Deep Sequent Systems for Modal Logic

Kai Brünnler

ABSTRACT. We see a systematic set of cut-free axiomatisations for all the basic normal modal logics formed from the axioms $\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$. They employ a form of deep inference but otherwise stay very close to Gentzen's sequent calculus, in particular they enjoy a subformula property in the literal sense. No semantic notions are used inside the proof systems, in particular there is no use of labels. All their rules are invertible, contraction is admissible and they allow for straightforward terminating proof search procedures.

Keywords: sequent calculus, modal logic, deep inference

1 Introduction

Numerous extensions of the sequent calculus have been proposed in order to give cut-free axiomatisations of modal logic. They are divided into two classes: *labelled* formalisms, which incorporate Kripke semantics in the proof system, and *unlabelled* formalisms, which do not. Prominent examples of unlabelled formalisms are the hypersequent calculus [1] and the display calculus [2, 10]. These and more can be found in the survey by Wansing [11]. A recent account of labelled sequent systems which also includes more references can be found in Negri [6].

The labelled approach seems to have become more prominent and according to several criteria has been more successful than the unlabelled approaches. It allows to capture a wide class of modal logics and does so systematically. In many important cases it yields systems which are natural and easy to use, which have good structural properties like contraction-admissibility and invertibility of all rules, and which give rise to decision procedures.

However, there are concerns about incorporating the semantics into the syntax of a proof system, see for example [1]. For motivating the present work I would just like to take it as a given that it is an interesting question whether something that has been achieved using labels can also be achieved without them. My goal here is to develop proof systems with the same good properties of Negri's labelled systems but to do so without using labels.

There is a closely related current research effort by Hein, Stewart and Stouppa which has the same aim [5, 7, 8]. To make the property of "not using labels" a bit more precise we call a proof system *pure* if each sequent has an equivalent formula. Sequent systems for modal logic are clearly pure: just read the comma on the left as conjunction, the comma on the right as

Paper submitted to Advances in Modal Logic 2006

disjunction, and the turnstile as implication. Hypersequents are also pure. A labelled sequent, on the other hand, does not generally have an equivalent modal formula.

Hein, Stewart and Stouppa use the *calculus of structures* [3, 4] to give pure systems for modal logics. This formalism is based on *deep inference*, which is the ability to apply rules deep inside of a formula. So far the calculus of structures has captured essentially those modal logics which can also be captured using the sequent calculus or hypersequents. In particular that does not include B and $\mathsf{K5}$. In the present work I introduce the formalism of *deep sequents* which uses deep inference (like the calculus of structures) but maintains tree-shaped derivations and a distinction between logical and structural connectives (like the sequent calculus). Deep sequent systems capture all the normal logics formed from the axioms $\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}$, thus in particular B and $\mathsf{K5}$. They can be easily embedded into corresponding systems in the calculus of structures, so this answers questions from [7].

The plan of this paper is as follows: after some preliminaries I present deep sequent systems and prove invertibility of rules and admissibility of contraction. Then I show that they are sound and complete for the respective Kripke semantics. The completeness proof constructs a countermodel from the failure of a terminating proof search procedure. Some discussion of related formalisms and of future work ends this paper.

2 Preliminaries

Formulas and models. Propositions p and their negations $\bar{p}$ are *atoms*, with $\bar{\bar{p}}$ defined to be p. Atoms are denoted by a, b, c and so on. *Formulas*, denoted by A, B, C, D are given by the grammar

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond A \mid \Box A \quad .$$

Given a formula A, its *negation* $\bar{A}$ is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\bar{A} \vee B$ and $\bot$ is defined as $p \wedge \bar{p}$ for some proposition p. A *frame* is a pair $(S, \rightarrow)$ of a nonempty set S of *states* and a binary relation $\rightarrow$ on it. A *model* $\mathcal{M}$ is a triple $(S, \rightarrow, V)$ where $(S, \rightarrow)$ is a frame and V is a a mapping which assigns a subset of S to each proposition, and which is called *valuation*. A model $\mathcal{M}$ as given above induces a relation $\models$ between states and formulas which is defined as usual. In particular we have $s \models p$ iff $s \in V(p)$, $s \models \bar{p}$ iff $s \notin V(p)$, $s \models A \vee B$ iff $s \models A$ or $s \models B$, $s \models A \wedge B$ iff $s \models A$ and $s \models B$, $s \models \Diamond A$ iff there is a state t such that $s \rightarrow t$ and $t \models A$, and $s \models \Box A$ iff for all t if $s \rightarrow t$ then $t \models A$. Further, a formula A is valid in a model $\mathcal{M}$, denoted $\mathcal{M} \models A$, iff for all states s of $\mathcal{M}$ we have $s \models A$. A formula A is valid in a frame $(S, \rightarrow)$, denoted $(S, \rightarrow) \models A$, iff for all valuations V we have $(S, \rightarrow, V) \models A$.

