Pseudo-Hierarchies in Admissible Set Theory without Foundation and Explicit Mathematics

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Introduction

Pseudo-hierarchies have become a powerful tool in several areas of mathematical logic. They were first applied in the context of hyperarithmetical theory by Spector [41], Gandy [16] and Feferman and Spector [13]. Especially in second order arithmetic, the potent and flexible technique of pseudo-hierarchy arguments seems nowadays virtually indispensable. A typical application for specific fixed point definitions is given in Avigad [2], and a rich fund of important results obtained by working with pseudo-hierarchies, e.g. the pairwise equivalence of (ATR), the Perfect Set Theorem and Σ^0_1 determinacy, can be found in Simpson [40]. Without even looking at the many more instances of pseudo-hierarchy arguments for second order arithmetic gathered in [40] and the second chapter of this thesis, it is not presumptuous to wish for an equally potent device in subsystems of set theory or explicit mathematics.

Making the unconfined application of pseudo-hierarchy arguments possible in subsystems of admissible set theory without foundation and explicit mathematics has been the leading goal of this thesis. After a careful analysis of pseudo-hierarchies is second order arithmetic, firstly to understand this method thoroughly with regard to its subsequent adaption to the aforementioned frameworks, and secondly to emphasis the versatility of this technique by applying it in order to achieve own results, we pinpoint the obstacles on the road to success and provide effective strategies to sidestep these difficulties. Besides, we research various fixed point theories: Among other things, we prove that operations induced by positive arithmetical formulas possess in general no Δ^1_1 definable fixed points, although there are well-know methods to obtain Σ^1_1 as well as Π^1_1 definable fixed points (see e.g. [1]). As a nice by-product of this line of research, we obtain an embedding of ID^*_1 (the theory obtained from ID_1 by restricting fixed point induction to formulas that contain fixed point constants only positively) into Σ^1_1-DC, and thereby answer an old question ask in Feferman's article on Hancock's conjecture [11] about the upper bound of ID^*_1. Claiming fixed points of monotone, Σ definable operations on the entire universe over KPi^0 leads to the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ with the same proof-theoretic ordinal as KPm^0, the standard theory of strength meta-predicative Mahlo in admissible set theory (cf. Jäger and Strahm [27]). This result depends crucially on the application of pseudo-hierarchy arguments. Further, we study a strengthening of KPi^0 (cf. Jäger [19]) where not only the admissibles distinguished by the predicate $\mathsf{Ad}(x)$ are linearly ordered by $\in$, but all sets reflecting the Kripke-Platek axioms. Surprisingly, this extension has already the strength of Δ^1_2-CA_0. Moreover, we elaborate on the connection between iteration and dependent choice: We demonstrate how the axiom of KPm^0 asserting that the class $\{x : \mathsf{Ad}(x)\}$ is linearly ordered by $\in$ is used to embed transfinite dependent choice, and that this axiom can be compensated by claiming Π_2 reflection on admissibles that additionally satisfy dependent choice, which nicely parallels the situation in second order arithmetic where extending ACA_0 by Π^1_2-reflection on models of Σ^1_1-DC leads also to a theory of strength meta-predicative Mahlo (cf. Rüede [37],[38]). Finally, we propose a form of dependent choice suitable for subsystems of explicit mathematics.

The concept of a hierarchy is based on the notion of an operation: In second order arithmetic, a formula $A(U,u)$ induces canonically an operation F^A acting on the powerset of the natural numbers and mapping X to $\{x : A(X,x)\}$. By iterating such an operation along a well-ordering $\prec$, a *proper* hierarchy G is obtained whose αth level $(G)_\alpha := F^A((G)_{\prec\alpha})$ is the F^A image of the disjoint union of the levels $(G)_\beta$ below α. Proper hierarchies on their own constitute a very powerful and useful concept which is implemented in Friedman's well-known theory ATR_0 (cf. e.g. Friedman [14], Friedman/McAloon/Simpson[15], Steel [42]) and, at least in second order arithmetic, are intrinsically tied to *pseudo*-hierarchies. A pseudo-hierarchy looks locally like a proper hierarchy, so again $(G)_\alpha = F^A((G)_{\prec\alpha})$, however, the underlying ordering $\prec$ is only a linear ordering but not well-founded.

In subsystems of second order arithmetic comprising arithmetical comprehension, the existence of a pseudo-hierarchy for A follows already if F^A can be iterated along arbitrary well-orderings. Moreover, such a pseudo-hierarchy can inherit a chosen Σ^1_1 definable property of the proper hierarchy: If $A(U,u)$ is an arithmetical and $B(U,V)$ a Σ^1_1 formula of second order arithmetic such that

$$\forall X[\mathsf{Wo}(X) \to \exists F(\mathsf{Hier}^A(F,X) \wedge B(F,X))],$$

then the well-known fact that being a well-ordering cannot be expressed by a Σ^1_1 formula forces the existence of a linear ordering $\prec$ that is not a well-ordering and a hierarchy G that meet $\mathsf{Hier}^A(G,\prec) \wedge B(G,\prec)$.

Each pseudo-hierarchy argument exploits that the field of the underlying ordering $\prec$ of the pseudo-hierarchy possesses a set $K \subseteq \mathsf{Field}(\prec)$ that is open at the bottom, i.e. K is non-empty, upward closed and has no $\prec$-least element. To apply its simplest form, we consider a pseudo-jumphierarchy G along the linear ordering $\prec$. Basically, $(G)_\alpha$ is the collection of all sets that are Π^0_1 in some level $(G)_\gamma$ for $\gamma \prec \alpha$. Then, for a set K open at the bottom, the collection M of all sets that are contained in each level $(G)_\alpha$ for $\alpha \in K$ is closed under arithmetical comprehension: If the set X is arithmetical in some set $Y \mathrel{\dot\in} M$, then X is already Π^0_n in Y for some $n \in \mathbb{N}$. Given a fixed but arbitrary $\alpha_0 \in K$, there is a sequence $\alpha_0 \succ \alpha_1 \succ \ldots \succ \alpha_n$ of elements of K. Since Y is also in $(G)_{\alpha_n}$, the definition of the jumphierarchy yields that X is an element of $(G)_{\alpha_0}$. Hence, the set X is in each $(G)_\gamma$ for $\gamma \in K$, thus also in M.

Often, one can boost the above argument by imposing additional properties on the pseudo-hierarchy. Avigad's argument in [2] for instance, considers a monotone operation F^A induced by an arithmetical formula. Then, there is a monotone pseudo-hierarchy G where $(G)_\alpha = F^A(\bigcup_{\beta \prec \alpha}(G)_\beta)$ and further, if $x \in (G)_\alpha$, then there is a $\prec$-least level α_0 where x appears first. For a set K open at the bottom, each $x \in Z := \bigcap_{\alpha \in K}(G)_\alpha$ enters the hierarchy at a least level α_0. Clearly, α_0 is not in K, otherwise there is a $\beta \in K$ with $\beta \prec \alpha_0$, contradicting that $x \in Z$. Therefore, each $x \in Z$ enters the hierarchy already at some level below K, thus $Z = \bigcup_{\alpha \prec K}(G)_\alpha$. Regarding Z as a union, the monotonicity of the operation and the hierarchy imply that $(G)_\alpha \subseteq F^A(Z)$ for all $\alpha \prec K$, whereas regarding Z as an intersection yields $F^A(Z) \subseteq (G)_\alpha$ for all $\alpha \in K$, thus $F^A(Z) = Z$.

A more general form of the condition imposed on the pseudo-hierarchy above is to stipulate that the underlying ordering $\prec$ of a pseudo-hierarchy G, which is not a well-ordering, looks like a well-ordering in a model D of ACA above G. If G is a pseudo-jumphierarchy conform with this condition, then the collection M from our first example is even a model of Σ^1_1-AC: If $\forall x \exists X A(X, x)$ holds in M for some arithmetical formula $A(U, u)$, then

$$\{\alpha \in \mathsf{Field}(\prec) : \forall x(\exists X \mathrel{\dot\in} (G)_\alpha) A(X, x)\}$$

is a superset of K in D. Therefore it has a least element α_0, which is already below K. Hence, $Y := (G)_{\alpha_0} \mathrel{\dot\in} M$, and for each x there is a least index e_x such that $A((Y)_{e_x}, x)$. Applying arithmetical comprehension in M yields a set $Z \mathrel{\dot\in} M$ with $(Z)_x = Y_{e_x}$, thus $\forall x A((Z)_x, x)$. The property that the underlying ordering $\prec$ of a pseudo-hierarchy G looks like a well-ordering in a model D of ACA above G is indeed so apt for pseudo-hierarchy arguments that we hardly ever need supplementary conditions.

Our declared aim is to establish an analogue situation with respect to pseudo-hierarchies in subsystems of admissible set theory without foundation and explicit mathematics. In admissible set theory, for instance, there exist initial segments of the constructible hierarchy $\mathcal{L}$ along arbitrary well-orderings, so we ask for a constructible hierarchy along a linear ordering that is not wellfounded. Obviously, this wish is not to fulfill in a well-founded universe, which highlights that the existence of pseudo-hierarchies is not provable without additional assumptions. In particular, neither in admissible set theory nor in explicit mathematics one can disprove that being a well-ordering is expressible by a Σ or a Σ^+ formula, respectively, which undermines the argument applied in second order arithmetic to prove the existence of pseudo-hierarchies and forces us to come up with new ideas.

In accordance with the setting in second order arithmetic, we base the concept of a hierarchy in admissible set theory upon the notion of an operation: A formula $A(u, v)$ induces an operation f^A, if for each set x there exists exactly one set y such that $A(x, y)$. The set y is then denoted by $f^A(x)$. When iterating an operation f^A along a well-ordering $\prec$, we obtain a proper hierarchy, namely a function g with domain $\mathsf{Field}(\prec)$ where $g(\alpha) = f^A(g{\restriction}\alpha)$ for each α in the domain of g. Again, a pseudo-hierarchy g looks locally like a proper hierarchy but its domain is a linear ordering that is not well-founded. Motivated by the previous example and assured by experience, we settle for the following *pseudo-hierarchy principle*:
For all Σ operations f^A, such that

$$\forall x[\mathsf{Wo}(x) \to \exists g\, \mathsf{hier}^A(g, x)],$$

there exists a linear ordering $\prec$ that is not a well-ordering and a hierarchy g that meet $\mathsf{hier}^A(g, \prec) \wedge \mathsf{Wo}^{g^+}(\prec)$, where g^+ denotes the least admissible above g.

This principle is not provable in a normal theory T of admissible sets, however, the following strategy offers an excellent workaround: We simply extend T by the above pseudo-hierarchy principle. Provided that $|\mathsf{T}| < |\Pi^1_1\text{-}\mathsf{CA}_0|$, then this extension is consistent, and moreover, has still the same proof-theoretic ordinal. Namely, if we can iterate a Σ operation f^A along arbitrary well-orderings but no pseudo-hierarchy for f^A exists, then $\forall x[\mathsf{Wo}(x) \leftrightarrow \exists g(\mathsf{Hier}^{f^A}(g, x) \wedge \mathsf{Wo}^{g^+}(x)]$. Exploiting the universal character of the formula $\mathsf{Wo}(x)$ and applying Δ separation then proves the translation of each instance of $(\Pi^1_1\text{-}\mathsf{CA})$ and thus also each ordinal α below $|\Pi^1_1\text{-}\mathsf{CA}_0|$, in particular $\mathsf{TI}_{\triangleleft}(\mathsf{U}, \alpha)$. Consequently, the pseudo-hierarchy principle is derivable in $\mathsf{T}^\dagger$, the extension of T by the axiom $\neg\mathsf{TI}_{\triangleleft}(\mathsf{U}, |\mathsf{T}|)$, stating that the relation U does not have a least element with respect to the underlying ordering $\triangleleft$ of our notation system. Finally, $|\mathsf{T}| = |\mathsf{T}^\dagger|$ is a consequence of an extension of Schütte's famous boundedness Theorem (cf. [39]) shown in Jäger and Probst [25] and summarized in subsection III.1.4.

In explicit mathematics, the setting is again slightly different, and additional problems have to be dealt with. The canonic notion of an operation specifying the transition from one level of a hierarchy to the next is now an individual term $(f : \Re \to \Re)$ that maps names to names. Also a hierarchy for the operation f along the ordering $\prec$ is represented by an individual term $(h : \mathsf{Field}(\prec) \to \Re)$ that assigns names of types to the elements of the field of $\prec$ in such a way that $h(\alpha) = f(\mathsf{j}(\{\beta : \beta \prec \alpha\}, h))$. Now the statement that h is a hierarchy for the operation f along the ordering $\prec$ is a Σ^+ formula, a Σ formula that contains the naming predicate $\Re$ only positively. Unfortunately, the lack of an appropriate form of Δ separation prevents to infer $(\Pi^1_1\text{-}\mathsf{CA})$

from the assumption that $\mathsf{Wo}(\prec)$ is expressible by a Σ^+ formula, hence the strategy enabling pseudo-hierarchies in admissible set theory is not directly applicable.

To circumvent this issue, we consider pseudo-hierarchy arguments in explicit mathematics only in the context of the theory EMA_0, basically the theory EMA, introduced and analyzed in Jäger and Strahm [27]. Given a name x and an operation $(f : \Re \to \Re)$ this theory provides a name $\mathsf{m}(x, f)$ of a universe, a type that contains only names and is closed under the basic type generators, so that $(f : \mathsf{m}(x, f) \to \mathsf{m}(x, f))$ and $x \mathrel{\dot\in} \mathsf{m}(x, f)$. This reflection principle (Mahlo axiom) then allows to prove that for a suitable closed term $\underline{\mathsf{hier}}$, $\underline{\mathsf{hier}}(f, w)$ represents a hierarchy for the operation $(f : \Re \to \Re)$ along a linear ordering w that looks like a well-ordering in a universe above $\mathsf{m}(w, f)$.

To comply with the nature of explicit mathematics, we aim for a *uniform* pseudo-hierarchy principle: There is a closed term $\underline{\mathsf{psh}}$, such that $(f : \Re \to \Re)$ implies that $\underline{\mathsf{psh}}\, f$ is a triple (h, w, k), where w is the name of a linear ordering that looks like a well-ordering in a universe above $\mathsf{m}(w, f)$, h represents a hierarchy for f along w and k is the name of a type K that is open at the bottom. Again, this principle is provable in the theory $\mathsf{EMA}_0^\dagger$. Its additional axiom, the assertion $\neg\mathsf{TI}_\lhd(\mathsf{U}, \varphi\omega 00)$, excludes that the class $\mathcal{O} := \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}(\lhd\restriction\alpha)\}$ is a type. Otherwise, $\mathcal{O}$ were the least type which is a fixed point of the accessible part operation mapping X to $\{\alpha \in \mathsf{Field}(\lhd) : (\forall\beta\lhd\alpha)(\beta \in X)\}$, and we could adapt the well-ordering proof for ID_1 to show $\mathsf{TI}_\lhd(\mathsf{U}, \varphi\omega 00)$. Given an operation $(f : \Re \to \Re)$, the Mahlo axiom provides a universe c with $(f : c \to c)$. Hence, for universes a, b with $c \mathrel{\dot\in} b \mathrel{\dot\in} a$, $\mathcal{O}^a$ is a proper subset of $\mathcal{O}^b$, simply because $\mathcal{O}^b$ is a type in a whereas $\mathcal{O}^a$ is not. Consequently, there is a least $\alpha_0 \in \mathsf{Field}(\lhd)$ w.r.t. the standard ordering on N, such that $\lhd\restriction\alpha_0$ looks like a well-ordering in the universe b above c, but is only a linear ordering. Therefore, $\underline{\mathsf{hier}}(f, \lhd\restriction\alpha_0)$ is a pseudo-hierarchy for f, and $K := \mathsf{Field}(\lhd) - \mathcal{O}^a$ is a type open at the bottom.

Another theme we focus onto is (transfinite) dependent choice. In second order arithmetic, (Σ^1_1-DC) claims the existence of choice sequences along $<_\mathsf{N}$ for Σ^1_1 formulas $A(U, V)$, provided that $\forall X \exists Y A(X, Y)$, whereas ($\Sigma^1_1$-$\mathsf{TDC}$) claims the existence of choice sequences along arbitrary well-orderings. In contrast to a hierarchy, where the αth level is uniquely determined by the levels below α and an operation F^A, a choice sequence has to satisfy $A((G)_{\prec\alpha}, (G)_\alpha)$, so out of the possibly many sets meeting $A((G)_{\prec\alpha}, Y)$, one has to be chosen. Thus, dependent choice can be seen as a combination of iteration and choice. This perception of dependent choice enables to construct choice sequences for arithmetical formulas $A(U, V)$ in KPi^0 extended by an appropriate iteration principle. The assertion that the admissible sets distinguished by the predicate $\mathsf{Ad}(x)$ are linearly ordered by $\in$ allows to select uniformly a witness Y for a given parameter X such that $A(X, Y)$ (cf. subsection III.1.6): The linearity of the class $\{x : \mathsf{Ad}(x)\}$ provides an $\in$-least admissible above X that still contains witnesses Z with $A(X, Z)$. However, such a witness corresponds to a path through a tree $T^A_X \in a$. The rightmost path through T^A_X that is left to all paths $\mathcal{F} \in a$ through T^A_X then yields a witness Y with $A(X, Y)$. Moreover, this witness is obtained in a uniform way exploiting the linearity of admissibles. So a Σ operation $f^{A'}$ can be found that assigns to X this particular set Y and the existence of a choice sequence now follows from the existence of a hierarchy for the operation $f^{A'}$.

The view of (transfinite) dependent choice as a combination of iteration and choice also motivates our implementation of dependent choice in explicit mathematics (cf. III.3.4) by dividing it into an axiom for (transfinite) iteration and one for choice: To state the choice rule, we extend the language by a constant ch, a term that is to choose a name of a fixed point of a term f, provided there exists one. For each finite set Γ of $\mathbb{L}_{\mathsf{ch}}$ formulas, we have:

$$(\mathsf{ch}) \qquad \frac{\Gamma, \exists x[\Re(x) \wedge fx \doteq x]}{\Gamma, \Re(\mathsf{ch}f) \wedge \mathsf{ch}f \doteq f(\mathsf{ch}f)}.$$

If for instance $A(U, V)$ is an elementary formula, then there is a closed term t such that $\Re(X, x)$ and $\Re(Y, y)$ imply

$$t(x, y) \doteq \{z : (A(X, Y) \wedge z \in Y) \vee (\neg A(X, Y) \wedge z \notin Y)\}.$$

Given $\Re(X, x)$ and $\Re(Y, y)$, we have $\lambda z.t(x, z)\, y \doteq y$ if and only if $A(X, Y)$. Moreover, $\mathsf{ch}\lambda z.t(x, z)$ names a specific witness if there is one: $\Re(W, \mathsf{ch}\lambda z.t(x, z))$ yields $A(X, W)$.

Indeed, the suggested form of (transfinite) dependent choice leads to theories of the expected strength. Combined with the iteration principle $(\mathsf{it_N})$ which allows to iterate operations along $<_{\mathsf{N}}$, we have $\mathsf{EETJ}_0 + (\mathsf{it_N}) + (\mathsf{ch}) = \varphi\omega 0 = |\Sigma^1_1\text{-}\mathsf{DC}_0|$, and replacing $(\mathsf{it_N})$ by (it) which provides hierarchies along arbitrary well-orderings yields $\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch}) = \varphi\omega 00 = |\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{TDC})|$. The lower bounds are obtained by embedding the corresponding theories of second order arithmetic, and the upper bounds are computed constructing (partial) models, making use of ideas and techniques developed in [27].

This thesis is organized in the following way: In chapter I, we fix the languages and theories. A notation system based on the ternary Veblen function is presented, some words on partial cut-elimination are said and the proof-theoretic ordinal of a theory is motivated and defined.

Chapter II gives an extensive introduction to pseudo-hierarchies in second order arithmetic and provides plenty of applications of pseudo-hierarchy arguments. First, we review how Σ^1_1 formulas relate to trees, which leads to normal forms of Σ^1_1 and Π^1_1 formulas and yields that being a well-ordering is not expressible by a Σ^1_1 formula, a crucial result for the existence of pseudo-hierarchies. Further, we develop the standard results about the jump-hierarchy and the hyperarithmetical sets. Then, we analyze a conservative extension of the iteration principle (ATR), namely an iteration principles (Δ^1_1-TR) for operations defined by a Δ formula. Next, we combine the fixed point construction from [2] with techniques developed in Jäger [21] to reason about fixed points of non-monotone operations. Our research on the relationship between fixed points and hyperarithmetical sets reveals that there are operations, given by positive arithmetical formulas, that have no fixed points in HYP and thus, due to the Kleene-Souslin Theorem, also no Δ^1_1 definable fixed points. Finally, we show that for a positive arithmetical formula $A(U^+, u)$, Σ^1_1-AC_0 proves that the Π^1_1 definable class $\mathsf{Fix}^A := \bigcap\{X : F^A(X) \subseteq X\}$ is a fixed point of the operation F^A, which gives rise to a new embedding of $\widehat{\mathsf{ID}}_1$ into Σ^1_1-AC. Its advantage over Aczel's embedding of $\widehat{\mathsf{ID}}_1$ into Σ^1_1-AC, known as Aczel's trick (cf. Aczel [1] and Feferman [11]), is that it extends to the initially mentioned embedding of ID^*_1 into Σ^1_1-DC. Moreover, Σ^1_1-DC proves that Fix^A is the least Π^1_1 definable fixed point of the operation F^A.

Chapter III eventually exhibits how pseudo-hierarchy arguments can be applied outside the framework of second order arithmetic. Section III.1 introduces a pseudo-hierarchy principle for admissible set theory which then is applied to analyze the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$. Further, we comment on the relationship between iteration, linearity of admissibles and dependent choice.
Section III.2 focuses on admissible sets. The language of KPi^0 is equipped with a relation symbol $\mathsf{Ad}(u)$ that distinguishes a class $\mathsf{Ad} := \{x : \mathsf{Ad}(x)\}$ of admissibles. To reason about the collection of all admissible sets, not just the ones distinguished by the predicate $\mathsf{Ad}(u)$, we introduce a Δ_0 formula $\mathsf{P}_{\mathsf{Ad}}(u)$ expressing that the set u is a transitive model of $\mathsf{KPu}^0 + (\mathsf{I}_{\mathsf{N}})$. A slight modification yields a Δ_0 formula $\mathsf{Ad}_{\mathsf{dc}}(u)$, which claims that u is admissible and in addition satisfies dependent choice. Relying heavily on pseudo-hierarchy arguments, we link the class

$$\mathsf{hyp}^x := \bigcap\{y : x \in y \wedge \mathsf{P}_{\mathsf{Ad}}(y)\},$$

the intersection of all admissibles above x, to the constructible hierarchy by showing that $\mathsf{hyp}^x = \bigcup_{\alpha \in \mathsf{on}(\mathsf{hyp}^x)} \mathcal{L}^x_\alpha$, where $\mathsf{on}(\mathsf{hyp}^x)$ denotes the set of all ordinals in hyp^x and $\mathcal{L}^x_\alpha$ is the αth level of the constructible hierarchy above x. As a consequence, we obtain that strengthening KPi^0 by an assertion that the class $\{x : \mathsf{P}_{\mathsf{Ad}}(x)\}$ is linearly ordered by $\in$, leads to a theory with the same strength as Δ^1_2-CA_0 and $\mathsf{KPi}^{\mathsf{r}}$, respectively (cf. Jäger [20]). Then, we consider an axiom (Δ_0-dc) corresponding to (Σ^1_1-DC) and argue that KPu^0 extended by Π_2 reflection on transitive models of $\mathsf{KPu}^0 + (\mathsf{I}_{\mathsf{N}}) + (\Delta_0\text{-}\mathsf{dc})$ is another theory of strength meta-predicative Mahlo.
In section III.3, we finally present a uniform pseudo-hierarchy principle for the theory EMA_0. Besides, we propose and analyze a form of dependent choice suitable for subsystems of explicit mathematics.

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Dieter Probst
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Chapter I

Languages, theories and provable ordinals

> Artists can color the sky red because they know it's blue. Those of us who aren't artists must color things the way they really are or people might think we're stupid.
>
> Jules Feiffer

The purpose of this chapter is mainly to fix the notation. After we agree on some general conventions, the languages and theories of this thesis are introduced and some of their basic properties are mentioned. Then, a notation system based on the ternary Veblen function is presented, some words on partial cut-elimination are said and the proof-theoretic ordinal of a theory is motivated and defined. You probably want to skip this chapter on your first reading and come back if you encounter unfamiliar terms. Precise references are provided in the index.

I.1 General conventions

In this section, we collect most of the meta-mathematical abbreviations and notational conventions that we make use of in the remainder of this thesis.

I.1.1 Mathematical reasoning and formal proofs

The objects of our research are formal theories. On the one hand, we argue within a formal theory to derive a particular formula of the language of some formal theory T, applying only the axioms and rules of T. To keep this thesis readable, we hardly ever give formal derivations in the strict sense, but *work informally in a formal theory* T. Rather then writing a proof in the language L of T, we use a mixture of English and L to present our arguments. However, for a reader with some experience with formal proofs, it should always be clear how to translate our demonstrations into pure formal derivations. On the other hand, we apply *general mathematical reasoning* to prove a statement about some formal theory. We forgo to specify the exact requirements of a *meta-theory* that subsumes the notion of general mathematical reasoning, but simply assume it is based on classical logic and strong enough to carry out all our arguments.

We always try to be clear whether we work in a formal theory or the meta-theory. To facilitate the distinction, we use $\mathbb{N}$ always for the set of natural numbers of the meta-theory, and given two mathematical meaningful statements A and B, $A \Longrightarrow B$ expresses that A implies B by general mathematical reasoning. Moreover, $A \Longleftrightarrow B$ means that $A \Longrightarrow B$ and $B \Longrightarrow A$.

I.1.2 Notational conventions

We use the vector notation $\vec{\varsigma}$ to denote finite strings $\varsigma_1, \ldots, \varsigma_n$ of symbols whose length is not important or given by the context. It is possible that $\vec{\varsigma}$ is the empty string ϵ. Sometimes, we stretch the vector notation a bit and write, for example, $\forall\vec{x}$ instead of $\forall x_1, \ldots, \forall x_n$ and $\vec{u} \in U$ for $u_1 \in U \wedge \ldots \wedge u_n \in U$. In connection with the vector notation, the letters i, j are often implicitly used as index variables: Let $\vec{u}$ such that u_i has the property P, is a lazy way to state that all symbols u of the string $\vec{u}$ have the property P. Given expressions $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, we write $\mathcal{X}[\mathcal{Y}/\mathcal{Z}]$ for the result of substituting simultaneously all occurrences of $\mathcal{Z}$ in $\mathcal{X}$ by $\mathcal{Y}$.
Square brackets are used for alternatives. For example, "$[\neg](t \in U)$ is a literal" is to express that both, $(t \in U)$ and $\neg(t \in U)$, are literals, and "A $[A']$ implies B $[B']$" is a shortcut for "A implies B" and "A' implies B'".

I.1.3 Sets, functions and relations

Regarding sets, functions and relations, we apply the standard terminology: $\mathbb{N}$ denotes the natural numbers and $\emptyset$ the empty set. For the ordered pair $\{\{x\}, \{x, y\}\}$ of x, y, we write (x, y), and $(x_1, \ldots, x_{n+1})$ is defined as $(x_1, (x_2 \ldots, x_{n+1}))$. Further, $x_1 \times \ldots \times x_n$ denotes the Cartesian product of $x_1, \ldots, x_n$, namely the set $\{(u_1, \ldots, u_n) : u_1 \in x_1, \ldots, u_n \in x_n\}$. In addition, $x^1 := x$, and $x^{n+1} := x \times x^n$. Other frequently used operations on sets are intersection, union and difference $x - y := \{u \in x : u \notin y\}$. For subsets of the natural numbers, usually denoted by $X, Y, Z, \ldots$, we also define the complement $\overline{X} := \{x \in \mathbb{N} : x \notin X\}$. The powerset $\mathcal{P}(x)$ consists of all subsets y of x. If two sets x and y contain the same elements, then we abbreviate this by $x = y$; $x \subset y$ and $x \subseteq y$ express that x is a subset of y and if x is a proper subset of y we indicate this by $x \subsetneq y$. Moreover, we call a set x *transitive*, in symbols $\mathsf{Tran}(x)$, if $(\forall y \in x)(y \subseteq x)$.

A subset R of $x_1 \times \ldots \times x_n$ is called an n-ary relation on $x_1, \ldots, x_n$. Instead of $(u_1, \ldots, u_n) \in R$, we often write $R(\vec{u})$. The field $\mathsf{Field}(R)$ of R is the set of all u, for which there exist $\vec{v}$ with $R(\vec{v})$ and $u = v_i$ for some $1 \leq i \leq n$. The range $\mathsf{Rng}(R)$ of R is the set $\{v : \exists \vec{u} R(\vec{u}, v)\}$ and the domain $\mathsf{Dom}(R)$ of R is the set $\{(u_1, \ldots, u_{n-1}) : \exists v R(\vec{u}, v)\}$. The restriction of a relation R to x, written $R{\restriction}x$, is simply the intersection of R with x. If $R \subseteq x^n$, we call R an n-ary relations on x. If, in addition, $y \subseteq x$, then we write $R{\restriction}y$ for $R{\restriction}y^n$.
A $n{+}1$-ary relation R is called an n-place function, if for all sets $\vec{x}, y$, $R(\vec{x}, y)$ and $R(\vec{x}, z)$ forces $y = z$. Then working in the meta-theory, functions are usually denoted by the letters f, g, h. Moreover, $(f : x \to y)$ states that f is a function with domain x whose range is a subset of y.

I.1.4 Orderings, trees and ordinals

Unless explicitly mentioned, the term ordering refers to linear irreflexive orderings. We call a pair (x, R) a linear irreflexive ordering, if $R \subseteq x \times x$, and for all $u, v, w \in x$, the following properties hold:

1. $R(u, v) \wedge R(u, w) \to R(u, w)$ (Transitivity),
2. $R(u, v) \vee u = v \vee R(v, u)$ (Comparability),
3. $\neg R(u, u)$ (Irreflexivity).

We consider mainly (linear irreflexive) orderings of the form $(\mathsf{Field}(R), R)$, and simply speak of the ordering R. In the sequel, orderings are usually denoted by $\prec$. Moreover, we write $\preceq$ for the reflexive closure of $\prec$, i.e. $u \preceq v$ if and only if $u \prec v \vee (u = v \wedge u \in \mathsf{Field}(\prec))$. By $\mathsf{Lin}(\prec)$, we express that $\prec$ is a linear ordering, and $\mathsf{Lin}_0(\prec)$ states that $\prec$ is a linear ordering with a least element, usually denoted by $0_\prec$ or just $\prec$.
If $\prec$ is an ordering and $u \in \mathsf{Field}(\prec)$, we write $\prec{\restriction}u$ for $\prec{\restriction}\{v : v \prec u\}$. Further, $\prec'$ is an initial segment of the ordering $\prec$, if $\prec' = \prec$ or if there exists an $u \in \mathsf{Field}(\prec)$, such that $\prec' = \prec{\restriction}u$. In the latter case, $\prec'$ is called a proper initial segment of $\prec$. A subset x of the field of an ordering $\prec$ is called downward [upward] closed, if $u \in x$ and $v \prec u$ [$u \prec v$] implies $v \in x$. Moreover, if $\mathsf{min}_\prec\{v : u \prec v\}$ exists, it is denoted by $u +_\prec 1$, or $u{+}1$ for short, also called the successor of u w.r.t. $\prec$. A function f is called a order-isomorphism between $\prec$ and $\prec'$, if f is a bijection from the field of $\prec$ to the field of $\prec'$, and

$$\forall u, v[u \prec v \leftrightarrow f(u) \prec' f(v)].$$

Two orderings $\prec$ and $\prec'$ are said to be *comparable*, if there is an order isomorphism f between one and an initial segment of the other. This order isomorphism is called the *comparison map between* $\prec$ and $\prec'$. Alternatively, we say that f *compares* $\prec$ and $\prec'$.

An ordering $\prec$ is a *well-ordering*, denoted by $\mathsf{Wo}(\prec)$, if each non-empty subset of its field has a $\prec$-least element, i.e.

$$(\forall u \subseteq \mathsf{Field}(\prec))[u \neq \emptyset \rightarrow (\exists v \in u)(\forall w \prec v)(w \notin u)].$$

Alternatively, if $x \subseteq \mathsf{Field}(\prec)$, we say that x is *well-ordered* by $\prec$ if $\prec\restriction x$ is a well-ordering. A more general notion that applies to arbitrary binary relations is *well-foundedness*. A binary relation $R \subseteq x \times x$ is called well-founded on x, denoted by $\mathsf{Wf}(R)$, if every non-empty subset of x has a R-minimal element:

$$(\forall y \subset x)[y \neq \emptyset \rightarrow (\exists z \in y)(\forall w \in y)((w, z) \notin R)].$$

A tree $T = (x, R)$ is a pair consisting of a set x of *nodes* and a relation $R \subseteq x \times x$ such that x has a R-least element, called the *root* of T, and for each $y \in x$ the set $\{z \in x : R(z, y)\}$ is well-ordered by R. The R-maximal elements of x, i.e. all $u \in x$ such that for no $v \in x$, $R(u, v)$ holds, are called *leafs of the tree* T. A *path* through tree $T = (x, R)$ is a function $(f : \mathbb{N} \rightarrow x)$ such that $f(0)$ is the root of T and for all $n \in \mathbb{N}$, $\{f(0), \ldots, f(n)\} = \{x : R(x, f(n{+}1)\}$. A tree is called *well-founded* if it has no path.

Later on, we use *ordinals* to measure the *proof-theoretic strength* of theories. An introduction to ordinals can be found e.g. in Pohlers [28]. Ordinals, usually denoted by lower case Greek letters $\alpha, \beta, \gamma, \ldots$, are hereditarily transitive sets that are well-founded by the elementhood relation $\in$. If $\prec$ is a well-ordering, there is precisely one ordinal α such that $\prec$ is isomorphic to $\in\restriction\alpha$, called the *ordertype* of $\prec$. The class of all ordinals is denoted by ON.

The successor of an ordinal α is $\alpha \cup \{\alpha\}$, also denote by $\alpha{+}1$. Ordinals of the form $\alpha{+}1$ are called successor ordinals, the other ordinals beside $\emptyset$ are called limits or limit ordinals, often denoted by λ. Finally, we write 0 for $\emptyset$, if we regard it as the least element of ON. We conclude this subsection by mentioning a well-known proof-technique, *transfinite induction* along a well-ordering $\prec$ or a well-founded relation R:

Lemma I.1.1 (Transfinite Induction) *If $\prec$ is a well-ordering and R a well-founded relation on* $\mathsf{Field}(R)$*, then we have:*

1. $(\forall u \in \mathsf{Field}(\prec))[(\forall v \prec u)(v \in x) \rightarrow (u \in x)] \rightarrow \mathsf{Field}(\prec) \subseteq x,$
2. $(\forall u \in \mathsf{Field}(R))[\forall v(R(v, u) \rightarrow v \in x) \rightarrow (u \in x)] \rightarrow \mathsf{Field}(R) \subseteq x.$

I.1.5 Recursive and primitive recursive functions and relations

In this paragraph we review the [primitive] recursive functions and relations on $\mathbb{N}$. The class $\mathcal{PRIM}$ of *primitive recursive functions* is defined inductively by the following clauses:

1. For all natural numbers $m, n \in \mathbb{N}$ and $0 \leq i < n$, the successor function $\mathsf{s}(x) := x{+}1$, the constant functions $\mathsf{cs}^n_m(x_1, \ldots, x_n) := m$ and the projections $\mathsf{pr}^n_i(x_1, \ldots, x_n) := x_i$ are in $\mathcal{PRIM}$.

2. If $(f : \mathbb{N}^m \to \mathbb{N})$ and $(\vec{g} : \mathbb{N}^n \to \mathbb{N})$ are elements of $\mathcal{PRIM}$, then also the composition $Comp(f, \vec{g})(\vec{x}) := f(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ of f and $\vec{g}$.

3. If $(f : \mathbb{N}^n \to \mathbb{N})$ and $(g : \mathbb{N}^{n+2} \to \mathbb{N})$ are elements of $\mathcal{PRIM}$, then also the function $(Rec(f, g) : \mathbb{N}^{n+1} \to \mathbb{N})$ obtained from f and g by applying the schema of primitive recursion: $Rec(f, g)(0, \vec{y}) := f(\vec{y})$ and in the successor case, $Rec(f, g)(x{+}1, \vec{y}) := g(Rec(f, g)(x, \vec{y}), x, \vec{y})$.

So $\mathcal{PRIM}$ is the smallest class of functions containing the basic functions s, cs^n_m and pr^n_i that is closed under composition and the schema of primitive recursion. Primitive recursive functions are total functions: Their domain is always of the form $\mathbb{N}^n$. In contrast, the domain of a n-ary *partial recursive function* f may be a proper subset of $\mathbb{N}^n$. Thus, for $\vec{x} \in \mathbb{N}$, $f(\vec{x})$ may not be defined, which is expressed by $f(\vec{x})\uparrow$, and models the fact that some computations do not terminate. If $f(\vec{x})$ returns a natural number, then we indicate this by $f(\vec{x})\downarrow$.

The class $\mathcal{REC}$ of partial recursive functions contains the same basic functions as $\mathcal{PRIM}$, but besides composition and primitive recursion, it is also closed under the μ-schema:

4. If $(f : \mathbb{N}^{n+1} \to \mathbb{N})$ is an element of $\mathcal{REC}$, then also the (partial) function $(\mu.f : \mathbb{N}^n \to \mathbb{N})$, given by the following case distinction:

$$\mu.f(\vec{x}) := \begin{cases} z & : \quad f(z, \vec{x}) = 0 \wedge (\forall y < z)[f(y, \vec{x})\downarrow \wedge f(y, \vec{x}) > 0], \\ \uparrow & : \quad \text{otherwise.} \end{cases}$$

A (total) function $(f : \mathbb{N} \to \{0, 1\})$ is often called a *characteristic function*. An n-ary relation $R(\vec{x})$ is called [primitive] recursive, if there is a [primitive] recursive characteristic function, such that $R(\vec{x})$ if and only if $f(\vec{x}) = 0$. A relation $R(\vec{x})$ is *recursively enumerable*, if there exists a partial recursive function f with domain R, i.e. if $R(\vec{x})$ holds, if and only if $f(\vec{x})\downarrow$.

I.1.6 Some primitive recursive functions and relations

Most of the languages used in this thesis provide function and relation symbols for all primitive recursive functions and relations. Below we distinguish some primitive recursive functions and relations that play an important role in the sequel:

1. $([\cdot,\cdot] : \mathbb{N}^2 \to \mathbb{N})$ is a bijective pairing function with associated projections $[\cdot]_0$ and $[\cdot]_1$, mapping natural numbers to natural numbers such that $[[x,y]]_0 = x$, $[[x,y]]_1 = y$ and $x = [[x]_0, [x]_1]$.

2. In order to code finite sequences of natural numbers we introduce for each $n \in \mathbb{N}$ a function $(\langle \cdot, \dots, \cdot \rangle : \mathbb{N}^n \to \mathbb{N})$, given by

$$\langle x_0, \dots, x_{n-1} \rangle := \begin{cases} 1, & \text{if} \quad n = 0, \\ \prod_{i=0}^{n-1} p(i)^{x_i+1}, & \text{if} \quad n > 0, \end{cases}$$

where in this paragraph, $p(i)$ denotes the ith prime, starting with $p(0) := 2$. Note, that the constant $\langle\rangle$ for the empty sequence in an other name for the constant 1. Further, $(lh : \mathbb{N} \to \mathbb{N})$ is a function satisfying $lh(\langle\rangle) = 0$ and $lh(\langle x_0, \dots, x_{n-1} \rangle) = n$, for all n and $\vec{x}$. The projection function $(\pi : \mathbb{N}^2 \to \mathbb{N})$ is to meet the condition $\pi(\langle x_0, \dots, x_{n-1} \rangle, i) = x_i$ for $0 \le i \le n$. Instead of $\pi(s,i)$, we write $(s)_i$, and $(s)_{i,j}$ is a shortcut for $((s)_i)_j$. The unary relation seq consists of the codes of the finite sequences,

$$x \in \mathsf{seq} :\Longleftrightarrow x = \prod_{i < lh(x)} p(i)^{(x)_i + 1},$$

and $\mathsf{seq}_{0,1}$ consists of all the codes x of the finite $0,1$-sequences, i.e. $x \in \mathsf{seq}$ and $(\forall i < \mathsf{lh}(x))((x)_i \in \{0,1\})$.

Often, we do not require the pairing function to be bijective. Then, we regard pairs as sequences of length 2.

3. By $*$ we denote a primitive recursive function that assigns to the codes of two finite sequences the code of the concatenation of these two sequences: For all $\vec{x}, \vec{y}$, $\langle \vec{x} \rangle * \langle \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$. $\langle \vec{x} \rangle \sqsubseteq \langle \vec{y} \rangle$ is the binary relation indicating that $\vec{x}$ is an initial segment of the sequence $\vec{y}$. The symbol $\sqsubset$ is used for proper initial segments.

4. The *Kleene-Brouwer ordering* $<_{\mathsf{KB}}$ is the following primitive recursive ordering on seq: For $x, y \in \mathsf{seq}$, $x <_{\mathsf{KB}} y$ if either $y \sqsubset x$, or if

$$(\exists j < \mathsf{min}\{\mathsf{lh}(x), \mathsf{lh}(y)\})[(x)_j < (y)_j \land (\forall i < j)(x)_i = (y)_i].$$

Thus, x is smaller than y w.r.t to the Kleene-Brouwer ordering, if x extends the sequence y, or if at the first position i where the sequences differ, $(x)_i$ is smaller than $(y)_i$. If we are only interested in the Kleene-Brouwer ordering as a set, we denote it simply by KB. Further, if S is a subset of seq, we write $\mathsf{KB}(S)$ or $<_{\mathsf{KB}(S)}$ for $<_{\mathsf{KB}}\restriction S$.

I.1.7 Indices for the [primitive] recursive functions and relations

The class [Prim] Rec of *indices of [primitive] recursive functions* is defined below. Thereby, we follow basically the presentation in [22]. Note that for an index e in [Prim] Rec, $(e)_1$ denotes the ariety of the corresponding function. For subsequent use, we include additional indices $\langle 10{+}n, 1\rangle$ ($n \in \mathbb{N}$). Later on, these indices are regarded as characteristic functions of set parameters, but for the time being, $\langle 10{+}n, 1\rangle$ are indices of the function cs^1_0.

1. The indices $\langle 0, 1\rangle$, $\langle 1, n, m\rangle$ and $\langle 2, n, i\rangle$ of the functions s, cs^n_m and pr^n_i are in [Prim] Rec. Further, for each $n \in \mathbb{N}$, $\langle 10{+}n, 1\rangle$ are also indices of the function cs^1_0. In subsection II.1.1 theses indices are regarded as characteristic functions of set parameters.

2. If $a, \vec{b}$ with $(a)_1 = m$ and $(b_i)_1 = n$ are indices in [Prim] Rec of f and $\vec{g}$, then also the index $\langle 3, n, a, \vec{b}\rangle$ of $Comp(f, \vec{g})$.

3. If a, b with $(a)_1 = n{+}2$ and $b = n$ are indices in [Prim] Rec of f and g, then also the index $\langle 4, n{+}1, a, b\rangle$ of $Rec(f, g)$.

4. If a with $(a)_1 = n{+}1$ is an index of f in Rec, then also the index $\langle 5, n, a\rangle$ of $\mu.f$.

In the next chapter, formulas that are universal for a certain class of formulas are required. Since such formulas are built upon Kleene's T-predicate, we recall briefly its definition.

Kleene's T-predicate is a primitive recursive relation $T(e, \langle\vec{y}\rangle, z)$, expressing that $e \in \mathsf{Rec}$ and that z is a "proof" that the computation of the (partial) recursive function with index e terminates on input $\vec{y}$ and yields $(z)_{0,2}$ as result. Thereby, a "termination proof" z is a finite sequence of triples of the form $\langle e', \langle\vec{x}\rangle, r\rangle$ with the intended meaning that the function with index e' yields the result r on input $\vec{x}$. The sequence z starts with the triple $\langle e, \langle\vec{y}\rangle, (z)_{0,2}\rangle$, and unless this triple expresses a true statement about the successor, a projection or a constant function, i.e. is an "axioms", all the subsequent triples are justifications of this first or subsequent statements which are not "axioms". For example, a "proof" of $\langle\langle 3, 1, a, b\rangle, \langle y\rangle, z\rangle$

consists of a "proof" of $\langle a, \langle w \rangle, z \rangle$ and $\langle b, \langle y \rangle, w \rangle$. This allows to regard each e as a partial recursive function by defining $\{e\}(\vec{x}) := y$ if there exists a sequence z, such that $T(e, \langle \vec{x} \rangle, z)$ and $(z)_{0,2} = y$, and $\{e\}(\vec{x})\uparrow$ otherwise.

I.2 Languages, theories and structures

In this section, we define the languages L_1 and L_2 of first and second order arithmetic, the language $\mathcal{L}^*$ of Kripke-Platek set theory and the language $\mathbb{L}$ of explicit mathematics. Then, we introduce Tait-style calculi and fix the axiomatizations of Peano Arithmetic PA and the semi-formal system PA^*, basic set theory BS^0, Kripke-Platek set theory KPu^0, the subsystems ACA and ACA_0 of second order arithmetic and the base system EETJ_0 of explicit mathematics. We specify our notion of proof, (standard) structures and models, and comment on issues like embeddings, the dispensability of primitive recursive function symbols and some syntactic extensions of our languages.

I.2.1 Languages, theories and structures

A *language* L is characterized by a set of symbols. The first order languages we consider comprise infinitely many symbols for variables of a first sort, which we denote by lower case Latin letters $a, b, c, d, e, k, l, m, n, u, v, w, x, y, z$, the second order languages comprise in addition infinitely many variables of a second sort, denoted by $F, G, H, K, L, M, N, S, U, V, W, X, Y, Z$. There are symbols for constants (0-ary function symbols) and symbols for functions and relations, logical symbols $\sim, \wedge, \vee, \forall, \exists$ and auxiliary symbols (,). Each variable of the first sort and each constant is a *term*, and if f is an n-ary function symbol and $t_1, \dots, t_n$ are terms, then also $f(t_1, \dots, t_n)$ is a term. In the sequel, we let r, s, t, range over terms.

The *atoms* of a language L are the expressions of the form $R(X_1, \dots, X_m, t_1, , \dots, t_n)$, where R is a relation symbol of the appropriate ariety for each sort. A *literal* is an atom A or its negation $\sim A$. All literals are *formulas*, and with A, B, also $(A \vee B)$, $(A \wedge B)$, $\mathcal{Q}xA$, and dependent on the existence of a second sort of variables, also $\mathcal{Q}XA$, where $\mathcal{Q}$ ranges over the quantifiers $\forall, \exists$. Henceforth, we us mainly the letters A, B, C, D to denote formulas.

The *negation* $\neg A$ of a formula A is defined by making use of De Morgan's laws and the law of double negation. As usual, $A \rightarrow B$ abbreviates $\neg A \vee B$ and $A \leftrightarrow B$ stands for $(A \rightarrow B) \wedge (B \rightarrow A)$. Also $\exists ! x A(x)$ is used to express that there is exactly one x with $A(x)$, i.e. $\exists x A(x) \wedge \forall x, y[A(x) \wedge A(y) \rightarrow x = y]$. $\exists ! X A(X)$ is defined accordingly. The set of *subformulas* $\mathsf{Sufo}(A)$ of a formula A of some language L is defined inductively by $\mathsf{Sufo}(A) := \{A\}$, if A is a positive or negative atom, if A is of the form $B \wedge C$ or $B \vee C$, then $\mathsf{Sufo}(A) := \mathsf{Sufo}(B) \cup \mathsf{Sufo}(C) \cup \{A\}$, and if A is the formula $\mathcal{Q}xB(x)$ [$\mathcal{Q}XB(X)$], then we set $\mathsf{Sufo}(A) := \{B(t) : t \text{ term of } \mathsf{L}\} \cup \{A\}$ [$\mathsf{Sufo}(A) := \{B(U) : U \text{ a variable of } \mathsf{L}\} \cup \{A\}$]. Subformulas of A that are different from A are called proper subformulas of A.

The set of all variables that occur in a term t is denoted by $FV(t)$. Its elements are called *free variables of* t. Each variable that occurs in a literal A *occurs free* in A, and no variable *occurs bound* in a literal. The set of variables that occur free [bound] in A is denoted by $FV(A)$ [$FB(A)$]. If A is of the form $B \vee C$ or $B \wedge C$, then $FV(A) := FV(B) \cup FV(C)$ and $FB(A) := FB(B) \cup FB(C)$. If A is of the form $\mathcal{Q}xB$ [$\mathcal{Q}XB$], then $FV(A) := FV(B) - \{x\}$ [$FV(A) := FV(B) - \{X\}$] and $FB(A) := FB(B) \cup \{x\}$ [$FB(A) := FB(B) \cup \{X\}$]. Note, that in general the sets $FV(A)$ and $FB(B)$ are not disjoint. A *contains* u *free* is used as a synonym for u *occurs free in* A. Sometime, we refer to variables *occurring free in* a formula also as *free variables* or as *number and set parameters*. Finally, a term without free variables is called *closed*, and a *sentence* is a formula that contains no free variables. Note, that we do not distinguish syntactically between free and bound variables. However, we manly use the letters X, Y, Z, x, y, z for bound variables and parameters.

Often, we introduce a formula as $B(\vec{U}, \vec{u})$. The formula $B(\vec{X}, \vec{x})$ is then obtained by replacing simultaneously $\vec{U}, \vec{u}$ by $\vec{X}, \vec{x}$. However, the notation $B(\vec{U}, \vec{u})$ does not imply that $B(\vec{U}, \vec{u})$ actually contains all the variables $\vec{U}, \vec{u}$ free or that it does not contain additional free variables. Finally, a formula $B(U)$ is called *positive* in U, or U-positive, denoted by $B(U^+)$, if B contains no subformula of the form $t \notin U$.

An L-*structure* for a first [second] order languages L is a pair $\mathcal{M} = (M, \mathcal{I})$ [a triple $\mathcal{M} = (M, \mathcal{S}_M, \mathcal{I})$], where M is the intended range of the first order variables and $\mathcal{S}_M$ the range of the second order variables. Thereby, we require that M and $\mathcal{S}_M$ are non-empty, disjoint and that $\mathcal{S}_M$ is a subset of $\mathcal{P}(M)$. $\mathcal{I}$ specifies the interpretation of the function and relation symbols: If f is an n-ary function symbol, then $(\mathcal{I}(f) : M^n \to M)$, and for a relation symbol $R(U_1, \ldots, U_m, u_1, \ldots, u_n)$, $\mathcal{I}(R)$ is a subset of $\mathcal{S}_M^m \times M^n$. Further, we say that the structure $\mathcal{M} = (M, \mathcal{I})$ [$\mathcal{M} = (M, \mathcal{S}_M, \mathcal{I})$] is countable, if M and $\mathcal{S}_M$ are countable.

Next, we explain the notion of *satisfaction*. A *valuation* $\mathcal{E}$ for $\mathcal{M}$ is a function that maps variable symbols of the first [second] sort to M [$\mathcal{S}_M$]. To simplify the notation, we write $f^{\mathcal{M}}$, $R^{\mathcal{M}}$, and $u^{\mathcal{E}}$, $U^{\mathcal{E}}$ instead of $\mathcal{I}(f)$, $\mathcal{I}(R)$ and $\mathcal{E}(u)$, $\mathcal{E}(U)$. Moreover, for $m \in M$, $\mathcal{E}[u = m]$ updates the evaluation $\mathcal{E}$ by replacing the pair $(u, u^{\mathcal{E}})$ in $\mathcal{E}$ by (u, m). If $m \in \mathcal{S}_M$, then $\mathcal{E}[U = m]$ is defined analogously. To each term t, we assign a value $\mathcal{M}_{\mathcal{E}}(t)$, namely $t^{\mathcal{E}}$ if t is a variable, $t^{\mathcal{M}}$ if t is a constant and $f^{\mathcal{M}}(\mathcal{M}_{\mathcal{E}}(\vec{s}))$ if t is the term $f(\vec{s})$. Inductively on the build-up of formulas, we define below when $\mathcal{M}$ satisfies A under the valuation $\mathcal{E}$, denoted by $(\mathcal{M}, \mathcal{E}) \models A$.
If $R(\vec{U}, \vec{t})$ is an atom, then $(\mathcal{M}, \mathcal{E}) \models A$, if $R^{\mathcal{M}}(\mathcal{M}_{\mathcal{E}}(\vec{t}), \vec{U}^{\mathcal{E}})$ holds, otherwise we

have $(\mathcal{M},\mathcal{E}) \models \sim A$. For formulas A, B, $(\mathcal{M},\mathcal{E}) \models A \vee B$ [$(\mathcal{M},\mathcal{E}) \models A \wedge B$] if $(\mathcal{M},\mathcal{E}) \models A$ or $(\mathcal{M},\mathcal{E}) \models B$ [$(\mathcal{M},\mathcal{E}) \models A$ and $(\mathcal{M},\mathcal{E}) \models B$], $(\mathcal{M},\mathcal{E}) \models \forall x A(x)$ [$(\mathcal{M},\mathcal{E}) \models \exists x A(x)$] if for all $m \in M$, $(\mathcal{M},\mathcal{E}[u = m]) \models A(u)$ [if there is an $m \in M$ such that $(\mathcal{M},\mathcal{E}[u = m]) \models A(u)$] and accordingly for $\forall X A(X)$ [$\exists X A(X)$]. If $(\mathcal{M},\mathcal{E}) \models A$ for all valuations $\mathcal{E}$, then we write $\mathcal{M} \models A$ and say that A holds in the structure $\mathcal{M}$.

I.2.2 Theories

In this thesis, a theory T is a set of Tait-style axioms and rules (cf. [44]), which are used to derive finite sets of formulas, denoted by Γ, Δ, Λ. The intended meaning of Γ is thereby the disjunction $\bigvee \Gamma$ of all its elements. Further, Γ, A is used as an abbreviation for $\Gamma \cup \{A\}$ and Γ, Δ stands for $\Gamma \cup \Delta$. The free variables $FV(\Gamma)$ of Γ is the set $\bigcup\{FV(A) : A \in \Gamma\}$.

We assume that the theory T is formulated in some language L. A Tait-style axiom is then a finite set Γ of formulas of L, and a rule $\mathcal{R}$ is a scheme of the form

$$(\mathcal{R}) \qquad \frac{\Gamma_i \text{ for all } i \in I}{\Gamma},$$

where Γ and Γ_i are finite sets of L formulas; the sets Γ_i $(i \in I)$ are called the premises and Γ the conclusion of the rule $\mathcal{R}$. A rule without premises is called *axiom*. A theory that consists of a recursive set of Tait-style axioms and rules, where all rules have only finitely many premises, is called a *formal theory*, a theory which possesses rules with infinitely many premises is called *semi-formal* (cf. [39]). If each formula in a Γ_i is the subformula of a formula in Γ, the theory T has the *subformula property*. For the theories considered in this thesis, axioms and rules are introduced in one of the following forms,

$$\Gamma, C_1, \ldots, C_n \quad \frac{\Gamma, A_1, \ldots, A_n}{\Gamma, C} \quad \text{or} \quad \frac{\Gamma, A_i \text{ for all } i \in I}{\Gamma, C}.$$

The distinguished formulas $C_1, \ldots, C_n$ in the axiom and the formula C in the conclusion of the rules are called *main formulas* of the corresponding axiom or rule, whereas the formulas in the sets Γ are referred to as *side formulas*. Often, we do not mention the side formulas then introducing an axiom or a rule. The rules of our formal theories only comprise rules with one or two premises, the semi-formal systems comprise in addition rules with infinitely many premises.

Below, we define the notion of *proof*, also called *derivation*, in a [semi-] formal theory T.

Definition I.2.1 (Proof) *A proof of a set of formulas Γ in a theory T is a well-founded tree $\mathcal{D} = (x, R)$ together with a labeling function f that assigns to the root* *of $\mathcal{D}$ the set Γ, all leafs of $\mathcal{D}$ are mapped to axioms and if $u \in x$ is not a leaf and I is the set of R-successors of u, then*

$$\frac{f(v) : v \in I}{f(u)}$$

is a rule of T.

A common complexity measure for a proof is the *depth* of the corresponding tree and its *cut-rank*. Thereto, we presuppose a rank function, that assigns to each formula A of L an ordinal $\mathsf{rk}(A)$, called the rank of A, such that for all formulas $\mathsf{rk}(A) = \mathsf{rk}(\neg A)$. A rule of the form

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

is then called a cut with cut formulas A and $\neg A$ and cut-rank $\mathsf{rk}(A)$.

Definition I.2.2 (Depth and cut-rank of proofs) *For all ordinal α and ρ, and all sets $*$ of L formulas closed under negation, we define $\mathsf{T} \mid\!\frac{\alpha}{\rho}\, \Gamma$ and $\mathsf{T} \mid\!\frac{\alpha}{*}\, \Gamma$ by recursion on α:*

(i) If Γ is an axiom of T, then $\mathsf{T} \mid\!\frac{\alpha}{*}\, \Gamma$ and $\mathsf{T} \mid\!\frac{\alpha}{\rho}\, \Gamma$ for all ordinals α and ρ.

(ii) If $\mathsf{T} \mid\!\frac{\alpha_i}{\rho}\, \Gamma_i$ [$\mathsf{T} \mid\!\frac{\alpha_i}{*}\, \Gamma_i$] and $\alpha_i < \alpha$ hold for all premises Γ_i of a rule that is not a cut, or a cut whose cut-rank is less than ρ [a cut, whose cut-formulas $A, \neg A$ are elements of the set $*$], then $\mathsf{T} \mid\!\frac{\alpha}{\rho}\, \Gamma_i$ holds for the conclusion of this rule.

We write $\mathsf{T} \vdash \Gamma$ if there are ordinals α', ρ' such that $\mathsf{T} \mid\!\frac{\alpha'}{\rho'}\, \Gamma$ and $\mathsf{T} \mid\!\frac{<\alpha}{<\rho}\, \Gamma$, if there are ordinals $\alpha' < \alpha$, $\rho' < \rho$, such that $\mathsf{T} \mid\!\frac{\alpha'}{\rho'}\, \Gamma$.

$\mathsf{T} \mid\!\frac{\alpha}{\rho}\, \Gamma$ expresses that there is a proof of the finite set Γ in T of depth α that only contains cuts with rank less than ρ, and $\mathsf{T} \mid\!\frac{\alpha}{*}\, \Gamma$ tells us that the proof of Γ has depth less than α and contains only cuts whose cut-formulas are in $*$. Note, that if T is a formal theory and $\mathsf{T} \vdash \Gamma$, then there is already an n such that $\mathsf{T} \mid\!\frac{n}{}\, \Gamma$. In other words, proofs in formal theories are finite trees.

If the theory T is formulated in the language L, we call the L-structure $\mathcal{M}$ a model of T, also written $\mathcal{M} \models \mathsf{T}$ if for all axioms Γ of T, $\mathcal{M} \models \Gamma$ and for each rule $\mathcal{R}$, if $\mathcal{M} \models \Gamma_i$ for each premise Γ_i of the rule $\mathcal{R}$, then also $\mathcal{M} \models \Gamma$ for the conclusion Γ of this rule. Further, we say that $\mathcal{M}$ is countable, if the structure $\mathcal{M}$ is countable.

I.2.3 The languages L_1 and L_2 of first and second order arithmetic

The languages L_1 and L_2 serve to speak about the natural numbers and subsets of the natural numbers, respectively. Our language L_1 of first order arithmetic consists of infinitely many *number variables*, function and a relation symbol of ariety $(e)_1$ for each index $e \in \mathsf{Prim}$ and in addition, two unary relation symbols U and V, required for technical reasons. Unless explicitly mentioned, the function symbol with the least index of the primitive recursive function f is meant, then referring to the function symbol for f, and a corresponding convention applies to the primitive recursive relation symbols. We use s, $+$ and $\cdot$ for the function symbols of successor, addition and multiplication. For each natural number we have a constant cs_n, but often write $0, 1, 2, \ldots$ instead of $\mathsf{cs}_0, \mathsf{cs}_1, \mathsf{cs}_2, \ldots$, unless we like to stress that cs_i is a closed term and not an element of $\mathbb{N}$. The relation symbol for the natural numbers is N; $<_\mathsf{N}$ and $=_\mathsf{N}$ are the symbols for the standard ordering and the equality relation on $\mathbb{N}$. Often however, we drop the subscript $_\mathsf{N}$. The letters $f, g, h, \ldots$, are meant to range over primitive recursive function symbols, whereas $Q, R, \ldots$ are to range over primitive recursive relation symbols. Our language L_2 for second order arithmetic extends L_1 by infinitely many set variables and a symbol $\in$ for elementhood, where $t \notin U$ is short for $\sim (t \in U)$. If R is a unary relation symbol, we sometimes write $t \in R$ and $t \notin R$ for $R(t)$ and $\sim R(t)$, respectively. $U = V$ stands for $\forall x[x \in U \leftrightarrow x \in V]$ and $U \neq V$ for $\neg(U = V)$. Further, if $A(U)$ and $B(u)$ are formulas of L_2, then we write $A(\{x : B(x)\})$ for the formula that is obtained from $A(U)$ by replacing each literal of the form $[\sim](t \in U)$ in A by $[\sim]B(t)$.

The L_1-structure $(\mathbb{N}, \cdot^\mathbb{N})$, where $\cdot^\mathbb{N}$ assigns to each primitive recursive function [relation] symbol the corresponding function [relation] and $\mathsf{U}^\mathbb{N} = \mathsf{V}^\mathbb{N} = \emptyset$, is called the *standard structure* for L_1, and the L_2-structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \cdot^\mathbb{N})$, where now $\in^\mathbb{N}$ is the standard elementhood relation, is the standard structure for L_2. Often, we simply use $\mathbb{N}$ to refer to either of these standard structures. Further, we say that a formula is *true* [*false*] if $\mathbb{N} \models A$ [$\mathbb{N} \not\models A$].

If $A(u)$ is a formula of L_1 or L_2, x a number variable and t a number term that does not contain x, then we use $(\forall x < t)A(x)$ [$(\exists x < t)A(x)$] as a shortcut for $\forall x(x < t \to A(x))$ [$\exists x(x < t \land A(x))$]. The quantifiers $(\forall x < t)$ and $(\exists x < t)$ are called (numerically) *bounded quantifiers*. Formulas that are built up from literals by means of the connectives $\land$ and $\lor$ and (numerically) bounded quantifiers are called Π^0_0, Σ^0_0 or also Δ^0_0 formulas. If A is a Π^0_k [Σ^0_k] formula, then A and $\exists x A$ [$\forall x A$] are Σ^0_{k+1} [Π^0_{k+1}] formulas. If A is a formula of L_2 that does not contain bound set variables, then we refer to A as an *arithmetical formula*, and sometimes also as Π^1_0, Σ^1_0 or also Δ^1_0 formula. If A is a Π^1_k [Σ^1_k] formula, then A and $\exists X A$ [$\forall X A$] are Σ^1_{k+1} [Π^1_{k+1}] formulas. The class of Π formulas of L_2 is then the smallest class containing the arithmetical formulas which is closed under conjunction, disjunction, number quantification and universal set quantification. If A is a Π formula, then $\neg A$ is a Σ formula.

Sets (in the meta-mathematical sense) are classified accordingly. If $\mathcal{M} = \langle M, \mathcal{S}_M \rangle$ is a structure for L_2 and $A(\vec{U}, u, \vec{v})$ is e.g. a Π^1_1 formula of L_2 with exactly the displayed variables free, then Y is called Π^1_1 in $\vec{X}$ w.r.t. $\mathcal{M}$, if there are $\vec{m} \in M$, such that

$$Y = \{x : (\mathcal{M}, \mathcal{E}[\vec{U} = \vec{X}, u = x, \vec{v} = \vec{m}]) \models A(\vec{U}, u, \vec{v})\}.$$

If $\vec{X}$ is the empty string, then Y is simply called Π^1_1. The number parameters in the defining formula $A(\vec{U}, u, \vec{v})$ are required, since we do not have constants for all elements in M but all the same want $\{m\}$ to be a definable set for each $m \in M$. For $k > 0$, a set is Δ^0_k in $\vec{X}$ [Δ^1_k in $\vec{X}$] w.r.t. $\mathcal{M}$, if it is Π^0_k [Π^1_k] and Σ^0_k [Σ^1_k] in $\vec{X}$. Occasionally, we also speak of Δ^0_k and Δ^1_k formulas. This notion does not syntactically describe a class of formulas, but is rather a manner of speaking: By $C(u)$ is Δ^1_1 for instance, we meant there is a Π^1_1 formula $A(u)$ and a Σ^1_1 formula $B(u)$, which both may contain additional free variables such that A, B and C are equivalent.

I.2.4 The language $\mathcal{L}^*$ of Kripke-Platek set theory

The intended use of the language $\mathcal{L}^*$ of Kripke-Platek set theory is to speak about a universe of sets with the natural numbers as *urelements*. Urelements are objects that are outright given to us. They do not contain any elements and are no sets. The language $\mathcal{L}^* = \mathsf{L}_1(\in, \mathcal{S})$ is an extension of L_1 by the membership relation symbol $\in$ and a unary relation symbols $\mathcal{S}$ to distinguish sets from urelements. Since primitive recursive function symbols hardly ever play a role in a formal argument in Kripke-Platek set theory, we also use the letters f, g, h to denote variables.

Then working in the language $\mathcal{L}^*$, we apply the following short cuts: Equality between objects is not represented by a primitive symbol, but defined by

$$(s = t) := \begin{cases} (s \in \mathsf{N} \wedge t \in \mathsf{N} \wedge (s =_{\mathsf{N}} t)) \vee \\ (\mathcal{S}(s) \wedge \mathcal{S}(t) \wedge (\forall x \in s)(x \in t) \wedge (\forall x \in t)(x \in s)). \end{cases}$$

Further, $u = \mathsf{N}$ abbreviates $\forall x[x \in u \leftrightarrow \mathsf{N}(x)]$ and $\vec{u} \in \mathcal{S}$ is an other way to say $\mathcal{S}(\vec{u})$. As usual, $u \subseteq v$ states that u is a subset of the set v, formally, $u \subseteq v$ is short for $\mathcal{S}(u) \wedge \mathcal{S}(v) \wedge (\forall x \in u)(x \in v)$. The quantifiers $(\forall x \in a)$ and $(\exists x \in a)$ are called (setwise) bounded quantifier. The formula A^u is then the result of replacing each *(setwise) unbounded quantifier* $\mathcal{Q}x(\ldots)$ in A by a *(setwise) bounded quantifier* $(\mathcal{Q}x \in u)(\ldots)$, where $(\forall x \in u)B$ is to abbreviate $\forall x(x \in u \to B)$ and $(\exists x \in u)B$

stands for $\exists x(x \in u \wedge B)$. Accordingly, for a finite set Γ of $\mathcal{L}^*$ formulas, Γ^u represents the set $\{A^u : A \in \Gamma\}$. Further, we use $\mathsf{Tran}(u)$ to express that u is a transitive set, $\mathsf{Fun}(u)$ says that u is a function, $\mathsf{Dom}(u)$ denotes the set $\{x : \exists y[(x, y) \in u]\}$ and $\mathsf{Rng}(u)$ the set $\{y : \exists x[(x, y) \in u]\}$. Then working in $\mathcal{L}^*$, we tend to use the variables f, g, h to range over functions, and the variables a, b, c as parameters.

Again, formulas that are built up from literals by means of the connectives $\wedge$ and $\vee$ and (setwise) bounded quantifiers are called Δ_0 formulas. The formula classes Π_k, Σ_k, Δ as well as Σ and Π are then defined in analogy to the corresponding classes of L_1 formulas.

I.2.5 The language $\mathbb{L}$ for explicit mathematics

As the language L_2, the language $\mathbb{L}$ for explicit mathematics has two sorts of variables: The lower case variables are called *individual variables* and the upper case variables *type variables*. $\mathbb{L}$ is obtained from L_2 in the following way: First we drop the symbols for the primitive recursive functions and relations, keeping only the constants cs_n and the unary relation symbols $\mathsf{N}, \mathsf{U}, \mathsf{V}$. Then additional constants k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and projections), s_N (successor), p_N (predecessor), d_N (definition by numerical cases) as well as constants called *generators*, which will be used for the uniform representation of types, namely, $\mathsf{nat}, \mathsf{cs}_\mathsf{U}, \mathsf{cs}_\mathsf{V}$ (natural numbers, U, V), id (identity), co (complement), int (intersection), dom (domain), inv (inverse image) and j (join) are added. Further, there is one binary function symbol $\cdot$ for (partial) application of individuals to individuals, a unary relation symbols $\downarrow$ (defined), a binary relation symbol $=$ (equality on individuals) and a binary relation symbol $\Re$ (naming, representation).

The atoms of $\mathbb{L}$ are $s\downarrow$, $\mathsf{N}(s)$, $s = t$, $s \in U$ and $\Re(U, s)$. Since we will work with a logic of partial terms, it is not guaranteed that all terms have values, and the intended meaning of $s\downarrow$ is *s is defined* or *s has a value.* Moreover, $\mathsf{N}(s)$ says that s is a natural number, and the formula $\Re(U, s)$ is used to express that the individual *s represents* the type U or *is a name of* U. We agree to abbreviate $s \cdot t$ simply as st and adopt the convention of association to the left, so that $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 s_2)\dots s_n)$. General n-tupling is defined by induction on $n \geq 2$ as follows: $(s_1, s_2) := \mathsf{p} s_1 s_2$ and $(s_1, \dots, s_{n+1}) := (s_1, (s_2, \dots, s_{n+1}))$. Further, we write $t{+}1$ for $\mathsf{s}_\mathsf{N} t$ and $t\uparrow$ for $\sim t\downarrow$ and define the notion of *partial equality* between individuals, $s \simeq t := s\downarrow \vee t\downarrow \rightarrow s = t$.

