ we are done since $S_1, \dots, S^l_{n+1} = S_1, \dots, S_{n+1}$. Otherwise we have to show that the last completion rule in the sequence $S_1, \dots, S^{l+1}_{n+1}$ can be applied to $\mathcal{A}$. But this is the case as consequence of Lemma 18 and because $\mathcal{A}$ is separable.

$\bigcup_{1 \leq i \leq n} \mathcal{A}^c_i$ is complete and clash free by hypothesis and $\mathcal{A}^c_{n+1}$ is complete and clash free by precondition. $\bigcup_{1 \leq i \leq n+1} \mathcal{A}^c_i = \bigcup_{1 \leq i \leq n} \mathcal{A}^c_i \cup \mathcal{A}^c_{n+1}$ is thus also complete and clash free as a consequence of Lemma 18 and because $\mathcal{A}$ is separable.

□

We momentarily extend our definition of an ABox such that it can contain negative role assertions of the form $\neg R(a, b)$. An interpretation $\mathcal{I}$ satisfies a negative role assertion $\neg R(a, b)$ iff $\mathcal{I}$ does not satisfy $R(a, b)$.

Lemma 21. *Let R be a role and for every $1 \leq i \leq n$ let C_i be a concept and a, b_i individual constants with $a \neq b_i$ and $b_i \neq b_j$ for $i \neq j$. Then the ABox*

$$\mathcal{A} = \{C_i(b_i) | 1 \leq i \leq n\}$$

is satisfiable iff each ABox

$$\mathcal{A}' = \mathcal{A} \cup \{R_i(a, b_i) | R_i \in \{R, \neg R\},\ 1 \leq i \leq n\}$$

is satisfiable .

Proof. For the if direction let $\mathcal{I} \models \mathcal{A}'$. Then $\mathcal{I} \models \mathcal{A}$.
For the other direction let $\mathcal{I} \models \mathcal{A}$. We define the interpretation $\mathcal{J}$ with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ where $z \notin \Delta^{\mathcal{I}}$, $a^{\mathcal{J}} = z$, $b_i^{\mathcal{J}} = b_i^{\mathcal{I}}$ for $1 \leq i \leq n$, $A^{\mathcal{J}} = A^{\mathcal{I}}$ for all atomic concepts A, $S^{\mathcal{J}} = S^{\mathcal{I}}$ for all roles $S \neq R$ and $R^{\mathcal{J}} = R^{\mathcal{I}} \cup \{(z, b_i^{\mathcal{I}})\ | R(a, b_i) \in \mathcal{A}'\}$.

i) $\mathcal{J}$ satisfies $R(a, b_i)$ for all $R(a, b_i) \in \mathcal{A}'$ and $\mathcal{J}$ satisfies $\neg R(a, b_i)$ for all $\neg R(a, b_i) \in \mathcal{A}'$.

ii) $\mathcal{J}$ satisfies $C_i(b_i)$ for all $C_i(b_i) \in \mathcal{A}'$ since for all $r \in \Delta^\mathcal{I}$ we have $r \in C_i^\mathcal{I}$ iff $r \in C_i^\mathcal{J}$ as we will show by induction on formula structure:

$r \in A^\mathcal{I} = A^\mathcal{J}$.

$\neg$: $r \in (\neg C)^\mathcal{I}$ iff $r \notin C^\mathcal{I}$ iff $r \notin C^\mathcal{J}$ iff $r \in (\neg C)^\mathcal{J}$.

$\sqcap$: $r \in (C \sqcap D)^\mathcal{I}$ iff $r \in C^\mathcal{I}$ and $r \in D^\mathcal{I}$ iff $r \in C^\mathcal{J}$ and $r \in D^\mathcal{J}$ iff $r \in (C \sqcap D)^\mathcal{J}$.

$\sqcup$: $r \in (C \sqcup D)^\mathcal{I}$ iff $r \in C^\mathcal{I}$ or $r \in D^\mathcal{I}$ iff $r \in C^\mathcal{J}$ or $r \in D^\mathcal{J}$ iff $r \in (C \sqcup D)^\mathcal{J}$.

$\forall$: $r \in (\forall R.C)^\mathcal{I}$ iff (by definition)
$\forall s \in \Delta^\mathcal{I} : (r, s) \notin R^\mathcal{I} \vee s \in C^\mathcal{I}$ iff (1)
$\forall s \in \Delta^\mathcal{I} : (r, s) \notin R^\mathcal{J} \vee s \in C^\mathcal{I}$ iff (2)
$\forall s \in \Delta^\mathcal{J} : (r, s) \notin R^\mathcal{J} \vee s \in C^\mathcal{I}$ iff (3)
$\forall s \in \Delta^\mathcal{J} : (r, s) \notin R^\mathcal{J} \vee s \in C^\mathcal{J}$ iff (by definition)
$r \in (\forall R.C)^\mathcal{J}$.

$\exists$: $r \in (\exists R.C)^\mathcal{I}$ iff (by definition)
$\exists s \in \Delta^\mathcal{I} : (r, s) \in R^\mathcal{I} \wedge s \in C^\mathcal{I}$ iff (1)
$\exists s \in \Delta^\mathcal{I} : (r, s) \in R^\mathcal{J} \wedge s \in C^\mathcal{I}$ iff (2)
$\exists s \in \Delta^\mathcal{J} : (r, s) \in R^\mathcal{J} \wedge s \in C^\mathcal{I}$ iff (3)
$\exists s \in \Delta^\mathcal{J} : (r, s) \in R^\mathcal{J} \wedge s \in C^\mathcal{J}$ iff (by definition)
$r \in (\exists R.C)^\mathcal{J}$.

In the cases $\forall$ and $\exists$ (1) holds because $\forall s \in \Delta^\mathcal{I} : (r, s) \in R^\mathcal{I} \Leftrightarrow (r, s) \in R^\mathcal{J}$, (2) holds because $(r, z) \notin R^\mathcal{J}$, and (3) holds because $\forall s \in \Delta^\mathcal{J} : (r, s) \in R^\mathcal{J} \implies s \in \Delta^\mathcal{I}$ and by hypothesis.

Thus $\mathcal{J} \models \mathcal{A}'$. $\square$

Theorem 22 (Independence)**.** *Let R be a role, C a concept with $C \not\equiv \bot$ and $C \not\equiv \top$, and for every $1 \leq i \leq n$ let a, b_i be individual constants with $a \neq b_i$ and $b_i \neq b_j$ for $i \neq j$. Then the ABox*

$$\mathcal{A} = \{R(a, b_i), C(b_i) | 1 \leq i \leq n\}$$

is independent in the sense that for every assertion $\alpha \in \mathcal{A}$

$$\mathcal{A} \setminus \{\alpha\} \not\models \alpha \text{ and } \mathcal{A} \setminus \{\alpha\} \not\models \neg\alpha.$$

Proof. $\mathcal{A} \setminus \{\alpha\} \not\models \alpha$ iff $\mathcal{A}_0 = \mathcal{A} \setminus \{\alpha\} \cup \{\neg\alpha\}$ is satisfiable. If α is the concept assertion $C(b_k)$ for some $1 \leq k \leq n$ then $\mathcal{A}_0$ is satisfiable by Lemma 21 iff $\{C(b_i) | 1 \leq i \leq n\} \setminus \{C(b_k)\} \cup \{\neg C(b_k)\}$ is satisfiable. The latter is satisfiable by Lemma 20 because by precondition $C \not\equiv \bot$ and $\neg C \not\equiv \bot$. If α is a role assertion the argument is similar. The proof for $\mathcal{A} \setminus \{\alpha\} \not\models \neg\alpha$ is similar. $\square$

Note that excluding $\bot$ and $\top$ in Theorem 22 is irrelevant to our aim. In the probabilistic sense the empty set and the universe are always independent from any other event.

1.2.2 A Pumping Lemma for $\mathcal{ALC}$

The following lemma basically states, that whenever an ABox $\mathcal{A}$ is satisfiable, then we can expand the domain of its models at will. That is, if $\mathcal{A}$ has a model with domain $\Delta^{\mathcal{I}}$ we can find another model for $\mathcal{A}$ whose domain is a proper superset of $\Delta^{\mathcal{I}}$.

Lemma 23 (Pumping)**.** *Let $\mathcal{A}$ be an ABox with $\mathcal{I} \models \mathcal{A}$ for some interpretation $\mathcal{I}$. Then there exists an interpretation $\mathcal{J}$ which agrees on $\mathcal{I}$ except for $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$, such that $\mathcal{J} \models \mathcal{A}$.*

Proof. We prove by structural induction that either $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \{z\}$ holds or $C^{\mathcal{J}} = C^{\mathcal{I}}$ holds for any concept C.

For atomic concepts A and roles R we have $A^{\mathcal{J}} = A^{\mathcal{I}}$ and $R^{\mathcal{J}} = R^{\mathcal{I}}$ by definition. Assume either $C^{\mathcal{J}} = C^{\mathcal{I}} \cup \{z\}$ or $C^{\mathcal{J}} = C^{\mathcal{I}}$ holds.

$$\neg\text{: } (\neg C)^{\mathcal{J}} = \begin{cases} (\Delta^{\mathcal{I}} \cup \{z\}) \setminus C^{\mathcal{I}} = (\neg C)^{\mathcal{I}} \cup \{z\} & \text{if } C^{\mathcal{J}} = C^{\mathcal{I}}, \\ (\Delta^{\mathcal{I}} \cup \{z\}) \setminus (C^{\mathcal{I}} \cup \{z\}) = (\neg C)^{\mathcal{I}} & \text{otherwise.} \end{cases}$$

$$\sqcup\text{: } (C \sqcup D)^{\mathcal{J}} = \begin{cases} (C^{\mathcal{I}} \cup D^{\mathcal{I}}) = (C \sqcup D)^{\mathcal{I}} & \text{if } C^{\mathcal{J}} = C^{\mathcal{I}},\ D^{\mathcal{J}} = D^{\mathcal{I}}, \\ (C^{\mathcal{I}} \cup D^{\mathcal{I}}) \cup \{z\} = (C \sqcup D)^{\mathcal{I}} \cup \{z\} & \text{otherwise.} \end{cases}$$

$\sqcap$: Analog to the $\sqcup$ case.

$$\begin{aligned}
\forall\text{: } (\forall R.C)^{\mathcal{J}} &= \{r \in \Delta^{\mathcal{J}} | \forall s \in \Delta^{\mathcal{J}} : (r,s) \in R^{\mathcal{J}} \to s \in C^{\mathcal{J}}\} \\
&= \{r \in \Delta^{\mathcal{I}} | \forall s \in \Delta^{\mathcal{J}} : (r,s) \in R^{\mathcal{I}} \to s \in C^{\mathcal{J}}\} \cup \\
&\quad \underbrace{\{r \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}} | \forall s \in \Delta^{\mathcal{J}} : (r,s) \in R^{\mathcal{I}} \to s \in C^{\mathcal{J}}\}}_{=\{z\} \text{ since } \forall s \in \Delta^{\mathcal{J}} : (z,s) \notin R^{\mathcal{I}}} \\
&= \{r \in \Delta^{\mathcal{I}} | (\forall s \in \Delta^{\mathcal{I}} : (r,s) \in R^{\mathcal{I}} \to s \in C^{\mathcal{J}}) \wedge \\
&\quad \underbrace{((r,z) \in R^{\mathcal{I}} \to s \in C^{\mathcal{J}})}_{\text{always holds, since } \forall r \in \Delta^{\mathcal{I}} : (r,z) \notin R^{\mathcal{I}}} \} \cup \{z\} \\
&= \{r \in \Delta^{\mathcal{I}} | \forall s \in \Delta^{\mathcal{I}} : (r,s) \in R^{\mathcal{I}} \to (s \in C^{\mathcal{I}} \vee s = z)\} \cup \{z\} \\
&= \{r \in \Delta^{\mathcal{I}} | \forall s \in \Delta^{\mathcal{I}} : (r,s) \in R^{\mathcal{I}} \to (s \in C^{\mathcal{I}})\} \cup \{z\} \\
&= (\forall R.C)^{\mathcal{I}} \cup \{z\}
\end{aligned}$$

$\exists$: Using $(\exists R.C)^{\mathcal{J}} = (\neg \forall R.\neg C)^{\mathcal{J}}$ for a reduction to the previous cases yields $(\exists R.C)^{\mathcal{J}} = (\exists R.C)^{\mathcal{I}}$.

