Explicit Mathematics with Positive Existential Comprehension and Join

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1 Introduction

Explicit Mathematics was introduced by Feferman [3] as a logical framework for formalising constructive mathematics. In this thesis we are following Feferman and Jäger [6], where explicit mathematics, namely elementary explicit type theory EET is presented in the logic of partial terms (due to Beeson [1]) with a two sorted language of *individuals* and *types*. The individuals are forming a partial combinatory algebra extended with natural numbers, and the types are intended to be *extensional* collections of individuals. Furthermore the two sorts are connected by a naming relation such that individuals can (*intensionally*) represent types, i.e. individuals are *names* of types. This crucial connection between individuals and types allows us to operate on types via their names, and in this way the computational power of the combinatory algebra is extended to both sorts. For example in EET we have a particular individual that, when applied to a name is generating a name of the complement of the original type.

It is this uniform creation of complements, or better to say the absence of it, we are dealing with in this thesis. We want to investigate a weakened version $\Sigma^+\mathsf{ET}$ of EET without uniform generation of complements. In these systems of explicit mathematics the existence of complement types is not guaranteed: A similar situation occurs in ordinary recursion theory, there we have recursively enumerable sets and their names are indices (programs) but the complements don't need to have names, i.e. aren't recursively enumerable.

Explicit Mathematics without complementation has already been investigated by Minari [8] where term models are constructed for theories with positive stratified comprehension, and by Cantini and Minari [2] where strong power types are refuted for theories with weak uniform comprehension.

The contribution of this thesis is a proof-theoretic characterisation of some particular systems, that is we are going to relate systems of Explicit Mathematics to systems of first order arithmetic (and to sets of functions $f{:}\mathbb{N}\to\mathbb{N}$). We will give the classification for $\Sigma^+\mathsf{ET}$ with type induction $(\mathsf{T\text{-}I_N})$ and formula induction $(\mathcal{F}\text{-}\mathsf{I_N})$ enriched by the following ontological extensions: disjoint union (J), everything is a number $(\forall\mathsf{N})$, everything is a name $(\forall\Re)$, positive existential stratified comprehension $(\Sigma^+\mathsf{S\text{-}C})$ and weak power types (Pow^-). The main results can be stated as follows:

$$\begin{array}{rcccl}
\Sigma^+\mathsf{ET}+(\mathsf{T\text{-}I_N}) & \equiv & \mathsf{PRA} & \equiv & \Sigma^+\mathsf{ET}+(\mathsf{T\text{-}I_N})+(\mathsf{J})+(\forall\mathsf{N})+(\forall\Re)+(\Sigma^+\mathsf{S\text{-}C})+(\mathsf{Pow}^-) \\
\Sigma^+\mathsf{ET}+(\mathcal{F}\text{-}\mathsf{I_N}) & \equiv & \mathsf{PA} & \equiv & \Sigma^+\mathsf{ET}+(\mathcal{F}\text{-}\mathsf{I_N})+(\mathsf{J})+(\forall\mathsf{N})+(\forall\Re)+(\Sigma^+\mathsf{S\text{-}C})+(\mathsf{Pow}^-)
\end{array}$$

In contrast to these results, we have the following classifications for EET

going back to [3], [4]:

$$\begin{array}{lcllcl} \mathsf{EET}+(\mathsf{T}\text{-}\mathsf{I_N}) & \equiv & \mathsf{PA} & \mathsf{EET}+(\mathcal{F}\text{-}\mathsf{I_N}) & \equiv & (\Pi^0_\infty\text{-}\mathsf{CA}) \\ \mathsf{EET}+(\mathsf{T}\text{-}\mathsf{I_N})+(\mathsf{J}) & \equiv & \mathsf{PA} & \mathsf{EET}+(\mathcal{F}\text{-}\mathsf{I_N})+(\mathsf{J}) & \equiv & (\Pi^0_\infty\text{-}\mathsf{CA})_{<\varepsilon_0} \end{array}$$

We can see that dropping complements is leading to a drastic reduction in strength, even if we have a lot of ontological principles at hand.

With respect to power types our result is somehow optimal: That is to say, we have *uniform* weak power types for $\Sigma^+\mathsf{ET}+(\forall\Re)$ and in our context we can't strengthen this principle any further because the theory $\Sigma^+\mathsf{ET}+(\mathsf{Pow}^+)$ with strong power types is inconsistent [2]. We also have that $\mathsf{EET}+(\mathsf{J})+(\forall\Re)$ is inconsistent by [7], but in this case we still have positive comprehension between $\Sigma^+\mathsf{ET}+(\mathsf{J})+(\forall\Re)$ and $\mathsf{EET}+(\mathsf{J})+(\forall\Re)$. Minari [8] has actually constructed models of positive *stratified* comprehension being also models of $\Sigma^+\mathsf{ET}+(\mathcal{F}\text{-}\mathsf{I_N})+(\mathsf{J})+(\forall\Re)+(\mathsf{Pow}^-)$. This shows that there is no need to restrict comprehension to existential formulas as we are doing it in this thesis, hence an interesting problem would be to give a proof-theoretic classification of Explicit Mathematics with positive (stratified) comprehension in combination with various ontological principles.

It is worth mentioning the approach we take to tackle the main problem in this thesis, i.e. embedding Explicit Mathematics into arithmetic with induction restricted to purely existential formulas. We follow the construction of generated models based on computation sequences, similar to the positive operator form for applicative theories in [5]. Instead of giving an operator form, we explicitly state the underlying primitive recursive predicate. In addition to the application function we also have to deal with the element relation. The finite axiomatisation of type generators for comprehension in [6] allows us to directly integrate these generators into the computation sequences. In this way we get one kind of unified computation sequences for the application function and the element relation.

The structure of this document is mainly built around the proof-theoretic method of "embeddings". Of course we first need to introduce the two systems we want to compare, this is first order arithmetic and Explicit Mathematics (section 2 and 4). Next we have to become acquainted with these two systems, i.e. we need to know some properties and concepts used to construct the embeddings (section 3 and 5). Finally we are ready to formulate the embeddings and state the proof-theoretic equivalences (section 6).

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Bern, November 2006 Jürg Krähenbühl

2 Systems of Arithmetic

We give a thorough definition of Peano arithmetic PA, primitive recursive arithmetic PRA and its extension with Σ_1-induction PRA^+. As usual the systems are formulated in classical first order predicate logic.

2.1 Syntax

First of all we need to define the language of arithmetic. It consists of symbols, terms and formulas. Based on this language we are able to define the logical axioms for equality, quantifiers and the propositional axioms. After adding rules of inference for modus ponens and for introducing quantifiers, we are ready to state the formal concept of proof (for proving arithmetic statements).

Definition 2.1 (Function Symbols $Prim_n$, $Prim$)**.** Function symbols are denoted by f, g, h (possibly with subscripts). Let p_i be an enumeration of all prime numbers in $\mathbb{N}$ such that $i < j \Rightarrow p_i < p_j$ and $p_0 = 2$. We use the following notation:

$$\begin{aligned} S &:= p_0 \\ Cs^x_y &:= {p_1}^x \cdot {p_2}^y \\ Pr^x_y &:= {p_2}^x \cdot {p_3}^y \\ Comp^x(y_1, \ldots, y_k) &:= {p_3}^x \cdot {p_{3+1}}^{y_1} \cdot \ldots \cdot {p_{3+k}}^{y_k} \\ Rec^x(y, z) &:= {p_4}^x \cdot {p_5}^y \cdot {p_6}^z \end{aligned}$$

$Prim_n \subset \mathbb{N}$ is inductively defined by the following rules:

(1) $S \in Prim_1$

(2) $\{Cs^n_i \mid i \in \mathbb{N}\} \subset Prim_n$

(3) $\{Pr^n_i \mid i < n\} \subset Prim_n$

(4) $f \in Prim_m \wedge g_1, \ldots, g_m \in Prim_n \;\Rightarrow\; Comp^n(f, g_1, \ldots, g_m) \in Prim_n$

(5) $f \in Prim_{n+1} \wedge g \in Prim_{n+3} \;\Rightarrow\; Rec^{n+2}(f, g) \in Prim_{n+2}$

The set of all function symbols is $Prim := \bigcup_{n>0} Prim_n$.

Definition 2.2 (Basic Symbols $\mathcal{S}^{\mathsf{A}}$)**.** The basic symbols $\mathcal{S}^{\mathsf{A}}$ consist of the following:

(1) Countably many variables. The set of all variables is denoted by $\mathcal{V}_A$ and the variables are denoted by a, b, c, i, j, k, u, v, w, x, y, z (possibly with subscripts).

(2) Constant: $\mathsf{0}$

(3) All function symbols in $Prim$.

(4) Relation Symbol: $=$

(5) Logical Symbols: $\neg$, $\vee$, $\wedge$, $\exists$, $\forall$

(6) Auxiliary Symbols:), (, ,

Definition 2.3 (Terms $\mathcal{T}^{\mathsf{A}}$)**.** Terms are denoted by r, s, t (possibly with subscripts).