A formula is called *elementary*, if it contains neither the relation symbol $\Re$ nor bound type variables. The Π and Σ formulas are defined as for the language L_2. Additional formula classes are the Σ^+ formulas, i.e. Σ formulas that contain no subformula of the form $\sim \Re(X, t)$ and the Π^- formulas, i.e. Π formulas that do not contain subformulas of the form $\Re(X, t)$.

I.2.6 First and second order predicate logic and the logic of partial terms

With the exception of the theories formulated in $\mathbb{L}$ or extensions thereof, which are built upon Beeson's [5] *logic of partial terms*, the underlying logic of the theories introduced later on is classical first or second order predicate logic. The axioms and rules of a theory T regarding the underlying logic are called the *logical axioms and rules* of T, all the other axioms and rules are referred to as *non-logical axioms and rules*. For a theory T, the formulas that T can reason about are called the formulas of T. Often, if T is formulated in some language L, then the formulas of T are all the L formulas, but sometimes they may consist of a proper subset, e.g. only the closed formulas of L. The sets of relation symbols, function symbols and terms of T are defined accordingly.

Below we list the logical axioms and rules of a theory T based on first or second order predicate logic. Thereby, Γ, Δ and Λ range over finite sets of formulas of T, and A, B, C range over formulas of T.

Basic axioms. For each Γ and each atom A:

$$\Gamma, A, \neg A.$$

Propositional rules. For each Γ and each A, B:

$$\frac{\Gamma, A}{\Gamma, A \vee B} \qquad \frac{\Gamma, B}{\Gamma, A \vee B} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

Quantifier rules. For each Δ, Γ with $u \notin FV(\Delta)$ and, if T comprises second order formulas also for each Λ with $U \notin FV(\Lambda)$, each A and each term s of T:

$$\frac{\Gamma, A(s)}{\Gamma, \exists x A(x)} \quad \frac{\Delta, A(u)}{\Delta, \forall x A(x)} \qquad \frac{\Gamma, A(U)}{\Gamma, \exists X A(x)} \quad \frac{\Lambda, A(U)}{\Lambda, \forall X A(X)}$$

Cut rules. For each Γ and each A:

$$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

The formulas A and $\neg A$ are the *cut formulas* of the cut.

The logic of partial terms comprises additional axioms for the relation symbol $\downarrow$ and has a different rule for the first order existential quantifier that takes into account the possible non-definedness of terms. All the other rules are inherited from predicate logic.

Axioms for $\downarrow$. For each Γ, all individual variables u, constants c, individual terms $\vec{t}$, relation symbols R and function symbols f of T:

$$\Gamma, u\downarrow, \quad \Gamma, c\downarrow, \quad \Gamma, {\sim}f(\vec{t})\downarrow, \vec{t}\downarrow, \quad \Gamma, {\sim}R(\vec{X}, \vec{t}), \vec{t}\downarrow,$$

where for $\vec{t} = t_1, \ldots, t_n$, $\vec{t}{\downarrow}$ abbreviates $t_1{\downarrow} \wedge \ldots \wedge t_n{\downarrow}$.

Existential number quantification. For each Γ and each individual term t of T:

$$\frac{\Gamma, A(t) \quad \Gamma, t{\downarrow}}{\Gamma, \exists x A(x)}.$$

To conclude this paragraph, we remark that the logical axioms and rules do not distinguish a symbol $=$ that is to be interpreted as equality on the first order objects. Instead, $=$ or $=_{\mathbb{N}}$ are binary relation symbols of our theories, and only the axioms and rules of T ensure that $=$ or $=_{\mathbb{N}}$ are interpreted as equivalence relations on their intended domain.

I.2.7 Peano Arithmetic

The most prominent first order theory is of course Peano Arithmetic PA. It is used to reason about the natural numbers. Its non-logical axioms and rules are part of most of the other theories we introduce, thus we treat them in detail. Besides the axioms and rules of first order predicate calculus, PA comprises axioms for all the primitive recursive functions and relations and induction along the natural numbers.

The theory PA is formulated in L_1 and the formulas of PA are all the L_1 formulas. Its non-logical axioms and rules are listed below:

Equality axioms. Making use of the shortcut $\vec{s} = \vec{t}$ for $\bigwedge_{1..n}(s_i = t_i)$, the equality axioms take the following form: For all finite sets Γ of L_1 formulas and all function and relation symbols f and R of L_1,

$$\Gamma, u = u, \qquad \Gamma, \vec{u} \neq \vec{v}, f(\vec{u}) = f(\vec{v}),$$

$$\Gamma, \vec{u} \neq \vec{v}, {\sim}R(\vec{u}), R(\vec{v}).$$

Axioms for the primitive recursive functions and relations. If $e \in \mathsf{Prim} \subseteq \mathbb{N}$, we let f_e be a function symbol of the function $\{(m,n) : \{e\}(m) = n\}$. Similarly, if $e \in \mathsf{Prim}$ is an index of a characteristic function, then R_e denotes the relation symbol of the corresponding relation. Below we write s for $f_{\langle 0,1\rangle}$.

For all finite sets Γ of L_1 formulas we have the following axioms:

1. The successor axioms:
   $\Gamma, 0 \neq \mathsf{s}(u)$ and $\Gamma, \mathsf{s}(u) = \mathsf{s}(v) \to u = v$ and for all $n \in \mathbb{N}$: $\Gamma, \mathsf{s}(\mathsf{cs}_n) = \mathsf{cs}_{n+1}$.

2. Axioms for the constant functions:
   $\Gamma, f_e(u_1, \ldots, u_n) = \mathsf{cs}_m$, if $e = \langle 1, n, m\rangle \in \mathsf{Prim}$.

3. Axioms for the projections:
   $\Gamma, f_e(u_1, \ldots, u_n) = u_i$, if $e = \langle 2, n, i \rangle \in \mathsf{Prim}$.

4. Axioms for the composition of functions:
   $\Gamma, f_e(u_1, \ldots, u_n) = f_a(f_{b_1}(u_1, \ldots, u_n), \ldots, f_{b_m}(u_1, \ldots, u_n))$,
   if $e = \langle 3, n, a, b_1, \ldots, b_m \rangle \in \mathsf{Prim}$.

5. Axioms for the schema of primitive recursion:
   $\Gamma, f_e(0, u_2, \ldots, u_n) = f_a(u_2, \ldots, u_n)$ and
   $\Gamma, f_e(\mathsf{s}(u_1), u_2, \ldots, u_n) = f_b(f_e(u_1, u_2, \ldots, u_n), u_1, u_2, \ldots, u_n)$,
   if $e = \langle 4, n, a, b \rangle \in \mathsf{Prim}$.

6. Axioms for the primitive recursive relations:
   $\Gamma, R_e(\vec{u}) \leftrightarrow f_e(\vec{u}) = 0$, if $e \in \mathsf{Prim}$ is an index of a characteristic function.

Induction along the natural numbers. For all finite sets Γ of L_1 formulas and all formulas $A(u)$ of L_1, we have:

$$(\mathsf{IND_N}) \qquad \Gamma, A(0) \wedge \forall x[A(x) \to A(x{+}1)] \to \forall x A(x).$$

I.2.8 The semi-formal systems PA^*

Due to Gödel's second incompleteness theorem, the formal theory PA is incomplete w.r.t. the standard structure $\mathbb{N}$, which means that there are true sentences that we cannot prove. On the other hand, the semi-formal system PA^* introduced below is designed in such a way, that the quantified number variables range always over $\mathbb{N}$, which forces completeness. The completeness of PA^* w.r.t. to $\mathbb{N}$ is shown easily applying the technique of deduction chains, cf. e.g. [28]. The price we pay to prove all true sentences is the presence of an ω-rule, a rule with ω many premises, which turns proofs into infinite objects; the depth of a proof may be greater than ω.

The semi-formal system PA^* is formulate in the language L_1. The formulas of PA^* are the closed formulas of L_1. In order to state the axioms and rules of PA^*, we assign to each closed number term t of L_1 its value $t^{\mathbb{N}}$ in the standard model and say that two literals are *numerically equivalent* if they are syntactically equivalent modulo subterms which have the same value. The *true [false] literals of* L_1 are the closed literals of L_1 that evaluate to true [false] in the standard model.

Axioms of PA^*. For Γ, all true literals A and all numerically equivalent literals B and C of PA^*:

$$\Gamma, A \qquad \text{and} \qquad \Gamma, \neg B, C$$

The rules of PA^* are the rules of first order predicate logic with the rule for the universal number quantifier replaced by the so-called *ω-rule*:

$$\frac{\Gamma, A(t) \text{ for all closed number terms } t}{\Gamma, \forall x A(x)} (\omega\text{-rule}).$$

I.2.9 The theories BS^0 and KPu^0

The theory KPu^0 is formulated in the language $\mathcal{L}^*$ and the formulas of KPu^0 are all the $\mathcal{L}^*$ formulas. Its non-logical axioms can be divided into the following groups.

Ontological axioms. We have for all terms $r, \vec{s}$ and t of L_1, all function symbols $\mathcal{H}$ and relation symbols $\mathcal{R}$ of L_1 and all finite sets of $\mathcal{L}^*$ formulas Γ:

1. $\Gamma, u \in \mathsf{N} \leftrightarrow \neg\mathcal{S}(u)$,
2. $\Gamma, \vec{u} \notin \mathsf{N}, \mathcal{H}(\vec{u}) \in \mathsf{N}$,
3. $\Gamma, \sim\!\mathcal{R}(\vec{u}), \vec{u} \in \mathsf{N}$,
4. $\Gamma, u \notin v, \mathcal{S}(v)$,
5. $\Gamma, \exists x(x = \mathsf{N})$.

Number-theoretic axioms. We have for all axioms $\Delta(\vec{u})$ of Peano arithmetic PA which are not instances of the schema of complete induction and whose free variables belong to the list $\vec{u}$ and all finite sets of $\mathcal{L}^*$ formulas Γ:

(Number theory) $\Gamma, \vec{u} \notin \mathsf{N}, \Delta^{\mathsf{N}}(\vec{u})$.

Equality axioms. For the natural numbers the equality axioms are inherited from PA. That x and y have the same elements if $x = y$ is due to the definition of $=$. Still, an equality axiom for sets is needed.

(Equality) $\Gamma, u = v \land u \in w \to v \in w$.

Kripke Platek axioms. We have for all Δ_0 formulas $A(u)$ and $B(u, v)$ of $\mathcal{L}^*$:

(Pair) $\Gamma, \exists x(w \in x \land v \in x)$,

(Tran) $\Gamma, \mathcal{S}(w) \to \exists x(w \subseteq x \land \mathsf{Tran}(x))$,

(Δ_0-Sep) $\Gamma, \exists y(\mathcal{S}(y) \land y = \{x \in w : A(x)\})$,

(Δ_0-Col) $\Gamma, (\forall x \in w)\exists y B(x, y) \to \exists z(\forall x \in w)(\exists y \in z)B(x, y)$.

Set induction. The only induction principle included in the axioms of KPu^0 is the following axiom of complete induction on the natural numbers for sets: For all finite sets of $\mathcal{L}^*$ formulas Γ,

($\mathcal{S}$-$\mathsf{I_N}$) $\Gamma, 0 \in u \land (\forall x \in \mathsf{N})(x \in u \to x{+}1 \in u) \to \mathsf{N} \subseteq u.$

The theory BS^0, called basic set theory, is KPu^0 without the axiom (Δ_0-Col) for Δ_0 collection. If we replace set induction by formula induction, i.e. for all finite sets Γ of $\mathcal{L}^*$ formulas and all formulas $A(u)$ of $\mathcal{L}^*$:

$$(\mathsf{I_N}) \qquad \Gamma, A(0) \land \forall x(A(x) \to A(x{+}1)) \to \forall x A(x),$$

then the resulting theories are simply named $\mathsf{BS}^0{+}(\mathsf{I_N})$ and $\mathsf{KPu}^0{+}(\mathsf{I_N})$, respectively. Later in this thesis, we also consider foundation as an additional axiom, claiming that each non-empty set has an $\in$-least element,

$$(\mathsf{I}_\in) \qquad \Gamma, \emptyset \subsetneq u \to (\exists x \in u)(x \cap u = \emptyset).$$

The theory $\mathsf{KPu}^0 + (\mathsf{I}_\in)$ goes under the name KPu^r (cf. [20]). Below, we gather some elementary properties of KPu^0. For proofs, we refer the reader to Barwise [3].

Lemma I.2.3 (Σ Reflection) *For each Σ formula A of $\mathcal{L}^*$, we have:*

$$\mathsf{KPu}^0 \vdash A \leftrightarrow \exists x A^x.$$

In fact, the Σ reflection principle is equivalent to (Δ_0-Col). Next, we mention two useful strengthenings of the separation and collection axioms.

Lemma I.2.4 (Δ Separation) *For each Σ formula $A(u)$ and each Π formula $B(u)$ of $\mathcal{L}^*$, the following is provable in KPu^0:*

$$(\forall x \in w)[A(x) \leftrightarrow B(x)] \to \exists y[y = \{z \in w : A(z)\}].$$

Lemma I.2.5 (Σ Collection) *For each Σ formula $A(u,v)$ of $\mathcal{L}^*$, the following is provable in KPu^0:*

$$(\forall x \in w)\exists y A(x,y) \to \exists z(\forall x \in w)(\exists y \in z)A(x,y)$$

Combining the previous to lemmas yields another neat consequence, which comes in handy then working in KPu^0.

Lemma I.2.6 (Σ Replacement) *For each Σ formula $A(u,v)$ of $\mathcal{L}^*$, the following is provable in KPu^0:*

$$(\forall x \in w)\exists! y A(x,y) \to \exists f[\mathsf{Fun}(f) \land \mathsf{Dom}(f) = w \land (\forall x \in w)A(x, f(x))].$$

I.2.10 ACA and ACA_0: Second order theories with arithmetical comprehension

Most of the theories of second order arithmetic that play a role in this thesis are extensions of ACA_0. The acronym ACA stands for *arithmetical comprehension*. The axioms of ACA assert the existence of subsets of $\mathbb{N}$ which are definable from given sets by *arithmetical formulas*. The non-logical axioms and rules of ACA_0 are the axioms and rules of PA without induction adapted to the language L_2, the aforementioned comprehension axioms and an axiom for set induction:

Arithmetical comprehension. For all finite sets Γ of L_2 formulas and each arithmetical formulas $A(u)$ of L_2 with $X, x \notin FV(A)$:

$$\Gamma, \exists X[\forall x(x \in X \leftrightarrow A(x))].$$

Set induction. For all finite sets Γ of L_2 formulas:

$$\Gamma, 0 \in U \wedge \forall x(x \in U \rightarrow x{+}1 \in U) \rightarrow \forall x(x \in U).$$

The theory ACA is obtained from ACA_0 by replacing the axiom for set induction by an axiom for *formula induction*.

Formula induction. For all finite sets Γ of L_2 formulas and all formulas $A(u)$ of L_2:

$$\Gamma, A(0) \wedge \forall x(A(x) \rightarrow A(x{+}1)) \rightarrow \forall x A(x).$$

Sometimes, we consider restrictions of formula induction. By ($\mathcal{K}$-IND_N) we denote the restriction of the above axiom schema, where the formula A has to be an element of the formula class $\mathcal{K}$.

Sets in theories comprising ACA_0 are meant to be subsets of $\mathbb{N}$. However, one often likes to speak about relations and functions, which are subsets of $\mathbb{N}^n$, or even sets of sets. A bit of coding makes this possible: For sets $\vec{X}$, we define the product $X_1 \times \ldots \times X_n$ to be the set $\{\langle x_1, \ldots, x_n\rangle : \bigwedge_{1 \leq i \leq n}(x_i \in X_i)\}$. Thus, an n-ary relation R can be identified with the set $\{\langle \vec{x}\rangle : R(\vec{x})\}$ and an n-ary function f is coded as $\{\langle \vec{x}, y\rangle : f(\vec{x}) = y\}$. Moreover, $(X)_k$ denotes the set $\{x : \langle x, k\rangle \in X\}$. Hence X can be seen as the collection of the sets $\{(X)_k : k \in \mathbb{N}\}$. This motivates the abbreviation $Y \dot{\in} X$ for the formula $\exists z[Y = (X)_z]$. Finitely many sets $X_1, \ldots, X_n$ are coded into a single one by forming their disjoint union $\oplus\vec{X} := \bigcup_{1 \leq i \leq n}\{\langle y, \mathsf{cs}_i\rangle : y \in X_i\}$. Further, $(\forall Y \dot{\in} X)A(X)$ $[(\exists Y \dot{\in} X)A(X)]$ is a shortcut for $\forall z A((X)_z)$ $[\exists z A((X)_z)]$, and $X \doteq Y$ is to express that $\forall Z[Z \dot{\in} X \leftrightarrow Z \dot{\in} Y]$. $X \dot{\notin} Y$ and $X \dot{\neq} Y$ are defined accordingly. To any formula A of L_2 with $Z \notin FV(A)$, we assign an arithmetical formula A^Z by replacing each second order quantifier $\forall X$ and $\exists X$ by $(\forall X \dot{\in} Z)$ and $(\exists X \dot{\in} Z)$, respectively. In A^Z, the range of the set quantifiers is restricted to elements of Z w.r.t. the $\dot{\in}$ relation. If $\prec$ is an ordering and $K \subseteq \mathsf{Field}(\prec)$, we denote by $(X)_{\prec K}$ the disjoint union of the sets $(X)_\beta$ for $\beta \prec K$, namely.

$$(X)_{\prec K} := \{\langle x, \beta \rangle : \beta \prec K \wedge x \in (X)_\beta\},$$

where $\beta \prec K$ abbreviates that β is $\prec$-smaller than all the elements of K. Finally, if $K \subseteq \mathsf{Field}(\prec)$ is of the form $\{\alpha\}$, we write $(X)_{\prec \alpha}$ instead of $(X)_{\prec \{\alpha\}}$.

I.2.11 The theory EETJ_0

Explicit mathematics has been introduced by Feferman [8, 9, 10] for the study of constructive mathematics. We will not work with Feferman's original formalization of these systems; instead we treat them as *theories of types and names* as developed by Jäger [18]. All the systems of explicit mathematics that will be used in the subsequent chapters are extensions of the base theory EETJ_0 of explicit elementary types and join introduced below, which is based on Beeson's logic of partial terms. The theory EETJ_0 is formulated in $\mathbb{L}$. To state its non-logical axioms and rules, we use the following abbreviations:

$$\begin{aligned}
\Re(s) &:= \exists X \Re(X, s), \\
s \,\dot{\in}\, t &:= \exists X(\Re(X, t) \wedge s \in X), \\
s \,\dot{\subseteq}\, t &:= (\forall x \,\dot{\in}\, s)(x \,\dot{\in}\, t), \\
s \doteq t &:= s \,\dot{\subseteq}\, t \wedge t \,\dot{\subseteq}\, s, \\
(f : \mathsf{N} \to \mathsf{N}) &:= \forall x(\mathsf{N}(x) \to \mathsf{N}(fx)),
\end{aligned}$$

Equality axioms. For all finite sets Γ of $\mathbb{L}$ formulas and all relation symbols R of $\mathbb{L}$:

$$\Gamma, u = u, \qquad \Gamma, u_1 = v_1 \wedge u_2 = v_2 \to u_1 u_2 \simeq v_1 v_2,$$

$$\Gamma, \vec{u} = \vec{v} \wedge R(\vec{U}, \vec{u}) \to R(\vec{U}, \vec{v}).$$

Axioms for the constants cs_n. For each natural number $n \in \mathbb{N}$ we have

$$\mathsf{cs}_{n+1} = \mathsf{cs}_n + 1.$$

Applicative axioms. These axioms formalize that the individuals form a partial combinatory algebra, that we have pairing and projection, the usual closure conditions on the natural numbers and definition by numerical cases.

1. $\mathsf{k}uv = u$,

2. $\mathsf{s}uv\downarrow \wedge\, \mathsf{s}uvw \simeq uw(vw)$,

3. $\mathsf{p_0}(u,v) = u \wedge \mathsf{p_1}(u,v) = v$,

4. $0 \in \mathsf{N} \wedge (\forall x \in \mathsf{N})(x{+}1 \in \mathsf{N})$,

5. $(\forall x \in \mathsf{N})(x{+}1 \neq 0 \wedge \mathsf{p_N}(x{+}1) = x)$,

6. $(\forall x \in \mathsf{N})(x \neq 0 \rightarrow \mathsf{p_N}x \in \mathsf{N} \wedge (\mathsf{p_N}x){+}1 = x)$,

7. $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u = v \rightarrow \mathsf{d_N}xyuv = x$,

8. $u \in \mathsf{N} \wedge v \in \mathsf{N} \wedge u \neq v \rightarrow \mathsf{d_N}xyuv = y$.

Explicit representation and extensionality. The following axioms state that each type has a name, that there are no homonyms and that types containing the same elements have the same names.

1. $\exists x \Re(U,x)$,

2. $\Re(U,u) \wedge \Re(V,u) \rightarrow U = V$,

3. $\Re(U,u) \wedge U = V \rightarrow \Re(V,u)$.

Basic type existence axioms. In the following we provide a finite axiomatization of uniform elementary comprehension plus join.

1. $\Re(\mathsf{nat}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{nat} \leftrightarrow \mathsf{N}(x))$,

2. $\Re(\mathsf{cs_U}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{cs_U} \leftrightarrow \mathsf{U}(x)) \wedge \mathsf{cs_U} \mathrel{\dot\subseteq} \mathsf{nat}$,

3. $\Re(\mathsf{cs_V}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{cs_V} \leftrightarrow \mathsf{V}(x)) \wedge \mathsf{cs_V} \mathrel{\dot\subseteq} \mathsf{nat}$,

4. $\Re(\mathsf{id}) \wedge \forall x(x \mathrel{\dot\in} \mathsf{id} \leftrightarrow \exists y(x = (y,y)))$,

5. $\Re(u) \rightarrow \Re(\mathsf{co}(u)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{co}(u) \leftrightarrow x \mathrel{\dot\notin} u)$,

6. $\Re(u) \wedge \Re(v) \rightarrow \Re(\mathsf{int}(u,v)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{int}(u,v) \leftrightarrow x \mathrel{\dot\in} u \wedge x \mathrel{\dot\in} v)$,

7. $\Re(u) \rightarrow \Re(\mathsf{dom}(u)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{dom}(u) \leftrightarrow \exists y((x,y) \mathrel{\dot\in} u))$,

8. $\Re(u) \rightarrow \Re(\mathsf{inv}(u,f)) \wedge \forall x(x \mathrel{\dot\in} \mathsf{inv}(u,f) \leftrightarrow fx \mathrel{\dot\in} u)$,

9. $\Re(u) \wedge (\forall x \mathrel{\dot\in} u)\Re(fx) \rightarrow \Re(\mathsf{j}(u,f)) \wedge A(u,f,\mathsf{j}(u,f))$.

In this last axiom the formula $A(u, v, w)$ expresses that w names the disjoint union of v over u, i.e.

$$A(u, v, w) := \forall x[x \mathrel{\dot\in} w \leftrightarrow \exists y, z(x = (y, z) \wedge y \mathrel{\dot\in} vz \wedge z \mathrel{\dot\in} u)].$$

Type induction. We have complete induction on N for types.

$$(\mathsf{T\text{-}I_N}) \qquad 0 \in U \wedge (\forall x \in \mathsf{N})(x \in U \rightarrow x{+}1 \in U) \rightarrow (\forall x \in \mathsf{N})(x \in U).$$

This are all the axioms of EETJ_0. If the axiom for join, the 9th of the basic type existence axioms is omitted, we are left we the theory EET_0.

In the original formulation of explicit mathematics, elementary comprehension is not dealt with by a finite axiomatization but directly as an infinite axiom scheme. The following result of Feferman and Jäger [12] shows that this scheme of uniform elementary comprehension is provable from our finite axiomatization.

Lemma I.2.7 (Elementary comprehension) *Suppose that $A(\vec{U}, \vec{u}, v)$ is an elementary formula of $\mathbb{L}$ with at most the displayed variables free. Then there exists a closed term t of $\mathbb{L}$ such that EETJ_0 proves:*

$$\forall \vec{X}, \vec{x}, \vec{y}[\Re(\vec{X}, \vec{x}) \rightarrow \Re(t(\vec{x}, \vec{y})) \wedge t(\vec{x}, \vec{y}) \doteq \{z : A(\vec{X}, \vec{y}, z)\}].$$

An other useful observation is that EETJ_0 proves λ-abstraction, a recursion theorem and a lemma about primitive recursion on N that allows us to model primitive recursive function by closed terms of $\mathbb{L}$.

Definition I.2.8 *Let t be a term of $\mathbb{L}$. Then $(\lambda x.t)$ is the term given by the following inductive definition:*

(i) $(\lambda x.t) := \mathsf{skk}$, *if* $t = x$,

(ii) $(\lambda x.t) := \mathsf{k}t$, *if x is not a free variable of t,*

(iii) $(\lambda x.t) := \mathsf{s}(\lambda x.s_1)(\lambda x.s_2)$, *if* $t = s_1 s_2$.

Lemma I.2.9 (λ-abstraction) *If s, t are terms of $\mathbb{L}$, then the following is provable in EETJ_0:*

$$(\lambda x.t)\downarrow \quad \textit{and} \quad (\lambda x.t)x \simeq t \quad \textit{and} \quad (\lambda x.t)s \simeq t[s/x].$$

Moreover, the free variables of $(\lambda x.t)$ are the free variables of t excluding x.

The next lemma helps us to find fixed points of operations in the specified sense.

Lemma I.2.10 (Recursion Theorem) *There is a closed term* $\underline{\mathsf{rec}}$ *of* $\mathbb{L}$ *such that* EETJ_0 *proves:*

$$\underline{\mathsf{rec}}y\downarrow \wedge \underline{\mathsf{rec}}yx \simeq y(\underline{\mathsf{rec}}y)x.$$

Applying the previous lemma helps to find a term modeling the schema of primitive recursion.

Lemma I.2.11 (Primitive recursion on N) *There is a closed term* $\underline{\mathsf{r}_\mathsf{N}}$ *of* $\mathbb{L}$ *such that* EETJ_0 *proves:*

(i) $(u : \mathsf{N} \to \mathsf{N}) \wedge (v : \mathsf{N}^3 \to \mathsf{N}) \to (\underline{\mathsf{r}_\mathsf{N}}uv : \mathsf{N}^2 \to \mathsf{N})$,

(ii) $(u : \mathsf{N} \to \mathsf{N}) \wedge (v : \mathsf{N}^3 \to \mathsf{N}) \wedge a, b \in \mathsf{N} \wedge w = \underline{\mathsf{r}_\mathsf{N}}uv \to$
$wa0 = ua \wedge wa(b{+}1) = vab(wab)$.

This allows us to assign to each primitive recursive function symbol f a closed term $\underline{\mathsf{f}}$ of $\mathbb{L}$ that represents this function in the following sense: Each main formula of an axiom for the primitive recursive functions of PA (cf. subsection I.2.7) becomes provable in EETJ_0, provided the function symbols f are replaced by the corresponding closed terms $\underline{\mathsf{f}}$. For instance, the L_1 formula $\mathsf{pr}^2_0(u, v) = u$ becomes the $\mathbb{L}$ formula $\underline{\mathsf{pr}^2_0}(u, v) = u$, where in the $\mathbb{L}$ formula, (u, v) denotes $\mathsf{p}uv$ as defined in subsection I.2.5.

Corollary I.2.12 *For each* $e \in \mathsf{Prim} \subseteq \mathbb{N}$*, there is a closed term* $\underline{\mathsf{f}_\mathsf{e}}$ *of* $\mathbb{L}$*, such that each main formula of an axiom for the primitive recursive functions of* PA *(cf. subsection I.2.7) becomes provable in* EETJ_0*, provided the function symbols* f_e *are replaced by the closed terms* $\underline{\mathsf{f}_\mathsf{e}}$.

Proofs of these theorems can be found in Beeson [5] or Feferman [8].

I.2.12 Translations and embeddings

Sometimes, we need to compare theories T and T' that are formulated in different languages L and L'. All the same, we may wish to establish a relation between the two theories, for instance that both theories prove "basically" the same formulas. To make such a statement formally precise, we introduce *translations* and *embeddings*.

A translation from L to L' is a function as described in the next paragraph, usually denoted by $\cdot^*$, that maps finite sets of formulas of L to finite sets of formulas of L'. We call a translation $\cdot^*$ an embedding of T into T', if for all finite sets Γ of L, $\mathsf{T} \vdash \Gamma$ implies $\mathsf{T}' \vdash \Gamma^*$. If this implication does not hold for all finite sets of L formulas but only for a finite sets of formulas from a certain class $\mathcal{K}$, when we say that $\cdot^*$ is an embedding of T into T' w.r.t. $\mathcal{K}$ formulas.

Many translations are induced by a *pretranslation* $\cdot^\star$ that maps variables, terms and atoms of L to variables, terms and formulas of L' and formulas $\mathcal{V}_i$, one for each sort of L variables, that specify the range of the corresponding sort of variables. Further, we assume that this pretranslation commutes with variable substitution, i.e. for atoms A we presuppose $(A[v/u])^\star = A^\star[v^\star/u^\star]$. Then, the pretranslation $\cdot^\star$ extends to all formulas of L in the expected way, e.g. for an atom A we define $(\sim A)^\star = \neg A^\star$, and for formulas A, B we agree that $(A \wedge B)^\star$ is $A^\star \wedge B^\star$, and $(\exists u A(u))^\star$ is $\exists u^\star(\mathcal{V}_1(u^\star) \wedge A^\star(u^\star))$. If L contains one sort of variables, the induced translations $\cdot^*$ assigns to the finite set $\Gamma(\vec{u})$ of L formulas containing exactly the variables $\vec{u}$ free, the finite set $\neg\mathcal{V}_1(\vec{u}^\star), \Gamma^\star(\vec{u}^\star)$ of L' formulas, and if L contains two sorts of variables, then we have that

$$(\Gamma(\vec{U}, \vec{u}))^* := \neg\mathcal{V}_1(\vec{U}^\star), \neg\mathcal{V}_2(\vec{u}^\star), \Gamma^\star(\vec{U}^\star, \vec{u}^\star).$$

Below, we define translations from the language L_2 into $\mathcal{L}^*$ and $\mathbb{L}$. Of course, this yields also translations from L_1 into $\mathcal{L}^*$ and $\mathbb{L}$. We will refer to these translations as standard translations throughout this thesis.

We start by giving a translation from L_2 into $\mathcal{L}^*$. Observe that L_2 and $\mathcal{L}^*$ comprise the same function and relation symbols with the exception of the relation symbol $\mathcal{S}$ of $\mathcal{L}^*$. Due to the above considerations it suffices to define the pretranslation and the formulas $\mathcal{V}_1(U)$ and $\mathcal{V}_2(u)$.

For number variables we set $u_i^\star := u_{2i}$, and for set variables, we define $U_i^\star := u_{2i+1}$. For number terms, $(f(\vec{t}))^\star$ is given by $f(\vec{t^\star})$. If R is a primitive recursive relation symbol, then the atom $R(\vec{t})$ is mapped to $R(\vec{t^\star})$ and to $t \in U$ we assign $t^\star \in U^\star$. The first order variables are to range over N, thus $\mathcal{V}_1(u) := \mathsf{N}(u)$ and the set variables are interpreted as subsets of N, therefore $\mathcal{V}_2(U) := U \subseteq \mathsf{N}$.

The translation $\cdot^*$ induced by the pretranslation $\cdot^\star$ is an embedding from PA or ACA_0 into KPu^0.

Lemma I.2.13 *For each finite set Γ of L_1 formulas and each finite set Δ of L_2 formulas, we have that*

$$\mathsf{PA} \vdash \Gamma \Longrightarrow \mathsf{KPu}^0 \vdash \Gamma^* \qquad \textit{and} \qquad \mathsf{ACA}_0 \vdash \Delta \Longrightarrow \mathsf{KPu}^0 \vdash \Delta^*.$$

Next, we are looking for a translation from L_2 to $\mathbb{L}$. This time, we define the pretranslation $\cdot^\star$ to map first order variables to individual variables and set variables to type variables. Moreover, $\cdot^\star$ is the identity on the constants cs_n. For terms t other than variables, $t^\star$ is defined inductively on the built-up of t: If $e \in \mathsf{Prim} \subseteq \mathbb{N}$, f_e is the function symbol for the primitive recursive function with index e and t is the term $f_e(\vec{s})$, then $t^\star := \underline{\mathsf{f_e}}(\vec{s^\star})$, where $\underline{\mathsf{f_e}}$ is the term provided by corollary I.2.12. As pointed out there, the expression $(\vec{s^\star})$ in the $\mathbb{L}$ term $\underline{\mathsf{f_e}}(\vec{s^\star})$ is the tuple $(s_1^\star, \dots, s_n^\star)$.

If R is a primitive recursive relation symbol, $\cdot^\star$ maps $R(\vec{t})$ to $\underline{\mathsf{ch_R}}(\vec{t}^\star) = 0$, where $\underline{\mathsf{ch_R}}$ is the closed term that represents the characteristic function of R. $t \in U$ is mapped to $t^\star \in U^\star$. Again $\mathcal{V}_1(u) := \mathsf{N}(u)$ and $\mathcal{V}_2(U) := U \subseteq \mathsf{N}$. This time we have:

Lemma I.2.14 *For each finite set Γ of L_1 formulas and each finite set Δ of L_2 formulas,*

$$\mathsf{PA} \vdash \Gamma \Longrightarrow \mathsf{EETJ}_0 \vdash \Gamma^* \qquad \textit{and} \qquad \mathsf{ACA}_0 \vdash \Delta \Longrightarrow \mathsf{EETJ}_0 \vdash \Delta^*.$$

I.2.13 On the dispensability of primitive recursive function symbols

Beside the language $\mathbb{L}$ which only contains the binary function symbol $\cdot$ and the constants cs_n, the function symbols of all the other languages introduced so far are those for the primitive recursive functions. However, when working within a theory T that comprises the axioms for the primitive recursive function and relation symbols, function symbols with an ariety greater than 0 become superfluous. Instead of working with a primitive recursive function symbol $f(\vec{u})$, we work with its graph $R_f := \{(\vec{u}, v) : f(\vec{u}) = v\}$, for which there is a relation symbol. In this way some technical arguments performed later in this thesis can be simplified. This paragraph shows how to remove primitive recursive function symbols by embedding T into a theory T^- whose formulas contain no primitive recursive function symbols and no constants.

It is straight forward to embed a theory T comprising the axioms for the primitive recursive function and relation symbols into a corresponding theory T^- whose formulas contain no primitive recursive function symbols except constants. Thereto, we first assign to each number term t of T a formula $\mathsf{Val}_t(u)$, expressing that u is the value of t. If t is a variable then $\mathsf{Val}_t(u) := (u = t)$. If $\vec{s}$ are number terms for which $\mathsf{Val}_{\vec{s}}(\vec{u})$ (our shortcut for $\mathsf{Val}_{s_1}(u_1) \wedge \ldots \wedge \mathsf{Val}_{s_n}(u_n)$) are already defined, and f a primitive recursive function symbol with ariety bigger than 0, then $\mathsf{Val}_{f(\vec{s})}(u) := (\exists \vec{y} \in \mathsf{N})(\mathsf{Val}_{\vec{s}}(\vec{y}) \wedge R_f(\vec{y}, u))$, where we take care that the variables $\vec{y}$ do not occur free in $\vec{s}$. This induces a pretranslation on atoms of T. We set $(R(\vec{U}, \vec{s}))^\star := (\exists \vec{x} \in \mathsf{N})[\mathsf{Val}_{\vec{s}}(\vec{x}) \wedge R(\vec{U}, \vec{x})]$. $\mathcal{V}_1(U)$ and $\mathcal{V}_2(u)$ are not required.

The theory T^- is then obtained from T by applying the translation induced by the pretranslation $\cdot^\star$ to all its axioms and rules. An induction on the depth of the proof of a finite set Γ of formulas of T reveals that $\mathsf{T} \vdash \Gamma$ if and only if $\mathsf{T}^- \vdash \Gamma^\star$.

We can also get rid of the constants cs_n in a formula by adding to the above definition of $\mathsf{Val}_t(u)$ the clause $\mathsf{Val}_{\mathsf{cs}_n}(u) := R_{\mathsf{cs}_n}(u)$, where R_{cs_n} is a relation symbol for the set $\{n\}$. The resulting translation $\cdot^{\star_c}$ then assigns to a formula A of T the formula $A^{\star_c}$ of T that contains neither constants nor function symbols. Moreover, for all finite sets Γ of formulas of T, the equivalence of $\bigvee \Gamma$ and $\bigvee \Gamma^{\star_c}$ is provable in T.

I.2.14 Syntactical extensions of L_2

To increase the readability of formal arguments, we enrich the expressibility of the formal language by adding set terms, sequence and function variables and course of value notations. Then working in a theory T formulated in L_2 that comprises the axioms and rules of ACA_0, it often proves useful to equip the language L_2 with sequence variables $\sigma, \tau, \rho, \ldots$ and function variables $\mathcal{F}, \mathcal{G}, \mathcal{H}, \ldots$. The resulting language is denoted by $\mathsf{L}_2^{\mathcal{F},\sigma}$. Simultaneously, we extend T to $\mathsf{T}^{\mathcal{F},\sigma}$ by axioms and rules for sequence and function variables.

All number and sequence variables are number terms of $\mathsf{L}_2^{\mathcal{F},\sigma}$ and if $s, \vec{t}$ are number terms of $\mathsf{L}_2^{\mathcal{F},\sigma}$, $\mathcal{F}$ a function variable and f an n-ary function symbol of L_2, then so are $\mathcal{F}(s)$ and $f(\vec{t})$, $\sigma[s]$, $U[s]$ and $\mathcal{F}[s]$. The atoms of $\mathsf{L}_2^{\mathcal{F},\sigma}$ are the formulas of the form $R(\vec{s})$ and $t \in U$, where R is a primitive recursive relation symbol and $\vec{s}, t$ range over terms of $\mathsf{L}_2^{\mathcal{F},\sigma}$. Formulas are built from literals as described in subsection I.2.1. The sets of free sequence and function variables in a formula A are defined analogously to the sets of free number and set variables.

The additional axioms for sequences and functions and the various forms of course of value notations are listed below. Thereby, we denote by $\mathsf{Fun}(U)$ the formula

$$(\forall x \in U)(x = \langle (x)_0, (x)_1 \rangle) \land \forall x \exists! y (\langle x, y \rangle \in U).$$

1. $\Gamma, \sigma \in \mathsf{seq}$ and $\Gamma, \mathsf{Fun}(\mathcal{F})$,
2. $\Gamma, \sigma[u] \sqsubseteq \sigma$ and $\Gamma, \mathsf{lh}(\sigma[u]) = \mathsf{min}\{u, \mathsf{lh}(\sigma)\}$,
3. $\Gamma, \mathcal{F}[0] = \langle\rangle$ and $\Gamma, \mathcal{F}[u{+}1] = \mathcal{F}[u] * \langle \mathcal{F}(u) \rangle$,
4. $\Gamma, U[0] = \langle\rangle \quad \Gamma, u \notin U, U[u{+}1] = U[u] * \langle 0 \rangle$ and $\Gamma, u \in U, U[u{+}1] = U[u] * \langle 1 \rangle$.

The quantifier axioms for the new sort of variables take the following form: For each Δ, Γ, Λ with $\sigma \notin FV(\Delta)$ and $\mathcal{F} \notin FV(\Lambda)$, each A and each term s of $\mathsf{L}_2^{\mathcal{F},\sigma}$:

$$\frac{\Gamma, s \in \mathsf{seq} \quad \Gamma, A(s)}{\Gamma, \exists \sigma A(\sigma)} \qquad \frac{\Delta, A(\sigma)}{\Delta, \forall \sigma A(\sigma)} \qquad \frac{\Gamma, \mathsf{Fun}(X) \quad \Gamma, A(X)}{\Gamma, \exists \mathcal{F} A(\mathcal{F})} \qquad \frac{\Lambda, A(\mathcal{F})}{\Lambda, \forall \mathcal{F} A(\mathcal{F})}$$

There is a straight forward embedding of $\mathsf{T}^{\mathcal{F},\sigma}$ into T: Similarly as in the previous subsection, we define for each number term of $\mathsf{L}_2^{\mathcal{F},\sigma}$ a formula $\mathsf{Val}_t(u)$ of L_2. Thereby we assume that the variables denoted by $u_\sigma, u_\tau, \ldots, U_\mathcal{F}, U_\mathcal{G}, \ldots$ are pairwise distinct and syntactically different from all variables without a subscript. Further, y does not occur in any of the terms s, t.

1. $\mathsf{Val}_t(u) := (u = t)$, if t is a term of L_2,
2. $\mathsf{Val}_\sigma(u) := (u = u_\sigma)$,

3. $\mathsf{Val}_{\mathcal{F}(s)}(u) := \exists y(\mathsf{Val}_s(y) \wedge \langle y, u\rangle \in U_{\mathcal{F}})$,

4. $\mathsf{Val}_{U[t]}(u) := \exists y[\mathsf{Val}_t(y) \wedge u \in \mathsf{seq}_{0,1} \wedge \mathsf{lh}(u) = y \wedge (\forall i < y)((u)_i = 0 \leftrightarrow i \in U)]$,

5. $\mathsf{Val}_{\mathcal{F}[t]}(u) := \exists y[\mathsf{Val}_t(y) \wedge u \in \mathsf{seq} \wedge \mathsf{lh}(u) = y \wedge (\forall i < y)(\langle i, (u)_i\rangle \in U_{\mathcal{F}})]$,

6. $\mathsf{Val}_{\sigma[t]}(u) := \exists y, z[\mathsf{Val}_t(y) \wedge \mathsf{Val}_\sigma(z) \wedge u \sqsubseteq z \wedge \mathsf{lh}(u) = \mathsf{min}\{y, \mathsf{lh}(z)\}]$.

Again, this yields a pretranslation $\cdot^\star$: Variables $\mathcal{F}, \sigma$ are mapped to variables $U_{\mathcal{F}}$ and u_σ of L_2 so that no conflicts arise. If R is a primitive recursive relation symbol, then $R(\vec{t})$ is mapped to $\exists\vec{x}[\mathsf{Val}_{\vec{t}}(\vec{x}) \wedge R(\vec{x})]$. Further, we have $\mathcal{V}_{\mathcal{F}}(U) := \mathsf{Fun}(U)$ and $\mathcal{V}_\sigma(u) := u \in \mathsf{seq}$. The induced translation $\cdot^*$ is now an embedding of T into $\mathsf{T}^{\mathcal{F},\sigma}$. This allows us to shift tacitly from T to $\mathsf{T}^{\mathcal{F},\sigma}$ an back.

We conclude this subsection by a lemma that allows us to define functions $\mathcal{F}$ by recursion within ACA_0.

Lemma I.2.15 *The following is provable in* ACA_0*: If* $A(\sigma, u)$ *is an arithmetical formula of* $\mathsf{L}_2^{\mathcal{F},\sigma}$*, then*

$$\forall\sigma\exists!xA(\sigma, x) \to \exists\mathcal{F}\forall yA(\mathcal{F}[y], \mathcal{F}(y)).$$

Proof: By arithmetical comprehension, the set

$$G := \{\langle x, y\rangle : \exists\sigma[\mathsf{lh}(\sigma) = x \wedge (\forall z < x)A(\sigma[z], (\sigma)_z) \wedge A(\sigma[x], y)]\}$$

exists. Now we show by set induction that $\mathsf{Fun}(G)$, i.e. $\forall x\exists!y\langle x, y\rangle \in G$. We call this function $\mathcal{F}$ and show by an other induction that $\forall xA(\mathcal{F}[x], \mathcal{F}(x))$. $\Box$

I.3 Proof-theoretic basics

Proof-theorists want to compare theories in terms of "proof-theoretic strength". To obtain a linear ordering on theories, they assign to each theory T its *proof-theoretic ordinal* $|\mathsf{T}|$, which has a striking property: The existence of the proof-theoretic ordinal $|\mathsf{T}|$, or more precisely, the existence of a well-ordering on $\mathbb{N}$ of ordertype $|\mathsf{T}|$, is equivalent to the consistency of T. In this subsection, we review the fundamentals necessary to detail and understand the concept of proof-theoretic ordinal. More on this subject can be found e.g in [28] and [39].

I.3.1 A notation system based on the ternary Veblen function

A *notation system* for an ordinal Φ is a primitive recursive relation $\prec$ such that there is an order-isomorphism $|\cdot|_\prec : (\mathbb{N}, \prec) \to (\Phi, \in)$, together with primitive recursive functions $+_\prec$, $\cdot_\prec$ and $\exp_\prec$, that perform addition, multiplication and exponentiation on the codes of the ordinals below Φ, e.g. $|\alpha{+}\beta|_\prec = |\alpha|_\prec +_\prec |\beta|_\prec$. A general theory of notation systems can be found e.g. in Rogers [35].

To denote all the ordinals relevant for this thesis, a notation system based on the ternary *Veblen function* suffices. Its development is sketched below.

The standard notation system up to the Feferman-Schütte ordinal Γ_0 makes use of the usual Veblen hierarchy generated by the binary function φ, starting off with the function $\varphi 0\beta = \omega^\beta$, cf. Pohlers [28] and Schütte [39]. For larger notations, one simply generalizes the definition principle of the usual binary φ function and generates the ternary φ function inductively as follows:

1. $\varphi 0\beta\gamma := \varphi\beta\gamma$,
2. if $\alpha > 0$, the $\varphi\alpha 0\gamma$ denotes the γth ordinal which is strongly critical w.r.t. all functions $\lambda\xi,\eta.\varphi\alpha'\xi\eta$ for $\alpha' < \alpha$,
3. if $\alpha > 0$ and $\beta > 0$, then $\varphi\alpha\beta\gamma$ denotes the γth common fixed point of the functions $\lambda\xi.\varphi\alpha\beta'\xi$ for $\beta' < \beta$.

For this thesis, we let Φ_0 be the least ordinal which is closed w.r.t. the ternary Veblen function. In [28] it is shown in detail how to obtain a primitive recursive order relation on $\mathbb{N}$ whose ordertype is closed under the binary φ function. Similarly, one constructs a primitive recursive order relation on $\mathbb{N}$ of ordertype Φ_0, that we denote in the sequel by $\lhd$. We agree that $|\cdot|_\lhd : \mathbb{N} \to \Phi_0$ is the corresponding order-isomorphism, and $\varphi_\lhd$, $+_\lhd$, $\cdot_\lhd$ and $\exp_\lhd$, are the primitive recursive functions performing the corresponding operations on the natural numbers $\mathbb{N}$ seen as codes of the ordinals below Φ_0.

Then working in a formal theory, we rarely distinguish between ordinals and their codes or the primitive recursive function $\varphi_\lhd$, $+_\lhd$, $\cdot_\lhd$, $\exp_\lhd$ and the corresponding functions $\varphi, +, \cdot, \exp$ acting on the ordinals: For example, if we regard x, y as elements of the field of $\lhd$, then we write $x{+}y$ for $x +_\lhd y$, and if α is the ordinal $|n|_\lhd$, then we often use α for the constant cs_n. To emphasis that we look upon numbers as elements of the field of $\lhd$, we denote them by lower case Greek letters $\alpha, \beta, \gamma, \ldots$.

I.3.2 Cut-Elimination

It is the cut rule that makes it so hard to detect whether a theory is consistent or not. At least for theories with the subformula property, a cut-free proof of $0 = 1$ is simply impossible, unless $0 = 1$ is itself an axiom. Therefore, one is interested in an effective procedure to transform a proof within a theory T into a *cut-free proof*. In general, one has to resort to semi-formal systems and accept infinite proofs, however, theses proofs can still be coded as recursive sets. If one can turn a proof of T into a proof of PA^*, then the Cut-Elimination Theorem for PA^* below yields a cut-free proof, which in turn implies the consistency of T.

Definition I.3.1 (Natural rank) *Let L be a language comprising one or two sorts of variables. To each formula of L we assign its natural rank in the following way:*

(i) $\mathsf{rk}(A) := 0$ *if A is a literal,*

(ii) $\mathsf{rk}(A \wedge B) := \mathsf{rk}(A \vee B) := \mathsf{max}\{\mathsf{rk}(A), \mathsf{rk}(B)\} + 1$,

(iii) $\mathsf{rk}(\mathcal{Q}xA) := \mathsf{rk}(A) + 1$ *and* $\mathsf{rk}(\mathcal{Q}XA) := \mathsf{rk}(A) + 1$ *if L comprises a second sort of variables.*

Then the following theorem is standard.

Theorem I.3.2 (Cut-Elimination I) *Let T_1 be the set of axioms and rules of first or second order predicate logic and and T_2 the axioms and rules of the logic of partial terms. Then for each finite set Γ of T_i formulas ($i \in \{1, 2\}$), the following holds:*

$$\mathsf{T}_1 \overset{n}{\underset{r+1}{\vdash}} \Gamma \Longrightarrow \mathsf{T}_1 \overset{2^n}{\underset{r}{\vdash}} \Gamma \quad \textit{and} \quad \mathsf{T}_2 \overset{n}{\underset{r+1}{\vdash}} \Gamma \Longrightarrow \mathsf{T}_2 \overset{3^n}{\underset{r}{\vdash}} \Gamma.$$

In particular, if $\mathsf{T}_i \vdash \Gamma$, then there exists a natural number n such that already $\mathsf{T}_i \overset{n}{\underset{0}{\vdash}} \Gamma$.

The two premises of the rule for the individual existential quantifier cause that the depth of proofs grows a bit faster than in the case of predicate logic. Cut-elimination holds also for the semi-formal theory PA^*.

Theorem I.3.3 (Cut-Elimination II) *For each finite set Γ of formulas of PA^*, the following holds:*

$$\mathsf{PA}^* \overset{\alpha}{\underset{r+1}{\vdash}} \Gamma \Longrightarrow \mathsf{PA}^* \overset{2^\alpha}{\underset{r}{\vdash}} \Gamma.$$

If a theory T comprises also non-logical axioms and rules, we no longer can eliminate all the cuts. However, in many cases *partial cut-elimination* is still possible: When we work in a theory whose underlying logic is predicate logic, and apply the standard procedure to eliminate a cut of the form

$$\frac{\Gamma, \forall x A(x) \qquad \Gamma, \exists x \neg A(x)}{\Gamma},$$

we need that for each finite set Γ of formulas of T, each term t of T and $u \notin FV(\Gamma)$,

$$(*) \qquad T \overset{n}{\underset{m}{\vdash}} \Gamma, A(u) \Longrightarrow T \overset{n}{\underset{m}{\vdash}} \Gamma, A(t).$$

While this holds for predicate logic, it may fail in the presence of a theory T. For example, if T is the theory PA as formulated in subsection I.2.6, then $u = u$ is an equality axiom, but $0 = 0$ is not an axiom of T. However, a slight modification of T fixes this problem. If Γ is an axiom of T, then for all terms $t_1, \dots, t_n$ of T, we call $\Gamma[\vec{t}/\vec{u}]$ a *substitution instance* of Γ and

$$\frac{\Gamma_i[\vec{t}/\vec{u}] \text{ for all } i \in I}{\Gamma[\vec{t}/\vec{u}]} \quad \text{is a substitution instance of} \quad \frac{\Gamma_i \text{ for all } i \in I}{\Gamma}.$$

Clearly, a substitution instance of an axiom Γ of T is provable in T. Similar, if for all $i \in I$ a substitution instance $\Gamma_i[\vec{t}/\vec{u}]$ of the premises of a non-logical rule of T are provable in T, then also $\Gamma[\vec{t}/\vec{u}]$, provided the rule has not to meet conditions on variables or side formulas.

We call a theory T closed under substitution, if T contains all substitution instances of its non-logical axioms and rules. In many cases, the extension of a theory T by all substitution instances of its axioms and rules does not prove more formulas than T, but in this extension $(*)$ holds and partial cut-elimination becomes provable.

Theorem I.3.4 (Partial cut-elimination I) *Let T be a formal theory which is closed under substitution and whose underlying logic is predicate logic. Further, we assume that $*$ is a set of formulas of T that is closed under negation and contains the main formulas of all instance of the non-logical axioms and rules. Moreover, we assume that each formula in $*$ has rank less than k. Then the following holds for all natural numbers n, k, r and each finite set Γ of formulas of the language of T:*

$$\mathsf{T} \vdash^{n}_{k+r+1} \Gamma \Longrightarrow \mathsf{T} \vdash^{2^n}_{k+r} \Gamma.$$

In particular, if $\mathsf{T} \vdash \Gamma$, then already $\mathsf{T} \vdash_{*} \Gamma$.

If the underlying logic of a theory T is the logic of partial terms, then partial cut-elimination is slightly more difficult. To eliminate a cut whose cut formula is of the form $\forall x A(x)$, we now require that for each finite set Γ of formulas of T, each term t of T and $u \notin FV(\Gamma)$,

$$(**) \qquad T \vdash^{n}_{m} \Gamma, A(u) \Longrightarrow T \vdash^{n}_{m} \Gamma, {\sim}t{\downarrow}, A(t).$$

Again, this holds for the logic of partial terms, but may fail in the presence of a theory T. This time, we solve this problem as done in Glass and Strahm [17]: If Γ is an axiom of T, then for all individual terms $\vec{t}$, we call ${\sim}\vec{t}{\downarrow}, \Gamma[\vec{t}/\vec{u}]$ a *faithful instance* of Γ and

$$\frac{{\sim}\vec{t}{\downarrow}, \Gamma_i[\vec{t}/\vec{u}] \text{ for all } i \in I}{\Gamma[\vec{t}/\vec{u}]} \quad \text{is a faithful instance of} \quad \frac{\Gamma_i \text{ for all } i \in I}{\Gamma}.$$

We call a theory T closed under faithful substitution, if T contains all faithful instances of its non-logical axioms and rules. As above, the extension of a theory T by all faithful instances of its non-logical axioms and rules does in general not prove more formulas than T, but $(**)$ holds and partial cut-elimination becomes provable.

Theorem I.3.5 (Partial cut-elimination II) *Let T be a formal theory that is closed under faithful substitution and whose underlying logic is the logic of partial terms. Further, we assume that $*$ is a set of formulas of T that is closed under negation and contains the main formulas of all instance of the non-logical axioms and rules. Moreover, we assume that each formula in $*$ has rank less than k. Then the following holds for all natural numbers n, k, r and each finite set Γ of formulas of the language of T:*

$$\mathsf{T} \mathrel{\vert\!\!\frac{\;n\;}{k+r+1}} \Gamma \Longrightarrow \mathsf{T} \mathrel{\vert\!\!\frac{\;3^n\;}{k+r}} \Gamma.$$

In particular, if $\mathsf{T} \vdash \Gamma$, then already $\mathsf{T} \mathrel{\vert\!\!\frac{}{*}} \Gamma$.

I.3.3 The proof-theoretic ordinal $|\mathsf{T}|$ of a theory T

The proof-theoretic ordinal $|\mathsf{T}|$ of a theory T is a measure of its proof-theoretic strength. There are several reasonable ways to assign such an ordinal to a theory T. Four, for many theories equivalent definitions, $|\mathsf{T}|_1, \ldots, |\mathsf{T}|_4$ are presented below. To be specific, we set $|\mathsf{T}| := |\mathsf{T}|_3$.

For the subsequent definitions, we assume that T is formulated in some language L and that there exists an embedding $\cdot^*$ of PA into T. Further, we presuppose a *Gödelization* $\ulcorner\cdot\urcorner$ for the terms and formulas of L_1, that assigns to each term t and formula A of L_1 its *Gödelnumber* $\ulcorner t\urcorner$ and $\ulcorner A\urcorner$, respectively. Moreover, we introduce for each primitive recursive well-ordering $\prec$, the following L_1 formulas:

$$\begin{aligned} \mathsf{Prog}_{\prec}(U) &:= (\forall x \in \mathsf{Field}(\prec))((\forall y \prec x)(y \in U) \rightarrow (x \in U)), \\ \mathsf{TI}_{\prec}(U, u) &:= \mathsf{Prog}_{\prec}(U) \rightarrow (\forall y \prec u)(y \in U). \end{aligned}$$

1. The set of axioms and rules of a formal T is recursive, hence there is a Δ^0_1 formula $\mathsf{Proof}_{\mathsf{T}}(u, v)$ of L_1 which expresses that u codes a proof of the formula with Gödelnumber v. Then, we have for each formula A of L_1 that

$$\mathsf{T} \vdash A^* \Longleftrightarrow \mathsf{T} \vdash (\exists x \mathsf{Proof}_{\mathsf{T}}(x, \ulcorner A\urcorner))^*.$$

   Thus, T is consistent if and only if $\mathbb{N} \models \mathsf{cons}_{\mathsf{T}}$, where $\mathsf{cons}_{\mathsf{T}}$ is the following L_1 sentence excluding that there is a proof of the statement $0 = 1$, namely $\forall x \neg \mathsf{Proof}_{\mathsf{T}}(x, \ulcorner \mathsf{cs}_0 = \mathsf{cs}_1 \urcorner)$.

It is well-known, that T or PA cannot prove $\mathsf{cons_T}$. However, it turns out that for many theories T, the sentence $\mathsf{cons_T}$ becomes provable in extensions of PA by certain instances of transfinite induction. Thus, as a first possible definition, we call the least ordinal α such that $\mathsf{PA} + \mathsf{TI}_\lhd(\mathsf{U}, \alpha) \vdash \mathsf{cons_T}$, the proof-theoretic ordinal of T, denoted by $|\mathsf{T}|_1$.

2. Gödel's first incompleteness theorem shows that T does not prove all true sentences. However, the stronger a theory T, the more true sentences become provable. We choose the sentences $\mathsf{TI}_\lhd(\mathsf{U}, \alpha)$ as reference sentences and call the least ordinal α such that $\mathsf{T} \nvdash (\mathsf{TI}_\lhd(\mathsf{U}, \alpha))^*$, the proof-theoretic ordinal of T, denoted by $|\mathsf{T}|_2$.

3. It could be dangerous to define the proof-theoretic ordinal with respect to a specific notation system as done in (i) and (ii). Therefore, we generalize the above setting and say that T proves an ordinal α, if there exists a primitive recursive well-ordering $\prec$ of ordertype α such that $\mathsf{T} \vdash (\forall x \mathsf{TI}_\prec(\mathsf{U}, x))^*$. The least ordinal α that is not provable in T is then the proof-theoretic ordinal of T, denoted by $|\mathsf{T}|_3$. This is the standard definition.

4. In contrast to a formal theory, the semi-formal system PA^* is complete w.r.t. the standard structure $\mathbb{N}$, i.e. it proves all true sentences. Moreover, each true sentence is already provable without applying the cut rule. This motivates to call the least ordinal α such that for all finite sets Γ of L_1 formulas,

$$\mathsf{T} \vdash \Gamma^* \Longrightarrow \mathsf{PA}^* \,|\!\frac{<\alpha}{0}\, \Gamma,$$

the proof-theoretic ordinal of T, denoted by $|\mathsf{T}|_4$. In this case, T is also called α-equivalent to PA^*, formally expressed by $\mathsf{T} \simeq_\alpha \mathsf{PA}^*$ (cf. [25]).

For all sensible theories, in particular for all theories treated in this thesis, the four aforementioned definitions of the proof-theoretic ordinal are equivalent. Below we sketch that $|\mathsf{T}|_2 \leq |\mathsf{T}|_3 \leq |\mathsf{T}|_4$ and $|\mathsf{T}|_2 \leq |\mathsf{T}|_1 \leq |\mathsf{T}|_4$. For the theories T appearing in this thesis, an inspection of their proof-theoretic analysis yields $|\mathsf{T}|_2 = |\mathsf{T}|_4$. Thus, $|\mathsf{T}|_1 = |\mathsf{T}|_2 = |\mathsf{T}|_3 = |\mathsf{T}|_4$.

For all relevant theories we have $|\mathsf{T}|_4 = \omega \cdot |\mathsf{T}|_4$. If $\prec$ is a primitive recursive well-ordering and $\Gamma = \neg\mathsf{Prog}_\prec(\mathsf{U}), \beta \in \mathsf{U}$, then $\mathsf{T} \vdash \Gamma^*$, implies $\mathsf{PA}^* \,|\!\frac{<\alpha}{0}\, \Gamma$ by the definition of $|\mathsf{T}|_4$. Now Schütte's famous Boundedness Lemma [39] yields that $\beta < |\mathsf{T}|_4$, which then implies $|\mathsf{T}|_3 \leq |\mathsf{T}|_4$. Trivially, we have $|\mathsf{T}|_2 \leq |\mathsf{T}|_3$.
Since T cannot prove its own consistency, T cannot prove $(\mathsf{TI}_\lhd(\mathsf{U}, |\mathsf{T}|_1))^*$, therefore $|\mathsf{T}|_2 \leq |\mathsf{T}|_1$. Moreover, if all closed L_1 sentences of T have a cut-free proof in PA^* of depth less than $|\mathsf{T}|_4$, the argument sketched on page 70 in [28] reveals that $\mathsf{PA} + \mathsf{TI}_\lhd(\mathsf{U}, \alpha) \vdash \mathsf{cons_T}$. Thus, $|\mathsf{T}|_1 \leq |\mathsf{T}|_4$.

For a further comment on the proof-theoretic ordinal, see also remark II.2.40.

Chapter II

Pseudo-hierarchies in second order arithmetic

Beware of the man who won't be bothered with details.
William Feather (1908 - 1976)

After reviewing the standard results about the jump-hierarchy and the hyperarithmetical sets, we give an extensive introduction to pseudo-hierarchy arguments and employ them to research various subsystems of second order arithmetic. We combine the fixed point construction from [2] with techniques developed in Jäger [21] to reason about fixed points of non-monotone operators and show that there are operations, given by positive arithmetical formulas, that have no fixed points in HYP and thus, by Kleene-Souslin Theorem, also no Δ^1_1 definable fixed points. Finally, we show that for a positive arithmetical formula $A(U^+, u)$, $\Sigma^1_1\text{-}\mathsf{AC}_0$ proves that the Π^1_1 definable class $\mathsf{Fix}^A := \bigcap\{X : F^A(X) \subseteq X\}$ is a fixed point of the operator F^A, which leads to a new embedding of $\widehat{\mathsf{ID}}_1$ into $\Sigma^1_1\text{-}\mathsf{AC}$, which extends to an embedding of ID^*_1 into $\Sigma^1_1\text{-}\mathsf{DC}$, settling an old question ask in Feferman's article on Hancock's conjecture [11] about the upper bound of ID^*_1.

II.1 Preliminaries

First, we review the notion of universal formulas. Then, we elaborate on the relationship between trees, Π^1_1 and Σ^1_1 formulas and make explicit how a path trough a tree representing a Σ^1_1 formula yields a witness for the existential quantified set variable. Next, coded finite axiomatizations of $\Sigma^1_1\text{-}\mathsf{AC}$ and $\Sigma^1_1\text{-}\mathsf{DC}$ are provided and N-models are defined. It follows a detailed analysis of the jump-hierarchy before we conclude with an introduction to hyperarithmetical sets. Most results are taken form Simpson [40], but often, we prove a bit more and use a different notation. In the sequel, we work mainly in subsystems of second order arithmetic and deal a lot with linear orderings. Therefore, we agree to use lower case Greek letters not only for elements in the field of the underlying ordering $\lhd$ of our notation system, but use $\alpha, \beta, \gamma, \dots$ also as number variables to emphasis that they ranges over the field of an ordering apparent from the context. The letter λ is then used for elements with no immediate predecessor.

II.1.1 Universal formulas

A formula $A(u,v)$ is called universal for a formula class $\mathcal{K}$, or universal $\mathcal{K}$ for short, if $A(u,v)$ is itself a member of $\mathcal{K}$ and if for each formula $B(u)$ in $\mathcal{K}$, there exists a natural number e such that

$$\forall x[A(x,e) \leftrightarrow B(x)].$$

Universal formulas are used to enumerate an entire class of formulas, a property that we require to perform diagonalization arguments or to state finite axiomatizations for various theories.

It is a standard result of basic recursion theory that if L is the language L_1 without the relation symbols U, V, then the primitive recursive sets are precisely the sets which are definable w.r.t. the standard model $\mathbb{N}$ by a Δ^0_0 formula, the recursive sets are exactly the ones which are Δ^0_1, and recursive enumerable sets correspond to the sets that are definable by a Σ^0_1 formula of L. Thus, we have that for each Σ^0_1 formula $A(u,v)$ of L, there exists an index e such that

$$\forall x[A(x,e) \leftrightarrow \{e\}(x)\!\downarrow].$$

Moreover, the L formulas $\{e\}(x)\!\downarrow$ and $\{e\}(x)\!\uparrow$ are universal Σ^0_1 and Π^0_1, respectively.

We want a corresponding result to be provable in ACA_0. It is clear, that we need to adjust the definition of $\{e\}$, since now indices for the characteristic functions of the relation symbols U and V are required. We even go a step further and define partial functions $\{e\}^{X_1,\dots,X_n}$. The idea is to regard $\langle 10,1\rangle, \langle 11,1\rangle, \langle 11{+}1,1\rangle, \dots, \langle 11{+}n,1\rangle$ as indices of the characteristic functions of $\mathsf{U}, \mathsf{V}, X_1, \dots, X_n$ (cf. subsection I.1.7). Analogously to the proof that Kleene's T-predicate is primitive recursive, one constructs a Δ^0_0 formula $T(U_1,\dots,U_n,u,v,w)$ of L_2 such that ACA_0 proves the following: For all $\vec{X}, e, \vec{y}, z$, if $T(\vec{X}, e, \langle\vec{y}\rangle, z)$, then $e \in \mathsf{Rec}$ and z is a "proof" that the computation of the (partial) function with index e terminates on input $\vec{y}$ and yields $(z)_{0,2}$ as result, where the indices $\langle 10,1\rangle, \langle 11{+}n,1\rangle$ are interpreted as mentioned above. Now, we redefine $\{e\}^{\vec{U}}(\vec{u})\!\downarrow$ to stand for the Σ^0_1 formula $\exists z[(z)_0 = \langle\vec{u}\rangle \wedge T(\vec{U}, e, (z)_0, (z)_1)]$ of L_2 and $\{e\}^{\vec{U}}(\vec{u})\!\uparrow$ for its negation. This leads to the following definition:

Definition II.1.1 (Universal Π^0_1 formulas) *For all $k, l \in \mathbb{N}$, set variables $\vec{U} = U_1, \dots, U_k$ and number variables $\vec{u} = u_1, \dots, u_l$, we set*

$$\pi^0_{1,k,l}(\vec{U}, \vec{u}, e) := \{e\}^{\vec{U}}(\vec{u})\uparrow.$$

The previously mentioned standard results from basic recursion theory carry over to the present context. By induction on the build-up of Π^0_1 formulas one easily shows the following lemma.

Lemma II.1.2 (Universal Π^0_1 formulas of L_2) *If $A(U_1, \dots, U_k, u_1, \dots, u_l, \vec{v})$ is a Π^0_1 formula of L_2 with at most the displayed variables free, then the following is provable in ACA_0:*

$$\forall \vec{y} \exists e \forall \vec{X}, \vec{x}[A(\vec{X}, \vec{x}, \vec{y}) \leftrightarrow \pi^0_{1,k,l}(\vec{X}, \vec{x}, e)].$$

Sometimes, the variant below is required.

Lemma II.1.3 *If $A(U_1, \dots, U_k, u_1, \dots, u_l)$ is a Π^0_1 formula of L_2 with at most the displayed variables free, then there is an $e \in \mathbb{N}$, such that ACA_0 proves*

$$\forall \vec{X}, \vec{x}[A(\vec{X}, \vec{x}) \leftrightarrow \pi^0_{1,k,l}(\vec{X}, \vec{x}, \mathsf{cs}_e)].$$

Having universal Π^0_1 formulas at hand enables us to speak within ACA_0 about sets that are Π^0_1, Σ^0_1 or Δ^0_1 in X. For instance, that Y is Π^0_1 in $\vec{X}$ is expressed in ACA_0 by $\exists e[Y = \{x : \pi^0_1(\vec{X}, x, e)\}]$, and that a set Y is Δ^0_1 in $\vec{X}$ is formulated as

$$\exists e, e'[Y = \{x : \pi^0_1(\vec{X}, x, e) = \neg\pi^0_1(\vec{X}, x, e')\}].$$

Observe also, that the definition of the universal formula $\pi^0_{1,k,l}(\vec{U}, \vec{u}, v)$ entails that $\neg\pi^0_{1,k,l}(\vec{X}, \vec{x}, e)$ is equivalent to $(\exists e \in \mathsf{Rec})(\{e\}^{\vec{X}}(\vec{x})\downarrow\})$. Moreover, ACA_0 proves that under the premise $\pi^0_{1,k,l}(\vec{X}, \vec{x}, e) \leftrightarrow \neg\pi^0_{1,k,l}(\vec{X}, \vec{x}, e')$, we have

$$(\exists e_0 \in \mathsf{Rec})(\{e_0\}^{\vec{X}} \text{ is the characteristic function of } \{x : \pi^0_{1,k,l}(\vec{X}, \vec{x}, e)\}).$$

In this sense, the sets Δ^0_1 in $\vec{X}$ are precisely the sets recursive in $\vec{X}$, and the sets Σ^0_1 in X correspond to the recursively enumerable sets in $\vec{X}$. Unfortunately, we do not have universal Δ^0_0 formulas. We help ourself by defining what we mean by Y *is primitive recursive in* $\vec{X}$, namely that there is an $e \in \mathsf{Prim}$, such that $\{e\}^{\vec{X}}$ is the characteristic function of Y. However, in a non-standard model of ACA_0, the set Prim may contains also non-standard indices that define functions for which there are no function symbols in L_2. Therefore, if $\mathcal{M} = (M, \mathcal{S}_M)$ is a model of ACA_0, there is in general an $e \in \mathsf{Prim}^{\mathcal{M}}$ such that for all Δ^0_0 formulas $A(u,v)$ of L_2 and each $m \in M$,

$$\mathcal{M} \not\models \forall x[\{e\}(x) = 0 \leftrightarrow A(x,m)].$$

Further, we denote by $\mathsf{TRec}^{\vec{X}}$ the set of indices $\{e : \forall x(\{e\}^{\vec{X}}(x)\downarrow\}$ of the total functions recursive in $\vec{X}$. Then ACA_0 proves that for an index $e \in \mathsf{TRec}^{\vec{X}}$, the set $\{x : \{e\}^{\vec{X}}(x) = 0\}$ is Δ^0_1 in $\vec{X}$.