Hence for any assertion $C(a) \in \mathcal{A}$ we have $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $a^{\mathcal{I}} \in C^{\mathcal{J}}$ iff $a^{\mathcal{J}} \in C^{\mathcal{J}}$ and thus $\mathcal{J} \models C(a)$ and $\mathcal{J} \models \mathcal{A}$. $\square$

Chapter 2

$\mathcal{PALC}$: A Probabilistic Variant of $\mathcal{ALC}$

In this chapter we introduce syntax and semantics of the language $\mathcal{PALC}$. $\mathcal{PALC}$ is a variant of $\mathcal{ALC}$ which allows us to express probabilistic knowledge about concept and role assertions. In $\mathcal{ALC}$ the semantics of assertions is defined via set membership. In contrast, for $\mathcal{PALC}$ we define the semantics of probabilistic assertions by means of random variables. Using the independence result from Section 1.2.1 we are able introduce a further restriction on the interpretation of existential restrictions and value restrictions. This will lead to a reduction of the solution space when it comes to reasoning in $\mathcal{PALC}$. Further Theorem 11 will be applicable on our probabilistic interpretations which will lead to a zero-one law for $\mathcal{PALC}$. Finally we will see that in the limit when only the probability 1 is involved $\mathcal{PALC}$ is equivalent to $\mathcal{ALC}$.

2.1 Syntax and Semantics

$\mathcal{PALC}$ extends the syntax and semantics of $\mathcal{ALC}$ such that we can state probabilistic assertions about the extensions of concepts and roles. To achieve this we need a set of *probabilistic constants* denoted by $p_0, p_1, \ldots$ which are of a different type than the individual constants and a symbol P for the *probability* *operator*. If C is a concept, R is a role, p_0, p_1 are probabilistic constants, and a, b, c are individual constants, then $\mathrm{P}(C(a)) \doteq p_0$ and $\mathrm{P}(R(b,c)) \doteq p_1$ are *probabilistic concept assertions* and *probabilistic role assertions*, respectively. In the sequel we will use $\rho, \sigma, \ldots$ to denote probabilistic $\mathcal{PALC}$ assertions and $\alpha, \beta, \ldots$ for classical $\mathcal{ALC}$ assertions. Also if α stands for $C(a)$ we will use $\neg\alpha$ to denote $\neg C(a)$. A *probabilistic ABox* $\mathcal{A}$ is a finite set of probabilistic assertions.

The semantics of probabilistic assertions is given by a *probabilistic interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ where the domain $\Delta^{\mathcal{I}}$ is a non-empty, countable set and $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$ is a probability space. The *probabilistic interpretation function* $\cdot^{\mathcal{I}}$ interprets concepts and individuals as in $\mathcal{ALC}$ and additionally assigns a real number to each probabilistic constant.

Definition 24. With $\sigma(\mathrm{cns}(\mathcal{PALC})^{\mathcal{I}}) \subseteq \mathcal{P}(\Delta^{\mathcal{I}})$ we denote the σ-algebra generated by the set of all concepts of $\mathcal{PALC}$ interpreted by the probabilistic interpretation $\mathcal{I}$. Similarly with $\sigma(\mathrm{rls}(\mathcal{PALC})^{\mathcal{I}}) \subseteq \mathcal{P}(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})$ we denote the σ-algebra generated by the set of all roles of $\mathcal{PALC}$ interpreted by $\mathcal{I}$.

We associate with each individual $r \in \Delta^{\mathcal{I}}$ the random variable $X_r : \Omega \to \Delta^{\mathcal{I}}$ for $\sigma(\mathrm{cns}(\mathcal{PALC})^{\mathcal{I}})$ and with each pair of individuals $(r,s) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ the random variable $X_{rs} : \Omega \to \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for $\sigma(\mathrm{rls}(\mathcal{PALC})^{\mathcal{I}})$. We require these random variables to be independent. That is, we require independence for the set $\{X_r | r \in \Delta^{\mathcal{I}}\} \cup \{X_{rs} | (r,s) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\}$. For conciseness of notation we write $\mathrm{P}(r \in C^{\mathcal{I}})$ and $\mathrm{P}((r,s) \in R^{\mathcal{I}})$ instead of $\mathrm{P}(X_r \in C^{\mathcal{I}})$ and $\mathrm{P}(X_{rs} \in R^{\mathcal{I}})$, respectively. We also write $(r,s) \notin R^{\mathcal{I}}$ for $(r,s) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus R^{\mathcal{I}}$. Finally we require for all $r \in \Delta^{\mathcal{I}}$, concepts C and roles R the following condition to hold:

$$\mathrm{P}(r \in (\forall R.C)^{\mathcal{I}}) = \mathrm{P}(\bigcap_{s \in \Delta^{\mathcal{I}}} (\{(r,s) \notin R^{\mathcal{I}}\} \cup \{s \in C^{\mathcal{I}}\})). \tag{$\forall$}$$

Remark 25. For any $\mathcal{ALC}$ interpretation $\mathcal{I}$ we have

$$r \in (\forall R.C)^{\mathcal{I}} \text{ iff } r \in \bigcap_{s \in \Delta^{\mathcal{I}}} (\{r \in \Delta^{\mathcal{I}} | (r,s) \notin R^{\mathcal{I}}\} \cup \{r \in \Delta^{\mathcal{I}} | s \in C^{\mathcal{I}}\}).$$

Condition ($\forall$) is an extrapolation of the classical to the probabilistic case.[1] Independence is motivated by the fact that in $\mathcal{ALC}$ assertions about concepts and roles which affect value restrictions are logically independent as shown by Theorem 22.

A probabilistic interpretation $\mathcal{I}$ satisfies a probabilistic concept assertion $\mathrm{P}(C(a)) \doteq p_0$ iff $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = {p_0}^{\mathcal{I}}$. Similarly $\mathcal{I}$ satisfies a probabilistic role assertion $\mathrm{P}(R(a,b)) \doteq p_0$ iff $\mathrm{P}((a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}) = {p_0}^{\mathcal{I}}$. We say $\mathcal{I}$ is a *model* of a probabilistic assertion ρ if $\mathcal{I}$ satisfies ρ and write $\mathcal{I} \models \rho$. $\mathcal{I}$ satisfies a probabilistic ABox $\mathcal{A}$ iff it satisfies every element of $\mathcal{A}$. We then call $\mathcal{I}$ a model of $\mathcal{A}$ and write $\mathcal{I} \models \mathcal{A}$.

The next lemma shows that the values assigned to concepts and roles by the probabilistic interpretation function are indeed probabilities.

Lemma 26. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ be a probabilistic interpretation where $\mathbf{\Omega}$ is the probability space (Ω, F, P).*

i) For each $r \in \Delta^{\mathcal{I}}$, the triple $(\Delta^{\mathcal{I}}, \sigma(\mathrm{cns}(\mathcal{PALC})^{\mathcal{I}}), \mathrm{P}_r)$ is a probability space with $\mathrm{P}_r : \Delta^{\mathcal{I}} \to \mathbb{R}$, $\mathrm{P}_r(C^{\mathcal{I}}) = \mathrm{P}(r \in C^{\mathcal{I}})$.

ii) For each $(r,s) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, the triple $(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, \sigma(\mathrm{rls}(\mathcal{PALC})^{\mathcal{I}}), \mathrm{P}_{rs})$ is a probability space with $\mathrm{P}_{rs} : \Delta^{\mathcal{I}} \to \mathbb{R}$, $\mathrm{P}_{rs}(R^{\mathcal{I}}) = \mathrm{P}((r,s) \in R^{\mathcal{I}})$.

Proof. By Definition. See Section 1.1. □

In the following we will write interpretation instead of probabilistic interpretation and similarly interpretation function instead of probabilistic interpretation function and membership function instead of probabilistic membership function if the meaning is clear from the context. To simplify notation we adhere to the following convention: for each real number $x \in \mathbb{R}$ we use its decimal representation - denoted by $\mathbf{x}$ - as a probabilistic constant and require

[1] Gaifman takes a similar however more general approach in [Gai64]. There the probability of an universal quantified first order formula $\forall x \varphi(x)$ is taken to be the infimum of the probabilities of all finite conjunctions of the form $\varphi(y_1) \wedge \cdots \wedge \varphi(y_n)$. The constants y_i do not occur in the initial language.

$\mathbf{x}^{\mathcal{I}} = x$ for an interpretation $\mathcal{I}$. Thus, instead of $\mathcal{I} \models \mathrm{P}(\alpha) \doteq p_0$ and ${p_0}^{\mathcal{I}} = x$ we simply write $\mathcal{I} \models \mathrm{P}(\alpha) \doteq \mathbf{x}$.

We are now in the position to prove that the independence assumption and Condition ($\forall$) imply additional constraints on the interpretation of probabilistic assertions for value restrictions.

Theorem 27. *A probabilistic interpretation $\mathcal{I}$ satisfies a probabilistic value restriction* $\mathrm{P}(\forall R.C(a)) \doteq p_0$ *iff*

$$ {p_0}^{\mathcal{I}} = \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})(1 - \mathrm{P}(s \in C^{\mathcal{I}})). $$

Proof. Let $s_1, s_2, \ldots$ be an enumerations of the elements of the domain $\Delta^{\mathcal{I}}$.

$$ \begin{aligned} {p_0}^{\mathcal{I}} = \mathrm{P}(a^{\mathcal{I}} \in (\forall R.C)^{\mathcal{I}}) &= \mathrm{P}(\bigcap_{s \in \Delta^{\mathcal{I}}} (\{(a^{\mathcal{I}}, s) \notin R^{\mathcal{I}}\} \cup \{s \in C^{\mathcal{I}}\})) \\ &= \lim_{n \to \infty} \mathrm{P}(\bigcap_{i=1}^{n} (\{(a^{\mathcal{I}}, s_i) \notin R^{\mathcal{I}}\} \cup \{s_i \in C^{\mathcal{I}}\})) \\ &= \lim_{n \to \infty} \prod_{i=1}^{n} \mathrm{P}(\{(a^{\mathcal{I}}, s_i) \notin R^{\mathcal{I}}\} \cup \{s_i \in C^{\mathcal{I}}\}) \\ &= \prod_{s \in \Delta^{\mathcal{I}}} \mathrm{P}(\{(a^{\mathcal{I}}, s) \notin R^{\mathcal{I}}\} \cup \{s \in C^{\mathcal{I}}\}) \\ &= \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}(\{(a^{\mathcal{I}}, s) \in R^{\mathcal{I}}\} \cap \{s \in (\neg C)^{\mathcal{I}}\}) \\ &= \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})(1 - \mathrm{P}(s \in C^{\mathcal{I}})). \end{aligned} $$

□

Example 28. Consider the ABox

$$ \mathcal{A} = \{\mathrm{P}(\forall R.C(a)) \doteq 1, \mathrm{P}(R(a,b)) \doteq \mathbf{x}, \mathrm{P}(C(b)) \doteq p\} $$

with $x > 0$ and neither $C \equiv \bot$ nor $C \equiv \top$. In absence of Condition ($\forall$) we could find a model $\mathcal{I}$ for $\mathcal{A}$ such that $0 \leq p^{\mathcal{I}} \leq 1$. Condition ($\forall$) however requires $1 \leq 1 - x(1 - p^{\mathcal{I}})$ for every interpretation $\mathcal{I}$ and thus $p^{\mathcal{I}} = 1$.