(1) $\mathcal{V}_A \cup \{\mathsf{0}\} \subset \mathcal{T}^{\mathsf{A}}$

(2) $n > 0 \wedge f \in Prim_n \wedge t_1, \dots, t_n \in \mathcal{T}^{\mathsf{A}} \;\Rightarrow\; f(t_1, \dots, t_n) \in \mathcal{T}^{\mathsf{A}}$

Definition 2.4 (Formulas $\mathcal{F}^{\mathsf{A}}$)**.** Formulas are denoted by ϕ, ψ, ξ (possibly with subscripts).

(1) $s, t \in \mathcal{T}^{\mathsf{A}} \;\Rightarrow\; (s = t) \in \mathcal{F}^{\mathsf{A}}$

(2) $\phi, \psi \in \mathcal{F}^{\mathsf{A}} \;\Rightarrow\; \neg\phi$, $(\phi \vee \psi)$, $(\phi \wedge \psi) \in \mathcal{F}^{\mathsf{A}}$

(3) $\phi \in \mathcal{F}^{\mathsf{A}}$, $x \in \mathcal{V}_A \;\Rightarrow\; (\exists x)\phi$, $(\forall x)\phi \in \mathcal{F}^{\mathsf{A}}$

Definition 2.5 (Abbreviations)**.** We use the following shorthand notations:

$$\begin{aligned} s \neq t &:= \neg(s = t) \\ \phi \rightarrow \psi &:= (\neg\phi \vee \psi) \\ \phi \leftrightarrow \psi &:= (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \end{aligned}$$

Definition 2.6 (Free Variables and Substitution)**.** The set of free variables of a term t we denote by $FV_A(t)$ (or $FV_A(\phi)$ for a formula ϕ), see (7.2). The simultaneous substitution of terms $t_0, \dots, t_n$ for the variables $x_0, \dots, x_n$ in a term s we denote by $s[t_0/x_0, \dots, t_n/x_n]$ and $s[\vec{t}/\vec{x}]$ (or $\phi[t_0/x_0, \dots, t_n/x_n]$ and $\phi[\vec{t}/\vec{x}]$ for a formula ϕ), see (7.3). Further we denote by $FT_A(x, \phi)$ the set of terms t such that no variable z in $FV_A(t)$ is in the scope of a quantifier $(\exists z)$ or $(\forall z)$ in case x is replaced by t in ϕ, see (7.4).

Definition 2.7 (Closed Terms, Sentences). A term $t \in \mathcal{T}^{\mathsf{A}}$ is closed if $FV_A(t) = \{\}$, analogous a formula $\phi \in \mathcal{F}^{\mathsf{A}}$ is closed if $FV_A(\phi) = \{\}$. Closed formulas are usually called sentences.

Definition 2.8 (Propositional Axioms $\mathcal{A}^{\mathsf{A}}_{Prop}$). For all formulas ϕ, ψ, $\xi \in \mathcal{F}^{\mathsf{A}}$ the following formulas are in $\mathcal{A}^{\mathsf{A}}_{Prop}$:

(1) $(\phi \wedge \psi) \rightarrow \phi$

(2) $(\phi \wedge \psi) \rightarrow \psi$

(3) $\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$

(4) $\phi \rightarrow (\phi \vee \psi)$

(5) $\psi \rightarrow (\phi \vee \psi)$

(6) $(\phi \rightarrow \xi) \rightarrow ((\psi \rightarrow \xi) \rightarrow ((\phi \vee \psi) \rightarrow \xi))$

(7) $\phi \rightarrow (\psi \rightarrow \phi)$

(8) $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow (\psi \rightarrow \xi)) \rightarrow (\phi \rightarrow \xi))$

(9) $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg\psi) \rightarrow \neg\phi)$

(10) $\neg\neg\phi \rightarrow \phi$

Definition 2.9 (Equality Axioms $\mathcal{A}^{\mathsf{A}}_{Equal}$). For all variables z, $x_0, \ldots, x_n$, $y_0, \ldots, y_n \in \mathcal{V}_A$ and terms s, $t \in \mathcal{T}^{\mathsf{A}}$ the following formulas are in $\mathcal{A}^{\mathsf{A}}_{Equal}$:

(1) $z = z$

(2) $x_0 = y_0 \wedge \ldots \wedge x_n = y_n \wedge s = t \rightarrow (s = t)[\vec{y}/\vec{x}]$

Definition 2.10 (Quantifier Axioms $\mathcal{A}^{\mathsf{A}}_{Quant}$). For all formulas $\phi \in \mathcal{F}^{\mathsf{A}}$, for all variables $x \in \mathcal{V}_A$ and terms $t \in FT_A(x, \phi)$ the following formulas are in $\mathcal{A}^{\mathsf{A}}_{Quant}$:

(1) $\phi[t/x] \rightarrow (\exists x)\phi$

(2) $(\forall x)\phi \rightarrow \phi[t/x]$

Definition 2.11 (Rules of Inference $\mathcal{R}^{\mathsf{A}}$). For all formulas ϕ, $\psi \in \mathcal{F}^{\mathsf{A}}$, for all variables x, $y \in \mathcal{V}_A$ such that $y \in FT_A(x, \phi) \setminus FV_A(\psi)$ the following rules are in $\mathcal{R}^{\mathsf{A}}$:

(1) $\dfrac{\phi[y/x] \rightarrow \psi}{(\exists x)\phi \rightarrow \psi}$

(2) $\dfrac{\psi \to \phi[y/x]}{\psi \to (\forall x)\phi}$

(3) $\dfrac{\phi \quad \phi \to \psi}{\psi}$

Definition 2.12 (T-Proof and $T \vdash \phi$)**.** For $T \subset \mathcal{F}^{\mathsf{A}}$ and $\phi_0, \ldots, \phi_n \in \mathcal{F}^{\mathsf{A}}$ the sequence $(\phi_0, \ldots, \phi_n)$ is a T-Proof in case all ϕ_i satisfy one of the following conditions:

(1) $\phi_i \in T$

(2) $\phi_i \in \mathcal{A}^{\mathsf{A}}_{Prop} \cup \mathcal{A}^{\mathsf{A}}_{Equal} \cup \mathcal{A}^{\mathsf{A}}_{Quant}$

(3) ϕ_i is the conclusion of a rule of inference in $\mathcal{R}^{\mathsf{A}}$ whose premises belong to $\{\phi_0, \ldots, \phi_{i-1}\}$.

Let $T \subset \mathcal{F}^{\mathsf{A}}$ and $\phi \in \mathcal{F}^{\mathsf{A}}$, if there are $\phi_0, \ldots, \phi_k$ such that $(\phi_0, \ldots, \phi_k)$ is a T-Proof with $\phi_k = \phi$ then we denote this by $T \vdash^k \phi$ or simply $T \vdash \phi$. Let $S \subset \mathcal{F}^{\mathsf{A}}$, if we have $T \vdash \phi$ for all $\phi \in S$ then we denote this by $T \vdash S$. We use the shorthand notations $\vdash^k \phi$ and $\vdash \phi$ and $\vdash S$ whenever $T = \{\}$.

2.2 Principles and Theories

Based on the language of arithmetic we define the non-logical axioms of arithmetic, these consist of the defining equations for all primitive recursive functions and different induction principles.

2.2.1 Induction Principles

Definition 2.13 (Formulas QF, Σ_1, Π_2)**.** If rule 2.4-(3) (introducing symbols $\exists$ and $\forall$) is not used for building the formula $\phi \in \mathcal{F}^{\mathsf{A}}$, then we say ϕ is *quantifier free*. We denote the set of all quantifier free formulas by QF, and based on QF we further define $\Sigma_1 := \{(\exists x)\phi \mid \phi \in QF,\, x \in \mathcal{V}_A\} \cup QF$ and $\Pi_2 := \{(\forall x)\phi \mid \phi \in \Sigma_1,\, x \in \mathcal{V}_A\} \cup \Sigma_1$.

Definition 2.14 (Induction Axioms QF-Ind, Σ_1-Ind, $\mathcal{F}$-Ind)**.** Let $\mathcal{X} \subset \mathcal{F}^{\mathsf{A}}$. For all $\phi \in \mathcal{X}$ and $x \in \mathcal{V}_A$ the following formula is in $\mathcal{X}$-Ind:

$$\phi[\mathsf{0}/x] \wedge (\forall x)(\phi \to \phi[S(x)/x]) \to (\forall x)\phi$$

QF-Ind, Σ_1-Ind and $\mathcal{F}^{\mathsf{A}}$-Ind (usually denoted by $\mathcal{F}$-Ind) are the most prominent induction axioms in the text below.

