Deep Inference and its Normal Form of Derivations

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Abstract. We see a notion of normal derivation for the calculus of structures, which is based on a factorisation of derivations and which is more general than the traditional notion of cut-free proof in this formalism.

1 Introduction

An inference rule in a traditional proof theoretical formalism like the sequent calculus or natural deduction only has access to the main connective of a formula. It does not have the feature of *deep inference*, which is the ability to access subformulas at arbitrary depth. Proof theoretical systems which do make use of this feature can be found as early as in Schütte [21], or, for a recent example, in Pym [19]. The *calculus of structures* is a formalism due to Guglielmi [11] which is centered around deep inference. Thanks to deep inference it drops the distinction between logical and structural connectives, a feature which already Schütte desired [20]. It also drops the tree-shape of derivations to expose a vertical symmetry which is in some sense new. One motivation of the calculus of structures is to find cut-free systems for logics which lack cut-free sequent systems. There are plenty of examples of such logics, and many are relevant to computer science: important modal logics like S5, many temporal and also intermediate logics. The logic that gave rise to the calculus of structures is the substructural logic BV which has connectives that resemble those of a process algebra and which can not be expressed without deep inference [26]. Systems in the calculus of structures so far have been studied for linear logic [24], non-commutative variants of linear logic [13,7], classical logic [2] and several modal logics [22].

In this paper we ask the question what the right notion of *cut-free*, *normal* or *analytic* proof should be in the calculus of structures, and we see one such notion which is a factorisation of derivations and which generalises the notion that is used in the works cited above. This factorisation has independently been discovered by McKinley in [18]. The existence of normal derivations follows easily from translations between sequent calculus and calculus of structures. Here we consider systems for classical predicate logic, i.e. system LK [9] and system SKSgq

[2] as examples, but it is a safe conjecture that this factorisation applies to any logic which has a cut-free sequent calculus.

After recalling a system for classical predicate logic in the calculus of structures, as well as the traditional notion of cut admissibility for this system, we see a more general notion based on factorisation as well as a proof that each derivation can be factored in this way. An outlook on some current research topics around the calculus of structures concludes this paper.

2 A Proof System for Classical Logic

The *formulas* for classical predicate logic are generated by the grammar

$$A ::= \mathsf{f} \mid \mathsf{t} \mid a \mid [\,A, A\,] \mid (\,A, A\,) \mid \exists x A \mid \forall x A \quad ,$$

where f and t are the units *false* and *true*, a is an *atom*, which is a predicate symbol applied to some terms, possibly negated, $[A, B]$ is a *disjunction* and (A, B) is a *conjunction*. Atoms are denoted by a, b, c, formulas are denoted by A, B, C, D. We define $\bar{A}$, the *negation* of the formula A, as usual by the De Morgan laws. There is a *syntactic equivalence relation* on formulas, which is the smallest congruence relation induced by commutativity and associativity of conjunction and disjunction, the capture-avoiding renaming of bound variables as well as the following equations:

$$\begin{array}{ll} [A, \mathsf{f}] = A & [\mathsf{t}, \mathsf{t}] = \mathsf{t} \\ (A, \mathsf{t}) = A & (\mathsf{f}, \mathsf{f}) = \mathsf{f} \end{array} \qquad \forall x A = A = \exists x A \quad \text{if } x \text{ is not free in } A \quad .$$

Thanks to associativity, we write $[A, B, C]$ instead of $[A, [B, C]]$, for example.