While reasoning in $\mathcal{ALC}$ consists of checking whether an ABox has a model, reasoning in $\mathcal{PALC}$ consists of deriving interpretations for the probabilistic constants. This can be regarded as finding suitable probabilities for the extensions of concepts. To give a flavor of reasoning in $\mathcal{PALC}$ we will derive some standard results of probability theory.

In the following let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, \mathbf{\Omega})$ with the probability space $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$ be an interpretation. Let C, D be concepts, a an individual constant, and $p_0, p_1, \ldots$ probabilistic constants.

Lemma 29. *Let $\mathcal{I} \models \mathrm{P}(\top(a)) \doteq p_0$. Then ${p_0}^\mathcal{I} = 1$.*

Proof. $\mathcal{I} \models \mathrm{P}(\top(a)) \doteq p_0$ iff ${p_0}^\mathcal{I} = \mathrm{P}(a^\mathcal{I} \in \Delta^\mathcal{I}) = \mathrm{P}(\{s \in \Omega | X_{a^\mathcal{I}}(s) \in \Delta^\mathcal{I}\}) = \mathrm{P}(\Omega) = 1$. □

Lemma 30. *Let $\mathcal{I} \models \mathrm{P}(C(a)) \doteq p_0$ and $\mathcal{I} \models \mathrm{P}(\neg C(a)) \doteq p_1$. Then ${p_0}^\mathcal{I} + {p_1}^\mathcal{I} = 1$.*

Proof. Since $C^\mathcal{I} \cap (\neg C)^\mathcal{I} = \emptyset$ and $C^\mathcal{I} \cup (\neg C)^\mathcal{I} = \Delta^\mathcal{I}$ we find $1 = \mathrm{P}(a^\mathcal{I} \in \Delta^\mathcal{I}) = \mathrm{P}(a^\mathcal{I} \in C^\mathcal{I} \cup (\neg C)^\mathcal{I}) = \mathrm{P}(a^\mathcal{I} \in C^\mathcal{I}) + \mathrm{P}(a^\mathcal{I} \in (\neg C)^\mathcal{I}) = {p_0}^\mathcal{I} + {p_1}^\mathcal{I}$. □

Lemma 31. *Let $\mathcal{I} \models \mathrm{P}(\bot(a)) \doteq p_0$. Then ${p_0}^\mathcal{I} = 0$.*

Proof. $\mathcal{I} \models \mathrm{P}(\neg\bot(a)) \doteq 1$ because $(\neg\bot)^\mathcal{I} = \top^\mathcal{I} = \Delta^\mathcal{I}$. Using Lemma 30 we find $\mathcal{I} \models \mathrm{P}(\bot \sqcup \neg\bot(a)) \doteq 1$ and thus $1 = {p_0}^\mathcal{I} + 1$ and ${p_0}^\mathcal{I} = 0$. □

Lemma 32. *Let $\mathcal{I} \models \mathrm{P}(C(a)) \doteq p_0$, $\mathcal{I} \models \mathrm{P}(D(a)) \doteq p_1$ and $C^\mathcal{I} = D^\mathcal{I}$. Then ${p_0}^\mathcal{I} = {p_1}^\mathcal{I}$.*

Proof. ${p_0}^\mathcal{I} = \mathrm{P}(a^\mathcal{I} \in C^\mathcal{I}) = \mathrm{P}(a^\mathcal{I} \in D^\mathcal{I}) = {p_1}^\mathcal{I}$. □

Lemma 33. *Let $\mathcal{I} \models \mathrm{P}(C(a)) \doteq p_0$, $\mathcal{I} \models \mathrm{P}(D(a)) \doteq p_1$ and $(C \sqcup D)^\mathcal{I} = D^\mathcal{I}$. Then ${p_0}^\mathcal{I} \leq {p_1}^\mathcal{I}$.*

Proof. First note that $(C \sqcup (D \sqcap \neg C))^\mathcal{I} = (C \sqcup D)^\mathcal{I}$. From Lemma 32 follows $\mathcal{I} \models \mathrm{P}(C \sqcup (D \sqcap \neg C)(a)) \doteq p_1$. Because of $(C \sqcap (D \sqcap \neg C))^\mathcal{I} = \emptyset$ we get ${p_1}^\mathcal{I} = {p_0}^\mathcal{I} + \mathrm{P}(a^\mathcal{I} \in (D \sqcap \neg C)^\mathcal{I})$ and thus ${p_0}^\mathcal{I} \leq {p_1}^\mathcal{I}$. □

Lemma 34. *Let* $\mathcal{I} \models \mathrm{P}(C \sqcup D(a)) \doteq p_0$*,* $\mathcal{I} \models \mathrm{P}(C(a)) \doteq p_1$*,* $\mathcal{I} \models \mathrm{P}(D(a)) \doteq p_2$ *and* $\mathcal{I} \models \mathrm{P}(C \sqcap D(a)) \doteq p_3$*. Then* ${p_0}^{\mathcal{I}} = {p_1}^{\mathcal{I}} + {p_2}^{\mathcal{I}} - {p_3}^{\mathcal{I}}$*.*

Proof. First note that:

i) $(C \sqcup D)^{\mathcal{I}} = (C \sqcup (\neg C \sqcap D))^{\mathcal{I}}$,

ii) $(C \sqcap (\neg C \sqcap D))^{\mathcal{I}} = \emptyset$,

iii) $((C \sqcap D) \sqcup (\neg C \sqcap D))^{\mathcal{I}} = D^{\mathcal{I}}$,

iv) $((C \sqcap D) \sqcap (\neg C \sqcap D))^{\mathcal{I}} = \emptyset$.

Now ${p_0}^{\mathcal{I}} = {p_1}^{\mathcal{I}} + \mathrm{P}(a^{\mathcal{I}} \in (\neg C \sqcap D)^{\mathcal{I}})$ by Lemma 32. Together with $\mathcal{I} \models \mathrm{P}(C \sqcap D(a)) \doteq p_3$ we find ${p_2}^{\mathcal{I}} = \mathrm{P}(a^{\mathcal{I}} \in (\neg C \sqcap D)^{\mathcal{I}}) + {p_3}^{\mathcal{I}}$ and thus ${p_0}^{\mathcal{I}} = {p_1}^{\mathcal{I}} + {p_2}^{\mathcal{I}} - {p_3}^{\mathcal{I}}$. □

Lemma 35. *Let* $\mathcal{I} \models \mathrm{P}(C(a)) \doteq p_0$*. Then* $0 \leq {p_0}^{\mathcal{I}} \leq 1$*.*

Proof. $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = {p_0}^{\mathcal{I}} \geq 0$. Now assume ${p_0}^{\mathcal{I}} > 1$. But then $\mathrm{P}(a^{\mathcal{I}} \in (C \sqcup \neg C)^{\mathcal{I}}) > 1$ which contradicts Lemma 29. □

With these lemmas at our disposal, we can now derive the semantics of existential restrictions as a corollary to Theorem 27.

Corollary 36. *A probabilistic interpretation* $\mathcal{I}$ *satisfies a probabilistic existential restriction* $\mathrm{P}(\exists R.C(a)) \doteq p_0$ *iff*

$$ {p_0}^{\mathcal{I}} = 1 - \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})\, \mathrm{P}(s \in C^{\mathcal{I}}). $$

Proof. Using $(\exists R.C)^{\mathcal{I}} = (\neg \forall R.\neg C)^{\mathcal{I}}$ from Lemma 13 and applying Lemma 32 we find $\mathcal{I} \models \mathrm{P}(\exists R.C(a)) \doteq p_0$ iff $\mathcal{I} \models \mathrm{P}(\neg \forall R.\neg C(a)) \doteq p_0$ by applying Lemma 30 and Theorem 27 iff

$$ \begin{aligned} {p_0}^{\mathcal{I}} &= 1 - \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})(1 - \mathrm{P}(s \notin C^{\mathcal{I}})) \\ &= 1 - \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})\, \mathrm{P}(s \in C^{\mathcal{I}}) \end{aligned} $$

as required. $\square$

Theorem 37 (Zero-one law[2])**.** *Let $\mathcal{A}$ be an ABox with* $\mathrm{P}(\mathsf{Q}R.C(a)) \doteq \mathbf{x} \in \mathcal{A}$ *where* $\mathsf{Q} \in \{\exists, \forall\}$ *and* $0 < x < 1$*. Then for any interpretation $\mathcal{I}$ we have*

$$\mathcal{I} \models \mathcal{A} \implies \Delta^{\mathcal{I}} \textit{ is finite.}$$

Proof. Assume $\mathcal{I} \models \mathcal{A}$ and $\Delta^{\mathcal{I}}$ is not finite. But then $\mathbf{x}^{\mathcal{I}} \in \{0, 1\}$ according to Theorem 11 which contradicts $0 < x < 1$. The assumption was thus wrong and $\Delta^{\mathcal{I}}$ is finite. $\square$

In [YR00] Bernard and Rousset give a somewhat related result. They consider probability distributions on sets of random assertions and show, that for large domains the probability of these sentences converge to either 0 or 1 in general.

2.2 $\mathcal{PALC}$ Compared to $\mathcal{ALC}$

In this section we examine the properties of $\mathcal{PALC}$ and contrast them with those of $\mathcal{ALC}$. The main result will be, that $\mathcal{PALC}$ is equivalent to $\mathcal{ALC}$ if we only allow the probability 1. That is, all assertions of a $\mathcal{PALC}$ ABox have probability 1 if and only if an $\mathcal{ALC}$ ABox containing the corresponding non probabilistic assertions is satisfiable. $\mathcal{PALC}$ is therefore sound and complete with respect to $\mathcal{ALC}$.

Definition 38 (Valuation)**.** A function j is a *valuation (of the probabilistic constants)* of $\mathcal{PALC}$ if j maps each probabilistic constant p to a real number. Additionally, we require $j(\mathbf{x}) = x$ where $\mathbf{x}$ is the decimal representation of $x \in \mathbb{R}$. An interpretation $\mathcal{I}$ respects a valuation j, denoted by $\mathcal{I}_j$, if $\mathcal{I}$ agrees with j on the interpretation of the probabilistic constants (that is $p^{\mathcal{I}} = j(p)$ for all probabilistic constants p).

[2] According to Kyburg [KT01, p.85], Carnap gives a similar result for universally quantified formula in first order logic in [Car51].

Definition 39 (j-Satisfiability)**.** Let $\mathcal{A}$ be a probabilistic ABox and j a valuation. $\mathcal{A}$ is *j-satisfiable* if there is an interpretation $\mathcal{I}_j$ such that $\mathcal{I}_j \models \mathcal{A}$.

Definition 40 (j-Entailment)**.** A probabilistic ABox $\mathcal{A}$ *entails* a probabilistic assertion ρ with respect to a valuation j, if every model $\mathcal{I}_j$ of $\mathcal{A}$ is also a model of ρ. We write $\mathcal{A} \models_j \rho$.