2.2.2 Primitive Recursive Arithmetic

Definition 2.15 (Numerals $\overline{n} \in \mathcal{T}^{\mathsf{A}}$)**.** We use the following notation:

$$\overline{n+1} \;:=\; S(\overline{n}) \quad [\overline{0} := \mathsf{0}]$$

Definition 2.16 (Defining Equations $\mathcal{A}^{\mathsf{A}}_{Prim}$)**.** For all $n > 0$ and function symbols Cs^n_i, Pr^n_i, $Comp^n(f, g_1, \ldots, g_m)$, $Rec^{n+1}(f, g) \in Prim$ and for all variables $x_1, \ldots x_n \in \mathcal{V}_A$ the following formulas are in $\mathcal{A}^{\mathsf{A}}_{Prim}$:

(1) $Cs^n_i(x_1, \ldots, x_n) = \overline{i}$

(2) $Pr^n_i(x_1, \ldots, x_n) = x_{i+1}$

(3) $Comp^n(f, g_1, \ldots, g_m)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$

(4) $Rec^{n+1}(f, g)(x_1, \ldots, x_n, \mathsf{0}) = f(x_1, \ldots, x_n)$

(5) $Rec^{n+1}(f, g)(x_1, \ldots, x_n, S(x)) = g(x_1, \ldots, x_n, x, Rec^{n+1}(f, g)(x_1, \ldots, x_n, x))$

Definition 2.17 (Primitive Recursive Arithmetic PRA)**.**

$$\mathsf{PRA} = \{S(x) \neq \mathsf{0} \mid x \in \mathcal{V}_A\} \cup \mathcal{A}^{\mathsf{A}}_{Prim} \cup QF\text{-}\mathsf{Ind}$$

Definition 2.18 (Extensions PRA^+ and PA)**.**

$$\mathsf{PRA}^+ = \mathsf{PRA} \cup \Sigma_1\text{-}\mathsf{Ind}$$
$$\mathsf{PA} = \mathsf{PRA} \cup \mathcal{F}\text{-}\mathsf{Ind}$$

2.3 Semantics

The language of arithmetic defined above can be interpreted in many different ways. In the following we give a definition of what we mean by an interpretation. Based on this definition we are able to further define what it means for a statement in the language of arithmetic to be true (for all interpretations) or to be a logical consequence of some theory.

Definition 2.19 (Structures of Arithmetic $\mathbb{M}^{\mathsf{A}}$)**.** A structure $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ consists of the following:

(1) A domain M of numbers.

(2) A constant $\mathsf{0}^{\mathcal{M}} \in M$.

(3) For every function symbol $f \in Prim$ a function $f^{\mathcal{M}}$, such that $f^{\mathcal{M}} : M^n \to M$ for $f \in Prim_n$.

We usually denote M by $|\mathcal{M}|$.

Definition 2.20 (Valuations $\mathbb{V}^{\mathcal{M}}$)**.** A valuation $\nu \in \mathbb{V}^{\mathcal{M}}$ for a structure $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ is a mapping $\nu : \mathcal{V}_A \to |\mathcal{M}|$. If $\nu \in \mathbb{V}^{\mathcal{M}}$ then $\nu[u{:}m] \in \mathbb{V}^{\mathcal{M}}$ denotes the following valuation (where m is in $|\mathcal{M}|$):

$$\nu[u{:}m](v) := \begin{cases} m & v = u \\ \nu(v) & \text{otherwise} \end{cases}$$

Definition 2.21 (Interpretations $\mathbb{I}^{\mathsf{A}}$)**.** An interpretation $\mathcal{M}_\nu \in \mathbb{I}^{\mathsf{A}}$ (for a structure $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ and a valuation $\nu \in \mathbb{V}^{\mathcal{M}}$) consists of the following:

(1) A mapping $\mathcal{M}_\nu : \mathcal{T}^{\mathsf{A}} \to |\mathcal{M}|$ such that:

$$\mathcal{M}_\nu(t) := \begin{cases} \mathsf{0}^{\mathcal{M}} & t = \mathsf{0} \\ \nu(t) & t \in \mathcal{V}_A \\ f^{\mathcal{M}}(\mathcal{M}_\nu(s_0), \ldots, \mathcal{M}_\nu(s_n)) & t = f(s_0, \ldots, s_n) \end{cases}$$

(2) A mapping $\mathcal{M}_\nu : \mathcal{F}^{\mathsf{A}} \to \{\mathbf{t}, \mathbf{f}\}$ such that:

$$\begin{array}{lcl}
\mathcal{M}_\nu(s = t) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(s) = \mathcal{M}_\nu(t) \\
\mathcal{M}_\nu(\neg\phi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{f} \\
\mathcal{M}_\nu(\phi \vee \psi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{t} \vee \mathcal{M}_\nu(\psi) = \mathbf{t} \\
\mathcal{M}_\nu(\phi \wedge \psi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{t} \wedge \mathcal{M}_\nu(\psi) = \mathbf{t} \\
\mathcal{M}_\nu((\exists x)\phi) = \mathbf{t} & :\Leftrightarrow & (\exists m \in |\mathcal{M}|)(\mathcal{M}_{\nu[x:m]}(\phi) = \mathbf{t}) \\
\mathcal{M}_\nu((\forall x)\phi) = \mathbf{t} & :\Leftrightarrow & (\forall m \in |\mathcal{M}|)(\mathcal{M}_{\nu[x:m]}(\phi) = \mathbf{t})
\end{array}$$

Definition 2.22. For $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$, $\phi \in \mathcal{F}^{\mathsf{A}}$ and $T, S \subset \mathcal{F}^{\mathsf{A}}$ we define the following relations:

$$\begin{array}{lcl}
\mathcal{M} \models \phi & :\Leftrightarrow & (\forall \nu \in \mathbb{V}^{\mathcal{M}})(\mathcal{M}_\nu(\phi) = \mathbf{t}) \\
\mathcal{M} \models T & :\Leftrightarrow & (\forall \phi \in T)(\mathcal{M} \models \phi) \\
T \models \phi & :\Leftrightarrow & (\forall \mathcal{M} \in \mathbb{M}^{\mathsf{A}})(\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi) \\
T \models S & :\Leftrightarrow & (\forall \phi \in S)(T \models \phi)
\end{array}$$

If $\mathcal{M} \models \phi$ holds then we say ϕ is valid in $\mathcal{M}$, and if we have $\mathcal{M} \models T$ then we say $\mathcal{M}$ is a model of T. Finally if we have $T \models \phi$ (or $T \models S$) then we say ϕ (or S) is a logical consequence of T.

Theorem 2.23 (Adequacy)**.** *Let $\phi \in \mathcal{F}^{\mathsf{A}}$ then we have*

$$T \models \phi \quad \Leftrightarrow \quad T \vdash \phi$$

Proof. By soundness and completeness for theories in classical predicate logic (see any introductory book about classical logic). $\square$

Example 2.24. Let $\mathcal{M}$ be the following structure:

(1) $|\mathcal{M}| := \mathbb{N}$

(2) $\mathsf{0}^{\mathcal{M}} := 0$

(3) $f^{\mathcal{M}} := [\![f]\!]$ for $f \in Prim$ (see 7.1 for the definition of $[\![f]\!]$)

$\mathcal{M}$ is a model of PA, i.e. we can prove $\mathcal{M} \models \mathsf{PA}$.

3 Preliminary Steps in Arithmetic

Having defined the language of arithmetic and the concept of a formal proof, we now give some useful examples of provable statements. In this way we are building a collection of frequently used tools.

3.1 Function Symbols

There are a lot of useful primitive recursive functions contained in the language of arithmetic. We name some common functions and show that their expected properties are provable in PRA.

3.1.1 Basic Functions

Definition 3.1 (Logical Connectives)**.** Let And, $Or \in Prim_2$ be function symbols such that PRA proves:

(1) $And(\mathsf{0}, y) = y$

(2) $And(S(x), y) = S(x)$

(3) $Or(\mathsf{0}, y) = \mathsf{0}$

(4) $Or(S(x), y) = y$

We usually write $x \mathbin{\dot\wedge} y$ for $And(x, y)$ and $x \mathbin{\dot\vee} y$ for $Or(x, y)$, and we use association to the left, hence $x_0 \mathbin{\dot\wedge} x_1 \mathbin{\dot\wedge} \ldots \mathbin{\dot\wedge} x_n$ stands for $(\ldots(x_0 \mathbin{\dot\wedge} x_1) \ldots \mathbin{\dot\wedge} x_n)$ and $x_0 \mathbin{\dot\vee} x_1 \mathbin{\dot\vee} \ldots \mathbin{\dot\vee} x_n$ stands for $(\ldots(x_0 \mathbin{\dot\vee} x_1) \ldots \mathbin{\dot\vee} x_n)$.

Lemma 3.2.