The *inference rules* in the calculus of structures are just rewrite rules known from term rewriting that work on formulas modulo the equivalence given above. There is the notational difference that here the context $S\{\ \}$, in which the rule is applied, is made explicit. Here are two examples of inference rules:

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[A, \bar{A}]} \qquad \text{and} \qquad \mathsf{i}\!\uparrow \frac{S(A, \bar{A})}{S\{\mathsf{f}\}} \quad .$$

The name of the rule on the left is $\mathsf{i}\!\downarrow$ (read *i–down* or *identity*), and seen from top to bottom or from premise to conclusion it says that wherever the constant t occurs inside a formula, it can be replaced by the formula $[A, \bar{A}]$ where A is an arbitrary formula. The rule on the right (read *i–up* or *co-identity* or *cut*), also seen from top to bottom, says that anywhere inside a formula the formula $(A, \bar{A})$ can be replaced by the constant f. The two rules are *dual* meaning that one is obtained from the other by exchanging premise and conclusion and replacing each connective by its De Morgan dual. Here is another example of an inference rule, which is called *switch* and which happens to be its own dual:

$$\mathsf{s} \frac{S([A, B], C)}{S[(A, C), B]} \quad .$$

A *derivation* is a finite sequence of instances of inference rules. For example

$$\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{([A,C],[B,D])}{[A,(C,[B,D])]}}{[A,B,(C,D)]} \quad .$$

The topmost formula in a derivation is its *premise* of the derivation, and the formula at the bottom is its *conclusion*. A *proof* is a derivation with the premise t. Dually, a *refutation* is a derivation with the conclusion f.

Figure 1 shows system SKSgq from [2]: a system for classical predicate logic. It is *symmetric* in the sense that for each rule in the system, the dual rule is also in the system. Like all systems in the calculus of structures it consists of two dual fragments: an up- and a down-fragment. The *down-fragment* is the system $\{\mathsf{i}{\downarrow}, \mathsf{s}, \mathsf{w}{\downarrow}, \mathsf{c}{\downarrow}, \mathsf{u}{\downarrow}, \mathsf{n}{\downarrow}\}$ and the *up-fragment* is the system $\{\mathsf{i}{\uparrow}, \mathsf{s}, \mathsf{w}{\uparrow}, \mathsf{c}{\uparrow}, \mathsf{u}{\uparrow}, \mathsf{n}{\uparrow}\}$. We also denote these two systems respectively by $\downarrow$ and $\uparrow$ and their union, the symmetric system, by $\updownarrow$. The letters w, c, u, n are respectively for *weakening*, *contraction*, *universal* and *instantiation*. It is proved in [2] that the down-fragment is complete in the sense that it has a proof for each valid formula, the up-fragment is complete in the sense that it has a refutation for each unsatisfiable formula and their union is complete also in the sense that for each valid implication it has a derivation from the premise to the conclusion of this implication.

3 Cut Elimination

The importance of cut-free proofs in the sequent calculus comes from the fact that they have the subformula property. Now, clearly the subformula property does not make sense for the calculus of structures in the same way as a "subsequent property" does not make sense for the sequent calculus. So the question for the calculus of structures is: what is is a cut-free proof?

Definition 1 (Cut-free Proof). A proof in the calculus of structures is *cut-free* if it does not contain any up-rules.

The cut elimination theorem takes the following form:

Theorem 2 (Up-fragment Admissibility). For each proof in the symmetric system there is a proof in the down-fragment with the same conclusion.

The above notion seems reasonable for our system for classical predicate logic, since it gives us the usual immediate consequences of Gentzen's Hauptsatz such as consistency and Herbrand's Theorem [3]. Craig Interpolation also follows, but it would be a bit of a stretch to call it an immediate consequence. It requires some work because rules are less restricted in the calculus of structures than in the sequent calculus.

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[A,\bar{A}]} \qquad \mathsf{i}{\uparrow}\,\frac{S(A,\bar{A})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([A,B],C)}{S[(A,C),B]}$$

$$\mathsf{u}{\downarrow}\,\frac{S\{\forall x[A,B]\}}{S[\forall xA,\exists xB]} \qquad \mathsf{u}{\uparrow}\,\frac{S(\forall xA,\exists xB)}{S\{\exists x(A,B)\}}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{A\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{A\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}{\downarrow}\,\frac{S[A,A]}{S\{A\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{A\}}{S(A,A)}$$

$$\mathsf{n}{\downarrow}\,\frac{S\{A[x/t]\}}{S\{\exists xA\}} \qquad \mathsf{n}{\uparrow}\,\frac{S\{\forall xA\}}{S\{A[x/t]\}}$$