Lemma 41. *Let α be an assertion, $\mathcal{A}$ an ABox, j a valuation and $\mathbf{x}$, $\mathbf{y}$ probabilistic constants. Then*

i) $\mathcal{A} \models_j \mathrm{P}(\alpha) \doteq \mathbf{x} \implies \mathcal{A} \cup \{\mathrm{P}(\alpha) \doteq \mathbf{y}\}$ *is not j-satisfiable for any $y \neq x$.*

ii) $\mathcal{A} \cup \{\mathrm{P}(\alpha) \doteq \mathbf{x}\}$ *not j-satisfiable* $\implies \mathcal{A} \models_j \mathrm{P}(\alpha) \doteq \mathbf{y}$ *for some $y \neq x$.*

Proof.

i) If $\mathcal{A}$ is not j-satisfiable then $\mathcal{A} \cup \{\mathrm{P}(\alpha) \doteq \mathbf{y}\}$ is not j-satisfiable for any y. Otherwise $\mathcal{I}_j \models \mathcal{A}$ for some interpretation $\mathcal{I}_j$ and thus $\mathcal{I}_j \models \mathrm{P}(\alpha) \doteq \mathbf{x}$. But then $\mathcal{I}_j \not\models \mathrm{P}(\alpha) \doteq \mathbf{y}$ for any $y \neq x$.

ii) If $\mathcal{A}$ is not j-satisfiable then $\mathrm{P}(\alpha) \doteq \mathbf{y}$ vacuously holds in every model of $\mathcal{A}$. Otherwise $\mathcal{I}_j \models \mathcal{A}$ for some interpretation $\mathcal{I}_j$ and thus $\mathcal{I}_j \not\models \mathrm{P}(\alpha) \doteq \mathbf{x}$. But then $\mathcal{I}_j \models \mathrm{P}(\alpha) \doteq \mathbf{y}$ for some $y \neq x$.

$\square$

Theorem 42 (Soundness and Completeness w.r.t. $\mathcal{ALC}$)**.** *Let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox and j any valuation. We define the probabilistic ABox $\mathcal{A}_p = \{\mathrm{P}(\alpha) \doteq 1 | \alpha \in \mathcal{A}\}$. Then*

$$\mathcal{A} \text{ is satisfiable} \quad \text{iff} \quad \mathcal{A}_p \text{ is } j\text{-satisfiable.}$$

Proof. Let $\mathcal{J} \models \mathcal{A}$. Define the probabilistic interpretation $\mathcal{I}$ which coincides with $\mathcal{J}$ on the non probabilistic part and set for all atomic concepts A, roles R and individuals r, s

$$\mathrm{P}(r \in A^{\mathcal{I}}) = \begin{cases} 1 & \text{if } r \in A^{\mathcal{J}}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{P}((r,s) \in R^{\mathcal{I}}) = \begin{cases} 1 & \text{if } (r,s) \in R^{\mathcal{J}}, \\ 0 & \text{otherwise.} \end{cases}$$

Claim: For any concept C and individual r

$$\mathrm{P}(r \in C^{\mathcal{I}}) = \begin{cases} 1 & \text{if } r \in C^{\mathcal{J}}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $C(a) \in \mathcal{A}$. Then $\mathcal{J} \models C(a)$ iff $a^{\mathcal{J}} \in C^{\mathcal{J}}$ which by the claim implies $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = 1$ and thus $\mathcal{I}_j \models \mathrm{P}(C(a)) \doteq 1$. Hence we find $\mathcal{I}_j \models A_p$.

Proof of the claim: For atomic concepts and roles the claim holds by definition. So assume the claim holds for a concept C.

- $\neg$: If $r \in (\neg C)^{\mathcal{J}}$ then $r \notin C^{\mathcal{J}}$. By hypothesis $\mathrm{P}(r \in C^{\mathcal{I}}) = 0$ and thus $\mathrm{P}(r \in (\neg C)^{\mathcal{I}}) = 1$. The case $r \notin (\neg C)^{\mathcal{J}}$ is similar.

- $\sqcup$: If $r \in (C \sqcup D)^{\mathcal{J}}$ then without loss of generality assume $r \in C^{\mathcal{J}}$. By hypothesis $\mathrm{P}(r \in C^{\mathcal{I}}) = 1$ and thus $\mathrm{P}(r \in (C \sqcup D)^{\mathcal{I}}) = 1$.

  If $r \notin (C \sqcup D)^{\mathcal{J}}$ then $r \notin C^{\mathcal{J}}$ and $r \notin D^{\mathcal{J}}$. By hypothesis $\mathrm{P}(r \in C^{\mathcal{I}}) = 0$ and $\mathrm{P}(r \in D^{\mathcal{I}}) = 0$ and thus $\mathrm{P}(r \in (C \sqcup D)^{\mathcal{I}}) = 0$.

- $\sqcap$: Similar to the $\sqcup$-case.

- $\forall$: If $r \in (\forall R.C)^{\mathcal{J}}$ then $\forall s \in \Delta^{\mathcal{J}} : ((r,s) \notin R^{\mathcal{J}} \vee s \in C^{\mathcal{J}})$. By hypothesis $\forall s \in \Delta^{\mathcal{I}} : (\mathrm{P}((r,s) \in R^{\mathcal{I}}) = 0 \vee \mathrm{P}(s \in C^{\mathcal{I}}) = 1)$ and thus by Theorem 27 $\mathrm{P}(r \in (\forall R.C)^{\mathcal{I}}) = 1$.

  If $r \notin (\forall R.C)^{\mathcal{J}}$ then $\exists s \in \Delta^{\mathcal{J}} : ((r,s) \in R^{\mathcal{J}} \wedge s \notin C^{\mathcal{J}})$. By hypothesis $\exists s \in \Delta^{\mathcal{I}} : (\mathrm{P}((r,s) \in R^{\mathcal{I}}) = 1 \wedge \mathrm{P}(s \in C^{\mathcal{I}}) = 0)$ and thus by Theorem 27 $\mathrm{P}(r \in (\forall R.C)^{\mathcal{I}}) = 0$.

- $\exists$: Similar to the $\forall$-case.

For the other implication let $\mathcal{I}_j \models \mathcal{A}_p$ with an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ and the probability space $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$. Then $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = 1$ and

$\mathrm{P}((b^{\mathcal{I}}, c^{\mathcal{I}}) \in \mathcal{R}^{\mathcal{I}}) = 1$ for all assertions $\mathrm{P}(C(a)) \doteq 1 \in \mathcal{A}_p$ and $\mathrm{P}(R(b,c)) \doteq 1 \in \mathcal{A}_p$. Therefore

$$\mathrm{P}(\bigcap_{\mathrm{P}(C(a))\doteq 1 \in \mathcal{A}_p} \{a^{\mathcal{I}} \in C^{\mathcal{I}}\} \cap \bigcap_{\mathrm{P}(R(b,c))\doteq 1 \in \mathcal{A}_p} \{(b^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}}\}) =$$
$$\mathrm{P}(\{s \in \Omega | \bigwedge_{\mathrm{P}(C(a))\doteq 1 \in \mathcal{A}_p} X_{a^{\mathcal{I}}}(s) \in C^{\mathcal{I}} \wedge \bigwedge_{\mathrm{P}(R(b,c))\doteq 1 \in \mathcal{A}_p} X_{b^{\mathcal{I}}c^{\mathcal{I}}}(s) \in R^{\mathcal{I}}\}) = 1$$

and hence

$$\mathcal{E} = \{s \in \Omega | \bigwedge_{\mathrm{P}(C(a))\doteq 1 \in \mathcal{A}_p} X_{a^{\mathcal{I}}}(s) \in C^{\mathcal{I}} \wedge \bigwedge_{\mathrm{P}(R(b,c))\doteq 1 \in \mathcal{A}_p} X_{b^{\mathcal{I}}c^{\mathcal{I}}}(s) \in R^{\mathcal{I}}\} \neq \emptyset.$$

Define the $\mathcal{ALC}$ interpretation J which coincides with the non probabilistic part of $\mathcal{I}$ except for the interpretation of the individual constants for which we set $a^{\mathcal{J}} = X_{a^{\mathcal{I}}}(s)$ and $(b^{\mathcal{J}}, c^{\mathcal{J}}) = X_{b^{\mathcal{I}}c^{\mathcal{I}}}(s)$ for some $s \in \mathcal{E}$. Then $\mathcal{J} \models C(a)$ for $\mathrm{P}(C(a)) \doteq 1 \in \mathcal{A}_p$ and $\mathcal{J} \models R(b,c)$ for $\mathrm{P}(R(b,c)) \doteq 1 \in \mathcal{A}_p$ and therefore $J \models \mathcal{A}$. □

Corollary 43. *Let α be an assertion and $\mathcal{A}$, $\mathcal{A}_p$, j as in Theorem 42. Then*

i) $\mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq 1 \implies \mathcal{A} \models \alpha$.

ii) $\mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq 0 \implies \mathcal{A} \models \neg\alpha$.

Proof.

i) Let $\mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq 1$. Then $\mathcal{A}_p \models_j \mathrm{P}(\neg\alpha) \doteq 0$ and by Lemma 41 $\mathcal{A}_p \cup \{\mathrm{P}(\neg\alpha) \doteq 1\}$ is not j-satisfiable. Thus $\mathcal{A} \cup \{\neg\alpha\}$ is not satisfiable by Theorem 42 and hence $\mathcal{A} \models \alpha$.

ii) Let $\mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq 0$. Then by Lemma 41 $\mathcal{A}_p \cup \{\mathrm{P}(\alpha) \doteq 1\}$ is not j-satisfiable. Thus $\mathcal{A} \cup \{\alpha\}$ is not satisfiable by Theorem 42 and hence $\mathcal{A} \models \neg\alpha$.

□

Corollary 44. *Let α be an assertion, $\mathbf{x}$ a probabilistic constant and $\mathcal{A}$, $\mathcal{A}_p$, j as in Theorem 42. Then*

i) $\mathcal{A} \models \alpha \implies \mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq \mathbf{x}$ *for some* $x > 0$.

ii) $\mathcal{A} \models \neg\alpha \implies \mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq \mathbf{x}$ *for some* $x < 1$.

Proof.

i) Let $\mathcal{A} \models \alpha$. Then $\mathcal{A} \cup \{\neg\alpha\}$ is not satisfiable. Thus $\mathcal{A}_p \cup \{\mathrm{P}(\neg\alpha) \doteq 1\}$ is not j-satisfiable by Theorem 42 and by Lemma 41 $\mathcal{A}_p \models_j \mathrm{P}(\neg\alpha) \doteq \mathbf{y}$ for a probabilistic constant $\mathbf{y}$ with $y < 1$. Hence $\mathcal{A}_p \models_j \mathrm{P}(\alpha) \doteq \mathbf{x}$ for some $x > 0$.

ii) Substitute $\neg\alpha$ for α in the first part of the proof.

□

Chapter 3

Reasoning in $\mathcal{PALC}$

After having introduced $\mathcal{PALC}$ as a probabilistic extension to $\mathcal{ALC}$ we now proceed to formally define the reasoning problem for $\mathcal{PALC}$. First, we clarify what reasoning in $\mathcal{PALC}$ actually means and contrast it with reasoning in $\mathcal{ALC}$. Having done so, we identify constraints which solutions of a reasoning problem for $\mathcal{PALC}$ must satisfy. We start by identifying some linear constraints stemming from the axioms of probability. Our method basically resembles the approach taken in [BCM+03, BKW03] and also mentioned in [Jae94]: For a given set of sentences a basis is constructed such that each sentence of the initial set is equivalent to the disjunction of some members of the basis. Because the elements of the basis are pairwise disjunct, the probability of each of the original sentences is uniquely determined by the probabilities of the elements of the basis. Following Nilsson [Nil86] these constraints can then be stated in the form of a matrix equation. Proceeding with the implications of the independence assumption and Condition ($\forall$), we will find more constraints on possible solution which are not linear anymore.

3.1 Reasoning Formalized

In this section we propose a formal definition of reasoning in $\mathcal{PALC}$ and state some important properties about it. While in $\mathcal{ALC}$ reasoning is completely determined by entailment, in $\mathcal{PALC}$ the situation is different. We define the notion of j-admissibility which will be the dual to j-entailment. We will note that in $\mathcal{PALC}$, j-entailment gives rather blunt results whereas j-admissibility allows us to relate the probabilities of different assertions. It also conforms with the way we expect reasoning with probabilities to work.