Deep sequents. A *(deep) sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]$ where Γ is a sequent. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi, \Sigma$. A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n] \quad ,$$

where, as usual, the comma denotes multiset union and there is no distinction between a singleton multiset and its element. The *corresponding formula* of the above sequent is $\bot$ if $m = n = 0$ and otherwise

$$A_1 \vee \cdots \vee A_m \vee \Box(D_1) \vee \cdots \vee \Box(D_n) \quad ,$$

where $D_1 \dots D_n$ are the corresponding formulas of the sequents $\Delta_1 \dots \Delta_n$. Often we do not distinguish between a sequent and its corresponding formula, e.g. a model of a sequent is a model of its corresponding formula. A sequent has a *corresponding tree* whose nodes are marked with multisets of formulas. The corresponding tree of the above sequent is

$$\begin{tikzcd}[column sep=small]
 & & \{A_1, \dots, A_m\} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\mathit{tree}(\Delta_1) & \mathit{tree}(\Delta_2) & \cdots & \mathit{tree}(\Delta_{n-1}) & \mathit{tree}(\Delta_n)
\end{tikzcd} \quad ,$$

where $\mathit{tree}(\Delta_1) \dots \mathit{tree}(\Delta_n)$ are the corresponding trees of $\Delta_1 \dots \Delta_n$. Often we do not distinguish between a sequent and its corresponding tree, e.g. the root of a sequent is the root of its corresponding tree. In particular, a sequent Δ is an *immediate subtree* of a sequent Γ if there is a sequent Λ such that $\Gamma = \Lambda, [\Delta]$. It is a *proper subtree* if it is an immediate subtree either of Γ or of a proper subtree of Γ, and it is a *subtree* if it is either a proper subtree of Γ or $\Delta = \Gamma$. The set of all subtrees of Γ is denoted by $\mathit{st}(\Gamma)$. A formula A is *in* a sequent Γ if $A \in \Gamma$ and it is *inside* Γ if there is a subtree Δ of Γ such that $A \in \Delta$. The *set sequent* of the above sequent is the underlying set of

$$A_1, \dots, A_m, [\Lambda_1], \dots, [\Lambda_n] \quad ,$$

where $\Lambda_1 \dots \Lambda_n$ are the set sequents of $\Delta_1 \dots \Delta_n$. Clearly the set sequent of a given sequent is again a sequent since a set is a multiset.

A *(sequent) context* is a sequent with exactly one occurrence of the symbol $\{\ \}$, the *hole*, which does not occur inside formulas. It is denoted by $\Gamma\{\ \}$. The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\ \}$ inside $\Gamma\{\ \}$ by Δ. A *double context*, denoted by $\Gamma\{\ \}\{\ \}$ is a triple of contexts $\Gamma_0\{\ \}, \Gamma_1\{\ \}, \Gamma_2\{\ \}$. Then $\Gamma\{\Delta\}\{\ \}$ denotes the context $\Gamma_0\{\Gamma_1\{\Delta\}, \Gamma_2\{\ \}\}$ and $\Gamma\{\ \}\{\Delta\}$ denotes the context $\Gamma_0\{\Gamma_1\{\ \}, \Gamma_2\{\Delta\}\}$. The *depth* of a context $\Gamma\{\ \}$, denoted $\mathit{depth}(\Gamma\{\ \})$ is defined as $\mathit{depth}(\Gamma, \{\ \}) = 0$ and $\mathit{depth}(\Gamma, [\Delta\{\ \}]) = \mathit{depth}(\Delta\{\ \}) + 1$.

3 The Modal Systems

Figure 1 shows the set of rules from which we form our deductive systems. *System* K includes the rules $\{\wedge, \vee, \Box, \mathsf{k}\}$. We will look at extensions of System K with sets of rules $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. The $\mathsf{5}$-rule carries the proviso that the depth of $\Gamma\{\ \}\{\emptyset\}$ is greater than zero.

$$\Gamma\{a,\bar{a}\} \qquad \wedge\frac{\Gamma\{A\}\quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \Box\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$$

$$\mathsf{k}\frac{\Gamma\{\Diamond A,[\Delta,A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathsf{t}\frac{\Gamma\{\Diamond A,A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}\frac{\Gamma\{[\Delta,\Diamond A],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$\mathsf{4}\frac{\Gamma\{\Diamond A,[\Delta,\Diamond A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathsf{5}\frac{\Gamma\{\Diamond A\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

Figure 1. System $\mathsf{K}+\{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$

$$\mathsf{nec}\frac{\Gamma}{[\Gamma]} \qquad \mathsf{wk}\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\frac{\Gamma\{A\}\qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Figure 2. Necessitation, weakening, contraction and cut

In an instance of an inference rule $\rho\frac{\Gamma}{\Delta}$ we call Γ its *premise* and Δ its *conclusion*, and similarly for the $\wedge$-rule, which has two premises. A *system*, denoted by $\mathcal{S}$, is a set of rules. A *derivation* in a system $\mathcal{S}$ is a upward-growing finite tree whose nodes are labelled with sequents and which is built according to the inference rules from $\mathcal{S}$. The sequent at the root is the *conclusion* and the sequents at the leaves which are not instances of the axiom $\Gamma\{a,\bar{a}\}$ are the *premises* of the derivation. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ and without premises. We write $\mathcal{S}\vdash\Gamma$ if there is a proof of Γ in system $\mathcal{S}$. An inference rule ρ is *(depth-preserving) admissible* for a system $\mathcal{S}$ if for each proof in $\mathcal{S}\cup\{\rho\}$ there is a proof of in $\mathcal{S}$ with the same conclusion (and with at most the same depth). For each rule ρ there is its *inverse*, denoted by $\bar{\rho}$, which is obtained by exchanging premise and conclusion. The $\bar{\wedge}$-rule allows both $\Gamma\{A\}$ and $\Gamma\{B\}$ as conclusions of $\Gamma\{A\wedge B\}$. An inference rule ρ is *(depth-preserving) invertible* for a system $\mathcal{S}$ if $\bar{\rho}$ is (depth-preserving) admissible for $\mathcal{S}$.