Fig. 1. Predicate logic in the calculus of structures

Since for classical predicate logic there is a cut-free sequent system, Theorem 2 can be proved easily: we first translate derivations from the calculus of structures into this sequent system, using the cut in the sequent system to cope with the deep applicability of rules. Then we apply the cut elimination theorem for the sequent system. Finally we translate back the cut-free proof into the calculus of structures, which does not introduce any up-rules. Details are in [2]. To give an idea of how derivations in the sequent calculus are translated into the calculus of structures and to justify why the $\mathsf{i}{\uparrow}$-rule is also named *cut*, we see the translation of the cut rule:

$$\mathsf{Cut}\,\frac{\Phi\vdash A,\Psi \quad \Phi',A\vdash\Psi'}{\Phi,\Phi'\vdash\Psi,\Psi'} \qquad \text{translates into} \qquad \dfrac{\mathsf{i}{\uparrow}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{([\bar{\Phi},A,\Psi],[\bar{\Phi}',\bar{A},\Psi'])}{[\bar{\Phi},\Psi,(A,[\bar{\Phi}',\Psi',\bar{A}])]}}{[\bar{\Phi},\bar{\Phi}',\Psi,\Psi',(A,\bar{A})]}}{[\bar{\Phi},\bar{\Phi}',\Psi,\Psi',\mathsf{f}]}}{[\bar{\Phi},\bar{\Phi}',\Psi,\Psi']}{=} \quad .$$

A natural question here is whether there is an *internal* cut elimination procedure, i.e. one which does not require a detour via the sequent calculus. Such a procedure was nontrivial to find, since the deep applicability of rules renders the techniques of the sequent calculus useless. It has been given in [2,1] for the propositional fragment and has been extended to predicate logic in [3].

Now we see a more general notion of cut-free or normal derivation. It is not characterised by the absence of certain inference rules, but by the the way in which the inference rules are composed:

Definition 3 (Normal Derivation). A derivation in the calculus of structures is *normal* if no up-rule occurs below a down-rule. To put it differently, a normal derivation has the form

$$\begin{array}{c} A \\ \Big\Vert \uparrow \\ B \\ \Big\Vert \downarrow \\ C \end{array} \quad .$$

This definition subsumes the definition of a cut-free proof: a proof is cut-free if and only if it is normal. Consider a proof, i.e. a derivation with premise syntactically equivalent to t, of the form given in the definition above. Since the conclusion of all rules in the up-fragment is equivalent to t if their premise is equivalent to t, then B has to be equivalent to t. We thus have a proof of C in the down-fragment. So the following theorem subsumes the admissibility of the up-fragment:

Theorem 4 (Normalisation). For each derivation in the symmetric system there is a normal derivation with the same premise and conclusion.

We will see a proof of this theorem shortly, but let us first have a look at an immediate consequence. Since no rule in the up-fragment introduces new predicate symbols going down and no rule in the down-fragment introduces new predicate symbols going up, the formula that connects the derivation in the up- with the derivation in the down-fragment is an interpolant:

Corollary 5 (Craig Interpolation). For each two formulas A, C such that A implies C there is a formula B such that A implies B, B implies C and all the predicate symbols that occur in B occur in both A and C.