Definition 45 (j-Admissible)**.** An assertion ρ is *j-admissible for an ABox $\mathcal{A}$* - in symbols $\mathcal{A} \underset{j}{\approx} \rho$ - iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I}_j \models \mathcal{A}$ and $\mathcal{I}_j \models \rho$.

Definition 46 (Negation)**.** Let ρ be a probabilistic assertion of the form $\mathrm{P}(C(a)) = p$. We call $\overline{\rho}$ the *negation* of ρ and define $\mathcal{I} \models \overline{\rho}$ iff $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) \neq p^{\mathcal{I}}$. The negation $\overline{\sigma}$ of a role assertion σ is defined correspondingly.

Definition 47. Let J be the set of all valuations j with $\mathrm{ran}(j) \subseteq [0,1]$, $\mathcal{A}$ an ABox and ρ an assertion. We define the sets

$$P_{\mathcal{A},\rho} = \{j \in J | \mathcal{A} \underset{j}{\approx} \rho\} \text{ and } N_{\mathcal{A},\rho} = \{j \in J | \mathcal{A} \models_j \rho\}.$$

Remark 48. Intuitively $P_{\mathcal{A},\rho} = \{j \in J | \mathcal{A} \underset{j}{\approx} \rho\}$ corresponds to the set of valuations which are possible for ρ given $\mathcal{A}$ whereas $N_{\mathcal{A},\rho} = \{j \in J | \mathcal{A} \models_j \rho\}$ corresponds to the set of valuations which are necessary for ρ given $\mathcal{A}$.

Example 49. Let $\rho = \mathrm{P}(C(a)) \doteq p_0$. Then

$$N_{\emptyset,\rho} = \{j \in J | \models_j \rho\} = \begin{cases} \{j \in J | j(p_0) = 1\} & \text{if } C \equiv \top, \\ \{j \in J | j(p_0) = 0\} & \text{if } C \equiv \bot, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$P_{\emptyset,\rho} = \{j \in J | \underset{j}{\approx} \rho\} = \begin{cases} \{j \in J | j(p_0) = 1\} & \text{if } C \equiv \top, \\ \{j \in J | j(p_0) = 0\} & \text{if } C \equiv \bot, \\ \{j \in J | 0 \leq j(p_0) \leq 1\} & \text{otherwise.} \end{cases}$$

$P_{\mathcal{A},\rho}$ is dual to $N_{\mathcal{A},\rho}$ as the following lemma shows:

Lemma 50. *Let $\mathcal{A}$ be an ABox and ρ an assertion. Then*

$$P_{\mathcal{A},\rho} = J \setminus N_{\mathcal{A},\overline{\rho}} \text{ and } N_{\mathcal{A},\rho} = J \setminus P_{\mathcal{A},\overline{\rho}}.$$

Proof. $J \setminus N_{\mathcal{A},\overline{\rho}} = \{j \in J | \mathcal{A} \not\models_j \overline{\rho}\} = \{j \in J | \mathcal{A} \mathrel{|\!\approx_j} \rho\} = P_{\mathcal{A},\rho}$. The proof for the second equality is similar. □

Example 51. Let $\mathcal{A} = \{\mathrm{P}(C(a)) \doteq p_1\}$ and $\rho = \mathrm{P}(\top(a)) \doteq p_0$ with neither $C \not\equiv \bot$ nor $C \not\equiv \top$. Assume $\mathcal{I}_j \models \mathcal{A}$ and $\mathcal{I}_j \models \overline{\rho}$. Then $0 \leq j(p_1) \leq 1$ and $j(p_0) < 1$ and thus $P_{\mathcal{A},\overline{\rho}} = \{j \in J | 0 \leq j(p_1) \leq 1, j(p_0) < 1\}$. Hence $N_{\mathcal{A},\rho} = J \setminus P_{\mathcal{A},\overline{\rho}} = \{j \in J | j(p_0) = 1\}$.

Definition 52 ($\mathcal{PALC}$ reasoning problem)**.** Let $\mathcal{A}$ be an ABox, ρ be the probabilistic assertion $\mathrm{P}(\alpha) \doteq p_0$ where p_0 does not occur in $\mathcal{A}$, and j a valuation.

i) The triple $\langle \mathcal{A}, \rho, j \rangle$ is a called a $\mathcal{PALC}$ *reasoning problem.*

ii) A real number $s \in \mathbb{R}$ is called a *solution* to the $\mathcal{PALC}$ reasoning problem $\langle \mathcal{A}, \rho, j \rangle$ iff $\mathcal{A} \mathrel{|\!\approx_{j[p_0=s]}} \rho$ where

$$j[p = s](x) = \begin{cases} j(x) & \text{if } x \neq p, \\ s & \text{otherwise.} \end{cases}$$

iii) The set $S_{\langle \mathcal{A},\rho,j \rangle} = \{s \in \mathbb{R} | \mathcal{A} \mathrel{|\!\approx_{j[p_0=s]}} \rho\}$ is called the *set of solutions* to the $\mathcal{PALC}$ reasoning problem $\langle \mathcal{A}, \rho, j \rangle$.

Entailment and j-admissibility in $\mathcal{PALC}$ can be reduced to finding the set of solutions of a $\mathcal{PALC}$ reasoning problem $\langle \mathcal{A}, \rho, j \rangle$. This is because the set of solutions $S_{\langle \mathcal{A},\rho,j \rangle}$ coincides with the set of valuations in a specific subset of $P_{\mathcal{A},\rho}$ evaluated at p_0 as the following lemma shows:

Lemma 53. *Let*

$$P^j_{\mathcal{A},\rho} = \{j'(p_0) \in \mathbb{R} | j' \in P_{\mathcal{A},\rho} \text{ and } j'(p) = j(p) \text{ for all } p \neq p_0\}$$

be the restriction of $P_{\mathcal{A},\rho}$ to j evaluated at p_0. Then

$$S_{\langle\mathcal{A},\rho,j\rangle} = P^j_{\mathcal{A},\rho}.$$

Proof. We prove each inclusion separately.

($\subseteq$) Let $s' \in S_{\langle\mathcal{A},\rho,j\rangle}$. Then $\mathcal{A} \models\!\!\!\!\approx_{j[p_0=s']} \rho$ and thus $j[p_0 = s'] \in P_{\mathcal{A},\rho}$. Because $j[p_0 = s'](p) = j(p)$ for all $p \neq p_0$ we conclude $j[p_0 = s'](p_0) = s' \in P^j_{\mathcal{A},\rho}$.

($\supseteq$) Let $s' \in P^j_{\mathcal{A},\rho}$. Then there is a valuation $j' \in P_{\mathcal{A},\rho}$ such that $j'(p_0) = s'$ and $j'(p) = j(p)$ for all $p \neq p'_0$. Thus $j' = j[p_0 = s']$ and hence $s' \in \{s \in \mathbb{R} | \mathcal{A} \models\!\!\!\!\approx_{j[p_0=s]} \rho\}$.

□

Example 54. Consider the ABox

$$\mathcal{A} = \{\mathrm{P}(C(b)) \doteq 1, \mathrm{P}(C(c)) \doteq 1, \mathrm{P}(R(a,b)) \doteq \mathbf{y}, \mathrm{P}(\forall R.\neg C(a)) \doteq \mathbf{x}\}$$

and $\rho = \mathrm{P}(R(a,c)) \doteq p_0$. Assume we are given an $\mathcal{I}_j$ such that $\mathcal{I}_j \models \mathcal{A}$. Then $\mathrm{P}(b^{\mathcal{I}} \in (\neg C)^{\mathcal{I}}) = \mathrm{P}(c^{\mathcal{I}} \in (\neg C)^{\mathcal{I}}) = 0$. Together with Condition ($\forall$) we find $x + \delta = (1-y)(1-p_0^{\mathcal{I}})$ with a parameter $\delta \geq 0$. For $y = 1$ this yields the solutions $0 \leq p_0^{\mathcal{I}} \leq 1$ if $x = 0$ and no solution otherwise. For $y < 1$ we find $p_0^{\mathcal{I}} = 1 - \frac{x+\delta}{1-y}$. Therefore

$$S_{\langle\mathcal{A},\rho,j\rangle} = \begin{cases} [0, 1 - \min\{1, \frac{x}{1-y}\}] & \text{if } y < 1, \\ [0,1] & \text{if } y = 1,\ x = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

This example demonstrates that Condition ($\forall$) generally poses non-linear constraints on the possible interpretations of the probabilistic constants. A simple linear reasoning approach will therefore not suffice for reasoning in $\mathcal{PALC}$. Although we have non-linear constraints on the solutions of $\mathcal{PALC}$ reasoning problems, we conjecture that the set of solutions forms a closed interval.

Conjecture 55. *The set of solutions $S_{\langle \mathcal{A},\rho,j\rangle}$ of a $\mathcal{PALC}$ reasoning problem is a closed interval. That is, there exist $p_l, p_u \in \mathbb{R}$ such that $S_{\langle \mathcal{A},\rho,j\rangle} = [p_l, p_u]$.*

Building on this, we formulate the reasoning task for $\mathcal{PALC}$ as follows:

Definition 56. Let $\langle \mathcal{A}, \rho, j\rangle$ be a $\mathcal{PALC}$ reasoning problem. The $\mathcal{PALC}$ *reasoning task* consists of finding $p_l, p_u \in \mathbb{R}$ such that $\mathcal{A} \underset{j[p_0=s]}{\approx} \rho$ iff $s \in [p_l, p_u]$.

3.2 Finding a Basis

In order to further characterize the $\mathcal{PALC}$ reasoning problem $\langle \mathcal{A}, \rho, j\rangle$ we identify conditions which must hold for any valuation j in $P_{\mathcal{A},\rho}$. The following lemma shows how determining $P_{\mathcal{A},\rho}$ can be transformed into the problem of finding all possible interpretations of the ABox $\mathcal{A} \cup \{\rho\}$.

Lemma 57. *Let $\mathcal{A}$ be an ABox and ρ an assertion. Then*

$$P_{\mathcal{A},\rho} = \{j \in J | \exists \mathcal{I} : \mathcal{I}_j \models \mathcal{A} \cup \{\rho\}\}$$

Proof. $j \in P_{\mathcal{A},\rho}$ iff $\mathcal{A} \underset{j}{\approx} \rho$ iff (by Definition 45) $\mathcal{I}_j \models \mathcal{A} \cup \{\rho\}$ for some interpretation $\mathcal{I}$. □

Assuming $\mathcal{I}_j \models \mathcal{A}$ for an ABox $\mathcal{A}$ we concentrate on the conditions which must be satisfied by the valuation j. We therefore introduce the mapping $\sigma_{\mathcal{I}}$ which maps ABox assertions to its corresponding elements of the σ-algebra F generated by the random variables of $\mathcal{I}$. By finding a basis for theses elements, we are able to define a set of linear constraints which probability distributions over F must obey. Via the mapping $\sigma_{\mathcal{I}}$ these constraints carry directly over to the valuation j. In later sections we will examine other constraints which are implied by our independence assumption and by Condition ($\forall$).