LEMMA 1. *For each system* $\mathsf{K}+\mathsf{X}$ *with* $\mathsf{X}\subseteq\{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *the following hold:*
(i) The necessitation and weakening rules are depth-preserving admissible.
(ii) All its rules are depth-preserving invertible.
(iii) The contraction rule is depth-preserving admissible.

Proof. (i) follows from a routine induction on the depth of the proof. The same works for the $\wedge,\vee$ and $\Box$-rules in (ii). The inverses of all other rules are just weakenings. For (iii) we also proceed by induction on the depth of the proof tree, using depth-preserving invertibility of the rules. The cases are easy for the propositional rules and for the $\Box,\mathsf{t}$-rules. For the k rule we consider the formula $\Diamond A$ from its conclusion $\Gamma\{\Diamond A, [\Delta]\}$ and its position inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have the cases 1) $\Diamond A$ is inside Σ or 2) $\Diamond A$ is inside $\Lambda\{\ \}$. We have three subcases for case 1: 1.1) $[\Delta]$ inside $\Lambda\{\ \}$, 1.2) $[\Delta]$ inside Σ, 1.3) Σ, Σ inside $[\Delta]$. There are two subcases of case 2: 2.1) $[\Delta]$ inside $\Lambda\{\ \}$ and 2.2) $[\Delta]$ inside Σ. All cases are either simpler than or similar to case 1.2, which is as follows:

$$\mathsf{ctr}\,\dfrac{\mathsf{k}\,\dfrac{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\Diamond A, \Sigma', [\Delta], \Sigma', [\Delta]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta]\}} \quad\rightsquigarrow\quad \mathsf{k}\,\dfrac{\mathsf{ctr}\,\dfrac{\bar{\mathsf{k}}\,\dfrac{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta]\}}{\Lambda'\{\Diamond A, \Sigma', [\Delta, A], \Sigma', [\Delta, A]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta, A]\}}}{\Lambda'\{\Diamond A, \Sigma', [\Delta]\}} \quad ,$$

where the instance of $\bar{\mathsf{k}}$ in the proof on the right is removed because it is depth-preserving admissible and the instance of contraction is removed by the induction hypothesis. The case for the $\mathsf{4}$-rule works the same way.

For the b-rule we make a case analysis based on the position of $[\Delta, \Diamond A]$ from its conclusion $\Gamma\{[\Delta, \Diamond A]\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have three cases: 1) $[\Delta, \Diamond A]$ inside $\Lambda\{\ \}$, 2) $[\Delta, \Diamond A]$ in Σ and 3) Σ, Σ inside $[\Delta, \Diamond A]$. Case 3 has two subcases: either $\Diamond A \in \Sigma$ or not. All cases are trivial except for case 2 where invertibility of the b-rule is used.

For the $\mathsf{5}$ rule we make a case analysis based on the positions of the sequent occurrences $\Diamond A$ and $\emptyset$ from its conclusion $\Gamma\{\Diamond A\}\{\emptyset\}$ inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have two cases: 1) $\emptyset$ inside $\Lambda\{\ \}$, 2) $\emptyset$ inside Σ. The first case is trivial, in the second we have two subcases: 1) $\Diamond A$ inside $\Lambda\{\ \}$ and 2) $\Diamond A$ inside Σ. Case 2.1 is similar to case 2.2 which is as follows:

$$\mathsf{ctr}\,\dfrac{\mathsf{5}\,\dfrac{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\emptyset\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}\}} \quad\rightsquigarrow\quad \mathsf{5}\,\dfrac{\mathsf{ctr}\,\dfrac{\bar{\mathsf{5}}\,\dfrac{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}{\Lambda\{\Sigma\{\Diamond A\}\{\Diamond A\}, \Sigma\{\Diamond A\}\{\Diamond A\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\Diamond A\}\}}}{\Lambda\{\Sigma\{\Diamond A\}\{\emptyset\}\}} \quad .$$

■

4 Soundness

Each name of a rule in $\{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ corresponds both to a frame condition and to a modal Hilbert-style axiom as shown in Figure 3. The k-rule corresponds to the modal axiom $\Box(A \vee B) \supset (\Box A \vee \Diamond B)$.

For a subset $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we call a frame an X-frame if it satisfies all the conditions determined by the names in X. A formula is X-*valid* if it is valid in all X-frames and just *valid* if it is valid in all frames.

t:	reflexive	$\forall s\ s \to s$	$A \supset \Diamond A$
b:	symmetric	$\forall st\ s \to t \supset t \to s$	$A \supset \Box\Diamond A$
4:	transitive	$\forall stu\ s \to t \wedge t \to u \supset s \to u$	$\Box A \supset \Box\Box A$
5:	euclidean	$\forall stu\ s \to t \wedge s \to u \supset t \to u$	$\Diamond A \supset \Box\Diamond A$

Figure 3. Frame conditions and modal axioms

LEMMA 2. *The* 5*-rule is derivable for* $\{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}, \mathsf{ctr}\}$*, where* 5a,5b,5c *are the rules*

$$\mathsf{5a}\ \frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \quad , \quad \mathsf{5b}\ \frac{\Gamma\{[\Delta], [\Lambda, \Diamond A]\}}{\Gamma\{[\Delta, \Diamond A], [\Lambda]\}} \quad , \quad \mathsf{5c}\ \frac{\Gamma\{[\Delta, [\Lambda, \Diamond A]]\}}{\Gamma\{[\Delta, \Diamond A, [\Lambda]]\}} \quad .$$