To prove Theorem 4 we go the easy route just as for Theorem 2, we use cut elimination for LK and translations that we see in the two lemmas that follow. However, there is a crucial difference between the translations used to obtain Theorem 2 and the translations that we are going to see now: while the former just squeeze a tree into a sequence by glueing together the branches, the latter will rotate the proof by ninety degrees. We use a version of LK, which works on multisets of formulas, has multiplicative rules and which is restricted to formulas in negation normal form. LK is cut-free, we denote the system with the cut rule as $\mathsf{LK} + \mathsf{Cut}$. It is easy to check that we preserve cut admissibility when we replace the negation rules by two additional axioms:

$$A, \bar{A} \vdash \qquad \text{and} \qquad \vdash A, \bar{A} \qquad .$$

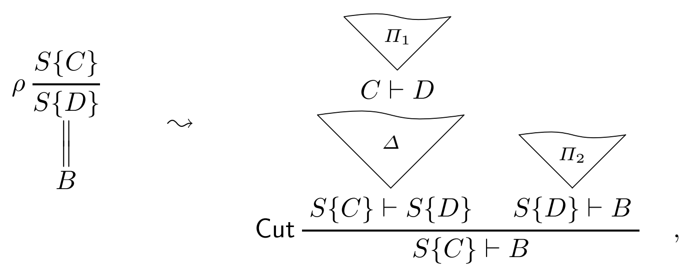
Formulas of the calculus of structures and sequents of the sequent calculus are easily translated into one another: for a sequent

$$\Phi \vdash \Psi \quad = \quad A_1, \ldots, A_m \vdash B_1, \ldots, B_n$$

we obtain two formulas that we denote by Φ and Ψ as well: $(A_1, \ldots, A_m)$ and $[B_1, \ldots, B_n]$. We identify an empty conjunction with t and an empty disjunction with f.

Lemma 6 (SKS to LK). For each derivation from A to B in SKSgq there is a proof of $A \vdash B$ in $\mathsf{LK} + \mathsf{Cut}$.

Proof. By induction on the length of the derivation, where we count applications of the equivalence as inference rules. The base case gives an axiom in the sequent calculus. The inductive case looks as follows:



where Π_2 exists by induction hypothesis, the existence of the derivation Δ can be easily shown for arbitrary formulas C, D and the existence of the proof Π_1 can be easily shown for each rule $\rho \in \mathsf{SKSgq}$ and for the equations which generate the syntactic equivalence. □

Lemma 7 (LK to SKS). For each proof of $\Phi \vdash \Psi$ in LK there is a normal derivation from Φ to Ψ in SKSgq.

Proof. By induction on the depth of the proof tree. All cases are shown in Figure 2 and Figure 3. In the cases of the $\vee_{\mathsf{L}}, \wedge_{\mathsf{R}}$-rules we get two normal derivations by induction hypothesis, and they have to be taken apart and composed in the right way to yield the normal derivation that is shown in the picture. In the cases of the $\exists_{\mathsf{L}}, \forall_{\mathsf{R}}$-rules the proviso on the eigenvariable is exactly what is needed to ensure the provisos of the syntactic equivalence. □

$$A \vdash A \quad\rightsquigarrow\quad A$$

$$A,\bar{A} \vdash \quad\rightsquigarrow\quad \mathsf{i}{\uparrow}\,\frac{(A,\bar{A})}{\mathsf{f}} \qquad\qquad \vdash A,\bar{A} \quad\rightsquigarrow\quad \mathsf{i}{\downarrow}\,\frac{\mathsf{t}}{[A,\bar{A}]} \quad .$$

$$\mathsf{c_L}\,\frac{\Phi, A, A \vdash \Psi}{\Phi, A \vdash \Psi} \quad\rightsquigarrow\quad \begin{array}{c} \mathsf{c}{\uparrow}\,\dfrac{\Phi, A}{(\Phi, A, A)} \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ \Psi \end{array} \qquad\qquad \mathsf{c_R}\,\frac{\Phi \vdash A, A, \Psi}{\Phi \vdash A, \Psi} \quad\rightsquigarrow\quad \begin{array}{c} \Phi \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ \mathsf{c}{\downarrow}\,\dfrac{[A, A, \Psi]}{[A, \Psi]} \end{array}$$

$$\mathsf{w_L}\,\frac{\Phi, \vdash \Psi}{\Phi, A \vdash \Psi} \quad\rightsquigarrow\quad \begin{array}{c} \mathsf{w}{\uparrow}\,\dfrac{(\Phi, A)}{\Phi} \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ \Psi \end{array} \qquad\qquad \mathsf{w_R}\,\frac{\Phi \vdash \Psi}{\Phi \vdash A, \Psi} \quad\rightsquigarrow\quad \begin{array}{c} \Phi \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ \mathsf{w}{\downarrow}\,\dfrac{\Psi}{[A, \Psi]} \end{array}$$