Definition 58. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ with the probability space $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$ be an interpretation and let $\mathfrak{A}$ denote the set of classical assertions of a given language $\mathcal{PALC}$. (i.e. $\mathfrak{A} = \{C(a)|C \in \text{cns}(\mathcal{PALC}), a \in \text{ind}(\mathcal{PALC})\}$). We define the mapping

$$\sigma_{\mathcal{I}} : \mathfrak{A} \rightarrow F,\ \sigma_{\mathcal{I}}(C(a)) = \{a^{\mathcal{I}} \in C^{\mathcal{I}}\}.$$

Lemma 59. *Let $\mathcal{A}$ be an ABox with $\mathrm{P}(\alpha) \doteq p_0 \in \mathcal{A}$ and $\mathcal{I}_j \models \mathcal{A}$. Then*

$$\mathrm{P}(\sigma_{\mathcal{I}}(\alpha)) = j(p_0).$$

Proof. Assume α is the concept assertion $C(a)$. Then $\mathcal{I}_j \models \mathrm{P}(\alpha) \doteq p_0$ iff $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = p_0^{\mathcal{I}} = j(p_0) = \mathrm{P}(\sigma_{\mathcal{I}}(C(a)))$. The proof for the case where α is a role assertions is similar. □

Using the mapping $\sigma_{\mathcal{I}}$ we can transform the problem of finding a valuation of the probabilistic constants of a given $\mathcal{PALC}$ ABox into the problem of finding a probability distribution on a σ-algebra F. We do so by identifying a subset of F - a basis - which elements are pairwise disjoint and which span the image of the elements of an ABox $\mathcal{A}$ under $\sigma_{\mathcal{I}}$. The probabilities of the elements of the ABox are then uniquely determined by the probabilities of the elements of its basis.

Definition 60 (Basis)**.** Let F be a σ-algebra over the set Ω and $\mathcal{E} \subseteq F$. The set $\mathcal{B} \subseteq F$ is called a *basis* for $\mathcal{E}$ iff

i) The elements of $\mathcal{B}$ are pairwise disjoint: $B \cap B' = \emptyset$ for all $B, B' \in \mathcal{B}$ with $B \neq B'$.

ii) The elements of $\mathcal{B}$ span the set $\mathcal{E}$: For each $E \in \mathcal{E}$ there exists $\mathcal{B}_E \subseteq \mathcal{B}$ such that $\bigcup_{B \in \mathcal{B}_E} B = \mathcal{E}$.

iii) $\mathcal{B}$ is exhaustive: $\bigcup_{B \in \mathcal{B}} B = \Omega$.

The following lemma shows that a basis $\mathcal{B}$ for $\mathcal{E}$ always exists if $\mathcal{E}$ is finite.

Lemma 61. *Let F be a σ-algebra over the set Ω and $\mathcal{E} \subseteq F$ finite. Then*

$$\mathcal{B} = \{\bigcap_{E \in \mathcal{E}} E' | E' \in \{E, \Omega \setminus E\}\}$$

is a basis for $\mathcal{E}$.

Proof. First note that $\mathcal{B} \subseteq F$ since F is closed under negation and countable union. Further $\mathcal{B}$ is a basis for $\mathcal{E}$ since

i) Let $B, B' \in \mathcal{B}$ with $B \neq B'$. Then there is a $E \in \mathcal{E}$ such that either $B \subseteq E$ and $B' \subseteq \Omega \setminus E$ or $B \subseteq \Omega \setminus E$ and $B' \subseteq E$. Thus $B \cap B' = \emptyset$.

ii) Let $E \in \mathcal{E}$ and $\mathcal{B}_E = \{B \in \mathcal{B} | B \subseteq E\}$. Then $E = \bigcup_{B \in \mathcal{B}_E} B$.

iii) Let $E \in \mathcal{E}$ and $\mathcal{B}_{\Omega \setminus E} = \{B \in \mathcal{B} | B \subseteq \Omega \setminus E\}$. Then $\Omega \setminus E = \bigcup_{B \in \mathcal{B}_{\Omega \setminus E}} B$. With $\mathcal{B}_E$ as above we find $\mathcal{B}_E \cup \mathcal{B}_{\Omega \setminus E} = \mathcal{B}$ and thus $\bigcup_{B \in \mathcal{B}} B = \Omega$.

□

Definition 62. For an ABox $\mathcal{A}$ we define the image of $\mathcal{A}$ under the mapping $\sigma_{\mathcal{I}}$ as

$$\sigma_{\mathcal{I}}(\mathcal{A}) = \{\sigma_{\mathcal{I}}(\alpha) | \, \mathrm{P}(\alpha) \doteq p_0 \in \mathcal{A}\}.$$

Lemma 63. *Let $\mathcal{A}$ be an ABox with $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ with the probability space $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$. Then*

i) There exists a basis $\mathcal{B} \subset F$ for $\sigma_{\mathcal{I}}(\mathcal{A})$.

ii) For each assertion $\mathrm{P}(\alpha) \doteq p_0 \in \mathcal{A}$ there exists $\mathcal{B}_E \subseteq \mathcal{B}$ such that $\sum_{B \in \mathcal{B}_E} \mathrm{P}(B) = j(p_0)$.

iii) $\sum_{B \in \mathcal{B}} \mathrm{P}(B) = 1$.

Proof.

i) $\mathcal{B} \subseteq F$ exists according to Lemma 61.

ii) Since $\mathcal{B}$ is a basis for $\sigma_{\mathcal{I}}$, there exists $\mathcal{B}_E \subseteq \mathcal{B}$ such that $\bigcup_{B \in \mathcal{B}_E} B = \sigma_{\mathcal{I}}(\alpha)$. The elements of $\mathcal{B}$ are pairwise disjoint. Hence $\mathrm{P}(\bigcup_{B \in \mathcal{B}_E} B) = \sum_{B \in \mathcal{B}_E} \mathrm{P}(B)$. Applying Lemma 59 finally $\mathrm{P}(\sigma_{\mathcal{I}}(\alpha)) = \sum_{B \in \mathcal{B}_E} \mathrm{P}(B) = j(p_0)$.

iii) Since $\mathcal{B}$ is exhaustive and its members are pairwise disjoint we find $\sum_{B\in\mathcal{B}} \mathrm{P}(B) = \mathrm{P}(\bigcup_{B\in\mathcal{B}} B) = \mathrm{P}(\Omega) = 1$.

$\square$

The following lemma establishes a connection between classic ABox satisfiability and the elements of a basis $\mathcal{B}$ for $\sigma_{\mathcal{I}}(\mathcal{A})$.

Lemma 64. *Let $\mathcal{A}$ be an $\mathcal{ALC}$ ABox. Then $\mathcal{A}$ is satisfiable iff there is an probabilistic interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega})$ with the probability space $\mathbf{\Omega} = (\Omega, F, \mathrm{P})$ such that*

$$\bigcap_{\alpha\in\mathcal{A}} \sigma_{\mathcal{I}}(\alpha) \neq \emptyset.$$

Proof. As a consequence to Theorem 42 $\mathcal{A}$ is satisfiable iff its probabilistic counterpart $\mathcal{A}_p = \{\mathrm{P}(\alpha) \doteq 1 | \alpha \in \mathcal{A}\}$ is j-satisfiable for any valuation j. Let $\mathcal{I} \models \mathcal{A}_p$ for some $\mathcal{I}$. Then $\mathrm{P}(a^{\mathcal{I}} \in C^{\mathcal{I}}) = \mathrm{P}(\sigma_{\mathcal{I}}(C(a))) = 1$ for all concept assertions $C(a) \in \mathcal{A}$ and $\mathrm{P}((a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}) = \mathrm{P}(\sigma_{\mathcal{I}}(R(a,b))) = 1$ for all role assertions $R(a,b) \in \mathcal{A}$. Hence $\mathrm{P}(\bigcap_{\alpha\in\mathcal{A}} \sigma_{\mathcal{I}}(\alpha)) = 1$ and thus $\bigcap_{\alpha\in\mathcal{A}} \sigma_{\mathcal{I}}(\alpha) \neq \emptyset$.

For the other implication assume there exists a probabilistic interpretation $\mathcal{I}$ such that $\bigcap_{\alpha\in\mathcal{A}} \sigma_{\mathcal{I}}(\alpha) \neq \emptyset$. Thus

$$\begin{aligned}\bigcap_{\alpha\in\mathcal{A}} \sigma_{\mathcal{I}}(\alpha) &= \bigcap_{C(a)\in\mathcal{A}} \{a^{\mathcal{I}} \in C^{\mathcal{I}}\} \cap \bigcap_{R(b,c)\in\mathcal{A}} \{(b^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}}\} \\ &= \{s \in \Omega | \bigwedge_{C(a)\in\mathcal{A}} X_{a^{\mathcal{I}}}(s) \in C^{\mathcal{I}} \wedge \bigwedge_{R(b,c)\in\mathcal{A}} X_{b^{\mathcal{I}}c^{\mathcal{I}}}(s) \in R^{\mathcal{I}}\} \neq \emptyset\end{aligned}$$

and therefore for all $C(a) \in \mathcal{A}$ and all $R(b,c) \in \mathcal{A}$ we find $X_{a^{\mathcal{I}}}(s) \in C^{\mathcal{I}}$ and $X_{b^{\mathcal{I}}c^{\mathcal{I}}}(s) \in R^{\mathcal{I}}$. From this we conclude that $\mathcal{A}$ is satisfiable. $\square$

We can bring these results into a more compact form which basically resembles the approach in [Nil86]. We therefore introduce a canonical representation of a basis for an ABox.

Definition 65. Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \ldots, \mathrm{P}(\alpha_n) \doteq p_n\}$ be an ABox with $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$. Let further $b(n,k) \in \{0,1\}$ denote the k-th digit from the right in the binary expansion of $n \in \mathbb{N}$ (i.e. $\sum_{k=1}^{\infty} b(n,k)2^{k-1} = n$).

We represent the elements B_i for $0 \leq i \leq 2^n - 1$ of the basis $\mathcal{B}$ of $\sigma_{\mathcal{I}}(\mathcal{A})$ as

$$B_i = \bigcap_{k=1}^{n} \sigma_{\mathcal{I}}(\alpha'_k) \text{ with } \alpha'_k = \begin{cases} \alpha_k & \text{if } b(i,k) = 1, \\ \neg\alpha_k & \text{otherwise.} \end{cases}$$

Similarly we define the $\mathcal{ALC}$ ABox $\mathcal{A}_i$ for each $0 \leq i \leq 2^n - 1$ as

$$\mathcal{A}_i = \{\alpha'_k | 1 \leq k \leq n,\ \alpha'_k = \begin{cases} \alpha_k & \text{if } b(i,k) = 1, \\ \neg\alpha_k & \text{otherwise.} \end{cases}\}.$$

Finally we define the $(2^n, n+1)$-matrix

$$\mathbf{B} = \begin{pmatrix} b(0,1) & \dots & b(0,n) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ b(2^n-1,1) & \dots & b(2^n-1,n) & 1 \end{pmatrix}.$$

Remark 66. $\mathcal{B} = \{B_i \in F | 0 \leq i \leq 2^n - 1\}$ is a basis for $\sigma_{\mathcal{I}}(\mathcal{A})$ by Lemma 61.

Theorem 67. *Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \dots, \mathrm{P}(\alpha_n) \doteq p_n\}$ be an ABox with $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$ and $\mathcal{B}$ a basis for $\sigma_{\mathcal{I}}(\mathcal{A})$ as in Definition 65. Then*

i) $(\mathrm{P}(B_0),\ \dots,\ \mathrm{P}(B_{2^n-1}))\,\mathbf{B} = (j(p_1),\ \dots,\ j(p_n),\ 1)$.

ii) $\mathrm{P}(B_i) = 0$ *iff* $\mathcal{A}_i$ *is not satisfiable* $(0 \leq i \leq 2^n - 1)$.

Proof.

i) For $1 \leq k \leq n$ we have

$$\sum_{i=0}^{2^n-1} \mathrm{P}(B_i) b(i,k) = \sum_{\substack{B \in \mathcal{B} \\ B \subseteq \sigma_{\mathcal{I}}(\alpha_k)}} \mathrm{P}(B) = j(p_k)$$

where the last equality holds because of Lemma 63 Part ii). Further we have

$$\sum_{i=0}^{2^n-1} \mathrm{P}(B_i) = \sum_{B' \in B} \mathrm{P}(B') = 1$$

where the last equality holds by Lemma 63 Part iii).

ii) With Lemma 64 we find $B_i = \emptyset$ iff $\mathcal{A}_i$ is not satisfiable thus proving the claim.