Proof. Seen bottom-up, the 5-rule allows to put a formula $\Diamond A$ which occurs at a node different from the root into an arbitrary node. We can use contraction to duplicate $\Diamond A$ and move one copy either to the root or to some child of the root by 5a. By 5b we can move it to any child of the root and by 5c into any descendant of a child of the root. ■

LEMMA 3. *For all contexts* $\Gamma\{\ \}$ *and formulas* A, B *the formula* $(A \supset B) \supset (\Gamma\{A\} \supset \Gamma\{B\})$ *is valid.*

Proof. By induction on $\Gamma\{\ \}$ using propositional reasoning and the implication $(A \supset B) \supset (\Box A \supset \Box B)$. ■

THEOREM 4 (Soundness). *Let* Γ, Γ', Δ *be sequents and let* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*. Then the following hold:*

(i) For any rule $\rho \in \mathsf{K}$ *if* $\rho\,\dfrac{\Gamma \quad (\Gamma')}{\Delta}$ *then* $\Gamma(\wedge\Gamma') \supset \Delta$ *is valid.*

(ii) For any rule $\rho \in \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *if* $\rho\,\dfrac{\Gamma}{\Delta}$ *then* $\Gamma \supset \Delta$ *is* $\{\rho\}$*-valid.*

(iii) If $\mathsf{K} + \mathsf{X} \vdash \Gamma$ *then* Γ *is* X*-valid.*

Proof. The axiom is valid in all frames which follows from an induction on $\Gamma\{\ \}$ where necessitation is used in the induction step. Thus (i) and (ii) imply (iii). Most cases of (i) are trivial, for the $\wedge$-rule it follows from an induction on the context and uses the implication $\Box A \wedge \Box B \supset \Box(A \wedge B)$. Lemma 3 used together with the k-axiom yields that the premise of the k-rule implies its conclusion. The cases from (ii) for the $\{\mathsf{t}, \mathsf{b}, \mathsf{4}\}$-rules are similar to the k-rule, using the corresponding modal axiom and for the corresponding frames.

For the soundness of the 5-rule we use Lemma 2 and show soundness of the rules $\mathsf{5a}, \mathsf{5b}, \mathsf{5c}$. For 5c we show that a euclidean countermodel for the conclusion is also a countermodel for the premise, the other cases are similar. A countermodel for $[\Delta, \Diamond A, [\Lambda]]$ has to contain states $s \to t \to u$ such that $t \not\models \Delta$, $u \not\models \Lambda$ and $v \not\models A$ for any v with $t \to v$. We need to show that for any w with $u \to w$ we have $w \not\models A$. By euclideanness we obtain, in this order: $t \to t, u \to t, t \to w$. Thus $w \not\models A$. ■

5 Completeness

We will not directly prove completeness of the systems $\mathsf{K}+\mathsf{X}$, but of different, equivalent systems $(\mathsf{K}+\mathsf{X})^*$ that we define now. For each rule $\rho \in \{\wedge, \vee, \Box, \mathsf{k}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we define a rule ρ' which keeps the main formula from the conclusion. For most rules $\rho = \rho'$ except for

$$\wedge' \frac{\Gamma\{A \wedge B, A\} \quad \Gamma\{A \wedge B, B\}}{\Gamma\{A \wedge B\}}$$

$$\vee' \frac{\Gamma\{A \vee B, A, B\}}{\Gamma\{A \vee B\}} \qquad \Box' \frac{\Gamma\{\Box A, [A]\}}{\Gamma\{\Box A\}} \quad .$$

Then for each rule $\rho \in \{\wedge, \vee, \Box, \mathsf{k}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we define a rule ρ^*, which is ρ' and carries the proviso that for all of its premises the set sequent is different from the set sequent of the conclusion. Given a system $\mathcal{S} \subseteq \{\wedge, \vee, \Box, \mathsf{k}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ the system $\mathcal{S}'(\mathcal{S}^*)$ is obtained by replacing each rule $\rho \in \mathcal{S}$ by $\rho'(\rho^*)$.

LEMMA 5. *For all systems* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and for all sequents* Γ

$$\mathsf{K}+\mathsf{X} \vdash \Gamma \qquad \textit{iff} \qquad (\mathsf{K}+\mathsf{X})' \vdash \Gamma \qquad \textit{iff} \qquad (\mathsf{K}+\mathsf{X})^* \vdash \Gamma \qquad .$$

Proof. The way right to middle is obvious, all other ways work by induction on the proof tree, using weakening admissibility for $\mathsf{K}+\mathsf{X}$ from left to middle and contraction admissibility for $\mathsf{K}+\mathsf{X}$ from middle to left. Middle to right uses contraction admissibility for $(\mathsf{K}+\mathsf{X})'$, which is established in the same way as for $\mathsf{K}+\mathsf{X}$. ■

Before proving completeness, we first need to characterise the euclidean and the transitive-euclidean closure of a relation.

DEFINITION 6. Let $\rightarrow$ be a binary relation on a set S. Then $\leftarrow$ denotes its inverse, $\leftrightarrow$ its symmetric closure, $\rightarrow^+$ its transitive closure and $\rightarrow^*$ its reflexive-transitive closure. For $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ $\rightarrow^{\mathsf{X}}$ denotes the smallest relation that includes $\rightarrow$ and has the properties in X.