Fig. 2. Axioms and structural rules

It is instructive to see how the cut translates, and why it does not yield a normal derivation:

$$\mathsf{Cut}\,\frac{\Phi \vdash A, \Psi \qquad \Phi', A \vdash \Psi'}{\Phi, \Phi' \vdash \Psi, \Psi'} \quad\rightsquigarrow\quad \begin{array}{c} (\Phi, \Phi') \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ \mathsf{s}\,\dfrac{(\Phi', [A, \Psi])}{[\Psi, (\Phi', A)]} \\ \Vert\,{\uparrow} \\ [\Psi, C'] \\ \Vert\,{\downarrow} \\ [\Psi, \Psi'] \end{array} \quad .$$

While the detour via the sequent calculus in order to prove the normalisation theorem is convenient, it is an interesting question whether we can do without. While it is easy to come up with local proof transformations that normalise a derivation if they terminate, the presence of contraction makes termination hard to show.

Problem 8. Find an internal normalisation procedure for classical logic in the calculus of structures.

$$\vee_{\mathsf{L}}\,\frac{A,\Phi\vdash\Psi \qquad B,\Phi'\vdash\Psi'}{A\vee B,\Phi,\Phi'\vdash\Psi,\Psi'} \quad\rightsquigarrow\quad \begin{array}{c} \mathsf{s}^2\,\dfrac{([A,B],\Phi,\Phi')}{[(A,\Phi),(B,\Phi')]}\\ \Big\|\uparrow\\ [C,C']\\ \Big\|\downarrow\\ [\Psi,\Psi']\end{array}$$

$$\wedge_{\mathsf{R}}\,\frac{\Phi\vdash A,\Psi \qquad \Phi'\vdash B,\Psi'}{\Phi,\Phi'\vdash A\wedge B,\Psi,\Psi'} \quad\rightsquigarrow\quad \begin{array}{c} (\Phi,\Phi')\\ \Big\|\uparrow\\ (C,C')\\ \Big\|\downarrow\\ \mathsf{s}^2\,\dfrac{([A,\Psi],[B,\Psi'])}{[(A,B),\Psi,\Psi']}\end{array}$$

$$\wedge_{\mathsf{L}}\,\frac{\Phi,A,B\vdash\Psi}{\Phi,A\wedge B\vdash\Psi} \quad\rightsquigarrow\quad \begin{array}{c} (\Phi,A,B)\\ \Big\|\uparrow\\ C\\ \Big\|\downarrow\\ \Psi\end{array} \qquad\qquad \vee_{\mathsf{R}}\,\frac{\Phi\vdash A,B,\Psi}{\Phi\vdash A\vee B,\Psi} \quad\rightsquigarrow\quad \begin{array}{c} \Phi\\ \Big\|\uparrow\\ C\\ \Big\|\downarrow\\ [A,B,\Psi]\end{array}$$

$$\forall_{\mathsf{L}}\,\frac{\Phi,A[x/\tau]\vdash\Psi}{\Phi,\forall xA\vdash\Psi} \quad\rightsquigarrow\quad \begin{array}{c} \mathsf{n}\!\uparrow\dfrac{(\Phi,\forall xA)}{(\Phi,A[x/\tau])}\\ \Big\|\uparrow\\ C\\ \Big\|\downarrow\\ \Psi\end{array} \qquad\qquad \exists_{\mathsf{R}}\,\frac{\Phi\vdash A[x/\tau],\Psi}{\Phi\vdash\exists xA,\Psi} \quad\rightsquigarrow\quad \begin{array}{c} \Phi\\ \Big\|\uparrow\\ C\\ \Big\|\downarrow\\ \mathsf{n}\!\downarrow\dfrac{[A[x/\tau],\Psi]}{[\exists xA,\Psi]}\end{array}$$