(1) $\mathsf{PRA} \vdash (x \mathbin{\dot\wedge} y) = \mathsf{0} \leftrightarrow (x = \mathsf{0} \wedge y = \mathsf{0})$

(2) $\mathsf{PRA} \vdash (x \mathbin{\dot\vee} y) = \mathsf{0} \leftrightarrow (x = \mathsf{0} \vee y = \mathsf{0})$

Definition 3.3 (Ordering)**.** Let $P \in Prim_1$ and $Ch_{\geq} \in Prim_2$ be function symbols such that PRA proves:

(1) $P(\mathsf{0}) = \mathsf{0}$

(2) $P(S(x)) = x$

(3) $Ch_{\geq}(\mathsf{0}, y) = y$

(4) $Ch_{\geq}(S(x), y) = P(Ch_{\geq}(x, y))$

We usually write $x \dot{\geq} y$ or $y - x$ for the term $Ch_{\geq}(x,y)$ and we write $x \geq y$ and $x < y$ for the formulas $Ch_{\geq}(x,y) = \mathsf{0}$ and $Ch_{\geq}(x,y) \neq \mathsf{0}$ respectively.

Lemma 3.4.

(1) $\mathsf{PRA} \vdash x \geq x$

(2) $\mathsf{PRA} \vdash x \geq z \wedge z \geq y \rightarrow x \geq y$

(3) $\mathsf{PRA} \vdash x \geq y \wedge y \geq x \rightarrow x = y$

(4) $\mathsf{PRA} \vdash x \geq y \vee y \geq x$

Proof. See the section about primitive recursive arithmetic in [10]. □

Definition 3.5 (Equality, Negation)**.** Let $Ch_{=} \in Prim_2$ and $Neg \in Prim_1$ be function symbols such that PRA proves:

(1) $Ch_{=}(x,y) = (x \dot{\geq} y) \dot{\wedge} (y \dot{\geq} x)$

(2) $Neg(x) = S(\mathsf{0}) - x$

We usually write $x \dot{=} y$ for $Ch_{=}(x,y)$ and $\dot{\neg}(x)$ for $Neg(x)$.

Lemma 3.6.

(1) $\mathsf{PRA} \vdash (x \dot{=} y) = \mathsf{0} \leftrightarrow x = y$

(2) $\mathsf{PRA} \vdash \dot{\neg}(x) = \mathsf{0} \leftrightarrow x \neq \mathsf{0}$

Proof. See the section about primitive recursive arithmetic in [10]. □

Definition 3.7 (Bounded Quantifiers)**.** For $n \leq k$ let $E_n^{k+1}, A_n^{k+1} : Prim_{k+1} \rightarrow Prim_{k+1}$ be such that PRA proves:

(1) $E_n^{k+1}(f)(x_0, \ldots, x_{n-1}, \mathsf{0}, x_{n+1}, \ldots, x_k) = S(\mathsf{0})$

(2) $E_n^{k+1}(f)(x_0, \ldots, x_{n-1}, S(x_n), x_{n+1}, \ldots, x_k) = E_n^{k+1}(f)(x_0, \ldots, x_k) \dot{\vee} f(x_0, \ldots, x_k)$

(3) $A_n^{k+1}(f)(x_0, \ldots, x_{n-1}, \mathsf{0}, x_{n+1}, \ldots, x_k) = \mathsf{0}$

(4) $A_n^{k+1}(f)(x_0, \ldots, x_{n-1}, S(x_n), x_{n+1}, \ldots, x_k) = A_n^{k+1}(f)(x_0, \ldots, x_k) \dot{\wedge} f(x_0, \ldots, x_k)$

We usually write $\dot{\exists}_n f$ for $E_n^{k+1}(f)$ and $\dot{\forall}_n f$ for $A_n^{k+1}(f)$.

Lemma 3.8.

(1) $\mathsf{PRA} \vdash \dot{\exists}_0 \ldots \dot{\exists}_n f(x_0, \ldots, x_n, \vec{y}) = \mathsf{0} \leftrightarrow$
$(\exists z_0) \ldots (\exists z_n)[z_0 < x_0 \wedge \ldots \wedge z_n < x_n \wedge f(z_0, \ldots, z_n, \vec{y}) = \mathsf{0}]$

(2) $\mathsf{PRA} \vdash \dot{\forall}_0 \ldots \dot{\forall}_n f(x_0, \ldots, x_n, \vec{y}) = \mathsf{0} \leftrightarrow$
$(\forall z_0) \ldots (\forall z_n)[z_0 < x_0 \wedge \ldots \wedge z_n < x_n \rightarrow f(z_0, \ldots, z_n, \vec{y}) = \mathsf{0}]$

3.1.2 Sequence Numbers

Definition 3.9 (Pairing)**.** Let π_0, $\pi_1 \in Prim_1$ and $\pi \in Prim_2$ be function symbols such that PRA proves:

(1) $\pi_0(\pi(x,y)) = x$

(2) $\pi_1(\pi(x,y)) = y$

We interchangeably write $|x|$ for $\pi_0(x)$.

Definition 3.10 (Sequence Numbers)**.** Let $Ins \in Prim_2$ be a function symbol such that PRA proves:

$$Ins(x,y) = \pi(S(|x|), \pi(y, \pi_1(x)))$$

We write $\langle\rangle$ for $\pi(0,0)$ and $\langle x_0, \dots, x_n\rangle$ for $Ins(\dots Ins(\langle\rangle, x_0)\dots, x_n)$.

Definition 3.11 (Projection)**.** Let $Iter^{k+2} : Prim_{k+1} \to Prim_{k+2}$ and $Proj \in Prim_2$ such that PRA proves:

(1) $Iter^{k+2}(f)(0, x, \vec{y}) = x$

(2) $Iter^{k+2}(f)(S(z), x, \vec{y}) = f(Iter^{k+2}(f)(z, x, \vec{y}), \vec{y})$

(3) $Proj(y,x) = \pi_0(Iter^2(\pi_1)(|x| - y, x))$

We usually write $(x)_{y_0,\dots,y_n}$ for $Proj(y_n, \dots Proj(y_0, x)\dots)$ and $(x)_{n_0,\dots,n_k}$ for $Proj(\overline{n_k}, \dots Proj(\overline{n_0}, x)\dots)$ and we also use a mix of this two notations.

Lemma 3.12.

(1) $\mathsf{PRA} \vdash (x)_i \neq (y)_i \to x \neq y$

(2) $\mathsf{PRA} \vdash i \geq |x| \to (x)_i = |x|$

Lemma 3.13. *For all i, $n \in \mathbb{N}$ such that $i < n$ we have*

(1) $\mathsf{PRA} \vdash (\langle x_1, \dots, x_n\rangle)_i = x_{i+1}$

(2) $\mathsf{PRA} \vdash |\langle x_1, \dots, x_n\rangle| = \overline{n}$

Definition 3.14 (Concatenation)**.** Let $Cat^{k+2} : Prim_{k+1} \to Prim_{k+2}$ be such that PRA proves:

(1) $Cat^{k+2}(f)(0, x_0, \dots, x_k) = x_k$

(2) $Cat^{k+2}(f)(S(z), x_0, \dots, x_k) = Ins(Cat^{k+2}(f)(z, x_0, \dots, x_k), f(z, x_0, \dots, x_{k-1}))$

We usually write $x{*}y$ for $Cat^3(Proj)(|y|, y, x)$ and $f^*(\vec{x})$ for $Cat^{k+2}(f)(\vec{x}, \langle\rangle)$.

Lemma 3.15.

(1) $\mathsf{PRA} \vdash |x * \langle y \rangle| = S(|x|)$

(2) $\mathsf{PRA} \vdash |x * y| \geq |x| \wedge |x * y| \geq |y|$

(3) $\mathsf{PRA} \vdash i < |x| \rightarrow (\exists j)(j < |x * y| \wedge (x)_i = (x * y)_j)$

(4) $\mathsf{PRA} \vdash i < |y| \rightarrow (\exists j)(j < |x * y| \wedge (y)_i = (x * y)_j)$

Lemma 3.16.

(1) $\mathsf{PRA} \vdash |f^*(x, \vec{y})| = x$

(2) $\mathsf{PRA} \vdash i < x \rightarrow (f^*(x, \vec{y}))_i = f(i, \vec{y})$

Lemma 3.17.

$$\mathsf{PRA} \vdash |x| = \overline{n} \rightarrow (x = \langle\rangle * x \leftrightarrow x = \langle (x)_0, \ldots, (x)_{n-1} \rangle)$$

This lemma shows that $x = \langle\rangle * x$ is a predicate for the sequence numbers.

3.2 Structural Properties

We are also interested in structural properties of proofs, i.e. generic proofs, rules for proofs or the existence of a whole class of proofs for a formula depending on a term or a function symbol. First we state some general results derivable by just using the logical axioms, then some statements about atomic formulas containing function symbols and numerals are proved in PRA. The section ends with the more intricate results of term extraction ($\exists$-inversion) and Parsons' Theorem, both stated without proof.