In the sequel, we no longer explicitly mention the number of free number and set variables in the universal Π^0_1 formulas and simply write π^0_1 for $\pi^0_{1,k,l}$. Given a universal Π^0_1 formula, it is straight forward to construct universal Π^0_n formulas: Suppose that $A(u,v,\vec{w})$ is a Δ^0_0 formula of L_2. Then there is an e such that for all $x, \vec{y}$, $\exists z A(x,\vec{y},z) \leftrightarrow \neg\pi^0_1(x,\vec{y},e)$ holds, hence $\forall x \exists z A(x,\vec{y},z)$ is equivalent to $\forall x \neg\pi^0_1(x,\vec{y},e)$, and this equivalence is already provable in ACA_0. Thus it makes sense to define

$$\pi^0_2(\vec{U},\vec{u},e) := \forall x \neg\pi^0_1(\vec{U},x,\vec{u},e) = \forall x[\{e\}^{\vec{U}}(x,\vec{u})\downarrow].$$

In the same way, universal Π^0_n formulas are constructed for all $n \in \mathbb{N}$.

II.1.2 Trees and normal forms of Π^1_1 and Σ^1_1 formulas of L_2

This subsection exhibits the close relationship between Π^1_1 and Σ^1_1 formulas of L_2 and trees. The notion of tree that we use in the framework of second order arithmetic is a special case of the definition given in subsection I.1.4: A *tree* is a set T of finite sequences closed under initial segments, i.e. $T \subseteq \mathsf{seq}$ and if $\tau \in T$ and $\sigma \sqsubset \tau$, then already $\sigma \in T$. Further, we say that the function $\mathcal{F}$ is a *path* through T, if for all n, the sequence $\mathcal{F}[n]$ is in T. Alternatively, we often say that T has a path or T has an *infinite branch*. A tree is *finitely branching*, if for each $\sigma \in T$, there are only finitely many numbers x_i such that $\sigma * \langle x_i \rangle \in T$.

The following result is a classic. For a proof, see e.g. Simpson [40], Theorem III.7.2.

Lemma II.1.4 (König's Lemma) *The following is provable in* ACA_0*: Every infinite, finitely branching tree has a path.*

A tree T is called well-founded, if it has no infinite branch. An other way to characterize the well-foundedness of T is revealed in the next lemma, Lemma V.1.3 in [40].

Lemma II.1.5 *The following is provable in* ACA_0*: If T is a tree, then*

$$\mathsf{Wo}(\mathsf{KB}(T)) \leftrightarrow \forall \mathcal{F} \exists n(\mathcal{F}[n] \notin T).$$

It follows the normal form theorem, which corresponds to lemma V.1.4 in [40]. We show, how to obtain a witness for the original formula form a witness of the normal form.

Lemma II.1.6 (Normal form lemma) *Let $A(\vec{U}, \vec{u})$ be a Σ^1_1 formula of L_2 with exactly the displayed variables free and assume that A is of the form $\exists Y A'(\vec{U}, Y, \vec{u})$ for some arithmetical formula A'. Then there exists a Δ^0_0 formula $B(\vec{\sigma}, \tau, \vec{u})$, such that ACA_0 proves:*

(i) $\forall \vec{X}, \vec{x}[A(\vec{X}, \vec{x}) \leftrightarrow \exists \mathcal{F} \forall n B(\vec{X}[n], \mathcal{F}[n], \vec{x})]$,

(ii) $\forall n B(\vec{X}[n], \mathcal{F}[n], \vec{x}) \to A'(\vec{X}, \{y : (\mathcal{F}(y))_0 = 0\}, \vec{x})$.

Proof: First, we prove (i) under the assumption that A is arithmetical. By the translation presented in subsection I.2.13, there exists a quantifier-free formula C of L_2 which does not contain function symbols and constants, such that $A(\vec{X}, \vec{x})$ is equivalent to

$$\forall y_1 \exists z_1 \dots \forall y_l \exists z_l C(\vec{X}, \vec{x}, y_1, z_1, \dots, y_l, z_l). \tag{1}$$

Since no function symbols appear in C, the set variables X_i appear in C only in the form $[\sim](t \in X_i)$, there t is one of the variables $\vec{x}, \vec{y}, \vec{z}$. Due to arithmetical comprehension, (1) holds if and only if there exist functions $\mathcal{G}_i$, such that

$$\forall y_1 \dots \forall y_l C(\vec{X}, \vec{x}, y_1, \mathcal{G}_1(y_1), \dots, y_l, \mathcal{G}_l(\langle y_1, \dots, y_l \rangle)).$$

The functions $\mathcal{G}_i$ can be coded into a single function $\mathcal{F}$ with $\mathcal{F}(\langle \mathsf{cs}_i, y_1, \dots, y_i \rangle) = \mathcal{G}_i(\langle y_1, \dots, y_i \rangle)$. Therefore, $A(\vec{X}, \vec{x})$ is also equivalent to $\exists \mathcal{F} \forall n D(\vec{X}, \mathcal{F}, \vec{x}, n)$, where $D(\vec{U}, \mathcal{F}, \vec{u}, v)$ is the Δ^0_0 formula

$$(\forall \vec{y} < v)[\bigwedge_{i=1..l} \langle \mathsf{cs}_i, y_1, \dots, y_i \rangle < v \wedge \mathcal{F}(\langle \mathsf{cs}_i, y_1, \dots, y_i \rangle) < v \wedge \vec{u} < v \to$$
$$C(\vec{U}, \vec{u}, y_1, \mathcal{F}(\langle \mathsf{cs}_1, y_1 \rangle), \dots y_l, \mathcal{F}(\langle \mathsf{cs}_l, y_1, \dots, y_l \rangle))].$$

To obtain a formula $E(\vec{\sigma}, \tau, \vec{u}, v)$ that only speaks about sequences, we replace in $D(\vec{U}, \mathcal{F}, \vec{u}, v)$ all expressions of the form $t \in U_i$ by $(\sigma_i)_t = 0$, $t \notin U_i$ by $(\sigma_i)_t = 1$ and $\mathcal{F}(t)$ by $(\tau)_t$. Then, $A(\vec{X}, \vec{x})$ is equivalent to $\exists \mathcal{F} \forall n E(\vec{X}[n], \mathcal{F}[n], \vec{x}, n)$. Note that all number variables appearing in E are bound by n. Finally, we let $B(\vec{\sigma}, \tau, \vec{u}) := E(\vec{\sigma}, \tau, \vec{u}, \mathsf{lh}(\tau))$.

Now we move to the case where A is of the form $\exists Y A'(\vec{X}, Y)$ for some arithmetical A'. By the special case we have proved above, there exists a Δ^0_0 formula $B'(\vec{\sigma}, \sigma', \tau)$ such that ACA_0 proves

$$\forall \vec{X}, Y[A'(\vec{X}, Y) \leftrightarrow \exists \mathcal{G} \forall n B'(\vec{X}[n], Y[n], \mathcal{G}[n])].$$

Next we choose $B(\vec{\sigma}, \tau)$ to be a Δ^0_0 formula which expresses that the sequence number τ is of the form $\langle\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\rangle$ so that $a_i \in \{0, 1\}$ for $1 \leq i \leq n$, and

$$B'(\vec{\sigma}, \langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_n\rangle).$$

Thus, the following holds for all $\vec{X}$: If

$$\mathcal{F}(y) = \begin{cases} \langle 0, \mathcal{G}(y)\rangle & : y \in Y \\ \langle 1, \mathcal{G}(y)\rangle & : y \notin Y \end{cases}$$

then $\forall n B'(\vec{X}[n], Y[n], \mathcal{G}[n])$ if and only if $\forall n B(\vec{X}[n], \mathcal{F}[n])$, which yields

$$\forall \vec{X}[\exists Y A'(\vec{X}, Y) \leftrightarrow \exists \mathcal{F} \forall n B(\vec{X}[n], \mathcal{F}[n])].$$

Hence, if $\mathcal{F}$ is a function such that $\forall n B(\vec{X}[n], \mathcal{F}[n])$, it follows form the construction of B that the set $Y := \{y : (\mathcal{F}(y))_0 = 0\}$ satisfies $A(\vec{X}, Y)$. □

In a next step, we use the Normal Form Lemma to transform the question whether some Σ^1_1 formula $A(\vec{U}, \vec{u})$ of L_2 holds, to the question whether the tree $T^A_{\vec{U},\vec{u}}$, that is uniform in the formula A and the parameters $\vec{U}, \vec{u}$, has a path. Moreover, a witness for $A(\vec{U}, \vec{u})$ is then easily obtained from a path through $T^A_{\vec{U},\vec{u}}$.

Lemma II.1.7 *Let $A(U_1, \ldots, U_k, u_1, \ldots, u_l)$ be a Σ^1_1 formula of L_2 with exactly the displayed variables free. Further, assume that $A(\vec{U}, \vec{u})$ is of the form $\exists Y A'(\vec{U}, Y, \vec{u})$ for some arithmetical formula A'. Then, there are Δ^0_0 formulas $\mathsf{TREE}^A(u)$, $\mathsf{WIT}^A(u)$ and $\mathsf{PROJ}^{k,l}(U, V_1, \ldots, V_k, u, v_1, \ldots, v_l)$ of L_2 with exactly the displayed variables free, such that the following is provable in ACA_0: The set $T^A := \{z : \mathsf{TREE}^A(z)\}$ is a tree, and for all $\vec{X}, \vec{x}$,*

(i) $T^A_{\vec{X},\vec{x}} := \{z : \mathsf{PROJ}^{k,l}(T^A, \vec{X}, z, \vec{x})\}$ *is a tree,*

(ii) $A(\vec{X}, \vec{x}) \leftrightarrow T^A_{\vec{X},\vec{x}}$ *has a path,*

(iii) if $\mathcal{F}$ is a path through $T^A_{\vec{X},\vec{x}}$, then $A'(\vec{X}, \{z : \mathsf{WIT}(\mathcal{F}, z)\}, \vec{x})$.

Proof: Suppose that $A(U_1, \ldots, U_k, u_1, \ldots, u_l) = \exists Y A'(\vec{U}, Y, \vec{u})$ is a Σ^1_1 formula of L_2, and that $B(\vec{\sigma}, \tau, \vec{u}, v)$ is a Δ^0_0 formula of L_2 satisfying (i) and (ii) of lemma II.1.6. In particularly, for all $\vec{X}, \vec{x}$,

$$A(\vec{X}, \vec{x}) \leftrightarrow \exists \mathcal{F} \forall n B(\vec{X}[n], \mathcal{F}[n], \vec{x}).$$

Now we choose $C(\rho)$ to be a Δ^0_0 formula which expresses that ρ is a sequence number of the form

$$\langle\langle a_{1,1}, \ldots, a_{k,1}, b_1, \vec{z}\rangle, \ldots \langle a_{1,n}, \ldots, a_{k,n}, b_n, \vec{z}\rangle\rangle,$$

so that $a_{i,j} \in \{0,1\}$ for $1 \leq i \leq n$, $1 \leq j \leq k$, and

$$B(\langle a_{1,1},\dots,a_{1,n}\rangle,\dots\langle a_{k,1},\dots,a_{k,n}\rangle,\langle b_1,\dots,b_n\rangle,\vec{z}).$$

Next, we define the tree

$$T^A := \{\rho : (\forall\rho' \sqsubseteq \rho)C(\rho')\}.$$

Suppose that $\mathcal{F}, X_1,\dots X_k, x_1,\dots,x_l$ are such that $\forall nB(\vec{X}[n],\mathcal{F}[n],\vec{x})$. Then, we define the function $\mathcal{G}$ by

$$\mathcal{G}(y) := \langle\xi_0,\dots,\xi_{k-1},\mathcal{F}(y),\vec{x}\rangle,$$

where for $0 \leq j < k$, $\xi_j = 0$ if $y \in X_j$ and $\xi_j = 1$ if $y \notin X_j$ and $\vec{z} = z_1,\dots,z_l$. We have that $\mathcal{G}$ is a path through T^A, i.e. $\forall nC(\mathcal{G}[n])$. Further, $\mathcal{G}$ is also a path through the tree $T^A_{\vec{X},\vec{x}}$ relevant for the parameters $\vec{X},\vec{x}$, consisting of all $\rho \in T^A$ which satisfy for all $n \leq \mathsf{lh}(\rho)$:

$$\begin{array}{rcl}
\langle(\rho)_{0,0},\dots,(\rho)_{n-1,0}\rangle & = & X_1[n] \\
& \vdots & \\
\langle(\rho)_{0,k-1},\dots,(\rho)_{n-1,k-1}\rangle & = & X_k[n], \\
\langle(\rho)_{0,k+1},\dots,(\rho)_{0,k+l}\rangle & = & \langle x_1,\dots,x_l\rangle, \\
& \vdots & \\
\langle(\rho)_{n-1,k+1},\dots,(\rho)_{n-1,k+l}\rangle & = & \langle x_1,\dots,x_l\rangle.
\end{array}$$

Moreover, the definition of C yields that if $\mathcal{G}$ is a path through $T^A_{\vec{X},\vec{x}}$, then the function $\mathcal{F}(y) := (\mathcal{G}(y))_k$ satisfies $\forall nB(\vec{X}[n],\mathcal{F}[n],\vec{x})$. Therefore, (ii) of lemma II.1.6 yields that $Y := \{z : (\mathcal{G}(z))_{k,0} = 0\}$ satisfies $A'(\vec{X},Y,\vec{x})$. □

In the sequel, we will stick to the notations introduced in the previous lemma. So if $A(U_1,\dots,U_k,u_1,\dots,u_l)$ is a Σ^1_1 formula of L_2, we denote by TREE^A, $\mathsf{PROJ}^{k,l}$ and WIT^A the formulas defined in the course of its proof. Moreover, we extend the operation induced by the formula $\mathsf{PROJ}^{k,l}$ to sets and let

$$X_{Y_1,\dots,Y_k,y_1,\dots,y_l} := \{x : \mathsf{PROJ}^{k,l}(X,Y_1,\dots,Y_k,x,y_1,\dots,y_l)\}.$$

Combining the previous lemma with lemma II.1.5 which states that that a tree T has a path if and only if $\mathsf{KB}(T)$ is not a well-ordering yields the following representation theorem for Π^1_1 formulas.

Theorem II.1.8 (Representation Theorem of Π^1_1 formulas) *Let $A(\vec{U},\vec{u})$ be a Π^1_1 formula of L_2 with exactly the displayed variables free and set $B := \neg A$. Then* ACA_0 *proves*

$$\forall\vec{X},\vec{x}[A(\vec{X},\vec{x}) \leftrightarrow \mathsf{Wo}(\mathsf{KB}(T^B_{\vec{X},\vec{x}}))].$$

The above characterization of Π^1_1 formulas leads to the insight that there are no models of ACA_0 where the formula $\mathsf{Wo}(X)$ is Δ^1_1. As we will see later, the existence of pseudo-hierarchies relies on this fact.

Theorem II.1.9 *For each Σ^1_1 formula $A(U,\vec{V},\vec{u})$ with exactly the displayed variables free,* ACA_0 *proves:*

$$\neg\exists\vec{Y},\vec{y},\forall X[A(X,\vec{Y},\vec{y}) \leftrightarrow \mathsf{Wo}(X)].$$

Proof: We assume that there exist a Σ^1_1 formula $A(U,\vec{V},\vec{u})$ with exactly the displayed variables free, and sets $\vec{Y}$ and numbers $\vec{y}$ such that

$$\forall X[A(X,\vec{Y},\vec{y}) \leftrightarrow \mathsf{Wo}(X)]. \tag{1}$$

Now there is a Σ^1_1 formula $B(U,\vec{V},\vec{u})$ which is equivalent to

$$U \text{ is a tree } \rightarrow A(\mathsf{KB}(U_{U,\vec{V},\vec{u}}),\vec{V},\vec{u}).$$

Applying Theorem II.1.8 to the formula $\neg B$ and using (1) yields for all X,

$$X \text{ is a tree } \wedge \neg\mathsf{Wo}(\mathsf{KB}(X_{X,\vec{Y},\vec{y}})) \leftrightarrow \mathsf{Wo}(\mathsf{KB}(T^B_{X,\vec{Y},\vec{y}})).$$

For $X := T^B$, this is a contradiction! □

A similar result holds also for the well-orderings that are primitive recursive, recursive or recursively enumerable in X. To speak about such orderings we introduce the following notation:

$$\prec^{\vec{X}}_e := \{\langle x,y\rangle : \{e\}^{\vec{X}}(x,y)=0\}.$$

Lemma II.1.10 ACA_0 *proves for all Σ^1_1 formulas $A(U,\vec{u},v)$ with exactly the displayed variables free: For each X and all $\vec{y}$, if S is one of the sets* N*,* TRec^X *or* Prim*, then*

$$\neg(\forall e \in S)[\mathsf{Wo}(\prec^X_e) \leftrightarrow A(X,\vec{y},e)].$$

Proof: We assume that there are $\vec{y}$, X and a Σ^1_1 formula $A(U,\vec{u},v)$, such that for all $e \in S$, $\mathsf{Wo}(\prec^X_e) \leftrightarrow A(X,\vec{y},e)$. Now we set $B(U,u) := \forall\mathcal{F}\neg\pi^0_1(U,\mathcal{F},u,u)$. Theorem II.1.8 yields that $B(X,x) \leftrightarrow \mathsf{Wo}(\mathsf{KB}(T^{\neg B}_{X,x}))$. Since $\mathsf{KB}(T^{\neg B}_{X,x})$ is Δ^0_0 in X, we conclude that for all x,

$$B(X,x) \leftrightarrow (\exists e \in S)[A(X,\vec{y},e) \wedge \prec^X_e \text{ is isomorphic to } \mathsf{KB}(T^{\neg B}_{X,x})].$$

The right hand side of this equivalence is Σ, thus the Normal Form Lemma II.1.6 and our observation about universal Π^0_1 formulas (cf. lemma II.1.2) provide an index e_0, such that

$$\forall x[B(X,x) \leftrightarrow \exists \mathcal{F} \pi^0_1(X,\mathcal{F},x,e_0)].$$

However, this implies

$$B(X,e_0) = \forall \mathcal{F} \neg \pi^0_1(X,\mathcal{F},e_0,e_0) \leftrightarrow \exists \mathcal{F} \pi^0_1(X,\mathcal{F},e_0,e_0).$$

A contradiction! □

A related result is stated in subsection III.1.4, lemma III.1.13. There, it is shown that the theory $\mathsf{ATR}_0 + \neg\mathsf{TI}_{\lhd}(\mathsf{U},\Gamma_0)$ proves: If $\mathsf{Wo}(\prec)$, then $\prec$ is a proper initial segment of $\lhd{\restriction}\Gamma_0$.

We conclude this subsection by putting on record the universal Π^1_1 and universal Σ^1_1 formulas one obtains from lemma II.1.2 and lemma II.1.6.

Corollary II.1.11 *For each Π^1_1 formula $A(\vec{U},\vec{u},\vec{v})$ of L_2 with exactly the displayed variables free, the following is provable in ACA_0:*

$$\forall \vec{y} \exists e \forall \vec{X},\vec{x}[A(\vec{X},\vec{x},\vec{y}) \leftrightarrow \forall \mathcal{F} \neg \pi^0_1(\vec{X},\mathcal{F},\vec{x},e)].$$

where $\pi^0_1(\vec{U},V,\vec{u},e)$ is the universal Π^0_1 formula from definition II.1.1.

Again, we also state the following variant.

Corollary II.1.12 *For each Π^1_1 formula $A(\vec{U},\vec{u})$ of L_2 with exactly the displayed variables free, there is an $e \in \mathbb{N}$, such that ACA_0 proves*

$$\forall \vec{X},\vec{x}[A(\vec{X},\vec{x}) \leftrightarrow \forall \mathcal{F} \neg \pi^0_1(\vec{X},\mathcal{F},\vec{x},\mathsf{cs}_e)].$$

For further use of universal Π^1_1 formulas, we define for each natural number k,l and $\vec{U} = U_1 \ldots, U_k$, $\vec{u} = u_1,\ldots,u_l$,

$$\pi^1_{1,k,l}(\vec{U},\vec{u},e) := \forall \mathcal{F} \neg \pi^0_1(\vec{X},\mathcal{F},\vec{x},e).$$

Then in need for universal Σ^1_1 formulas, we simply resort to $\neg\pi^1_{1,k,l}(\vec{U},\vec{u},e)$. As with the universal Π^0_1 formulas $\pi^0_{1,k,l}$, we forgo to explicitly mention the number of number parameters, and write $\pi^1_{1,k,l}(\vec{U},\vec{u},e)$ instead of $\pi^1_1(\vec{U},\vec{u},e)$.

II.1.3 Hierarchies and choice sequences

For set and number parameters $\vec{X}, \vec{y}$, a formula $A(U, \vec{V}, u, \vec{v})$ of L_2 defines canonically an *operator* $F^A_{\vec{Y},\vec{y}}$ on the powerset of the natural numbers, namely

$$F^A_{\vec{Y},\vec{y}}(X) := \{x : A(X, \vec{Y}, x, \vec{y})\}.$$

An *iteration principle* is an axiom schema, asserting that we can iterate a certain class of operators along a certain class of well-orderings. This means that there exists a *hierarchy*, i.e. a sequence of sets with domain $\prec$ such that the αth element of this sequence is obtained by applying the operator to the disjoint union of all the elements below α. Thereby, α itself may appear as a parameter of the operator, which proves convenient regarding applications of the iteration principle, however, does not affect its strength.

Moving towards a formal definition, we call F an A-hierarchy along $\prec$ for the parameters $\vec{Y}, \vec{y}$, denoted by $\mathsf{Hier}^A(F, \vec{Y}, \prec, \vec{y})$, if $A(U, \vec{V}, W, u, \vec{v}, w)$ contains exactly the displayed variables free, and F meets the following properties:

(i) $\mathsf{Lin}_0(\prec)$,

(ii) $(\forall x \in F)[x = \langle (x)_0, (x)_1 \rangle]$,

(iii) $\forall x[(F)_x \neq \emptyset \rightarrow x \in \mathsf{Field}(\prec)]$,

(iv) $(\forall \alpha \in \mathsf{Field}(\prec))[(F)_\alpha = F^A_{\vec{Y},\prec,\vec{y},\alpha}((F)_{\prec\alpha})]$.

If in addition, $\prec$ is a well-ordering, we call F a *proper hierarchy*. If $\prec$ is only a linear ordering with a least element, but not well-ordered, F is baptized a *pseudo-hierarchy*,

$$\mathsf{PSH}^A(F, \vec{Y}, \prec, \vec{y}) := \neg\mathsf{Wo}(\prec) \wedge \mathsf{Hier}^A(F, \vec{Y}, \prec, \vec{y}).$$

Immediately from the definition of a hierarchy, we derive that $\mathsf{Hier}^A(F, \prec)$ implies $\mathsf{Hier}^A((F)_{\prec\alpha}, \prec\restriction\alpha)$ for all α in the field of the ordering $\prec$. If the context implies that α is an element of the field of $\prec$, then $\mathsf{Hier}^A(G, \alpha)$ is short for $\mathsf{Hier}^A(G, \prec\restriction\alpha)$.

An important property of proper hierarchies is stated below.

Lemma II.1.13 *For any formula $A(U, \vec{V}, u, \vec{v})$ of L_2, the following is provable in ACA_0: If $\prec$ is a well-ordering, then*

$$\mathsf{Hier}^A(F, \vec{Z}, \prec, \vec{z}) \wedge \mathsf{Hier}^A(G, \vec{Z}, \prec, \vec{z}) \rightarrow F = G.$$

Proof: By transfinite induction along the well-ordering $\prec$ one easily shows that the set $\{\alpha \in \mathsf{Field}(\prec) : (F)_\alpha = (G)_\alpha\}$ is already the entire field of $\prec$. □

Finally, a word on hierarchies whose αth level does not explicitly depend on α and the ordering $\prec$. Suppose for instance, that the set F meets conditions (i) and (ii) of the aforementioned definition, and further satisfies $(F)_\alpha = F^A((F)_{\prec\alpha})$, where $A(U,u)$ contains only the displayed variables free. In the strict sense, F is not an A-hierarchy, but only a B-hierarchy for the formula $B(U,V,u,v) := A(U,u) \land V = V \land v = v$, otherwise, the operator F^A could not take the parameters $\prec$ and α. However, in this case, the superfluous parameters $\prec$ and α are discarded, and we call F an A-hierarchy all the same.

In the literature, iteration principles are often called *recursion principles*, since the hierarchy claimed to exist can be viewed as a function defined via recursion. In the sequel, we distinguish between iteration along the natural numbers (recursion along $<_{\mathsf{N}}$) and iteration along arbitrary well-orderings (*transfinite recursion*).

For each formula $A(U,\vec{V},W,u,\vec{v},w)$ of L_2 belonging to the class $\mathcal{K}$, we define

($\mathcal{K}$-TR) $$\mathsf{Wo}(\prec) \to \exists F \mathsf{Hier}^A(\vec{Y},\prec,\vec{y}).$$

The instance, where $\mathcal{K}$ is the class of arithmetical formulas of L_2 is probably the best know iteration principle in second order arithmetic and is also denoted by (ATR), standing for *arithmetical transfinite recursion*. The corresponding theory ATR_0, i.e. $\mathsf{ACA}_0 + (\mathsf{ATR})$, goes back to Friedman [14]. A key reference for ATR_0 is Friedman, McAloon, Simpson [15].

Another class of axiom schemas are *choice principles* and *dependent choice principles*, which claim the existence of *choice sequences*. In contrast to hierarchies, where the αth level is uniquely determined by the levels below α and an operator, the αth element of a choice sequence has to be chosen. If the possible choices for the αth level depends only on α, we speak of a choice principle; if the αth level depends also on the disjoint union of the levels below α, we speak of dependent choice. As with iteration, we distinguish *dependent choice* and *transfinite dependent choice*.

For all L_2 formulas $A(U,\vec{V},\vec{u},v)$, $B(U,V,\vec{W},\vec{u})$ and $C(U,V,\vec{W},\vec{u},v)$ of the class $\mathcal{K}$ we have:

($\mathcal{K}$-AC) $$\forall n \exists X A(X,\vec{Y},\vec{y},n) \to \exists F \forall n A((F)_n,\vec{Y},\vec{y},n),$$

($\mathcal{K}$-DC) $$\forall X \exists Y B(X,Y,\vec{Z},\vec{y}) \to \exists F[(F)_0 = W \land \forall n B((F)_n,(F)_{n+1},\vec{Z},\vec{y})],$$

($\mathcal{K}$-TDC) $$\forall\alpha\forall X \exists Y C(X,Y,\vec{Z},\vec{y},\alpha) \land \mathsf{Wo}(\prec) \to \exists F \forall\alpha C((F)_{\prec\alpha},(F)_\alpha,\vec{Z},\vec{y},\alpha).$$

Well-known choice principles are (Σ^1_1-AC) and (Σ^1_1-DC). Together with ACA_0 or ACA they form the theories Σ^1_1-AC_0, Σ^1_1-DC_0 or Σ^1_1-AC, Σ^1_1-DC, respectively. Although ACA_0 and Σ^1_1-AC_0 prove the same L_1 formulas and therefore are proof-theoretically equivalent (see e.g [4] or [7]), the theory Σ^1_1-AC_0 proves more L_2 formulas. For instance, over Σ^1_1-AC_0, the classes of Σ^1_1 [Π^1_1] definable and Σ [Π] definable sets coincide as an induction on the build-up of the formula reveals.

Lemma II.1.14 *For each* Π *formula* C *of* L_2 *there is a* Π^1_1 *formula* C' *of* L_2 *containing the same free variables as* C*, such that* Σ^1_1-AC_0 *proves:* $C \leftrightarrow C'$*.*

Another observation is recorded below:

Lemma II.1.15 *Over* ACA_0 *the schema* (Σ^1_1-DC) *implies* (Σ^1_1-AC)*.*

Proof: Assume that $A(U,u)$ is a Σ^1_1 formula of L_2 such that $\forall x \exists X A(X,x)$. Then, we also have $\forall X \exists Y B(X,Y)$, there $B(U,V)$ a the Σ^1_1 formula equivalent to

$$\forall n[(U)_0 = \{n\} \to (V)_0 = \{n{+}1\} \wedge A((V)_1), n].$$

Now (Σ^1_1-DC) yields a set Z such that $(Z)_0 = \{\langle 0,0\rangle\}$ and $\forall n B((Z)_n,(Z)_{n+1})$. We let $X := \{\langle x,n\rangle : x \in (Z)_{n+1,1}\}$ and show by induction that $\forall n[(Z)_{n,0} = \{n\}]$, thus $\forall x A((X)_x, x)$. □

An interesting transfinite dependent choice principle is (Σ^1_1-TDC). The theory ACA + (Σ^1_1-TDC) is introduced and analyzed in Rüede [36, 37, 38]. In the next chapter, we also mention a similar system for admissible set theory and discuss how hierarchies and choice sequences are related. Further, we antedate a result, which is a consequence of theorem II.2.21, namely that over ACA_0 the principle (ATR) implies the apparently stronger iteration principle (Δ-TR).

II.1.4 N-models of Σ^1_1-AC and Σ^1_1-DC

In theories comprising ACA_0 it is possible to talk about sets of sets. Therefore, we can talk about models of theories formulated in L_2. Thereby, we say that a set M is an N-*model*, or simply a model of T, if M contains only pairs, that is $x \in M$ implies $x = \langle (x)_0, (x)_1\rangle$, and for all instances of an axiom or rule of T with premises Γ_i and conclusion Γ, A^M holds, where A is the universal closure of $\bigvee \Gamma_1 \wedge \ldots \wedge \bigvee \Gamma_n \to \bigvee \Gamma$. Further, we say that a formula B of L_2 is a finite axiomatization of T, if B^M implies that M is a model of T, in the sense specified above.

The theorem below tells us that there are finite axiomatizations for ACA, Σ^1_1-AC and Σ^1_1-DC:

Theorem II.1.16 (Finite axiomatizations for ACA, Σ^1_1-AC and Σ^1_1-DC) *Consider the formulas listed below:*

(i) $\forall X, Y \exists Z (Z = X \oplus Y)$,

(ii) $\forall Z, z, e \exists Y \forall x [x \in Y \leftrightarrow \pi^0_1(Z,x,z,e)]$,

(iii) $\forall Z, z, e[\forall x \exists X \pi^0_2(X,Z,x,z,e) \to \exists Y \forall x \pi^0_2((Y)_x, Z, x, z, e)]$,

(iv) $\forall V, W, z, e[\forall X, Y \pi^0_2(X, Y, W, z, e) \rightarrow$
$\exists Z \forall n((Z)_0 = V \wedge \pi^0_2((Z)_n, (Z)_{n+1}, W, z, e))]$.

Then the conjunction of (i) and (ii), denoted by Ax_{ACA} *is a finite axiomatization of* ACA*, the conjunction of (i),(ii) and (iii), denoted by* $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$*, is a finite axiomatization of* Σ^1_1-AC *and the conjunction (i),(ii) and (iv), denoted by* $Ax_{\Sigma^1_1\text{-}\mathsf{DC}}$*, is a finite axiomatization of* Σ^1_1-DC.

In (iii) and (iv) we actually need a universal Π^0_2 formula. Since one finds claims in the literature (cf. [36, 38]) that already (Π^0_1-AC) and (Π^0_1-DC) imply (Σ^1_1-AC) and (Σ^1_1-DC), respectively, we comment on the pitfall wherein one is easily caught. A proof of the above theorem then emerges from this considerations. First, we show that the aforementioned claim is wrong.

Lemma II.1.17 (Strict Π^1_1-reflection) *For each* Δ^0_0 *formula* $A(U, \vec{V}, \vec{u}, \vec{v})$ *of* L_2*, the following is provable in* ACA_0*: If* $\vec{Y}, \vec{y}$ *are such that* $\forall X \exists \vec{x} A(X, \vec{Y}, \vec{x}, \vec{y})$*, then there exits an* n_0 *with*

(i) $\forall X (\exists \vec{x} < n_0) A(X, \vec{Y}, \vec{x}, \vec{y})$,

(ii) $\forall X (\forall \vec{x} < n_0)[A(X, \vec{Y}, \vec{x}, \vec{y}) \leftrightarrow A(X{\restriction}n_0, \vec{Y}, \vec{x}, \vec{y})]$,

where $X{\restriction}n_0$ *denotes the set* $\{x \in X : x < n_0\}$.

Proof: By the translation presented in subsection I.2.13 and the fact that each term has a unique value, i.e. for each term t, $\forall x[\mathsf{Val}_t(x) \rightarrow \sim R(x)]$ is equivalent to $\exists x[\mathsf{Val}_t(x) \wedge \sim R(x)]$, we may assume that there is a Δ^0_0 formula $B(U, \vec{V}, \vec{u}, \vec{v}, w)$ that does not contain function symbols such that $A(U, \vec{V}, \vec{u}, \vec{v})$ is equivalent to $\exists z B(U, \vec{V}, \vec{u}, \vec{v}, z)$. Therefore,

$$(*) \qquad \forall X, \vec{Y}, \vec{y}[\exists \vec{x} A(X, \vec{Y}, \vec{x}, \vec{y}) \leftrightarrow \exists z (\exists \vec{x}, w < z)(\vec{y} < z \wedge C(X[z], \vec{Y}, \vec{x}, \vec{y}, w))],$$

where $C(\sigma, \vec{V}, \vec{u}, \vec{v}, w)$ is obtained form $B(U, \vec{V}, \vec{u}, \vec{v}, w)$ by replacing each expression of the form $t \in U$ by $(\sigma)_t = 0$. Note that t is a variable, and that substituting $t \in U$ by $(t < z) \wedge (\sigma)_t = 0$ would lead to an equivalent formula. Next, we fix the parameters $\vec{Y}, \vec{y}$ and consider the set

$$T := \{\sigma \in \mathsf{seq}_{0,1} : \neg[(\exists \vec{x}, w < \mathsf{lh}(\sigma))(\vec{y} < \mathsf{lh}(\sigma) \wedge C(\sigma, \vec{Y}, \vec{x}, \vec{y}, w))]\}.$$

If T is a tree, then $\forall X \exists \vec{x} A(X, \vec{Y}, \vec{x}, \vec{y})$ is equivalent to $\forall X \exists z (X[z] \notin T)$. Next, we argue that T is indeed a tree: Suppose that $\sigma \in T$ and $\tau \sqsubset \sigma$. Now $\tau \in T$ follows, if $\vec{x}, w, \vec{y} < \mathsf{lh}(\tau) \wedge \neg C(\tau, \vec{Y}, \vec{x}, \vec{y}, w)$ implies $\neg C(\sigma, \vec{Y}, \vec{x}, \vec{y}, w)$. But this follows, since $C(\sigma, \vec{Y}, \vec{x}, \vec{y}, w)$ is a Δ^0_0 formula that does not contain function symbols (also no constants) and contains σ only in expressions of the form $(\sigma)_t = 0$ and $(\sigma)_t \neq 0$. Because t is a variable bound by $\mathsf{lh}(\tau)$, and $t < \mathsf{lh}(\tau) \to [(\sigma)_t = 0 \leftrightarrow (\tau)_t = 0]$ we conclude that $C(\tau, \vec{Y}, \vec{x}, \vec{y}, w)$.

Now we assume that $\forall X \exists \vec{x} A(X, \vec{Y}, \vec{x}, \vec{y})$ holds. So also $\forall X \exists z (X[z] \notin T)$, which expresses that T has no path. Therefore, König's Lemma provides an n_0 such that $\forall X (\exists z \leq n_0)(X[z] \notin T)$. Using $(*)$, (i) and (ii) easily follow. □

Corollary II.1.18 ACA_0 *proves each instance of* $(\Pi^0_1\text{-}\mathsf{AC})$.

Proof: We show the contraposition of $(\Pi^0_1\text{-}\mathsf{AC})$. Suppose that $A(U, u, v)$ is Δ^0_0 and that $\forall X \exists x \exists y A((X)_x, x, y)$. Applying lemma II.1.17, provides an n_0 with

(i) $\forall X (\exists x, y < n_0) A((X)_x, x, y)$,

(ii) $\forall X (\forall x, y < n_0)[A(X, x, y) \leftrightarrow A(X{\restriction}n_0, x, y)]$.

Now suppose for a moment that

$$(*) \qquad (\forall x < n_0)(\exists X \subseteq \{0, \ldots, n_0{-}1\})(\forall y < n_0) \neg A(X, x, y).$$

Now we assume that $\prec_{n_0}$ is a well-ordering on the 2^{n_0} many subsets of $\{0, \ldots, n_0{-}1\}$, which allows us to define a set Z such that $(Z)_k$ is the $\prec_{n_0}$-least $X \subseteq \{0, \ldots n_0{-}1\}$ with $(\forall y < n_0) \neg A(X, k, y)$ if $k < n_0$, and $(Z)_k = \emptyset$ otherwise. According to (i), such a Z cannot exist. Therefore, the negation of $(*)$ holds, i.e.

$$(\exists x < n_0)(\forall X \subseteq \{0, \ldots, n_0{-}1\})(\exists y < n_0) A(X, x, y).$$

Now (ii) yields $\exists x \forall X \exists y A(X, x, y)$. □

Remark II.1.19 *Let us try to prove that* $(\Pi^0_1\text{-}\mathsf{AC})$ *implies already* $(\Sigma^1_1\text{-}\mathsf{AC})$ *and observe what goes wrong. So we suppose that* $A(\vec{U}, V, \vec{u}, v)$ *is a* Σ^1_1 *formula of* L_2. *Further, assume that* $\vec{X}, \vec{x}$ *are such that* $\forall n \exists Y A(\vec{X}, Y, \vec{x}, n)$. *The Normal Form Lemma II.1.6 provides a* Δ^0_0 *formula* $B(\vec{\sigma}, \tau, \vec{u}, v, w)$ *with*

$$(*) \qquad \forall n \exists \mathcal{F} \forall m B(\vec{X}[m], \mathcal{F}[m], \vec{x}, n, m).$$

By $(\Pi^0_1\text{-}\mathsf{AC})$ *we obtain a* Z *such that* $\forall n \mathsf{Fun}((Z)_n)$ *and*

$$\forall n [\mathcal{F} = (Z)_n \to \forall m B(\vec{X}[m], \mathcal{F}[m], \vec{x}, n, m)].$$

By the second part of the Normal Form Lemma we conclude that the set

$$W := \{\langle y, n \rangle : \exists z [\langle y, \langle 0, z \rangle\rangle \in (Z)_n]\}$$

satisfies $\forall n A(\vec{X}, (W)_n, \vec{x}, n)$.

However, if we formulate $(*)$ *without function variables, we obtain*

$$\forall n \exists F[\mathsf{Fun}(F) \wedge \mathsf{Dom}(F) = \mathsf{N} \wedge \forall m B(\vec{X}[m], \mathcal{F}[m], \vec{x}, n, m)].$$

Since $\mathsf{Dom}(F) = \mathsf{N}$ *is* Π^0_2, *we actually need* $(\Pi^0_2\text{-}\mathsf{AC})$ *to infer* $(\Sigma^1_1\text{-}\mathsf{AC})$.

Only the following is correct:

Lemma II.1.20 *Over* ACA_0, $(\Pi^0_2\text{-}\mathsf{AC})$ *implies* $(\Sigma^1_1\text{-}\mathsf{AC})$.

Similarly, one obtains:

Lemma II.1.21 *Over* ACA_0, $(\Pi^0_2\text{-}\mathsf{DC})$ *implies* $(\Sigma^1_1\text{-}\mathsf{DC})$.

We conclude by stating an important property of model of ACA. Within models of ACA we have enough comprehension to adapt Russell's argument which yields that the collection of all sets is not a set itself.

Lemma II.1.22 *The following is provable in* ACA_0*:*

$$(Ax_{\mathsf{ACA}})^M \to M \dot{\notin} M.$$

Proof: Suppose that $M \dot{\in} M$, and apply arithmetical comprehension within M to obtain the Russell set

$$R := \{\langle x, e\rangle : (M)_e \dot{\notin} (M)_e \wedge x \in (M)_e\}.$$

Now $R \dot{\in} M$ implies that there is an index r with $R = (M)_r$. Moreover, $X := \{\langle 0, 0\rangle\}$ is a set in M and meets $X \dot{\notin} X$, so we know that $R \neq \emptyset$.

On the one hand, we have to refute the assumption that $R \dot{\notin} R$, since otherwise, the definition of R yields

$$\forall x[\langle x, r\rangle \in R \leftrightarrow R \dot{\notin} R \wedge x \in R \leftrightarrow x \in R],$$

which expresses $R = (R)_r$, thus $R \dot{\in} R$. On the other hand, there is also no index a such that $R = (R)_a$, for otherwise, again by the definition of R, we obtain

$$(*) \qquad \forall x[x \in R \leftrightarrow \langle x, a\rangle \in R \leftrightarrow (M)_a \dot{\notin} (M)_a \wedge x \in (M)_a].$$

Because R is not empty, we conclude $(M)_a \dot{\notin} (M)_a$. However, then $(*)$ expresses that $R = (M)_a$, which in turn yields $R \notin R$. Therefore, we have to refute the supposition that $M \dot{\in} M$. □

II.1.5 The jump-hierarchy

A prominent hierarchy is the so-called *jump-hierarchy*. The first level of a jump-hierarchy above X consists of the complements of the sets recursively enumerable in X, and the αth level contains the complements of all the sets that are recursively enumerable in some level β below α.

For Turing's *jump* formula

$$\mathcal{J}(U, V, u) := \exists y, z, e[u = \langle y, \langle z, e\rangle\rangle \wedge \pi^0_1((U)_z, V, y, e)],$$

an ordering $\prec$ and a set X, a set F satisfying $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ is called a jump-hierarchy above X along $\prec$. Such an F is in the sequel also denoted by $\mathcal{J}^X_\prec$. If the context implies that $\mathcal{J}^X_\prec$ exists and α is an element of the field of $\prec$, we use $\mathcal{J}^X_\alpha$ for the αth level of this jump-hierarchy. Provided that the underlying ordering of a jump-hierarchy is a well-ordering, there is, provable in ACA_0, exactly one jump-hierarchy $\mathcal{J}^X_\prec$. Hence, the formula $x \in \mathcal{J}^X_\alpha$ is Δ^1_1.

To compare sets in different stages of the jump-hierarchy, we resort to the following notation. We write e.g. $Y \leq_{\Delta^0_1} X$ to state that Y is Δ^0_1 in X, and $X =_{\Delta^0_1} Y$ to express $X \leq_{\Delta^0_1} Y$ and $Y \leq_{\Delta^0_1} X$. Observe that $X =_{\Delta^0_1} Y$ and $Y =_{\Delta^0_1} Z$ implies $X =_{\Delta^0_1} Z$, and that $Y \leq_{\Delta^0_1} X$ can be expressed by the arithmetical formula $\exists e, e'[Y = \{x : \pi^0_1(X, x, e\} \wedge \overline{Y} = \{x : \pi^0_1(X, x, e')\}$.

Some basic facts concerning the jump-hierarchy are collected in the lemma below.

Lemma II.1.23 *There are numbers $a, b, c, d \in \mathbb{N}$ (used only in claim (iv)) such that the following is provable in ACA_0: If $\mathsf{Lin}_0(\prec)$ and $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ and $\alpha \in \mathsf{Field}(\prec)$, then*

(i) For all z, $(F)_\alpha = \{\langle y, \langle z, e\rangle\rangle : \{e\}^{((F)_{\prec\alpha})_z, X}(y)\uparrow\}$ and for $z \not\prec \alpha$, $((F)_{\prec\alpha})_z = \emptyset$.

(ii) $(F)_\alpha$ is the union of $G := \{\langle y, \langle \beta, e\rangle\rangle : \beta \prec \alpha \wedge \pi^0_1((F)_\beta, X, y, e)\}$ and
$H := \{\langle y, \langle z, e\rangle\rangle : z \not\prec \alpha \wedge \pi^0_1(\emptyset, X, y, e)\}$. Further, $H \dot{\subseteq} G$.

(iii) If $\beta \prec \alpha$, then $\mathsf{Hier}^{\mathcal{J}}((F)_{\prec\beta}, X, \prec\restriction\beta)$.

(iv) $X = (F)_{\alpha,\langle 0,\mathsf{cs}_a\rangle}$, $(F)_{\prec\alpha,z} = (F)_{\alpha,\langle z,\mathsf{cs}_b\rangle}$, $\overline{(F)_{\prec\alpha,z}} = (F)_{\alpha,\langle z,\mathsf{cs}_c\rangle}$ and
$\mathsf{N} = (F)_{\alpha,\langle 0,\mathsf{cs}_d\rangle}$. Further, if $z \not\preceq \alpha$, then $(F)_z = (F)_{\alpha,\langle z,\mathsf{cs}_b\rangle}$, and
$\overline{(F)_z} = (F)_{\alpha,\langle z,\mathsf{cs}_c\rangle}$.

(v) If $\gamma \prec \beta \preceq \alpha$, then $(F)_{\alpha,\langle\gamma,e\rangle} = (F)_{\beta,\langle\gamma,e\rangle}$. In particular, we have $(F)_\beta \dot{\subseteq} (F)_\alpha$.

(vi) If $\beta \prec \alpha$ and $Y =_{\Pi^0_1} (F)_\beta$, then $Y \dot{\in} (F)_\alpha$. Thus, $(F)_\alpha$ is transitive:

$Z \dot{\in} Y \dot{\in} (F)_\alpha$ *implies* $Z \dot{\in} (F)_\alpha$.

(vii) If $\beta \prec \alpha$, then $(F)_\alpha$ is not Δ^0_1 in $(F)_\beta$. In particular, $(F)_\alpha \dot{\notin} (F)_\beta$.

Proof: (i),(ii) and (iii) are immediate from the definition. Note, that if $X \dot{\in} (F)_\alpha$, then (ii) yields that $X = (F)_{\alpha,\langle\gamma,e\rangle}$ for a $\gamma \prec \alpha$. We will use this observation tacitly in the arguments below.

For (iv), recall that $\pi^0_1(U, V, u, v)$ is the formula $\{u\}^{U,V}(v)\uparrow$. Now let f, g be the partial recursive functions with $f(0)\uparrow$ and $f(x) = 0$ for $x \neq 0$; $g(1)\uparrow$ and $g(x) = 0$ for $x \neq 1$, and let $e_f, e_g \in \mathsf{Rec}$ be indices of f and g. Then (cf. the definition of Rec in subsection II.1.1) for $a = \langle 3, 1, e_f, \langle 13, 1\rangle\rangle$, $b = \langle 3, 1, e_f, \langle 12, 1\rangle\rangle$, $c = \langle 3, 1, e_g, \langle 12, 1\rangle\rangle$, and an index $d \in \mathsf{Rec}$ of $f \circ ch_{\mathsf{N}}$ we have that $\{a\}^{U,V}$ does not depend on U, $\{b\}^{U,V}$ and $\{c\}^{U,V}$ do not depend on V and $\{d\}^{U,V}$ is independent of U and V. Hence

$$\begin{array}{ccccc} X & = & \{y : \{a\}^{\emptyset,X}(y)\uparrow\} & = & (F)_{\alpha,\langle 0,a\rangle}, \\ ((F)_{\prec\alpha})_z & = & \{y : \{b\}^{((F)_{\prec\alpha})_z,X}(y)\uparrow\} & = & (F)_{\alpha,\langle z,b\rangle}, \\ \overline{(F)_z} & = & \{y : \{c\}^{((F)_{\prec\alpha})_z,X}(y)\uparrow\} & = & (F)_{\alpha,\langle z,c\rangle}, \\ \mathsf{N} & = & \{y : \{d\}^{\emptyset,X}(y)\uparrow\} & = & (F)_{\alpha,\langle 0,d\rangle}. \end{array}$$

For $z \not\prec \alpha$, we have $(F)_z = ((F)_{\prec\alpha})_z$: If z is not in the field of $\prec$, then both sides are empty. The second part of the claim follows.

For (v), we observe that provided $\gamma \prec \beta \preceq \alpha$,

$$\begin{array}{rcl} y \in (F)_{\alpha,\langle\gamma,e\rangle} & \leftrightarrow & \langle y, \langle\gamma, e\rangle\rangle \in (F)_\alpha, \\ & \leftrightarrow & \pi^0_1(((F)_{\prec\alpha})_\gamma, X, y, e), \\ & \leftrightarrow & \pi^0_1(((F)_{\prec\beta})_\gamma, X, y, e), \\ & \leftrightarrow & \langle y, \langle\gamma, e\rangle\rangle \in (F)_\beta, \\ & \leftrightarrow & y \in (F)_{\beta,\langle\gamma,e\rangle}. \end{array}$$

Next we show (vi). Suppose $\beta \prec \alpha$ and that $A(U, u)$ is a Π^0_1 formula of L_2 such that $Y = \{y : A((F)_\beta, y)\}$. Clearly, there is also a Π^0_1 formula $B(U, V, u)$ with $Y = \{y : B((F)_\beta, X, y)\}$. Then there is an e such that $Y = \{y : \pi^0_1((F)_\beta, X, y, e)\}$, thus $Y = (F)_{\alpha,\langle\beta,e\rangle}$. Further, if $Z \dot{\in} Y \dot{\in} (F)_\alpha$, we have $Y := \{y : \pi^0_1((F)_\gamma, X, y, e)\}$ for some e and $\gamma \prec \alpha$. Hence Y is Π^0_1 in $(F)_\gamma$, which yields that also Z is Π^0_1 in $(F)_\gamma$. Therefore $Z \dot{\in} (F)_\alpha$, according to the argument given above.

A standard diagonalization argument yields (vii). Consider the set

$$D := \{y : \{y\}^{(F)_\beta,X}(y)\uparrow\}.$$

Of course, D is not Δ^0_1 in $(F)_\beta$, otherwise, $D = \{y : \{e_0\}^{(F)_\beta, X}(y)\downarrow\}$ for some index e_0, which implies $e_0 \in D \leftrightarrow e_0 \notin D$. However, if $(F)_\alpha$ were Δ^0_1 in $(F)_\beta$, then so were D, because $y \in D \leftrightarrow \{y\}^{((F)_{\prec\alpha})_\beta, X}(y)\uparrow \leftrightarrow \langle y, \langle \beta, y\rangle\rangle \in (F)_\alpha$. □

An immediate consequence of the previous lemma and the choice of b is that a jump-hierarchy allows us to regain the underlying ordering.

Corollary II.1.24 *The following is provable in* ACA_0*: If* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$*, then we have*

$$\prec = \{\langle \alpha, \beta\rangle : (F)_\alpha \neq \emptyset \wedge (F)_\beta \neq \emptyset \wedge (F)_{\beta, \langle \alpha, \mathsf{cs}_b\rangle} \neq \emptyset, \}$$

where $b \in \mathbb{N}$ *is as in lemma II.1.23.*

Proof: $(F)_\alpha \neq \emptyset$ and $(F)_\beta \neq \emptyset$ are equivalent to $\alpha, \beta \in \mathsf{Field}(\prec)$ and by (iv) of the previous lemma, $(F)_{\beta, \langle \alpha, \mathsf{cs}_b\rangle} \neq \emptyset$ is equivalent to $(F)_{\prec\beta, \alpha} \neq \emptyset$ which in turn is equivalent to $\alpha \prec \beta$. □

Next, we start to compare stages of jump-hierarchies. If two sets X and Y are recursive in each other, then so are the next levels of the jump-hierarchy above X and Y. First, an auxiliary lemma:

Lemma II.1.25 *The following is provable in* ACA_0*: If* $Y =_{\Delta^0_1} X$*, then*

$$F := \{\langle x, e\rangle : \pi^0_1(X, x, e)\} =_{\Delta^0_1} \{\langle x, e\rangle : \pi^0_1(Y, x, e)\} =: G.$$

Proof: Since $Y =_{\Delta^0_1} X$, there are Π^0_1 formulas $A(U, u)$ and $B(U, u)$, such that we have $Y = \{x : A(X, x)\}$ and $\overline{Y} = \{x : B(X, x)\}$. By replacing in $\pi^0_1(Y, x, e)$ all literals of the form $t \in Y$ by $A(X, t)$ and $t \notin Y$ by $B(X, t)$, we obtain the formula $D(X, x, e)$ which is equivalent to a Π^0_1 formula. Next, we let $D'(U, u) := u = \langle (u)_0, (u)_1\rangle \wedge D(U, (u)_0, (u)_1)$. There is an index e_0 such that for all Y and y, $D(Y, y)$ is equivalent to $\pi^0_1(Y, y, e_0)$. Thus,

$$y \in (F)_{e_0} \leftrightarrow \exists x, e[y = \langle x, e\rangle \wedge D(X, x, e)] \leftrightarrow \exists x, e[y = \langle x, e\rangle \wedge \pi^0_1(Y, y, e_0)] \leftrightarrow y \in G$$

Hence $G = (F)_{e_0}$ and $G <_{\Delta^0_1} F$. Switching the roles of F and G yields $F <_{\Delta^0_1} G$, i.e. $F =_{\Delta^0_1} G$. □

On a linear ordering $\prec$, we cannot define addition. However, if there is a sequence $\alpha_0 \prec \ldots \prec \alpha_k$, then we may say that α_k is at least k levels above α_0. In this sense, if a set Z is Π^0_{k+1} in some level α of a jump-hierarchy, and β is at least $k+1$ levels above α, then Z is an element of level β. Since the levels of a jump-hierarchy are not closed under complements w.r.t. $\mathbb{N}$, this fails for Π^0_0 formulas.

Lemma II.1.26 *For* $k \in \mathbb{N}$ *and each* Π^0_{k+1} *formula* $A(U, V, \vec{u}, v)$ *of* L_2 *with exactly the displayed variables free, the following is provable in* ACA_0*: If* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$*, then there exists for all* $\vec{y}$ *an* e*, such that for each sequence* $\alpha_0 \prec \alpha_1 \prec \ldots \prec \alpha_{\mathsf{cs}_{k+1}}$*,*

$$\{x : A((F)_{\alpha_0}, X, \vec{y}, x)\} = (F)_{\alpha_{\mathsf{cs}_{k+1}}, \langle \alpha_{\mathsf{cs}_k}, e\rangle}.$$

Proof: We prove the claim by meta-induction on $l \leq k$: For $l = 0$, lemma II.1.2 tells us, that for all $\vec{y}$, there is an e, such that

$$\{x : A((F)_{\alpha_0}, X, \vec{y}, x)\} = \{x : \pi^0_1((F)_{\alpha_0}, X, x, e)\} = (F)_{\alpha_1,\langle \alpha_0, e\rangle}.$$

Now suppose that the lemma holds for $l < k$ and that $A(U, V, \vec{u}, v)$ is the formula $\forall z B(U, V, \vec{u}, v, z)$ and B is Σ^0_{l+1}. The I.H. now provides an e', such that

$$\{x : x = \langle (x)_0, (x)_1\rangle \wedge \neg B((F)_{\alpha_0}, X, \vec{y}, (x)_0, (x)_1)\} = (F)_{\alpha_{\mathsf{cs}_{l+1}},\langle \alpha_{\mathsf{cs}_l}, e'\rangle}.$$

Thus

$$A((F)_{\alpha_0}, X, \vec{y}, x) \leftrightarrow \forall z[\langle x, z\rangle \notin (F)_{\alpha_{\mathsf{cs}_{l+1}},\langle \alpha_{\mathsf{cs}_l}, e'\rangle}].$$

Now there is an e, such that

$$\{x : A((F)_{\alpha_0}, X, \vec{y}, x)\} = \{x : \pi^0_1((F)_{\alpha_{\mathsf{cs}_{l+1}}}, X, x, e)\} = (F)_{\alpha_{\mathsf{cs}_{l+2}},\langle \alpha_{\mathsf{cs}_{l+1}}, e\rangle}.$$

□

For proper hierarchies we obtain the following corollary:

Corollary II.1.27 *For $k \in \mathbb{N}$ and each Π^0_{k+1} formula $A(U, V, \vec{u}, v)$ of L_2 with exactly the displayed variables free, the following is provable in ACA_0: If $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ and $\mathsf{Wo}(\prec)$, then*

$$\forall \vec{y} \exists e \forall \beta[\beta + \mathsf{cs}_{k+1} \in \mathsf{Field}(\prec) \rightarrow \{x : A((F)_\beta, X, \vec{y}, x)\} = (F)_{\beta+\mathsf{cs}_{k+1},\langle \beta+\mathsf{cs}_k, e\rangle}].$$

Since over ACA_0 each arithmetical formula is equivalent to some Π^0_n formula, the ωth stage of the jump-hierarchy above X is a model of ACA above X.

Corollary II.1.28 *The following is provable in ACA_0: If there exists an F with $\mathsf{Hier}^J(F, X, \lhd\restriction\omega{+}1)$, then $(F)_\omega$ is a model of ACA above X.*

For linear orderings a similar result holds:

Corollary II.1.29 *The following is provable in ACA_0: If $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ and $\prec$ is an ordering without a top element, then*

$$M := \{\langle y, \langle \beta, e\rangle\rangle : \exists \alpha(\beta \prec \alpha \wedge y \in (F)_{\alpha,\langle \beta, e\rangle})\}$$

is a model of ACA.

Proof: Suppose that the jump-hierarchy F and the set M are as specified above. An $x \in M$ is of the form $\langle y, \langle \beta, e\rangle\rangle$. Thus, if $Y \dot{\in} M$, then

$$Y = (M)_{\langle \beta, e\rangle} = (F)_{\alpha_0,\langle \beta, e\rangle},$$

for some $\beta \prec \alpha_0$, by definition of M and lemma II.1.23, (iv). Suppose now, that Z is Π^0_{k+1} in Y. Then Z is also Π^0_{k+1} in $(F)_{\alpha_0}$. Since $\prec$ has no top element, there is a sequence $\alpha_0 \prec \ldots \prec \alpha_{\mathsf{cs}_{k+1}}$. By lemma II.1.26, $Z \dot{\in} (F)_{\alpha_{\mathsf{cs}_{k+1}}}$. □

The following is kind of a limit version of lemma II.1.26.

Corollary II.1.30 *For each arithmetical formula $A(U, u, \vec{v})$, the following is provable in* ACA_0*: If* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ *and* $\prec$ *is an ordering without a top element, then*

$$Z := \{\langle\langle x, \vec{y}\rangle, \alpha\rangle : \alpha \in \mathsf{Field}(\prec) \wedge A((F)_\alpha, x, \vec{y})\} =_{\Delta^0_1} F.$$

Proof: By the previous lemma there is a $k \in \mathbb{N}$, such that ACA_0 proves: There is an e, such that for each $\beta \in \mathsf{Field}(\prec)$,

$$\forall z[z \in (Z)_\beta \leftrightarrow B(F, z, \beta, \mathsf{cs}_k, e)],$$

where $B(F, z, \beta, \mathsf{cs}_k, e)$ is a Σ^0_1 formula which expresses that there exists an $s \in \mathsf{seq}$ such that $\mathsf{lh}(s) = \mathsf{cs}_k{+}1$ and $\beta = (s)_0 \prec \ldots \prec (s)_{\mathsf{cs}_k+1}$ and $z \in (F)_{(s)_{\mathsf{cs}_{k+1}}, \langle (s)_{\mathsf{cs}_k}, e\rangle}$. Observe, that $B(F, z, \beta, \mathsf{cs}_k, e)$ is also expressible by a Π^0_1 formula. We have

$$Z = \{\langle\langle x, \vec{y}\rangle, \alpha\rangle : \alpha \in \mathsf{Field}(\prec) \wedge B(F, \langle x, \vec{y}\rangle, \alpha, \mathsf{cs}_k, e)\}.$$

The claim follows, if $\mathsf{Field}(\prec)$ is Δ^0_1 in F: By the definition of $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ we have that $\alpha \in \mathsf{Field}(\prec)$ exactly if $(F)_\alpha \neq \emptyset$, and lemma II.1.23 provides a $d \in \mathbb{N}$ such that $\alpha \in \mathsf{Field}(\prec)$ exactly if $(F)_{\alpha, \langle 0_\prec, \mathsf{cs}_d\rangle} = \mathsf{N}$. □

Corollary II.1.31 *For each arithmetical formula $A(U, u, \vec{v})$, the following is provable in* ACA_0*: If* $\mathsf{Wo}(\prec)$ *and there is an F with* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ *and* $\lambda \in \mathsf{Field}(\prec)$ *is a limit, then*

$$\{\langle\langle x, \vec{y}\rangle, \alpha\rangle : \alpha \prec \lambda \wedge A((F)_\alpha, x, \vec{y})\} =_{\Delta^0_1} (F)_{\prec\lambda}.$$

II.1.6 The hyper-arithmetical sets HYP

A set is *hyperarithmetical in X* if it appears in some level of a proper jump-hierarchy above X. The main result of this subsection is the Kleene-Souslin Theorem which state that Y is hyper-arithmetical in X if and only if Y is Δ^1_1 in X. More on HYP can be found in the next section.

Definition II.1.32 (Hyperarithmetical in $\vec{X}$) *We say that Y is hyperarithmetical in $\vec{X}$, or $Y \in \mathsf{HYP}^{\vec{X}}$ for short, if there exists an index a with $\mathsf{Wo}(\prec^{\vec{X}}_a)$, an $\alpha \in \mathsf{Field}(\prec^{\vec{X}}_a)$ and a hierarchy F, such that $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^{\vec{X}}_a)$ and $Y \mathrel{\dot\in} (F)_\alpha$. If Y is hyperarithmetical in $\emptyset$ we call Y hyperarithmetical. The class of all hyperarithmetical sets in $\vec{X}$ is denoted by $\mathsf{HYP}^{\vec{X}}$ and $\mathsf{HYP}^{\emptyset}$ is just* HYP.

If two well-orderings are arithmetical in X and of the same ordertype, then the corresponding jump-hierarchies above X are recursive in each other.

Lemma II.1.33 *The following is provable in* ACA_0*: If $F, X, \prec, \prec'$ and Z are such that* *(i)* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$,

(ii) $\prec$ *and* $\prec'$ *are well-orderings arithmetical in* X,

(iii) Z *is an order isomorphism between* $\prec$ *and* $\prec'$,

then there exists a G *with* $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec')$ *and* $F =_{\Delta^0_1} G$.

Proof: We assume that F satisfies $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ and let Z be an order isomorphism between $\prec$ and $\prec'$. In this proof, α is assumed to be an element of $\mathsf{Field}(\prec)$, and β an element of $\mathsf{Field}(\prec')$. By transfinite induction along $\prec$, we show that the set

$$\{\alpha : \exists\beta(\langle\alpha,\beta\rangle \in Z) \wedge \exists! G[(G =_{\Delta^0_1} (F)_{\prec\alpha}) \wedge \mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\beta)]\}$$

is the entire field of $\prec$.

For $\alpha = 0_\prec$, the unique set G satisfying $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction 0_{\prec'})$ is the empty set. Since $(F)_{\prec\alpha}$ is empty as well, the claim holds trivially.

Next, we consider the successor case and assume that $\langle\alpha,\beta\rangle \in Z$ and that there exists a unique $G =_{\Delta^0_1} (F)_{\prec\alpha}$ such that $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\beta)$. Observe, that for $z \not\prec' \beta$, $((G)_{\prec'\beta})_z = \emptyset$ and thus for all z, $((G)_{\prec'\beta})_z = (G)_z$. Now the unique set H that satisfies $\mathsf{Hier}^{\mathcal{J}}(H, X, \prec'\restriction\beta{+}1))$ has the form $H := G \cup G'$, where

$$G' := \{\langle x, \beta\rangle : x = \langle y, \langle z, e\rangle\rangle \wedge \pi^0_1((G)_z, X, y, e)\}.$$

Since also $(F)_{\prec\alpha+1}$ is of the form $(F)_{\prec\alpha} \cup F'$, where

$$F' := \{\langle x, \alpha\rangle : x = \langle y, \langle z, e\rangle\rangle \wedge \pi^0_1(((F)_{\prec\alpha})_z, X, y, e)\},$$

$H =_{\Delta^0_1} (F)_{\prec\alpha+1}$ is now due to the I.H. and lemma II.1.25.

Suppose now, that λ is a limit, $\langle\lambda,\lambda'\rangle \in Z$ and that for all $\alpha \prec \lambda$, there exists a unique set H so that $H =_{\Delta^0_1} (F)_{\prec\alpha} \mathrel{\dot\in} (F)_\lambda$ and $\langle\alpha,\beta\rangle \in Z$ implies $\mathsf{Hier}^{\mathcal{J}}(H, X, \prec'\restriction\beta)$. Also, if $H' =_{\Delta^0_1} (F)_{\prec\alpha}$ and $\mathsf{Hier}^{\mathcal{J}}(H', X, \prec'\restriction\beta')$, then $\beta = \beta'$: Otherwise, if e.g. $\beta \prec' \beta'$, then $H' = \mathcal{J}^X_{\prec'\restriction\beta'}$ is Δ^0_1 in $H = \mathcal{J}^X_{\prec'\restriction\beta}$, hence $\mathcal{J}^X_{\beta'}$ is Δ^0_1 in $\mathcal{J}^X_\beta$ by lemma II.1.25, which contradicts lemma II.1.23 (vii).

Next, we let

$$W := \{\langle\alpha, \langle y, \beta\rangle\rangle : \alpha \prec \lambda \wedge \exists H[H =_{\Delta^0_1} (F)_{\prec\alpha} \wedge \mathsf{Hier}^{\mathcal{J}}(H, X, \prec'\restriction\beta) \wedge y \in H].$$

Since $\prec$ is arithmetical in X and λ is a limit, $\prec$ is clearly Δ^0_1 in $(F)_{\prec\lambda}$. Further, $(F)_{\prec\alpha}$ is arithmetical in $(F)_\alpha$. Corollary II.1.30 yields that $W =_{\Delta^0_1} (F)_{\prec\lambda}$.
If $\langle\alpha, \langle y, \beta\rangle\rangle \in W$, then $\langle\alpha,\beta\rangle \in Z$ and $y \in \mathcal{J}^X_{\prec'\restriction\beta}$. Because $\beta \prec' \beta' \prec \lambda'$ yields $\mathcal{J}^X_{\prec'\restriction\beta} \subseteq \mathcal{J}^X_{\prec'\restriction\beta}$, the set $G := \{(x)_{1,1} : x \in W\}$ satisfies $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\lambda')$. □

As a corollary we obtain that the comparison map of two well-orderings that are arithmetical in X are recursive in either jump-hierarchy above X.

Corollary II.1.34 ACA_0 *proves: If* $\prec$ *and* $\prec'$ *are well-orderings arithmetical in* X*, and* $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$*, then* $\prec$ *and* $\prec'$ *are comparable. Moreover, the comparison map is* Δ^0_1 *in* F*.*

Proof: We let

$$Z := \{\langle \alpha, \gamma\rangle : \alpha \in \mathsf{Field}(\prec) \wedge \gamma \in \mathsf{Field}(\prec') \wedge (\exists G =_{\Delta^0_1} (F)_{\prec\alpha})\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\gamma)\}$$

If $\prec$ has no top element, then corollary II.1.30 implies that Z is Δ^0_1 in F. If $\prec$ has a top element, let λ be the largest limit in the field of $\prec$ if such a limit exists and $0_\prec$ otherwise, and Z' the restriction of Z to $\prec\restriction\lambda$. As before, we obtain that $Z' \leq_{\Delta^0_1} (F)_{\prec\lambda}$. Since Z extends Z' only be finitely many pairs, also $Z \leq_{\Delta^0_1} F$.

Using lemma II.1.33 we show by transfinite induction that for all $\alpha \in \mathsf{Field}(\prec)$, $Z\restriction\alpha$ compares $\prec\restriction\alpha$ and $\prec'$: Assume that $S := \mathsf{Rng}(Z\restriction\alpha)$ is not the entire field of $\prec'$. Then S has a supremum γ w.r.t. $\prec'$ and $Z\restriction\alpha \cup \{\langle\alpha, \gamma\rangle\}$ compares $\prec\restriction\alpha{+}1$ and $\prec'$. Since $Z\restriction\alpha$ is an order isomorphism between $\prec\restriction\alpha$ and $\prec'\restriction\gamma$, there exists a G with $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\gamma)$ and $G =_{\Delta^0_1} (F)_{\prec\alpha}$. It follows that $Z\restriction\alpha{+}1 = Z\restriction\alpha \cup \{\langle\alpha, \gamma\rangle\}$. If S is already the entire field of $\prec'$, then $Z\restriction\alpha$ has range $\mathsf{Field}(\prec')$ and for each $\gamma \in \mathsf{Field}(\prec')$ there is a $\beta \prec \alpha$ with $\langle\beta, \gamma\rangle \in Z$. So $G =_{\Delta^0_1} (F)_{\prec\alpha}$ and $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec'\restriction\gamma)$ would imply that $(F)_{\prec\alpha}$ is Δ^0_1 in $(F)_{\prec\beta}$, which contradicts lemma II.1.23 (vii). Therefore $Z = Z\restriction\alpha$.