DEFINITION 7. Let $\rightarrow$ be a binary relation on a set S and let $s, t \in S$. A *euclidean connection* for $\rightarrow$ from s to t is a nonempty sequence $s_1 \dots s_n$ of elements of S such that we have

$$s \leftarrow s_1 \leftrightarrow s_2 \leftrightarrow \cdots \leftrightarrow s_n \rightarrow t \qquad .$$

A *transitive-euclidean connection* is defined likewise but such that

$$s = s_1 \leftrightarrow s_2 \leftrightarrow \cdots \leftrightarrow s_n \rightarrow t \qquad .$$

We write $s \rightarrow_{(4)5}$ if there is a (transitive-)euclidean connection for $\rightarrow$ from s to t.

LEMMA 8. *Let* $\rightarrow$ *be a binary relation on a set* S*. Then the following hold:*
(i) For all $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *the relation* $\rightarrow^{\mathsf{X}}$ *is well-defined.*

(ii) The relation $\to \cup \to_5$ is the least euclidean relation that contains $\to$.
(iii) The relation $\to_{45}$ is the least transitive and euclidean relation that contains $\to$.

Proof. (i) is easy to check except for the cases for $\{\mathsf{5}\}$ and $\{\mathsf{4},\mathsf{5}\}$, which follow from (ii) and (iii).

(ii) Euclideanness is easy to check. For leastness we show that any euclidean relation $\longrightarrow$ that includes $\to$ also includes $\to_5$. If $s \to_5 t$ then $s \longrightarrow_5 t$. We show $s \longrightarrow_5 t$ for a euclidean connection of length n implies $s \longrightarrow t$ by induction on n. Assume there is an s_i in the euclidean connection such that $s_{i-1} \longrightarrow s_i \longleftarrow s_{i+1}$. Then we have two smaller euclidean connections to which we apply the induction hypothesis and obtain $s \longrightarrow t$ by euclideanness. If there is no such s_i then the euclidean connection looks as follows:

$$s = s_0 \longleftarrow s_1 \longleftarrow \ldots \longleftarrow s_j \longrightarrow \ldots \longrightarrow s_n \longrightarrow s_{n+1} = t \quad ,$$

and by euclideanness we have $s_{j-1} \longrightarrow s_{j+1}$ and thus removing s_j yields a smaller euclidean connection from s to t which by induction hypothesis implies $s \longrightarrow t$.

(iii) Euclideanness and transitivity are easy to check. For leastness we show that any transitive-euclidean relation $\longrightarrow$ that includes $\to$ also includes $\to_{45}$. If $s \to_{45} t$ then $s \longrightarrow_{45} t$. If there is no s_i in the transitive-euclidean such that $s_i \longleftarrow s_{i+1}$, then $s \longrightarrow t$ follows by transitivity. Otherwise, choose the first such s_i. We have a euclidean connection from s_i to t, thus similarly to (ii) obtain $s_i \longrightarrow t$ and by transitivity $s \longrightarrow s_i$ and $s \longrightarrow t$. ■

DEFINITION 9. A leaf of a sequent is called *cyclic* if there is an inner node in the sequent that carries the same set of formulas. A leaf is called *extensible* if it contains a formula of the form $\Box A$. We define a procedure *prove*(Γ, X), which takes a sequent Γ and a system $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ and builds a derivation tree for Γ by applying rules from $(\mathsf{K} + \mathsf{X})^*$ to non-axiomatic leaves in a bottom-up fashion as follows:

1. simultaneously apply the $\Box^*$ rule once wherever possible,

2. keep applying the rules in $((\mathsf{K} + \mathsf{X}) \setminus \Box)^*$ as long as possible.

Repeat the above until each non-axiomatic leaf in the derivation tree carries a sequent such that each extensible leaf in the sequent tree is cyclic. If *prove*(Γ, X) terminates and all leaves in the derivation tree are axiomatic then it *succeeds* and if it terminates and there is a non-axiomatic leaf then it *fails*.

LEMMA 10. *For all sets $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ and for all sequents Γ the procedure prove(Γ, X) terminates.*

Proof. Consider a sequence of sequents along a given branch of the derivation and starting from the root. A rule application in step 2 does not create new nodes in the sequent and causes the set of formulas at some node in the sequent to strictly grow. By the subformula property only finitely many formulas can occur in a node, so step 2 terminates. If there is an extensible leaf in a sequent then the size of the sequent strictly grows in step 1. Since there are only finitely many sets of formulas that can occur, once the sequent is large enough each extensible leaf has to be cyclic. ■

The current set of modal rules does not allow a modular completeness result of the form "if Γ is X-valid then $\mathsf{K}+\mathsf{X} \vdash \Gamma$". In particular we have $\mathsf{K}+\{\mathsf{t},\mathsf{5}\} \nvdash \Box A \supset \Box\Box A$ and $\mathsf{K}+\{\mathsf{b},\mathsf{4}\} \nvdash \Diamond A \supset \Box\Diamond A$. However, we obtain a weaker form of modularity. We define a set $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ to be *maximal* if it is closed under implication when the elements are read as frame conditions, or, more precisely, for each $\rho \in \{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\} \setminus \mathsf{X}$ there is an X-frame which does not satisfy ρ. The set $\{\mathsf{b},\mathsf{4}\}$ is not maximal, for example, while $\{\mathsf{b},\mathsf{4},\mathsf{5}\}$ is. Our completeness result will hold for maximal X.