3.2.1 Properties in General

Lemma 3.18. *For all $\phi \in \mathcal{F}^{\mathsf{A}}$ and $x_0, \ldots, x_n \in \mathcal{V}_A$ and $s_0, \ldots, s_n$, $t_0, \ldots, t_n \in \mathcal{T}^{\mathsf{A}}$ such that s_i, $t_i \in FT_A(x_i, \phi)$, we have*

$$\vdash s_0 = t_0 \wedge \ldots \wedge s_n = t_n \wedge \phi[\vec{s}/\vec{x}] \rightarrow \phi[\vec{t}/\vec{x}]$$

Proof. By induction on the complexity of $\phi \in \mathcal{F}^{\mathsf{A}}$. ∎

Lemma 3.19. *For all $T \subset \mathcal{F}^{\mathsf{A}}$ and $\phi \in \mathcal{F}^{\mathsf{A}}$ we have*

$$T \vdash \phi \quad \Rightarrow \quad T \vdash (\forall x)\phi$$

Proof. Let $\psi := (y = y) \in \mathcal{A}^{\mathsf{A}}_{Equal}$. We have $(\phi \to (\psi \to \phi)) \in \mathcal{A}^{\mathsf{A}}_{Prop}$ and $T \vdash \phi$, hence $T \vdash \psi \to \phi$. Applying $\dfrac{\psi \to \phi[x/x]}{\psi \to (\forall x)\phi}$ we get $T \vdash \psi \to (\forall x)\phi$ and finally $T \vdash (\forall x)\phi$ by modus ponens. $\square$

Lemma 3.20. *For all $t \in \mathcal{T}^{\mathsf{A}}$ and $\phi \in \mathcal{F}^{\mathsf{A}}$ we have*

(1) $\vdash t = t$

(2) $\vdash \phi \to \phi$

Proof. (1) We have $x = x \in \mathcal{A}^{\mathsf{A}}_{Equal}$ hence $\vdash (\forall x)x = x$ by (3.19). Applying modus ponens to $((\forall x)x = x \to t = t) \in \mathcal{A}^{\mathsf{A}}_{Quant}$ yields $\vdash t = t$. (2) Let $\psi := (\phi \vee \phi)$ then $\phi \to \psi \in \mathcal{A}^{\mathsf{A}}_{Prop}$ and $\phi \to (\psi \to \phi) \in \mathcal{A}^{\mathsf{A}}_{Prop}$ and $(\phi \to \psi) \to ((\phi \to (\psi \to \phi)) \to (\phi \to \phi)) \in \mathcal{A}^{\mathsf{A}}_{Prop}$. Applying modus ponens twice yields $\vdash \phi \to \phi$. $\square$

3.2.2 Properties for Theories

Lemma 3.21. *Let f be a function symbol in $Prim_k$, then we have*

$$(\forall n_1, \ldots, n_k \in \mathbb{N}) \qquad \mathsf{PRA} \vdash f(\overline{n_1}, \ldots, \overline{n_k}) = \overline{[\![f]\!](n_1, \ldots, n_k)}$$

Proof. By induction on the function symbols $f \in Prim$. $\square$

Corollary 3.22. *Let $t \in \mathcal{T}^{\mathsf{A}}$ be a closed term, then there is a number $n \in \mathbb{N}$ such that*

$$\mathsf{PRA} \vdash t = \overline{n}$$

Proof. By induction on the complexity of the closed term $t \in \mathcal{T}^{\mathsf{A}}$. $\square$

Lemma 3.23. *Let $t \in \mathcal{T}^{\mathsf{A}}$ and $\{x_1, \ldots, x_n\} \supset FV_A(t)$, then there is a function symbol $f \in Prim_n$ such that*

$$\mathsf{PRA} \vdash t = f(x_1, \ldots, x_n)$$

Proof. By induction on the complexity of the term $t \in \mathcal{T}^{\mathsf{A}}$. $\square$

Lemma 3.24. *For $\phi \in \mathcal{F}^{\mathsf{A}}$ and $y \in FT_A(x_i, \phi)$ we have*

$$\mathsf{PRA} \vdash (\exists x_0) \ldots (\exists x_n)\phi \leftrightarrow (\exists y)\phi[(y)_0/x_0, \ldots, (y)_n/x_n]$$

Proof.

(1) Let $\psi := \phi[(y)_0/x_0, \ldots, (y)_n/x_n]$ and $t := \langle x_0, \ldots, x_n \rangle$. We have

$$\mathsf{PRA} \vdash x_0 = (t)_0 \wedge \ldots \wedge x_n = (t)_n \rightarrow (\phi \rightarrow \psi[t/y])$$

by (3.18). From $\mathsf{PRA} \vdash x_i = (t)_i$ we get $\mathsf{PRA} \vdash \phi \rightarrow \psi[t/y]$. We have $(\psi[t/y] \rightarrow (\exists y)\psi) \in \mathcal{A}^{\mathsf{A}}_{Quant}$ hence we can deduce $\mathsf{PRA} \vdash \phi \rightarrow (\exists y)\psi$. Finally we introduce the quantifiers on the left side by the corresponding rule in $\mathcal{R}^{\mathsf{A}}$.

(2) The following formulas are in $\mathcal{A}^{\mathsf{A}}_{Quant}$:

$$\begin{aligned} \psi &\rightarrow (\exists x_n)\phi[(y)_0/x_0, \ldots, (y)_{n-1}/x_{n-1}] \\ (\exists x_n)\phi[(y)_0/x_0, \ldots, (y)_{n-1}/x_{n-1}] &\rightarrow (\exists x_{n-1})(\exists x_n)\phi[(y)_0/x_0, \ldots, (y)_{n-2}/x_{n-2}] \\ &\vdots \\ (\exists x_1) \ldots (\exists x_n)\phi[(y)_0/x_0] &\rightarrow (\exists x_0) \ldots (\exists x_n)\phi \end{aligned}$$

hence we can deduce $\mathsf{PRA} \vdash \psi \rightarrow (\exists x_0) \ldots (\exists x_n)\phi$ and we finally introduce the quantifier on the left side using the rule in $\mathcal{R}^{\mathsf{A}}$.

□

Theorem 3.25 (Term Extraction)**.** *Let $\phi \in QF$ and $\mathsf{PRA} \vdash (\exists x)\phi$, then there exists a term $t \in \mathcal{T}^{\mathsf{A}}$ such that*

(1) $\mathsf{PRA} \vdash \phi[t/x]$

(2) $FV_A(t) = FV_A(\phi) \backslash \{x\}$

Proof. See for example [9] for this intricate result. □

Corollary 3.26. *Let $\phi \in QF$ and $\mathsf{PRA} \vdash (\exists x_0) \ldots (\exists x_n)\phi$, then there exists a term $t \in \mathcal{T}^{\mathsf{A}}$ such that*

(1) $\mathsf{PRA} \vdash \phi[(t)_0/x_0, \ldots, (t)_n/x_n]$

(2) $FV_A(t) = FV_A(\phi) \backslash \{x_0, \ldots, x_n\}$

Proof. Using (3.24). □

Theorem 3.27 (Parsons' Theorem)**.** *Let $\phi \in \Sigma_1$ then*

$$\mathsf{PRA}^{+} \vdash \phi \quad \Leftrightarrow \quad \mathsf{PRA} \vdash \phi$$

Proof. See for example [9]. □

4 Systems of Explicit Mathematics

We give a thorough definition of the applicative theory of basic operations and numbers BON^- and the explicit type theories $\Sigma^+\mathsf{ET}$ and EET^-. We will formulate these systems in the logic of partial terms (due to Beeson [1]) similar to [5], [6].

4.1 Syntax

First of all we need to define the language of Explicit Mathematics. It consists of symbols, terms and formulas. Based on this language we are able to define the logical axioms for definedness, equality, quantifiers and the propositional axioms. After adding rules of inference for modus ponens and for introducing quantifiers, we are ready to state the formal concept of proof (for proving statements in Explicit Mathematics).

Definition 4.1 (Basic Symbols $\mathcal{S}^{\mathsf{E}}$)**.** The basic symbols $\mathcal{S}^{\mathsf{E}}$ consist of the following:

(1) Countably many individual variables. The set of all individual variables is denoted by $\mathcal{V}_I$ and the variables are denoted by a, b, c, f, g, h, u, v, w, x, y, z (possibly with subscripts).

(2) Countably many type variables. The set of all type variables is denoted by $\mathcal{V}_T$ and the variables are denoted by A, B, C, U, V, W, X, Y, Z (possibly with subscripts).

(3) Constants k, s, p, p_0, p_1, 0, s_N, p_N, d_N, nat, id, neg, con, dis, dom, inv, j. The set of all constants is denoted by $\mathcal{C}^{\mathsf{E}}$.

(4) Function Symbol: $\cdot$ (centered dot)

(5) Relation Symbols: $\downarrow$, N, $\in$, $=$, $\Re$

(6) Logical Symbols: $\neg$, $\vee$, $\wedge$, $\exists$, $\forall$

(7) Auxiliary Symbols:), (, ,

Definition 4.2 (Terms $\mathcal{T}^{\mathsf{E}}$)**.** Terms are denoted by r, s, t (possibly with subscripts).