If λ is a limit and $Z\restriction\alpha$ compares $\prec\restriction\alpha$ and $\prec'$ for all $\alpha \prec \lambda$, then $Z\restriction\lambda$ compares already $\prec\restriction\lambda$ and $\prec'$. □

The following lemma helps to establish the Kleene-Souslin Theorem. It states that the ordertypes of a Σ^1_1 definable family of recursively enumerable well-orderings is bounded by the ordertype of a recursively enumerable well-ordering.

Lemma II.1.35 ACA_0 *proves: If* X *and* Y *are such that,*

(i) any two well-orderings of the form $\prec^X_a$ *and* $\prec^X_b$ *are comparable,*

(ii) Y *is* Σ^1_1 *in* X*,*

(iii) $(\forall e \in Y)\mathsf{Wo}(\prec^X_e)$*,*

then there exists an index a *with* $\mathsf{Wo}(\prec^X_a)$*, such that for each* $e \in Y$*,* $\prec^X_e$ *is isomorphic to a proper initial segment of* $\prec^X_a$*.*

Proof: Otherwise, we had

$$\forall a[\mathsf{Wo}(\prec^X_a) \leftrightarrow \exists F, e[e \in Z \wedge A(F, X, a, e)],$$

where $A(\mathcal{F}, X, a, e)$ is a Σ^1_1 formula of L_2 expressing that F is an isomorphism between $\prec^X_a$ and an initial segment of $\prec^X_e$. This, however, contradicts lemma II.1.10. □

We conclude this section with the well-know Kleene-Souslin theorem, which states that ACA_0 proves that a set Y is in HYP^X if and only if Y is Δ^1_1 in X, provided that the jump-hierarchy above X exists for a sufficiently large class of well-ordering.

Theorem II.1.36 (Kleene-Souslin) *The following is provable in* ACA_0*:*

$$\forall a[\mathsf{Wo}(\prec^X_a) \to \exists F \mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_a)] \to [Y \in \mathsf{HYP}^X \leftrightarrow Y \leq_{\Delta^1_1} X].$$

Proof: If Y is hyper-arithmetical in X, there are a, α and e such that

$$\begin{aligned} x \in Y &\leftrightarrow \exists F[\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_a) \wedge x \in (F)_{\alpha,e}] \\ &\leftrightarrow \forall F[\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_a) \to x \in (F)_{\alpha,e}]. \end{aligned}$$

Thus, Y is also Δ^1_1 in X.

For the other direction, assume that there are indices e, e', such that

$$\forall x[\pi^1_1(X, x, e) \leftrightarrow \neg\pi^1_1(X, x, e')].$$

We have to show that the set $Y := \{x : \pi^1_1(X, x, e)\}$ is hyper-arithmetical in X. Theorem II.1.8 yields there exists a set Z which is Δ^0_0 in X, such that

$$\forall x[\mathsf{Wo}((Z)_x) \leftrightarrow \pi^1_1(X, x, e) \leftrightarrow \neg\pi^1_1(X, x, e')].$$

The index set $W := \{a : \exists x[\prec^X_a = (Z)_x \wedge \neg\pi^1_1(X, x, e')]\}$ is Σ^1_1 in X. Since the jump-hierarchy exists along every well-ordering of the form $\prec^X_a$ and the proof of corollary II.1.34 implies the comparability of well-orderings of this form, we are in the position to apply lemma II.1.35. So we obtain an index b, such that $\mathsf{Wo}(\prec^X_b)$ is a well-ordering without a top element and a limit $\lambda \in \mathsf{Field}(\prec^X_b)$ such that for each $a \in W$, $\prec^X_a$ is isomorphic to a proper initial segment of $\prec^X_b \restriction \lambda$. Let G be the set satisfying $\mathsf{Hier}^{\mathcal{J}}(G, X, \prec^X_b)$. Then corollary II.1.34 enables us to describe the set Y as the collection of all x such that

$$\exists \beta(\exists F \mathrel{\dot\in} (G)_\lambda)[F \text{ is an order-isomorphism between } (Z)_x \text{ and } \prec^X_b \restriction \beta)].$$

Thus, Y is arithmetical in $(G)_\lambda$, hence by corollary II.1.26 there is a $k \in \mathbb{N}$, such that $Y \mathrel{\dot\in} (G)_{\lambda + \mathsf{cs}_k}$. □

II.2 Pseudo-hierarchy arguments

A pseudo-hierarchy F looks locally like a proper hierarchy, however, the underlying ordering $\prec$ is not a well-ordering, so that there is a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element. The careful design of the pseudo-hierarchy then implies that if a selected arithmetical property $A(\alpha)$ holds for all $\alpha \in K$, then there is a β below K such that $A(\beta)$ still holds, which essentially accounts for the expedient closure properties of the intersection $\bigcap_{\alpha \in K}(F)_\alpha$ of all the levels $\alpha \in K$ of the pseudo-hierarchy F.

Pseudo-hierarchies have become a powerful tool in several areas of mathematical logic. They were first applied in the context of hyperarithmetical theory by Spector [41], Gandy [16] and Feferman and Spector [13]. Especially in second order arithmetic, the potent and flexible technique of pseudo-hierarchy arguments seems nowadays virtually indispensable. A typical application for specific fixed point definitions is given in Avigad [2], and a rich fund of important results obtained by working with pseudo-hierarchies is found in Simpson [40].

First, we review results from [40]. We will adapt many of them to the context of admissible set theory after we have developed the necessary tools to apply pseudo-hierarchies also in this framework. Then, we combine the fixed point construction from [2] with techniques developed in Jäger [21] to reason about fixed points of non-monotone operators. Next, we introduce the theory FP_0^- to research the relationship between fixed points and hyperarithmetical sets. We prove that there are operators, given by a positive arithmetical formula, that have no fixed points in HYP. Finally, we reveal an interesting property of models of Σ^1_1-DC_0. Given a positive arithmetical formula $A(U^+, u)$, then the class $\mathsf{Cl}^A := \bigcap\{X : F^A(X) \subseteq X\}$ is the least Π^1_1 definable fixed point of the operator F^A. This result leads immediately to the answers an old question asked by Feferman in his paper on Hancock's conjecture [11] about the strength of ID^*_1.

II.2.1 On HYP

The existence of pseudo-hierarchies is a direct consequence of the fact that being a well-ordering is not expressible by a Σ^1_1 formula of L_2. Moreover, if a proper hierarchy and its underlying well-ordering satisfy a Σ^1_1 definable property, then this property holds already for some pseudo-hierarchy and its underlying ordering.

Theorem II.2.1 (Existence of pseudo-hierarchies) *Assume that $A(U,u)$ is an arithmetical and $B(U,V)$ is a Σ^1_1 formula of L_2. Then ACA_0 proves: If*

$$(1) \qquad \forall X[\mathsf{Wo}(X) \to \exists F(\mathsf{Hier}^A(F,X) \land B(F,X))],$$

then there exists an ordering $\prec$ and a G such that $\mathsf{PSH}^A(G,\prec)$ and $B(G,\prec)$.

Proof: Since $A(U,u)$ is arithmetical, the formula $\exists F(\mathsf{Hier}^A(F,U)\wedge B(F,U))$ is equivalent to a Σ^1_1 formula of L_2. Theorem II.1.9 and the assumption (1) yield the claim. $\square$

Of course, we can relativize this theorem to well-orderings that are primitive recursive, Δ^0_1 or Σ^0_1 in some set X, by applying lemma II.1.10 in place of theorem II.1.9.

Lemma II.2.2 *Assume that $A(U,u)$ is an arithmetical and $B(U,V,u)$ a Σ^1_1 formula of L_2 with at most the displayed set variables free. Then ACA_0 proves: If S is one of the sets N, TRec^X or Prim, and*

$$(\forall e\in S)[\mathsf{Wo}(\prec^X_e)\rightarrow\exists F(\mathsf{Hier}^A(F,\prec^X_e)\wedge B(F,X,e))],$$

then there exists an $e'\in S$ and a G such that $\mathsf{PSH}^A(G,\prec^X_{e'})$ and $B(G,X,e')$.

In Σ^1_1-AC_0, there the formula classes Σ^1_1 and Σ coincide, we can strengthen the above results as follows:

Corollary II.2.3 *Let $A_0(U,\vec{V},W,u,\vec{v},w)$ be a Π formula and $A_1(U,\vec{V},W,u,\vec{v},w)$, $B(U,V)$ be Σ formulas of L_2. Then Σ^1_1-AC_0 proves: If*

(i) $\forall F,Z,x,\alpha[A_0(F,\vec{Y},Z,x,\vec{y},\alpha)\leftrightarrow A_1(F,\vec{Y},Z,x,\vec{y},\alpha)]$,

(ii) $\forall Z[\mathsf{Wo}(Z)\rightarrow\exists F(\mathsf{Hier}^{A_0}(F,\vec{Y},Z,\vec{y})\wedge B(F,X))]$,

then there exists an ordering $\prec$ and a G such that $\mathsf{PSH}^{A_0}(G,\vec{Y},\prec,\vec{y})$ and $B(G,\prec)$.

Proof: Assumption (i) yields that $\exists F(\mathsf{Hier}^{A_0}(F,\vec{Y},Z,\vec{y})\wedge B(F,X))$ is equivalent to a Σ^1_1 formula of L_2. $\square$

Next, we employ pseudo-hierarchy arguments to learn more about the class HYP.

Definition II.2.4 (Hyperarithmetically closed sets) *We say that M is hyperarithmetically closed, if $X\,\dot{\in}\,M$ and $Y\in\mathsf{HYP}^X$ imply $Y\,\dot{\in}\,M$.*

A first observation is that each model M of Σ^1_1-AC is hyperarithmetically closed. The following theorem is a first step towards a better understanding of the hyperarithmetical sets. It displays how a pseudo-jumphierarchy gives rise to a hyperarithmetically closed set.

Theorem II.2.5 *The following is provable in ACA_0: Suppose that $\mathsf{PSH}^{\mathcal{J}}(F,X,\prec)$, that $K\subseteq\mathsf{Field}(\prec)$ is non-empty, upwards closed and has no $\prec$-least element. Then, for each $\alpha_0\in K$, the set*

$$M:=\{\langle x,\langle\alpha_0,e\rangle\rangle:(\forall\alpha\in K)\exists e'[(F)_{\alpha_0,e}=(F)_{\alpha,e'}\wedge x\in(F)_{\alpha_0,e}]\}$$

contains X and is hyperarithmetically closed.

By lemma II.1.23, (v), a jump-hierarchy is monotone, i.e. if $\beta \prec \alpha$ and $Y \dot{\in} (F)_\beta$ then $Y \dot{\in} (F)_\alpha$. Therefore, we have that $X \dot{\in} M$ if and only if $(\forall \alpha \in K)(X \dot{\in} (F)_\alpha)$. In this sense, M is the intersection of the $(F)_\alpha$ with $\alpha \in K$. Moreover, M does not depend on the choice of α_0.

Proof: First, we show that M satisfies arithmetical comprehension. So we assume that $A(\vec{U}, u)$ is a Π^0_k formula and that $\vec{Y}$ are in M. Now we choose an arbitrary $\alpha \in K$. Since K has no $\prec$-least element, there are $\vec{\alpha} \in K$ with $\alpha \succ \alpha_1 \succ \ldots \succ \alpha_k$. Since $\vec{Y} \dot{\in} (F)_{\alpha_k}$, lemma II.1.26 implies that $Z := \{x : A(\vec{Y}, x)\} \dot{\in} (F)_\alpha$. Because $\alpha \in K$ was arbitrary, we have $Z \dot{\in} (F)_\beta$ for all $\beta \in K$, thus $Z \dot{\in} M$.

Next, we assume that $\prec', Y$ are in M with $\mathsf{Wo}(\prec')$. By means of transfinite induction, we show that

$$\{\beta \in \mathsf{Field}(\prec') : (\exists H \dot{\in} M)\mathsf{Hier}^{\mathcal{J}}(H, Y, \prec' \restriction \beta)\}$$

is already the entire field of $\prec'$.

Since M is closed under arithmetical comprehension, we only have to consider the limit case. So assume that for all $\beta \prec' \lambda$, there is a (unique) $H \dot{\in} M$ with $\mathsf{Hier}^{\mathcal{J}}(H, Y, \prec' \restriction \beta)$. Thus, we have for an arbitrary $\alpha \in K$ that the set

$$G := \{\langle x, \beta\rangle : \beta \prec' \lambda \wedge \exists e[\mathsf{Hier}^{\mathcal{J}}((F)_{\alpha,e}, Y, \prec' \restriction \beta{+}1) \wedge \langle x, \beta\rangle \in (F)_{\alpha,e}]\}$$

satisfies $\mathsf{Hier}^{\mathcal{J}}(G, Y, \prec' \restriction \lambda)$. Again lemma II.1.26 yields a $k \in \mathbb{N}$ such that for each sequence $\alpha = \alpha_0 \prec \ldots \prec \alpha_k$, $G \dot{\in} (F)_{\alpha_k}$. Since K has no $\prec$-least element, this implies $G \dot{\in} M$. □

As a corollary, we obtain an additional characterization of HYP.

Corollary II.2.6 *The following is provable in* ACA_0*: Provided* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then* $Y \in \mathsf{HYP}^X$ *if and only if* $Y \dot{\in} M$ *for each hyperarithmetically closed set* M *above* X*.*

Proof: If $Y \in \mathsf{HYP}^X$, then Y is clearly in each hyperarithmetically closed set M above X. For the converse direction, we assume that $Y \notin \mathsf{HYP}^X$ and show that there is a hyperarithmetically closed set M above X that does not contain Y.
Because $\mathcal{J}^X_{\prec^X_a}$ exists for each $a \in \mathsf{Prim}$ with $\mathsf{Wo}(\prec^X_a)$, we have

$$(\forall a \in \mathsf{Prim})[\mathsf{Wo}(\prec^X_a) \to \exists F(\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_a) \wedge B(F, X, a))],$$

where $B(U, V, u)$ is the Σ^1_1 formula

$$\exists Q[Q = \{\langle x, e\rangle : \pi^0_1(U, x, e)\} \wedge \mathsf{Wo}^Q(\prec^V_u)].$$

Now lemma II.2.2 provides a $b \in \mathsf{Prim}$ and an F, such that $\mathsf{PSH}^{\mathcal{J}}(F, X, \prec^X_b)$ and $B(F, X, b)$. For further reference, we set $\prec := \prec^X_b$. Next, we distinguish the following two cases:

(i) $(\exists\alpha \in \mathsf{Field}(\prec))[Y \dot{\notin} (F)_\alpha \wedge \neg\mathsf{Wo}(\prec\restriction\alpha)]$,

(ii) $(\forall\alpha \in \mathsf{Field}(\prec))[Y \dot{\notin} (F)_\alpha \rightarrow \mathsf{Wo}(\prec\restriction\alpha)]$.

If the first case applies, then there is already a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element and an $\alpha \in K$ with $Y \notin (F)_\alpha$. According to theorem II.2.5, this gives rise to a hyperarithmetically closed set M above X that does not contain Y.

In the second case, there is an $\beta \in \mathsf{Field}(\prec)$ with $Y \in (F)_\beta$, because $\prec$ is not a well-ordering. Since $(F)_{\prec\beta}$ is Δ^0_1 in $(F)_\beta$, the set

$$\{\alpha \prec \beta : Y \dot{\in} (F)_{\prec\beta,\alpha}\}$$

is Π^0_1 in F, therefore it has a least element α_0. However, then α_0 is also the minimum of the set $S := \{\alpha \in \mathsf{Field}(\prec) : Y \in (F)_\alpha\}$ which is impossible: Due to the assumption $Y \notin \mathsf{HYP}^X$, $Y \dot{\in} (F)_\alpha$ yields that $\neg\mathsf{Wo}(\prec\restriction\alpha_0)$. Thus, there is a $\beta \prec \alpha$ with $\neg\mathsf{Wo}(\prec\restriction\beta)$ and $Y \dot{\notin} (F)_\beta$. However, this contradicts (ii). Hence, case (ii) never holds. □

Consequently, if Y is in HYP^X, then Y is in each model M of Σ^1_1-AC or Σ^1_1-DC above X. To show the converse is our next goal.

Theorem II.2.1 allows us to impose Σ^1_1 definable conditions on the underlying ordering $\prec$ of a pseudo-jumphierarchy F. It proves extremely useful, to force this underlying ordering to look like a well-ordering in an appropriate collection Q of sets, i.e. $\mathsf{Wo}^Q(\prec)$. When working in ACA_0, $Q =: \{\langle x, e\rangle : \pi^0_1(F, x, e)\}$ is a good choice, as we have seen above. If we work in stronger theories that prove the existence of $\mathcal{J}^F_\omega$, we simplify things and set $Q := \mathcal{J}^F_\omega$ instead. This property is indeed so apt for pseudo-hierarchy arguments that we hardly ever omit it. The reason is, that this condition implies the inseparability of a non-empty $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element with arithmetical formulas:

Lemma II.2.7 (Inseparability) *For each arithmetical formula $A(\vec{U}, u)$ of L_2 with at most the displayed set variables free, the following is provable in ACA_0: If*

(i) M is a model of ACA and $\vec{X} \dot{\in} M$,

(ii) $\mathsf{Wo}^M(\prec)$,

(iii) $K \subseteq \mathsf{Field}(\prec)$ is non-empty, upward closed and has no $\prec$-least element,

then

(iv) $(\forall\alpha \in K)A(\vec{X}, \alpha) \rightarrow (\exists\beta \prec K)A(\vec{X}, \beta)$, and

(v) $(\forall\alpha \prec K)A(\vec{X},\alpha) \rightarrow (\exists\beta \in K)A(\vec{X},\beta)$.

Proof: We show (iv). (v) is the contraposition of *(iv)*. If $\vec{X} \dot{\in} M$, then also the set $S := \{\alpha \in \mathsf{Field}(\prec) : A(\vec{X},\alpha)\} \dot{\in} M$. The premise $(\forall\alpha \in K)A(\vec{X},\alpha)$ entails that $K \subseteq S$ and (ii) implies that S has a $\prec$-least element β. Therefore (iii) yields that $\beta \prec K$. □

We collect some interesting instances of this lemma:

Corollary II.2.8 ACA_0 *proves: If* $\forall\alpha,\beta[\alpha \prec \beta \rightarrow (F)_\alpha \subseteq (F)_\beta]$, *$M$ is a model of* ACA *above F and* Wo^M, *then*

$$\bigcup_{\alpha \prec K}(F)_\alpha = \bigcap_{\alpha \in K}(F)_\alpha.$$

If we regard quantified number variables that range over the field of an ordering as a fresh sort of variables, we can look upon the next corollary as a kind of Σ reflection for this fresh sort of variables.

Corollary II.2.9 *For each arithmetical formula $A(U^+,\vec{V})$ of* L_2 *with at most the displayed set variables free, the following is provable in* ACA_0*: If*

(i) $\mathsf{PSH}^A(F,\vec{Y},\prec)$,

(ii) $(Ax_{\mathsf{ACA}})^M$, $\mathsf{Wo}^M(\prec)$ *and* $F,\vec{Y} \dot{\in} M$,

(iii) $K \subseteq \mathsf{Field}(\prec)$ *is non-empty, upward closed and has no $\prec$-least element,*

then

$$A((F)_{\prec K},\vec{Y}) \rightarrow (\exists\alpha \prec K)A((F)_{\prec\alpha},\vec{Y}).$$

Proof: Since $A((F)_{\prec K},\vec{Y})$ implies $(\forall\alpha \in K)A((F)_{\prec\alpha},\vec{Y})$, the previous lemma yields the claim. □

Another application is given below. If $\mathsf{PSH}(F,\prec)$, and $K \subseteq \mathsf{Field}(\prec)$ has no $\prec$-least element, K divides the pseudo-hierarchy into an upper and lower part that both contain the same sets.

Corollary II.2.10 ACA_0 *proves: If* $\mathsf{PSH}^{\mathcal{J}}(F,X,\prec)$, *$M$ is a model of* ACA *above X and* $\mathsf{Wo}^M(\prec)$, *then we have for each upward closed* $K \subseteq \mathsf{Field}(\prec)$*:*

$$\forall X[(\exists\alpha \prec K)(X \dot{\in} (F)_\alpha) \leftrightarrow (\forall\alpha \in K)(X \dot{\in} (F)_\alpha)].$$

Proof: Suppose that $(\forall\alpha \in K)(X \dot{\in} (F)_\alpha)$. By lemma II.2.7 there is a $\beta \prec K$ with $X \dot{\in} (F)_\beta$. The other direction is due to the monotonicity of jump-hierarchies (cf. lemma II.1.23, (v)). □

Next, we show how a pseudo-jumphierarchy G whose underlying ordering looks like a well-ordering in an appropriate collection of sets Q gives rise to a model of Σ^1_1-DC.

Theorem II.2.11 (Models of Σ^1_1-DC) *The following is provable in* ACA_0*: If*

(i) $\mathsf{PSH}^{\mathcal{J}}(F, X, \prec)$,

(ii) $\mathsf{Wo}^{Q}(\prec)$, *where* $Q := \{\langle x, e\rangle : \pi^0_5(F, \prec, x, e)\}$,

(iii) $K \subseteq \mathsf{Field}(\prec)$ *is non-empty, upward closed and has no* $\prec$*-least element,*

then

$$M := M^F_{\prec K} := \{\langle x, \langle \gamma, e\rangle\rangle : \gamma \prec K \wedge \langle x, \langle \gamma, e\rangle\rangle \in (F)_{\gamma+1}\},$$

is a model of Σ^1_1-DC*. Further, if* $\prec' \dot{\in} M$ *is a well-ordering, then there exists an order-isomorphism* $Z \dot{\in} M$ *between* $\prec'$ *and a proper initial segment of* $\prec$*.*

Since $\mathsf{Wo}^{Q}(\prec)$ and K has no $\prec$-least element, $\gamma+1$ is well-defined for all $\gamma \prec K$. By lemma II.1.23, $\gamma \prec \beta \prec \alpha$ and $\langle x, \langle \gamma, e\rangle\rangle \in (F)_\beta$ implies $\langle x, \langle \gamma, e\rangle\rangle \in (F)_\alpha$. Similar to the previous lemma, we now infer that $X \dot{\in} M$ if and only if $(\forall \alpha \in K)(X \dot{\in} (F)_\alpha)$. Hence, M is hyperarithmetically closed by theorem II.2.5.

Proof: To verify that M satisfies (Σ^1_1-DC), it suffices to show dependent choice for Π^0_2 formulas (cf. lemma II.1.21). So let $A(U, V)$ be a Π^0_2 formula of L_2 and assume that

$$(1) \qquad (\forall X \dot{\in} M)(\exists Y \dot{\in} M)A(X, Y).$$

If $X \dot{\in} M$, then there exists an index a such that $X = (M)_a$. The definition of M implies that a is of the form $\langle \gamma, e\rangle$, where e is a natural number and γ an element of the field of $\prec$. Now, we set

$$\mathsf{I} := \{\langle \gamma, e\rangle : e \in \mathsf{N} \wedge \gamma \in \mathsf{Field}(\prec)\},$$

and order I by $<_{\mathsf{I}}$, letting $\langle \gamma, e\rangle <_{\mathsf{I}} \langle \delta, e'\rangle$ if $\gamma \prec \delta$, or $\gamma = \delta$ and $e <_{\mathsf{N}} e'$. Note, that $\langle \gamma, e\rangle \in \mathsf{I}$ and $\gamma \not\prec K$ implies $(M)_{\langle \gamma, e\rangle} = \emptyset$. Therefore, (1) becomes equivalent to the formula $(\forall y \in \mathsf{I})(\exists z \in \mathsf{I})A((M)_y, (M)_z)$. Moreover, for each $y \in \mathsf{I}$, the set $\{z \in \mathsf{I} : A((M)_y, (M)_z)\}$ has a $<_{\mathsf{I}}$-least element. To see this, we pick an $\alpha_0 \in K$ and let

$$M' := M^F_{\prec\alpha_0} := \{\langle x, \langle \gamma, e\rangle\rangle : \gamma \prec \alpha_0 \wedge \langle x, \langle \gamma, e\rangle\rangle \in (F)_{\prec\alpha_0, \gamma+1}\}.$$

Then, we have that

$$S_1 := \{z \in \mathsf{I} : A((M)_y, (M)_z)\} \subseteq \{z \in \mathsf{I} : A((M')_y, (M')_z)\} =: S_2.$$

Since I is Δ^0_1 in $\prec$, M' is Δ^0_1 in $(F)_{\alpha_0}$ and S_2 is Π^0_2 in M', we have that $S_2 \dot{\in} Q$. Hence S_2 has a $\prec$-least element. This is also the minimum of the set S_1, because $z \in S_1$, $y \in S_2$ and $y <_{\mathsf{I}} z$ yields already $y \in S_1$. Therefore, we conclude that

$(\forall y \in \mathsf{I})(\exists! z \in \mathsf{I})A'(M, y, z)$, where A' is a Π^0_3 formula of L_2 expressing that z is the least index w.r.t. our index ordering $<_{\mathsf{I}}$, such that $A((M)_y, (M)_z)$ holds.
Next, we fix an index $w \in \mathsf{I}$ with $(w)_0 \prec K$. and show that there exists a choice sequence $Z \dot{\in} M$, such that $(Z)_0 = (M)_w$ and $\forall n A((Z)_n, (Z)_{n+1})$. First, we look for initial segments of such a choice sequence. In the present setting, this is a finite sequence σ of indices such that

$$\mathsf{ChSeq}_{A'}(M, \sigma, w, n) := \mathsf{lh}(\sigma) = n{+}1 \wedge (\sigma)_0 = w \wedge (\forall m < n)A'(M, (\sigma)_m, (\sigma)_{m+1}).$$

Assumption (1) allows us to prove by set induction that $\forall n \exists! \sigma \mathsf{ChSeq}_{A'}(M, \sigma, w, n)$. Note that $\mathsf{ChSeq}_{A'}$ is equivalent to a Π^0_3 formula. Again, we pick an $\alpha_0 \in K$. Since $\gamma \prec K$ implies $(M)_{\langle\gamma,e\rangle} = (M^F_{\prec\alpha})_{\langle\gamma,e\rangle}$ for each $\alpha \in K$, the set

$$\{\alpha \prec \alpha_0 : \forall n \exists \sigma \mathsf{ChSeq}_{A'}(M^F_{\prec\alpha}, \sigma, w, n)\}$$

is not empty. Moreover, it is Π^0_5 in $(F)_{\alpha_0}$, so it has a least element β_0. Since $\beta_0 \prec K$,

$$Z := \{\langle x, n\rangle : \exists \sigma[\mathsf{ChSeq}_{A'}(M^F_{\prec\beta_0}, \sigma, w, n) \wedge x \in (M^F_{\prec\beta_0})_{(\sigma)_n}]\}$$

is a set in M and serves as a witness for our sought for choice sequence.

Now, we turn to the second part of the theorem. Let $\prec' \dot{\in} M$ be a well-ordering. We show that $\prec'$ is isomorphic to a proper initial segment of $\prec$. Thereto, we choose an arithmetical formula $A(U, u)$ expressing that

$$u \in \mathsf{Field}(\prec) \wedge u \notin U \wedge$$
$$(\exists \alpha \in \mathsf{Field}(\prec'))[U \text{ is an order isomorphism between } \prec\restriction u \text{ and } \prec'\restriction\alpha].$$

Since M is a model of Σ^1_1-DC, there is an $F \dot{\in} M$ with $\mathsf{Hier}^A(F, \prec')$. By transfinite induction along $\prec'$, we show that for each $\beta \in \mathsf{Field}(\prec')$, $(F)_{\prec'\beta}$ is an order isomorphism between an initial segment of $\prec$ and $\prec'\restriction\beta$: If $\beta = 0_{\prec'}$ there is nothing to show. So assume that $(F)_{\prec'\beta}$ is an order isomorphism between an initial segment of $\prec$ and $\prec'\restriction\beta$. Since $\mathsf{Wo}^{\mathcal{J}^F_\omega}(\prec)$, $\mathsf{Dom}((F)_{\prec'\beta})$ has a least upper bound α w.r.t. the ordering $\prec$. The definition of A yields that $(F)_{\prec'\beta+1} = (F)_{\prec'\beta} \cup \{\langle\alpha, \beta\rangle\}$, which is an order isomorphism between an initial segment of $\prec$ and $\prec'\restriction\beta{+}1$. For a limit $\lambda \in \mathsf{Field}(\prec')$, observe that $(F)_{\prec'\lambda} = \bigcup_{\beta\prec'\lambda}(F)_{\prec'\beta}$. This easily yields the limit case. So F compares $\prec$ and $\prec'$. However, $\prec$ cannot be an initial segment of $\prec'$, since then the inverse image of K under F would be a subset of $\mathsf{Field}(\prec')$ without a $\prec'$-least element. □

The previous theorem tells us how to construct a model M of Σ^1_1-DC, if we have a pseudo-jumphierarchy F along an ordering $\prec$ with $\mathsf{Wo}^Q(\prec)$, for a suitable collection Q of sets. The next lemma shows that there are such pseudo-jumphierarchies, provided $\mathcal{J}_{\prec}^X$ exists for all well-orderings $\prec$ that are primitive recursive in X. In the same go, we show that if Y is not in HYP^X, then there are models of Σ^1_1-DC above X which do not contain Y.

Lemma II.2.12 *The following is provable in* ACA_0*: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$ *and* Y *is an arbitrary set, then there is an* F *and an index* $e \in \mathsf{Prim}$ *such that for* $Q := \{\langle x, e\rangle : \pi^0_5(F, x, e)\}$,

$$\mathsf{PSH}^{\mathcal{J}}(F, X, \prec^X_e) \quad \textit{and} \quad \mathsf{Wo}^Q(\prec^X_e) \quad \textit{and} \quad Y \notin \mathsf{HYP}^X \to \forall\alpha(Y \dot{\notin} (F)_\alpha).$$

Proof: This is proven as corollary II.2.6. □

This yields our aimed at characterization of HYP via models of Σ^1_1-AC or Σ^1_1-DC.

Corollary II.2.13 ACA_0 *proves: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then* $Y \in \mathsf{HYP}^X$ *if and only if* Y *is in each model of* Σ^1_1-AC *or* Σ^1_1-DC *above* X

Therefore, if the jumphierarchy along primitive recursive well-orderings exists, then there exist also A-hierarchies along arbitrary well-orderings in HYP for all arithmetical formulas $A(U, u)$.

Corollary II.2.14 *The following is provable in* ACA_0*: If* $A(U, \vec{V}, W)$ *is an arithmetical formula of* L_2 *and* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then*

$$\vec{Y}, \prec \in \mathsf{HYP}^X \wedge \mathsf{Wo}(\prec) \to \exists F \mathsf{Hier}^A(F, \vec{Y}, \prec).$$

Proof: If $\vec{Y}, \prec \in \mathsf{HYP}^X$, then there is a model M of Σ^1_1-DC with $X, \vec{Y}, \prec \dot{\in} M$. Since $\mathsf{Wo}(\prec)$, $(\exists F \dot{\in} M)\mathsf{Hier}^A(F, \vec{Y}, \prec)$ follows by transfinite induction along $\prec$. □

Although, we cannot prove that HYP is a set, we obtain that HYP is a model of Σ^1_1-AC, i.e. the class HYP satisfies each instance of $(\Sigma^1_1$-$\mathsf{AC})$.

Lemma II.2.15 *The following is provable in* ACA_0*: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then* HYP^X *is a model of* Σ^1_1-AC *above* X.

Proof: We just show that HYP^X satisfies $(\Sigma^1_1$-$\mathsf{AC})$. Lemma II.2.12 yields an index e and an F such that $\mathsf{PSH}^{\mathcal{J}}(F, X, \prec^X_e)$ and $\mathsf{Wo}^Q(\prec^X_e)$, where Q is the collection $\{\langle x, e\rangle : \pi^0_4(F, x, e)\}$. To simplify the notation we set $\prec := \prec^X_e$. Now assume that $A(U, V, u)$ is a Π^0_2 formula of L_2 and that $Z \in \mathsf{HYP}^X$ is such that $\forall n(\exists Y \in \mathsf{HYP}^X)A(Y, Z, n)$. Since Z is in each model of Σ^1_1-DC above X, there is an $\alpha_0 \in \mathsf{Field}(\prec)$ such that $Z \dot{\in} (F)_{\alpha_0}$. Moreover, there is even an β_0 with $Z \dot{\in} (F)_{\beta_0}$ and $\mathsf{Wo}(\prec\restriction\beta_0)$: Otherwise, we have

$$(\forall\alpha \in \mathsf{Field}(\prec))[\mathsf{Wo}(\prec\restriction\alpha) \to Z \dot{\notin} (F)_\alpha].$$

However, this leads to a contradiction. Since

$$\{\alpha \prec \alpha_0 : Z \dot{\in} (F)_{\prec\alpha_0,\alpha}\}$$

is in Q, it has a $\prec$-least element γ_0. But $Z \dot{\in} (F)_{\gamma_0}$ implies that $\neg\mathsf{Wo}(\prec\restriction\gamma_0)$, hence there is a $\gamma \prec \gamma_0$ with $\neg\mathsf{Wo}(\prec\restriction\gamma)$ and $Z \dot{\notin} (F)_\gamma$. Now a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec\restriction\gamma)$ leads to a model of Σ^1_1-DC above X that does not contain Z.

Next, we pick an $\alpha_0 \in \mathsf{Field}(\prec)$ with $\neg\mathsf{Wo}(\prec\restriction\alpha_0)$ and observe that the set

$$\{\alpha \prec \alpha_0 : \forall n(\exists Y \dot{\in} (F)_{\prec\alpha_0,\alpha})A(Y, Z, n)\}$$

is Π^0_4 in $(F)_{\alpha_0}$ and thus has a $\prec$-least element δ_0. Further, $\mathsf{Wo}(\prec\restriction\delta_0)$. Otherwise, there is a model M of Σ^1_1-DC with $\neg\forall n(\exists Y \dot{\in} M)A(Y, Z, n)$. Such a model is constructed using a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element that contains a $\gamma \prec \delta_0$. □

Since models of Σ^1_1-DC are hyperarithmetically closed, the second part of theorem II.2.11 yields that each well-ordering that is hyperarithmetical in X is already surpassed by a well-ordering that is primitive recursive in X.

Corollary II.2.16 *The following is provable in* ACA_0*: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*,* $\prec \in \mathsf{HYP}^X$ *and* $\mathsf{Wo}(\prec)$*, then there exists an index* $e \in \mathsf{Prim}$*, such that* $\prec$ *is isomorphic to an initial segment of* $\prec^X_e$*.*

Proof: By lemma II.2.12 and theorem II.2.11 there is a model of Σ^1_1-DC above X of the form $M := M^F_{\prec^X_b K}$. Since $\prec \in \mathsf{HYP}^X$, $\prec$ is also in M. So $\mathsf{Wo}(\prec)$ together with the second part of theorem II.2.11 yields that $\prec$ is isomorphic to a proper initial segment of $\prec^X_b$. □

This gives yet another characterization of HYP.

Corollary II.2.17 ACA_0 *proves: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then* $Y \in \mathsf{HYP}^X$ *if and only if*

$(*)$ $\quad (\exists e \in \mathsf{Prim})[\mathsf{Wo}(\prec^X_e) \wedge (\exists \alpha \in \mathsf{Field}(\prec^X_e))\exists F(\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_e) \wedge Y \dot{\in} (F)_\alpha)].$

Proof: Assume that Y is an element of HYP^X, but fails to meet $(*)$. Lemma II.2.2 provides an index $b \in \mathsf{Prim}$ and an F such that $\mathsf{PSH}^{\mathcal{J}}(F, X, \prec^X_b)$ and $\mathsf{Wo}^Q(\prec^X_b)$ for the set $Q := \{\langle x, e\rangle : \pi^0_5(F, \prec, x, e)\}$. Now $Y \dot{\in} (F)_\alpha$ yields $\neg\mathsf{Wo}(\prec^X_b\restriction\alpha)$. Lemma II.2.12 yields a model of Σ^1_1-DC above X that does not contain Y. □

From the next lemma emerges a well-ordering test: In order to verify whether an ordering $\prec$ in HYP is well-founded, it suffices to check if a jump-hierarchy F along $\prec$ is in HYP.

Lemma II.2.18 *The following is provable in* ACA_0*: If* $\mathcal{J}^X_{\prec^X_a}$ *exists for each* $a \in \mathsf{Prim}$ *with* $\mathsf{Wo}(\prec^X_a)$*, then we have for all orderings* $\prec \in \mathsf{HYP}$*,*

$$\mathsf{Wo}(\prec) \leftrightarrow (\exists F \in \mathsf{HYP}^X)\mathsf{Hier}^{\mathcal{J}}(F, X, \prec).$$

In particular, no pseudo-jumphierarchy above X *is in* HYP^X*.*

Proof: First we prove the direction from left to right. Since $\mathcal{J}^X_{\prec^X_a}$ exists for each $a \in \mathsf{Prim}$ with $\mathsf{Wo}(\prec^X_a)$, lemma II.2.12 and theorem II.2.11 imply that there are models of Σ^1_1-DC above X. Now $\mathsf{Wo}(\prec)$ and $\prec \in \mathsf{HYP}^X$ yields that the hierarchy F with $\mathsf{Hier}^{\mathcal{J}}(F, X, \prec)$ is in each model of Σ^1_1-DC above X, thus also in HYP^X.
For the converse direction no additional assumptions are required. Assume that there is an $F \in \mathsf{HYP}^X$ with $\mathsf{PSH}^{\mathcal{J}}(F, X, \prec)$. However, then there is a $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element. By theorem II.2.5, we obtain a hyperarithmetically closed set M such that

$$\forall Y[Y \dot{\in} M \leftrightarrow (\forall \alpha \in K)(Y \dot{\in} (F)_\alpha)].$$

Lemma II.1.23 yields that if $\beta{+}1 \in \mathsf{Field}(\prec)$, then $(F)_{\beta+1} \dot{\notin} (F)_\beta$, thus if $\alpha_0 \in K$, $(F)_{\alpha_0} \dot{\notin} M$. Therefore, $(F)_{\alpha_0}$ is not hyperarithmetical in X. This contradicts $F \in \mathsf{HYP}^X$. □

We close this subsection by presenting the Hyperarithmetical Quantifier Theorem (cf. e.g. [40]). It generalizes the situation of lemma II.2.18. The quest for a witness Y for an arithmetical formula $A(X, Y)$ can be reduced to check an arithmetical property for all sets in HYP^X.

Theorem II.2.19 (Hyperarithmetical Quantifiers)
For each Σ *formula* $A(U, V)$*, there is a* Σ *formula* $B(U)$*, and for each arithmetical formula* $C(\vec{U}, V)$*, there is an arithmetical formula* $D(\vec{U}, V)$*, such that* ACA_0 *proves: If* $\mathcal{J}^X_\prec$ *exists for each well-ordering* $\prec$ *that is primitive recursive in* X*, then*

(i) $(\forall Y \in \mathsf{HYP}^X)A(X, Y) \leftrightarrow B(X)$*,*

(ii) $\exists Y C(\vec{X}, Y) \leftrightarrow (\forall Y \in \mathsf{HYP}^{\vec{X}})D(\vec{X}, Y)$*.*

Proof: For (i), observe that $(\forall Y \in \mathsf{HYP}^X)A(X, Y)$ is equivalent to

$$\forall a, \alpha, b[\mathsf{Wo}(\prec^X_a) \to \exists F(\mathsf{Hier}^{\mathcal{J}}(F, X, \prec^X_a) \wedge A(X, (F)_{\alpha,b}))].$$

For (ii), let $C'(\vec{U}) := \exists Y C(\vec{U}, Y)$. By Theorem II.1.8 we have that $C'(\vec{X})$ is equivalent to $\neg\mathsf{Wo}(\mathsf{KB}(T^{C'}_{\vec{X}}))$. There is an index b such that $\mathsf{KB}(T^{\vec{X}}_{C'}) = \prec^{\vec{X}}_b$, and by lemma II.2.18, $C'(\vec{X})$ is equivalent to

$$\neg(\exists F \in \mathsf{HYP}^{\vec{X}})\mathsf{Hier}^{\mathcal{J}}(F, \vec{X}, \prec^{\vec{X}}_b).$$

□

We have seen, that HYP is a model of Σ^1_1-AC. To prove that HYP is a model of Σ^1_1-DC requires $(\Pi^1_1\text{-}\mathsf{IND_N})$.

Lemma II.2.20 *The following is provable in* $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{IND_N})$*: If* $\mathcal{J}^X_\prec$ *exists for each well-ordering* $\prec$ *that is primitive recursive in* X*, then* HYP^X *is a model of* Σ^1_1-DC *above* X*.*

Proof: We just show that HYP^X satisfies $(\Sigma^1_1\text{-}\mathsf{DC})$. So assume that $A(U,V)$ is a Π^0_2 formula of L_2 such that $(\forall Y \in \mathsf{HYP}^X)(\exists Z \in \mathsf{HYP}^X)A(Y,Z)$. By lemma II.2.12, there is an F and an index $e \in \mathsf{Prim}$ such that for $Q := \{\langle x,e\rangle : \pi^0_5(F,x,e)\}$,

$$\mathsf{PSH}^{\mathcal{J}}(F,X,\prec^X_e) \quad \text{and} \quad \mathsf{Wo}^Q(\prec^X_e).$$

Next, we let $\prec := \prec^X_e$ and fix a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element and denote by $M := M^F_{\prec^X_e K}$ the model of Σ^1_1-DC defined in theorem II.2.11. Also the index set I and the corresponding ordering $<_\mathsf{I}$ are defined as there.

Now we proceed similar as in the proof of theorem II.2.11. If $W \in \mathsf{HYP}^X$, then the proof of lemma II.2.15 tells us that $W = (M)_w$ for an index w with $\mathsf{Wo}(\prec\restriction(w)_0)$. Then $(\Pi^1_1\text{-}\mathsf{IND_N})$ induction yields

$$\forall n \exists! \sigma[\mathsf{ChSeq}_{A'}(M,\sigma,w,n) \wedge (\forall n < \mathsf{lh}(\sigma))\mathsf{Wo}(\prec\restriction(\sigma)_n)].$$

Note that $\mathsf{ChSeq}_{A'}$ is equivalent to a Π^0_3 formula. Again, we pick an $\alpha_0 \in K$. Since $\gamma \prec K$ implies $(M)_{\langle\gamma,e\rangle} = (M^F_{\prec\alpha})_{\langle\gamma,e\rangle}$ for each $\alpha \succ \gamma$, the set

$$\{\alpha \prec \alpha_0 : \forall n \exists \sigma \mathsf{ChSeq}_{A'}(M^F_{\prec\alpha},\sigma,w,n)\}$$

is not empty. Further, it is Π^0_5 in $(F)_{\alpha_0}$, so it has a least element β_0. Moreover, $\mathsf{Wo}(\prec\restriction\beta_0)$. Therefore,

$$Z := \{\langle x,n\rangle : \exists\sigma[\mathsf{ChSeq}_{A'}(M^F_{\prec\beta_0},\sigma,w,n) \wedge x \in (M^F_{\prec\beta_0})_{(\sigma)_n}]\}$$

is a set in HYP^X and serves as a witness for our sought for choice sequence. □

II.2.2 The theory $\mathsf{ACA}_0 + (\Delta\text{-}\mathsf{TR})$

In this subsection we combine theorem II.2.11 and the Kleene-Souslin Theorem to show that the iteration principle $(\Delta\text{-}\mathsf{TR})$ is provable in ATR_0. By $(\Delta\text{-}\mathsf{TR})$ we denote the iteration principle where the operator F^A to iterate is specified by a formula that is in a certain sense Π and Σ. More precisely, $(\Delta\text{-}\mathsf{TR})$ is the axiom schema that claims for each Σ formula $A(U,\vec{V},W,u,\vec{v},w)$ and each Π formula $B(U,\vec{V},W,u,\vec{v},w)$ of L_2 the following:

If $\mathsf{Wo}(\prec)$ and $\vec{Y},\vec{y}$ are so that

$$(\forall\alpha \in \mathsf{Field}(\prec))\forall X,x[A(X,\vec{Y},\prec,x,\vec{y},\alpha) \leftrightarrow B(X,\vec{Y},\prec,x,\vec{y},\alpha)],$$

then there exists an F with $\mathsf{Hier}^A(F,\vec{Y},\prec,\vec{y})$.

Another way of looking at (Δ-TR) is to see it as a restricted form of the choice principle (Σ^1_1-TDC), whose thorough analysis is carried out in Rüede [37] and [38]. If we remove the choice aspect from this principle by postulating a functional character of the formula defining the choice sequence, we obtain the following axiom schema: For each Σ^1_1 formula $C(U,V,\vec{W},\vec{u},v)$ of L_2,

$$(*) \qquad \forall\alpha\forall X\exists! Y C(X,Y,\vec{Z},\vec{y},\alpha),$$

and $\mathsf{Wo}(\prec)$ imply the existence of a choice sequence F with

$$(\forall\alpha \in \mathsf{Field}(\prec))C((F)_{\prec\alpha},(F)_\alpha,\vec{Z},\vec{y},\alpha).$$

This principle, however, is an immediate consequence of (Δ^1_1-TR): We fix sets $\vec{Z}$ and numbers $\vec{y}$ and assume that $(*)$ holds. Then for the formulas

$$\begin{aligned} A(U,\vec{V},u,\vec{v},w) &:= \exists Y[C(U,Y,\vec{V},\vec{v},w) \land u \in Y], \\ B(U,\vec{V},u,\vec{v},w) &:= \forall Y[C(U,Y,\vec{V},\vec{v},w) \to u \in Y], \end{aligned}$$

we have

$$(\forall\alpha \in \mathsf{Field}(\prec))\forall X,x[A(X,\vec{Z},x,\vec{y},\alpha) \leftrightarrow B(X,\vec{Z},x,\vec{y},\alpha)].$$

Moreover, $Y := F^A_{\vec{Z},\vec{y},\alpha}(X)$ yields $C(X,Y,\vec{Z},\vec{y},\alpha)$. Given $\mathsf{Wo}(\prec)$, an application of the iteration principle (Δ-TR) yields the existence of a hierarchy F such that $\mathsf{Hier}^A(F,\vec{Z},\prec,\vec{y})$. Now we conclude

$$(\forall\alpha \in \mathsf{Field}(\prec))C((F)_{\prec\alpha},(F)_\alpha,\vec{Z},\vec{y},\alpha).$$

Here we supplement the proof of (Δ-TR) in ATR_0.

Theorem II.2.21 *For each formula Π formula $A(U,\vec{V},W,u,\vec{v},w)$ and each Σ formula $B(U,\vec{V},W,u,\vec{v},w)$ of L_2, the following is provable in ATR_0: If $\mathsf{Wo}(\prec)$ and $\vec{Y},\vec{y}$ are such that for all $\alpha \in \mathsf{Field}(\prec)$ and all X,x,*

$$A(X,\vec{Y},\prec,x,\vec{y},\alpha) \leftrightarrow B(X,\vec{Y},\prec,x,\vec{y},\alpha)$$

then we have

$$(\exists F \in \mathsf{HYP}^{\vec{Y},\prec})\mathsf{Hier}^A(F,\vec{Y},\prec,\vec{y}).$$

Proof: We show that the F with $\mathsf{Hier}^A(F, \vec{Y}, \prec, \vec{z})$ is a set in each model M of Σ^1_1-AC above $\vec{Y}, \prec$. Theorem II.2.11 then yields that F is in $\mathsf{HYP}^{\vec{Y},\prec}$.

We suppose that M is a model of Σ^1_1-AC above $\vec{Y}, \prec$ and prove by transfinite induction along $\prec$ that the set

$$S := \{\alpha \in \mathsf{Field}(\prec) : (\exists F \mathrel{\dot{\in}} M)\mathsf{Hier}^A(F, \vec{Y}, \prec\restriction\alpha, \vec{y})\}$$

is already the entire field of $\prec$. So assume that all $\beta \prec \alpha$ are in S. Since M is a model of Σ^1_1-AC and proper hierarchies are unique, also $(F)_{\prec\alpha} \mathrel{\dot{\in}} M$. Now

$$(F)_\alpha = \{x : A((F)_{\prec\alpha}, \vec{Y}, \prec\restriction\alpha, \vec{y}, \alpha)\}$$

is already Δ^1_1 in $\vec{Y}, \prec$. By the Kleene-Souslin Theorem II.1.36 and because M is hyperarithmetically closed, $\alpha \in S$. Hence $S = \mathsf{Field}(\prec)$. Similarly, it follows that $F \mathrel{\dot{\in}} M$. □

II.2.3 Fixed points of monotone and non-monotone operators

In this subsection, we apply pseudo-hierarchy arguments to construct fixed points of monotone and non-monotone operators. We start by introducing some notation. Suppose that $A(U, u)$ is such that F^A is a monotone operator. When iterating this operator along an ordering $\prec$, it seems natural to form the αth level by applying F^A to the union, rather than the disjoint union of all the levels $\beta \prec \alpha$. This motivates the definition of a *fixed point hierarchy*: If $A(U, \vec{V}, u, \vec{v})$ is a formula of L_2, then the set F is called a fixed point hierarchy along $\prec$ for A w.r.t. the parameters $\vec{Y}, \vec{y}$, denoted by $\mathsf{FHier}^A(F, \vec{Y}, \prec, \vec{y})$, if

(i) $\mathsf{Lin}_0(\prec)$,

(ii) $(\forall x \in F)[x = \langle (x)_0, (x)_1 \rangle]$,

(iii) $\forall x[(F)_x \neq \emptyset \to x \in \mathsf{Field}(\prec)]$,

(iv) $(\forall \alpha \in \mathsf{Field}(\prec))[(F)_\alpha = F^A_{\vec{Y},\vec{y}}(\bigcup_{\beta\prec\alpha}(F)_\beta)]$.

Again, a fixed point hierarchy is called proper, if the underlying ordering is a well-ordering, and a pseudo fixed point hierarchy otherwise, denoted by $\mathsf{FPSH}^A(F, \prec)$.

Before we turn to non-monotone operators, we review Avigad's result given in [2], namely that ATR_0 proves the fixed point principle (FP) claiming the existence of fixed points of monotone operators induced by arithmetical formulas.

Theorem II.2.22 *If $A(U, \vec{V}, \vec{v})$ is an arithmetical formula of L_2 with exactly the displayed variables free, then the following is provable in ATR_0:*

$$(\mathsf{FP}) \qquad \forall X, Y[X \subseteq Y \to F^A_{\vec{Z},\vec{z}}(X) \subseteq F^A_{\vec{Z},\vec{z}}(Y)] \to \exists S[S = F^A_{\vec{Z},\vec{z}}(S)].$$

Proof: As usual, theorem II.2.1 provides a F and an ordering $\prec$ such that

$$\mathsf{FPSH}^A(F, \vec{Z}, \prec, \vec{z}) \wedge \mathsf{Wo}^{\mathcal{J}^F_\omega}(\prec).$$

Note, that $\mathsf{Wo}^{\mathcal{J}^F_\omega}(\prec)$ enables us to show by transfinite induction that the hierarchy is monotone, i.e. $\alpha \prec \beta$ implies $(F)_\alpha \subseteq (F)_\beta$. Hence, for a non-empty, upward closed subset $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element, we have by corollary II.2.8 that the sets $S := \bigcup_{\alpha \prec K}(F)_\alpha$ and $S' := \bigcap_{\alpha \in K}(F)_\alpha$ are equal. By the monotonicity of the operator F^A we obtain $S \subseteq F^A(S)$ and $F^A(S') \subseteq S'$, thus $F^A(S) = S$. □

Combining the above argument with techniques developed in Jäger [21] allows us also to deal with non-monotone operators. Of course, non-monotone operators have in general no fixed points. Therefore, we assign to each arithmetical formula $A(U, u)$ of L_2 the formula $A^\circ(U, u) := A(U, u) \vee u \in U$, hence $F^{A^\circ}(X) = F^A(X) \cup X$. The operator F^{A° is still not monotone, however, it is *inclusive*, i.e. we have for all sets X that $X \subseteq F^{A^\circ}(X)$. For sure, inclusive operators have fixed points, namely $\mathbb{N} = F^{A^\circ}(\mathbb{N})$. However, such fixed points are not very interesting. What we are interested in is not so much the fixed point itself, but rather its step-by-step build-up. There exists in general no well-ordering long enough to reach the fixed point from below. Looking for the next best thing, we try to reach the fixed point from below via a fixed point hierarchy F along an ordering that looks like a well-ordering at least in $\mathcal{J}^F_\omega$. As we learn from theorem II.2.22, this works for a monotone operator induced by arithmetical formula, and as the next theorem exhibits, it works also for a non-monotone operator F^{A° induced by a Π^0_1 formula $A(U, u)$ of L_2. For its prove, we borrow an auxiliary lemma 6 from [21].

Lemma II.2.23 *Let $A(U^+, V^+, u)$ be a Π^0_1 formula of L_2. Then ATR_0 proves: If $\prec$ is an ordering, $\gamma \succ 0$ and G satisfies $\forall \alpha, \beta[\beta \prec \alpha \to (G)_\beta \subseteq (G)_\alpha]$, then*

$$(\forall \alpha \prec \gamma)(\exists \beta \prec \gamma)A((G)_\beta, \overline{(G)_\alpha}, x) \to A(\bigcup_{\xi \prec \gamma}(G)_\xi, \overline{\bigcup_{\xi \prec \gamma}(G)_\xi}, x).$$

Proof: Due to the positivity of U in $A(U^+, V^+, u)$ and the monotonicity of G we conclude that

$$(\forall \alpha \prec \gamma)(\exists \beta \prec \gamma)[A((G)_\beta, \overline{(G)_\alpha}, x) \to (\forall \alpha \prec \gamma)(A(\bigcup_{\xi \prec \gamma}(G)_\xi, \overline{(G)_\alpha}, x))].$$

It remains to show that

$$(\forall\alpha \prec \gamma)(A(\bigcup_{\xi\prec\gamma}(G)_\xi, \overline{(G)_\alpha}, x)) \to A(\bigcup_{\xi\prec\gamma}(G)_\xi, \overline{\bigcup_{\xi\prec\gamma}(G)_\xi}, x),$$

which is done by induction the build-up of $A(U^+, V^+, u)$.

If V does not occur in $A(U, V, u)$, then there is nothing to prove. If $A(U, V, u)$ is $t \in V$, the claim follows since $(\forall\alpha \prec \gamma)(t \notin (G)_\alpha)$ implies $t \notin \bigcup_{\alpha\prec\gamma}(G)_\alpha$. If $A(U, V, u)$ is a conjunction, a disjunction or begins with a unbounded or bounded universal number quantifier, the claim follows easily form the I.H.

Hence, it remains to consider the case where the formula $A(U, V, u)$ is of the form $(\exists y < t)B(U, V, u, y)$. The I.H. implies that

$$(\exists y < t)(\forall\alpha \prec \gamma)[B(\bigcup_{\xi\prec\gamma}(G)_\xi, \overline{(G)_\alpha}, x, y)] \to A(\bigcup_{\xi\prec\gamma}(G)_\xi, \overline{\bigcup_{\xi\prec\gamma}(G)_\xi}, x).$$

Now we simplify the notation by setting $C(\alpha, x, y) := B(\bigcup_{\xi\prec\gamma}(G)_\xi, \overline{(G)_\alpha}, x, y)$. and show by complete induction along N that for each $t \in \mathsf{N}$,

$$(\forall\alpha \prec \gamma)(\exists y < t)C(\alpha, y) \to (\exists y < t)(\forall\alpha \prec \gamma)C(\alpha, y).$$

Assume that the claim holds for t and that $(\forall\alpha \prec \gamma)(\exists y < t+1)C(\alpha, y)$. We proceed by a case distinction: If $(\forall\alpha \prec \gamma)C(\alpha, t)$ we are done. In the other case, we have $(\exists\alpha \prec \gamma)\neg C(\alpha, t)$. But this implies $(\forall\alpha \prec \gamma)(\exists y < t)C(\alpha, y)$ and the I.H. can be applied: If there is a $\beta \prec \lambda$ such that $(\forall y < t)\neg C(\beta, y)$ but $C(\beta, t)$, then also $(\forall\alpha \preceq \beta)C(\alpha, t)$. Since t is not a witness for all α below γ, there is a β' with $\beta \prec \beta' \prec \gamma$ such that $\neg C(\beta', t)$. But then, there is a $y < t$ with $C(\beta', y)$, hence already $(\forall\alpha \preceq \beta)C(\alpha, y)$. A contradiction. □

Also in the case of suitable non-monotone operators, a pseudo fixed-point-hierarchy leads to a fixed point. In addition, we have some kind of fixed point induction.

Lemma II.2.24 *For each Π^0_1 formula $A(U, u)$ of L_2, the following is provable in ACA_0: If $\mathsf{FPSH}^{A^\circ}(G, \prec)$, $\mathcal{J}^G_\omega$ exists and $\mathsf{Wo}^{\mathcal{J}^G_\omega}(\prec)$, then $Z := \bigcup_{\alpha\prec K}(G)_\alpha$ is a fixed point of F^{A° for each non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element. Moreover, if $X \mathrel{\dot\in} \mathcal{J}^G_\omega$ with $F^A(X) \subseteq (X)$, then $Z \subseteq X$.*

In particular, if x belongs to the fixed point Z, then there is a $\prec$-least level α_0 such that $x \in (G)_{\alpha_0}$. Otherwise, $K' := \{\alpha \in \mathsf{Field}(\prec) : x \in (G)_\alpha\}$ were a non-empty set in $\mathcal{J}^G_\omega$, and thus $Z' := \bigcup_{\beta\prec K'}(G)_\beta$ were a fixed point in $\mathcal{J}^G_\omega$ properly contained in Z. Hence $A((G)_{\prec\alpha_0}, x)$, or in other words, x belongs to the fixed point for a specific reason.

Proof: We choose a Π^0_1 formula $B(U^+, V^+, u)$ of L_2 such that for all numbers x and sets X, $A^\circ(X, x) \leftrightarrow B(X, \overline{X}, x)$, and aim to show that $F^{A^\circ}(Z) \subseteq Z$. So we pick an $x \in F^A(Z)$ and argue that $x \in Z$.

If $\alpha \prec K$, the positivity of B in both arguments yields $(\forall \beta \in K) B((G)_\beta, \overline{(G)_\alpha}, x)$. The inseparability of K (lemma II.2.7) and lemma I.2.15 provide a function $\mathcal{F} \dot\in \mathcal{J}^G_\omega$ with $\mathcal{F}(0) = 0_\prec$ and

$$\mathcal{F}(n{+}1) = \mathsf{min}_\prec\{\beta \succ \mathcal{F}(n) : B((G)_\beta, \overline{(G)_{\mathcal{F}(n)}}, x)\}.$$

Moreover, $\forall n(\mathcal{F}(n) \prec K)$ is easily shown by induction. Thus, the $\prec$-least element of the set $\{\alpha \in \mathsf{Field}(\prec)) : (\forall n \in \mathsf{N})(\mathcal{F}(n) \prec \alpha)\}$ exists and is below K. Therefore,

$$(\forall \alpha \prec \lambda)(\exists \beta \prec \lambda)[B((G)_\beta, \overline{(G)_\alpha}, x)].$$

Now lemma II.2.23 yields

$$B(\bigcup_{\xi \prec \lambda}(G)_\xi, \overline{\bigcup_{\xi \prec \lambda}(G)_\xi}, x).$$

By the choice of B, we have

$$x \in F^{A^\circ}(\bigcup_{\xi \prec \lambda}(G)_\xi) = (G)_\lambda \subseteq Z.$$

Finally, if $X \dot\in \mathcal{J}^G_\omega$ with $F^A(X) \subseteq (X)$, then it follows by transfinite induction along $\prec$ that $(\forall \alpha \in \mathsf{Field}(\prec))((G)_\alpha \subseteq X)$. □

Working in ATR_0, we can state the following result:

Theorem II.2.25 *Let $A(U, \vec{V}, u)$ be a Π^0_1 formula of L_2 with exactly the displayed set variables free. Then ATR_0 proves: There is a set Z such that $F^{A^\circ}_{\vec{Y}}(Z) = Z$ and for all $Z' \in \mathsf{HYP}^{\vec{Y}}$ with $F^A_{\vec{Y}}(Z') \subseteq Z'$ we have $Z \subseteq Z'$.*

Proof: Given sets $\vec{Y}$, there is a model M of Σ^1_1-DC with $\vec{Y} \dot\in M$. Further, there are $G, \prec \dot\in M$ such that $\mathsf{FPSH}^A(G, \vec{Y}, \prec)$ and $\mathsf{Wo}^M(\prec)$. Because $Z' \dot\in \mathsf{HYP}^{\vec{Y}}$ entails $Z' \dot\in M$, the claim follows by the previous lemma. □

II.2.4 Fixed points and hyperarithmetical sets

In this subsection we research the question of how complex fixed points are in terms of definability in L_2. For that purpose, we introduce the theory FP^-_0, an extension of ACA_0 which formalizes the existence of fixed points of operators defined by positive arithmetical formulas without set parameters. We will see that HYP is not a model of any theory comprising FP_0^-. As we will see later, FP_0^- proves a parameter free version of (Σ^1_1-AC). Therefore, the following serves as motivation to research the question of how complex fixed points are.

Aczel's embedding of $\widehat{\mathsf{ID}}_1$ into Σ^1_1-AC (cf. Aczel [1] and Feferman [11]) introduces what become known as Aczel's trick: Due to lemma II.1.14, there is for each Π formula C of L_2 a Π^1_1 formula C' of L_2 such that Σ^1_1-AC_0 proves $C \leftrightarrow C'$. Hence, there is a universal Π^1_1 formula $\pi^1_1(u, v, w)$ of L_2 (see corollary II.1.12), such that for each Π formula $B(u, v)$ of L_2 without free set variables, there exists an $e \in \mathbb{N}$ such that

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \vdash B(x, y) \leftrightarrow \pi^1_1(x, y, \mathsf{cs}_e).$$

This means in particular, that for an arithmetical formula $A(U^+, u)$ with exactly the displayed variables free, there is an $e_A \in \mathbb{N}$ such that

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \vdash A(\{z : \pi^1_1(x, x, z)\}, y) \leftrightarrow \pi^1_1(x, y, \mathsf{cs}_{e_A}).$$

Letting $C(u)$ be the Π^1_1 formula $\pi^1_1(u, \mathsf{cs}_{e_A}, \mathsf{cs}_{e_A})$, Σ^1_1-AC_0 proves:

$$\begin{aligned} A(\{z : C(z)\}, x) &\leftrightarrow A(\{z : \pi^1_1(z, \mathsf{cs}_{e_A}, \mathsf{cs}_{e_A})\}, x) \\ &\leftrightarrow \pi^1_1(x, \mathsf{cs}_{e_A}, \mathsf{cs}_{e_A}) \\ &\leftrightarrow C(x). \end{aligned}$$

This means that $\{z : C(z)\}$ is a Π^1_1 definable fixed point of the operator F^A. Using a universal Σ^1_1 formula instead yields a Σ^1_1 definable fixed point of F^A. So the question arises, whether, by some ingenious trick, one could obtain a fixed point which is both, Π^1_1 and Σ^1_1 at the same time?

We will answer this question negatively. Fixed points are in general not Δ^1_1 definable sets: If a theory T of second order arithmetic comprises for each U positive, arithmetical formula $A(U^+, u)$ an axiom that asserts the existence of a fixed point of the operator F^A, then there is a U positive, arithmetical formula $C(U^+, u)$ such that no fixed point of the operator F^C is Δ^1_1.

The theory FP_0^- is formulated in the language L_2 and comprises, besides the axioms of ACA_0, for each U positive, arithmetical formula $A(U^+, u)$ with exactly the displayed variable free, an axiom asserting the existence of a fixed point of the operator F^A.

(FP^-) $$\exists X \forall x[x \in X \leftrightarrow A(X, x)].$$

To facilitate the subsequent proof-theoretic treatment of FP_0^- we did not permit number parameters in the formula $A(U^+, u)$ specifying the operator F^A for which we claim the existence of fixed points. However, the existence of fixed points of operators defined by arithmetical formulas $A(U^+, u, \vec{v})$ containing number parameters is easily proved:

Lemma II.2.26 *For each U positive, arithmetical formula $A(U^+, u, \vec{v})$ with exactly the displayed variables free, the following is provable in FP_0^-:*

$$\forall \vec{y} \exists X [X = F^A_{\vec{y}}(X)].$$

Proof: Suppose that $A(U^+, u, \vec{v})$ contains exactly the displayed variables free. Now we define

$$B(U^+, u) := \exists x, \vec{y}[u = \langle x, \langle \vec{y} \rangle\rangle \land A((U)_{\langle \vec{y} \rangle}, x, \vec{y})].$$

By (FP^-) we obtain a fixed point F of the operator F^B. The definition of B implies readily that $(F)_{\langle \vec{y} \rangle} = F^A_{\vec{y}}((F)_{\langle \vec{y} \rangle})$ for all $\vec{y}$. □

Next, we apply an argument given in Avigad [2] that shows how the existence of fixed points allows to construct hierarchies:

Lemma II.2.27 *Let $A(U, u)$ be an arithmetical formula of L_2 with exactly the displayed set variable free. Then the following is provable in FP_0^-: If $\mathsf{Lin}_0(\prec^{\emptyset}_a)$, then there exists a downward closed subset $S \subseteq \mathsf{Field}(\prec^{\emptyset}_a)$ and an F such that*

(i) $\mathsf{Hier}^A(F, \prec^{\emptyset}_a \restriction S)$,

(ii) $\mathsf{Wo}(\prec^{\emptyset}_a \restriction \alpha) \to \alpha \in S$.

In particular, this implies that $\mathsf{Wo}(\prec^{\emptyset}_a) \to \exists F \mathsf{Hier}^A(F, \prec^{\emptyset}_a)$.

Proof: Suppose that $\prec := \prec^{\emptyset}_a$ is a linear ordering with a least element denoted by 0, and that $A(U, u)$ is an arithmetical formula of L_2 containing no other set variables than U. Our aim is to construct sets S and F meeting the properties (i) and (ii). Let us explore the proof idea first. It is not hard to see that a fixed point F of the operator

$$X \mapsto \{\langle x, \alpha \rangle : A((X)_{\prec \alpha}, x)\}$$

satisfies $\mathsf{Hier}^A(F, \prec)$. Unfortunately, this operator is in general not even monotone, and therefore the existence of its fixed point is not guaranteed by the axioms of FP_0^-. Therefore we attempt to obtain the characteristic function of the hierarchy F from a fixed point of an operator F^C for some U positive, arithmetical formula $C(U, u)$. First we let $B_1(U^+, V^+, u)$ be the formula that we obtain from $A(U, u)$ by replacing all literals of the form $t \notin U$ by $t \in V$, and $B_2(U^+, V^+, u)$ the formula we obtain from $\neg A(U, u)$ by replacing all literals of the form $t \notin U$ by $t \in V$. Observe, that we have

$$(*) \qquad \{x : B_1(X, \overline{X}, x)\} = F^A(X) \quad \text{and} \quad \{x : B_2(X, \overline{X}, x)\} = \mathsf{N} - F^A(X).$$

Note that for any X, the set $(X)_{1, \prec \alpha}$ contains only pairs of the form $\langle y, \beta \rangle$ with $\beta \prec \alpha$. Suppose that for all $\beta \prec \alpha$, $(X)_{1,\beta}$ is the complement of $(X)_{0,\beta}$. To obtain the complement of $(X)_{0, \prec \alpha}$, we have to add to $(X)_{1, \prec \alpha}$ the elements of the set

$$H_\alpha := \{\langle y, z \rangle : z \not\prec \alpha\} \cup \{x : x \neq \langle (x)_0, (x)_1 \rangle\}.$$

Next, we consider the operator that assigns to a set X the union of the two sets given below:

$$\{\langle\langle x,\alpha\rangle,0\rangle \ : \ B_1((X)_{0,\prec\alpha},(X)_{1,\prec\alpha}\cup H_\alpha,x)\},$$
$$\{\langle\langle x,\alpha\rangle,1\rangle \ : \ B_2((X)_{0,\prec\alpha},(X)_{1,\prec\alpha}\cup H_\alpha,x)\}.$$

Clearly, this operator can be defined by a U positive arithmetical formula $C(U^+,u)$.

Let G be a fixed point of F^C and set $S := \{\alpha \in \mathsf{Field}(\prec) : (\forall\beta \preceq \alpha)[(G)_{0,\beta} = \overline{(G)_{1,\beta}}]\}$ and $F := \{\langle x,\alpha\rangle \in (G)_0 : \alpha \in S\}$. (i) follows now directly from the definition of F^C and (ii) is shown by transfinite induction: Suppose that $\mathsf{Wo}(\prec\restriction\alpha)$. For $\beta = 0$, observe that for any set Z, $(Z)_{0,\prec\beta} = \emptyset$ and $H_0 = \mathsf{N}$. Hence $(G)_{0,0} = F^A(\emptyset)$ and $(G)_{1,0} = F^{\bar{A}}(\emptyset) = \mathsf{N} - F^A(\emptyset)$, thus $0 \in S$. Similarly, if all $\beta' \prec \beta$ are elements of S, then $\langle y,\beta'\rangle$ is in $(G)_{0,\prec\beta}$ if and only if it is not in $(G)_{1,\prec\beta}$. Hence the complement of $(G)_{0,\beta}$ is given by $(G)_{1,\beta} \cup H_\beta$. Thus, the definition of F^C and $(*)$ yield $\beta \in S$. □

Knowing that the jump-hierarchy above the empty set along a Σ^0_1 definable well-ordering exists, we can apply lemma II.2.12 and theorem II.2.11 to obtain the following:

Corollary II.2.28 *The following is provable in* FP_0^-*:*

(i) There exists a model M of Σ^1_1-DC.