THEOREM 11 (Completeness). *For all maximal sets* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ *and for all sequents* Γ *the following hold:*
(i) If Γ *is* X*-valid then* $\mathsf{K}+\mathsf{X} \vdash \Gamma$.
(ii) If $\mathit{prove}(\Gamma,\mathsf{X})$ *fails then* Γ *is not* X*-valid.*

Proof. The contrapositive of (i) follows from (ii): if $\mathsf{K}+\mathsf{X} \nvdash \Gamma$ then by Lemma 5 also $(\mathsf{K}+\mathsf{X})^* \nvdash \Gamma$ and thus in particular $\mathit{prove}(\Gamma,\mathsf{X})$ cannot yield a proof and by Lemma 10 has to fail. For (ii) we define a model $\mathcal{M}$ on an X-frame for which we prove that it is a countermodel for Γ. Let Γ^* be the set sequent of the non-axiomatic sequent obtained with all extensible leaves cyclic. Let Y be the set of all extensible leaves in Γ^*. Let $S = \mathit{st}(\Gamma^*) \setminus Y$. Let $f : Y \to S$ be some function which maps an extensible leaf to a sequent in S whose root carries the same set of formulas and extend f to $\mathit{st}(\Gamma^*)$ by the identity on S. Define a binary relation $\to$ on S such that $\Delta \to \Lambda$ iff either 1) Λ is an immediate subtree of Δ or 2) Δ has an immediate subtree $\Sigma \in Y$ and $f(\Sigma) = \Lambda$. Let $V(p) = \{\Delta \in S \,|\, \bar{p} \in \Delta\}$. Let $\mathcal{M} = (S, \to^{\mathsf{X}}, V)$. We prove three claims about $\mathcal{M}$, each claim depending on the next. Since all rules seen top-down preserve countermodels Claim 1 implies that $\mathcal{M} \not\models \Gamma$.

Claim 1 $\quad \forall \Delta \in \mathit{st}(\Gamma^*) \quad \mathcal{M}, f(\Delta) \not\models \Delta$

By induction on the depth of Δ. For depth zero this follows from Claim 2 and the fact that a formula is in Δ iff it is in $f(\Delta)$. So let $\Delta = A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n]$ and $n > 0$. Then $f(\Delta) = \Delta$. We have $\mathcal{M}, f(\Delta) \not\models A_i$ for all $i \leq m$ by Claim 2 and $\mathcal{M}, \Delta \not\models [\Delta_i]$ because $\Delta \to f(\Delta_i)$ and by induction hypothesis $\mathcal{M}, f(\Delta_i) \not\models \Delta_i$.

Claim 2 $\quad \forall \Delta \in S \; \forall A \in \Delta \quad \mathcal{M}, \Delta \not\models A$

By induction on the depth of A. For atoms it is clear from the definition of $\mathcal{M}$ and since Γ^* is not axiomatic. For the propositional connectives it is clear from the shape of the $\wedge, \vee$-rules. If $A = \Box B$ then by the $\Box$-rule we have some $[\Lambda] \in \Delta$ with $B \in \Lambda$. By induction hypothesis we have $\mathcal{M}, \Lambda \not\models B$ and thus $\mathcal{M}, \Delta \not\models \Box B$. If $A = \Diamond B$ then by Claim 3 we have $B \in \Lambda$ for all Λ with $\Delta \to^{\mathsf{X}} \Lambda$, and thus $\mathcal{M}, \Lambda \not\models B$. Thus $\mathcal{M}, \Delta \not\models \Diamond B$.

Claim 3 For all Δ, Λ with $\Delta \to^{\mathsf{X}} \Lambda\ \forall A \quad \Diamond A \in \Delta \implies A \in \Lambda$

K $\mathsf{X} = \emptyset$: By the definition of $\to$ there is an immediate subtree of Δ whose root node carries the same set of formulas as the root node of Λ. By the k-rule we have A in (the root node of) all immediate subtrees of Δ.

T $\mathsf{X} = \{\mathsf{t}\}$: $\Delta \to^{\{\mathsf{t}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Delta = \Lambda$. In the second case $A \in \Lambda$ follows from the t-rule.

KB $\mathsf{X} = \{\mathsf{b}\}$: $\Delta \to^{\{\mathsf{b}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Lambda \to \Delta$. In the second case $A \in \Lambda$ follows by the b-rule.

K4 $\mathsf{X} = \{4\}$: $\Delta \to^{\{4\}} \Lambda$ iff there is a sequence

$$\Delta = \Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots \to \Delta_n = \Lambda\ ,$$

with $n \geq 1$. An induction on i gives us that $\Diamond A \in \Delta_i$ for $0 \leq i \leq n$ by the $\mathsf{4}$-rule. That $A \in \Delta_i$ for $1 \leq i \leq n$ follows from that by the k-rule.

K5 $\mathsf{X} = \{5\}$: By Lemma 8 we have $\Delta \to^{\{5\}} \Lambda$ iff $\Delta \to \Lambda$ or there is a euclidean connection from Δ to Λ. In the second case there are sequents Π, Σ such that $\Pi \to \Delta$ and $\Sigma \to \Lambda$. Thus there is a subtree Δ' of Π with the same formulas as Δ and a subtree Λ' of Σ with the same formulas as Λ. Since $\Diamond A \in \Delta$ we have $\Diamond A \in \Delta'$ and since $\Delta' \neq \Gamma^*$ by the $\mathsf{5}$-rule we have $\Diamond A \in \Sigma$. Thus by the k-rule we have A in Λ' and thus in Λ.