(1) $\mathcal{V}_I \cup \mathcal{C}^{\mathsf{E}} \subset \mathcal{T}^{\mathsf{E}}$

(2) $s, t \in \mathcal{T}^{\mathsf{E}} \ \Rightarrow \ \cdot(s,t) \in \mathcal{T}^{\mathsf{E}}$

For $\cdot(s,t)$ we write $(s{\cdot}t)$ or simply st and we use association to the left, that is $s_0 s_1 \ldots s_n$ and $s_0{\cdot}s_1{\cdot}\ldots{\cdot}s_n$ stand for $(\ldots(s_0{\cdot}s_1)\ldots{\cdot}s_n)$.

Definition 4.3 (Atomic Formulas $\mathcal{F}_0^{\mathsf{E}}$)**.**

$$s, t \in \mathcal{T}^{\mathsf{E}},\ X \in \mathcal{V}_T \ \Rightarrow\ =(s,t),\ {\downarrow}(t),\ \mathsf{N}(t),\ \in(t,X),\ \Re(t,X) \in \mathcal{F}_0^{\mathsf{E}}$$

For $=(s,t)$, ${\downarrow}(t)$, $\mathsf{N}(t)$, $\in(t,X)$ we usually write $s=t$, $t{\downarrow}$, $t{\in}\mathsf{N}$, $t{\in}X$ respectively.

Definition 4.4 (Formulas $\mathcal{F}^{\mathsf{E}}$)**.** Formulas are denoted by ϕ, ψ, ξ (possibly with subscripts).

(1) $\mathcal{F}_0^{\mathsf{E}} \subset \mathcal{F}^{\mathsf{E}}$

(2) $\phi, \psi \in \mathcal{F}^{\mathsf{E}} \ \Rightarrow\ \neg\phi,\ (\phi \vee \psi),\ (\phi \wedge \psi) \in \mathcal{F}^{\mathsf{E}}$

(3) $\phi \in \mathcal{F}^{\mathsf{E}},\ x \in \mathcal{V}_I,\ X \in \mathcal{V}_T \ \Rightarrow\ (\exists x)\phi,\ (\forall x)\phi,\ (\exists X)\phi,\ (\forall X)\phi \in \mathcal{F}^{\mathsf{E}}$

We use the shorthand notations $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ for formulas in $\mathcal{F}^{\mathsf{E}}$ analogous to definition (2.5). For $(\exists x_0)\ldots(\exists x_n)\phi$ and $(\exists X_0)\ldots(\exists X_n)\phi$ we also write $(\exists\vec{x})\phi$ and $(\exists\vec{X})\phi$ respectively. The same notation we use for $\forall$.

Definition 4.5 (Free Variables and Substitution)**.** The set of free individual variables of a term t we denote by $FV_I(t)$ (or $FV_I(\phi)$ for a formula ϕ), see (7.5). The set of free type variables of a formula ϕ we denote by $FV_T(\phi)$, see (7.6). The simultaneous substitution of terms $t_0, \ldots, t_n$ for individual variables $x_0, \ldots, x_n$ in a term s we denote by $s[t_0/x_0, \ldots, t_n/x_n]$ and $s[\vec{t}/\vec{x}]$ (or $\phi[t_0/x_0, \ldots, t_n/x_n]$ and $\phi[\vec{t}/\vec{x}]$ for a formula ϕ), see (7.7). The simultaneous substitution of type variables $Y_0, \ldots, Y_n$ for type variables $X_0, \ldots, X_n$ in a formula ϕ we denote by $\phi[Y_0/X_0, \ldots, Y_n/X_n]$ and $\phi[\vec{Y}/\vec{X}]$, see (7.8). Further we denote by $FT(x,\phi)$ the set of terms t such that no variable z in $FV_I(t)$ is in the scope of a quantifier $(\exists z)$ or $(\forall z)$ in case x is replaced by t in ϕ (analogously with $FT(X,\phi)$ for type variables), see (7.9)+(7.10).

Definition 4.6 (Closed Terms, Sentences)**.** A term $t \in \mathcal{T}^{\mathsf{E}}$ is closed if $FV_I(t) = \{\}$, analogous a formula $\phi \in \mathcal{F}^{\mathsf{E}}$ is closed if $FV_I(\phi) = \{\}$. Closed formulas are usually called sentences.

Definition 4.7 (Propositional Axioms $\mathcal{A}^{\mathsf{E}}_{Prop}$)**.** We define $\mathcal{A}^{\mathsf{E}}_{Prop} \subset \mathcal{F}^{\mathsf{E}}$ analogous to $\mathcal{A}^{\mathsf{A}}_{Prop}$ in (2.8).

Definition 4.8 (Equality Axioms $\mathcal{A}^{\mathsf{E}}_{Equal}$)**.** For all atomic formulas $\phi \in \mathcal{F}_0^{\mathsf{E}}$ and variables z, $x_0, \ldots, x_n$, $y_0, \ldots, y_n \in \mathcal{V}_I$ the following formulas are in $\mathcal{A}^{\mathsf{E}}_{Equal}$:

(1) $z = z$

(2) $x_0 = y_0 \wedge \ldots \wedge x_n = y_n \wedge \phi \to \phi[\vec{y}/\vec{x}]$

Definition 4.9 (Quantifier Axioms $\mathcal{A}^{\mathsf{E}}_{Quant}$)**.** For all formulas $\phi \in \mathcal{F}^{\mathsf{E}}$, for all variables $x \in \mathcal{V}_I$ and terms $t \in FT(x, \phi)$ and for all variables $X \in \mathcal{V}_T$ and $Y \in FT(X, \phi)$ the following formulas are in $\mathcal{A}^{\mathsf{E}}_{Quant}$:

(1) $\phi[t/x] \wedge t\downarrow \to (\exists x)\phi$

(2) $(\forall x)\phi \wedge t\downarrow \to \phi[t/x]$

(3) $\phi[Y/X] \to (\exists X)\phi$

(4) $(\forall X)\phi \to \phi[Y/X]$

Definition 4.10 (Definedness Axioms $\mathcal{A}^{\mathsf{E}}_{Def}$)**.** For all variables and constants $r \in \mathcal{V}_I \cup \mathcal{C}^{\mathsf{E}}$, for all atomic formulas $\phi \in \mathcal{F}^{\mathsf{E}}_0$, for all variables $x_0, \ldots, x_n \in \mathcal{V}_I$ such that $x_i \in FV_I(\phi)$ and for all terms s, t, $t_0, \ldots, t_n \in \mathcal{T}^{\mathsf{E}}$ the following formulas are in $\mathcal{A}^{\mathsf{E}}_{Def}$:

(1) $r\downarrow$

(2) $(s \cdot t)\downarrow \to s\downarrow \wedge t\downarrow$

(3) $\phi[\vec{t}/\vec{x}] \to t_0\downarrow \wedge \ldots \wedge t_n\downarrow$

Definition 4.11 (Rules of Inference $\mathcal{R}^{\mathsf{E}}$)**.** For all formulas ϕ, $\psi \in \mathcal{F}^{\mathsf{E}}$, for all variables x, $y \in \mathcal{V}_I$ such that $y \in FT(x, \phi) \setminus FV_I(\psi)$ and for all variables $X \in \mathcal{V}_T$ and $Y \in FT(X, \phi) \setminus FV_T(\psi)$ the following rules are in $\mathcal{R}^{\mathsf{E}}$:

(1) $\dfrac{\phi[y/x] \to \psi}{(\exists x)\phi \to \psi}$

(2) $\dfrac{\psi \to \phi[y/x]}{\psi \to (\forall x)\phi}$

(3) $\dfrac{\phi[Y/X] \to \psi}{(\exists X)\phi \to \psi}$

(4) $\dfrac{\psi \to \phi[Y/X]}{\psi \to (\forall X)\phi}$

(5) $\dfrac{\phi \quad \phi \to \psi}{\psi}$

Definition 4.12 (T-Proof and $T \vdash \phi$)**.** For $T \subset \mathcal{F}^{\mathsf{E}}$ and $\phi_0, \ldots, \phi_n \in \mathcal{F}^{\mathsf{E}}$ the sequence $(\phi_0, \ldots, \phi_n)$ is a T-Proof in case all ϕ_i satisfy one of the following conditions:

(1) $\phi_i \in T$

(2) $\phi_i \in \mathcal{A}^{\mathsf{E}}_{Prop} \cup \mathcal{A}^{\mathsf{E}}_{Equal} \cup \mathcal{A}^{\mathsf{E}}_{Quant} \cup \mathcal{A}^{\mathsf{E}}_{Def}$

(3) ϕ_i is the conclusion of a rule of inference in $\mathcal{R}^{\mathsf{E}}$ whose premises belong to $\{\phi_0, \ldots, \phi_{i-1}\}$.