□

Example 68. Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \ldots, \mathrm{P}(\alpha_4) \doteq p_4\}$ be an ABox with $\alpha_1 = \forall R.C \sqcap D(a)$, $\alpha_2 = R(a,b)$, $\alpha_3 = \neg C \sqcap D(b)$, $\alpha_4 = C \sqcap \neg D(b)$ and let $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$.

With Theorem 67 Part i) we find

$$(\mathrm{P}(B_0),\ \ldots,\ \mathrm{P}(B_{15}))\,\mathbf{B} = (j(p_1),\ \ldots,\ j(p_4),\ 1)$$

with

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ & & \ldots\ldots\ldots & & \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since for example the ABox $\mathcal{A}_{12} = \{\alpha_1, \alpha_2, \neg\alpha_3, \neg\alpha_4\}$ is not satisfiable, applying Theorem 67 Part ii) we find $\mathrm{P}(B_{12}) = 0$. Checking satisfiability for all ABoxes $\mathcal{A}_i$ $(0 \leq i \leq 15)$, we find that only A_1, A_2, A_5, A_6, A_9 and A_{10} are satisfiable. We can thus eliminate all but the corresponding rows from the matrix $\mathbf{B}$ which leaves us with the matrix

$$\mathbf{B}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and

$$(\mathrm{P}(B_1),\ \mathrm{P}(B_2),\ \mathrm{P}(B_5),\ \mathrm{P}(B_6),\ \mathrm{P}(B_9),\ \mathrm{P}(B_{10}))\,\mathbf{B}' = (j(p_1),\ \ldots,\ j(p_4),\ 1).$$

3.3 Constraints Implied by Independence

So far we left out any implication of our assumption that the random variables of a probabilistic interpretation are independent. The following example shows that there indeed are more constraints arising from this assumption.

Example 69. Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \mathrm{P}(\alpha_2) \doteq p_2\}$ be an ABox with $\alpha_1 = \forall R.\bot(a)$, $\alpha_2 = R(a,b)$ and let $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$.

With Theorem 67 Part i) we find

$$(\mathrm{P}(B_0),\ \ldots,\ \mathrm{P}(B_3))\,\mathbf{B} = (j(p_1),\, j(p_2),\, 1)$$

with

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The ABox $\mathcal{A}_3 = \{\alpha_1, \alpha_2\}$ is not satisfiable and thus $P(B_3) = 0$ which enables us to infer $j(p_1) + j(p_2) \leq 1$. However since $\mathcal{A}_3$ is not satisfiable we have by Lemma 64 that $\sigma_{\mathcal{I}}(\alpha_1) \cap \sigma_{\mathcal{I}}(\alpha_2) = \emptyset$ for every interpretation $\mathcal{I}$. Because the events $\sigma_{\mathcal{I}}(\alpha_1)$ and $\sigma_{\mathcal{I}}(\alpha_2)$ are independent we find $\mathrm{P}(\sigma_{\mathcal{I}}(\alpha_1))\,\mathrm{P}(\sigma_{\mathcal{I}}(\alpha_2)) = 0$.

Which by applying Lemma 59 yields the additional constraint $j(p_1)j(p_2) = 0$ on the valuation j. Thus $j(p_1) = 1 \implies j(p_0) = 0$ and $j(p_0) = 1 \implies j(p_1) = 0$ which is in analogy to the classical case where $\mathcal{I} \models \forall R.\bot(a) \implies \mathcal{I} \not\models R(a,b)$ and $\mathcal{I} \not\models R(a,b) \implies \mathcal{I} \models \forall R.\bot(a)$ for an interpretation $\mathcal{I}$.

Theorem 70. *Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \ldots, \mathrm{P}(\alpha_n) \doteq p_n\}$ with $\mathcal{I}_j \models \mathcal{A}$. Let further $\mathcal{A}' \subseteq \mathcal{A}_{2^n-1}$ such that $\{C(a), D(b)\} \subseteq \mathcal{A}' \implies a \neq b$. Then $\mathcal{A}'$ not satisfiable implies*

$$\prod_{\mathrm{P}(\alpha)=p_i \in \mathcal{A}'} j(p_i) = 0.$$

Proof. If $\mathcal{A}'$ is not satisfiable then by Lemma 64 $\bigcap_{\alpha \in \mathcal{A}'} \sigma_{\mathcal{I}}(\alpha) = \emptyset$ for any interpretation $\mathcal{I}$. Using independence and applying Lemma 59 we find

$$\mathrm{P}(\bigcap_{\alpha \in \mathcal{A}'} \sigma_{\mathcal{I}}(\alpha)) = \prod_{\alpha \in \mathcal{A}'} \mathrm{P}(\sigma_{\mathcal{I}}(\alpha)) = \prod_{\mathrm{P}(\alpha)=p_i \in \mathcal{A}'} j(p_i) = 0.$$

$\square$

For an ABox $\mathcal{A}$ where no proper subset $\mathcal{A}' \subset \mathcal{A}$ is not satisfiable we might still be able to apply Theorem 70 by first expanding $\mathcal{A}$ in a specific way.

Lemma 71. *Let $\mathcal{A}$ be a probabilistic ABox with $\{\mathrm{P}(C(a)) \doteq p_0, \mathrm{P}(D(a)) \doteq p_1\} \subseteq \mathcal{A}$ and let j, j' be valuations such that $j(p) = j'(p)$ for all probabilistic constants p occurring in $\mathcal{A}$. Then $\mathcal{A}$ is j-satisfiable iff $\mathcal{A} \cup \{\mathrm{P}(C \sqcap D(a)) \doteq p_3\}$ is j'-satisfiable where p_3 is a new probabilistic constant not occurring in $\mathcal{A}$.*

Proof. Let $\mathcal{I}_{j'} \models \mathcal{A} \cup \{\mathrm{P}(C \sqcap D(a)) \doteq p_3\}$. Then $\mathcal{I}_j \models \mathcal{A}$. For the other implication let $\mathcal{I}_j \models \mathcal{A}$. Let $\mathcal{J}$ be the interpretation which agrees on $\mathcal{I}$ except for $p_3^{\mathcal{J}} = \mathrm{P}(a^{\mathcal{I}} \in (C \sqcap D)^{\mathcal{I}})$ and let $j'(p_3) = p_3^{\mathcal{J}}$. Then $\mathcal{J}_{j'} \models \mathcal{A} \cup \{\mathrm{P}(C \sqcap D(a)) \doteq p_3\}$. $\square$

Example 72. Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \ldots, \mathrm{P}(\alpha_3) \doteq p_3\}$ be an ABox with $\alpha_1 = \forall R.C(a)$, $\alpha_2 = \forall R.\neg C(a)$, $\alpha_3 = R(a,b)$, and let $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$. While $\mathcal{A}_3 = \{\alpha_1, \ldots, \alpha_3\}$ is not satisfiable, non of its proper subsets is not satisfiable so Theorem 70 cannot be applied at first. However with

Lemma 71 we find $\mathcal{A}$ is j-satisfiable for some j iff $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(\alpha_4) \doteq p_4\}$ with $\alpha_4 = (\forall R.C \sqcap \forall R.\neg C)(a)$ is j'-satisfiable. Now $\{\alpha_3, \alpha_4\}$ is not satisfiable which, by applying Theorem 70, yields the constraint $j(p_3)j(p_4) = 0$ on the valuation j'.

3.4 Constraints Implied by Condition $(\forall)$

There are still more constraints on valuations j when we consider Condition $(\forall)$ which we left out so far.

Lemma 73. *Let $\mathcal{A}$ be an ABox with $\mathrm{P}(\forall R.C(a)) = p_0 \in \mathcal{A}$ and for $1 \leq i \leq n$ for some $n \in \mathbb{N}$ let $\{\mathrm{P}(R(a, b_i) = p_i, \mathrm{P}(C(b_i) = q_i\} \subseteq \mathcal{A}$. Let further $\mathcal{I}_j \models \mathcal{A}$. Then*

$$j(p_0) \leq \prod_{i=1}^{n} 1 - j(p_i)(1 - j(q_i)).$$

Proof. By Theorem 27

$$\begin{aligned}
j(p_0) = {p_0}^{\mathcal{I}} &= \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})(1 - \mathrm{P}(s \in C^{\mathcal{I}})) \\
&\leq \prod_{i=1}^{n} 1 - \mathrm{P}((a^{\mathcal{I}}, b_i^{\mathcal{I}}) \in R^{\mathcal{I}})(1 - \mathrm{P}(b_i^{\mathcal{I}} \in C^{\mathcal{I}})) \\
&= \prod_{i=1}^{n} 1 - j(p_i)(1 - j(q_i)).
\end{aligned}$$

□

Lemma 74. *Let $\mathcal{A}$ be an ABox with $\mathrm{P}(\exists R.C(a)) = p_0 \in \mathcal{A}$ and for $1 \leq i \leq n$ for some $n \in \mathbb{N}$ let $\{\mathrm{P}(R(a, b_i) = p_i, \mathrm{P}(C(b_i) = q_i\} \subseteq \mathcal{A}$. Let further $\mathcal{I}_j \models \mathcal{A}$. Then*

$$j(p_0) \geq 1 - \prod_{i=1}^{n} 1 - j(p_i)j(q_i).$$

Proof. By Corollary 36

$$\begin{aligned} j(p_0) = {p_0}^{\mathcal{I}} &= 1 - \prod_{s \in \Delta^{\mathcal{I}}} 1 - \mathrm{P}((a^{\mathcal{I}}, s) \in R^{\mathcal{I}})\, \mathrm{P}(s \in C^{\mathcal{I}}) \\ &\geq 1 - \prod_{i=1}^{n} 1 - \mathrm{P}((a^{\mathcal{I}}, b_i^{\mathcal{I}}) \in R^{\mathcal{I}})\, \mathrm{P}(b_i^{\mathcal{I}} \in C^{\mathcal{I}}) \\ &= 1 - \prod_{i=1}^{n} 1 - j(p_i) j(q_i). \end{aligned}$$

□

When ABoxes contain nested value restrictions or nested existential restrictions (for example $\forall R.\exists C(a)$) the above lemmas cannot be readily applied since there is no probabilistic constant for the inner formula. In order to account for constraints arising from such situations we have to expand ABoxes to include witnesses for its inner assertions. Before doing so however, we need a pumping lemma for $\mathcal{PALC}$ to account for additional individual constants introduced along with the witnesses.