K45 $\mathsf{X} = \{4, 5\}$: By Lemma 8 we have $\Delta \to^{\{4,5\}} \Lambda$ iff $\Delta \to \Lambda$ or there is a transitive-euclidean connection from Δ to Λ. In the second case there is a sequent Σ such that $\Sigma \to \Lambda$ and thus a subtree Λ' of Σ with the same formulas as Λ. Since $\Diamond A \in \Delta$, by the $\mathsf{5}$- and $\mathsf{4}$-rules we have $\Diamond A$ in every subtree of Γ^* and thus also in Σ, and by the k-rule we have A in Λ' and thus in Λ. (The $\mathsf{5a}$-rule instead of the $\mathsf{5}$-rule is enough.)

KB5 $\mathsf{X} = \{\mathsf{b}, 4, 5\}$: $\Delta \to^{\{\mathsf{b},4,5\}} \Lambda$ iff $\Delta \leftrightarrow^+ \Lambda$. Thus there is a sequent Σ such that either $\Sigma \to \Lambda$ or $\Sigma \leftarrow \Lambda$. Rule $\mathsf{4}, \mathsf{5}$ imply that $\Diamond A$ is in every subtree of Γ^* and thus in particular in Σ. We have $A \in \Lambda$ in the first case by the k-rule and in the second case by the b-rule.

KTB $\mathsf{X} = \{\mathsf{b}, \mathsf{t}\}$: $\Delta \to^{\{\mathsf{b},\mathsf{t}\}} \Lambda$ iff $\Delta \to \Lambda$ or $\Delta \leftarrow \Lambda$ or $\Delta = \Lambda$. In these cases $A \in \Lambda$ respectively follows from the k- or b- or t-rule.

S4 $\mathsf{X} = \{\mathsf{t}, 4\}$: $\Delta \to^{\{\mathsf{t},4\}} \Lambda$ iff $\Delta \to^+ \Lambda$ or $\Delta = \Lambda$. In the first case $A \in \Lambda$ follows from the $\mathsf{4}$-rule and in the second case from the t-rule.

S5 $\mathsf{X} = \{\mathsf{t}, \mathsf{b}, 4, 5\}$: $\Delta \to^{\{\mathsf{t},\mathsf{b},4,5\}} \Lambda$ iff $\Delta \leftrightarrow^* \Lambda$. We have $\Diamond A$ in all subtrees of Γ^* by the rules $\mathsf{4}, \mathsf{5}$ and thus also A by the t-rule. (Again, the $\mathsf{5a}$-rule is enough and the b-rule is not needed.) ■

6 Discussion

Our goal was to give pure proof systems with the good properties of Negri's labelled sequent systems. To some extent we have succeeded: while we do not (yet) have cut-free systems for all the logics which have cut-free labelled systems, we have captured all logics formed from t,b,$\mathsf{4}$,$\mathsf{5}$ and thus several important cases. Our systems are systematic in the sense that there is a one-to-one correspondence between the modal rules and the frame conditions considered, they enjoy invertibility, contraction admissibility and a terminating proof search procedure. The main conceptual difference between labelled and deep sequents is that the structural level in labelled systems is more general: it can form an arbitrary graph, while deep sequents are always trees. I hope that this restriction will help in using deep sequent systems for interpolation proofs, for which labelled systems do not seem to be well-suited.

Relation to hypersequents. Deep sequents are a natural generalisation of (modal) hypersequents, in allowing arbitrary nestings of boxed disjunctions instead of just a disjunction of boxed disjunctions. I am not aware of hypersequent systems for $\mathsf{K5}$ or B nor of hypersequent systems with invertible rules and contraction admissibility for the modal logics treated here. In fact, deep sequent systems came out of unsuccessfully trying to design an invertible (hyper)sequent system for $\mathsf{S4}$. A notational simplification I enjoy with respect to hypersequent systems is that the two kinds of context in inference rules (sequent context and hypersequent context) are merged into one.

Relation to the display calculus. Display sequents are closely related, in particular the idea of simply allowing $\Box$ as a structural connective is common to display sequents and deep sequents. However, the proof systems are rather different. Loosely speaking, in the display calculus one has to make a formula bubble up to the top by using the structural rules in order to apply a logical rule to it, while in deep sequent systems one can apply the rule on the spot. This leads to deductive systems with fewer rules and shortens derivations. On the other hand the display calculus so far has captured more modal logics than deep sequents and also enjoys a general cut elimination result, which for deep sequents is subject of current research. As with hypersequents, I am not aware of display systems with invertible rules and contraction admissibility.

We now turn to some future work and open questions.

Seriality. The set of modal axioms treated does not include seriality. I chose to put it aside for the time being, not because it is particularly problematic, but because it does not quite follow the same scheme as the other rules. The candidate for the d-rule is

$$\mathsf{d}\,\frac{\Gamma\{\Diamond A,[A]\}}{\Gamma\{\Diamond A\}} \qquad .$$

Clearly it increases the size of the sequent going up, so in the proof search procedure it has to be applied together with $\Box$. The notion of serial closure also has to be defined differently from the other closures. The conjecture is as follows:

CONJECTURE 12. For each sequent Γ and each maximal $\mathsf{X} \subseteq \{\mathsf{d},\mathsf{t},\mathsf{b},\mathsf{4},\mathsf{5}\}$ we have $\mathsf{K}+\mathsf{X} \vdash \Gamma$ iff Γ is X-valid.