Let $T \subset \mathcal{F}^{\mathsf{E}}$ and $\phi \in \mathcal{F}^{\mathsf{E}}$, if there are $\phi_0, \ldots, \phi_k$ such that $(\phi_0, \ldots, \phi_k)$ is a T-Proof with $\phi_k = \phi$ then we denote this by $T \vdash^k \phi$ or simply $T \vdash \phi$. Let $S \subset \mathcal{F}^{\mathsf{E}}$, if we have $T \vdash \phi$ for all $\phi \in S$ then we denote this by $T \vdash S$. We use the shorthand notations $\vdash^k \phi$ and $\vdash \phi$ and $\vdash S$ whenever $T = \{\}$.

4.2 Principles and Theories

Based on the language of Explicit Mathematics we define the non-logical axioms for theories, several induction principles and some ontological principles.

4.2.1 Operations and Numbers

The theory of basic operations and numbers BON^- consists of axioms for a partial combinatory algebra, pairing and projection, natural numbers with successor and predecessor, and definition by numerical cases.

Definition 4.13 (Abbreviations)**.** We use the following shorthand notations:

$$
\begin{array}{rcll}
s \neq t & := & \neg s = t \wedge s{\downarrow} \wedge t{\downarrow} & \\
s \simeq t & := & (s{\downarrow} \vee t{\downarrow}) \rightarrow s = t & \\
\langle s_0, \ldots, s_n \rangle & := & (\mathsf{p}\langle s_0, \ldots, s_{n-1}\rangle s_n) & [\langle s_0 \rangle := s_0] \\
(\exists x {\in} \mathsf{N})\phi & := & (\exists x)(x {\in} \mathsf{N} \wedge \phi) & \\
(\forall x {\in} \mathsf{N})\phi & := & (\forall x)(x {\in} \mathsf{N} \rightarrow \phi) & \\
t {\in} (\mathsf{N}^{k+1} {\rightarrow} \mathsf{N}) & := & (\forall x_0 {\in} \mathsf{N}) \ldots (\forall x_k {\in} \mathsf{N}) t x_0 \ldots x_k {\in} \mathsf{N} & [i \neq j \Rightarrow x_i \neq x_j] \\
t {\in} (\mathsf{N} {\rightarrow} \mathsf{N}) & := & t {\in} (\mathsf{N}^1 {\rightarrow} \mathsf{N}) &
\end{array}
$$

Remark: There is an ambiguity in the choice of $x_0, \ldots, x_k \in \mathcal{V}_I$ in the formula $t{\in}(\mathsf{N}^{k+1}{\rightarrow}\mathsf{N})$.

Definition 4.14 (Basic Operations and Numbers BON^-)**.** For all individual variables u, v, x, y, $z \in \mathcal{V}_I$ the following formulas are in BON^-:

(1) $(\mathsf{k}x)y = x$

(2) $\mathsf{s}xy\downarrow \wedge (\mathsf{s}xy)z \simeq (xz)(yz)$

(3) $\mathsf{p}_0\langle x, y\rangle = x \wedge \mathsf{p}_1\langle x, y\rangle = y$

(4) $\mathsf{0}{\in}\mathsf{N} \wedge \mathsf{s}_\mathsf{N}{\in}(\mathsf{N}{\to}\mathsf{N})$

(5) $(\forall x{\in}\mathsf{N})(\mathsf{s}_\mathsf{N}x \neq \mathsf{0} \wedge \mathsf{p}_\mathsf{N}(\mathsf{s}_\mathsf{N}x) = x)$

(6) $(\forall x{\in}\mathsf{N})(x \neq \mathsf{0} \to \mathsf{p}_\mathsf{N}x{\in}\mathsf{N} \wedge \mathsf{s}_\mathsf{N}(\mathsf{p}_\mathsf{N}x) = x)$

(7) $x{\in}\mathsf{N} \wedge y{\in}\mathsf{N} \wedge x = y \to (\mathsf{d}_\mathsf{N}uv)xy = u$

(8) $x{\in}\mathsf{N} \wedge y{\in}\mathsf{N} \wedge x \neq y \to (\mathsf{d}_\mathsf{N}uv)xy = v$

4.2.2 Explicit Types

The non-logical axioms for explicit type theories consist of axioms for the naming relation and the axioms for type generators (name generators). The first is stating that every type has a name and names behave well (no homonyms), the latter guarantees the existence of particular types and the uniform creation of their names.

Definition 4.15 (Abbreviations)**.** We use the following shorthand notations:

$$\begin{array}{rcl} X \subset Y & := & (\forall x)(x{\in}X \to x{\in}Y) \\ X = Y & := & (\forall x)(x{\in}X \leftrightarrow x{\in}Y) \\ \Re(\vec{t}, \vec{X}) & := & \Re(t_0, X_0) \wedge \ldots \wedge \Re(t_n, X_n) \\ \Re(t) & := & (\exists X)\Re(t, X) \\ s\dot{\in}t & := & (\exists X)(\Re(t, X) \wedge s{\in}X) \end{array}$$

Definition 4.16 (Representation Axioms $\mathcal{A}_{Rep}$)**.** For all individual variables $x \in \mathcal{V}_I$ and for all type variables X, $Y \in \mathcal{V}_T$ the following formulas are in $\mathcal{A}_{Rep}$:

(1) $(\exists x)\Re(x, X)$

(2) $\Re(x, X) \wedge \Re(x, Y) \to X = Y$

(3) $X = Y \wedge \Re(x, X) \to \Re(x, Y)$

Definition 4.17 (Generator Axioms $\mathcal{A}_{\mathsf{E}}$, $\mathcal{A}_{\Sigma^+\mathsf{E}}$)**.** For all individual variables $a, b, f, x, y \in \mathcal{V}_I$ the formulas (1) ... (7) are in $\mathcal{A}_{\mathsf{E}}$ and the formulas (1) ... (6) are in $\mathcal{A}_{\Sigma^+\mathsf{E}}$:

(1) $\Re(\mathsf{nat}) \wedge (\forall x)(x\dot{\in}\mathsf{nat} \leftrightarrow x\in\mathsf{N})$

(2) $\Re(\mathsf{id}) \wedge (\forall x)(x\dot{\in}\mathsf{id} \leftrightarrow (\exists y)x = \langle y, y\rangle)$

(3) $\Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{con}\langle a, b\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{con}\langle a, b\rangle \leftrightarrow x\dot{\in}a \wedge x\dot{\in}b)$

(4) $\Re(a) \wedge \Re(b) \rightarrow \Re(\mathsf{dis}\langle a, b\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{dis}\langle a, b\rangle \leftrightarrow x\dot{\in}a \vee x\dot{\in}b)$

(5) $\Re(a) \rightarrow \Re(\mathsf{dom}\langle a\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{dom}\langle a\rangle \leftrightarrow (\exists y)\langle x, y\rangle\dot{\in}a)$

(6) $\Re(a) \rightarrow \Re(\mathsf{inv}\langle a, f\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{inv}\langle a, f\rangle \leftrightarrow fx\dot{\in}a)$

(7) $\Re(a) \rightarrow \Re(\mathsf{neg}\langle a\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{neg}\langle a\rangle \leftrightarrow \neg x\dot{\in}a)$

Definition 4.18 (Disjoint Union J (Join))**.** For all individual variables a, f, x, y, $z \in \mathcal{V}_I$ the following formula is in J:

$$\begin{aligned}&\Re(a) \wedge (\forall x\dot{\in}a)\Re(fx) \rightarrow \Re(\mathsf{j}\langle a, f\rangle) \wedge \\ &\qquad (\forall x)[x\dot{\in}\mathsf{j}\langle a, f\rangle \leftrightarrow (\exists y)(\exists z)(x = \langle y, z\rangle \wedge y\dot{\in}a \wedge z\dot{\in}fy)]\end{aligned}$$

Definition 4.19 (Explicit Types $\Sigma^+\mathsf{ET}$ and EET^-)**.**

(1) $\Sigma^+\mathsf{ET} = \mathsf{BON}^- \cup \mathcal{A}_{Rep} \cup \mathcal{A}_{\Sigma^+\mathsf{E}}$ (Positive Existential Explicit Types)

(2) $\mathsf{EET}^- = \mathsf{BON}^- \cup \mathcal{A}_{Rep} \cup \mathcal{A}_{\mathsf{E}}$ (Elementary Explicit Types)

4.2.3 Induction Principles

In (5.8) and (5.10) we see that the following induction principles are ordered by their strength, formula induction of course being the strongest form.