Lemma 75 (Pumping)**.** *Let $\mathcal{A}$ be a probabilistic ABox with $\mathcal{I}_j \models \mathcal{A}$ for some interpretation $\mathcal{I}$ and some valuation j. Then there exists an interpretation $\mathcal{J}$ with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$, such that for all assertions $\mathrm{P}(\alpha) \doteq p_i \in \mathcal{A}$*

$$\mathcal{I}_j \models \mathrm{P}(\alpha) \doteq p_i \textit{ iff } \mathcal{J}_j \models \mathrm{P}(\alpha) \doteq p_i.$$

Proof. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathbf{\Omega}_{\mathcal{I}})$ with the probability space $\mathbf{\Omega}_{\mathcal{I}} = (\Omega_{\mathcal{I}}, F_{\mathcal{I}}, \mathrm{P}_{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}}, \mathbf{\Omega}_{\mathcal{J}})$ with the probability space $\mathbf{\Omega}_{\mathcal{J}} = (\Omega_{\mathcal{J}}, F_{\mathcal{J}}, \mathrm{P}_{\mathcal{J}})$. Set $\cdot^{\mathcal{J}} = \cdot^{\mathcal{I}}$, for any individual constant a. For any concept C set $\mathrm{P}_{\mathcal{J}}(a^{\mathcal{J}} \in C^{\mathcal{J}}) = \mathrm{P}_{\mathcal{I}}(a^{\mathcal{I}} \in C^{\mathcal{I}})$ and for any individual constants b, c and any role R set $\mathrm{P}_{\mathcal{J}}((b^{\mathcal{J}}, c^{\mathcal{J}}) \in R^{\mathcal{J}}) = \mathrm{P}_{\mathcal{I}}((b^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}})$. Finally set $\mathrm{P}_{\mathcal{J}}(z \in C^{\mathcal{J}}) = 0$ for any concept C. Now it is sufficient to show, that $P_{\mathcal{J}}$ is indeed a probability distribution: Let $r \in \Delta^{\mathcal{J}}$ then

i) Using $C^{\mathcal{J}} \setminus \{z\} = \mathcal{C}^{\mathcal{I}}$ from Lemma 23 we find $\mathrm{P}_{\mathcal{J}}(r \in C^{\mathcal{J}}) = \mathrm{P}_{\mathcal{I}}(r \in C^{\mathcal{J}} \setminus \{z\}) = \mathrm{P}_{\mathcal{I}}(r \in C^{\mathcal{I}}) \geq 0$ for any concept C.

ii) Let C_k be a concept for any k in a countable index set K and let $C_i^{\mathcal{J}} \cap C_k^{\mathcal{J}} = \emptyset$ if $i \neq k$. Using Lemma 23 again, we find

$$\begin{aligned}
\mathrm{P}_{\mathcal{J}}(r \in \bigcup_{k \in K} C_k^{\mathcal{J}}) &= \mathrm{P}_{\mathcal{I}}(r \in \underbrace{(\bigcup_{k \in K} C_k^{\mathcal{J}}) \setminus \{z\}}_{=\bigcup_{k \in K} C_k^{\mathcal{I}}}) \\
&= \mathrm{P}_{\mathcal{I}}(r \in (\bigcup_{k \in K} \underbrace{C_k^{\mathcal{J}} \setminus \{z\}}_{=C_k^{\mathcal{I}}})) \\
&= \sum_{k \in K} \mathrm{P}_{\mathcal{I}}(r \in C_k^{\mathcal{I}}) \\
&= \sum_{k \in K} \mathrm{P}_{\mathcal{J}}(r \in C_k^{\mathcal{J}}).
\end{aligned}$$

iii) $\mathrm{P}_{\mathcal{J}}(r \in \top^{\mathcal{J}}) = \mathrm{P}_{\mathcal{I}}(r \in \top^{\mathcal{I}}) = 1$.

□

We can now proceed by introducing witnesses for certain assertions in an ABox.

Lemma 76. *Let $\mathcal{A}$ be a probabilistic ABox and j, j' valuations such that $j(p) = j'(p)$ for all probabilistic constants p occurring in $\mathcal{A}$. Let further p_2 and p_3 be probabilistic constants which do not occur in $\mathcal{A}$ and c an individual constant with $c \notin \mathrm{ind}(\mathcal{A})$.*

i) If $\mathrm{P}(\forall R.C(a)) \doteq p_0 \in \mathcal{A}$ then the ABox $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(R(a,c)) \doteq p_2, \mathrm{P}(C(c)) \doteq p_3\}$ is j'-satisfiable iff $\mathcal{A}$ is j-satisfiable.

ii) If $\{\mathrm{P}(\forall R.C(a)) \doteq p_0, \mathrm{P}(R(a,b)) \doteq p_1\} \subseteq \mathcal{A}$ then the ABox $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(C(b)) \doteq p_2\}$ is j'-satisfiable iff $\mathcal{A}$ is j-satisfiable.

iii) If $\mathrm{P}(\exists R.C(a)) \doteq p_0 \in \mathcal{A}$ then the ABox $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(R(a,c)) \doteq p_2, \mathrm{P}(C(c)) \doteq p_3\}$ is j'-satisfiable iff $\mathcal{A}$ is j-satisfiable.

iv) If $\{\mathrm{P}(\exists R.C(a)) \doteq p_0, \mathrm{P}(R(a,b)) \doteq p_1\} \subseteq \mathcal{A}$ then the ABox $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(C(b)) \doteq p_2\}$ is j'-satisfiable iff $\mathcal{A}$ is j-satisfiable.

Proof. Let $\mathcal{I}_{j'} \models \mathcal{A}'$ then in any of the four cases $\mathcal{I}_j \models \mathcal{A}$. For the other implication assume $\mathcal{I}_j \models \mathcal{A}$. Applying Lemma 75 we construct a new interpretation $\mathcal{J}$ for each case:

i) Let $\mathcal{J}$ be an interpretation with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$ such that $\mathcal{J}$ agrees on $\mathcal{I}$ except for $c^{\mathcal{J}} = z$, $p_2^{\mathcal{J}} = \mathrm{P}((a^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}})$, and $p_3^{\mathcal{J}} = \mathrm{P}(c^{\mathcal{I}} \in C^{\mathcal{I}})$. Let further $j'(p_2) = p_2^{\mathcal{J}}$ and $j'(p_3) = p_3^{\mathcal{J}}$. Then $\mathcal{J}_{j'} \models \mathcal{A}'$.

ii) Let $\mathcal{J}$ be an interpretation with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$ such that $\mathcal{J}$ agrees on $\mathcal{I}$ except for $c^{\mathcal{J}} = z$ and $p_2^{\mathcal{J}} = \mathrm{P}(b^{\mathcal{I}} \in C^{\mathcal{I}})$. Let further $j'(p_2) = p_2^{\mathcal{J}}$. Then $\mathcal{J}_{j'} \models \mathcal{A}'$.

iii) Let $\mathcal{J}$ be an interpretation with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$ such that $\mathcal{J}$ agrees on $\mathcal{I}$ except for $c^{\mathcal{J}} = z$, $p_2^{\mathcal{J}} = \mathrm{P}((a^{\mathcal{I}}, c^{\mathcal{I}}) \in R^{\mathcal{I}})$ and $p_3^{\mathcal{J}} = \mathrm{P}(c^{\mathcal{I}} \in C^{\mathcal{I}})$. Let further $j'(p_2) = p_2^{\mathcal{J}}$ and $j'(p_3) = p_3^{\mathcal{J}}$. Then $\mathcal{J}_{j'} \models \mathcal{A}'$.

iv) Let $\mathcal{J}$ be an interpretation with $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}} \cup \{z\}$ for some $z \notin \Delta^{\mathcal{I}}$ such that $\mathcal{J}$ agrees on $\mathcal{I}$ except for $c^{\mathcal{J}} = z$ and $p_2^{\mathcal{J}} = \mathrm{P}(b^{\mathcal{I}} \in C^{\mathcal{I}})$. Let further $j'(p_2) = p_2^{\mathcal{J}}$. Then $\mathcal{J}_{j'} \models \mathcal{A}'$.

□

In conclusion to the above we are now in a position to formulate further constraints on valuations:

Definition 77. A probabilistic ABox $\mathcal{A}$ is *witness complete* if

i) For any assertion $\mathrm{P}(\forall R.C(a)) \doteq p_0 \in \mathcal{A}$ there is an individual constant b such that $\mathcal{W} = \{\mathrm{P}(R(a, b)) \doteq p_1, \mathrm{P}(C(b)) \doteq p_2\} \subseteq \mathcal{A}$ and $b \notin \mathrm{ind}(\mathcal{A} \setminus \mathcal{W})$.

ii) For any assertion $\mathrm{P}(\exists R.C(a)) \doteq p_0 \in \mathcal{A}$ there is an individual constant b such that $\mathcal{W} = \{\mathrm{P}(R(a, b)) \doteq p_1, \mathrm{P}(C(b)) \doteq p_2\} \subseteq \mathcal{A}$ and $b \notin \mathrm{ind}(\mathcal{A} \setminus \mathcal{W})$.

Theorem 78. *Let* $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \ldots, \mathrm{P}(\alpha_n) \doteq p_n\}$ *be an ABox which is witness complete and let* $\mathcal{I}_j \models \mathcal{A}$ *for an interpretation* $\mathcal{I}$.

i) If $\{\mathrm{P}(\forall R.C(a)) = p_0, \mathrm{P}(R(a,b_i)) = p_i, \mathrm{P}(C(b_i)) = q_i\} \subseteq \mathcal{A}$ *for* $1 \leq i \leq n$ *and some* $n \in \mathbb{N}$ *then*

$$j(p_0) \leq \prod_{i=1}^{n} 1 - j(p_i)(1 - j(q_i)).$$

ii) If $\{\mathrm{P}(\exists R.C(a)) = p_0, \mathrm{P}(R(a,b_i)) = p_i, \mathrm{P}(C(b_i)) = q_i\} \subseteq \mathcal{A}$ *for* $1 \leq i \leq n$ *and some* $n \in \mathbb{N}$ *then*

$$j(p_0) \geq 1 - \prod_{i=1}^{n} 1 - j(p_i)j(q_i).$$

Proof. The theorem follows directly from Lemma 73 and Lemma 74. □

Example 79. Let $\mathcal{A} = \{\mathrm{P}(\alpha_1) \doteq p_1, \mathrm{P}(\alpha_2) \doteq p_2\}$ be an ABox with $\alpha_1 = \forall R.\exists S.\bot(a)$, $\alpha_2 = R(a,b)$ and let $\mathcal{I}_j \models \mathcal{A}$ for an interpretation $\mathcal{I}$. Then according to Lemma 76 we can introduce witnesses such that the ABox $\mathcal{A}' = \mathcal{A} \cup \{\mathrm{P}(\alpha_3) \doteq p_3, \ldots, \mathrm{P}(\alpha_5) \doteq p_5\}$ with $\alpha_3 = \exists S.\bot(b)$, $\alpha_4 = S(b,c)$ and $\alpha_5 = \bot(c)$ is j'-satisfiable for a valuation j' with $j'(p_i) = j(p_i)$, $(i \in \{1,2\})$. Applying Theorem 78 we then find $j'(p_1) \leq 1 - j'(p_2)(1 - j(p_3'))$ and $j'(p_3) \leq j'(p_4)j'(p_5)$. Using $j'(p_5) = 0$ we finally find $j'(p_1) + j'(p_2) \leq 1$.

Chapter 4

Conclusion and Outlook

With $\mathcal{PALC}$ we presented a novel description logic framework to deal with ABoxes containing probabilistic assertions. We introduced syntax and semantics for $\mathcal{PALC}$ and we have defined the corresponding reasoning problem. Previous approaches to probabilistic assertions only considered probabilistic concept assertions. In contrast, $\mathcal{PALC}$ also allows for probabilistic role assertions. Further, the identification of some specific sets of independent assertions yields additional constraints on the interpretation of probabilistic constants. We have shown that $\mathcal{PALC}$ is equivalent to $\mathcal{ALC}$ if we only allow the probability 1. Therefore $\mathcal{PALC}$ is sound and complete with respect to $\mathcal{ALC}$.

We investigated the structure of solutions to a $\mathcal{PALC}$ reasoning problem and identified constraints which are imposed on such solutions. The linear constraints of Theorem 67 originate directly in the axioms of probability. All candidates for which these constraints hold are solutions of a linear optimization problem as described in [Nil86] and also mentioned in [Jae94, BCM$^+$03, BKW03]. The space of such candidate solutions will therefore be convex. The additional constraints of Theorem 70 are non linear. They have their cause in the independence of the random variables of probabilistic interpretations. Finally taking Condition (∀) into consideration, we identified constraints on the interpretations of value restrictions and existential restrictions in Theorem 78. Again these constraints are non linear.

As yet it is an open question whether the constraints identified in this work fully characterize the solution space of a $\mathcal{PALC}$ reasoning problem. Specifically it is an open question whether the solution space is convex or even connected as was conjectured in Conjecture 55.

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