Modularity. Our systems are not modular in the sense that each combination of modal rules is complete for the corresponding class of frames (labelled systems are modular in that sense). The modal rules presented are all $\Diamond$-rules, in the sense that the active formula in the conclusion has

$$\mathsf{ser}\,\frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{refl}\,\frac{\Gamma\{\Sigma,[\Sigma]\}}{\Gamma\{\Sigma\}} \qquad \mathsf{sym}\,\frac{\Gamma\{\Sigma,[\Delta,\Sigma]\}}{\Gamma\{\Sigma,[\Delta]\}}$$

$$\mathsf{trans}\,\frac{\Gamma\{[\Delta,[\Sigma,\Lambda]],[\Sigma]\}}{\Gamma\{[\Delta,[\Sigma,\Lambda]]\}} \qquad \mathsf{euc}\,\frac{\Gamma\{[\Delta,[\Sigma]],[\Lambda,\Sigma]\}}{\Gamma\{[\Delta],[\Lambda,\Sigma]\}} \qquad .$$

Figure 4. Modal axioms as structural rules

$\Diamond$ as main connective. I am not sure whether modularity can be achieved with this style of rules. However, there is also the possibility of formulating the modal rules as structural rules (structural in the sense of not affecting connectives of formulas), shown in Figure 4. I conjecture that these systems are modular. Notice the absence of the word "maximal":

CONJECTURE 13. For each sequent Γ and $\mathsf{X} \subseteq \{\mathsf{ser},\mathsf{refl},\mathsf{sym},\mathsf{trans},\mathsf{euc}\}$ we have $\mathsf{K}+\mathsf{X} \vdash \Gamma$ iff Γ is X-valid.

Robert Hein independently came up with essentially the same rules and the same conjecture for the calculus of structures.

Syntactic cut elimination. A cut elimination procedure is easily defined in the cases where the rules $\mathsf{4}$ and $\mathsf{5}$ are absent. It works like the standard procedure for system $\mathsf{G3}$ for first-order predicate logic, see for example [9]. Invertibility is used in all the passive cases and in the active case for the k-rule the proof

$$\mathsf{cut}\,\dfrac{\Box\,\dfrac{\overset{1}{\nabla}\atop{\Gamma\{[\Delta],[A]\}}}{\Gamma\{[\Delta],\Box A\}} \qquad \mathsf{k}\,\dfrac{\overset{2}{\nabla}\atop{\Gamma\{[\Delta,\bar{A}],\Diamond\bar{A}\}}}{\Gamma\{[\Delta],\Diamond\bar{A}\}}}{\Gamma\{[\Delta]\}}$$

is replaced by the proof

$$\mathsf{cut}\,\dfrac{\mathsf{ctr}\,\dfrac{\mathsf{wk}^2\,\dfrac{\overset{1}{\nabla}\atop{\Gamma\{[\Delta],[A]\}}}{\Gamma\{[\Delta,A],[\Delta,A]\}}}{\Gamma\{[\Delta,A]\}} \qquad \mathsf{cut}\,\dfrac{\mathsf{wk},\Box\,\dfrac{\overset{1}{\nabla}\atop{\Gamma\{[\Delta],[A]\}}}{\Gamma\{[\Delta,\bar{A}],\Box A\}} \qquad \overset{2}{\nabla}\atop{\Gamma\{[\Delta,\bar{A}],\Diamond\bar{A}\}}}{\Gamma\{[\Delta,\bar{A}]\}}}{\Gamma\{[\Delta]\}}$$

where the lower cut has a lower rank and the upper cut decreased in the sum of the depths of its two subproofs. The cases for t and b also work, but for $\mathsf{4}$ and $\mathsf{5}$ some more refined measure is needed which is subject of current research.

BIBLIOGRAPHY

[1] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss,

editors, *Logic: from foundations to applications. Proc. Logic Colloquium, Keele, UK, 1993*, pages 1–32. Oxford University Press, New York, 1996.

[2] Nuel D. Belnap, Jr. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.

[3] Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs*. PhD thesis, Technische Universität Dresden, September 2003.

[4] Alessio Guglielmi. A system of interaction and structure. Technical Report WV-02-10, Technische Universität Dresden, 2002. To appear in ACM Transactions on Computational Logic.

[5] Robert Hein and Charles Stewart. Purity through unravelling. In Paola Bruscoli, François Lamarche, and Charles Stewart, editors, *Structures and Deduction*, pages 126–143. Technische Universität Dresden, 2005.

[6] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5 – 6):507 – 544, 2005.

[7] Charles Stewart and Phiniki Stouppa. A systematic proof theory for several modal logics. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, *Advances in Modal Logic*, volume 5 of *King's College Publications*, pages 309–333, 2005.

[8] Phiniki Stouppa. A deep inference system for the modal logic S5. To appear in Studia Logica, 2006.

[9] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.

[10] Heinrich Wansing. *Displaying Modal Logic*, volume 3 of *Trends in Logic Series*. Kluwer Academic Publishers, Dordrecht, 1998.

[11] Heinrich Wansing. Sequent systems for modal logics. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, 2nd edition*, volume 8, pages 61–145. Kluwer, Dordrecht, 2002.

Kai Brünnler

Institut für angewandte Mathematik und Informatik,
Neubrückstr. 10, CH – 3012 Bern, Switzerland

kai@iam.unibe.ch