(ii) There exists an index $a \in \mathsf{Prim}$ *with* $\mathsf{Lin}_0(\prec^\emptyset_a)$*, such that for all* $\prec \in \mathsf{HYP}$, $\mathsf{Wo}(\prec)$ *implies that there is an order isomorphism in* HYP *that maps* $\prec$ *onto a proper initial segment of* $\prec^\emptyset_a$,

(iii) $(\forall X,\prec \in \mathsf{HYP})[\mathsf{Wo}(\prec) \to (\exists F \in \mathsf{HYP})\mathsf{Hier}^{\mathcal{J}}(F,X,\prec)]$.

Proof: Lemma II.2.12 provides an index e and an F such that $\mathsf{PSH}^{\mathcal{J}}(F,\emptyset,\prec^\emptyset_e)$ and $\mathsf{Wo}^Q(\prec^\emptyset_e)$ for $Q := \{\langle x,e\rangle : \pi^0_5(F,x,e)\}$. Now theorem II.2.11 yields a model $M^F_{\prec^\emptyset_e K}$ of Σ^1_1-DC. Further, if $\prec \in \mathsf{HYP}$, $\prec$ is isomorphic to an initial segment of $\prec^\emptyset_e$ and the corresponding order isomorphism Z is in M. If Z were not in HYP, then there is an $\alpha_0 \in \mathsf{Field}(\prec^\emptyset_e)$ (cf. the proof of lemma II.2.15) such that $Z \notin (F)_{\alpha_0}$ and $\neg\mathsf{Wo}(\prec^\emptyset_e\restriction\alpha_0)$. Thus there is a non-empty, upward closed $K' \subseteq \mathsf{Field}(\prec^\emptyset_e)$ that contains α_0. But then, $M^F_{\prec^\emptyset_e K'}$ is a model of Σ^1_1-DC which does not contain Z. If $X,\prec \in \mathsf{HYP}$ and $\mathsf{Wo}(\prec)$, then a hierarchy F with $\mathsf{Hier}^{\mathcal{J}}(F,\emptyset,\prec)$ exists in each model of Σ^1_1-DC, thus F is already in HYP. □

Next we show that FP_0^- implies already (Σ^1_1-AC) restricted to formulas without set parameters.

Lemma II.2.29 *For each Σ^1_1 formula $A(U,u)$ of L_2 with at most the displayed set variable free, the following is provable in FP_0^-:*

$$\forall x \exists X A(X,x) \to \exists Y \forall y A((Y)_y, y).$$

Let us first explain the idea of the proof. The normal form theorem lets us reformulate the assumption $\forall x \exists X A(X,x)$ as follows: For each n, the tree T^A_n has a path. Our task is to select a path through each of these trees. Thereto, we define an operator F^B that collects the leafs of trees. In general, the operator F^B has no least fixed point, so it may collect some infinite branches. However, we can assure that not the entire tree ends up in the fixed point of F^B. After the trees T^A_n are stripped of their leafs, we can pick their leftmost branches.

Proof: Suppose that $A(U,\vec{u},v)$ is a Σ^1_1 formula of L_2 with no other free set variable than U. Further, assume that the numbers $\vec{x}$ are such that $\forall y \exists X A(X,\vec{x},y)$ holds. Due to theorem II.1.8, we also have that for each y, there is a path trough the tree $T^A_{\vec{x},y}$. Now we set $S := \{\langle \sigma, n\rangle : \sigma \in T^A_{\vec{x},n}\}$. Next, we define an operator that collects the leafs of the trees $(S)_n$:

$$B(U,u) := \exists n[u = \langle \sigma, n\rangle \wedge \sigma \in (S)_n \wedge (\forall \tau \in (S)_n)(\tau \supset \sigma \to \tau \in (U)_n)].$$

Note, that we can replace the parameter S in the definition of the formula $B(U,u)$ by its Δ^0_0 definition. Provided that $\mathsf{Wo}(\prec^\emptyset_a)$ holds, lemma II.2.27 yields an F such that $\mathsf{Hier}^B(F, \prec^\emptyset_a)$. Moreover, since each of the trees $(S)_n$ has an infinite path, our leaf collector never picks the root of $(S)_n$,

$$\forall n (\forall \beta \in \mathsf{Field}(\prec^\emptyset_a))(\langle\langle\rangle, n\rangle \notin (F)_\beta).$$

A consequence of lemma II.2.27 is also that $\mathcal{J}^F_\omega$ exists. Recall that $\mathcal{J}^F_\omega$ is a model of ACA above F. Due to theorem II.2.1 there is an index b and a G, such that $\mathsf{PSH}^B(G, \prec^\emptyset_b)$ and also $\forall n(\forall \beta \in \mathsf{Field}(\prec^\emptyset_b))(\langle\rangle, n\rangle \notin (G)_\beta)$, i.e. of each tree $(S)_n$, an infinite part remains. Moreover, $(G)_\beta \subseteq (G)_\alpha$ for $\beta \prec^\emptyset_b \alpha$. Of course, there is also a $K \subseteq \mathsf{Field}(\prec^\emptyset_b)$ without $\prec^\emptyset_b$-least element, and as in the proof of theorem II.2.22 we obtain that

$$L := \{\langle \sigma, n\rangle : (\exists \alpha \prec^\emptyset_b K)(\sigma \in (G)_{\alpha,n})\}$$

is a fixed point of the operator F^B. Thus, each of the trees $(S)_n - (L)_n$ contains no more leafs, therefore we can select their leftmost branches: Let W be such that $(W)_n$ is the function

$$\{\langle m, \sigma\rangle : \mathsf{lh}(\sigma) = m \wedge \sigma \in ((S)_n - (L)_n) \wedge (\forall \tau \in (S)_n)(\tau <_{\mathsf{KB}((S)_n)} \sigma \to \tau \supset \sigma)\}.$$

Lemma II.1.7 now tells us that if $\mathcal{F} = (W)_n$, then the set Y with

$$(Y)_n := \{y : \mathsf{WIT}^B(\mathcal{F}, y)\}$$

constitutes a choice sequence satisfying $\forall n A((Y)_n, \vec{x}, n)$. □

As a consequence, we obtain that a Δ^1_1 definable class is already a set.

Corollary II.2.30 *The following is provable in FP^-_0: If $A(u,\vec{v})$ and $B(u,\vec{v})$ are Π^1_1 formulas of L_2 with exactly the displayed variables free, then*

$$\forall x[A(x,\vec{y}) \leftrightarrow \neg B(x,\vec{y})] \to \exists Y[Y = \{x : A(x,\vec{y})\}].$$

Proof: First, we observe that the formula

$$(U = \{0\} \wedge \neg A(u,\vec{v})) \vee (U = \{1\} \wedge \neg B(u,\vec{v}))$$

is equivalent to a Σ^1_1 formula $C(U,u,\vec{v})$ of L_2. Further, $\forall x[A(x,\vec{y}) \leftrightarrow \neg B(x,\vec{y})]$ implies that $\forall x \exists X C(X,x)$. The previous lemma yields a set Z such that $\forall x C((Z)_x, x)$. So $Y := \{x : 1 \in (Z)_x\} = \{x : A(x,\vec{y})\}$. □

Now the stage is set to prove the main theorem of this subsection.

Theorem II.2.31 *There is a U positive, arithmetical formula $C(U^+,u)$ of L_2 with exactly the displayed set variables free, such that no fixed point of the operator F^C is Δ^1_1. In other words: For each Π^1_1 formula $A(u)$ and each Σ^1_1 formula $B(u)$ of L_2 without free set variables, the following is provable in FP^-_0:*

$$\forall x[A(x) \leftrightarrow B(x)] \to F^C(\{x : A(x)\}) \neq \{x : A(x)\}.$$

Proof: Let $a \in \mathsf{Prim}$ be an index as provided by corollary II.2.28, such that $\prec^{\emptyset}_a$ is an ordering that is longer than any well-ordering $\prec$ in HYP. If each operator given by an arithmetical formula $C(U^+,u)$ with only the displayed set variable free had a fixed point in HYP, then the proof of lemma II.2.27 would provide a downward closed subset $S \in \mathsf{HYP}$ of the field of $\prec^{\emptyset}_a$ and an $F \in \mathsf{HYP}$, such that $\mathsf{Hier}^{\mathcal{J}}(F, \prec^{\emptyset}_a\restriction S)$ as well as $\mathsf{Wo}(\prec^{\emptyset}_a\restriction\alpha) \to \alpha \in S$. However, lemma II.2.18 tells us that F is a proper hierarchy. Hence, by the choice of $\prec^{\emptyset}_a$ and corollary II.2.28, an ordering $\prec^{\emptyset}_b$ is a well-ordering exactly if it is isomorphic to an initial segment of $\prec^{\emptyset}_a$ whose field is a subset of S. This contradicts lemma II.1.10. □

The theory FP^-_0 also proves the existence of fixed points of monotone operators.

Lemma II.2.32 *For each arithmetical formula $A(U,u)$ of L_2 with exactly the displayed set variables free, the following is provable in FP^-_0:*

$$\forall X, Y[X \subseteq Y \to F^A(X) \subseteq F^A(Y)] \to \exists Z[F^A(Z) = Z].$$

Proof: Lemma II.2.27 and lemma II.1.10 provide an $a \in \mathsf{Prim}$ and a G such that $\mathsf{PSH}^A(G, \prec_a)$. Now, for a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec_a)$ without a $\prec_a$-least element, $Z := \bigcup_{\alpha \prec_a K}(G)_\alpha$ is a fixed point of F^A. This is show as theorem II.2.22. □

II.2.5 The proof-theoretic analysis of FP_0^-

In this subsection, we show that the proof-theoretic ordinal $|\mathsf{FP}_0^-|$ of the theory FP_0^- equals $\varphi\varepsilon_0 0$, the proof-theoretic ordinal of Σ^1_1-AC (cf. e.g. [6]). An easy induction on the depth of the proof immediately yields the following:

Lemma II.2.33 *For each finite set $\Gamma(\vec{U})$ of L_2 formulas with exactly the set variables $\vec{U}$ free, we have:*

$$\Sigma^1_1\text{-}\mathsf{AC} \vdash \Gamma(\vec{U}) \Longrightarrow \mathsf{FP}_0^- \vdash \neg(Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M, \neg(\vec{U} \mathrel{\dot{\in}} M), \Gamma^M.$$

By corollary II.2.28, FP_0^- proves the existence of models M of (Σ^1_1-AC) above sets in HYP, therefore we conclude that Σ^1_1-AC proves the same arithmetical formulas without free set variables as FP_0^-.

Theorem II.2.34 *If Γ is a finite set of arithmetical formulas without free set variables, then the following holds:*

$$\Sigma^1_1\text{-}\mathsf{AC} \vdash \Gamma \Longrightarrow \mathsf{FP}_0^- \vdash \Gamma.$$

Remark II.2.35 *For the reader familiar with well-ordering proofs we point out that for each $\alpha < \varphi\varepsilon_0 0$, Σ^1_1-$\mathsf{AC} \vdash \mathsf{TI}_\triangleleft(X, \alpha)$, but only $\mathsf{FP}_0^- \vdash X \notin \mathsf{HYP}, \mathsf{TI}_\triangleleft(X, \alpha)$: There are sets $X \notin \mathsf{HYP}$, for which $\mathsf{TI}_\triangleleft(X, \alpha)$ fails. Therefore, the restriction to arithmetical formulas without set variables in the previous theorem cannot be omitted.*

It remains to show that $\varphi\varepsilon_0 0$ is also a lower bound. We use the occasion to exhibit a new method to interpret fixed points into Σ^1_1-AC.

The standard way to perform this embedding would consist in applying *Aczel's trick* to gain Σ definitions of the fixed points of the operators in the fixed point axioms (FP^-), and then proceed similar to the embedding of $\widehat{\mathsf{ID}}_1$ into ACA_0, confer e.g. Feferman [11]. It seems to us that there is a more natural way to interpret fixed points, namely by the Π^1_1 definition of their least fixed point. Of course, some work is required to prove in Σ^1_1-AC that this class indeed satisfies the fixed point equation. The advantage of our approach is that one has much more information about this intuitive least fixed point definition than then applying Aczel's trick, where the fixed point property stems from a diagonalization, and not much more can be extracted from this argument. The Π^1_1 translation proves to be superior in cases their one has to model additional properties of fixed points, see Probst [30].

The canonic candidate to interpret the fixed point of the operator F^A, provided $A(U^+, u)$ is an arithmetical formula of L_2 that contains only the displayed variables free, is the intersection of all A-closed sets, namely the Π^1_1-definable class

$$\mathsf{Fix}^A := \bigcap\{X : F^A(X) \subseteq X\}.$$

Of course, we cannot prove in Σ^1_1-AC that Fix^A is a set, yet $F^A(\mathsf{Fix}^A) \subseteq \mathsf{Fix}^A$ is still immediate: For all A-closed sets X, the U-positivity of $A(U^+, u)$ allows us to conclude $F^A(\mathsf{Fix}^A) \subseteq F^A(X) \subseteq X$. For the other direction, though, we can no longer argue that $F^A(\mathsf{Fix}^A)$ is A-closed, and therefore a superset of Fix^A. To show that Σ^1_1-AC_0 proves $\mathsf{Fix}^A \subseteq F^A(\mathsf{Fix}^A)$, a more refined argument is required.

We prove $F^A(\mathsf{Fix}^A) = \mathsf{Fix}^A$ in a slightly more general context. For an U-positive, arithmetical formula $A(U^+, \vec{V}, \vec{v}, u)$ of L_2, we set

$$\begin{aligned} \mathsf{Cl}^A_{\vec{Y},\vec{y}}(U) &:= \forall x(A(U, \vec{Y}, \vec{y}, x) \to x \in U), \\ \mathsf{Fix}^A_{\vec{Y},\vec{y}} &:= \{x : \forall X[\mathsf{Cl}^A_{\vec{Y},\vec{y}}(X) \to x \in X]\}. \end{aligned}$$

Often, we do not explicitly mention the parameters in the formula A, and write $\mathsf{Cl}^A(X)$, Fix^A and F^A instead of $\mathsf{Cl}^A_{\vec{Y},\vec{y}}(X)$, $\mathsf{Fix}^A_{\vec{Y},\vec{y}}$ and $F^A_{\vec{Y},\vec{y}}$. The context provides always enough information to identify the dropped parameters. Below, we prove within Σ^1_1-AC_0 that for each arithmetical formula $A(U^+, u)$, Fix^A is a fixed point of the operator F^A. The direction from right to left is again immediate. For the other direction, the following lemma almost handles the job.

Lemma II.2.36 (Separation Lemma) *For all arithmetical, U-positive formulas $A(U^+, \vec{V}, u, \vec{v})$ and $B(U^+, \vec{u})$ of L_2, Σ^1_1-AC_0 proves:*

$$\forall X[\mathsf{Cl}^A_{\vec{Y},\vec{y}}(X) \to B(X, \vec{x})] \to B(\mathsf{Fix}^A_{\vec{Y},\vec{y}}, \vec{x}).$$

Proof: We prove the lemma by induction on the build-up of the formula $B(U^+)$. If U does not occur in B, then there is nothing to prove, and if B is the formula $t \in U$, then the claim follows from the definition of Fix^A. If B is a conjunction or a disjunction, a similar argument applies as in the cases treated below.

(i) $B(U)$ is of the form $\exists y B_1(U^+, y)$. We assume $\forall X[\mathsf{Cl}^A(X) \to B(X)]$ and $\forall y \neg B_1(\mathsf{Fix}^A, y)$, and argue for a contradiction. The contraposition of the I.H. reads
$$\neg B_1(\mathsf{Fix}^A, y) \to \exists X[\mathsf{Cl}^A(X) \wedge \neg B_1(X, y)],$$
hence our assumptions yield that
$$\forall y \exists X[\mathsf{Cl}^A(X) \wedge \neg B_1(X, y)].$$
Applying (Σ^1_1-AC) gives us a set X such that
$$\forall y[\mathsf{Cl}^A((X)_y) \wedge \neg B_1((X)_y, y)].$$
Now we set
$$Z := \{z : \forall y(z \in (X)_y)\},$$

and observe that $\mathsf{Cl}^A(Z)$: From $A(Z,z)$ we conclude that $\forall y A((X)_y, z)$, and so $\forall y \mathsf{Cl}^A((X)_y)$ yields $\forall y(z \in (X)_y)$. Hence, by the positivity of B_1, we have

$$\mathsf{Cl}^A(Z) \land \forall y \neg B_1(Z, x, y),$$

which contradicts our assumptions.

(ii) $B(U)$ is of the form $\forall y B_1(U, y)$. Now $\forall X[\mathsf{Cl}^A(X) \to B(X)]$ implies that $\forall y \forall X[\mathsf{Cl}^A(X) \to B_1(X, y)]$, and the claim follows by the I.H.

□

Our claim is now obtained effortlessly.

Lemma II.2.37 *For all arithmetical U-positive formulas $A(U^+, \vec{V}, u, \vec{v})$ of L_2, the theory Σ^1_1-AC_0 proves:*

$$\forall x[x \in \mathsf{Fix}^A_{\vec{Y},\vec{y}} \leftrightarrow A(\mathsf{Fix}^A_{\vec{Y},\vec{y}}, \vec{Y}, x, \vec{y})].$$

Proof: It remains to show that $x \in \mathsf{Fix}^A$ implies $A(\mathsf{Fix}^A, x)$. Due to lemma II.2.36 it suffices to show that $x \in \mathsf{Fix}^A$ implies $\forall X[\mathsf{Cl}^A(X) \to A(X, x)]$. Assume for a moment, that there is an $x \in \mathsf{Fix}^A$ and a set Z with $\mathsf{Cl}^A(Z)$ and $x \notin F^A(Z)$. Because also $F^A(Z)$ is A-closed, this contradicts $x \in \mathsf{Fix}^A$. □

An upper bound for FP^-_0 is provided by the following lemma:

Lemma II.2.38 *We let a formula A belong to the set $*$, exactly if A or $\neg A$ is the main formula of an instance of a non-logical axiom of FP^-_0. Then, for all finite sets $\Gamma(U_1, \ldots, U_n)$ of arithmetical formulas of L_2 and all formlas $C_1(u), \ldots, C_n(u)$ of L_2 which may contain other free variables,*

$$\mathsf{FP}^-_0 \mathrel{\mid\!\frac{n}{*}} \Gamma(\vec{U}) \Longrightarrow \Sigma^1_1\text{-}\mathsf{AC} \vdash \Gamma[\vec{\mathcal{C}}/\vec{U}],$$

where $\mathcal{C}_i$ denotes the set term $\{x : C_i(x)\}$, $(1 \leq i \leq n)$.

Proof: We prove the lemma by (meta-) induction on n. If Γ is an axiom, then the claim is due to formula induction and the fact that Σ^1_1-$\mathsf{AC} \vdash C(u), \neg C(u)$ for all formulas C of L_2. For rules, the only cases where the claim does not follow directly by the I.H. is then $\Gamma(\vec{U})$ was obtained by a cut with an instance of a comprehension axiom or a fixed point axiom. So assume that for an $n' < n$,

$$\mathsf{FP}^-_0 \mathrel{\mid\!\frac{n'}{*}} \Gamma(\vec{U}), \neg\exists X[X = \{x : B(\vec{U}, x)\}].$$

$\forall$-inversion yields $\mathsf{FP}^-_0 \mathrel{\mid\!\frac{n'}{*}} \Gamma(\vec{U}), V \neq \{x : B(\vec{U}, x)\}$, for a $V \notin FV(\Gamma, B(\vec{U}, u))$. The I.H. now yields that

$$\Sigma^1_1\text{-}\mathsf{AC} \vdash \Gamma[\vec{\mathcal{C}}/\vec{U}], \{x : B(\vec{\mathcal{C}}, x)\} \neq \{x : B(\vec{\mathcal{C}}, x)\},$$

where V has been replaced by $\{x : B(\vec{\mathcal{C}}, x)\}$. Since, $\{x : D(x)\} = \{x : D(x)\}$ is provable in Σ^1_1-AC for all formulas D of L_2, the claim follows by a cut.
If the last inference was a cut with a fixed point axiom, there is an $n' < n$, so that $\mathsf{FP}_0^- \overset{n'}{\underset{*}{\vdash}} \Gamma(\vec{U}), \neg\exists X[X = F^A(X)]$, for an arithmetical formula $A(U^+, u)$ with at most the displayed variables free. Then $\forall$-inversion yields $\mathsf{FP}_0^- \overset{n'}{\underset{*}{\vdash}} \Gamma(\vec{U}), V \neq F^A(V)$, for a $V \notin FV(\Gamma(\vec{U}))$. The I.H. now yields that Σ^1_1-$\mathsf{AC} \vdash \Gamma[\vec{\mathcal{C}}/\vec{U}], \mathsf{Fix}^A \neq F^A(\mathsf{Fix}^A)$. Lemma II.2.37 and a cut yield the claim. □

Note that the above lemma fails in Σ^1_1-AC_0. $\Gamma(U)$ might be an instance of set induction.

If $\mathsf{FP}_0^- \vdash \Gamma$ for a finite set Γ of L_2 formulas, then lemma I.3.4 tells us that there is also a $k \in \mathbb{N}$ such that $\mathsf{FP}_0^- \overset{k}{\underset{*}{\vdash}} \Gamma$, where $*$ is the set from the previous lemma. Together with theorem II.2.34 we conclude:

Theorem II.2.39 *The theories FP_0^- and Σ^1_1-AC prove the same arithmetical formulas of L_2 without free set variables. Moreover,*

$$|\mathsf{FP}_0^-| = |\Sigma^1_1\text{-}\mathsf{AC}| = \varphi\varepsilon_0 0.$$

Remark II.2.40 *For the reader familiar with well-ordering proofs, we like to comment on the definition of proof-theoretic ordinal again. It is well-known that there is a Π^0_1 formula $A(U, u)$ of L_2 such that*

$$\mathsf{ACA}_0 \vdash \mathsf{TI}_\lhd(F^A(X), \alpha) \to \mathsf{TI}_\lhd(X, \omega^\alpha).$$

Thus, if we set $\omega_0 := 0$ and $\omega_{n+1} := \omega^{\omega_n}$, we obtain immediately that for each $n \in \mathbb{N}$,

$$\mathsf{FP}_0^- \vdash \mathsf{Wo}(\lhd\restriction\omega_n) \quad \textit{and thus} \quad \mathsf{FP}_0^- \vdash (\forall X \in \mathsf{HYP})\exists F \mathsf{Hier}^{\mathcal{J}}(F, X, \omega_n).$$

Hence, standard well-ordering techniques yield $\mathsf{FP}_0^- \vdash \mathsf{Wo}^{\mathsf{HYP}}(\lhd\restriction\varphi\omega_n 0)$, in particular $\mathsf{FP}_0^- \vdash \mathsf{TI}_\lhd(\mathsf{U}, \varphi\omega_n 0)$. Note, that $\mathsf{Wo}^{\mathsf{HYP}}(\lhd\restriction\alpha)$ does not imply $\exists F \mathsf{Hier}^{\mathcal{J}}(F, \alpha)$. Observe in particular, that FP_0^- does not prove $\forall X \mathsf{TI}_\lhd(X, \varepsilon_0)$ which is $\mathsf{Wo}(\lhd\restriction\varepsilon_0)$, otherwise we had also $\mathsf{FP}_0^- \vdash \mathsf{TI}_\lhd(\mathsf{U}, \varphi\varepsilon_0 0)$, contradicting $|\mathsf{FP}_0^-| = \varphi\varepsilon_0 0$. We only have that for each $n \in \mathbb{N}$, $\mathsf{FP}_0^- \vdash \mathsf{Wo}(\lhd\restriction\omega_n)$. This point is also addressed in [23].

II.2.6 Additional results on the class Fix^A

The theory Σ^1_1-AC_0 proves that for an arithmetical formula $A(U^+, u)$ of L_2, the class Fix^A is a subclass of every A-closed set. When we move to the slightly stronger theory Σ^1_1-DC_0 we can even prove that Fix^A is contained in every A-closed, Π^1_1-definable class. As a consequence, we also obtain induction along the natural numbers for Π^1_1 formulas. Also, the aforementioned embedding of $\widehat{\mathsf{ID}}_1$ into Σ^1_1-AC extends to an embedding of ID_1^* into Σ^1_1-DC which yields a sharp upper bound and finally answers an old question concerning the proof-theoretic strength of ID_1^* (cf.[30]).

Theorem II.2.41 *For all arithmetical U-positive formulas $A(U^+, \vec{V}, u, \vec{v})$ of L_2 and each Π^1_1 formula $C(u)$ of L_2, the following is provable in Σ^1_1-DC_0:*

$$\mathsf{Cl}^A_{\vec{Y},\vec{y}}(\{x : C(x)\}) \to \mathsf{Fix}^A_{\vec{Y},\vec{y}} \subseteq \{x : C(x)\}.$$

Before we give the proof, we consider a simpler case to illustrate the proof idea: Suppose that $A(U^+, u)$ and $B(U^+, u)$ are arithmetical formulas and that Fix^B is A-closed. We assume that there is an $x \in \mathsf{Fix}^A$ that is not an element of Fix^B, and argue for a contradiction. Thereto, we construct a sequence $V_0 \supseteq V_1 \supseteq \ldots$ of B-closed sets, such that for all $n \in \mathsf{N}$, we have $x \notin V_n$ and $F^A(V_n) \subseteq V_{n+1}$. Then $W := \bigcap_{n \in \mathsf{N}} V_n$ is A-closed, but does not contain x.

To apply this argument in the general case, we require that every Π^1_1-definable class $\{x : C(x)\}$ is Δ^0_0 in a fixed point.

Lemma II.2.42 (Representation Lemma) *For each Π^1_1 formula $C(U, u)$ of L_2, there exists an U-positive arithmetical formula $A(U^+, V, u)$ and an U-positive Δ^0_0 formula $D(U^+, u)$ of L_2, such that Σ^1_1-AC_0 proves: For all sets Y, there exists a set S, such that*

$$\forall x[D(\mathsf{Fix}^A_S, x) \leftrightarrow C(Y, x)].$$

Proof: Theorem II.1.8 provides a set S, depending on the number and set parameters occurring in C, such that for all n,

$$(S)_n \text{ is a tree,} \qquad \text{and} \qquad C(Y, n) \leftrightarrow [(S)_n \text{ is well-founded.}]$$

As in the proof of lemma II.2.29, we set

$$A(U^+, u) := \exists n, \sigma[u = \langle \sigma, n \rangle \wedge \sigma \in (S)_n \wedge (\forall \tau \in (S)_n)(\tau \sqsupset \sigma \to \tau \in (U)_n)].$$

Recall that the operator F^A picks the leafs of the trees $(S)_n$. If the tree $(S)_n$ is well-founded, then the root $\langle\rangle$ of the tree $(S)_n$ is an element of Fix^A, otherwise the infinite branches and therefore the root do not enter Fix^A. It is now easy to see that

$$\forall x[\langle\langle\rangle, x\rangle \in \mathsf{Fix}^A_S \leftrightarrow C(Y, x)].$$

□

Next we return to the proof of theorem II.2.41.
Proof: We assume $\mathsf{Cl}^A(\{x : C(x)\})$, and aim to prove that $x \in \mathsf{Fix}^A$ implies $C(x)$. Lemma II.2.42 provides a set S, an arithmetical formula $B(U^+, V, u)$ and a Δ^0_0 formula $D(U^+, u)$ of L_2 such that

$$\forall x[D(\mathsf{Fix}^B_S, x) \leftrightarrow C(x)].$$

Hence our assumption reads $\mathsf{Cl}^A(\{x : D(\mathsf{Fix}^B_S, x)\})$. We show that this implies

(1) $\forall X \exists Z[F^D(X) \neq \mathsf{N} \wedge \mathsf{Cl}^B_S(X) \to \mathsf{Cl}^B_S(Z) \wedge Z \subseteq X \wedge F^A \circ F^D(Z) \subseteq F^D(X)]$,

where $F^A \circ F^D(Z)$ is an alternative notation for $F^A(F^D(Z))$. Fix an arbitrary X such that $\mathsf{Cl}^B_S(X)$ and suppose that $F^D(X)$ does not contain all natural numbers. If $x \notin F^D(X)$, then $x \notin F^D(\mathsf{Fix}^B_S)$, so our assumption yields $x \notin F^A \circ F^D(\mathsf{Fix}^B_S)$, and lemma II.2.36 provides a set Y that is B-closed with respect to S, such that we have $x \notin F^A \circ F^D(Y)$. If $\mathsf{Cl}^B_S(X)$ and $\mathsf{Cl}^B_S(Y)$ then also $\mathsf{Cl}^B_S(X \cap Y)$, thus we may assume that $Y \subseteq X$. Summarizing, we obtain

$$\forall x \exists Y[x \notin F^D(X) \to \mathsf{Cl}^B_S(Y) \wedge Y \subseteq X \wedge x \notin F^A \circ F^D(Y)].$$

Now $(\Sigma^1_1\text{-}\mathsf{AC})$ gives us a set Y such that for all $x \notin F^D(X)$

$$\mathsf{Cl}^B_S((Y)_x) \wedge (Y)_x \subseteq X \wedge x \notin F^A \circ F^D((Y)_x).$$

Therefore, if we set

$$Z := \bigcap_{x \notin F^D(X)} (Y)_x,$$

we have $\mathsf{Cl}^B_S(Z)$ and $Z \subseteq X$ and

$$\forall x[x \notin F^D(X) \to x \notin F^A \circ F^D(Z)],$$

which means $F^A \circ F^D(Z) \subseteq F^D(X)$. Thus we have shown claim (1).

Now we suppose that there is an $x \in \mathsf{Fix}^A$ that is not an element of $x \notin F^D(\mathsf{Fix}^B_S)$ and argue for a contradiction. Again, lemma II.2.36 provides a set Q that is B-closed with respect to S and $x \notin F^D(Q)$. Applying $(\Sigma^1_1\text{-}\mathsf{DC})$ to (1) gives us a set V such that $(V)_0 = Q$ and

$$\forall n[\mathsf{Cl}^B_S((V)_n) \to \mathsf{Cl}^B_S((V)_{n+1}) \wedge (V)_{n+1} \subseteq (V)_n \wedge F^A \circ F^D((V)_{n+1}) \subseteq F^D((V)_n)].$$

One easily proves by induction that

$$\forall n[\mathsf{Cl}^B_S((V)_n) \wedge (V)_{n+1} \subseteq (V)_n \wedge F^A \circ F^D((V)_{n+1}) \subseteq F^D((V)_n)].$$

Hence, for $W := \bigcap_{n \in \mathsf{N}} (V)_n$, we have that

$$F^A \circ F^D(W) \subseteq \bigcap_{n \in \mathsf{N}} F^D((V)_n) = F^D(\bigcap_{n \in \mathsf{N}} (V)_n) = F^D(W).$$

The second but last equality follows from the fact that D is Δ^0_0. So $W \subseteq Q$ and $\mathsf{Cl}^A(F^D(W))$, i.e. $\mathsf{Fix}^A \subseteq F^D(W)$. Now $x \notin F^D(Q)$ yields $x \notin F^D(W)$, thus $x \notin \mathsf{Fix}^A$. A contradiction! □

The following corollary is an immediate consequence of theorem II.2.41. To enhance readability, we let $\mathsf{Fix}^{\vec{B}}$ stands for $\mathsf{Fix}^{B_1}, \dots, \mathsf{Fix}^{B_n}$.

Corollary II.2.43 *For all arithmetical formulas $A(U^+, u)$ and $\vec{B}(U^+, u)$ and each $\vec{U}$-positive arithmetical formula $C(\vec{U}^+, u)$ of L_2, Σ^1_1-DC_0 proves:*

$$\mathsf{Cl}^A(\{x : C(\mathsf{Fix}^{\vec{B}}, x)\}) \to \mathsf{Fix}^A \subseteq \{x : C(\mathsf{Fix}^{\vec{B}}, x)\}.$$

Proof: Note that $C(\mathsf{Fix}^{\vec{B}}, x)$ is equivalent to a Π^1_1 formula of L_2. □

Remark II.2.44 *Consider the formula*

$$A(U^+, \prec, u) := (\forall x \prec u)(u \in U].$$

Observe, that $\mathsf{Cl}^A_\prec(X)$ is the formula $\mathsf{Prog}_\prec(X)$ and the formula $\mathsf{Wo}(\prec)$ can be written as $\forall Y[\mathsf{Cl}^A_\prec(Y) \to \mathsf{Field}(\prec) \subseteq Y]$. It is immediate, that $\mathsf{Wo}(\prec)$ is equivalent to $\mathsf{Fix}^A_\prec = \mathsf{Field}(\prec)$. Due to theorem II.2.41, Σ^1_1-DC_0 proves for each Π^1_1 formula $C(u)$ of L_2 that

$$\mathsf{Wo}(\prec) \to [\mathsf{Cl}^A_\prec(\{z : C(z)\}) \to \mathsf{Field}(\prec) \subseteq \{z : C(z)\}],$$

which is normally written as

$$(\Pi^1_1\text{-}\mathsf{TI}) \qquad \mathsf{Wo}(\prec) \to \mathsf{TI}_\prec(\{z : C(z)\}).$$

It is shown, e.g. in [40], that $(\Pi^1_1$-$\mathsf{TI})$ is provable in Σ^1_1-DC_0. In this sense, corollary II.2.41 is a generalization of this result.

By the above corollary we obtain an embedding of ID^*_1 into Σ^1_1-DC. The theory ID^*_1 is formulated in the language $\mathsf{L}_{\mathsf{Fix}}$ that extends L_1 by fixed point constants P^A for each U-positive arithmetical formula $A(U^+, u,)$ of L_2 with exactly the displayed variables free. Technically, we treat fixed point constants as unary relation symbols, but write $t \in \mathsf{P}^A$ instead of $\mathsf{P}^A(t)$. The axioms of ID^*_1 consist of the axioms of PA without induction, complete induction along the natural numbers for all formulas of $\mathsf{L}_{\mathsf{Fix}}$ as well as the following two fixed point axioms: For each arithmetical formula $A(U^+, u)$ of L_2 with exactly the displayed variables, we have

$$(\mathsf{FIX}) \qquad \forall x[A(\mathsf{P}^A, x) \leftrightarrow x \in \mathsf{P}^A],$$

and for all arithmetical formulas $A(U^+, u)$, $A_1(U^+, u), \dots, A_n(U^+, u)$ of L_2 with exactly the displayed free variables, and each $\vec{V}$-positive formula $B(\vec{V}, u, \vec{v})$ of L_2 with exactly the displayed free variables, we have

$$(\mathsf{IND}^+_{\mathsf{FIX}}) \quad \forall x[A(\{z : B(\mathsf{P}^{\vec{A}}, z, \vec{y})\}, x) \to B(\mathsf{P}^{\vec{A}}, x, \vec{y})] \to \forall x[x \in \mathsf{P}^A \to B(\mathsf{P}^{\vec{A}}, x, \vec{y})].$$

Note that we wrote $\mathsf{P}^{\vec{A}}$ for the string $\mathsf{P}^{A_1}, \dots, \mathsf{P}^{A_n}$ and that A may be syntactically identical to some A_i. The axiom (FIX) asserts that P^A is indeed a fixed point of the operator F^A and $(\mathsf{IND}^+_{\mathsf{FIX}})$ is the scheme for proof by induction on P^A restricted to formulas of $\mathsf{L}_{\mathsf{Fix}}$ that contain fixed point constants only positively.

If we translate an $\mathsf{L}_{\mathsf{Fix}}$ formula B to a L_2 formula B^* by substituting each fixed point constant P^A by the Π^1_1-definable class Fix^A, then the following is due to the previous corollary:

Theorem II.2.45 *For each finite set* Γ *of* $\mathsf{L}_{\mathsf{Fix}}$ *formulas,*

$$\mathsf{ID}^*_1 \vdash \Gamma \Longrightarrow \Sigma^1_1\text{-}\mathsf{DC} \vdash \Gamma^*.$$

Since $\widehat{\mathsf{ID}}_1$ is contained in ID^*_1 and $|\widehat{\mathsf{ID}}_1| = \varphi\varepsilon_0 0 = |\Sigma^1_1\text{-}\mathsf{DC}|$, this answers the question for a sharp upper bound of ID^*_1:

Corollary II.2.46

$$|\mathsf{ID}^*_1| = \varphi\varepsilon_0 0.$$

Chapter III

Pseudo-hierarchy arguments in admissible set theory without foundation and explicit mathematics

> Some luck lies in not getting what you thought you wanted but getting what you have, which, once you have got it, you may be smart enough to see is what you would have wanted had you known. Garrison Keillor (1942 -)

The previous chapter has hinted at the many possibilities that pseudo-hierarchies offer to fruitfully investigate subsystems of second order arithmetic. There is no doubt that pseudo-hierarchies would serve as a potent device in admissible set theory without foundation and explicit mathematics as well, once a way to adapt this technique to these frameworks has been discovered.

In subsystems of second order arithmetic, the existence of pseudo-hierarchies follows from the theorem that "being a well-ordering" is not expressible by a Σ formula of L_2. In the standard models of admissible set theories, initial segments of the constructible hierarchy $\mathcal{L}$, an ordering $\prec$ is well-ordered if there exists a collapsing function for $\prec$. Therefore, it is consistent to assume that "$\prec$ is a well-ordering" is expressible by a Σ formula. Hence, the existence of pseudo-hierarchies cannot be inferred by the methods used in second order arithmetic.

In this chapter, we start by presenting a method to apply pseudo-hierarchy arguments in theories for admissible sets. This technique also sheds new light on the situation in second order arithmetic and opens additional ways to apply pseudo-hierarchy arguments there, which we will briefly discuss. Then, as a first application, we establish the equivalence of a fixed point principle and an iteration principle

over a conservative extension of KPi^0. Further, we comment on the relationship of iteration, linearity of admissibles and dependent choice.
Next, we define a Δ_0 formula $\mathsf{P_{Ad}}(u)$ of $\mathcal{L}^*$ expressing that a set satisfies the axioms of $\mathsf{KPu}^0 + (\mathsf{I_N})$. Results about the class hyp^x, the intersection of all models of $\mathsf{KPu}^0 + (\mathsf{I_N})$ above x follow, before we have a look at an extension of KPu^0 with the same proof-theoretic strength as Δ^1_2-CA_0. In this extension, models of $\mathsf{KPu}^0 + (\mathsf{I_N})$ exists above arbitrary sets and in addition, are linearly ordered by $\in$.
The results gathered so far allow as to treat dependent choice in admissible set theory. We consider an axiom (Δ_0-dc) corresponding to (Σ^1_1-DC) and argue that KPu^0 extended by Π_2 reflection on models of $\mathsf{KPu}^0 + (\mathsf{I_N}) + (\Delta_0\text{-}\mathsf{dc})$ is another theory of strength meta-predicative Mahlo. Finally, we consider pseudo-hierarchies and dependent choice in explicit mathematics.

III.1 Pseudo-hierarchies in admissible set theory

In this section, we first specify what exactly we aim to achieve under the heading *pseudo-hierarchies in admissible set theory*. After a method is presented on how to apply pseudo-hierarchy arguments, we use them to establish the equivalence of an iteration principle (Σ-tr) that allows to iterate an operation specified by a Σ formula along a well-ordering and a fixed point principle (Σ-fp') that claims the existence of fixed points of monotone operators acting on the entire universe. It turns out, that the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ is inconsistent with foundation, however, proves the existence of pseudo-hierarchies. We conclude giving an embedding of $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{TDC})$ into $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})$.

III.1.1 Hierarchies and pseudo-hierarchies

The concepts of hierarchies and pseudo-hierarchies are adapted straight forward to the framework of Kripke-Platek set theory. Again, a hierarchy is induced by an operation that determines a certain level of the hierarchy given its predecessors. In the present context, we associate with a formula $A(u_1, \dots u_n, v, \vec{w})$ of a language comprising $\mathcal{L}^*$ and parameters $\vec{a}$ with the property that

$$\mathsf{Op}^n_A(\vec{a}) := (\forall x_1, \dots, x_n \in \mathcal{S})(\exists! y \in \mathcal{S}) A(\vec{x}, y, \vec{a}),$$

an n-ary operator $f^A_{\vec{a}}$. Strictly speaking, we use $f^A_{\vec{a}}(\vec{x}) = y$ as an abbreviation for $A(\vec{x}, y, \vec{a})$, and $B(f^A_{\vec{a}}(\vec{x}))$ is seen as a short cut for $\exists y[f^A_{\vec{a}}(\vec{x}) = y \wedge B(y)]$, which is equivalent to $\forall y[f^A_{\vec{a}}(\vec{x}) = y \rightarrow B(y)]$ under the assumption $\mathsf{Op}^n_A(\vec{a})$. If $A(u, v, \vec{w})$ is a Σ formula and $\mathsf{Op}^n_A(\vec{a})$, then $f^A_{\vec{a}}$ is called a Σ operation.

An operation f^A then induces an A-hierarchy. We call a function g an A-hierarchy along $\prec$ for the parameters $\vec{a}$, denoted by $\mathsf{hier}^A(g, \vec{a}, \prec)$, if $A(u_1, u_2, v, \vec{w}, z)$ contains exactly the displayed variables free, $\vec{a}$ are such that $\mathsf{Op}^2_A(\vec{a}, \prec)$, and g meets the following properties:

(i) $\mathsf{Fun}(g) \land \mathsf{Dom}(g) = \mathsf{Field}(\prec) \land \mathsf{Lin}_0(\prec)$,

(ii) $(\forall \alpha \in \mathsf{Field}(\prec))[g(\alpha) = f^A_{\vec{a},\prec}(g{\restriction}\alpha, \alpha)]$.

Thereby, $g{\restriction}\alpha$ denotes the restriction of g to the elements of its domain below α, namely the function $\{(\beta, g(\beta)) : \beta \prec \alpha\}$. As in second order arithmetic, if the αth level does not explicitly depend on α, i.e. if $\mathsf{Op}^1_A(\vec{a}, \prec)$, then (ii) is adjusted accordingly. In case that $\mathsf{Op}^2_A(\vec{a})$ is not guaranteed, $g(\alpha) = f^A_{\vec{a},\prec}(g{\restriction}\alpha, \alpha)$ in the formula (ii) above is read as $A(g{\restriction}\alpha, \alpha, g(\alpha), \vec{a}, \prec)$, hence in any case, if A is Σ, then so is the formula $\mathsf{hier}^A(f, \prec)$.

Also, we use lowercase Greek letters to range over the field of an ordering. As before, a hierarchy g along $\prec$ is called proper if $\prec$ is a well-ordering and a pseudo-hierarchy, $\mathsf{psh}^A(g, \vec{y}, \prec)$, otherwise.

III.1.2 Admissible sets and the theories KPi^0, KPi^r and KPm^0

The standard approach to talk about admissible sets is to extend the language $\mathcal{L}^*$ to the language $\mathcal{L}^*_{\mathsf{Ad}}$ by a new relation symbol $\mathsf{Ad}(u)$ that is to distinguish admissible sets, i.e. transitive models of $\mathsf{KPu}^0 + (\mathsf{I}_\mathsf{N})$. This approach is realized with the theory KPi^0, introduced and analyzed in Jäger [19], where it is also shown that $|\mathsf{KPi}^0| = \Gamma_0$. The theory KPi^0 comprises, besides the axioms and rules of KPu^0 adapted to the new language, additional axioms for the predicate $\mathsf{Ad}(u)$. First of all, the sets distinguished by $\mathsf{Ad}(u)$ are transitive sets above N that reflect the axioms of $\mathsf{KPu}^0 + (\mathsf{I}_\mathsf{N})$. Thus, for all Kripke-Platek axioms $A(\vec{u})$ (cf. subsection I.2.9), whose free variables belong to the list $\vec{u}$, we have

$$\mathsf{Ad}(a) \to (\forall \vec{x} \in a) A^a(\vec{x}) \quad \text{and} \quad \mathsf{Ad}(a) \to (\mathsf{N} \in a \land \mathsf{Tran}(a)).$$

Moreover, the admissibles in the class $\mathsf{Ad} := \{x : \mathsf{Ad}(x)\}$ are linearly ordered by $\in$. Particularly in extensions of KPi^0, this axiom proves very convenient, although, by itself, does not increase the proof-theoretic strength. As shown in Jäger [20], KPi^0 without the axiom (lin) has still the ordinal Γ_0.

(lin) $$\mathsf{Ad}(a) \land \mathsf{Ad}(b) \to a \in b \lor a = b \lor b \in a.$$

Finally, the limit axiom guarantees the existence of admissibles above arbitrary sets,

(lim) $$\exists x (a \in x \land \mathsf{Ad}(x)).$$

The linearity axiom (lin) also plays an important role in the standard theory KPm^0 of strength meta-predicative Mahlo in admissible set theory. KPm^0 has been introduced and analyzed by Jäger and Strahm [27]. It is formulated in the language $\mathcal{L}^*_{\mathsf{Ad}}$ and extends KPi^0 by an axiom for Π_2 reflection on admissibles:

For each Δ_0 formula $A(u,v,\vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$,

$$(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad}} \qquad \forall x \exists y A(x,y,\vec{z}) \to \exists a[\mathsf{Ad}(a) \wedge \vec{z} \in a \wedge (\forall x \in a)(\exists y \in a) A(x,y,\vec{z})].$$

Finally, the theory $\mathsf{KPi}^0 + (\mathsf{I}_\in)$ is named KPi^r. It is introduce and analyzed in [20]. There, it is also shown that $|\mathsf{KPi}^\mathsf{r}| = |\Delta^1_2\text{-}\mathsf{CA}_0|$.

Some basic properties of the theory KPi^0 are gathered in the next paragraph. We start by mentioning a result due to Jäger.

Lemma III.1.1 *Let $A(u,\vec{v})$ be a Δ_0 formula of $\mathcal{L}^*_{\mathsf{Ad}}$ with at most the variables $u,\vec{v}$ free. Then KPi^0 proves that the class*

$$\mathcal{C} := \{x : A(x,\vec{a}) \wedge \vec{a} \in x \wedge \mathsf{Ad}(x)\}$$

is empty or has a $\in$-least element.

Proof: We assume that $\mathcal{C}$ is not empty and has no $\in$-least element, and argue for a contradiction. Since admissibles are linearly ordered by $\in$, we have for $b,c \in \mathcal{C}$ with $b \in c$ that the set $z := \bigcap(\mathcal{C} \cap b)$ equals $\bigcap \mathcal{C}$ and that $z \in c$. Since $\mathcal{C}$ has no $\in$-least element, z is in an element of each admissible in $\mathcal{C}$, thus also the set $r := \{u \in z : u \notin u\}$ is an element of each admissible in $\mathcal{C}$ and therefore in z. However, $r \in z$ implies $r \in r \leftrightarrow r \notin r$. □

The previous lemma implies for example that in KPi^0, for any set x, the intersection of all the sets y with $x \in y$ and $\mathsf{Ad}(y)$ is an element of the class Ad itself. In the sequel, we denote this least admissible in Ad above x by x^+,

$$x^+ := \bigcap \{y : x \in y \wedge \mathsf{Ad}(y)\}.$$

Also, the hierarchy that iterates the operation $\cdot^+$ will play an important role. Therefore, we write $\mathsf{hier}^+(f,z,\prec)$ for $\mathsf{hier}^A(f,z,\prec)$, when $A(u,v,w)$ is the formula $v = \{u,w\}^+$.

Another property of KPi^0 is that we can iterate Σ definable operations on admissibles along well-orderings.

Lemma III.1.2 *Let a $A(u_1,u_2,v,\vec{w},z)$ be a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$. Then the following is provable in KPi^0: Suppose that $\vec{a},\prec$ are such that $\mathsf{Op}^2_A(\vec{a},\prec)$. Further, assume that b is admissible, $\vec{a},\prec \in b$ and $[\mathsf{Op}^2_A(\vec{a},\prec)]^b$. Then*

$$\mathsf{Wo}^{b^+}(\prec) \to (\exists! f \in b)\mathsf{hier}^A(f,\vec{a},\prec).$$

Proof: Using transfinite induction, we aim to show that

$$(*) \qquad \{\alpha \in \mathsf{Field}(\prec) : (\exists f \in b)\mathsf{hier}^{A^b}(f, \vec{a}, \prec\restriction\alpha)\} \in b^+$$

is already the entire field of $\prec$. The claim then follows by persistence. So suppose that we have shown that

$$(\forall \beta \prec \alpha)(\exists! g \in b)\mathsf{hier}^{A^b}(g, \vec{a}, \prec\restriction\beta),$$

Then, we define

$$c := \{g \in b : (\exists \beta \prec \alpha)\mathsf{hier}^{A^b}(g, \vec{a}, \prec\restriction\beta)\}.$$

The I.H. yields that $\bigcup c$ is a function with domain $\mathsf{Field}(\prec\restriction\alpha)$, and once we have shown that $c \in b$, we obtain that $h := f^{A^b}_{\vec{a},\prec}(\bigcup c, \alpha) \in b$ and $\mathsf{hier}^{A^b}(h, \vec{a}, \prec\restriction\alpha)$. To show that $c \in b$, we set

$$B(u_1, u_2, v, \vec{w}, z) := \forall x[A(u_1, u_2, x, \vec{w}, z) \to x = v].$$

Observe that B is a Π formula of $\mathcal{L}^*_{\mathsf{Ad}}$ and that $[\mathsf{Op}^2_A(\vec{a}, \prec)]^b$ yields

$$(\forall x_1, x_2, y \in b)[A^a(x_1, x_2, y, \vec{a}, \prec) \leftrightarrow B^b(x_1, x_2, y, \vec{a}, \prec)],$$

which in turn yields

$$(\forall \beta \prec \alpha)(\forall g \in b)[\mathsf{hier}^{A^b}(g, \vec{a}, \prec\restriction\beta) \leftrightarrow \mathsf{hier}^{B^b}(g, \vec{a}, \prec\restriction\beta)].$$

That $c \in b$ follows now by Δ separation within b. This shows $(*)$. Hence, there is for each $\alpha \in \mathsf{Field}(\prec)$ a hierarchy $f \in b$ such that $\mathsf{hier}^{A^b}(f, \vec{a}, \prec\restriction\alpha)$. Therefore, we have for all $\beta \prec \alpha$ that $A^b(f\restriction\beta, \beta, f(\beta), \vec{a}, \prec)$. Persistence yields $A(f\restriction\beta, \beta, f(\beta), \vec{a}, \prec)$ and $\mathsf{Op}^2_A(\vec{a}, \prec)$ implies $f(\beta) = f^A(f\restriction\beta, \beta, \vec{a}, \prec)$. This yields $\mathsf{hier}^A(f, \vec{a}, \prec\restriction\alpha)$. The lemma follows now by a similar argument. □

As a consequence, the standard translation of every instance of the axiom (ATR) is provable in KPi^0.

As mentioned before, the existence of pseudo-hierarchies is not provable in KPi^0 or common extensions thereof that are valid in the standard model of Kripke-Platek set theory. In KPi^0, we call a set x an ordinal, denoted by $\mathsf{Ord}(x)$, if

$$\mathsf{Tran}(x) \wedge (\forall y \in x)\mathsf{Tran}(y) \wedge \mathsf{Wo}(\in\restriction x).$$

For every well-ordering $\prec$, there exists an ordinal x, such that $\prec$ is order-isomorphic to $\in\restriction x$. The corresponding order isomorphism f is called the *collapse* of $\prec$,

$$\mathsf{Clp}(f, \prec) := \begin{cases} \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \mathsf{Field}(\prec) \wedge \\ (\forall x \in \mathsf{Field}(\prec))(f(x) = \{f(y) : y \prec x\}). \end{cases}$$

The following lemma states that for each well-ordering $\prec$, there exists exactly one collapsing function f.

Corollary III.1.3 *The following is provable in* KPi^0*:*

$$\mathsf{Wo}(\prec) \rightarrow \exists! f[\mathsf{Clp}(f, \prec) \wedge \mathsf{Ord}(\mathsf{Rng}(f))].$$

Proof: Let $A(u, v)$ be a Δ_0 formula of $\mathcal{L}^*$ such that $A(x, y)$ implies $y = \mathsf{Rng}(x)$. If $\mathsf{Wo}(\prec)$, then the previous lemma provides an f such that $\mathsf{hier}^A(f, \prec)$. By transfinite induction along $\prec$ we obtain that $(f : \mathsf{Field}(\prec) \rightarrow \mathsf{Rng}(f))$ is an order-isomorphism and that $\mathsf{Rng}(f)$ is an ordinal. □

An extension of KPi^0 equipped with foundation, for instance KPi^r, proves $\mathsf{Wo}(x{\restriction} \in)$, thus $\mathsf{Ord}(x)$ becomes equivalent to $\mathsf{Tran}(x) \wedge (\forall y \in x)\mathsf{Tran}(y)$ and $\mathsf{Wo}(\prec)$ becomes equivalent to a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$.

Corollary III.1.4 *The following is provable in* KPi^r*:*

$$\mathsf{Wo}(\prec) \leftrightarrow \exists f[\mathsf{Clp}(f, \prec) \wedge \mathsf{Tran}(\mathsf{Rng}(f)) \wedge (\forall y \in \mathsf{Rng}(f))\mathsf{Tran}(y)].$$

Further, we remark that an admissible set does not contain itself.

Lemma III.1.5 *The following is provable in* KPi^0*:*

$$\mathsf{Ad}(a) \rightarrow a \notin a.$$

Proof: Russell's argument applies: If $a \in a$, then $r := \{x \in a : x \notin x\} \in a$ by Δ_0 separation in a. It follows $r \in r \leftrightarrow r \notin r$. □

III.1.3 A pseudo-hierarchy principle for KPi^0

Of course, there exist pseudo-hierarchies also in admissible set theory. Applying the standard translation $\cdot^*$ to a theorem of ATR_0 from the previous chapter yields a theorem of KPi^0. What we mean, however, when we speak about showing the existence of pseudo-hierarchies in admissible set theory, is a bit more. Whenever we have a Σ operation f^A that we can iterate along an arbitrary well-ordering, then we ask for a pseudo-hierarchy g whose underlying ordering looks like a well-ordering in g^+. For each Σ formula $A(u_1, u_2, v, \vec{w}, z)$ of $\mathcal{L}^*_{\mathsf{Ad}}$,

$$(\mathsf{psh}') \quad \forall x \mathsf{Op}^2_A(\vec{a}, x) \wedge \forall x[\mathsf{Wo}(x) \rightarrow \exists f \mathsf{hier}^A(f, \vec{a}, x)] \rightarrow \exists g, y[\mathsf{psh}^A(g, \vec{a}, y) \wedge \mathsf{Wo}^{g^+}(y)].$$

The pseudo-hierarchy principle is adequate for theories formulated in the language $\mathcal{L}^*_{\mathsf{Ad}}$ comprising the axioms and rules of KPi^0. The requirement that the underlying ordering of the pseudo-hierarchy g looks like a well-ordering in g^+ is motivated by the condition that we have usually imposed on the underlying ordering $\prec$ of a pseudo-hierarchy G in second order arithmetic, namely that $\mathsf{Wo}^{\mathcal{J}^G_\omega}(\prec)$. It is actually sufficient to demand that the underlying ordering of g looks like a well-ordering in the ωth level of the constructible hierarchy above g, but for the time being, the above pseudo-hierarchy principle serves its purpose. Only later, then we have introduced the constructible hierarchy and are working in theories formulated in the language $\mathcal{L}^*$ that does not contain the predicate $\mathsf{Ad}(u)$ and where the existence of a least admissible above some set x is not provable, we will consider this more refined variant of the pseudo-hierarchy principle.

Sometimes, we can apply pseudo-hierarchy arguments without actually using a particular instance of (psh'). For example, assume that we can iterate a Σ operation f^A along an arbitrary well-ordering, and that the existence of sets $\prec, g$ such that $\mathsf{psh}^A(g, \prec)$, would imply our claim. The existence of such a pseudo-hierarchy is not provable, but the supposition that there are no $\prec, g$ with $\mathsf{psh}^A(g, \prec)$, turns $\mathsf{Wo}(\prec)$ into a Σ definable predicate. In some cases, this implies our claim as well, and we are done. For an application of this procedure, see subsection III.2.3.

Unfortunately, this method is not always applicable. However, if the theory T is an extension of KPi^0 with $|\mathsf{T}| < \Phi_0$, then $\mathsf{T} + (\mathsf{psh}')$ is consistent, and moreover, $|\mathsf{T}| = |\mathsf{T} + (\mathsf{psh}')|$. This is the main result of the next subsection.

III.1.4 Extending theories by (psh')

In order to show that extending a subsystem of admissible set theory by the principle (psh') does not increase its proof-theoretic ordinal, we apply a more general result by Jäger and Probst [25]. In this article, an extension of Schütte's famous Boundedness Theorem (cf. [39]) is proved, which then yields that $|\mathsf{T}| = |\mathsf{T}^\dagger|$ for a wide range of theories, where $\mathsf{T}^\dagger$ is the theory $\mathsf{T} + \neg\mathsf{TI}^*_\triangleleft(\mathsf{U}, |\mathsf{T}|)$. For completeness' sake and since we use a slightly different definition of the proof-theoretic ordinal of a theory T, we summarize the ideas and theorems from [25].

Schütte's Boundedness Theorem states that there is a close relationship between the cut-free provability of the assertion $\mathsf{TI}_\prec(\mathsf{U}, t)$ within PA^* and the ordinal $|t|_\prec$, i.e. the ordertype of the primitive recursive well-ordering $\prec\restriction t^\mathbb{N}$, where $t^\mathbb{N}$ is the value of the closed term t in the standard model. A cut-free PA^* proof of depth, more or less, $|t|_\prec$ is required in order to establish within PA^* that the initial segment of $\prec\restriction t$ is well-ordered.

Theorem III.1.6 (Boundedness Theorem) *Let $\prec$ be some primitive recursive well-ordering. For any closed number term t of L_1 and any ordinal α we have that*

$$\mathsf{PA}^* \overset{\alpha}{\underset{0}{\vdash}} \mathsf{TI}_\prec(t) \Longrightarrow |t|_\prec \leq \omega\alpha.$$

The proof of this lemma is given in Schütte [39] in all details; alternatively it can also be found in Pohlers [28].

Now we turn to the variation or extension of Schütte's theorem. The crucial step is the following lemma whose proof is tailored according to a corresponding lemma in Schütte [39].

Lemma III.1.7 *Let $\prec$ be a primitive recursive well-ordering, $\mathbb{F}$ the set of all false literals of PA^* and α, β ordinals less than the ordertype $|{\prec}|$ of $\prec$. Further, suppose that we are given two sets Γ and Δ of closed formulas of L_1 and two finite sets M_+ and M_- of closed number terms of L_1, so that the following assumptions are satisfied:*

(i) $M_+ \neq \emptyset$ *and* $\beta = \mathsf{min}\{|r|_\prec : r \in M_+\}$,

(ii) $\{|r|_\prec : r \in M_+\} \cap \{|r|_\prec : r \in M_-\} = \emptyset$,

(iii) $\Delta \subseteq \{\neg \mathsf{Prog}_\prec(\mathsf{V})\} \cup \{r \in \mathsf{V} : r \in M_+\} \cup \{r \notin \mathsf{V} : r \in M_-\} \cup \mathbb{F}$,

(iv) the relation symbol V *does not occur in* Γ,

(v) $\mathsf{PA}^* \overset{\alpha}{\underset{0}{\vdash}} \Gamma, \Delta$ *and* $\omega\alpha \leq \beta$.

Then we even have that $\mathsf{PA}^* \overset{\alpha}{\underset{0}{\vdash}} \Gamma$.

Proof: Almost literally as the corresponding proof in [25]. □

To state the following theorem we introduce the notion of a *normal* theory.

Definition III.1.8 *A theory T is called normal, if $|\mathsf{T}|$ is ω-closed, i.e. $\alpha < |\mathsf{T}|$ implies that $\omega\alpha < |\mathsf{T}|$, and if for each finite set Γ of closed formulas of L_1*

$$\mathsf{T} \vdash \Gamma^* \Longrightarrow \mathsf{PA}^* \overset{<|\mathsf{T}|}{\underset{0}{\vdash}} \Gamma,$$

where $\cdot^*$ is the standard translation from L_1 to the language of T.

Each theory T that we treat in this thesis is normal according to the above definition. This claim is proved as soon as it becomes relevant, i.e. then we consider the theory $\mathsf{T}^\dagger$.

For our purposes, the following variant of the main theorem in [25] suffices.

Theorem III.1.9 *For each normal theory we have that $|\mathsf{T}| = |\mathsf{T}^\dagger|$.*

Proof: Suppose that $\Phi_0 > |\mathsf{T}^\dagger| > |\mathsf{T}| =: \lambda$ and that T is a normal theory. Then there is a primitive recursive well-ordering $\prec$ and an $l \in \mathbb{N}$, such that $|\mathsf{cs}_l|_\prec = \lambda$ and $\mathsf{T}^\dagger$ proves $\mathsf{TI}^*_\prec(\mathsf{V}, \mathsf{cs}_l)$. Since T is a normal theory, we also obtain that

$$\mathsf{PA}^* \overset{<\lambda}{\underset{0}{\vdash}} \mathsf{TI}_\lhd(\mathsf{U}, \lambda), \mathsf{TI}_\prec(\mathsf{V}, \mathsf{cs}_l),$$

which readily implies that

$$\mathsf{PA}^* \overset{<\lambda}{\underset{0}{\vdash}} \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \lambda \in \mathsf{U}, \neg\mathsf{Prog}_{\prec}(\mathsf{V}), \mathsf{cs}_l \in \mathsf{V}.$$

An application of the main lemma yields thus

$$\mathsf{PA}^* \overset{<\lambda}{\underset{0}{\vdash}} \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \lambda \in \mathsf{U},$$

hence $\mathsf{PA}^* \overset{<\lambda}{\underset{0}{\vdash}} \mathsf{TI}_{\lhd}(\mathsf{U}, \lambda)$, which contradicts the definition of $|\mathsf{T}|$. □

Next, we show that KPi^0 extended by $\neg\mathsf{TI}_{\lhd}(\mathsf{U}, |\mathsf{KPi}^0|)$ proves the principle (psh'), which implies $|\mathsf{KPi}^0| = |\mathsf{KPi}^0 + (\mathsf{psh}')|$. That KPi^0 is a normal theory follows form its proof-theoretic analysis carried out in [19].

Lemma III.1.10 *Let* T *be a normal theory formulated in* $\mathcal{L}^*_{\mathsf{Ad}}$ *that comprises the axioms and rules of* KPi^0 *with* $|\mathsf{T}| < \Phi_0$. *Then, for all* Σ *formulas* $A(u, \vec{v})$ *of the language of* T, *the following is provable in the theory* $\mathsf{T}^\dagger$*:*

$$\forall \vec{y} \neg \forall x [A(x, \vec{y}) \leftrightarrow \mathsf{Wo}(x)].$$

Proof: The claim cannot hold, otherwise we could prove the ordinal $|\mathsf{T}|$ in $\mathsf{T}^\dagger$: Assume, that there is a Σ formula $A(u, \vec{v})$ and sets $\vec{z}$ such that for all orderings $\prec$, we have $A(\prec, \vec{z}) \leftrightarrow \mathsf{Wo}(\prec)$. However, this implies that the standard translation of $(\Pi^1_1\text{-}\mathsf{CA})$ becomes provable in $\mathsf{T}^\dagger$: If $B(\vec{V}, u, \vec{v})$ is a Π^1_1 formula of L_2, then the representation theorem II.1.8 for Π^1_1 formulas yields that for all $\vec{Y} \subseteq \mathsf{N}$ and $\vec{y} \in \mathsf{N}$,

$$X := \{x \in \mathsf{N} : B(\vec{Y}, x, \vec{y})\} = \{x \in \mathsf{N} : \mathsf{Wo}(\mathsf{KB}(T^{\neg B}_{\vec{Y}, x, \vec{y}}))\}.$$

Our assumption and $(\Delta\text{-}\mathsf{Sep})$ imply that X is a set. But $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA})$ proves $|\mathsf{T}|$ (cf. e.g. [6]), hence also $\mathsf{T}^\dagger$. □

A slightly more sophisticated argument yields that we have pseudo-hierarchies along our notation system.

Lemma III.1.11 *Let* T *be a normal theory with* $|\mathsf{T}| < \Phi_0$ *formulated in a language comprising* $\mathcal{L}^*$. *Further, we assume that* T *proves the standard translation of each axiom and rule of* ACA_0. *Then the following is provable in* $\mathsf{T}^\dagger$*:*

$$\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}(\lhd \restriction \alpha)\} \textit{ is not a set.}$$

Proof: Since ACA_0 is contained in T, we prove the claim in $\mathsf{ACA}_0 + \neg\mathsf{TI}_{\lhd}(\mathsf{U}, |\mathsf{T}|)$. Assume that $S := \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}(\lhd \restriction \alpha)\}$ is a set. Then $|\mathsf{T}| < \Phi_0$ implies that $S \subsetneq \mathsf{Field}(\lhd)$. But now, we have that S is the least fixed point Fix^A of the accessible part operator induced by $A(U, u) := (\forall \beta \lhd u)(\beta \in U)$. Moreover, we have $\mathsf{Wo}(\lhd \restriction S)$.

By induction on n, we now show that $S_n := \{\alpha : (\forall\beta \in S)(\varphi n\alpha\beta \in S)\} \supseteq S$. To perform the induction step, it suffices to show $\mathsf{Prog}_{\lhd}(S_{n+1})$. We fix an α and assume that $(\forall\alpha' \lhd \alpha)(\alpha' \in S_{n+1})$ and then show that $\mathsf{Prog}_{\lhd}(\{\beta \in S : \varphi(n{+}1)\alpha\beta \in S\})$, which yields $\alpha \in S_{n+1}$. We just consider the case where α is of the form $\alpha'{+}1$. Then $\xi := \varphi(n{+}1)\alpha 0 \in S$ follows because ξ is the limit of the sequence $\gamma_0 = 0$ and $\gamma_{n+1} := \varphi(n{+}1)\alpha'\gamma_n$. That $\eta := \varphi(n{+}1)\alpha(\beta{+}1) \in S$ follows because η is the limit of the sequence $\gamma_0 = \varphi(n{+}1)\alpha\beta{+}1$ and $\gamma_{n+1} := \varphi(n{+}1)\alpha'\gamma_n$. Finally, $\varphi(n{+}1)\alpha\lambda \in S$ for a limit λ follows from the continuity of the function $\xi \mapsto \varphi(n{+}1)\alpha\xi$. So we have that $\mathsf{T}^\dagger$ proves $(\forall\alpha < \varphi\omega 00)(\alpha \in S)$. Since $\mathsf{Wo}(\lhd\restriction S)$, this yields $(\forall\alpha < \varphi\omega 00)\mathsf{TI}_{\lhd}(\mathsf{U}, \alpha)$, which in turn forces $\mathsf{TI}_{\lhd}(\mathsf{U}, \varphi\omega 00)$.

Similarly, one shows that $\Phi_0 \subseteq S$. If we had introduced a notation system with ordertype $|\mathsf{ID}_1|$, we could prove for each ordinal below $|\mathsf{ID}_1|$ that its corresponding notation is in S by following the well-ordering proof for ID_1 given in [28]. Because $|\mathsf{T}| < \Phi_0 < |\mathsf{ID}_1|$, $\mathsf{T}^\dagger$ proves $\mathsf{TI}_{\lhd}(\mathsf{U}, |\mathsf{T}|)$, contradiction $\neg\mathsf{TI}_{\lhd}(\mathsf{U}, |\mathsf{T}|)$. □

The existence of pseudo-hierarchies along $\lhd$ and the principle (psh') are immediate from the lemma below.

Lemma III.1.12 *Let* T *be a normal theory formulated in* $\mathcal{L}^*_{\mathsf{Ad}}$ *that comprises the axioms and rules of* KPi^0 *with* $|\mathsf{T}| < \Phi_0$. *Further, assume that* $A(u_1, u_2, v, \vec{w}, z)$ *and* $B(u, \vec{w}, z)$ *are* Σ *formulas of* $\mathcal{L}^*$ *or* $\mathcal{L}^*_{\mathsf{Ad}}$, *respectively. Then* $\mathsf{T}^\dagger$ *proves: If* $\forall x \mathsf{Op}^2_A(\vec{a}, x)$ *and*

(i) if for all orderings $\prec$, $\mathsf{Wo}(\prec) \to \exists f(\mathsf{hier}^A(f, \vec{a}, \prec) \land B(f, \vec{a}, \prec))$, *then there exists an ordering* $\prec'$ *and a function* g *such that* $\mathsf{psh}^A(g, \vec{a}, \prec')$ *and* $B(g, \vec{a}, \prec')$.

(ii) if for all $\alpha \in \mathsf{Field}(\lhd)$, $\mathsf{Wo}(\alpha) \to \exists f(\mathsf{hier}^A(f, \vec{a}, \alpha) \land B(f, \vec{a}, \alpha))$, *then there exists a* $\beta \in \mathsf{Field}(\lhd)$ *and a function* g *such that* $\mathsf{psh}^A(g, \vec{a}, \lhd\restriction\beta)$ *and* $B(g, \vec{a}, \beta)$.