$$\exists_{\mathsf{L}}\,\frac{\Phi,A[x/y]\vdash\Psi}{\Phi,\exists xA\vdash\Psi} \quad\rightsquigarrow\quad \begin{array}{c} = \dfrac{(\Phi,\forall xA)}{\mathsf{u}\!\uparrow\dfrac{(\forall y\Phi,\exists yA[x/y])}{\exists y(\Phi,A[x/y])}}\\ \Big\|\uparrow\\ \exists yC\\ \Big\|\downarrow\\ = \dfrac{\exists y\Psi}{\Psi}\end{array} \qquad\qquad \forall_{\mathsf{R}}\,\frac{\Phi\vdash A[x/y],\Psi}{\Phi\vdash\forall xA,\Psi} \quad\rightsquigarrow\quad \begin{array}{c} = \dfrac{\Phi}{\forall y\Phi}\\ \Big\|\uparrow\\ \forall yC\\ \Big\|\downarrow\\ \mathsf{u}\!\downarrow\dfrac{\forall y[A[x/y],\Psi]}{= \dfrac{[\forall yA[x/y],\exists y\Psi]}{[\forall xA,\Psi]}}\end{array}$$

Fig. 3. Logical rules

The point of proving with different means the same theorem is of course that a solution might give us a clue on how to attack the next problem:

Problem 9. Prove the normalisation theorem for logics which do not have a cut-free sequent calculus but which do have cut-free systems in the calculus of structures, such as BV or the modal logic S5.

4 Outlook

The problems above illustrate one direction of research around the calculus of structures: developing a proof theory which carries over to logics which do not have cut-free sequent systems. Examples are modal logics which can not be captured in the (plain vanilla) sequent calculus, like S5. Hein, Stewart and Stouppa are working on the project of obtaining modular proof systems for modal logic in the calculus of structures [15,22,23].

Another research thread is that of proof semantics. There is still the question of the right categorical axiomatisation of classical proofs. For predicate logic there is an approach by McKinley [18] which is derived from the concept of *classical category* by Führmann and Pym [8] and which is partly inspired by the shape of inference rules in the calculus of structures. A second approach is based on the notion of a *boolean category* by Lamarche and Straßburger [17,25]. It is also involves concepts from the calculus of structures, in particular the fact that contraction can be reduced to atomic form and the so-called *medial* rule [5], which achieves that reduction.

The proof complexity of systems in the calculus of structures is also a topic of current research. The cut-free calculus of structures allows for an exponential speedup over the cut-free sequent calculus, as Bruscoli and Guglielmi [6] show using Statman's tautologies. Among the many open questions is whether there are short proofs for the pigeonhole principle. Short cut-free proofs in the calculus of structures of course come with a price: there is much more choice in applying rules than in the sequent calculus, which is an obstacle to implementation and applications. Work by Kahramanoğullari [16] is attacking this issue.

Finally there is a war against bureaucracy, which is also known as the quest for *deductive proof nets*, due to Guglielmi [12]. We say that a formalism contains bureaucracy if it allows to form two different derivations that differ only due to trivial rule permutations and are thus morally identical. Proof nets, for example, do not contain bureaucracy, while the sequent calculus and the calculus of structures do. Deductive proof nets, which still do not exist, should not contain bureaucracy (and thus be like proof nets and unlike sequent calculus), but should also have a locally and/or easily checkable correctness criterion (and thus be like sequent calculus and unlike proof nets). Approaches to the identification and possibly elimination of bureaucracy can be found in Guiraud [14] and Brünnler and Lengrand [4].

This has been a subjective and incomplete outlook, but more open problems and conjectures can be found on the calculus of structures website [10].

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