Proof: The failure of (i) contradicts lemma III.1.10. If (ii) fails for some Σ formulas A, B, then (Δ-Sep) yields that $\{\alpha : \mathsf{Wo}(\lhd\restriction\alpha)\}$ is a set, contradicting the lemma above. □

In $\mathsf{T}^\dagger$, we not only can apply *standard pseudo-hierarchy arguments*, i.e. arguments that involve (psh'), but also perform pseudo-hierarchy arguments on the field of $\lhd$, as justified by (ii) of the previous lemma. These kind of pseudo-hierarchy arguments prove also useful in subsystems of second order arithmetic. For example, we immediately obtain the following lemma:

Lemma III.1.13 $\mathsf{ATR}^\dagger_0$ *proves:*

$$\mathsf{Wo}(\prec) \to \exists\alpha[\prec \text{ is isomorphic to } \lhd\restriction\alpha].$$

Proof: Suppose that $\mathsf{Wo}(\prec)$. Lemma II.2.12 in combination with theorem II.2.11 yields a model M of Σ^1_1-AC above $\prec$. Since $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}(\lhd\restriction\alpha)\}$ is not a set, there is an $\alpha \in \mathsf{Field}(\lhd)$ such that $\mathsf{Wo}^M(\lhd\restriction\alpha)$ but $\neg\mathsf{Wo}(\lhd\restriction\alpha)$. As in the proof of theorem II.2.11, we conclude that $\prec$ is isomorphic to a proper initial segment of $(\lhd\restriction\alpha)$. The claim follows. □

Moreover, we can define proper initial segments of the natural numbers without a top element which enable us to carry out yet another kind of pseudo-hierarchy arguments. As an example, we work in Σ^1_1-$\mathsf{AC}^\dagger_0$ and construct a "very small" Σ^1_1 definable class which is a fixed point of the accessible part operator F^A induced by $A(U,u) := (\forall\beta \lhd u)(\beta \in U)$. In particular, this Σ^1_1 definable fixed point is smaller than the Π^1_1 definable fixed point $\mathsf{Fix}^A := \{x : \forall X[\mathsf{Cl}^A(X) \to x \in X]\}$ from subsection II.2.5.

The notation introduced in subsection II.2.3 comes in handy. Recall that we write $\mathsf{FHier}^A(F,n)$ to express that for all $m < n$, $(F)_m = F^A(\bigcup_{m'<m}(F)_{m'})$, and that $(F)_k \neq \emptyset$ implies $k < n$. Now we consider the class

$$\mathfrak{N} \;:=\; \{n : \exists F \mathsf{Hier}^{\mathcal{J}}(F,n)\}.$$

Clearly, $\mathfrak{N}$ is inductive, it contains 0 and $n \in \mathfrak{N}$ implies that $n{+}1 \in \mathfrak{N}$. Moreover, $\mathfrak{N} \neq \mathsf{N}$, for otherwise we had that $\exists F \mathsf{Hier}^{\mathcal{J}}(F,\omega)$, which in turn yields $\mathsf{TI}_\lhd(\varepsilon_0)$. Since $|\Sigma^1_1\text{-}\mathsf{AC}_0| = \varepsilon_0$, this contradicts $\neg\mathsf{TI}_\lhd(\mathsf{U}, |\Sigma^1_1\text{-}\mathsf{AC}_0|)$. Therefore, $\mathfrak{N}$ is a proper subclass of N.

Using lemma I.2.15, Σ^1_1-AC_0 proves that there exists an F with $\mathsf{FHier}^A(F, \lhd\restriction\omega)$. It follows that

$$\mathsf{FIX}^A := \{x : (\exists n \in \mathfrak{N})\exists F \mathsf{FHier}^A(F,n)\}$$

is a fixed point of F^A: The monotonicity of the operator F^A yields $\mathsf{FIX}^A \subseteq F^A(\mathsf{FIX}^A)$. However, if there is an $x \in F^A(\mathsf{FIX}^A)$, again by the monotonicity of F^A, we have $x \in (F)_n$ for each $n > \mathfrak{N}$. But $\{n : x \in (F)_n\}$ has a least element n_0 which is in $\mathfrak{N}$. Therefore $\mathsf{FIX}^A = F^A(\mathsf{FIX}^A)$. Clearly, FIX^A is a proper subclass of Fix^A.

If we work in $\mathsf{KPi}^0 + (\mathsf{psh}')$, then $\mathfrak{N} := \{n \in \mathsf{N} : \exists f \mathsf{hier}^+(f,\emptyset,n)\}$ is a proper subclass of N, that allows to perform similar arguments. The class $\mathfrak{N}$ cannot equal N, for otherwise we could prove the ordinal Γ_0.

To illustrate the use of standard pseudo-hierarchy arguments, we show that there is a close connection between fixed point and iteration principles.

III.1.5 Fixed point principles vs. iteration principles

In this subsection, we investigate the relationship between an iteration and a fixed point principle over KPi^0. A similar question has been researched by Avigad [2]. In this article, Avigad shows among other things, that over ACA_0 the iteration principle (ATR) and the fixed point principle (FP) are equivalent. The following material is partly taken from Probst [32].

The iteration principle that we consider allows us to iterate a Σ operation f^A along an arbitrary well-ordering: For all Σ formulas $A(u_1, u_2, v, \vec{w}, z)$ of $\mathcal{L}^*_{\mathsf{Ad}}$ that contain at most the displayed variables free,

$$(\Sigma\text{-}\mathsf{tr}) \qquad \mathsf{Wo}(\prec) \wedge \mathsf{Op}^2_A(\vec{a}, \prec) \rightarrow \exists f \, \mathsf{hier}^A(f, \vec{a}, \prec).$$

An equivalent iteration principle is analyzed in Jäger and Probst [24]. It is shown there, that $|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})| = \varphi\omega 00$.

The corresponding fixed point principle (Σ-fp') claims the existence of fixed points of monotone Σ operations acting on the entire universe. Looking for a compact formulation, we set for all formulas $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$,

$$\mathsf{Mon}_A(\vec{a}) := \forall x, y[x \subseteq y \rightarrow f^A_{\vec{a}}(x) \subseteq f^A_{\vec{a}}(y)],$$

to express that $f^A_{\vec{a}}$ is monotone. The principle (Σ-fp'), takes the following form: For each Σ formula $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$ with at most the variables $u, v, \vec{w}$ free, we have

$$(\Sigma\text{-}\mathsf{fp}') \qquad \mathsf{Op}^1_A(\vec{a}) \wedge \mathsf{Mon}_A(\vec{a}) \rightarrow \exists x[\mathcal{S}(x) \wedge f^A_{\vec{a}}(x) = x].$$

Below, we argue that over $\mathsf{KPi}^0 + (\mathsf{psh}')$ the iteration principle (Σ-tr) and the fixed point principle (Σ-fp') are equivalent.

Theorem III.1.14 *For each Σ formula $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$ with at most the variables $u, v, \vec{w}$ free, we have that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr}) + (\mathsf{psh}')$ proves:*

$$\mathsf{Op}^1_A(\vec{a}) \wedge \mathsf{Mon}_A(\vec{a}) \rightarrow \exists x[\mathcal{S}(x) \wedge f^A_{\vec{a}}(x) = x].$$

Proof: Suppose that $A(u, v, \vec{w})$ is a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$ and $\vec{a}$ is such that $\mathsf{Op}^1_A(\vec{a})$ and $\mathsf{Mon}_A(\vec{a})$. In order to obtain a hierarchy g where the αth level $g(\alpha)$ is the result of applying $f^A_{\vec{a}}$ to the union $\bigcup_{\beta \prec \alpha} g(\beta)$ of the levels below α rather than to $g{\restriction}\alpha$, we let the formula $B(u, v, \vec{w})$ be the formula $\exists y[y = \bigcup \mathsf{Rng}(u) \wedge A(y, v, \vec{w})]$, which implies that $f^B_{\vec{a}}(x) = f^A_{\vec{a}}(\bigcup \mathsf{Rng}(x))$. Now, the pseudo-hierarchy principle (psh') yields a g and an ordering $\prec$ such that

$$\mathsf{psh}^B(g, \vec{a}, \prec) \wedge \mathsf{Wo}^{g^+}(\prec).$$

The choice of B implies that for each $\alpha \in \mathsf{Field}(\prec)$, $g(\alpha) = f^A_{\vec{a}}(\bigcup_{\beta \prec \alpha} f(\beta))$. Since that set

$$\{\alpha : (\forall \beta \prec \alpha)(g(\beta) \subseteq g(\alpha))\}$$

is in g^+, transfinite induction along $\prec$ in g^+ yields that the pseudo-hierarchy is still monotone, i.e. that we have $g(\alpha) \subseteq g(\beta)$ if $\alpha \prec \beta$. For a non-empty, upward closed $k \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element, we now set $b := \bigcup_{\alpha \prec k} g(\alpha)$ and argue that b is a fixed point of $f^A_{\vec{a}}$: By the monotonicity of the hierarchy we have for each $\alpha \prec k$ and each $\beta \in k$ that $g(\alpha) \subseteq b \subseteq g(\beta)$. By the monotonicity of the operation $f^A_{\vec{a}}$ we conclude that $b \subseteq f^A_{\vec{a}}(b) \subseteq g(\alpha)$, for each $\alpha \in k$. On the other hand, if $x \in f^A_{\vec{a}}(b)$, then $\{\alpha : x \in g(\alpha)\}$ is a non-empty set in g^+ that has a $\prec$-least element α_0. Since k has no least element, we infer $\alpha_0 \prec k$, thus $x \in b$. □

For the converse direction we need an auxiliary lemma.

Lemma III.1.15 *There is a* Σ *formula* $A(u, v)$ *of* $\mathcal{L}^*_{\mathsf{Ad}}$ *with only the displayed variables free, such that* KPi^0 *proves:* Op^1_A *and* Mon_A *and*

$$\forall x[f^A(x) = \bigcap\{z : x \subseteq z \wedge \mathsf{Ad}(z)\}].$$

This justifies the notation x° for the set $\bigcap\{z : x \subseteq z \wedge \mathsf{Ad}(z)\}$.

Proof: There is a Σ formula $A(u, v)$ of $\mathcal{L}^*$ such that $A(x, y)$ implies that

$$y = \bigcap\{z \in (x^+)^+ : x \subseteq z \wedge \mathsf{Ad}(z)\}.$$

Since $x^+ \in (x^+)^+$ and $x \subseteq x^+ \wedge \mathsf{Ad}(x^+)$ we are not forming the intersection of the empty set. Because admissibles are linearly ordered by $\in$, an admissible z that is not an element of $(x^+)^+$ satisfies already $x \subseteq x^+ \subseteq z$. Thus, $A(x, y)$ implies $y = x^\circ$. Op^1_A and Mon_A are now obvious. □

The set x° is the intersection of admissible sets, therefore it is a model of (Δ_0-Sep). However, $\mathsf{Ad}(x^\circ)$ may not hold, although, as we will prove later (see lemma III.2.36), x° is a model of (Δ_0-Col). For instance, let f and $\prec$ be such that $\mathsf{psh}^+(f, \prec)$. Then, for a non-empty $k \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element, $a := \bigcup_{\alpha \prec k} f(\alpha)$ does not satisfy $\mathsf{Ad}(a)$, because otherwise, a were an element of each admissible $f(\alpha)$ for $\alpha \in k$, therefore also $a \in a$. Due to (lin), there is for each admissible b with $a \subseteq b$, an $\alpha \in k$ with $f(\alpha) \in b$, hence we conclude $a^\circ = a$. As we will prove later, (see lemma III.2.36), x° is also a model of (Δ_0-Col).

Lemma III.1.16 $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ *proves each instance of* $(\Sigma\text{-}\mathsf{tr})$.

Proof: Suppose that $A(u_1, u_2, v, \vec{w}, z)$ is a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$ and $\vec{a}, \prec$ are sets, such that $\mathsf{Op}^2_A(\vec{a}, \prec)$ and $\mathsf{Wo}(\prec)$ holds. We aim to show $\exists f \mathsf{hier}^A(f, \vec{a}, \prec)$.

By the previous lemma, there is a Σ formula $B(u, v, \vec{w}, z)$ such that $\mathsf{Op}^1_B(\vec{a}, \prec)$ and $\mathsf{Mon}_B(\vec{a}, \prec)$ and

$$f^B_{\vec{a},\prec}(x) = (\{f^A_{\vec{a},\prec}(y_1, y_2) : y_1 \in x \wedge y_2 \in \mathsf{Field}(\prec)\} \cup \{\prec\})^\circ.$$

The principle (Σ-fp') yields a set b that is a fixed point of $f^B_{\vec{a},\prec}$. By transfinite induction along $\prec$, we show that

$$(*) \qquad (\forall \alpha \in \mathsf{Field}(\prec))(\exists! f \in b)\mathsf{hier}^A(f, \vec{a}, \prec\restriction\alpha).$$

So assume that for each $\beta \prec \alpha$, there is exactly one $g \in b$ with $\mathsf{hier}^A(g, \vec{a}, \prec\restriction\beta)$. Since $\mathsf{Wo}(\prec\restriction\beta)$, we also know that there is exactly one g with $g \in b$ and $\mathsf{hier}^A(g, \vec{a}, \prec\restriction\beta)$. By Σ replacement, there is a unique function h with domain α, such that for all $\beta \prec \alpha$, we have $h(\beta) \in b$ and $\mathsf{hier}^A(h(\beta), \vec{a}, \prec\restriction\beta)$. Further, h is an element of each admissible c with $b \subseteq c$, thus $h \in b$. The I.H. yields also that $\bigcup_{\xi \prec \alpha} h(\xi)$ is a function with domain $\mathsf{Field}(\prec\restriction\alpha)$. For $f := f^A_{\vec{a},\prec}(\bigcup_{\xi\prec\alpha} h(\xi), \xi)$ we thus have $\mathsf{hier}^A(f, \vec{a}, \prec\restriction\alpha)$. Moreover, the choice of b implies that $f \in b$. This shows $(*)$. Similarly, we obtain $\exists f \mathsf{hier}^A(f, \vec{a}, \prec)$. □

Since $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})$ is easily embedded into KPm^0, an upper bound is immediate. That $|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})| = \varphi\omega 00$ is shown in [24]. Moreover, the normality of the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})$ follows from the normality of KPm^0, which is sketched in [27] or is obtained by methods used in Rathjen [34]. Hence we obtain:

Theorem III.1.17

$$|\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})| = |\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')| = |\mathsf{KPm}^0| = \varphi\omega 00.$$

The fixed point principle (Σ-fp') that claims the existence of fixed points of monotone Σ operations on the universe is somewhat problematic: If we extend the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ e.g. by ($\mathsf{I}_\in$) to $\mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{fp}')$, then monotone operations become definable by Σ formulas of $\mathcal{L}^*_{\mathsf{Ad}}$ which cannot have fixed points. However, this drawback comes along with an additional feature: All instances of (psh') are provable in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$.

Below, we define a Π^1_1 operation that has no fixed point:

$$\mathsf{o}(x) := \bigcup\{\alpha \cup \{\alpha\} : \alpha \in x \wedge \mathsf{Ord}(\alpha)\}.$$

Lemma III.1.18 KPi^0 *proves that the operation* $x \mapsto \mathsf{o}(x) \cup \{\mathsf{o}(x)\}$ *is monotone but has no fixed point.*

Proof: Note that $\mathsf{o}(x)$ is an ordinal and that $\mathsf{o}(\alpha) = \alpha$ for all ordinals α. Ordinals are linearly ordered by the $\in$ relation, hence the operation is monotone. Since ordinals are well-founded by $\in$, the above operation has no fixed point. □

We have already mentioned in subsection III.1.2, that in $\mathsf{KPi}^{\mathsf{r}}$, being an ordinal is equivalent to a Δ_0 formula of $\mathcal{L}^*$. Thus, the aforementioned operation becomes definable by a Σ formula of $\mathcal{L}^*$, which implies the inconsistency of $\mathsf{KPi}^{\mathsf{r}} + (\Sigma\text{-}\mathsf{fp}')$.

Moreover, the operation $x \mapsto \mathsf{o}(x) \cup \{\mathsf{o}(x)\}$ is definable by a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$ in any extension of KPi^0 where there exists a Σ formula $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$ such that

$$\exists \vec{y} \forall x[\mathsf{Wo}(x) \leftrightarrow A(x, \vec{y})]$$

is provable. Due to the previous lemma, such an extension cannot consistently be further extended by the principle (Σ-fp').

We proceed by relating the consistency of $\mathsf{T} + (\Sigma\text{-}\mathsf{fp}')$ to the consistency of $\mathsf{T} + (\Sigma\text{-}\mathsf{tr})$ for theories T comprising KPi^0. Namely, one can consistently extend a theory T by the principle (Σ-fp') if and only if it is consistent to assume that $\mathsf{Wo}(\prec)$ is not expressible by a Σ formula of $\mathsf{T} + (\Sigma\text{-}\mathsf{tr})$.

Lemma III.1.19 *Let* T *be a theory that comprises* KPi^0*. Then* $\mathsf{T} + (\Sigma\text{-}\mathsf{fp}')$ *is consistent if and only if there is no* Σ *formula* $A(\vec{u}, v, w)$ *of* $\mathcal{L}^*$ *for which* $\mathsf{T} + (\Sigma\text{-}\mathsf{tr})$ *proves* $\exists \vec{y} \forall x[A(x, \vec{y}) \leftrightarrow \mathsf{Wo}(x)]$.

Proof: Suppose that $\mathsf{T} + (\Sigma\text{-}\mathsf{fp}')$ is consistent. Then lemma III.1.16 yields (Σ-tr). If $\mathsf{Wo}(\prec)$ is equivalent to some Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$, then the operation of lemma III.1.18 is definable by a Σ formula of $\mathcal{L}^*_{\mathsf{Ad}}$ as well, which contradicts (Σ-fp').
If we assume the right hand side, we can consistently extend $\mathsf{T} + (\Sigma\text{-}\mathsf{tr})$ by a principle that claims that we have for every Σ formula $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$ that

$$\forall \vec{y} \neg \forall x[A(x, \vec{y}) \leftrightarrow \mathsf{Wo}(x)].$$

But this implies (psh'), thus we can use the proof of theorem III.1.14 to show (Σ-fp'). □

A reformulation of the previous lemma yields that $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ proves each instance of (psh').

Theorem III.1.20 *For each* Σ *formula* $A(u_1, u_2, v, \vec{w}, z)$ *of* $\mathcal{L}^*_{\mathsf{Ad}}$*, the following is provable in* $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$*:*

$$\forall x \mathsf{Op}^2_A(\vec{a}, x) \to \exists g, x[\mathsf{psh}^A(g, \vec{a}, x) \land \mathsf{Wo}^{g^+}(x)].$$

Proof: Suppose that there is a Σ formula A such that the theorem fails. But then, the extension of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ by

$$\exists \vec{a}[\forall x \mathsf{Op}^2_A(\vec{a}, x) \land \forall g, x(\mathsf{hier}^A(g, \vec{a}, x) \to (\mathsf{Wo}(x) \lor \neg \mathsf{Wo}^{g^+}(x))]$$

is consistent. Since $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ proves (Σ-tr), this extension of $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ proves also

$$\exists \vec{a} \forall x[\mathsf{Wo}(x) \leftrightarrow \exists g(\mathsf{hier}^A(g, \vec{a}, x) \land \mathsf{Wo}^{g^+}(x))],$$

which contradicts the previous lemma. □

As a further consequence, we observe that in $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$ the class Ad is not well-ordered. Otherwise, we have that $\mathsf{Wo}(\prec)$ is equivalent to $\exists f \mathsf{hier}^+(f, \prec)$, due to lemma III.1.16. More surprisingly, if we close the class Ad under union, the resulting class $\overline{\mathsf{Ad}} := \{\bigcup x : x \subseteq \mathsf{Ad}\}$ is not linearly ordered by $\in$:

Lemma III.1.21 *The following is provable in* $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp}')$*:*

$$\neg \forall a, b[a \in \overline{\mathsf{Ad}} \land b \in \overline{\mathsf{Ad}} \to a \in b \lor a = b \lor b \in a].$$

Proof: If $\overline{\mathsf{Ad}}$ is linearly ordered by $\in$, then the operation

$$a \mapsto \hat{a} \cup \{\hat{a}\},$$

where $\hat{a} := \bigcup \{x^+ : x \in a\}$, is Σ definable and monotone. But if $a = \hat{a} \cup \{\hat{a}\}$, a is in $\overline{\mathsf{Ad}}$ and contains itself. A contradiction. □

To avoid the aforementioned problems, a restricted fixed point principle is considered in [32], that only claims the existence of fixed points for monotone, Σ definable operations on the power set of the natural numbers. There, we set for all formulas $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$,

$$\begin{aligned}
\mathsf{Op}^{\mathsf{N}}_A(\vec{a}) &:= (\forall x \subseteq \mathsf{N})(\exists! y \subseteq \mathsf{N})A(x, y, \vec{a}),\\
\mathsf{Mon}^{\mathsf{N}}_A(\vec{a}) &:= (\forall x_1, x_2, y_1, y_2 \subseteq \mathsf{N})[A(x_1, y_1, \vec{a}) \land A(x_2, y_2, \vec{a}) \land x_1 \subseteq x_2 \to y_1 \subseteq y_2].
\end{aligned}$$

The restricted fixed point principle $(\Sigma\text{-}\mathsf{fp})$ then takes the following form: For each Σ formula $A(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$ with at most the variables $u, v, \vec{w}$ free, we have

$$(\Sigma\text{-}\mathsf{fp}) \qquad \mathsf{Op}^{\mathsf{N}}_A(\vec{a}) \land \mathsf{Mon}^{\mathsf{N}}_A(\vec{a}) \to (\exists x \subseteq \mathsf{N})A(x, x, \vec{a}).$$

Although, the theory $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{fp})$ no longer proves the iteration principle $(\Sigma\text{-}\mathsf{tr})$, it still has the proof-theoretic strength of KPm^0 (cf. [32]).

III.1.6 On linearity, iteration and choice

We end this section by illustrating how the linearity of the class Ad in combination with some form of iteration is a substitute for dependent choice. Its two constituents, iteration and choice become visible.

Suppose that $A(\vec{W}, U, V)$ is an arithmetical formula of L_2 and that the sets $\vec{Z}$ are such that $\forall X \exists Y A(\vec{Z}, X, Y)$. The requirement for a choice axiom stems from the fact that there is no uniform way to select a witness Y for a given set X, such that $A(\vec{Z}, X, Y)$. However, in the theory KPi^0 we can, thanks to the axiom (lin). To simplify the notation, we identify below L_2 formulas with their standard translation into $\mathcal{L}^*$, and upper case variables are to range over subsets of N.

We fix the set X and the parameters $\vec{Z}$. Now lemma III.1.1 tells us that there is a unique least admissible a that still contains a witness Y:

$$a := \bigcap\{b : \mathsf{Ad}(b) \wedge \vec{Z}, X \in b \wedge (\exists Y \in b)A(\vec{Z}, X, Y)\}.$$

By lemma II.1.7 we have that

$$(\exists Y \in a)A(\vec{Z}, X, Y) \leftrightarrow (\exists \mathcal{F} \in a)\forall n(\mathcal{F}[n] \in T^A_{\vec{Z},X}).$$

Now we define a specific path $\mathcal{F}_0$ through $T^A_{\vec{Z},X}$ by $\mathcal{F}_0(0) := \langle\rangle$ and $\mathcal{F}_0(n{+}1) := m$, where m is the least natural number such that

$$(\exists \mathcal{G} \in a)(\mathcal{G} \text{ is a path through } T^A_{\vec{Z},X} \wedge \mathcal{G}(n{+}1) = \mathcal{F}_0(n) * \langle m\rangle).$$

Clearly, $\mathcal{F}_0 \in a^+$ is a path through $T^A_{\vec{Z},X}$ that is left to all paths $\mathcal{G} \in a$ through $T^A_{\vec{Z},X}$. Again by lemma II.1.7 we conclude that $Y := \{y : \mathsf{WIT}^A(\mathcal{F}_0, y)\}$ satisfies $A(\vec{Z}, X, Y)$.

It is not hard to find a Σ formula $B(u, v, \vec{w})$ of $\mathcal{L}^*_{\mathsf{Ad}}$, such that $\mathsf{Op}^{\mathsf{N}}_B(\vec{Z})$ and moreover, $f^B_{\vec{Z}}(x)$ is the unique witness Y obtained by the above procedure. Depending on the iteration principles available in T, we can derive different forms of dependent choice. For instance $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{I_N})$ proves each instance of $(\Sigma^1_1\text{-}\mathsf{DC})$, and accordingly $\mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr})$ proves each instance of $(\Sigma^1_1\text{-}\mathsf{TDC})$.

Theorem III.1.22 *Let $\cdot^*$ be our standard translation form L_2 to $\mathcal{L}^*$. Then the following holds for all finite sets Γ of L_2 formulas:*

(i) $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC}) \vdash \Gamma \Longrightarrow \mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{I_N}) \vdash \Gamma^*$,

(ii) $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{TDC}) \vdash \Gamma \Longrightarrow \mathsf{KPi}^0 + (\Sigma\text{-}\mathsf{tr}) \vdash \Gamma^*$.

III.2 Admissible sets and linearity

The language of KPi^0 is equipped by a relation symbol $\mathsf{Ad}(u)$ that distinguishes a class $\mathsf{Ad} := \{x : \mathsf{Ad}(x)\}$ of admissibles which are linearly ordered by $\in$ due to the axiom (lin). In this section, we introduce a Δ_0 formula $\mathsf{P_{Ad}}(u)$ of $\mathcal{L}^*$ expressing that the set u is admissible, i.e. a transitive model of $\mathsf{KPu}^0 + (\mathsf{I_N})$. We study ways to construct admissible sets and examine the class $\mathsf{hyp}^x := \bigcap\{y : x \in y \wedge \mathsf{P_{Ad}}(y)\}$. Then, we analyse what happens if we claim the class $\{x : \mathsf{P_{Ad}}(x)\}$ of all the admissible sets to be linearly ordered by $\in$. Finally, we extend KPu^0 by an axiom $(\Delta_0\text{-}\mathsf{dc})$ for dependent choice together with a formula $\mathsf{Ad_{dc}}(u)$ stating that u is a transitive model of $\mathsf{KPu}^0 + (\mathsf{I_N}) + (\Delta_0\text{-}\mathsf{dc})$, which enables us to study Π_2 reflection on the class $\mathsf{Ad_{dc}} := \{x : \mathsf{Ad_{dc}}(x)\}$. For all this, we need the constructible hierarchy $\mathcal{L}$.

III.2.1 The constructible hierarchy $\mathcal{L}$

In chapter two, the jump-hierarchy played a dominating role. Its counterpart in admissible set theory is the constructible hierarchy. Although the constructible hierarchy plays a very similar role as the jump-hierarchy in many respects, its definition is more complicated. This reflects the issue that we are no longer restricted to subsets of N but also have to deal with sets containing sets, sets of sets and possibly also non-wellfounded sets.

The step from one level of the constructible hierarchy to the next is determined by an operation $f^{\mathcal{D}}(u)$ that is defined by a Δ formula $\mathcal{D}(u,v)$ of KPu^0, which in turn is composed of a couple of relatively simple operations f^{A_i} that are again defined by Δ formulas $A_i(u,v,w)$ of $\mathcal{L}^*$, where i ranges over an appropriate index set I. We borrow the formulas A_i, the definition of the constructible hierarchy and most of the proofs concerning its properties from Barwise [3], p.57ff. Only small changes are required to adapt things to our set-up. Given the formulas A_i, there is a Δ formula $\mathcal{D}(u,v)$ of $\mathcal{L}^*$ such that

$$f^{\mathcal{D}}(u) = u \cup \bigcup_{i \in I} \{f^{A_i}(v,w) : v, w \in u\}.$$

A constructible hierarchy h along the ordering $\prec$ above the set x is then a function which meets the conditions:

(i) $\mathsf{Dom}(h) = \mathsf{Field}(\prec) \wedge h(0) = x \cup \mathsf{N}$,

(ii) $(\forall \alpha \in \mathsf{Field}(\prec))[\alpha{+}1 \in \mathsf{Dom}(h) \to h(\alpha{+}1) = f^{\mathcal{D}}(\bigcup h(\alpha) \cup h(\alpha) \cup \{h(\alpha)\})]$,

(iii) $(\forall \lambda \in \mathsf{Field}(\prec))[\lambda \in \mathsf{Dom}(h) \wedge \mathsf{limit}(\lambda) \to h(\lambda) = \bigcup_{\alpha \prec \lambda} h(\alpha)]$.

To be in sync with the notation in [3] we handle the limit case separately. If $h(\alpha)$ is transitive, then $\bigcup h(\alpha) \subseteq h(\alpha)$ and the union with $\bigcup h(\alpha)$ adds nothing new to the argument of $f^{\mathcal{D}}$ of (ii). However, if we build the constructible hierarchy above a set x that is not transitive, then (ii) ensures that the level $h(\omega)$ is transitive.

Next, we choose a Σ formula $\mathcal{L}(u_1,u_2,v,w)$ of $\mathcal{L}^*$ such that for all sets y,z we have $\mathsf{Op}^2_{\mathcal{L}}(y,z)$, and moreover, for all sets g, $f^{\mathcal{L}}_{x,\prec}(g,0_\prec) = x$, if $\alpha{+}1$ is a successor in the field of the linear ordering $\prec$ then $f^{\mathcal{L}}_{x,\prec}(g,\alpha{+}1) = f^{\mathcal{D}}(\bigcup g(\alpha) \cup g(\alpha) \cup \{g(\alpha)\})$, and $f^{\mathcal{L}}_{x,\prec}(g,\lambda) = \bigcup \mathsf{Rng}(g)$, if $\lambda \in \mathsf{Field}(\prec)$ is a limit. With this setting, we obtain that h is a constructible hierarchy along $\prec$ above x according to the aforementioned definition exactly if $\mathsf{hier}^{\mathcal{L}}(h,x,\prec)$.

In analogy to the jump-hierarchy, we write $\mathcal{L}^x_\prec$ for a constructible hierarchy above x along $\prec$. Again, if the context implies that α is an element of the field of $\prec$, then we use $\mathcal{L}^x_\alpha$ for the αth level of this constructible hierarchy. Provided that the underlying ordering of a constructible hierarchy is a well-ordering, there is, provable in KPu^0, exactly one jump-hierarchy $\mathcal{L}^x_\prec$. Hence, the formula $x \in \mathcal{L}^x_\alpha$ is Δ.

The formulas A_i have been chosen such that the following lemmas hold. Their proofs are contained in the proofs of Lemma 6.1 and Theorem 6.4 on page 62ff in [3]. To be precise, we note that Barwise works in a language without function symbols, whereas we have function symbols for all the primitive recursive functions. Observe however, that the translation introduced in subsection I.2.13 can be applied to get rid of these function symbols.

Lemma III.2.1 *The following is provable in* KPu^0*: If* $\prec$ *is a linear ordering and* h *is such that* $\mathsf{hier}^{\mathcal{L}}(h, x, \prec)$ *holds, then*

(i) $\forall \alpha, \beta[\alpha \prec \beta \to h(\alpha) \subseteq h(\beta)]$,

(ii) $(\forall \alpha \in \mathsf{Field}(\prec))[h(\alpha) \in h(\alpha{+}1)]$,

(iii) $(\forall \alpha \in \mathsf{Field}(\prec))[\mathsf{Tran}(h(\alpha)) \to \mathsf{Tran}(h(\alpha{+}1))]$.

The following lemma corresponds to lemma II.1.26.

Lemma III.2.2 *For each* Δ_0 *formula* $B(u, \vec{v})$ *of* $\mathcal{L}^*$ *there exists an* $n \in \mathbb{N}$*, such that the following is provable in* KPu^0*: If* $\prec$ *is a linear ordering without a top element and* h *is such that* $\mathsf{hier}^{\mathcal{L}}(h, x, \prec)$ *holds, then we have for all sequences* $\alpha \prec \alpha_1 \prec \ldots \prec \alpha_{\mathsf{cs}_n}$ *that* $a, \vec{b}, c \in h(\alpha)$ *implies*

$$\{y \in a : B(y, \vec{b})\} \in h(\alpha_{\mathsf{cs}_n}) \quad \textit{and} \quad \{a, c\} \in h(\alpha_1) \quad \textit{and} \quad \bigcup a \in h(\alpha_1).$$

Corollary III.2.3 *For each* Δ_0 *formula* $A(u, \vec{v})$ *of* $\mathcal{L}^*$*, there is an* $n \in \mathbb{N}$ *such that* KPu^0 *proves:*

$$\mathsf{hier}^{\mathcal{L}}(h, x, \lhd{\restriction}\omega) \wedge m \lhd \omega \wedge a, \vec{b} \in h(m) \to \{y \in a : A(y, \vec{b})\} \in h(m{+}\mathsf{cs}_n).$$

Moreover, we have that the ωth level of a constructible hierarchy is always transitive, no matter with what set x we start.

Lemma III.2.4 *The following is provable in* KPu^0*: For any set* x*,* $\mathcal{L}^x_\omega$ *is a transitive set.*

Proof: Suppose that $z \in y \in \mathcal{L}^x_\omega$. Then there in an $n \in \mathsf{N}$ with $y \in \mathcal{L}^x_n$. This yields $z \in \bigcup \mathcal{L}^x_n$, hence $z \in \mathcal{L}^x_{n+1}$. □

Remark III.2.5 *Please note, that for a non-transitive* x*,* $\mathcal{L}^x_\omega$ *is in general not a model of* $\mathsf{BS}^0 + (\mathsf{I_N})$*. For instance, if we define* $z^0 := z$ *and* $z^{n+1} := \{z^n\}$*, and let* $x := \{n^n : n \in \mathsf{N}\}$*, then* $\mathcal{L}^x_\omega$ *does not contain a transitive hull of* x*.*

Now we are ready to give a more refined version of the pseudo-hierarchy principle (psh') for theories T with $|\mathsf{T}| < \Phi_0$ that comprise KPu^0 and prove the existence of $\mathcal{L}^x_\omega$ for each set x. For each Σ formula $A(u_1, u_2, v, \vec{w}, z)$ of $\mathcal{L}^*$,

$$(\mathsf{psh})\ \ \forall x \mathsf{Op}^2_A(\vec{a}, x) \wedge \forall x[\mathsf{Wo}(x) \rightarrow \exists f \mathsf{hier}^A(f, \vec{a}, x)] \rightarrow \exists g, y[\mathsf{psh}^A(g, \vec{a}, y) \wedge \mathsf{Wo}^{\mathcal{L}^g_\omega}(y)].$$

Examining the proofs in the previous section, we find that using (psh) instead of (psh') does not affect the arguments given there. Also the following lemma is obtained in the same way as lemma III.1.12.

Lemma III.2.6 *If* T *is a theory with* $|\mathsf{T}| < \Phi_0$ *that comprises* KPu^0 *and proves for each set* x *the existence of* $\mathcal{L}^x_\omega$*, then* $|\mathsf{T}| = |\mathsf{T} + (\mathsf{psh})|$.

However, we need more detailed information about the build-up of the constructible hierarchy when analyzing dependent choice in admissible set theory in subsection III.2.5. Thus, we cannot avoid the explicit mentioning of the formulas $A_i(u, v, w)$. These are the formulas from [3] specialized to the present situation. For each i from the set $I := \{1, \ldots, 16\}$, we choose $A_i(u, v, w)$ such that the conditions below are satisfied.

$$\begin{array}{lll} f^{A_1}(x, y) = \{x, y\} & f^{A_2}(x, y) = \bigcup x & f^{A_3}(x, y) = x - y \\ f^{A_4}(x, y) = x \times y & f^{A_5}(x, y) = \mathsf{Dom}(x) & f^{A_6}(x, y) = \mathsf{Rng}(x) \end{array}$$

The next two operations are needed for technical reasons to handle the asymmetry in the forming of ordered pairs (cf. [3] p. 63 ff).

$$\begin{array}{rcl} f^{A_7}(x, y) & = & \{(u, v, w) : (u, v) \in x \wedge w \in y\} \\ f^{A_8}(x, y) & = & \{(u, w, v) : (u, v) \in x \wedge w \in y\} \end{array}$$

For the primitive recursive relations, we have the operation

$$f^{A_9}(x, y) = \{\langle u_1, \ldots, u_n\rangle : \vec{u} \in \mathsf{N} \wedge x \in \mathsf{Prim} \wedge \pi(x, 1) = n \wedge \{x\}(\vec{u}) = 0\},$$

if $\pi(x, 1) > 1$, and otherwise

$$f^{A_9}(x, y) = \{u : u \in \mathsf{N} \wedge x \in \mathsf{Prim} \wedge \pi(x, 1) = n \wedge \{x\}(u) = 0\},$$

where $\pi(x, y)$ is the projection functions I.1.5 and Prim the set of indices of the primitive recursive function defined in subsection I.1.7. Recall from subsection II.1.1, that $\langle 10, 1\rangle$ and $\langle 11, 1\rangle$ are indices of the characteristic functions of U and V, respectively. The non-primitive recursive relation symbols of $\mathcal{L}^*$ are taken care of by the next group of operations.

$$f^{A_{10}}(x, y) = \{(u, v) \in x \times y : u = v\}, \qquad f^{A_{11}}(x, y) = \{(u, v) \in x \times y : u \in v\}.$$

In order to satisfy claim (iii) of lemma III.2.2 above, we need some additional operations. Hence we choose $A_{12}(u,v,w), \dots A_{16}(u,v,w)$ such that $f^{A_{12}}(x,y) = (x,y)$ and

$$f^{A_{13}}(x,y) = \begin{cases} (u,v,y) & \text{if } x = (u,v) \\ \emptyset & \text{otherwise} \end{cases} \qquad f^{A_{14}}(x,y) = \begin{cases} \{u,(v,y)\} & \text{if } x = (u,v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$f^{A_{15}}(x,y) = \begin{cases} (u,y,v) & \text{if } x = (u,v) \\ \emptyset & \text{otherwise} \end{cases} \qquad f^{A_{16}}(x,y) = \begin{cases} \{u,(y,v)\} & \text{if } x = (u,v) \\ \emptyset & \text{otherwise} \end{cases}$$

So much to the constructible hierarchy for the moment. We will come back to it in the next subsection where we reason about the class hyp^x, i.e. the intersection of all models of $\mathsf{KPu}^0 + (\mathsf{I_N})$ above x. Thereto, we require a Δ_0 formula $\mathsf{P_{Ad}}(u)$ of $\mathcal{L}^*$ expressing that u satisfies the axioms of $\mathsf{KPu}^0 + (\mathsf{I_N})$. This formula will play the role of the formula $\mathrm{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}}$ of L_2 in second order arithmetic.

III.2.2 A Δ_0 formula expressing admissibility

The goal of this subsection is to reformulate the theory KPi^0 stripped by the axiom (lin) in the language $\mathcal{L}^*$. Since $\mathcal{L}^*$ does not contain the relation symbol $\mathsf{Ad}(u)$, we will define a Δ_0 formula $\mathsf{P_{Ad}}(a)$ of $\mathcal{L}^*$ that expresses that the set a is admissible. To be precise, $\mathsf{P_{Ad}}(a)$ is to imply that a is a transitive set containing N, and that a reflects the Kripke-Platek axioms: For each instance $A(\vec{u})$ of a Kripke-Platek axiom (cf. subsection I.2.9), we have that $\mathsf{KPu}^0 \vdash \mathsf{P_{Ad}}(a) \land \vec{x} \in a \to A^a(\vec{x})$. To obtain such a formula $\mathsf{P_{Ad}}(u)$, we make use of the techniques developed in Barwise [3] to define a universal Σ formula. However, our job is a bit harder, since we are working in a considerably weaker theory.

To begin with, we let $(\Delta_0\text{-}\mathsf{Sep}')$ be the sentence

$$\bigwedge_{i \in I} \forall x, y \exists z (z = f^{A_i}(x,y)),$$

where for $i \in I = \{1, \dots, 16\}$, $A_i(u,v,w)$ is the formula chosen in the previous subsection while specifying the build-up of the constructible hierarchy. The following is then an immediate consequence of corollary III.2.3.

Lemma III.2.7 *For each instance $A(\vec{u})$ of $(\Delta_0\text{-}\mathsf{Sep})$, the following is provable in KPu^0:*

$$\vec{x} \in a \land (\Delta_0\text{-}\mathsf{Sep}')^a \to A^a(\vec{x}).$$

In the course of the subsequent argument, we require transitive sets that are closed under $(\Delta_0\text{-}\mathsf{Sep}')$, (Pair) and union. We distinguish sets with slightly stronger closure properties, namely models of BS^0. The Δ_0 formula

$$\mathsf{P_{BS^0}}(u) := \mathsf{Tran}(u) \land \mathsf{N} \in u \land (\Delta_0\text{-}\mathsf{Sep}')^u \land (\forall x \in u)(\exists y \in u)(x \subseteq y \land \mathsf{Tran}(y)).$$

is to state that u is a model of $\mathsf{BS}^0 + (\mathsf{I_N})$.

If $n \in \mathsf{N}$, we denote by $\mathsf{N}{\restriction}n$ the set $\{x \in \mathsf{N} : x < n\}$. Further, we say that $\mathsf{card}(x) = n$ if there exists a bijective function $(f : \mathsf{N}{\restriction}n \to x)$, and x is called finite if there is an $n \in \mathsf{N}$ with $\mathsf{card}(x) = n$. The set x is finite in b, if there exists a bijection $(f : \mathsf{N}{\restriction}n \to x)$ with $f \in b$. Below, we observe that a set a with $\mathsf{P}_{\mathsf{BS}^0}(a)$ contains all its finite subsets.

Lemma III.2.8 *The following is provable in* KPu^0*:*

$$\mathsf{P}_{\mathsf{BS}^0}(a) \land y \subseteq a \land \mathsf{card}(y) = n \to y \in a.$$

Proof: Suppose that there is an $n \in \mathsf{N}$ such that $(f : \mathsf{N}{\restriction}n \to y)$ is a bijection. Using induction, one shows that $(\forall m <_{\mathsf{N}} n{+}1)(\exists g \in a)(g = f{\restriction}(\mathsf{N}{\restriction}m))$. □

Therefore, if a, b are transitive sets satisfying $(\Delta_0\text{-}\mathsf{Sep}')$ with $x \in a \in b$, then we have for each $n \in \mathsf{N}$, that the set of all subsets of x with n elements,

$$\{y \subseteq x : \mathsf{card}(y) = n\} = \{y \in a : y \subseteq x \land (\exists f \in a)[(f : \mathsf{N}{\restriction}n \to y) \text{ is a bijection}]\}$$

is an element of b.

Our next step is to define formulas $\Delta_0\text{-}\mathsf{Sat}(u, \vec{v})$ of $\mathcal{L}^*$ (for each ariety one), such that for all Δ_0 formulas $A(\vec{v})$ of $\mathcal{L}^*$ that contain no other function symbols than the constants cs_n, the following is provable in KPu^0: If $\mathsf{P}_{\mathsf{BS}^0}(a)$, $\mathsf{P}_{\mathsf{BS}^0}(b)$ and $a \in b$, then

$$(\forall \vec{x} \in a)[A(\vec{x}) \leftrightarrow \Delta_0\text{-}\mathsf{Sat}(\vec{x}, \ulcorner A \urcorner)],$$

where $\ulcorner A \urcorner \in \mathbb{N}$ is the Gödelnumber of the formula A. Among other things, we assume that Fml is the primitive recursive set $\{\ulcorner A \urcorner : A \text{ is a formula of } \mathcal{L}^*\}$, and that there is a function $\mathsf{FV}(\ulcorner A \urcorner)$ that returns the set of the Gödelnumbers of the free variables occurring in A, i.e. $\mathsf{FV}(\ulcorner A \urcorner) = \{\ulcorner u \urcorner : u \in FV(A)\}$. Further, we make use of a function Val satisfying $\mathsf{Val}(\ulcorner \mathsf{cs}_n \urcorner) = \mathsf{cs}_n$.

We say that c is a structure for $\mathcal{L}^*$, denoted by $\mathsf{Struct}(c)$, if c is a transitive set that contains N. Given a structure c and a formula A of $\mathcal{L}^*$, we denote by

$$\mathsf{EV}(c, \ulcorner A \urcorner) := \{g \cup \mathsf{Val} : \mathsf{Fun}(g) \land \mathsf{Dom}(g) = \mathsf{FV}(\ulcorner A \urcorner) \land \mathsf{Rng}(g) \subseteq c\}$$

the set of possible evaluations of the free variables of A in c. Because $\mathsf{EV}(c, \ulcorner A \urcorner)$ is a collection of subsets of $\mathsf{N} \times c$ with the cardinality of $\mathsf{FV}(\ulcorner A \urcorner)$, the previous lemma yields the following:

Lemma III.2.9

$$\mathsf{KPu}^0 \vdash c \in a \in b \land \mathsf{P}_{\mathsf{BS}^0}(a) \land \mathsf{P}_{\mathsf{BS}^0}(b) \to (\forall n \in \mathsf{Fml})(\mathsf{EV}(c, n) \in b).$$

Next, we define the set $\mathsf{Sat}(c, \ulcorner A \urcorner)$. It is supposed to contain all the valuations $s \in \mathsf{EV}(c, \ulcorner A \urcorner)$ under which A evaluates to true. The idea is to define $\mathsf{EV}(c, \ulcorner A \urcorner)$ inductively on the build-up of A. More precisely, $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the unique set satisfying the equations listed below.

The following clauses are to be read as formulas of $\mathcal{L}^*$. So for instance "A is the formula $R_e(t)$" translates to $n \in \mathsf{Fml}$ is of the form $\langle 0, e, s \rangle$ and $s \in \mathsf{Term}$", assuming that Term is the primitive recursive set $\{\ulcorner s \urcorner : s \text{ is a term of } \mathsf{L}_1\}$ and $\langle 0, e, \ulcorner s \urcorner \rangle$ is the Gödelnumber of the formula $R_e(s)$.

1. If A is the formula $[\sim]R_e(\vec{t})$, where R_e is a primitive recursive relation symbol of the relation $\{\vec{x} : \{e\}(\vec{x}) = 0\}$, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : [\neg](s(\ulcorner \vec{t} \urcorner) \in \mathsf{N} \wedge \{e\}(s(\ulcorner \vec{t} \urcorner)) = 0)\}.$$
2. If A is the formula $[\sim]R(t)$, where R is one of the relation symbols $\mathsf{U}, \mathsf{V}, \mathcal{S}$, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : [\sim]R(s(\ulcorner t \urcorner))\}.$$
3. If A is the formula $[\sim](t_1 \in t_2)$, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : [\sim](s(\ulcorner t_1 \urcorner) \in s(\ulcorner t_2 \urcorner))\}.$$
4. If A is of the form $B \vee C$, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : s{\restriction}\mathsf{FV}(\ulcorner B \urcorner) \in \mathsf{Sat}(c, \ulcorner B \urcorner) \vee s{\restriction}\mathsf{FV}(\ulcorner C \urcorner) \in \mathsf{Sat}(c, \ulcorner C \urcorner)\}.$$
5. If A is of the form $B \wedge C$, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : s{\restriction}\mathsf{FV}(\ulcorner B \urcorner) \in \mathsf{Sat}(c, \ulcorner B \urcorner) \wedge s{\restriction}\mathsf{FV}(\ulcorner C \urcorner) \in \mathsf{Sat}(c, \ulcorner C \urcorner)\}.$$
6. If A is of the form $\exists x B$ and x occurs free in B, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : (\exists y \in c)(s \cup \{(\ulcorner x \urcorner, y)\}) \in \mathsf{Sat}(c, \ulcorner B \urcorner)\}.$$
If x does not occur free in B, then $\mathsf{Sat}(c, \ulcorner A \urcorner) := \mathsf{Sat}(c, \ulcorner B \urcorner)$.
7. If A is of the form $\forall x B$ and x occurs free in B, then $\mathsf{Sat}(c, \ulcorner A \urcorner)$ is the set
$$\{s \in \mathsf{EV}(c, \ulcorner A \urcorner) : (\forall y \in c)(s \cup \{(\ulcorner x \urcorner, y)\}) \in \mathsf{Sat}(c, \ulcorner B \urcorner)\}.$$
If x is not free in B, then $\mathsf{Sat}(c, \ulcorner A \urcorner) := \mathsf{Sat}(c, \ulcorner B \urcorner)$.

The relevant property of $\mathsf{Sat}(c, \ulcorner A \urcorner)$ in view of the definition of the formulas Δ_0-Sat is put on record below.

Lemma III.2.10 KPu^0 *proves:*

$$c \in a \in b \wedge \mathsf{P}_{\mathsf{BS}^0}(a) \wedge \mathsf{P}_{\mathsf{BS}^0}(b) \to (\forall n \in \mathsf{Fml})(\mathsf{Sat}(c, n) \in b).$$

Proof: We only give a rough sketch: Assume that we have a primitive recursive function rk that assigns the to the Gödelnumber of a formula its natural rank. Then we show by induction that

$$(\forall n \in \mathsf{N})(\forall m \in \mathsf{Fml})[\mathsf{rk}(m) <_{\mathsf{N}} n \to \mathsf{Sat}(c, m) \in b].$$

□

Next, we require that if $x \in a \in b$ with $\mathsf{P}_{\mathsf{BS}^0}(a)$ and $\mathsf{P}_{\mathsf{BS}^0}(b)$, then the transitive closure $\mathsf{TC}(x)$ of x,

$$\mathsf{TC}(x) := \bigcap \{y : \mathsf{Tran}(y) \wedge x \subseteq y\},$$

is an element of b.

Lemma III.2.11

$$\mathsf{KPu}^0 \vdash x \in a \in b \wedge \mathsf{P}_{\mathsf{BS}^0}(a) \wedge \mathsf{P}_{\mathsf{BS}^0}(b) \to \mathsf{TC}(x) \in b.$$

Proof: Let a be such that $x \in a$ and $\mathsf{P}_{\mathsf{BS}^0}(a)$. Induction yields that for all $n \in \mathsf{N}$, there exists a unique $f \in a$ such that

$$A(f, n) := \mathsf{Fun}(f) \wedge \mathsf{Dom}(f) = \mathsf{N}\restriction(n{+}1) \wedge f(0) = x \wedge (\forall m < n) f(m{+}1) = \bigcup f(m).$$

It follows that $\mathsf{TC}(x) = \{y \in a : (\exists n \in \mathsf{N})(\exists f \in a)(A(f, n) \wedge y \in \bigcup \mathsf{Rng}(f))\}$ is an element of b. □

Given a formula $A(u_1, \dots, u_n)$ of $\mathcal{L}^*$ and sets $x_1, \dots, x_n$, we write $s^A_{\vec{x}}$ for the valuation that maps the Gödelnumbers of the free variables in A to the set $\{x_1, \dots, x_n\}$.

Definition III.2.12 *Let* Δ_0-$\mathsf{Sat}(\vec{u}, v)$ *be the following formula of* $\mathcal{L}^*$*:*

$$v \in \mathsf{Fml} \wedge \mathsf{Dom}(s^A_{\vec{u}}) = \mathsf{FV}(v) \wedge s^A_{\vec{u}} \in \mathsf{Sat}(\mathsf{TC}(\{\mathsf{N}, \vec{u}\}), v).$$

Clearly, Δ_0-$\mathsf{Sat}(\vec{u}, v)$ is equivalent to a Σ formula of $\mathcal{L}^*$. A (meta-) induction on the build-up of the formula A now yields:

Lemma III.2.13 *Let* $A(\vec{u})$ *be a* Δ_0 *formula of* $\mathcal{L}^*$*. Then* KPu^0 *proves: If* $\mathsf{P}_{\mathsf{BS}^0}(a)$, $\mathsf{P}_{\mathsf{BS}^0}(b)$ *and* $a \in b$*, then*

$$(\forall \vec{x} \in a)[A(\vec{x}) \leftrightarrow \Delta_0\text{-}\mathsf{Sat}(\vec{x}, \ulcorner A \urcorner)].$$

This leads immediately to a universal Σ formula of $\mathcal{L}^*$:

Lemma III.2.14 (Universal Σ formula for $\mathcal{L}^*$) *For each Σ formula $A(\vec{u})$ of $\mathcal{L}^*$, there is an $e \in \mathsf{Fml} \subseteq \mathbb{N}$, such that $\mathsf{KPu}^0 + \forall x \exists y(x \in y \land \mathsf{P}_{\mathsf{BS}^0}(y))$ proves:*

$$\forall \vec{x}[A(\vec{x}) \leftrightarrow \exists y \Delta_0\text{-}\mathsf{Sat}(\vec{x}, y, \mathsf{cs}_e)].$$

Towards the definition of our Δ_0 formula $\mathsf{P}_{\mathsf{Ad}}(u)$ of $\mathcal{L}^*$ expressing admissibility, we let (Δ_0-Col') be the sentence

$$(\forall e \in \mathsf{Fml})\forall a, b[(\forall x \in a)\exists y \Delta_0\text{-}\mathsf{Sat}_3(x, y, b, e) \to \exists z(\forall x \in a)(\exists y \in z)\Delta_0\text{-}\mathsf{Sat}_3(x, y, b, e)],$$

and finally define define

$$\mathsf{P}_{\mathsf{Ad}}(u) := \mathsf{P}_{\mathsf{BS}^0}(u) \land (\forall y \in u)(\exists z \in u)(y \in z \land \mathsf{P}_{\mathsf{BS}^0}(z)) \land (\Delta_0\text{-}\mathsf{Col}')^u.$$

Clearly, $\mathsf{P}_{\mathsf{Ad}}(u)$ is Δ_0, and moreover, satisfies the other expected properties.

Lemma III.2.15 *If $A(\vec{u})$ is an instance of a Kripke-Platek axiom, then the theory KPu^0 proves:*

$$\vec{x} \in a \land \mathsf{P}_{\mathsf{Ad}}(a) \to A^a(\vec{x}).$$

Proof: This follows since $\mathsf{P}_{\mathsf{Ad}}(a)$ implies that $(\forall x \in a)(\exists y \in a)(x \in y \land \mathsf{P}_{\mathsf{BS}^0}(y))$, hence KPu^0 proves

$$\mathsf{P}_{\mathsf{Ad}}(a) \to [\forall x(\Delta_0\text{-}\mathsf{Sat}(\vec{x}, \ulcorner A \urcorner) \leftrightarrow A(\vec{x}))]^a,$$

for all Δ_0 formulas A of $\mathcal{L}^*$. The claim follows. □

Of course, a distinguished admissibles $\mathsf{Ad}(a)$ satisfies $\mathsf{P}_{\mathsf{Ad}}(a)$ as well.

Lemma III.2.16

$$\mathsf{KPu}^0 \vdash \mathsf{Ad}(a) \to \mathsf{P}_{\mathsf{Ad}}(a).$$

Proof: Clearly, $\mathsf{Ad}(a)$ implies $\mathsf{P}_{\mathsf{BS}^0}(a)$ and $(\forall y \in x)(\exists z \in x)(y \in z \land \mathsf{P}_{\mathsf{BS}^0}(z))$. And $(\Delta_0\text{-}\mathsf{Col}')^a$ follows by Σ collection in a. □

The efforts taken in this subsection led us to a Δ_0 formula $\mathsf{Ad}(u)$ of $\mathcal{L}^*$ that allows us to talk about admissible sets, which enables us to formulate the theory KPi^0 without the linearity axiom in the language $\mathcal{L}^*$. We call this reformulation KPj^0. It extends KPu^0 by the following axiom:

(lim′) $$\exists y(x \in y \land \mathsf{P}_{\mathsf{Ad}}(y)).$$

The next theorem states that the theory KPi^0 stripped by the axiom (lin), below denoted by KPi^-, proves the same $\mathcal{L}^*$ formulas as KPj^0. A formula A of $\mathcal{L}^*_{\mathsf{Ad}}$ translates to the $\mathcal{L}^*$ formula $A[\mathsf{P}_{\mathsf{Ad}}/\mathsf{Ad}]$ that is obtained from A by substituting every literal of the form $\mathsf{Ad}(t)$ by $\mathsf{P}_{\mathsf{Ad}}(t)$ and every literal of the form $\sim\!\mathsf{Ad}(t)$ by $\neg\mathsf{P}_{\mathsf{Ad}}(t)$.

Theorem III.2.17 *For all finite sets* Γ *of* $\mathcal{L}^*$ *formulas and all finite sets* Δ *of* $\mathcal{L}^*_{\mathsf{Ad}}$ *formulas we have:*

(i) $\mathsf{KPj}^0 \vdash \Gamma \Longrightarrow \mathsf{KPi}^- \vdash \Gamma$,

(ii) $\mathsf{KPi}^- \vdash \Delta \Longrightarrow \mathsf{KPj}^0 \vdash \Delta[\mathsf{P}_{\mathsf{Ad}}/\mathsf{Ad}]$.

Proof: (i) follows from lemma III.2.15. (ii) follows from lemma III.2.16 and the fact that $\mathsf{P}_{\mathsf{Ad}}(x)$ is Δ_0. □

It seems natural to ask, how the assertion that the class $\{x : \mathsf{P}_{\mathsf{Ad}}(x)\}$ is linearly ordered by $\in$, affects the proof-theoretic strength of the theory KPj^0. Since the class of all admissible sets $\{x : \mathsf{P}_{\mathsf{Ad}}(x)\}$ is in general bigger than the class Ad of admissibles distinguished by the predicate $\mathsf{Ad}(u)$, this assertion is much stronger than the axiom (lin). After the next subsection, we learn that KPj^0 extended by such a linearity axiom has the same proof-theoretic strength as Δ^1_2-CA_0. A closer look at the class hyp leads to this result.

III.2.3 On hyp

The formula $\mathsf{P}_{\mathsf{Ad}}(u)$ introduced in the previous subsection allows us to speak about admissible sets. Whenever we say that a set x is admissible, we mean that $\mathsf{P}_{\mathsf{Ad}}(x)$ holds. In this subsection we examine the class hyp^x, the intersection of all admissible sets that contain x,

$$\mathsf{hyp}^x := \bigcap\{a : x \in a \wedge \mathsf{P}_{\mathsf{Ad}}(a)\}.$$

A similar class $\mathbb{H}\mathsf{YP}_{\mathcal{M}}$, the intersection of all admissibles above $\mathcal{M}$, has been studied exhaustively in Barwise [3]. However, Barwise's notion of an admissible is different: He calls a set a admissible if it is a transitive model of $\mathsf{KPu}^0 + (\mathsf{I_N})$ plus full $\in$ induction, i.e. for all formulas $A(u)$ of $\mathcal{L}^*_{\mathsf{Ad}}$,

$$\forall x[(\forall y \in x)A(y) \to A(x)] \to \forall x A(x).$$

Moreover, he assumes that a is an element of the cumulative hierarchy and that the interpretation of the element relation in a is the restriction of the $\in$ relation of the cumulative hierarchy to a. In particular, his admissibles are well-founded.

In the present context, admissibles can be seen as equivalents of models of Σ^1_1-AC in second order arithmetic. Indeed, hyp^x shares many of the properties of HYP^X presented in subsection II.2.1. Our first result is that already KPu^0 proves that hyp^x is an admissible class, which corresponds to theorem II.2.20. To structure the proof, we distinguish between the following three cases: There is no admissible above x, hyp^x is a set, and there are admissibles above x but hyp^x is not a set. We start with the first case which is trivial:

Lemma III.2.18 *The following is provable in* KPu^0*: If there is no admissible above* x*, then* hyp^x *is an admissible class.*

Proof: If there is no admissible set above x, then $\mathsf{hyp}^x = \{u : u = u\}$, which is obviously an admissible class. □

In the remaining cases, there are admissibles above x, hence for each well-ordering $\prec \in \mathsf{hyp}^x$, also the constructible hierarchy $\mathcal{L}^x_\prec$ exists and is an element of hyp^x.

Pseudo-hierarchies play an important role in the subsequent arguments. However, a pseudo-hierarchy principle is not required: On the one hand, hyp^x is admissible follows from the assumption that hyp^x is a set. On the other hand, the assumption that hyp^x is not a set, implies the existence of a pseudo-hierarchy which helps us to prove that hyp^x is an admissible class. How pseudo-hierarchies are used to construct admissible sets is the subject of the next lemma.

Lemma III.2.19 (Construction of admissible sets) *The following is provable in* KPu^0*: Suppose that* $\mathsf{psh}^{\mathcal{L}}(f, x, \prec)$*, that* $\mathcal{L}^f_\omega$ *exists and* $\mathsf{Wo}^{\mathcal{L}^f_\omega}(\prec)$*. For a non-empty* $k \subseteq \mathsf{Field}(\prec)$ *without a* $\prec$*-least element,*

$$a := \bigcup_{\xi \prec k} f(\xi)$$

is an admissible set that does not contains f*. Moreover, if* $\prec' \in a$ *is a well-ordering, then* $\prec'$ *is isomorphic to a proper initial segment of* $\prec$*, and the corresponding order isomorphism is an element of the admissible set* a*.*

Proof: Since the field of $\prec\restriction k$ has no biggest element, a is a transitive set that satisfies (Δ_0-Sep') by lemma III.2.2. To show that a satisfies each instance of (Δ_0-Col), let $A(u, v, \vec{w})$ be a Δ_0 formula of $\mathcal{L}^*_{\mathsf{Ad}}$ and $b, \vec{c} \in a$ such that

$$(\forall y \in b)(\exists z \in a)A(y, z, \vec{c}).$$

Then k is a subset of

$$\{\alpha \in \mathsf{Field}(\prec) : (\forall y \in b)(\exists z \in f(\alpha))A(y, z, \vec{c})\}.$$

Now $\mathsf{Wo}^{\mathcal{L}^f_\omega}(\prec)$ yields that this set has a $\prec$-least element $\alpha_0 \prec k$, which yields that $(\forall y \in b)(\exists z \in f(\alpha_0))A(y, z, \vec{c})$. If f were an element of a, then $\bigcup \mathsf{Rng}(f) \in a$ and $\bigcup \mathsf{Rng}(f) = a$. However, an admissible cannot contain itself. The second part of the lemma is proved as the corresponding claim of theorem II.2.11.

□

Lemma III.2.20 *The following is provable in* KPu^0*: If there is an admissible above* x*, then we have for all orderings* $\prec \in \mathsf{hyp}^x$*,*

$$\mathsf{Wo}(\prec) \leftrightarrow (\exists f \in \mathsf{hyp}^x)[\mathsf{hier}^{\mathcal{L}}(f, x, \prec) \wedge \mathsf{Wo}^{\mathcal{L}^f_\omega}(\prec)].$$

Proof: Since the existence of admissibles above x is assumed, $f \in \mathsf{hyp}^x$ yields that f is in an admissible above x, thus $\mathcal{L}^f_\omega$ exists.
The direction from left to right is straight forward. If the converse direction fails, then there are $\prec, f \in \mathsf{hyp}^x$ with $\mathsf{psh}^{\mathcal{L}}(f, x, \prec)$ and $\mathsf{Wo}^{\mathcal{L}^f_\omega}(\prec)$. Lemma III.2.19 provides now an admissible that does not contain f, contradicting the definition of hyp^x. □

Next, we consider the case where hyp^x is a set and show that this implies that hyp^x is admissible. In this case, the ordinals in hyp^x provide a lot of information about hyp^x. Thereto, we define

$$\mathsf{on}(a) := \{\alpha \in a : \mathsf{Ord}(\alpha)\}.$$

In the next paragraphs, ordinals appear quite frequently. If the context implies that α is an ordinal, we often write $\mathsf{hier}^{\mathcal{L}}(f, x, \alpha)$ instead of $\mathsf{hier}^{\mathcal{L}}(f, x, \in\restriction\alpha)$, where $\in\restriction\alpha$ is the set $\{(y, z) : y \in z \wedge y, z \in \alpha\}$. In connection with ordinals, we also speak of hereditarily transitive sets, sets that are transitive and contain only transitive sets: $\mathsf{HTran}(u) := \mathsf{Tran}(u) \wedge (\forall x \in u)\mathsf{Tran}(u)$.

Lemma III.2.21 *The following is provable in* KPu^0*: If* hyp^x *is a set, then* $\mathsf{on}(\mathsf{hyp}^x)$ *is a set as well. Moreover,* $\bigcup \mathsf{on}(\mathsf{hyp}^x)$ *is the least ordinal not in* hyp^x*.*

Proof: If hyp^x is a set, then, by the previous lemma, $\mathsf{on}(\mathsf{hyp}^x)$ is a set by Δ separation. Thus $\alpha := \bigcup \mathsf{on}(\mathsf{hyp}^x)$ is an ordinal. Because an ordinal does not contain itself, $\alpha \notin \mathsf{hyp}^x$. □

If a is an admissible and $\mathsf{on}(a)$ is a set, then $\lambda := \bigcup \mathsf{on}(a)$, the least ordinal not in a, is called *the ordinal of* a. The ordinal λ is admissible in the following sense:

Lemma III.2.22 *The following is provable in* KPu^0*: If* $x \in a \wedge \mathsf{P}_{\mathsf{Ad}}(a)$ *and* $\mathsf{on}(a)$ *is a set, then* $\lambda := \bigcup \mathsf{on}(a)$ *is an ordinal and* $\mathcal{L}^x_\lambda$ *is an admissible subset of* a*.*

Proof: Since λ is a limit, $\mathcal{L}^x_\lambda$ satisfies (Δ_0-Sep'), and it remains to show that $\mathcal{L}^x_\lambda$ satisfies (Δ_0-Col). First, we show this under the assumption that for all $\alpha \in a$,

$$(*) \qquad \mathsf{Ord}(\alpha) \leftrightarrow (\exists f \in a)[\mathsf{hier}^{\mathcal{L}}(f, x, \alpha) \wedge \mathsf{Ord}^{\mathcal{L}^f_\omega}(\alpha)].$$

Suppose that $A(u, v)$ is Δ_0, that $b \in \mathcal{L}^x_\lambda$ and $(\forall y \in b)(\exists z \in \mathcal{L}^x_\lambda)A(y, z)$. This implies readily that $(\forall y \in b)(\exists \alpha \in a)(\mathsf{Ord}(\alpha) \wedge (\exists z \in \mathcal{L}^x_\alpha)A(y, z)$. Using $(*)$ and applying Σ collection in a then yields an ordinal $\gamma \in a$ such that

$$(\forall y \in b)(\exists \alpha \in \gamma)(\exists z \in \mathcal{L}^x_\alpha)A(y, z).$$

If the direction from right to left of $(*)$ fails, then there are $f, \alpha \in a$ with $\mathsf{Ord}^{\mathcal{L}^f_\omega}(\alpha)$ and $\mathsf{psh}^{\mathcal{L}}(f, x, \alpha)$. Further, the second part of lemma III.2.19 yields $\lambda \subseteq \alpha$. Now again, suppose that $A(u, v)$ is a Δ_0 formula of $\mathcal{L}^*$, that $b \in \mathcal{L}^x_\lambda$ and moreover, $(\forall y \in b)(\exists z \in \mathcal{L}^x_\lambda)A(y, z)$. Now

$$\{\gamma \in \alpha : (\forall y \in b)(\exists z \in f(\gamma))A(y, z)\}$$

has a least $\in$-element γ_0. Since $\alpha - \gamma := \{\beta \in \alpha : \beta \notin \gamma\}$ has no $\in$-least element, γ_0 is already below λ, so $f(\gamma_0) = \mathcal{L}^x_{\gamma_0} \in \mathcal{L}^x_\lambda$.
Because λ is an ordinal, $\alpha \in \lambda$ implies $\mathcal{L}^x_\alpha \in a$. $\mathcal{L}^x_\lambda \subseteq a$ follows. □

As a consequence, we obtain the following corollary, which settles case two.

Corollary III.2.23 KPu^0 *proves: If* hyp^x *is set, then* hyp^x *is admissible and for* $\lambda := \bigcup \mathsf{on}(\mathsf{hyp}^x)$*, we have* $\mathsf{hyp}^x = \mathcal{L}^x_\lambda$.

To show that hyp^x is an admissible class in the case where hyp^x is not a set, we first observe that in this case all ordinals are already contained in hyp^x.

Lemma III.2.24 *The following is provable in* KPu^0*: If there is an admissible above* x*, and* hyp^x *is not a set, then*

$$\mathsf{Ord}(\alpha) \to \alpha \in \mathsf{hyp}^x.$$

Proof: We assume that there is an ordinal $\beta \notin \mathsf{hyp}^x$ and argue for a contradiction. Since there are admissible sets above x, there is also an admissible a above x with $\beta \notin a$. Then,

$$\mathsf{on}(a) = \{\alpha \in \beta : (\exists f \in a)\mathsf{hier}^{\mathcal{L}}(f, x, \alpha)\}.$$

For $\lambda := \bigcup \mathsf{on}(a)$, lemma III.2.22 yields that $\mathcal{L}^x_\lambda$ is an admissible above x. Hence, the set

$$\{\gamma \in \beta{+}1 : \mathsf{P}_{\mathsf{Ad}}(\mathcal{L}^x_\gamma)\}$$

is not empty, thus it has a $\in$-least element γ_0. But then, $\mathsf{hyp}^x = \mathcal{L}^x_{\gamma_0}$, contradicting the assumption that hyp^x is not a set. □

Now we are in the position to clear case three. Note, that the conclusion of the next lemma follows also without the assumption "hyp^x is not a set", as we have seen in the above corollary. However, it is required for the proof.

Lemma III.2.25 *The following is provable in* KPu^0*: If there is an admissible above* x*, and* hyp^x *is not a set, then*

$$\mathsf{hyp}^x = \bigcup_{\alpha \in \mathsf{on}(\mathsf{hyp}^x)} \mathcal{L}^x_\alpha,$$

and hyp^x *is an admissible class.*

Proof: First, we argue that the two classes are equal: It is easy to see that the class on the right hand side is a subclass of hyp^x. For the other direction, we pick a y such that for all ordinals $\alpha \in \mathsf{hyp}^x$, $y \notin \mathcal{L}^x_\alpha$ and establish that then also $y \notin \mathsf{hyp}^x$. If α is an ordinal, then $\alpha \in \mathsf{hyp}^x$ by the previous lemma. Further, in each admissible above x, there exists and f with $\mathsf{hier}^{\mathcal{L}}(f, x, \alpha)$. Since such an f is unique, we have $f \in \mathsf{hyp}^x$, hence $y \notin \mathsf{Rng}(f)$. Therefore, if a is an admissible above x, we have for all sets $\alpha \in a$,

$$(*) \qquad \mathsf{Ord}(\alpha) \to (\exists f \in a)[\mathsf{hier}^{\mathcal{L}}(f, x, \alpha) \land \mathsf{Ord}^{\mathcal{L}^f_\omega}(\alpha) \land y \notin \mathsf{Rng}(f)].$$

Next, we argue, that the implication from right to left of $(*)$ fails: Otherwise, $\mathsf{on}(a)$ were a set by Δ separation. Then, $\lambda := \bigcup \mathsf{on}(a)$ were an ordinal not in a, contradicting the previous lemma. Hence, there are $\alpha_0, g \in a$, such that $\mathsf{psh}^{\mathcal{L}}(g, x, \alpha_0)$, $\mathsf{Ord}^{\mathcal{L}^g_\omega}(\alpha_0)$ and $y \notin \mathsf{Rng}(g)$. Moreover, there is a non-empty $k \subset \alpha_0$ without an $\in$-least element. But then, according to lemma III.2.19, $b := \bigcup_{\xi \in \bigcap k} g(\xi)$ is an admissible above x that does no contain y.

Next, we make use of the above g and α_0 to show that hyp^x an admissible class. The second part of lemma III.2.19 yields $\mathsf{Ord}(\beta) \to \beta \in \alpha_0$, thus also $\mathcal{L}^x_\beta = g(\beta)$. To see that hyp^x satisfies (Δ_0-Col), let $A(u, v)$ be a Δ_0 formula of $\mathcal{L}^*$ and c an element of hyp^x, such that we have $(\forall y \in c)(\exists z \in \mathsf{hyp}^x)A(y, z)$. By the first part of this lemma and the above considerations, we conclude that

$$(\forall y \in c)(\exists \gamma \in \mathsf{on}(\mathsf{hyp}^x))(\exists z \in g(\gamma))A(y, z).$$

Since all ordinals are elements of α_0, the set

$$\{\gamma \in \alpha_0 : (\forall y \in c)(\exists z \in g(\gamma))A(y, z)\}.$$

is not empty and has a $\in$-least element γ_0. It follows that γ_0 is an ordinal. Otherwise, there were a $\beta \in \gamma_0$ that is also not an ordinal, contradicting the choice of γ_0. □

Hence, we managed to prove in all three cases that hyp^x is an admissible class. This result is summarized in the next theorem.

Theorem III.2.26 *The following is provable in* KPu^0*:* hyp^x *is an admissible class. If there is an admissible above* x*, then* $\mathsf{hyp}^x = \bigcup_{\alpha \in \mathsf{on}(\mathsf{hyp}^x)} \mathcal{L}^x_\alpha$.

Next, we strengthen lemma III.2.20 such that it corresponds to lemma II.2.18. For its proof, we require *weak admissibles.*

Definition III.2.27 (Weak admissible sets) *We call a transitive set* b *weakly admissible, if it satisfies* (Δ_0-Sep') *and if* $x, \prec \in b$ *and* $\mathsf{Wo}(\prec)$ *imply* $\mathcal{L}^x_\prec \in b$.

Lemma III.2.28 *The following is provable in* KPu^0*: Suppose that* $\mathsf{psh}^{\mathcal{L}}(f, x, \prec)$ *and that* $k \subseteq \mathsf{Field}(\prec)$ *is non-empty, upward closed and has no* $\prec$*-least element. Then the set*

$$a := \bigcap_{\xi \in k} f(\xi),$$

is weakly admissible and does not contain f*.*

Proof: Similar to the proof of theorem II.2.5. We only show that

$$x, \prec \in a \wedge \mathsf{Wo}(\prec) \rightarrow \mathcal{L}^x_{\prec} \in a.$$

This is done by transfinite induction on $\prec$. Assume that for all $\beta \prec \alpha$ we have $\mathcal{L}^x_{\prec\restriction\beta} \in f(\xi)$ for all $\xi \in k$. Since $\prec \in a$ yields that $\prec \in f(\xi)$ for all $\xi \in k$, lemma III.2.2 yields an $n \in \mathbb{N}$ such that for each sequence $\alpha_n \prec \ldots \prec \alpha_1$ with elements from k, $\mathcal{L}^x_{\prec\restriction\alpha} \in f(\alpha_1)$. Since k has no least element, $\mathcal{L}^x_{\prec\restriction\alpha} \in a$ follows. If f were in a, then we had $a = \mathsf{Rng}(f)$, so $a \in a$. However, the proof of lemma III.1.5 tells us that $(\Delta_0\text{-}\mathsf{Sep}')^a$ yields $a \notin a$. □

Lemma III.2.29 *The following is provable in* KPu^0*: If there is an admissible above* x*, then*

$$\mathsf{hyp}^x = \bigcap\{a : a \textit{ is weakly admissible }\}.$$

Proof: If $y \in \mathsf{hyp}^x$, then theorem III.2.26 yields an ordinal $\alpha \in \mathsf{hyp}^x$ such that $y \in \mathcal{L}^x_{\alpha}$. Thus, y is an element of each weak admissible set above x. For the converse direction, assume that $y \notin \mathsf{hyp}^x$. But then there is already an admissible a that with $y \notin a$. Since each admissible is also weakly admissible, the claim follows. □

Lemma III.2.30 *The following is provable in* KPu^0*: If there is an admissible above* x*, then*

$$\alpha \in \mathsf{hyp}^x \wedge \mathsf{Ord}(\alpha) \leftrightarrow \mathsf{HTran}(\alpha) \wedge (\exists f \in \mathsf{hyp}^x)\mathsf{hier}^{\mathcal{L}}(f, x, \alpha).$$

Proof: It remains to show the direction from right to left. Assume that there is a hereditarily transitive α and an $f \in \mathsf{hyp}^x$ with $\mathsf{psh}^{\mathcal{L}}(f, x, \alpha)$. Then lemma III.2.28 yields a weak admissible above x that does not contain f. A contradiction to the assumption $f \in \mathsf{hyp}^x$. □

We conclude this subsection by drawing some conclusions on how admissibles relate to each other, depending on whether hyp^x is a set.

Lemma III.2.31 *The following is provable in* KPu^0*: If there is an admissibles above* x*, and* hyp^x *is not a set, then there are admissibles* a, b *above* x *such that we have* $a \notin b \wedge a \neq b \wedge b \notin a$*.*

Proof: Otherwise, the admissibles above x were linearly ordered by $\in$, and we could adapt the proof of lemma III.1.1 to show that hyp^x is a set. □

Lemma III.2.32 *The following is provable in* KPu^0*: If there is an admissible above* x*, and* hyp^x *is not a set, then no admissible above* x *is well-founded.*

Proof: If hyp^x is not, then an admissible a above x contains already each ordinal. Further, $\mathsf{on}(a)$ is not a set, due to theorem III.2.26. Hence, there is a hereditarily transitive set α and a function f with $\alpha, f \in a$ and $\mathsf{psh}^{\mathcal{L}}(f, x, \alpha)$ and $\mathsf{Ord}^a(\alpha)$. But then, an admissible constructed according to lemma III.2.19 contains a β such that $\beta \in \alpha$ with $\mathsf{Ord}^a(\beta)$ and $\neg\mathsf{Ord}(\beta)$. □

Lemma III.2.33 *The following is provable in* KPu^0*: If* hyp^x *is a set, then* x *is well-founded if and only if* hyp^x *is well-founded.*

Proof: Recall that $\mathsf{hyp}^x = \bigcup_{\alpha \in \mathsf{on}(\mathsf{hyp}^x)} \mathcal{L}^x_\alpha$. The claims follows now by transfinite induction on $\lambda := \bigcup \mathsf{on}(\mathsf{hyp}^x)$. □

Then working in $\mathsf{KPj}^0 + (\mathsf{psh})$ we can strengthen e.g. lemma III.2.24:

Lemma III.2.34 *The following is provable in* $\mathsf{KPj}^0 + (\mathsf{psh})$*: If* $\mathsf{Wo}(\prec)$*, then* $\prec$ *is isomorphic to an initial segment of* $\lhd$*.*

Proof: Due to lemma III.1.10 (ii), there is an $\alpha \in \mathsf{Field}(\lhd)$ and an f such that $\mathsf{psh}^{\mathcal{L}}(f, \prec, \lhd\restriction\alpha)$ and $\mathsf{Wo}^{\mathcal{L}^f_\omega}(\lhd\restriction\alpha)$. The claim now follows by the second part of lemma III.2.19. □

This allows us to characterize hyp^x as follows:

Corollary III.2.35

$$\mathsf{KPj}^0 + (\mathsf{psh}) \vdash \mathsf{hyp}^x = \bigcup\{\mathcal{L}^x_\alpha : \alpha \in \mathsf{Field}(\lhd) \wedge \mathsf{Wo}(\lhd\restriction\alpha)\}.$$

Finally, we answer a question asked in the previous section. There, we claimed that x° is admissible, although KPi^0 does not prove $\mathsf{Ad}(x^\circ)$.

Lemma III.2.36 *We have that*

$$\mathsf{KPi}^0 \vdash \mathsf{P}_{\mathsf{Ad}}(x^\circ).$$

Proof: It remains to show that x° satisfies (Δ_0-Col). We only need consider the case where $\neg\mathsf{Ad}(x^\circ)$. Then, the set $s := \{a \in (x^+)^+ : \mathsf{Ad}(a) \wedge x \subseteq a\}$ has no $\in$-least element. Now let $A(u, v)$ be a Δ^0 formula of $\mathcal{L}^*_{\mathsf{Ad}}$ and assume that $b \in x^\circ$ is such that $(\forall y \in b)(\exists z \in x^\circ)A(y, z)$. Lemma III.1.1 tells us that

$$\{c : (\forall y \in b)(\exists z \in c)A(y, z) \wedge b \in c \wedge \mathsf{Ad}(c)\}$$

has an $\in$-least element c_0 which is admissible, contains b and moreover, satisfies $(\forall y \in c_0)(\exists z \in c_0)A(y, z)$. Because s has no $\in$-least element, we conclude that $c_0 \in x^\circ$. □

III.2.4 Admissibles linearly ordered by $\in$

In this subsection, we analyze what happens if we force admissible sets to be linearly ordered by $\in$, i.e. we are interested in the extension of KPj^0 by the axiom:

$$(\mathsf{lin}') \qquad \mathsf{P}_{\mathsf{Ad}}(a) \land \mathsf{P}_{\mathsf{Ad}}(b) \to a \in b \lor a = b \lor b \in a.$$

In KPi^0, the admissibles of the class Ad are linearly ordered by $\in$. In general, however, there are many more admissible sets. Thus, it is no surprise that KPj^0 is much stronger as KPi^0. In fact, it turns out that $|\mathsf{KPj}^0 + (\mathsf{lin}')| = |\Delta^1_2\text{-}\mathsf{CA}_0|$.

To obtain this result, we show in a first step, that well-foundedness of a set is – provably in $\mathsf{KPj}^0 + (\mathsf{lin}')$ – equivalent to a Σ formula of $\mathcal{L}^*$. This allows to embed $\mathsf{KPj}^0+(\mathsf{I}_\in)$ into $\mathsf{KPj}^0+(\mathsf{lin}')$. On the other hand, there is an asymmetric interpretation of $\mathsf{KPj}^0+(\mathsf{lin}')$ into KPi^r, which yields $|\mathsf{KPj}^0 + (\mathsf{I}_\in)| = |\mathsf{KPi}^\mathsf{r}|$. That $|\mathsf{KPi}^\mathsf{r}| = |\Delta^1_2\text{-}\mathsf{CA}_0|$ is shown in Jäger [20].

We start with the observation that due to lemma III.2.31, (lin') implies that hyp^x is a set. Hence $\mathsf{KPj}^0 + (\mathsf{lin}')$ proves the existence of a least admissible $y = \mathsf{hyp}^x$ above each set x. Then working in $\mathsf{KPj}^0 + (\mathsf{lin}')$, we denote the set hyp^x also by x^+. Since the relation symbol $\mathsf{Ad}(u)$ is not part of the language $\mathcal{L}^*$ of $\mathsf{KPj}^0+(\mathsf{lin}')$, this should not conflict with our previous use of $\cdot^+$ in theories formulated in $\mathcal{L}^*_{\mathsf{Ad}}$.

Next, we conclude by lemma III.2.30 that in $\mathsf{KPj}^0 + (\mathsf{lin}')$, $\mathsf{Ord}(\alpha)$ is equivalent to the Σ formula of $\mathcal{L}^*$ asserting that $\mathsf{HTran}(\alpha)$ and $(\exists f \in \alpha^+)\mathsf{hier}^{\mathcal{L}}(f, \emptyset, \in\restriction\alpha)$. This leads then to a Σ formula of $\mathcal{L}^*$ that is equivalent to $\mathsf{Wf}(\in\restriction x)$, asserting that x is well-founded with respect to $\in$. To define such a formula, we extend the notion of collapse to sets:

$$\mathsf{Clp}'(f, x) := \begin{cases} \mathsf{Fun}(f) \land \mathsf{Dom}(f) = x \land \\ (\forall y \in x)(f(x) = \{f(z) : z \in y\}). \end{cases}$$

Lemma III.2.37 *The following is provable in* $\mathsf{KPj}^0 + (\mathsf{lin}')$*:*

$$\mathsf{Wf}(\in\restriction x) \leftrightarrow (\exists f \in x^+)[\mathsf{Clp}'(f, x) \land \mathsf{Ord}(\mathsf{Rng}(f))].$$

Proof: Assume that x is well-founded. Then $\mathsf{Clp}'(f, x) \land \mathsf{Clp}'(g, x)$ implies $f = g$. Otherwise, there were an $\in$-least element $x_0 \in x$ for which $f(x_0) \neq g(x_0)$. But then already $f(y) \neq g(y)$ for some $y \in x_0$.
Using this and Σ replacement in x^+, we similarly obtain that for each $y \in \mathsf{TC}(\{x\})$, there exists an $f \in x^+$ with $\mathsf{Clp}'(f, y)$ and $\mathsf{Ord}(\mathsf{Rng}(f))$.

If $\mathsf{Clp}'(f, x)$ and $y \subseteq x$ has no $\in$-least element, then also $\{f(z) : z \in y\}$ has no $\in$-least element, thus $\mathsf{Rng}(f)$ is not an ordinal. □

Therefore, if a formula is derivable in $\mathsf{KPj}^0 + (\mathsf{I}_\in)$, it holds also in the well-founded part of $\mathsf{KPj}^0 + (\mathsf{lin}')$. In the lemma below, Wf denotes the class $\{x : \mathcal{S}(x) \wedge \mathsf{Wf}(\in\restriction x)\}$ of well-founded sets, and A^{Wf} is the formula obtained form A by relativizing bound set variables to the class Wf.

Lemma III.2.38 *For all finite sets $\Gamma(\vec{u})$ of $\mathcal{L}^*$ formulas with exactly the displayed variables free, we have:*

$$\mathsf{KPj}^0 + (\mathsf{I}_\in) \vdash \Gamma(\vec{u}) \Longrightarrow \mathsf{KPj}^0 + (\mathsf{lin}') \vdash \vec{u} \notin \mathsf{Wf}, \Gamma^{\mathsf{Wf}}(\vec{u}).$$

Proof: The lemma is proved by induction on the depth of the proof in $\mathsf{KPj}^0 + (\mathsf{I}_\in)$. Since $\mathsf{Wf}(\in\restriction x)$ is Δ, the Kripke-Platek axioms cause no problems. By lemma III.2.33, $\mathsf{Wf}(\in\restriction x)$ yields $\mathsf{Wf}(\in\restriction \mathsf{hyp}^x)$, hence (lim') holds when relativized to the class Wf. Because well-founded sets only contain well-founded sets, also the relativization of the Kripke-Platek axioms to well-founded admissibles are provable in $\mathsf{KPj}^0 + (\mathsf{lin}')$. And $(\mathsf{I}_\in)$ comes for free in Wf. □

Observe that $\mathsf{KPj}^0 + (\mathsf{I}_\in)$ is the reformulation of KPi^r in the language $\mathcal{L}^*$. Analogously to theorem III.2.17, we obtain:

Lemma III.2.39 *For all finite sets Γ of $\mathcal{L}^*$ formulas and all finite sets Δ of $\mathcal{L}^*_{\mathsf{Ad}}$ formulas we have:*

(i) $\mathsf{KPj}^0 + (\mathsf{I}_\in) \vdash \Gamma \Longrightarrow \mathsf{KPi}^\mathsf{r} \vdash \Gamma$,

(ii) $\mathsf{KPi}^\mathsf{r} \vdash \Delta \Longrightarrow \mathsf{KPj}^0 + (\mathsf{I}_\in) \vdash \Delta[\mathsf{P}_{\mathsf{Ad}}/\mathsf{Ad}]$.

The converse direction of lemma III.2.38 fails, since $(\mathsf{I}_\in)$ does not imply (lin'), however, an asymmetric interpretation does the job. Thereby we write $B^{a,b}$ for the formula that is obtained from B by relativizing all unbounded universal quantifiers in B to a and all unbounded existential quantifiers to b.
It remains to adjust a technical detail. To successfully perform the asymmetric interpretation, we require that a proof of a finite set Γ of $\mathcal{L}^*$ formulas can be transformed into a proof of Γ, where the cut rule is applied only to Σ and Π formulas of $\mathcal{L}^*$. According to Theorem I.3.4, cuts that are neither Π nor Σ can be eliminated, provided the main formulas of each axiom and rule are only Σ or Π formulas of $\mathcal{L}^*$. To achieve this, we reformulate $(\Delta_0\text{-}\mathsf{Col})$ as a rule: For each Δ_0 formula $B(u, v)$ of $\mathcal{L}^*$,

$$\frac{\Gamma, (\forall x \in w)\exists y B(x, y)}{\Gamma, \exists z (\forall x \in w)(\exists y \in z) B(x, y)}.$$

The asymmetric interpretation is now straight forward.

Lemma III.2.40 *We let $*$ be the set of all Π and Σ formulas of $\mathcal{L}^*$ and assume that $\Gamma(\vec{u})$ is a finite set of formulas of $\mathcal{L}^*$, such that for some natural number $n \in \mathbb{N}$, $\mathsf{KPj}^0 + (\mathsf{lin}') \mathrel{|\!\frac{n}{*}} \Gamma(\vec{u})$. Then, we have for all natural numbers $m \in \mathbb{N}$:*

$$\mathsf{KPj}^0 + (\mathsf{I}_\in) \vdash \neg\mathsf{hier}^+(f, \emptyset, m{+}2^n), \vec{u} \notin f(m), \Gamma^{f(m),f(m+2^n)}(\vec{u}).$$

For each natural number $n \in \mathbb{N}$, $\mathsf{KPj}^0 + (\mathsf{lin}')$ proves the existence of a hierarchy f with $\mathsf{hier}^+(f, \emptyset, n)$. This immediately implies that $\mathsf{KPj}^0 + (\mathsf{lin}')$ and $\mathsf{KPj}^0 + (\mathsf{I}_\in)$ prove the same Π_2 formulas of $\mathcal{L}^*$. Putting things together, we conclude that also KPi^r and $\mathsf{KPj}^0 + (\mathsf{lin}')$ prove the same Π_2 formulas of $\mathcal{L}^*$. Applying the aforementioned result in [20], we can state the following theorem.

Theorem III.2.41

$$|\mathsf{KPi}^\mathsf{r}| = |\mathsf{KPj}^0 + (\mathsf{lin}')| = |\Delta^1_2\text{-}\mathsf{CA}_0|$$

III.2.5 Dependent choice in admissible set theory

In subsection III.1.6, we have seen that the axiom (lin), asserting that the elements of the class Ad are linearly ordered by $\in$, is some substitute for dependent choice. In this subsection, we take this matter further. We introduce the theory KPd^0, that extends KPu^0 by an axiom for dependent choice and then prove in KPm^0, making use of the axiom (lin) and pseudo-hierarchy arguments, the existence of so-called n-inaccessible models $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$. This yields that KPu^0 extended by an axiom for Π_2 reflection on models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$ is another theory with the meta-predicative Mahlo ordinal $\varphi\omega00$.

We start by introducing the theory KPd^0, which is formulated in $\mathcal{L}^*$ and extends KPu^0 by an axiom for dependent choice: For each Δ_0 formula $A(u, v, \vec{w})$ of $\mathcal{L}^*$,

$$(\Delta_0\text{-}\mathsf{dc}) \quad \forall x \exists y A(x, y, \vec{z}) \to \exists f[\mathsf{Fun}(f) \land f(0) = a \land (\forall n \in \mathsf{N}) A(f(n), f(n{+}1), \vec{z})].$$

It is easy to show that KPd^0 proves already the following form of $(\Sigma\text{-}\mathsf{dc})$.

Lemma III.2.42 *For each Σ formula $A(u, v, n)$ of $\mathcal{L}^*$, the following is provable in KPd^0:*

$$(\forall n \in \mathsf{N}) \forall x \exists y A(x, y, n) \to$$
$$\exists f[\mathsf{Fun}(f) \land \mathsf{Dom}(f) = \mathsf{N} \land f(0) = a \land \forall n A(f(n), f(n{+}1), n)].$$

To speak about models of KPd^0, we define a Δ_0 formula $\mathsf{Ad}_\mathsf{dc}(u)$ of $\mathcal{L}^*$, expressing that u is a model of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$:

$$\mathsf{Ad}_\mathsf{dc}(u) := \mathsf{P}_\mathsf{Ad}(u) \land (\forall e \in \mathsf{Fml})(\forall a, b \in u)[\forall x \exists y \Delta_0\text{-}\mathsf{Sat}(x, y, a, e) \to$$
$$\exists f(\mathsf{Fun}(f) \land f(0) = b \land (\forall n \in \mathsf{N}) \Delta_0\text{-}\mathsf{Sat}(f(n), f(n{+}1), a, e))]^u.$$

The properties of the formula Δ_0-Sat (cf. lemma III.2.13) guarantee that $\mathsf{Ad_{dc}}(u)$ distinguishes admissibles that satisfy (Δ_0-dc):

Lemma III.2.43 *For each instance $A(\vec{u})$ of (Δ_0-dc), the following is provable in KPu^0:*

$$\mathsf{Ad_{dc}}(a) \wedge \vec{z} \in a \to A^a(\vec{z}).$$

Now we are ready to present the theory KPdm^0. It is also formulated in $\mathcal{L}^*$ and extends KPu^0 by axioms for Π_2 reflection on models of $\mathsf{KPd}^0 + (\mathsf{I_N})$: For each Δ_0 formula of $\mathcal{L}^*$

$$(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad_{dc}}} \quad \forall x \exists y A(x,y,\vec{z}) \to \exists a[\mathsf{Ad_{dc}}(a) \wedge \vec{z} \in a \wedge (\forall x \in a)(\exists y \in a) A(x,y,\vec{z})].$$

We will show that KPdm^0 is yet another theory with $|\mathsf{KPdm}^0| = \varphi\omega 00$. This theory corresponds to the theory $\mathsf{ATR}_0 + (\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$ of second order arithmetic, introduced and analyzed in [37, 38] that extends ATR_0 by the following axiom:

$$(\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}} \quad \forall X \exists Y A(X,Y,\vec{Z}) \to$$
$$\exists M[(\mathrm{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \vec{Z} \,\dot{\in}\, M \wedge (\forall X \,\dot{\in}\, M)(\exists Y \,\dot{\in}\, M) A(X,Y,\vec{Z})].$$

Also $\mathsf{ATR}_0 + (\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$ has strength $\varphi\omega 00$, and it seems that this theory is contained in KPdm^0, which is indeed the case as we will see below. However, it is not as simple as it appears at first sight: If $A(U,V,\vec{W})$ is an arithmetical formula of L_2 and $\vec{z} \subseteq \mathsf{N}$ are such that $(\forall x \subseteq \mathsf{N})(\exists y \subseteq \mathsf{N}) A^*(x,y,\vec{z})$, then $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad_{dc}}}$ provides a set a above $\vec{z}$ with $\mathsf{Ad_{dc}}(a)$ and $[(\forall x \subseteq \mathsf{N})(\exists y \subseteq \mathsf{N}) A^*(x,y,\vec{z})]^a$. Also, if $\mathsf{Ad_{dc}}(a)$, then $a \cap \mathcal{P}(\mathsf{N})$ satisfies each instance of (Σ^1_1-DC). But for the translation of $(\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$ to hold, we need a subset of N that codes such a model, i.e. an $M \subseteq \mathsf{N}$ such that $X \in a \cap \mathcal{P}(\mathsf{N}) \leftrightarrow \exists e[(M)_e = X]$. The existence of such an M depends on the existence of a function f that enumerates $a \cap \mathcal{P}(\mathsf{N})$.

When we establish an upper bound of KPdm^0, we observe that strengthening KPdm^0 by an axiom asserting that each set is enumerable does not increase its proof-theoretic ordinal. All the same, to prove the standard translation of an instance of $(\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$, such an additional assertion is not required. We alternatively succeed by applying a nice argument that goes back to Jäger and Strahm [26] and results from [37]. But let us make some remarks on well-orderings first.

If $\preceq_1$ and $\preceq_2$ are reflexive orderings, then we use $\preceq_1 + \preceq_2$ for the ordering with field $\preceq_1 \times \{0\} \cup \preceq_2 \times \{1\}$ where (α, m) is smaller than (β, n) if $m <_\mathsf{N} n$ or if $m = n = 0$ and $\alpha \prec_1 \beta$ or if $m = n = 1$ and $\alpha \prec_2 \beta$. Since in KPdm^0 well-orderings are comparable, it makes sense to define a partial binary operation $+_\prec$ on the field of a well-ordering $\prec$ in the following way: For $\alpha, \beta \in \mathsf{Field}(\prec)$, we let $\alpha +_\prec \beta$ be the

$\prec$-least γ such that the ordering $\preceq\restriction\alpha+\preceq\restriction\beta$ is isomorphic to $\preceq\restriction\gamma$, provided such a γ exists. If such a γ exists for all $\alpha, \beta \in \mathsf{Field}(\prec)$, then we call the operation $+_\prec$ total.

Theorem III.2.44 *For each Σ formula $A(u,v)$ of $\mathcal{L}^*$, the following is provable in* KPdm^0*: If $\prec\subseteq \mathsf{N}\times\mathsf{N}$ is a well-ordering so that the operation $+_\prec$ is total and $\prec'$ is a proper initial segment of $\prec$, then*

$$\forall x\exists y A(x,y) \to \exists f[\mathsf{Fun}(f)\wedge \mathsf{Dom}(f)=\mathsf{Field}(\prec')\wedge(\forall\alpha\in\mathsf{Field}(\prec'))A(f\restriction\alpha, f(\alpha))].$$

Proof: The restriction to well-orderings whose field is a subset of N stems form the need for fundamental sequences. For each limit λ in the field of $\prec$, we depend on a function l_λ with domain N such that for all n, we have $l_\lambda(n)\prec\lambda$, but for each $\beta\prec\lambda$ there is a m such that $l_\lambda(m)\succ\beta$. It is not hard to see, that for each limit $\lambda\in\mathsf{Field}(\prec)$, where exists a unique fundamental sequence l_λ for λ that meets the conditions below:

(i) $l_\lambda(0)=0_\prec$,

(ii) if $n{+}1\prec\lambda$, then $l_\lambda(n{+}1)=\mathsf{min}_\prec\{\alpha\in\mathsf{Field}(\prec) : n{+}1\prec\alpha\wedge l_\lambda(n)\prec\alpha\}$,

(iii) if $n{+}1\not\prec\lambda$, then $l(n{+}1)=\mathsf{min}_\prec\{\alpha\in\mathsf{Field}(\prec) : l_\lambda(n)\prec\alpha\}$.

In the sequel, we write $\lambda[n]$ for $l_\lambda(n)$ and use $\lambda^-[n]$ to denote the unique γ such that $\lambda[n]+_\prec\gamma=\lambda[n{+}1]$.

Next, we introduce some auxiliary formulas to reason about choice sequences. If $B(u,v)$ is a Σ formula of $\mathcal{L}^*$, then

$$\mathsf{ChSeq}_B(f,x,\prec,\alpha) := \mathsf{Fun}(f)\wedge\mathsf{Dom}(f)=\mathsf{Field}(\preceq\restriction\alpha)\wedge(\forall\beta\prec\alpha)B(f\restriction\beta,f(\beta)),$$

is to express that f is a choice sequence for B along $\preceq\restriction\alpha$. The next formula states that if f is a choice sequence for B along $\preceq\restriction\alpha$, then g is a choice sequence for B along $\preceq\restriction(\alpha+_\prec\beta)$ that extends f.

$$H^B(f,g,\prec,\alpha,\beta) := \mathsf{ChSeq}^B(f,\prec,\alpha)\to[\mathsf{ChSeq}^B(g,\prec,\alpha+_\prec\beta)\wedge(f\restriction\alpha=g\restriction\alpha)].$$

This concludes the preparation and we can start to prove the theorem: Assume that $\forall x\exists y A(x,y)$. By $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad_{dc}}}$, we obtain a set b with $\mathsf{Ad_{dc}}(b)$ that contains $\prec$ and the parameters appearing in A, such that $(\forall x\in b)(\exists y\in b)A^b(x,y)$. Then transfinite induction along $\prec$ yields that

$$S := \{\beta\in\mathsf{Field}(\prec) : (\forall\alpha\in\mathsf{Field}(\prec))(\forall f\in b)(\exists g\in b)H^{A^b}(f,g,\prec,\alpha,\beta)\}$$

is the entire field of $\prec$: The successor case is straight forward; we just treat the limit case. So assume that $\lambda \in \mathsf{Field}(\prec)$ is a limit and that for all $n \in \mathsf{N}$, $\lambda[n] \in S$. We fix an α and conclude

$$(\forall n \in \mathsf{N})(\forall f \in b)(\exists g \in b)H^{A^b}(f, g, \prec, \alpha +_\prec \lambda[n], \lambda^-[n]).$$

Now we choose f_0 such that $\mathsf{ChSeq}^A(f_0, \alpha)$. Using dependent choice in the form of lemma III.2.42 yields a choice function h such that induction along N yields

$$(\forall n \in \mathsf{N})[\mathsf{ChSeq}^{A^b}(h(n), \prec, \alpha +_\prec \lambda[n]) \wedge (h(n)\restriction(\alpha +_\prec \lambda[n]) = h(n{+}1)\restriction(\alpha +_\prec \lambda[n]))].$$

For $g := \bigcup_{n\in\mathsf{N}} h(n)$ we have $\mathsf{ChSeq}^{A^b}(g, \prec, \alpha +_\prec \lambda)$, thus $\lambda \in S$. By persistence, we obtain that

$$\mathsf{Field}(\prec) = \{\beta \in \mathsf{Field}(\prec) : (\forall \alpha \in \mathsf{Field}(\prec))(\forall f \in b)(\exists g \in b)H^A(f, g, \prec, \alpha, \beta)\}.$$

The claim follows. □

The restriction in the previous theorem to orderings $\prec$ where the operation $+_\prec$ is total is not really an issue. As detailed in [36], we can assign to each well-ordering $\prec$ a well-ordering $\prec'$ such that $+_{\prec'}$ is total and $\prec$ is isomorphic to an initial segment of $\prec'$. The field of the ordering $\prec'$ consists of all the finite sequences $(f : \mathsf{N}\restriction(n+1) \to \mathsf{Field}(\prec))$, where $(\forall m <_\mathsf{N} n)(f(m{+}1) \prec f(m))$. And f is $\prec'$-smaller than g, if either the sequence g properly extends f, or if there is an $n \in \mathsf{Dom}(f) \cap \mathsf{Dom}(g)$ such that $f(n) \prec g(n)$ and $f\restriction n = g\restriction n$. It is now straight forward to embed $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{TDC})$ into KPdm^0: Given a Σ^1_1 formula $A(U, V, W)$ of L_2 and a set Z such that (the translation of) $\forall X \exists Y A(X, Y, Z)$ holds, we have to find for each well-ordering $\prec$ on N a choice sequence F that satisfies (the translation of) $(\forall \alpha \in \mathsf{Field}(\prec))A((F)_{\prec\alpha}, (F)_\alpha, Z)$. By the above comment, there is an ordering $\prec'$ that extends $\prec$ and $+_{\prec'}$ is total. Thus, a choice sequence F as desired is obtained using the previous lemma. The slightly more general form of $(\Sigma^1_1\text{-}\mathsf{TDC})$ given in subsection II.1.3 then follows.

Further, it is shown in [37] that the theory $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{TDC})$ proves each instance of the rule $(\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$. Therefore we conclude:

Lemma III.2.45 *The standard translation of each instance of* $(\Pi^1_2\text{-}\mathsf{REF})^{\Sigma^1_1\text{-}\mathsf{DC}}$ *is provable in* KPdm^0.

Next, we determine the upper bound of KPdm^0. The standard way to handle Π_2 reflection on admissibles, as demonstrated e.g. by Jäger and Strahm in [27], is to reduce the reflection axiom to axioms asserting the existence of n-inaccessibles, admissible sets, that contain with a set x also an $n{-}1$-inaccessible above x. We aim to adapt this strategy to reduce Π_2 reflection on models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$ to n-inaccessible models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$. Then, in a next step, we look for a theory where we can build such n-inaccessible models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$. To do so, we need to figure out how to handle the base case, the construction of models of (Δ_0-dc).

In second order arithmetic, we have seen that a pseudo jumphierarchy may give rise to a model of Σ^1_1-DC. On the other hand, a similar construction in admissible set theory, where we replaced the jump-hierarchy by the constructible hierarchy, only lead to an admissible set. The extra ingredient available in second order arithmetic is that every set in some level of a jump-hierarchy can be addressed by an index, and picking the set with the least index helps to handle dependent choice.

We adapt this idea to the present context and resort to enumerable sets. Thereby, we say that f *enumerates* x, if f is a function with $\mathsf{Rng}(f) = x$ and $\mathsf{Dom}(f) \subseteq \mathsf{N}$. The set x is then called *enumerable*. Further, we call a set a *locally countable*, denoted by $\ell\mathsf{c}(a)$, if each set $x \in a$ is enumerable by an function $f \in a$.

This leads to the following notion of n-inaccessibility. For each $n \in \mathbb{N}$, we define a Δ_0 formula $\mathsf{Ia}_n(a)$ that declares a as a locally countable n-inaccessible model of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$:

(i) $\mathsf{Ia}_0(u) := \mathsf{Ad}_\mathsf{dc}(u) \wedge \ell\mathsf{c}(u)$,

(ii) $\mathsf{Ia}_{n+1}(u) := \mathsf{Ad}_\mathsf{dc}(u) \wedge \ell\mathsf{c}(u) \wedge (\forall x \in u)(\exists y \in u)(x \in y \wedge \mathsf{Ia}_n(y))$.

This set-up allows us even to reduce the following stronger form of Π_2 reflection on locally countable models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$. The extension of KPu^0 by the rule below is called $\mathsf{KPdm}\ell\mathsf{c}^0$.
For all finite sets of $\mathcal{L}^*$ formulas and each Δ_0 formula $A(u, v, \vec{w})$,

$$\frac{\Gamma, \forall x \exists y A(x, y, \vec{z})}{\Gamma, \exists a[\vec{z} \in a \wedge \mathsf{Ad}_\mathsf{dc}(a) \wedge \ell\mathsf{c}(a) \wedge (\forall x \in a)(\exists y \in a)A(x, y, \vec{z})]}.$$

Note, that this rule implies in particular that all sets are enumerable.

To obtain an upper bound for the theory $\mathsf{KPdm}\ell\mathsf{c}^0$, we first show, making once again use of pseudo-hierarchies and exploiting the axiom (lin), that for each $n \in \mathbb{N}$,

$$\mathsf{KPm}^0 + \neg\mathsf{TI}_\triangleleft(\mathsf{U}, |\mathsf{KPm}^0|) \vdash \exists a\, \mathsf{Ia}_n(a).$$

Next, we show that if A is a Σ sentence of $\mathcal{L}^*$ and $\mathsf{KPdm}\ell\mathsf{c}^0 \overset{n}{\vdash} A$, then already $\mathsf{KPu}^0 \vdash \neg\mathsf{Ia}_n(a), A$. Combining the two steps yields

$$|\mathsf{KPdm}^0| \leq |\mathsf{KPdm}\ell\mathsf{c}^0| \leq |\mathsf{KPm}^0 + \neg\mathsf{TI}_\triangleleft(\mathsf{U}, |\mathsf{KPm}^0|)| = |\mathsf{KPm}^0|.$$

A level $(F)_\alpha$ of the jump-hierarchy is basically an enumeration of that level via $e \mapsto (F)_{\alpha,e}$. Towards the construction of locally countable models of $\mathsf{KPd}^0 + (\mathsf{I}_\mathsf{N})$

we proceed accordingly and build a hierarchy h such that for each α in the field of the underlying ordering $\prec$ on N, $h(\alpha)$ is a function that enumerates – more or less – the αth level of the constructible hierarchy. At limit levels, we also add the enumerating functions as sets, i.e. if $f := h(\lambda)$, then there is for each $\alpha \prec \lambda$ an $e \in \mathsf{N}$ with $f(e) = h(\alpha)$. We build this hierarchy only along orderings on the natural numbers to ensures that we do not run out of codes for our sets. Further, we choose the enumeration functions $h(\alpha)$ so that each index of a set x that enters the hierarchy first at level α is of the form $\langle \alpha, e\rangle$, where $e \in \mathsf{N}$ is a sequence number with $\mathsf{lh}(e) = 2$ or $\mathsf{lh}(e) = 3$. In the case that $\mathsf{lh}(e) = 3$, e is of the form $\langle i, m_1, m_2\rangle$, indicating that $\langle \alpha, e\rangle$ is a code of the set $f^{A_i}(y, z)$, provided that m_1 and m_2 are codes of y and z, respectively. If $\mathsf{lh}(e) = 2$, then $\langle \alpha, e\rangle$ is a code of a set added for another reason: So $\langle \alpha{+}1, \langle 2, 0\rangle\rangle$ is a code of the set enumerated by $h(\alpha)$, namely $\mathsf{Rng}(h(\alpha))$, and if α is below the limit λ, then $\langle \lambda, \langle 3, \alpha\rangle\rangle$ is a code of the function $h(\alpha)$. The first level $h(0_\prec)$ is such that $\langle 0_\prec, \langle 0, n\rangle\rangle$ is a code of the natural number n (cf. the definition of f_N in the definition below) and $\langle 0_\prec, \langle 1, n\rangle\rangle$ is a code of $g(n)$, where g is a given enumeration. Moreover, we take care that if for all $\beta \prec \alpha$, $h(\beta)$ enumerates a transitive set, then also $h(\alpha)$ enumerates a transitive set.

That a function h is such a hierarchy above an enumeration g along an ordering $\prec$ on N is described by the formula $\mathsf{hier}^{\mathcal{LF}}(h, g, \prec)$. Thereby, we let $\mathcal{LF}(u_1, u_2, v, w)$ be a Σ formula of $\mathcal{L}^*$, such that $\mathsf{hier}^{\mathcal{LF}}(h, g, \prec)$ implies for all α and limits λ with $\alpha{+}1, \lambda \in \mathsf{Field}(\prec)$,

(i) $h(0_\prec) = \{(\langle 0_\prec, \langle 1, n\rangle\rangle, g(n))) : n \in \mathsf{Dom}(g)\} \cup f_\mathsf{N}$,

(ii) $h(\alpha{+}1) = f^{\mathcal{D}'}(h(\alpha) \cup \{(\langle \alpha{+}_\prec 1, \langle 2, 0\rangle\rangle, \mathsf{Rng}(h(\alpha)))\}, \alpha)$,

(iii) $h(\lambda) = \bigcup_{\alpha \prec \lambda} h(\alpha) \cup \bigcup_{\alpha \prec \lambda} \{(\langle \lambda, \langle 3, \alpha\rangle\rangle, h(\alpha))\}$,

where $f_\mathsf{N} := \{(\langle 0_\prec, \langle 0, n\rangle\rangle, n) : n \in \mathsf{N}\}$ is an enumeration of N and $\mathcal{D}'(u, v, w)$ is a Σ formula of $\mathcal{L}^*$ such that $\mathsf{Op}^2_{\mathcal{D}'}$ and

$$f^{\mathcal{D}'}(u, \alpha) = u \cup \bigcup_{i \in I} \{(\langle \alpha{+}_\prec 1, \langle i, v_0, w_0\rangle\rangle, f^{A_i}(v_1, w_1)) : (v_0, v_1), (w_0, w_1) \in u\}.$$

I is again the set $\{1, \ldots, 16\}$, and for each $i \in I$, the formula $A_i(u, v, w)$ is as chosen in subsection III.2.1. If the underlying ordering $\prec$ is a well-ordering, then there is exactly one h with $\mathsf{hier}^{\mathcal{LF}}(h, g, \prec)$. As with the jump-hierarchy and the constructible hierarchy, we then denote this hierarchy by $\mathcal{LF}^g_\prec$ and if α is an element of $\mathsf{Field}(\prec)$, then we write $\mathcal{LF}^g_\alpha$ for its αth level, i.e. $z = \mathcal{LF}^g_\alpha$ means that $\mathsf{hier}^{\mathcal{LF}}(h, g, \prec)$ and $z = h(\alpha)$.

The next lemma collects the relevant properties of this hierarchy.

Lemma III.2.46 *The following is provable in* KPu^0*: Let* $\prec$ *be an ordering on the natural numbers. Suppose, that the function* f *enumerates a transitive set* x*, and*

(i) $\mathsf{hier}^{\mathcal{LF}}(g, f, \prec)$ *and* $\mathsf{Wo}^{g^+}(\prec)$*,*

(ii) h *is the function with* $\mathsf{Dom}(h) = \mathsf{Field}(\prec)$ *and* $h(\alpha) = \mathsf{Rng}(g(\alpha))$*,*

Then we have for all $\alpha \in \mathsf{Field}(\prec)$ *that*

(iv) $h(\alpha)$ *is transitive,* $g(\alpha)$ *enumerates* $h(\alpha)$ *and* $h(\alpha{+}1) = f^{\mathcal{D}}(h(\alpha))$*,*

(v) if $\lambda \in \mathsf{Field}(\prec)$ *is a limit with* $\alpha \prec \lambda$*, then* $g(\alpha) \in h(\lambda)$*,*

(vi) $z \in \mathsf{Dom}(g(\alpha)) - \bigcup_{\beta \prec \alpha} \mathsf{Dom}(g(\beta))$ *implies* $z = \langle \alpha, e \rangle$ *with* $e \in \mathsf{N}$*.*

Proof: We show the claims simultaneously by transfinite induction along $\prec$. Clearly, $g(0_\prec)$ is a function that enumerates $x \cup \mathsf{N}$ which is a transitive set and the elements of $\mathsf{Dom}(g(0_\prec))$ have the requested form.

Suppose that we have shown the claims for α. Since $\langle \alpha{+}1, \langle 2, 0 \rangle \rangle$ is not in the domain of $g(\alpha)$, the addition of the pair $\{(\langle \alpha{+}1, \langle 2, 0\rangle\rangle, \mathsf{Rng}(g(\alpha)))\}$ does not destroy the functional character of g. Applying the definition of $f^{\mathcal{D}'}$, we obtain

$$\mathsf{Rng}(g(\alpha{+}1)) = \mathsf{Rng}(g(\alpha)) \cup \bigcup_{i \in I} \{f^{A_i}(v, w) : v, w \in \mathsf{Rng}(g(\alpha))\}.$$

The I.H. yields that $\mathsf{Rng}(g(\alpha)) = h(\alpha)$ is transitive, hence $\bigcup h(\alpha) \subseteq h(\alpha)$, thus the definition of $f^{\mathcal{D}}$ implies $h(\alpha{+}1) = f^{\mathcal{D}}(h(\alpha))$, which in turn yields by lemma III.2.1 that $h(\alpha{+}1)$ is transitive. The other properties easily follow.

For the limit case, we have to show that the addition of the enumerations for the levels $\alpha \prec \lambda$ does not affect the transitivity of $h(\lambda)$. But if $y \in g(\alpha) \in h(\lambda)$, then y is of the form (e, z) with $e \in \mathsf{N}$ and $z \in h(\alpha)$. By the definition of the hierarchy g, this yields $(e, z) \in h(\alpha{+}2) \subseteq h(\lambda)$. ☐

The following lemma exploits the linearity of the class Ad to obtain in a uniform way a non-empty, upward closed set $k \subseteq \mathsf{Field}(\lhd)$ without a $\lhd$-least element. Again, $\lhd$ is the underlying ordering of our notation system.

Lemma III.2.47 *The following is provable in* $\mathsf{KPm}^0 + \neg\mathsf{TI}_\lhd(\mathsf{U}, |\mathsf{KPm}^0|)$*: Suppose that* $\mathsf{Ad}(a)$ *and* $g \in a$ *enumerates the transitive set* x*. Then we have*

(i) $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^a(\lhd\restriction\alpha)\} \notin a$*,*

(ii) $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\} \subsetneq \{\alpha \in \mathsf{Field}(\lhd) : (\exists h \in a)\mathsf{hier}^{\mathcal{LF}}(h, g, \alpha)\}$*,*

(iii) $\{\alpha \in \mathsf{Field}(\lhd) : (\exists h \in a)\mathsf{hier}^{\mathcal{LF}}(h, g, \alpha) \wedge \neg\mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\}$ *is non-empty and has no* $\lhd$*-least element.*

Proof: (i) is a consequence of lemma III.1.11. The inclusion in (ii) follows from lemma III.1.2, and that the inclusion is proper follows from Δ separation in a^+ and the fact that $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\}$ is not a set in a^+. (iii) is due to (ii) and the observation that $\{\alpha \in \mathsf{Field}(\lhd) : \neg\mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\}$ cannot have a least element. □

Another auxiliary observation: $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad}}$ implies also the following form of Π_2 reflection.

Lemma III.2.48 *The following is provable in* KPm^0*: If* $A(u, v, \vec{w})$ *is a* Σ *formula of* $\mathcal{L}^*_{\mathsf{Ad}}$*, then* KPm^0 *proves:*

$$\forall x\exists y A(x, y, \vec{z}) \rightarrow \exists a[\mathsf{Ad}(a) \wedge \vec{z} \in a \wedge (\forall x \in a)(\exists y \in a)A^a(x, y, \vec{z})].$$

Proof: Σ reflection yields that $\forall x\exists y A(x, y, z) \leftrightarrow \forall x\exists y[y = (y_0, y_1) \wedge A^{y_1}(x, y_0, z)]$. Applying $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad}}$ and persistence yields the claim. □

Now the stage is set to adapt the construction of models of Σ^1_1-DC from Theorem II.2.11 to the present context. The lemma below tells us, under which circumstances a pseudo-hierarchy gives rise to a model of $\mathsf{KPd}^0 + (\mathsf{I_N})$. In the lemma below, h is a pseudo-hierarchy and g is the corresponding hierarchy of functions where $g(\alpha)$ enumerates the set $h(\alpha)$.

Lemma III.2.49 *The following is provable in* KPu^0*: Suppose that* $k \subseteq \mathsf{Field}(\lhd)$ *is non-empty, upward closed and has no* $\lhd$*-least element. Further,* a *is an admissible that contains the functions* g, h *with* $\mathsf{Dom}(g) = \mathsf{Dom}(h) \supseteq \mathsf{Field}(\lhd) - k$ *and* $\mathsf{Wo}^a(\mathsf{Dom}(g))$*. Moreover, for all* $\alpha, \beta \in \mathsf{Dom}(g)$*:*

(i) $g(\alpha)$ *enumerates* $h(\alpha)$ *and* $\alpha \lhd \beta \rightarrow g(\alpha) \subseteq g(\beta) \wedge f^{\mathcal{D}}(h(\alpha)) \subseteq h(\beta)$*,*

(ii) $z \in \mathsf{Dom}(g(\alpha)) - \bigcup_{\beta\prec\alpha} \mathsf{Dom}(g(\beta))$ *implies* $z = \langle\alpha, e\rangle$ *with* $e \in \mathsf{N}$*.*

Then $b := \bigcup_{\alpha\lhd k} h(\alpha)$ *is a model of* $\mathsf{KPd}^0 + (\mathsf{I_N})$*.*

Proof: The set $\{\alpha : \alpha \lhd k\}$ has no top element. Together with the assertion that $\alpha \lhd \beta$ implies $f^{\mathcal{D}}(h(\alpha)) \subseteq h(\beta)$, this yields that b satisfies $(\Delta_0\text{-}\mathsf{Sep}')$. It remains to show that b is a model of $(\Delta_0\text{-}\mathsf{dc})$. The assertion that $g(\alpha)$ enumerates $h(\alpha)$ allows us to adapt the proof of theorem II.2.11.
Let $A(u, v)$ be a Δ_0 formula of $\mathcal{L}^*$ and assume that

$$(1) \qquad (\forall x \in b)(\exists y \in b)A(x, y).$$

We let $\mathsf{I} := \bigcup_{\alpha\in\mathsf{Dom}(g)} \mathsf{Dom}(g(\alpha))$. Observe that $i \in \mathsf{I}$ is of the form $\langle\gamma, e\rangle$, where $e \in \mathsf{N}$ and $\gamma \in \mathsf{Field}(\lhd)$, and that $\langle\gamma, e\rangle \in \mathsf{Dom}(g(\alpha))$ implies $\gamma \preceq \alpha$. Moreover,

$\mathsf{I} \in a$. Now we order I by $<_{\mathsf{I}}$, letting $\langle \gamma, e\rangle <_{\mathsf{I}} \langle \delta, e'\rangle$ if $\gamma \lhd \delta$, or $\gamma = \delta$ and $e <_{\mathsf{N}} e'$. Further, let $\mathsf{I}' := \bigcup_{\alpha \lhd k} \mathsf{Dom}(g(\alpha))$. Note that $\mathsf{I}' \subseteq \mathsf{I}$, but I' may not be in a.

Assumption (i) ensures that $g^* := \bigcup \mathsf{Rng}(g) \in a$ is a function with domain I so that we have $g^*(\langle \gamma, e\rangle) = g(\gamma)(\langle \gamma, e\rangle)$ for all $\langle \gamma, e\rangle \in \mathsf{I}$. Now (1) becomes equivalent to the formula $(\forall y \in \mathsf{I}')(\exists z \in \mathsf{I}')A(g^*(y), g^*(z))$. Moreover, for each $y \in \mathsf{I}'$, the set $\{z \in \mathsf{I} : A(g^*(y), g^*(z))\}$ has a $<_{\mathsf{I}}$-least element z_0, which is already in I'. Hence, we conclude that $(\forall y \in \mathsf{I}')(\exists! z \in \mathsf{I})A'(g^*, y, z)$, where $A'(g^*, y, z)$ expresses that z is the least index w.r.t. our index ordering $<_{\mathsf{I}}$, such that $A(g^*(y), g^*(z))$ holds. Next, we fix an index z with $g^*(z) \in b$ and show that there exists a function $f \in b$, such that $f(0) = g^*(z)$ and $(\forall n \in \mathsf{N})A(f(n), f(n{+}1))$. First, we look for initial segments of choice sequences $\sigma \in \mathsf{seq}$, such that $\mathsf{ChSeq}_{A'}(g^*, \sigma, z, n)$, where $\mathsf{ChSeq}_{A'}(g^*, \sigma, z, n)$ is the formula

$$n \in \mathsf{N} \wedge \mathsf{lh}(\sigma) = n{+}1 \wedge (\sigma)_0 = z \wedge (\forall m <_{\mathsf{N}} n)A'(g^*, (\sigma)_m, (\sigma)_{m+1}).$$

Then, assumption (1) allow us to prove by set induction along N that

$$(\forall n \in \mathsf{N})[(\exists! \sigma \in \mathsf{seq})\mathsf{ChSeq}_{A'}(g^*, \sigma, z, n)].$$

So the set

$$\{\alpha \in \mathsf{Field}(\lhd) : (\forall n \in \mathsf{N})[(\exists \sigma \in \mathsf{seq})\mathsf{ChSeq}_{A'}(g^*, \sigma, z, n)]\}$$

has a least element $\alpha_0 \lhd k$. If we set $g^*_{\alpha_0} := \bigcup_{\beta \lhd \alpha_0} g(\beta)$, then the function

$$f := \{(n, x) : (\exists \sigma \in \mathsf{seq})[\mathsf{ChSeq}_{A'}(g^*_{\alpha_0}, \sigma, z, n) \wedge g^*_{\alpha_0}((\sigma)_n) = x]\}$$

is a set in b and serves as a witness for our sought for choice sequence. □

The above lemma enables us to construct locally countable n-inaccessible models of $\mathsf{KPd}^0 + (\mathsf{I_N})$. We even prove a stronger statement, namely that for each $n \in \mathbb{N}$, there is a Σ formula $A_n(u, v)$ of $\mathcal{L}^*$, such that Op^1_A, and whenever f enumerates a transitive set x, then $\mathsf{Ia}_n(\mathsf{Rng}(f^{A_n}(f)))$.

Lemma III.2.50 *For each $n \in \mathbb{N}$, there is a Σ formula $A_n(u, v)$ of $\mathcal{L}^*_{\mathsf{Ad}}$, such that* $\mathsf{KPm}^0 + \neg\mathsf{TI}_{\lhd}(\mathsf{U}, |\mathsf{KPm}^0|)$ *proves:*

(i) $\forall x \exists! y A_n(x, y)$,

(ii) if x is transitive and f enumerates x, then $A_n(f, g)$ implies $f \subseteq g$, $\mathsf{Ia}_n(\mathsf{Rng}(g))$ and $x \in \mathsf{Rng}(g)$.

Proof: We proof the claim by (meta-) induction on n making use of a uniform pseudo-hierarchy argument.

Let us first consider the base case $n = 0$. We aim to find a Σ formula $A_0(u, v)$ such that whenever the function f enumerates a transitive set x, then $A_0(f, g)$ implies that g is a function whose range contains x and is a model of $\mathsf{KPd}^0 + (\mathsf{I_N})$. Thereto, we let $a := f^+$ and set

$$\{\alpha \in \mathsf{Field}(\lhd) : (\exists h \in a)\mathsf{hier}^{\mathcal{LF}}(h, f, \alpha) \wedge \neg\mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\}.$$

Lemma III.2.47 tells us that k has no $\lhd$-least element. Then, we set

$$g := \bigcup_{\alpha \lhd k} \{h \in a : \mathsf{hier}^{\mathcal{LF}}(h, f, \alpha)\}.$$

The lemma previous lemma together with III.2.46 implies that $\mathsf{Ia}_0(\mathsf{Rng}(g))$. The uniformity of the construction leads to a Σ formula $A_0(u, v)$ as desired.

The induction step is performed similarly. Assume that f enumerates the transitive set x. Then, by $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad}}$ and lemma III.1.1, we know that there is a least admissible $a \in \mathsf{Ad}$ such that $f \in a$ and $(\forall y \in a)(\exists! z \in a)A_n^a(y, z)$. Now we choose a Σ formula $B_n(u, v, w)$ such that $\mathsf{Op}^1_{B_n}(f)$ and $\mathsf{Hier}^{B_n}(h, f, \lhd)$ implies $h(0) = f$, $h(\alpha{+}1) = f^{A_n}(h(\alpha))$ and $h(\lambda) = \bigcup_{\alpha\lhd\lambda} h(\alpha)$, for limits $\lambda \in \mathsf{Field}(\lhd)$. This time, we set

$$k := \{\alpha \in \mathsf{Field}(\lhd) : \exists h\, \mathsf{hier}^{B_n}(h, f, \alpha) \wedge \neg\mathsf{Wo}^{a^+}(\lhd\restriction\alpha)\}$$

and conclude again by the previous lemma that

$$g := \{z : (\exists \alpha \lhd k)(\exists h \in a)[\mathsf{hier}^{B_n}(h, f, \alpha) \wedge z \in h]\}$$

enumerates an $n{+}1$-inaccessible model of $\mathsf{KPd}^0 + (\mathsf{I_N})$ above x. Again, the uniformity of the above construction yields to a Σ formula $A_{n+1}(u, v)$ with the required properties. □

So for each $n \in \mathbb{N}$, $\mathsf{KPm}^0 + \neg\mathsf{TI}_\lhd(\mathsf{U}, |\mathsf{KPm}^0|)$ proves the existence sets a with $\mathsf{Ia}_n(a)$. This concludes the first step of our reduction. For the next step, we replace the axiom $(\Delta_0\text{-}\mathsf{Col})$ by a rule, as we did it for lemma III.2.40. Due to partial cut-elimination we now may assume that in a proof in the theory $\mathsf{KPdm}\ell\mathsf{c}^0$ the cut rule applies only to cut-formulas which are Σ or Π. The next theorem corresponds to Theorem 9 in [27].

Theorem III.2.51 *Let $*$ be the set of all Π and Σ formulas of $\mathcal{L}^*$ and assume that $\Gamma(\vec{u})$ is a finite set of Σ formulas of $\mathcal{L}^*$ with exactly the displayed variables free. Then*

$$\mathsf{KPdm}\ell\mathsf{c}^0 \mathrel{\vert\!\!\frac{\;n\;}{*}} \Gamma(\vec{u}) \Longrightarrow \mathsf{KPu}^0 \vdash \vec{u} \notin a, \neg\mathsf{Ia}_n(a), \Gamma^a(\vec{u}).$$

Proof: The claim is shown by (meta-) induction on n. The only interesting cases are cut and reflection.

If $\Gamma(\vec{u})$ is the conclusion of a cut, then there are $n_0, n_1 < n$ and a Δ_0 formula $B(v, \vec{u})$ such that

$$\mathsf{KPdm\ell c}^0 \mathrel{|\!\!\frac{n_0}{*}} \Gamma(\vec{u}), \exists x B(x, \vec{u}) \qquad \text{and} \qquad \mathsf{KPdm\ell c}^0 \mathrel{|\!\!\frac{n_1}{*}} \Gamma(\vec{u}), \neg B(v, \vec{u}),$$

for $v \notin FV(\Gamma)$. The I.H. yields

(i) $\mathsf{KPu}^0 \vdash \vec{u} \notin a, \neg \mathsf{Ia}_{n_0}(a), \Gamma^a(\vec{u}), (\exists x \in a)B(x, \vec{u}),$

(ii) $\mathsf{KPu}^0 \vdash \vec{u} \notin a, \neg \mathsf{Ia}_{n_1}(a), \Gamma^a(\vec{u}), v \notin a, \neg B(v, \vec{u}).$

Since $\neg \mathsf{Ia}_n(a), \mathsf{Ia}_m(a)$ is provable in KPu^0 for each $m <_{\mathsf{N}} n$, the claim follows.

If $\Gamma(\vec{u})$ is the conclusion of reflection, then there is an $n_0 < n$ and a Δ_0 formula $B(v, w, \vec{u})$ such that

$$\mathsf{KPdm\ell c}^0 \mathrel{|\!\!\frac{n_0}{*}} \Gamma(\vec{u}), \exists y B(v, y, \vec{u}),$$

for $v \notin FV(\Gamma)$. Applying the I.H. and quantifying v yields

$$\text{(1)} \qquad \mathsf{KPu}^0 \vdash \vec{u} \notin a, \neg \mathsf{Ia}_{n_0}(a), \Gamma^a(\vec{u}), (\forall x \in a)(\exists y \in a)B(x, y, \vec{u}).$$

Due to the definition of the formula $\mathsf{Ia}_n(u)$, we have that

$$\text{(2)} \qquad \mathsf{KPu}^0 \vdash \vec{u} \notin b, \neg \mathsf{Ia}_n(b), (\exists a \in b)[\mathsf{Ia}_{n_0}(a) \land \vec{u} \in a].$$

Combining (1) and (2) and applying persistence yields then that KPu^0 derives

$$\vec{u} \notin b, \neg \mathsf{Ia}_n(b), \Gamma^b(\vec{u}), (\exists a \in b)[\mathsf{Ad}_{\mathsf{dc}}(a) \land \ell\mathsf{c}(a) \land \vec{u} \in a \land (\forall x \in a)(\exists y \in a)B(x, y, \vec{u})].$$

□

So if A is a Σ sentence of $\mathcal{L}^*$ that is provable in $\mathsf{KPdm\ell c}^0$, then according to the above theorem, there is an $n \in \mathbb{N}$ such that $\mathsf{KPu}^0 \vdash \neg \mathsf{Ia}_n(a), A^a$, therefore also $\mathsf{KPu}^0 \vdash \neg \mathsf{Ia}_n(a), A$. Since $\mathsf{KPm}^0 + \neg \mathsf{TI}_{\triangleleft}(\mathsf{U}, |\mathsf{KPm}^0|)$ proves the existence of an n-inaccessible model of $\mathsf{KPd}^0 + (\mathsf{I_N})$, we conclude that $\mathsf{KPm}^0 + \neg \mathsf{TI}_{\triangleleft}(\mathsf{U}, |\mathsf{KPm}^0|) \vdash A$. This completes our analysis of the theory $\mathsf{KPdm\ell c}^0$.

Theorem III.2.52

$$|\mathsf{KPdm\ell c}^0| = |\mathsf{KPdm}^0| = |\mathsf{KPm}^0|.$$

Remark III.2.53 *The above considerations also yield the ordinal of the theory* $\mathsf{KPd}^0 + (\mathsf{I_N})$. *Clearly,* $\mathsf{KPd}^0 + (\mathsf{I_N})$ *contains* Σ^1_1-DC. *For an upper bound, we interpret* $\mathsf{KPd}^0 + (\mathsf{I_N})$ *into the theory* $\mathsf{KPu}^0 + (\mathsf{Ad}_1) + (\mathsf{psh})$, *that extends* $\mathsf{KPu}^0 + (\mathsf{psh})$ *by an axiom asserting the existence of one admissible set. It is formulated in the language* $\mathcal{L}^*$ *and extends* KPu^0 *by the axiom* $\exists x \mathsf{P_{Ad}}(x)$. *It follows from results in [20] that* $|\mathsf{KPu}^0 + (\mathsf{Ad}_1)| = \varphi\varepsilon_0 0$ *and that it is a normal theory. Using the methods above,* $\mathsf{KPu}^0 + (\mathsf{Ad}_1) + (\mathsf{psh})$ *proves the existence of set a with* $\mathsf{Ia}_0(a)$. *Moreover, if* $*$ *is the set of all* Π *and* Σ *formulas of* $\mathcal{L}^*$, *then we have for each finite set* $\Gamma(\vec{u})$ *of* Σ *formulas of* $\mathcal{L}^*$ *with exactly the displayed variables free,*

$$\mathsf{KPd}^0 + (\mathsf{I_N}) \underset{*}{\vdash} \Gamma(\vec{u}) \Longrightarrow \mathsf{KPu}^0 \vdash \neg \mathsf{Ia}_0(a), \vec{u} \notin a, \Gamma^a(\vec{u}).$$

Thus, it follows that $|\mathsf{KPd}^0 + (\mathsf{I_N})| = \varphi\varepsilon_0 0$. *Similarly, we can interpret* KPd^0 *into the theory* $\mathsf{T}^\dagger$, *where* T *is* $\mathsf{KPu}^0 + (\Sigma\text{-}\mathsf{I_N})$. *We remark that* $|\mathsf{KPu}^0 + (\Sigma\text{-}\mathsf{I_N})| = \varphi\omega 0$. *So* $\mathsf{T}^\dagger$ *proves that*

$$\mathcal{C} := \{\alpha \lhd \omega^\omega : \exists g\, \mathsf{hier}^{\mathcal{LF}}(g, \emptyset, \lhd\restriction\alpha)\}$$

is a proper subclass of $\mathcal{C}$. *Adapting the proof of lemma III.2.49 allows us to show that the class*

$$\mathcal{M} := \{x : (\exists \alpha \in \mathcal{C})\exists g[\mathsf{hier}^{\mathcal{LF}}(g, \emptyset, \lhd\restriction\alpha) \wedge x \in \mathsf{Rng}(g(\alpha))]\}$$

is model of KPd^0. $(\Sigma\text{-}\mathsf{I_N})$ *is required to show* $(\forall n \in \mathsf{N})[(\exists! f \in \mathcal{M})\mathsf{ChSeq}_{A'}(f, z, n)]$.

III.3 Pseudo-hierarchies in explicit mathematics

In section III.3, we present a uniform pseudo-hierarchy principle for a suitable subsystem of explicit mathematics. Then, we apply it to derive a uniform fixed point principle. Finally, we propose a form of (transfinite) dependent choice for explicit mathematics and verify that it leads to theories of the expected strength.

III.3.1 Hierarchies and pseudo-hierarchies

In contrast to second order arithmetic or admissible set theory, a hierarchy is no longer represented by a function, but by an individual term, that maps the elements of the field of an ordering to names of types. Also, the canonic notion of an operator specifying the transition from one level to the next, is now an individual term that maps names to names. The concept of a hierarchy is adjusted accordingly: If b is the name of an ordering and $(f : \Re \to \Re)$, then we say that the pair (h, b) is a hierarchy for f along the ordering b, denoted by $\mathsf{Hier}^f(h, b)$, if the following conditions are met: There exists a type $\prec$ such that

(i) $\Re(\prec, b) \wedge \mathsf{Lin_0}(\prec) \wedge (h : \mathsf{Field}(\prec) \to \Re)$,

(ii) $(\forall \alpha \in \mathsf{Field}(\prec))[h(\alpha) = f(\mathsf{j}(\{\beta : \beta \prec \alpha\}, h))]$.

To enhance the readability, we only mentioned the type $\{\beta : \beta \prec \alpha\}$ in (ii), instead of its name $\mathsf{int}(\underline{\mathsf{field}}(b), \mathsf{inv}(b, \lambda y.(y, \alpha)))$, where $\underline{\mathsf{field}}$ is a closed term of $\mathbb{L}$ that assigns to the name of an ordering a name of its field. Such a term exists by lemma I.2.7. Again, (h, b) is a proper hierarchy if b names a well-ordering and a pseudo-hierarchy, denoted by $\mathsf{PSH}^f(h, b)$, if $\mathsf{Hier}^f(h, b)$ and $\neg\mathsf{Wo}(b)$, where $\mathsf{Wo}(b)$ is short for $\exists X(\Re(X, b) \wedge \mathsf{Wo}(X))$. Note, that f may contain free variables. Observe also, that $\mathsf{Hier}^f(h, b)$ is a Σ^+ formula of $\mathbb{L}$. If b is the name of an ordering, we write $x \prec_b y$ for $(x, y) \dot{\in} b$ and 0_b for the least element of the field of b, provided it exists. As before, we use α, β, γ for individual variables that are meant to range over the field of some ordering.

A non-uniform version of a pseudo-hierarchy principle might take the form

$$\forall b[\mathsf{Wo}(b) \wedge (f : \Re \to \Re) \to \exists g\, \mathsf{Hier}^f(g, b)] \to \exists h, c\, \mathsf{PSH}^f(h, c).$$

However, in explicit mathematics, we have an additional difficulty to overcome when attempting to show that adding a pseudo-hierarchy principle to a theory T with $|\mathsf{T}| < \Phi_0$ does not result in a theory with a bigger proof-theoretic ordinal. The assumption that for a certain operation $(f : \Re \to \Re)$ only proper hierarchies exist,

$$(*) \qquad \mathsf{Wo}(b) \leftrightarrow \exists h\, \mathsf{Hier}^f(h, b),$$

no longer permits us to derive (Π^1_1-CA), due to the lack of an appropriate form of comprehension or separation involving Σ^+ formulas, unless we have a strong enough reflection principle at hand: Suppose that $A(U, u)$ is a Π^1_1 formula of L_2 and X an arbitrary set. If we can find a universe a (a type that contains only names and is closed under the type generators of EETJ_0) which contains a name x of the type X and the name b, and satisfies $(f : a \to a)$, then (Π^1_1-CA) follows from $(*)$: Let t be a closed term such that for all $n \in \mathsf{N}$, $t(x, n)$ is a name of the type $\mathsf{KB}(T^A_{X,n})$. Assuming the above equivalence, we conclude that

$$\{n : A(X, n)\} = \{n \in \mathsf{N} : \mathsf{Wo}(t(x, n))\} = \{n \in \mathsf{N} : \exists h\, \mathsf{Hier}^f(h, t(x, n))\}.$$

Due to the choice of a, $(h : \underline{\mathsf{field}}(t(x, n)) \to \Re)$ is equivalent to $(h : \underline{\mathsf{field}}(t(x, n)) \to a)$, thus the collection on the right is a type.
Therefore, we analyze the pseudo-hierarchy principle only in connection with the theory EMA_0, which provides an suitable reflection principle and is introduced in the next subsection.

III.3.2 The theory EMA_0

The theory EMA_0 extends the theory EETJ_0 by a reflection principle that corresponds to the Mahlo axiom $(\Pi_2\text{-}\mathsf{Ref})^{\mathsf{Ad}}$ of admissible set theory. EMA_0 is basically the theory EMA, introduced in Jäger and Strahm [27], but we omit some axioms for the type generators and set aside an assertion that claims linearity and connectivity for so-called normal universes. The proof-theoretic analysis of EMA is carried out in detail in [27] and Strahm [43], which entails that $|\mathsf{EMA}| = \varphi\omega 00$. We will argue below that EMA_0 has still the same proof-theoretic ordinal as EMA.

To state the reflection principle of EMA_0, we have to introduce the notion of *universes*, types that contain only names and are closed under the type generators of EETJ_0. The precise closure condition is expressed by the formula $C(U,u)$ of $\mathbb{L}$, which is the disjunction of the $\mathbb{L}$ formulas given below:

1. $u = \mathsf{nat} \vee u = \mathsf{cs}_\mathsf{U} \vee u = \mathsf{cs}_\mathsf{V} \vee u = \mathsf{id}$,
2. $\exists x(u = \mathsf{co}(x) \wedge x \in U)$,
3. $\exists x, y(u = \mathsf{int}(x,y) \wedge x \in U \wedge y \in U)$,
4. $\exists x(u = \mathsf{dom}(x) \wedge x \in U)$,
5. $\exists x, f(u = \mathsf{inv}(x,f) \wedge x \in U)$,
6. $\exists x, f[u = \mathsf{j}(x,f) \wedge x \in U \wedge (\forall y \,\dot{\in}\, x)(fy \in U)]$.

Thus, the formula $\forall x(C(X,x) \to x \in X)$ states that X is a type which is closed under the type generators of EETJ_0. If, in addition, all elements of X are names, we call X a universe, in symbols, $\mathcal{U}(X)$. Moreover, we write $\dot{\mathcal{U}}(x)$ to express that the individual x is the name of a universe.

$$
\begin{aligned}
\mathcal{U}(U) &:= \forall x(C(U,x) \to x \in U) \wedge (\forall x \in U)\Re(x), \\
\dot{\mathcal{U}}(u) &:= (\exists X)(\Re(X,u) \wedge \mathcal{U}(X)).
\end{aligned}
$$

The theory EMA_0 is now formulated in the language $\mathbb{L}_\mathsf{m}$ that extends $\mathbb{L}$ by the constant m and extends EETJ_0 by the so-called Mahlo axiom

$$
\begin{aligned}
& \Re(x) \wedge (f : \Re \to \Re) \to \\
(f\text{-}\mathsf{Ref})^{\dot{\mathcal{U}}} \qquad & \dot{\mathcal{U}}(\mathsf{m}(x,f)) \wedge x \,\dot{\in}\, \mathsf{m}(x,f) \wedge (f : \mathsf{m}(x,f) \to \mathsf{m}(x,f)).
\end{aligned}
$$

For instance, if a is a name, then $\mathsf{m}(a, \lambda x.x)$ is the name of a universe that contains a. The Mahlo axiom implies that we can iterate operations on names along arbitrary well-orderings. Moreover, we can do this in a uniform way:

Lemma III.3.1 *There exists a closed term* $\underline{\mathsf{hier}}$ *of* $\mathbb{L}$ *such that the following is provable in* EMA_0*:*

$$(f : \Re \to \Re) \wedge \Re(b) \wedge a = \mathsf{m}(b, f) \wedge a' = \mathsf{m}(a, \lambda x.x) \wedge \mathsf{Wo}^{a'}(b) \to \mathsf{Hier}^f(\underline{\mathsf{hier}}(f, b), b).$$

Proof: Using λ-abstraction and the recursion theorem, we find a closed term $\underline{\mathsf{hier}}$ such that EMA_0 proves:

(i) $\forall f, b(\underline{\mathsf{hier}}(f, b)\downarrow)$,

(ii) $\forall f, b, c[\underline{\mathsf{hier}}(f, b)(c) \simeq f(\mathsf{j}(\underline{\mathsf{prec}}\, bc, \underline{\mathsf{hier}}(f, b)))]$,

where $\underline{\mathsf{prec}}$ is a closed term so that $\underline{\mathsf{prec}}\, bc$ names the type $\{y : (y, c) \dot{\in} b\}$, given b is a name. For the construction of the term $\underline{\mathsf{hier}}$, we set $t := \lambda xyz.(y)_0(\mathsf{j}(\underline{\mathsf{prec}}\, (y)_1 z, xy))$. By lemma I.2.10, we know that for $\underline{\mathsf{hier}} := \underline{\mathsf{rec}}\, t$,

$$\underline{\mathsf{hier}}(f, b) \simeq (t\, \underline{\mathsf{hier}})(f, b) \simeq \lambda z.f(\mathsf{j}(\underline{\mathsf{prec}}\, bz, \underline{\mathsf{hier}}(f, b))).$$

Due to lemma I.2.9, $\lambda x.s$ is defined for all terms s, thus also $\underline{\mathsf{hier}}(f, b)\downarrow$ for all terms f and names b.

It remains to show that $(\underline{\mathsf{hier}}(f, b) : \underline{\mathsf{field}}(b) \to a)$. Since

$$\{x \dot{\in} \underline{\mathsf{field}}(b) : \underline{\mathsf{hier}}(f, b)(x) \dot{\in} a\} \dot{\in} a',$$

this follows by transfinite induction along b. □

III.3.3 A pseudo-hierarchy principle for EMA_0

In this subsection, we show that the theory EMA_0 can be conservatively extended by a uniform pseudo-hierarchy principle (u-psh). The proof of this result relies heavily on the Mahlo axiom.

Given an operation $(f : \Re \to \Re)$ and a name b of an ordering $\prec$ that is not a well-ordering but looks like a wellordering in a universe $\mathsf{m}(a, \lambda x.x)$ above the universe $a := \mathsf{m}(b, f)$, lemma III.3.1 yields that $h := \underline{\mathsf{hier}}(f, b)$ is a pseudo-hierarchy along the ordering $\prec$, i.e. $\mathsf{PSH}^f(h, b)$. However, when performing an argument involving a pseudo-hierarchy, also a non-empty, upward closed $K \subseteq \mathsf{Field}(\prec)$ without a $\prec$-least element is required. Therefore, we regard the name k of such a set K as an integral part of a uniform pseudo-hierarchy. Since we can iterate any operation on names along an arbitrary well-ordering, we suggest the following uniform pseudo-hierarchy principle: For the closed term $\underline{\mathsf{psh}}$ constructed in the next lemma,

(u-psh) $(f : \Re \to \Re) \to \exists h, b, k[$

$$\underline{\mathsf{psh}}\, f = (h, b, k) \wedge \mathsf{PSH}^f(h, b) \wedge \mathsf{Wo}^{\mathsf{m}(\mathsf{m}(b,f),\lambda x.x)}(b) \wedge$$

$k \dot{\subseteq} \underline{\mathsf{field}}(b)$ is non-empty, upward closed and has no b-least element].

Again, the principle $(\mathsf{u\text{-}psh})$ provable in the theory $\mathsf{EMA}_0^\dagger$ that extends EMA_0 by the axiom $\mathsf{TI}^*_\lhd(\mathsf{U}, |\mathsf{EMA}_0|)$, where $\cdot^*$ is the standard translation from L_2 to $\mathbb{L}$. Further, we will choose the term $\underline{\mathsf{psh}}$ so that it yields a pseudo-hierarchy whose underlying ordering is an initial segment of the underlying ordering $\lhd$ of our notation system. Strictly speaking, the language $\mathbb{L}$ is not equipped with a relation symbol for the primitive recursive relation $\lhd$. However, corollary I.2.12 provides a closed term $\underline{\mathsf{f}}_\lhd$ such that $\forall x, y(x \lhd y \leftrightarrow \underline{\mathsf{f}}_\lhd xy = 0)$. In the sequel, we regard $\lhd$ as the type $\{(x, y) : \underline{\mathsf{f}}_\lhd xy = 0\}$.

The main ingredient beside the assertion $\mathsf{TI}^*_\lhd(\mathsf{U}, |\mathsf{EMA}_0|)$ used to prove the pseudo-hierarchy principle $(\mathsf{u\text{-}psh})$ is lemma III.1.11, which implies that for universes a and b with $a \,\dot{\in}\, b$, there is an $\alpha \in \mathsf{Field}(\lhd)$ such that $\lhd\restriction\alpha$ looks like a well-ordering in a but not in b.

Lemma III.3.2 (Pseudo-hierarchies principle) *There is a closed term* $\underline{\mathsf{psh}}$ *of* $\mathbb{L}_\mathsf{m}$*, such that* $\mathsf{EMA}_0^\dagger$ *proves* $(\mathsf{u\text{-}psh})$*. Moreover, if* $(f : \Re \to \Re)$ *and* $\underline{\mathsf{psh}}\, f = (h, b, k)$ *then* b *is an initial segment of* $\lhd$*.*

Proof: We assume that $(f : \Re \to \Re)$ and let $a := \mathsf{m}(\mathsf{nat}, f)$, $a' := \mathsf{m}(a, \lambda x.x)$ and $a'' := \mathsf{m}(a', \lambda x.x)$. By lemma III.3.1, it suffices to select in a uniform way an $\alpha_0 \in \mathsf{Field}(\lhd)$ such that $\mathsf{Wo}^{a'}(\lhd\restriction\alpha_0)$ and $\neg\mathsf{Wo}^{a''}(\lhd\restriction\alpha_0)$. Since a universe satisfies the standard translation of each axiom and rule of ACA, we conclude by lemma III.1.11 that the type $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a'}(\lhd\restriction\alpha)\}$ has no name in a', which in turn yields that

$$\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a''}(\lhd\restriction\alpha)\} \subsetneq \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a'}(\lhd\restriction\alpha)\}.$$

Therefore, the type $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a'}(\lhd\restriction\alpha) \wedge \neg\mathsf{Wo}^{a''}(\lhd\restriction\alpha)\}$ is not empty and has a $<_\mathsf{N}$-least element α_0. It is now straight forward to construct a term $\underline{\mathsf{psh}}$ such that $\underline{\mathsf{psh}}\, f = (h, b, k)$, where b is a name of the type $\lhd\restriction\alpha_0$, $h = \underline{\mathsf{hier}}(f, b)$ and k is a name of the type $K := \{\alpha \in \mathsf{Field}(\lhd) : \neg\mathsf{Wo}^{a''}(\lhd\restriction\alpha)\}$. Again, if K had a $\lhd$-least element β_0, then $\{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^{a''}(\lhd\restriction\alpha)\} = \{\alpha : \alpha \lhd \beta_0\}$ were a type in a'', which is impossible by the argument given above. □

As a first application of the pseudo-hierarchy principle, we prove that there is a closed term $\underline{\mathsf{fix}}$ that assigns to a monotone operation the name of one of its fixed points.

Lemma III.3.3 (Uniform fixed points) *There is a closed term* $\underline{\mathsf{fix}}$ *of* $\mathbb{L}_\mathsf{m}$ *such that the following is provable in* $\mathsf{EMA}_0 + (\mathsf{u\text{-}psh})$*:*

$$(f : \Re \to \Re) \wedge \forall x, y(x \,\dot{\subseteq}\, y \to fx \,\dot{\subseteq}\, fy) \to \Re(\underline{\mathsf{fix}} f) \wedge \underline{\mathsf{fix}} f \doteq f(\underline{\mathsf{fix}} f).$$

Proof: Suppose that $(f : \Re \to \Re)$ is a monotone operation. To obtain a hierarchy, where the αth level is the union, rather than the disjoint union, of all the stages below α, we consider the operation $g := \lambda x.f(\mathsf{union}\, x)$, where union is a closed term of $\mathbb{L}$ such that for each name a, we have

$$\mathsf{union}\, a \doteq \{x : \exists y[(x, y) \dot{\in} a]\}.$$

Recall, that then forming a hierarchy, f is applied to a name of the form $\mathsf{j}(t, f)$. The corresponding type consists of pairs (x, y) where $x \dot{\in} ft$ and $y \dot{\in} t$. Therefore, the type with name $\mathsf{union}\,\mathsf{j}(t, f)$ is indeed the union of the extensions of the types fy for $y \dot{\in} t$. With $(f : \Re \to \Re)$, we have that $(g : \Re \to \Re)$, and if $\mathsf{Wo}(b)$, then $\underline{\mathsf{hier}}(g, b)$ is monotone, i.e. $\alpha \prec_b \beta$ implies $\underline{\mathsf{hier}}(g, b)(\alpha) \dot{\subseteq} \underline{\mathsf{hier}}(g, b)(\beta)$.

Now let $a := \mathsf{m}(\mathsf{nat}, f)$, $a' := \mathsf{m}(a, \lambda x.x)$ and $a'' := \mathsf{m}(a', \lambda x.x)$. The pseudo-hierarchy principle provides a name b with $\mathsf{Wo}^{a'}(b)$ and $\neg\mathsf{Wo}^{a''}(b)$, a term h such that $\mathsf{Hier}^g(h, b)$ as well as a type with name k without a $\prec_b$-least element. Moreover, $(\mathsf{Hier}^g(h, b) : \underline{\mathsf{field}}(b) \to a)$. As in the proof of theorem II.2.22, one shows that if K is the type with name k, then $X := \{x : (\exists \alpha \prec_b K)(x \dot{\in} h(\alpha))\}$ is a fixed point of the operator f. The uniformity of the fixed point construction allows to extract a term $\underline{\mathsf{fix}}$ such that $\underline{\mathsf{fix}} f$ is a name of the fixed point X. □

It remains the question whether EMA_0 is a normal theory. So far, we only know that $\varphi\omega 00$ is an upper bound. Maybe, the omission of the axioms for uniqueness of generators and the linearity and connectivity of normal universes weakens the theory EMA. However, this is not the case. The theory EMA_0 proves the ordinal $\varphi\omega 00$. Since uniform pseudo-hierarchy arguments cannot be applied in EMA_0, we make use of a case distinction. Thereto, let C be the following sentence of $\mathbb{L}_\mathsf{m}$:

$$\forall a, b[\dot{\mathcal{U}}(a) \wedge \dot{\mathcal{U}}(b) \wedge a \dot{\in} b \to \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^b(\lhd\restriction\alpha)\} \subsetneq \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^a(\lhd\restriction\alpha)\}].$$

We argue that EMA_0 proves for each $n \in \mathbb{N}$,

(i) $C \to \mathsf{TI}^*_\lhd(\mathsf{U}, \varphi n 00)$,

(ii) $\neg C \to \mathsf{TI}^*_\lhd(\mathsf{U}, \varphi n 00)$.

Case (ii) is taken care of by lemma III.1.11: If there are universes a and b with $a \dot{\in} b$, so that

$$S := \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^b(\lhd\restriction\alpha)\} = \{\alpha \in \mathsf{Field}(\lhd) : \mathsf{Wo}^a(\lhd\restriction\alpha)\}$$

then the type S has a name in b. Following the proof of lemma III.1.11 yields $\mathsf{TI}^*_\lhd(\mathsf{U}, \varphi\omega 00)$. To show case (i), we first argue that $|\mathsf{EMA}_0 + C| = |\mathsf{EMA}_0 + (\mathsf{u\text{-}psh})|$:

A look at the proof of lemma III.3.2 reveals that for each instance A of $(\mathsf{u\text{-}psh})$, $\mathsf{EMA}_0 + C \vdash A$. Next, we embed for each $n \in \mathbb{N}$, the theory $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ into $\mathsf{EMA}_0 + (\mathsf{u\text{-}psh})$. In [38], it is shown that $|\mathsf{I}_n\text{-}\mathsf{RFN}_0| = \varphi n00$, which settles case (i).

To introduce the theories $\mathsf{I}_n\text{-}\mathsf{RFN}_0$, we define for each $n \in \mathbb{N}$ an arithmetical formula $\mathsf{I}_n(U)$:

$$\begin{aligned} \mathsf{I}_0(U) &:= (\mathrm{Ax}_{\Sigma^1_1\text{-}\mathsf{AC}})^U, \\ \mathsf{I}_{n+1}(U) &:= (\mathrm{Ax}_{\Sigma^1_1\text{-}\mathsf{DC}})^U \wedge (\forall X \dot{\in} U)(\exists Y \dot{\in} U)(X \dot{\in} Y \wedge \mathsf{I}_n(Y). \end{aligned}$$

We call a set Y satisfying $\mathsf{I}_n(Y)$ n-inaccessible. The theory $\mathsf{I}_n\text{-}\mathsf{RFN}_0$ formalizes the existence of n-inaccessible sets Y above each set X. It extends ACA_0 by the axiom

$$(\mathsf{I}_n\text{-}\mathsf{RFN}) \qquad \forall X \exists Y [X \dot{\in} Y \wedge \mathsf{I}_n(Y)].$$

Our strategy to perform the aforementioned embedding is to define for each $n \in \mathbb{N}$ a closed term i_n of $\mathbb{L}_\mathsf{m}$, that yields a name of an n-inaccessible Y above X, provided that $X \subseteq \mathsf{N}$.

Lemma III.3.4 *For each $n \in \mathbb{N}$, there exists a closed term i_n of $\mathbb{L}_\mathsf{m}$, such that $\mathsf{EMA}_0 + (\mathsf{u\text{-}psh})$ proves:*

$$\Re(X, x) \wedge X \subseteq \mathsf{N} \to (\exists Y \subseteq \mathsf{N})[\Re(Y, \mathsf{i}_n x) \wedge (X \dot{\in} Y)^\star \wedge \mathsf{I}^\star_n(Y)],$$

where $\cdot^\star$ is our standard pretranslation from L_2 to $\mathbb{L}$ of subsection I.2.12.

Proof: The lemma is proven by (meta-) induction on n. We start with the case $n = 0$. Lemma I.2.7 provides a closed term t such that for $\Re(X, x)$, $\Re(Y, y)$ and $X, Y \subseteq \mathsf{N}$,

$$t(y, x) \doteq \{z : \mathcal{J}^\star(Y, X, z)\},$$

where $\mathcal{J}^\star(U, V, u)$ is pretranslation of Turing's Jump formula from subsection II.1.5 to the language $\mathbb{L}$. To build the jump-hierarchy, we have to consider the different forming of disjoint unions in second order arithmetic and explicit mathematics: Thereto, we let s be a closed term of $\mathbb{L}$, such that for $\Re(V, v)$ and $\Re(W, w)$,

$$sw \doteq \{u : \exists x, y(u = \langle x, y\rangle \wedge (x, y) \in W)\}.$$

In the above formula $\langle u, v\rangle = w$ is to read as $\underline{\mathsf{f}}_{\langle\rangle} uv = w$, where $\underline{\mathsf{f}}_{\langle\rangle}$ is a closed term that represents the primitive recursive function $\langle\cdot,\cdot\rangle$. So sw is a name of a type that codes the pairs in W using the function $\langle\cdot,\cdot\rangle$ instead of $(\cdot,\cdot)$. Thus, if $\Re(X, x)$, $X \subseteq \mathsf{N}$, $\Re(b, \prec)$, $\mathsf{Wo}(\prec)$ and $f := \lambda z.t(sz, x)$, then $\underline{\mathsf{hier}}(f, b)(0_b)$ is a name of $f(\mathsf{j}(\emptyset, \underline{\mathsf{hier}}(f, b))) = t(\emptyset, x)$. If $0_b \neq \alpha \in \mathsf{Field}(\prec)$, then $\underline{\mathsf{hier}}(f, b)(\alpha)$ equals

$f(\mathsf{j}(\{\beta : \beta \prec \alpha\}, \underline{\mathsf{hier}}(f,b)))$, which in turn is $t(z,x)$, where z is a name of the type $\{\langle y,\gamma\rangle : \gamma \prec \alpha \wedge y \mathrel{\dot\in} \underline{\mathsf{hier}}(f,b)(\gamma)\}$. Hence, if for each $\beta \prec \alpha$, $\underline{\mathsf{hier}}(f,b)(\beta)$ is the name of the type $\mathcal{J}^X_\beta$, then $\underline{\mathsf{hier}}(f,b)(\alpha)$ is the name of the type $\mathcal{J}^X_\alpha$.

If x is a name, then $(f : \Re \to \Re)$ and $\underline{\mathsf{psh}}\, f$ equals a term of the form (g,c,k) where the ordering with name c is an initial segment of $\lhd$ and k names a set K without a $\lhd$-least element. As in the proof of theorem II.2.11, one shows that

$$M_0 := \{\langle z,\langle\gamma,e\rangle\rangle : \gamma \lhd K \wedge \langle z,\langle\gamma,e\rangle\rangle \mathrel{\dot\in} g(\gamma{+}1)\}$$

is a model of Σ^1_1-AC above X. The uniformity of the construction yields a term $(\mathsf{i}_0 : \Re \to \Re)$ and if x is a name of the type $X \subseteq \mathsf{N}$, then $\mathsf{i}_0 x$ is a name of the model M_0 of Σ^1_1-AC above X.

Suppose now, that we already have a term $(\mathsf{i}_n : \Re \to \Re)$ that maps a name x to a name $\mathsf{i}_n x$ of an n-inaccessible above the type named x. To build a hierarchy of n-inaccessibles above a type X, we let r be a closed term of $\mathbb{L}$, such that for $\Re(V,v)$ and $\Re(W,w)$,

$$r(v,w) \doteq \{u : (u \in V \wedge W = \emptyset) \vee (\exists x,y(u = \langle x,y\rangle \wedge (x,y) \in W) \wedge W \neq \emptyset)\}.$$

So $r(v,w)$ is a name of V if w names the empty type, and a name of a type that codes the pairs in W using the function $\langle\cdot,\cdot\rangle$ instead of $(\cdot,\cdot)$, otherwise. Thus, if $\Re(X,x)$, $X \subseteq \mathsf{N}$, $\Re(b,\prec)$, $\mathsf{Wo}(\prec)$ and $h := \lambda z.\mathsf{i}_n(r(x,z))$, then $\underline{\mathsf{hier}}(h,b)(0_b)$ is a name of $h(\mathsf{j}(\emptyset, \underline{\mathsf{hier}}(h,b))) \doteq h(\emptyset)$, which is the name of an n-inaccessibles above X. And if $0_b \neq \alpha \in \mathsf{Field}(\prec)$, then $\underline{\mathsf{hier}}(h,b)(\alpha)$ equals $h(\mathsf{j}(\{\beta : \beta \prec \alpha\}, \underline{\mathsf{hier}}(h,b)))$, which in turn is $\mathsf{i}_n(z)$, where z is a name of the type $\{\langle x,\gamma\rangle : \gamma \prec \alpha \wedge x \mathrel{\dot\in} \underline{\mathsf{hier}}(h,b)(\gamma)\}$. In particular, $\underline{\mathsf{hier}}(h,b)(\alpha)$ is a name of an n-inaccessibles above the type named $\underline{\mathsf{hier}}(f,b)(\beta)$ for each $\beta \prec \alpha$. Again, if x is a name, then $(h : \Re \to \Re)$ and $\underline{\mathsf{psh}}\, h$ equals a term of the form (g,c,k) where the ordering with name c is an initial segment of $\lhd$ and k names a set K without a $\lhd$-least element. The methods used in the proof of theorem II.2.11 readily imply that the type

$$M_{n+1} := \{\langle z,\langle\gamma,e\rangle\rangle : \gamma \lhd K \wedge \langle z,\langle\gamma,e\rangle\rangle \mathrel{\dot\in} g(\gamma{+}1)\}$$

is a model of Σ^1_1-DC. Its construction yields that it is $n{+}1$-inaccessible. A term i_{n+1} as desired is obtained as before. □

Corollary III.3.5

$$|\mathsf{EMA}_0| = |\mathsf{EMA}| = \varphi\omega 00.$$

So we conclude that EMA_0 is a normal theory: Clearly, $|\mathsf{EMA}_0|$ is ω-closed. Further, if Γ is a finite set of L_1 formulas such that $\mathsf{EMA}_0 \vdash \Gamma^*$, then Theorem 6 in combination with Theorem 11 of [27] yields $\mathsf{PA}^* \mathrel{|\!\frac{<\varphi\omega 00}{0}} \Gamma$.

Theorem III.3.6

$$|\mathsf{EMA}_0| = |\mathsf{EMA}_0 + (\mathsf{u\text{-}psh})|.$$

III.3.4 Choice in explicit mathematics

In subsection III.1.6 we argued that a dependent choice principle constitutes of a "choice" part and an "iteration" part. This view motivates us to propose a form of choice for explicit mathematics, which, together with an axiom for iteration along N, yields a form of dependent choice. Combined with an axiom for iteration along arbitrary well-orderings, a form of transfinite dependent choice is the result.

To formalize the choice rule, we extend the language $\mathbb{L}$ by a constant ch, a term that is to choose a name of a fixed point of a term f, provided there exists one. For each finite set Γ of $\mathbb{L}_{\mathsf{ch}}$ formulas, we have:

$$(\mathsf{ch}) \qquad \frac{\Gamma, \exists x[\Re(x) \wedge ux \doteq x]}{\Gamma, \Re(\mathsf{ch}u) \wedge \mathsf{ch}u \doteq u(\mathsf{ch}u)}.$$

If for instance $A(U,V)$ is an elementary formula of $\mathbb{L}_{\mathsf{m}}$, then there is a closed term t such that $\Re(X,x)$ and $\Re(Y,y)$ imply

$$t(x,y) \doteq \{z : (A(X,Y) \wedge z \in Y) \vee (\neg A(X,Y) \wedge z \notin Y)\}.$$

Given $\Re(X,x)$ and $\Re(Y,y)$, we have $\lambda z.t(x,z)\, y \doteq y$ if and only if $A(X,Y)$. Moreover, $\mathsf{ch}\lambda z.t(x,z)$ names a specific witness if there is one: $\Re(W, \mathsf{ch}\lambda z.t(x,z))$ yields $A(X,W)$.

Nonetheless, without an iteration principle, the choice rule (ch) does not strengthen the theory EETJ_0. For a proof, we refer to the next subsection.

Lemma III.3.7

$$|\mathsf{EETJ}_0| = |\mathsf{EETJ}_0 + (\mathsf{ch})| = \varepsilon_0.$$

Over ACA_0, dependent choice, $(\Sigma^1_1\text{-}\mathsf{DC})$, and weak dependent choice, claiming that for each Σ^1_1 formula $A(U,V)$ of L_2,

$$\text{weak-}(\Sigma^1_1\text{-}\mathsf{DC}) \qquad \forall X \exists! Y A(X,Y) \rightarrow \exists F[(F)_0 = Q \wedge \forall n A((F)_n, (F)_{n+1})],$$

lead to theories of the same strength. Only in combination with transfinite recursion, (ATR), the two principles are separated: $\mathsf{ACA}_0 + (\mathsf{ATR}) + \text{weak-}(\Sigma^1_1\text{-}\mathsf{DC})$ is provable in the theory $\mathsf{ACA}_0 + (\Delta\text{-}\mathsf{TR})$ analyzed in subsection II.2.2 which proves only the ordinals below Γ_0, whereas the theory $|\mathsf{ACA}_0 + (\mathsf{ATR}) + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi 1 \omega 0$ as shown in Jäger and Strahm [26]. In explicit mathematics, we have an analogue situation. An iteration principle corresponding to weak-$(\Sigma^1_1\text{-}\mathsf{DC})$ has been analyzed in [31]. In the formulation given below, we wrote $\mathsf{nat}_<$ for the name the type $\{(m,n) : m <_{\mathsf{N}} n\}$ and $\underline{\mathsf{hier}}$ is the closed term constructed in the proof of lemma III.3.1. For each finite set Γ of $\mathbb{L}$ formulas, we have:

$$(\mathsf{it}_{\mathsf{N}}) \qquad \frac{\Gamma, (u : \Re \rightarrow \Re)}{\Gamma, \mathsf{Hier}^u(\underline{\mathsf{hier}}(u, \mathsf{nat}_<), \mathsf{nat}_<)}.$$

An extension of the language by the constant ch and the addition of the choice rule (ch) does not strengthen the theory. The asymmetric interpretation justifying this result is given in the next subsection.

Lemma III.3.8

$$|\mathsf{EETJ}_0 + (\mathsf{it_N})| = |\mathsf{EETJ}_0 + (\mathsf{it_N}) + (\mathsf{ch})| = \varphi\omega 0.$$

Only in the presence of a principle corresponding to (ATR) that allows to iterate operations along well-orderings, the choice rule (ch) unfolds its power.

$$(\mathsf{it}) \qquad (u : \Re \to \Re) \wedge \mathsf{Wo}(b) \to \mathsf{Hier}^u(\underline{\mathsf{hier}}(u,b), b),$$

The proof-theoretical ordinal of the theory $\mathsf{EETJ}_0 + (\mathsf{it})$ is that of ATR_0, (cf. [29]). But the theory $\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch})$ formulated in $\mathbb{L}_{\mathsf{ch}}$, proves each ordinal below $\varphi\omega 00$.

Lemma III.3.9

$$|\mathsf{EETJ}_0 + (\mathsf{it})| = \Gamma_0 \qquad |\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch})| = \varphi\omega 00.$$

Proof: For a lower bound, observe that $(\Sigma^1_1\text{-}\mathsf{TDC})$ is contained in $\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch})$. An upper bound is computed in the subsection below. □

III.3.5 EMA_0, OMA and asymmetric interpretations

In this subsection we supplement the embeddings and asymmetric interpretations referred to in the previous subsection. Thereby, we make constant use of ideas and techniques developed in Jäger and Strahm [27]. In particular the embedding of $\mathsf{EMA}_0 + (\mathsf{ch})$ into EMA is a direct consequence of Theorem 6 in [27]. We subsume the argument and refer to the aforementioned article for additional informations.

In all the interpretations performed below, the first order part plus the type variables of $\mathbb{L}$ are translated into L_1: The number and type variables of $\mathbb{L}$ are mapped into the number variables of L_1 such that no conflicts arise. In the sequel, type variables of $\mathbb{L}$ are often identified with their translations. Application $u \cdot v$ of $\mathbb{L}$ is translated to $\{u\}(v)$ in L_1. It is then possible to assign pairwise different numerals to the constants k, s, p, p_0, p_1, $\mathsf{s_N}$, $\mathsf{p_N}$ and $\mathsf{d_N}$ so that the applicative axioms of EETJ_0 are satisfied. Further, the constants cs_n of $\mathbb{L}$ translate to the corresponding constants of L_1 and $\mathsf{s_N}u$ of $\mathbb{L}$ becomes $u{+}1$ in L_1. In addition, we let pairing and projections of $\mathbb{L}$ go over into the primitive recursive pairing function $\langle\cdot,\cdot\rangle$ and the associated projections. Similar as in subsection I.2.13, we assign to each term t of $\mathbb{L}$ a formula $\mathsf{Val}_t(u)$ of L_2 expressing that u is the value of t under the interpretation described above. Accordingly, the atomic formulas $t\downarrow$, $s = t$ and $\mathsf{N}(t)$ are given their obvious interpretations in L_2 with the translation of N ranging over all natural numbers.

To the generators, we assign constants $\underline{\mathsf{nat}}, \underline{\mathsf{id}}, \underline{\mathsf{co}}, \underline{\mathsf{int}}, \underline{\mathsf{dom}}, \underline{\mathsf{inv}}$ and $\underline{\mathsf{j}}$ such that the following properties are met:

$$\underline{\mathsf{nat}} = \langle 0,0\rangle, \quad \underline{\mathsf{cs_U}} = \langle 1,0\rangle, \quad \underline{\mathsf{cs_V}} = \langle 2,0\rangle, \quad \underline{\mathsf{id}} = \langle 3,0\rangle, \quad \{\underline{\mathsf{co}}\}(u) = \langle 4,u\rangle,$$
$$\{\underline{\mathsf{int}}\}(\langle u,v\rangle) = \langle 5,u,v\rangle, \quad \{\underline{\mathsf{dom}}\}(u) = \langle 6,u\rangle, \quad \{\underline{\mathsf{inv}}\}(\langle u,v\rangle) = \langle 7,u,v\rangle,$$
$$\{\underline{\mathsf{j}}\}(\langle u,v\rangle) = \langle 8,u,v\rangle, \quad \{\underline{\mathsf{m}}\}(u) = \langle 9,u\rangle, \quad \{\underline{\mathsf{ch}}\}(u) = \langle 10,u\rangle, \quad \{e_0\}(u) \neq e_1$$

for all natural numbers u, v and all e_0 and e_1 from the set ranging over the constants $\underline{\mathsf{nat}}, \underline{\mathsf{cs_U}}, \underline{\mathsf{cs_V}}, \underline{\mathsf{id}}, \underline{\mathsf{co}}, \underline{\mathsf{int}}, \underline{\mathsf{dom}}, \underline{\mathsf{inv}}, \underline{\mathsf{j}}, \underline{\mathsf{m}}$ and $\underline{\mathsf{ch}}$. In order to model the names and the extensions of the corresponding types, we define a collection $\mathcal{C}$ of triples (u,v,w) and translate $\Re(u)$ as $(u,0,0) \in \mathcal{C}$ and $\Re(u) \wedge v \mathrel{\dot\in} u$ as $(u,v,1) \in \mathcal{C}$. The means we use to define the collection $\mathcal{C}$ and the exact implementation of the ideas sketched above depend on the theory we are embedding into.

First, we show that $\varphi\omega 00$ is also an upper bound for $\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch})$. Thereto, we embed $\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch})$ into the theory OMA, which is introduced in [27]. OMA is a first order theory with ordinals, tailored for dealing with non-monotone inductive definitions. The theory OMA is formulated in the language $\mathsf{L}_1^{\mathbb{O}}$ which extends L_1 by countably many ordinal variables $\alpha, \beta, \gamma, \ldots$, a new binary relation symbol $\prec_{\mathbb{O}}$ for the less relation on ordinal variables, a unary relation symbol $\mathsf{Ad}(\alpha)$ to distinguish admissible ordinals and a $n{+}1$-ary relation symbol $\mathsf{P}^A(\alpha, u_1, \ldots, u_n)$ for each formula $A(\mathsf{P}, u_1, \ldots, u_n)$ of $\mathsf{L}_1(\mathsf{P})$ with at most $u_1, \ldots, u_n$ free. In the sequel, $\mathsf{L}_1(P)$ formulas are referred to as operator forms. The atoms of $\mathsf{L}_1^{\mathbb{O}}$ are the atoms of L_1 together with the expressions of the form $\alpha \prec_{\mathbb{O}} \beta$, $\alpha = \beta$ and $\mathsf{P}^A(\alpha, \vec{u})$, which is usually written $\mathsf{P}^A_\alpha(\vec{u})$. Further, $\mathsf{P}^A_{\prec_{\mathbb{O}}\alpha}(\vec{u})$ abbreviates $(\exists \beta \prec_{\mathbb{O}} \alpha)\mathsf{P}^A_\beta(\vec{u})$. The subscript $_{\mathbb{O}}$ of the relation $\prec_{\mathbb{O}}$ is subsequently omitted. The $\Delta_0^{\mathbb{O}}$ formulas of $\mathsf{L}_1^{\mathbb{O}}$ are the formulas that contain only bounded ordinal quantifiers and the $\Sigma^{\mathbb{O}}$ formulas are the formulas without unbounded universal ordinal quantifiers. Moreover, if A is a formula of $\mathsf{L}_1^{\mathbb{O}}$, then A^α is the formula obtained from A by replacing each unbounded ordinal quantifier $\forall \beta$ and $\exists \beta$ in A by $(\forall \beta \prec \alpha)$ and $(\exists \beta \prec \alpha)$, respectively.

Beside the axioms of PA and an axiom assuring that $\prec$ is a linear ordering on the ordinal variables, the theory OMA comprises an axiom $\mathsf{P}^A_\alpha(\vec{u}) \leftrightarrow A(\mathsf{P}^A_{\prec\alpha}, \vec{u})$ for each relation symbol P^A, and an axiom claiming Σ reflection for each $\Sigma^{\mathbb{O}}$ formula A of $\mathsf{L}_1^{\mathbb{O}}$, i.e. $A \to \exists \alpha A^\alpha$. Further, induction on N is available for $\Delta_0^{\mathbb{O}}$ formulas, whereas induction on ordinals is omitted completely. Moreover, we have the following axioms for admissible ordinals: For each $\Delta_0^{\mathbb{O}}$ formula $A(\alpha, \beta, \vec{\gamma})$ and each $\Sigma^{\mathbb{O}}$ formula $B(\vec{\xi})$

of $\mathsf{L}_1^{\mathbb{O}}$ with exactly the displayed variables free, we have

$$\forall\alpha\exists\beta A(\alpha,\beta,\vec{\gamma}) \to \exists\delta[\mathsf{Ad}(\delta) \land \vec{\gamma} < \delta \land (\forall\alpha < \delta)(\exists\beta < \delta)A(\alpha,\beta,\vec{\gamma})],$$

$$\mathsf{Ad}(\delta) \land \vec{\xi} < \delta \land B^{\delta}(\vec{\xi}) \to (\exists\alpha < \delta)B^{\alpha}(\vec{\xi}).$$

To describe the aforementioned class $\mathcal{C}$ specifying the interpretation of the elementhood and naming predicates, we use the class $\{(x,y,z) : \exists\alpha\mathsf{P}_{\alpha}^{A}(x,y,z)\}$ for an appropriate formula $A(\mathsf{P},u,v,w)$. Thereby, we basically use the operator form $A(\mathsf{P},u,v,w)$ introduced in section 4 of [27]. Towards its definition, we first introduce the auxiliary formula $A_0(u,v,w)$. It is given by the disjunction of the clauses (1)–(23). The clauses (19)–(21) deal with the constant ch.

(1) $u = \langle 0,0\rangle \land v = 0 \land w = 0,$

(2) $u = \langle 0,0\rangle \land w = 1,$

(3) $u = \langle 1,0\rangle \land v = 0 \land w = 0,$

(4) $u = \langle 1,0\rangle \land \mathsf{U}(v) \land w = 1,$

(5) $u = \langle 2,0\rangle \land v = 0 \land w = 0,$

(6) $u = \langle 2,0\rangle \land \mathsf{V}(v) \land w = 1,$

(7) $u = \langle 3,0\rangle \land v = 0 \land w = 0,$

(8) $u = \langle 3,0\rangle \land \exists x(v = \langle x,x\rangle) \land w = 1,$

(9) $\exists x[u = \langle 4,x\rangle \land \mathsf{P}(x,0,0)] \land v = 0 \land w = 0,$

(10) $\exists x[u = \langle 4,x\rangle \land \mathsf{P}(x,0,0) \land {\sim}\mathsf{P}(x,v,1)] \land w = 1,$

(11) $\exists x,y[u = \langle 5,x,y\rangle \land \mathsf{P}(x,0,0) \land \mathsf{P}(y,0,0)] \land v = 0 \land w = 0,$

(12) $\exists x,y[u = \langle 5,x,y\rangle \land \mathsf{P}(x,0,0) \land \mathsf{P}(y,0,0) \land \mathsf{P}(x,v,1) \land \mathsf{P}(y,v,1)]$
$\land\, w = 1,$

(13) $\exists x[u = \langle 6,x\rangle \land \mathsf{P}(x,0,0)] \land v = 0 \land w = 0,$

(14) $\exists x,y[u = \langle 6,x\rangle \land \mathsf{P}(x,0,0) \land \mathsf{P}(x,\langle v,y\rangle,1)] \land w = 1,$

(15) $\exists x,f[u = \langle 7,x,f\rangle \land \mathsf{P}(x,0,0)] \land v = 0 \land w = 0,$

(16) $\exists x,f[u = \langle 7,x,f\rangle \land \mathsf{P}(x,0,0) \land \mathsf{P}(x,\{f\}(v),1)] \land w = 1,$

(17) $\exists x, f[u = \langle 8, x, f\rangle \wedge \mathsf{P}(x,0,0) \wedge \forall y(\mathsf{P}(x,y,1) \to \mathsf{P}(\{f\}(y),0,0))]$
$\wedge\, v = 0 \wedge w = 0$,

(18) $\exists x, f[u = \langle 8, x, f\rangle \wedge \mathsf{P}(x,0,0) \wedge \forall y(\mathsf{P}(x,y,1) \to \mathsf{P}(\{f\}(y),0,0))$
$\wedge\, \exists a, z(v = \langle a, z\rangle \wedge \mathsf{P}(x,z,1) \wedge \mathsf{P}(\{f\}(z),a,1))] \wedge w = 1$,

The clauses (19)–(21) below handle the translation of the constant ch.

(19) $\mathsf{P}(u,0,0) \wedge \mathsf{P}(\{u\}(v),0,0) \wedge \forall y[\mathsf{P}(v,y,1) \leftrightarrow \mathsf{P}(\{u\}(v),y,1)] \wedge w = 2$,

(20) $u = \langle 10, f\rangle \wedge \exists x[\mathsf{P}(f,x,2) \wedge (\forall y < x){\sim}\mathsf{P}(f,y,2)] \wedge v = 0 \wedge w = 0$,

(21) $u = \langle 10, f\rangle \wedge \exists x[\mathsf{P}(f,x,2) \wedge (\forall y < x){\sim}\mathsf{P}(f,y,2) \wedge \mathsf{P}(x,v,1)] \wedge w = 1$,

To express that the names given by P form a universe, the abbreviation

$$\mathsf{Univ}(\mathsf{P}) := \forall x, y, z[A(\mathsf{P},x,y,z) \to \mathsf{P}(x,y,z)]$$

is introduced. $A_1(\mathsf{P},u,v,w)$ is then the disjunction of $A_0(\mathsf{P},u,v,w)$ with the following clauses for the constant m:

(22) $\exists x, f[u = \langle 9, x, f\rangle \wedge \mathsf{P}(x,0,0) \wedge \forall y(\mathsf{P}(y,0,0) \to \mathsf{P}(\{f\}(y),0,0))]$
$\wedge\, \mathsf{Univ}(\mathsf{P}) \wedge v = 0 \wedge w = 0$,

(23) $\exists x, f[u = \langle 9, x, f\rangle \wedge \mathsf{P}(x,0,0) \wedge \forall y(\mathsf{P}(y,0,0) \to \mathsf{P}(\{f\}(y),0,0))]$
$\wedge\, \mathsf{Univ}(\mathsf{P}) \wedge \mathsf{P}(v,0,0) \wedge w = 0$.

To ensure that each triple belongs the a unique level of P^A, the operator form $A(\mathsf{P},u,v,w)$ takes the following form:

$$A(\mathsf{P},u,v,w) := A_1(\mathsf{P},u,v,w) \wedge {\sim}\mathsf{P}(u,v,w).$$

Exactly as in [27], we set

$$\mathsf{Rep}(u) := \exists \alpha \mathsf{P}^A_\alpha(u,0,0), \qquad \mathsf{E}(v,u) := \exists \alpha \mathsf{P}^A_\alpha(u,v,1),$$

and let the type variables of $\mathbb{L}$ range over Rep. The translation $\cdot^\star$ of the atoms of $\mathbb{L}_{\mathsf{ch},\mathsf{m}}$ involving types is as follows:

$$\begin{aligned}
\Re(U,t)^\star &:= \exists x[\mathsf{Val}_t(x) \wedge \mathsf{Rep}(x) \wedge \mathsf{Rep}(U) \wedge \forall y(\mathsf{E}(y,x) \leftrightarrow \mathsf{E}(y,U))],\\
(t \in U)^\star &:= \exists x[\mathsf{Val}_t(x) \wedge \mathsf{E}(x,U)],\\
(U = V)^\star &:= \forall x[\mathsf{E}(x,U) \leftrightarrow \mathsf{E}(x,V)].
\end{aligned}$$

The following is basically Theorem 6 in [27]. In this article, it is also shown that $|\mathsf{OMA}| = \varphi\omega 00$. Hence, lemma III.3.9 follows.

Theorem III.3.10 *For each finite set* $\Gamma(\vec{U},\vec{u})$ *of* $\mathbb{L}_{\mathsf{ch,m}}$ *formulas with exactly the variables* $\vec{U},\vec{u}$ *free,*

$$\mathsf{EMA}_0 + (\mathsf{ch}) \vdash \Gamma(\vec{U},\vec{u}) \Longrightarrow \mathsf{OMA} \vdash \neg\mathsf{Rep}(\vec{U}), \Gamma^\star(\vec{U},\vec{u}).$$

Proof: To see that the above theorem is still correct, we have to verify that the translation of the choice rule (ch) holds: So assume that there is a name x such that $ux \doteq x$. This translates to $\mathsf{Rep}(x) \wedge \forall y[\mathsf{E}(y,x) \leftrightarrow \mathsf{E}(y,\{u\}(x))]$. Thus, there is an ordinal α such that $\mathsf{P}^A_{\prec\alpha}(x,0,0)$ and $\mathsf{P}^A_{\prec\alpha}(\{u\}(x),0,0)$, hence $A_1(\mathsf{P}^A_{\prec\alpha},u,x,2)$ holds. Therefore, $\mathsf{P}^A_{\prec\beta}(u,x,2)$ for some ordinal β. Because OMA is equipped with $\Delta^{\mathbb{O}}_0$ induction on the natural numbers, $\{y : \mathsf{P}^A_{\prec\beta}(u,y,2)\}$ has a $<_{\mathsf{N}}$-least element. Thus $A_1(\mathsf{P}^A_{\prec\beta},\langle 10,u\rangle,0,0)$ holds and so either $\mathsf{P}^A_{\beta}(\langle 10,u\rangle,0,0)$ or $\mathsf{P}^A_{\prec\beta}(\langle 10,u\rangle,0,0)$. Now there is a unique ordinal γ with $\mathsf{P}^A_{\gamma}(\langle 10,u\rangle,0,0)$. The definition of A yields that $\mathsf{P}^A_{\gamma}(\langle 10,u\rangle,y,1)$ if and only if $\mathsf{P}^A_{\prec\gamma}(a,y,1)$, where a is the $<_{\mathsf{N}}$-least element of $\{z : \mathsf{P}^A_{\prec\gamma}(u,z,2)\}$. Moreover, we have that $\mathsf{P}^A_{\prec\gamma}(a,y,1)$ exactly if $\mathsf{P}^A_{\prec\gamma}(\{u\}(a),y,1)$. Thus, the translation of $\mathsf{ch}u \doteq u(\mathsf{ch}u)$ holds. □

Next, we consider the asymmetric interpretations of the theories $\mathsf{EETJ}_0 + (\mathsf{ch})$ and $\mathsf{EETJ}_0 + (\mathsf{it_N}) + (\mathsf{ch})$ into ACA_0 and Σ^1_1-DC_0, respectively. The translation of the first order part of $\mathbb{L}_{\mathsf{ch}}$ into L_2 is exactly the the translation $\cdot^\star$ described above. The translation of formulas involving type variables works similar, however, instead of the relation P^A, a hierarchy F along an initial segment of $\lhd$ is used. We modify the setup in such a way that $\mathsf{P}^A_\alpha(x,y,z)$ and $\mathsf{P}^A_{\prec\alpha}(x,y,z)$ correspond to the L_2 formulas $(\langle x,y,z\rangle \in (F)_\alpha)$ and $(\exists\beta \lhd \alpha)(\langle x,y,z\rangle \in (F)_\beta)$, respectively. To obtain such a hierarchy, we let $B_1(\mathsf{P},u,v,w)$ be the disjunction of the clauses (1)–(21) and $B(\mathsf{P},u,v,w) := B_1(\mathsf{P},u,v,w) \wedge {\sim}\mathsf{P}(u,v,w)$. Now we transform the $\mathsf{L}_1(\mathsf{P})$ formula $B(\mathsf{P},u,v,w)$ to the L_2 formula $A(U,u)$, by defining

$$A(U,u) := \exists x,y,z[u = \langle x,y,z\rangle \wedge A_1(U,x,y,z)],$$

where $A_1(U,u,v,w)$ is obtained from $B(\mathsf{P},u,v,w)$ by replacing all literals of the form $[{\sim}]\mathsf{P}(r,s,t)$ by $[{\sim}](\langle r,s,t\rangle \in U)$. Below, F will be so that $\mathsf{Hier}^A(F,\emptyset,\lhd\restriction\gamma)$ for some γ in the field of $\lhd$. Note also, that if $\beta\lhd\gamma$, then $\langle x,y,z\rangle \in (F)_\beta \leftrightarrow A((F)_{\lhd\alpha},\langle x,y,z\rangle)$, which corresponds to the axiom $\mathsf{P}^A_\alpha(\vec{u}) \leftrightarrow A(\mathsf{P}^A_{\prec\alpha},\vec{u})$ of OMA. The definition of A also entails, that if $\mathsf{Hier}^A(F,\emptyset,\lhd\restriction\gamma)$, then all the levels $(F)_\beta$ for $\beta\lhd\gamma$ are disjoint. To prepare for the definition of the asymmetric interpretation into L_2, we let

$$\begin{aligned}\mathsf{Rep}(U,u,w) &:= (\exists\alpha \lhd w)(\langle u,0,0\rangle \in (U)_\alpha),\\ \mathsf{E}(U,v,u,w) &:= (\exists\alpha \lhd w)(\langle u,v,1\rangle \in (U)_\alpha).\end{aligned}$$

Then $\{x : \mathsf{Rep}(F,x,\beta)\}$ is the set of codes of names below the βth level of the hierarchy. Similar to the translation of $\mathbb{L}_{\mathsf{ch,m}}$ into $\mathsf{L}^{\mathbb{O}}_1$, we now define for each formula B of

$\mathbb{L}_{\mathsf{ch}}$ its asymmetric translation $B^{\alpha,\beta,F}$. The idea is to let range universally bounded type variables over $\{u : \mathsf{Rep}(F,u,\alpha)\}$ and existentially bounded type variables over $\{u : \mathsf{Rep}(F,u,\alpha{+}\beta)\}$ for a suitable β. Recall that type variables of $\mathbb{L}$ go over to number variables of L_2. If we want to be precise, we write $U^\star$ for the number variable of L_2 that is the translation of the type variable U.

To specify the asymmetric interpretation, we extend the language L_2 by constants p_u for each number variable u of L_2. The constants p_u serve as placeholders and are later replaced by elements of the field of $\lhd$. If B is a formula that does not contain type variables then $B^{\alpha,\beta,F} := B^\star$.

$$\begin{aligned}
\Re(U,t)^{\alpha,\beta,F} &:= \exists x[\mathsf{Val}_t(x) \wedge \mathsf{Rep}(F,x,\mathsf{p}_{U^\star}) \wedge \\
&\qquad\qquad \forall y(\mathsf{E}(F,y,x,\mathsf{p}_{U^\star}) \leftrightarrow \mathsf{E}(F,y,U,\mathsf{p}_{U^\star}))],\\
(t \in U)^{\alpha,\beta,F} &:= \exists x[\mathsf{Val}_t(x) \wedge \mathsf{E}(F,x,U,\mathsf{p}_{U^\star})],\\
(U = V)^{\alpha,\beta,F} &:= \forall x(\mathsf{E}(F,x,U,\mathsf{p}_{U^\star}) \leftrightarrow \mathsf{E}(F,x,V,\mathsf{p}_{V^\star})).
\end{aligned}$$

The asymmetric interpretation is distributive over conjunction, disjunction and number quantification. Further, the translation of the negation of an atom is the negation of the translation of the atom. Quantification over type variables is handled as follows:

$$\begin{aligned}
(\forall X^\star B)^{\alpha,\beta,F} &:= \forall X^\star(\mathsf{Rep}(F,X^\star,\alpha) \to B^{\alpha,\beta,F}[\alpha/\mathsf{p}_{X^\star}]),\\
(\exists X^\star B)^{\alpha,\beta,F} &:= \exists X^\star(\mathsf{Rep}(F,X^\star,\beta) \wedge B^{\alpha,\beta}[\beta/\mathsf{p}_{X^\star}]).
\end{aligned}$$

Now the following is easily proved by induction on the build-up of formulas.

Lemma III.3.11 (Persistence) *Assume that $A(U,u)$ is the formula defined above. Then, for each Σ^+ formula $B(\vec{V})$ of $\mathbb{L}$ with exactly the displayed type variables free, the following is provable in ACA_0: If $\mathsf{Hier}^A(F,\emptyset,\lhd\restriction\gamma)$ and $\alpha \unlhd \alpha' \unlhd \beta' \unlhd \beta \lhd \gamma$, then*

$$\vec{y} \in \{x : \mathsf{Rep}(F,x,\alpha)\} \wedge B^{F,\alpha,\beta}(\vec{y}) \to B^{F,\alpha',\beta'}(\vec{y}).$$

To carry out the asymmetric interpretations, we require that derivations in $\mathsf{EETJ}_0 + (\mathsf{ch})$ and $\mathsf{EETJ}_0 + (\mathsf{ch}) + (\mathsf{it_N})$ can be transformed to derivations applying the cut rule only to formulas that are Σ^+ or Π^-. Such a partial cut-elimination is guaranteed by theorem I.3.5, if the main formulas of all axioms and rules are Σ^+ or Π^-. This is not the case with the axiomatization given in subsection I.2.11. Most of the basic type existence axioms are not Σ^+ or Π^- formulas. However, with the exception of the axiom for join, it is obvious how the replace it by a rule whose main formula is Σ^+. For join, we use a rule similar to the one given in Glass and Strahm [17]. Thereto, we define $B(U,V,v)$ to be the formula

$$\forall x(x \in V \leftrightarrow x = (\mathsf{p}_0 x, \mathsf{p}_1 x) \wedge \mathsf{p}_0 x \in U \wedge \exists X(\Re(X, v(\mathsf{p}_0 x)) \wedge \mathsf{p}_1 x \in X)).$$

Note, that if $\Re(X,x)$, $(f : X \to \Re)$ and $\Re(Y, \mathsf{j}(x,f))$, then we have $B(X,Y,f)$. Further, we define $A_1(U,V,u,v) := \Re(V,\mathsf{j}(u,v)) \wedge B(U,V,v)$. It is now immediate that $\Re(U,u) \wedge (f : U \to \Re) \to \exists Y A_1(U,Y,u,f)$ is equivalent to the axiom for join given in I.2.11. Moreover, we let $A_2(U,V,u,v)$ be the following Σ^+ formula of $\mathbb{L}$:

$$\begin{aligned} &\exists X, Y \forall z [\, \Re(V, \mathsf{j}(u,v)) \wedge \\ &\qquad (z \in V \to z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in U \wedge \Re(X, f(\mathsf{p}_0 z)) \wedge \mathsf{p}_1 z \in X) \wedge \\ &\qquad\quad (z = (\mathsf{p}_0 z, \mathsf{p}_1 z) \wedge \mathsf{p}_0 z \in U \wedge (\Re(Y, f(\mathsf{p}_0 z)) \to \mathsf{p}_1 z \in Y) \to z \in V)\,]. \end{aligned}$$

Lemma III.3.12 *The following is provable in* EET_0*:*

$$(f : U \to \Re) \to [\exists Z A_1(U,Z,y,f) \leftrightarrow \exists Z A_2(U,Z,y,f)].$$

Therefore, we can replace the axiom for join by the following rule: For each finite set Γ of $\mathbb{L}$ formulas,

$$\frac{\Gamma, \Re(U,u) \wedge (f : U \to \Re)}{\Gamma, \exists Z A_2(U,Z,u,f)}.$$

So we can assume for the following, that we have reformulations of $\mathsf{EETJ}_0 + (\mathsf{ch})$ and $\mathsf{EETJ}_0 + (\mathsf{ch})(\mathsf{it}_\mathsf{N})$ that allow to eliminated all cuts that are neither Σ^+ nor Π^-.

Lemma III.3.13 *Suppose that $A(U,u)$ is the L_2 formula defined above. Then, for each finite set Γ of Σ^+ and Π^- formulas of $\mathbb{L}_{\mathsf{ch}}$, the following holds for all $m,n \in \mathbb{N}$ and all ordinals $\alpha < \omega^\omega$:*

$$\mathsf{EETJ}_0 + (\mathsf{ch}) \mid\!\frac{n}{*}\, \Gamma(\vec{U}) \implies \mathsf{ACA}_0 \vdash \neg B(F, \vec{U}^\star, m, n), \Gamma^{m,m+2^n,F}[\vec{m}/\vec{\mathsf{p}}_{U^\star}],$$

$$\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch}) \mid\!\frac{n}{*}\, \Gamma(\vec{U}) \implies \mathsf{ACA}_0 \vdash \neg C(F, \vec{U}^\star, \alpha, n), \Gamma^{\alpha,\alpha+\omega^n,F}[\vec{\alpha}/\vec{\mathsf{p}}_{U^\star}],$$

where $*$ stands of the set of Σ^+ and Π^- formulas of $\mathbb{L}_{\mathsf{ch}}$, and

$$\begin{aligned} B(F,\vec{u},m,n) &:= \mathsf{Hier}^A(F,\emptyset,\lhd\!\restriction\! m{+}2^n) \wedge \vec{u} \in \{z : \mathsf{Rep}(F,z,m)\}, \\ C(F,\vec{u},\alpha,n) &:= \mathsf{Hier}^A(F,\emptyset,\lhd\!\restriction\! \alpha{+}\omega^n) \wedge \vec{u} \in \{z : \mathsf{Rep}(F,z,\alpha)\}. \end{aligned}$$

Further, if $\vec{U} = U_1, \ldots U_l$, then $\Gamma[\vec{m}/\vec{\mathsf{p}}_{U^\star}]$ is short for $\Gamma[m,\ldots,m/\mathsf{p}_{U_1^\star},\ldots \mathsf{p}_{U_l^\star}]$, and $\Gamma[\vec{\alpha}/\vec{\mathsf{p}}_{U^\star}]$ is defined accordingly.

Proof: Both claims are show by (meta-) induction on n. For the first statement, we consider the choice rule (ch), for the second statement we have a look at the iteration rule. The cut is treated in the standard way exploiting persistence. Below, we let T_1 be the theory $\mathsf{EET}_0 + (\mathsf{ch})$ and T_2 the theory $\mathsf{T}_1 + (\mathsf{it}_\mathsf{N})$.

So assume that $\mathsf{T}_1 \overset{n}{\underset{*}{\vdash}} \Gamma, \mathsf{ch}f \doteq f(\mathsf{ch}f)$ was obtained by applying a faithful instance of the choice rule. Thus, there is an $n' < n$ and a term f of $\mathbb{L}_{\mathsf{ch}}$, so that we have $\mathsf{T}_1 \overset{n'}{\underset{*}{\vdash}} \Gamma, \sim f\downarrow, \exists x[\Re(x) \wedge fx \doteq x]$. By the I.H. we conclude that ACA_0 proves

$$\neg B(F, \vec{U}^\star, m, n'), \neg\exists x \mathsf{Val}_f(x), \Gamma^{m,m+2^{n'},F}[\vec{m}/\vec{\mathsf{p}}_{U^\star}], (\exists x[\Re(x) \wedge fx \doteq x])^{m,m+2^{n'},F}.$$

Now we assume that $B(F, \vec{U}^\star, m, n')$, $\mathsf{Val}_f(e)$ and $(\exists x[\Re(x) \wedge fx \doteq x])^{m,m+2^{n'},F}$. This implies that there are $k_1 \leq k_2 < m+2^{n'}$ and a z, such that $\mathsf{Rep}(F, z, k_1)$, $\mathsf{Rep}(F, \{e\}(z), k_2)$ and $\forall y[\mathsf{E}(F, y, z, k_2) \leftrightarrow \mathsf{E}(F, y, \{e\}(z), k_2)]$. Arguing analogously to the proof of theorem III.3.10, the definition of the hierarchy F yields that $\mathsf{Rep}(F, \langle 10, e\rangle, k_2{+}1)$ and that $\forall y[\mathsf{E}(F, y, z, k_2{+}1) \leftrightarrow \mathsf{E}(F, y, \langle 10, e\rangle, k_2{+}1)]$.

If $\mathsf{T}_2 \overset{n}{\underset{*}{\vdash}} \Gamma, \mathsf{Hier}^f(\underline{\mathsf{hier}}(f, \mathsf{nat}_<), \mathsf{nat}_<)$ was obtained by the application of faithful instance of the iteration rule, then we have $\mathsf{T}_2 \overset{n'}{\underset{*}{\vdash}} \sim f\downarrow, \Gamma, (f : \Re \to \Re)$ for some $n' < n$. By the I.H. we conclude that ACA_0 proves

$$\neg C(F, \vec{U}^\star, \alpha, n'), \neg\exists x \mathsf{Val}_f(x), \Gamma^{\alpha,\alpha+\omega^{n'},F}[\vec{\alpha}/\vec{\mathsf{p}}_{U^\star}], (f : \Re \to \Re)^{\alpha,\alpha+\omega^{n'},F}.$$

This time, we assume that $C(F, \vec{U}^\star, \alpha, n')$, $\mathsf{Val}_f(e)$ and $(f : \Re \to \Re)^{\alpha,\alpha+\omega^{n'},F}$. Due to the definition of the hierarchy F, we have $\forall x[\mathsf{Rep}(F, x, \alpha) \to \mathsf{Rep}(F, \{e\}(x), \alpha{+}\omega^{n'})]$. Observe, that

$$a := \underline{\mathsf{hier}}(f, \mathsf{nat}_<)(0) = f(\mathsf{j}(\mathsf{int}(\underline{\mathsf{field}}(\mathsf{nat}_<), \mathsf{inv}(\mathsf{nat}_<, \lambda y.(y, 0))), \underline{\mathsf{hier}}(f, \mathsf{nat}_<)))$$

is a name and that $\mathsf{Val}_a(y)$ yields $\mathsf{Rep}(F, y, k{+}\omega^{n'})$ for some $k \in \mathsf{N}$. Next, we let e' so that $\mathsf{Val}_{\underline{\mathsf{hier}}(f,\mathsf{nat}_<)}(e')$ and show by set induction that

$$(\forall x \in \mathsf{N})[\mathsf{Rep}(F, \{e'\}(x), k{+}\omega^{n'}{\cdot}(1{+}x))].$$

Thus, $(\forall x \in \mathsf{N})[\mathsf{Rep}(F, \{e'\}(x), \alpha{+}\omega^n)]$, which is $(\mathsf{Hier}^f(\underline{\mathsf{hier}}(f, \mathsf{nat}_<), \mathsf{nat}_<))^{\alpha,\alpha+\omega^n,F}$.
□

If we identify each natural number $n \in \mathbb{N}$ with the notation of the ordinal n, then we can state the following lemma:

Lemma III.3.14

$$\mathsf{ACA}_0 \vdash \exists F \mathsf{Hier}^A(F, \emptyset, \lhd\restriction n) \quad \textit{and} \quad \Sigma^1_1\text{-}\mathsf{DC}_0 \vdash \exists F \mathsf{Hier}^A(F, \emptyset, \lhd\restriction\omega^n).$$

Together with the previous lemma, this yields the upper bounds for $\mathsf{EETJ}_0 + (\mathsf{ch})$ and $\mathsf{EETJ}_0 + (\mathsf{ch}) + (\mathsf{it}_\mathsf{N})$ mentioned in the lemmas III.3.7 and III.3.8.

Theorem III.3.15 *For each finite set Γ of Σ^+ of $\mathbb{L}$ without free type variables,*

$$\mathsf{EETJ}_0 + (\mathsf{ch}) \vdash \Gamma \implies \mathsf{ACA}_0 \vdash \Gamma^\star,$$

$$\mathsf{EETJ}_0 + (\mathsf{it}) + (\mathsf{ch}) \vdash \Gamma \implies \Sigma^1_1\text{-}\mathsf{DC}_0 \vdash \Gamma^\star.$$

Bibliography

[1] Peter Aczel, *The strength of Martin-Löf's type theory with one universe*, Tech. report, Dept. of Philosophy, University of Helsinki, 1977.

[2] Jeremy Avigad, *On the relationship between ATR_0 and $\widehat{ID}_{<\omega}$*, The Journal of Symbolic Logic **61** (1996), no. 3, 768–779.

[3] Jon Barwise, *Admissible sets and structures: An approach to definability theory*, Springer, Berlin, 1975.

[4] Jon Barwise and Jon Schlipf, *On recursively saturated models of arithmetic*, Model Theory and Algebra, Lecture Notes in Mathematics, vol. 498, Springer, 1975, pp. 42–55.

[5] Michael J. Beeson, *Foundations of constructive mathematics: Metamathematical studies*, Springer, Berlin, 1985.

[6] Wilfried Buchholz, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg, *Iterated inductive definitions and subsystems of analysis: Recent proof-theoretical studies*, Lecture Notes in Mathematics, vol. 897, Springer, Berlin, 1981.

[7] Andrea Cantini, *On the relationship between choice and comprehension principles in second order arithmetic*, Journal of Symbolic Logic **51** (1986), 360–373.

[8] Solomon Feferman, *A language and axioms for explicit mathematics*, Algebra and Logic (J.N. Crossley, ed.), Lecture Notes in Mathematics, vol. 450, Springer, Berlin, 1975, pp. 87–139.

[9] ______, *Recursion theory and set theory: a marriage of convenience*, Generalized recursion theory II, Oslo 1977 (J. E. Fenstad, R. O. Gandy, and G. E. Sacks, eds.), Stud. Logic Found. Math, vol. 94, North Holland, Amsterdam, 1978, pp. 55–98.

[10] ______, *Constructive theories of functions and classes*, Logic Colloquium '78 (M. Boffa, D. van Dalen, and K. McAloon, eds.), North Holland, Amsterdam, 1979, pp. 159–224.

[11] ———, *Iterated inductive fixed-point theories: application to Hancock's conjecture*, The Patras Symposion (G. Metakides, ed.), North Holland, Amsterdam, 1982, pp. 171–196.

[12] Solomon Feferman and Gerhard Jäger, *Systems of explicit mathematics with non-constructive μ-operator. Part II*, Annals of Pure and Applied Logic **79** (1996), no. 1, 37–52.

[13] Solomon Feferman and Clifford Spector, *Incompleteness along paths in progressions of theories*, The Journal of Symbolic Logic **27** (1960), no. 4, 383–390.

[14] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians, Vancouver 1974, vol. 1, Canadian Mathematical Congress, 1975, pp. 235–242.

[15] Harvey Friedman, Kenneth McAloon, and Stephen Simpson, *A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis*, Patras Symposion (G. Metakides, ed.), North Holland, Amsterdam, 1982, pp. 197–230.

[16] Robin O. Gandy, *Proof of Mostowski's conjecture*, Bulletin de l'Academie Polonaise des Sience, Série des Siences Mathematiques, Astronomiques et Physiques **8** (1960), 571–575.

[17] Thomas Glass and Thomas Strahm, *Systems of explicit mathematics with non-constructive μ-operator and join*, Annals of Pure and Applied Logic **82** (1996), 193–219.

[18] G. Jäger, *Induction in the elementary theory of types and names*, Computer Science Logic '87 (E. Börger, H. Kleine Büning, and M.M. Richter, eds.), Lecture Notes in Computer Science, vol. 329, Springer, Berlin, 1988, pp. 118–128.

[19] Gerhard Jäger, *Die konstruktible hierarchie als hilfsmittel zur beweistheoretischen untersuchung von teilsystemen der mengenlehre und analysis*, Ph.D. thesis, Universität München, 1979.

[20] ———, *Theories for admissible sets: A unifying approach to proof theory*, Bibliopolis, Napoli, 1986.

[21] ———, *First order theories for nonmonotone inductive definitions: Recursive and inaccessible mahlo*, Journal of Symbolic Logic **66** (2001), no. 3, 1073–1089.

[22] ———, *Einführung in die theoretische informatik*, 2004, Lecture Notes.

[23] Gerhard Jäger and Barbara Primo, *About the proof-theoretic ordinals of weak fixed point theories*, Journal of Symbolic Logic **57** (1992), no. 3, 1108–1119.

[24] Gerhard Jäger and Dieter Probst, *Iterating Σ operations in admissible set theory without foundation: a further aspect of metapredicative Mahlo*, One Hundred Years of Russell's Paradox: Mathematics, Logic, Philosophy (Godehard Link, ed.), de Gruyter, Berlin, 2004, pp. 119–134.

[25] ———, *Variation on a theme of Schütte*, Mathematical Logic Quarterly **50** (2004), no. 3, 258–264.

[26] Gerhard Jäger and Thomas Strahm, *Fixed point theories and dependent choice*, Archive for Mathematical Logic **39** (2000), 493–508.

[27] ———, *Upper bounds for metapredicative Mahlo in explicit mathematics and admissible set theory*, The Journal of Symbolic Logic **66** (2001), no. 2, 935–958.

[28] Wolfram Pohlers, *Proof theory: An introduction*, Lecture Notes in Mathematics, vol. 1407, Springer, Berlin, 1989.

[29] Dieter Probst, *Iteration, choice and reflection in explicit mathematics*, In preparation.

[30] ———, *The proof-theoretic analysis of transfinitely iterated quasi least fixed points*, To appear in the Journal of Symbolic Logic.

[31] ———, *Dependent choice in explicit mathematics*, Master's thesis, Institut für Informatik und angewandte Mathematik, 1999.

[32] ———, *On the relationship between fixed points and iteration in admissible set theory without foundation*, Archive for Mathematical Logic (2005), no. 44, 561–580.

[33] Dieter Probst and Thomas Studer, *How to normalize the jay*, Theoretical Computer Science **1–2** (2001), no. 254, 677–681.

[34] Michael Rathjen, *The strength of Martin-Löf type theory with a superuniverse. Part II*, Archive for Mathematical Logic **40** (2001), no. 3, 207–233.

[35] H. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.

[36] Christian Rüede, *Metapredicative subsystems of analysis*, Ph.D. thesis, Institut für Informatik und angewandte Mathematik, Univeristät Bern, 2000.

[37] ———, *Transfinite dependent choice and omega-model reflection*, Journal of Symbolic Logic **67** (2002), no. 3, 1153–1168.

[38] ———, *The proof-theoretic analysis of* Σ^1_1 *transfinite dependent choice*, Annals of Pure and Applied Logic **121** (2003), no. 1, 195–234.

[39] Kurt Schütte, *Proof theory*, Springer, Berlin, 1977.

[40] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag, 1998.

[41] Clifford Spector, *Hyperarithmetic quantifiers*, Fundamenta Mathematicae **48** (1959), 313–320.

[42] John R. Steel, *Determinanteness and subsystems of analysis*, Ph.D. thesis, University of California at Berkely, 1976.

[43] Thomas Strahm, *Wellordering proofs for metapredicative Mahlo*, The Journal of Symbolic Logic **67** (2002), no. 1, 260–278.

[44] William Tait, *Normal derivability in classical logic*, The Syntax and Semantics of Infinitary Languages (Jon Barwise, ed.), Springer, Berlin, 1968, pp. 204–236.

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