Definition 4.20 (Set Induction $\mathsf{S}\text{-}\mathsf{I}_{\mathsf{N}}$)**.** For all individual variables x, $f \in \mathcal{V}_I$ the following formula is in $\mathsf{S}\text{-}\mathsf{I}_{\mathsf{N}}$:

$$f\in(\mathsf{N}\rightarrow\mathsf{N}) \wedge f0 = 0 \wedge (\forall x\in\mathsf{N})(fx = 0 \rightarrow f(\mathsf{s}_{\mathsf{N}}x) = 0) \rightarrow (\forall x\in\mathsf{N})(fx = 0)$$

Definition 4.21 (Value Induction $\mathsf{V}\text{-}\mathsf{I}_{\mathsf{N}}$)**.** For all individual variables x, $f \in \mathcal{V}_I$ the following formula is in $\mathsf{V}\text{-}\mathsf{I}_{\mathsf{N}}$:

$$f0\in\mathsf{N} \wedge (\forall x\in\mathsf{N})(fx\in\mathsf{N} \rightarrow f(\mathsf{s}_{\mathsf{N}}x)\in\mathsf{N}) \rightarrow f\in(\mathsf{N}\rightarrow\mathsf{N})$$

Definition 4.22 (Type Induction $\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}}$)**.** For all individual variables $x \in \mathcal{V}_I$ and type variables $X \in \mathcal{V}_T$ the following formula is in $\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}}$:

$$0{\in}X \land (\forall x{\in}\mathsf{N})(x{\in}X \rightarrow \mathsf{s_N}x{\in}X) \rightarrow (\forall x{\in}\mathsf{N})x{\in}X$$

Definition 4.23 (Formula Induction $\mathcal{F}\text{-}\mathsf{I_N}$)**.** For all individual variables $x \in \mathcal{V}_I$ and formulas $\phi \in \mathcal{F}^{\mathsf{E}}$ the following formula is in $\mathcal{F}\text{-}\mathsf{I_N}$:

$$\phi[0/x] \land (\forall x{\in}\mathsf{N})(\phi \rightarrow \phi[\mathsf{s_N}x/x]) \rightarrow (\forall x{\in}\mathsf{N})\phi$$

4.2.4 Ontological Principles

Definition 4.24 (All Individuals are Names of Types $\forall\Re$)**.** For all individual variables $x \in \mathcal{V}_I$ the formula $(\forall x)\Re(x)$ is in $\forall\Re$.

Definition 4.25 (All Individuals are Numbers $\forall\mathsf{N}$)**.** For all individual variables $x \in \mathcal{V}_I$ the formula $(\forall x)\mathsf{N}(x)$ is in $\forall\mathsf{N}$.

Definition 4.26 (Uniform Comprehension)**.** Let $\mathcal{X}$ be a subset of $\mathcal{F}^{\mathsf{E}}$ then for all formulas $\phi \in \mathcal{X}$ the following formula with $\{X_0, \ldots, X_n\} := FV_T(\phi)$ and $\{y_0, \ldots, y_m\} := FV_I(\phi)\backslash\{z\}$ is in $\mathcal{X}\text{-}\mathsf{C}$:

$$\begin{aligned}(\exists f)(\forall \vec{x})(\forall \vec{y})(\forall \vec{X})[\ &\Re(\vec{x}, \vec{X}) \rightarrow \\ &\Re(fx_0 \ldots x_n y_0 \ldots y_m) \land (\forall z)(z\dot{\in} fx_0 \ldots x_n y_0 \ldots y_m \leftrightarrow \phi)\ ]\end{aligned}$$

Similar to the generator axioms this formula is also stating the existence of a generator, but now for arbitrary $\phi \in \mathcal{X}$.

Definition 4.27 (Abbreviations)**.**

$$\begin{aligned} Ext(X) :=& (\forall a)(\forall b)[(\exists Z)(\Re(a, Z) \land \Re(b, Z)) \rightarrow (a{\in}X \rightarrow b{\in}X)] \\ \overline{Pow}(X, Y) :=& (\forall Z)(Z \subset Y \rightarrow (\exists a)(a{\in}X \land \Re(a, Z))) \land \\ & (\forall a)(a{\in}X \rightarrow (\exists Z)(Z \subset Y \land \Re(a, Z))) \\ \overset{+}{Pow}(X, Y) :=& \overline{Pow}(X, Y) \land Ext(X) \end{aligned}$$

If $Ext(X)$ holds then we have for every type Z, that X either contains all names of Z or none of them. If $\overline{Pow}(X, Y)$ holds then for every subtype Z of Y there is at least one name of Z contained in X, further we have that X contains nothing else but names of subtypes of Y. Hence $\overset{+}{Pow}(X, Y)$ holds if X contains all names of all subtypes of Y, and X contains nothing else.

Definition 4.28 (Weak Power Types $\overline{\mathsf{Pow}}$)**.** For all type variables $X, Y \in \mathcal{V}_T$ the formulas of the form $(\forall Y)(\exists X)\overline{Pow}(X, Y)$ are in $\overline{\mathsf{Pow}}$.

Definition 4.29 (Strong Power Types $\overset{+}{\mathsf{Pow}}$)**.** For all type variables $X, Y \in \mathcal{V}_T$ the formulas of the form $(\forall Y)(\exists X)\overset{+}{Pow}(X, Y)$ are in $\overset{+}{\mathsf{Pow}}$.

4.3 Semantics

The language of Explicit Mathematics defined above can be interpreted in many different ways. In the following we give a definition of what we mean by an interpretation. Based on this definition we are able to further define what it means for a statement in the language of Explicit Mathematics to be true (for all interpretations) or to be a logical consequence of some theory.

Definition 4.30 (Structures of Explicit Mathematics $\mathbb{M}^{\mathsf{E}}$)**.** A structure $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ consists of the following:

(1) A domain of individuals M and an extra individual $\infty \notin M$.

(2) A domain of types $T \subset \mathbf{P}(M)$.

(3) For every constant $c \in \mathcal{C}^{\mathsf{E}}$ a constant $c^{\mathcal{M}} \in M$.

(4) A (restricted) binary operation $\cdot^{\mathcal{M}}$ on $M^{\infty} := M \cup \{\infty\}$, that is $\cdot^{\mathcal{M}} : M^{\infty} \times M^{\infty} \to M^{\infty}$ such that $(\forall x \in M^{\infty})(\infty \cdot^{\mathcal{M}} x = x \cdot^{\mathcal{M}} \infty = \infty)$.

(5) A unary relation $\mathsf{N}^{\mathcal{M}} \subset M$ and a binary relation $\Re^{\mathcal{M}} \subset M \times T$.

We usually denote M, T, and M^{∞} by $|\mathcal{M}|^{I}$, $|\mathcal{M}|^{T}$ and $|\mathcal{M}|^{\infty}$ respectively.

Definition 4.31 (Valuations $\mathbb{V}^{\mathcal{M}}$)**.** A valuation $\nu \in \mathbb{V}^{\mathcal{M}}$ for a structure $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ is a mapping $\nu : \mathcal{V}_I \cup \mathcal{V}_T \to |\mathcal{M}|^{I} \cup |\mathcal{M}|^{T}$ such that $x \in \mathcal{V}_I \Rightarrow \nu(x) \in |\mathcal{M}|^{I}$ and $X \in \mathcal{V}_T \Rightarrow \nu(X) \in |\mathcal{M}|^{T}$. If $\nu \in \mathbb{V}^{\mathcal{M}}$ then $\nu[u{:}m] \in \mathbb{V}^{\mathcal{M}}$ denotes the following valuation (where m is in $|\mathcal{M}|^{I}$ or in $|\mathcal{M}|^{T}$ according to u):

$$\nu[u{:}m](v) := \begin{cases} m & v = u \\ \nu(v) & \text{otherwise} \end{cases}$$

Definition 4.32 (Interpretations $\mathbb{I}^{\mathsf{E}}$)**.** An interpretation $\mathcal{M}_\nu \in \mathbb{I}^{\mathsf{E}}$ (for a structure $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ and a valuation $\nu \in \mathbb{V}^{\mathcal{M}}$) consists of the following:

(1) A mapping $\mathcal{M}_\nu : \mathcal{T}^{\mathsf{E}} \to |\mathcal{M}|^{\infty}$ such that:

$$\mathcal{M}_\nu(t) := \begin{cases} t^{\mathcal{M}} & t \in \mathcal{C}^{\mathsf{E}} \\ \nu(t) & t \in \mathcal{V}_I \\ \mathcal{M}_\nu(r) \cdot^{\mathcal{M}} \mathcal{M}_\nu(s) & t = (r \cdot s) \end{cases}$$

(2) A mapping $\mathcal{M}_\nu : \mathcal{F}^{\mathsf{E}} \to \{\mathbf{t}, \mathbf{f}\}$ such that:

$$
\begin{array}{lll}
\mathcal{M}_\nu(s = t) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(s) = \mathcal{M}_\nu(t) \neq \infty \\
\mathcal{M}_\nu(t{\downarrow}) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(t) \neq \infty \\
\mathcal{M}_\nu(t{\in}\mathsf{N}) = \mathbf{t} & :\Leftrightarrow & \mathsf{N}^{\mathcal{M}}(\mathcal{M}_\nu(t)) \\
\mathcal{M}_\nu(t{\in}X) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(t) \in \nu(X) \\
\mathcal{M}_\nu(\Re(t, X)) = \mathbf{t} & :\Leftrightarrow & \Re^{\mathcal{M}}(\mathcal{M}_\nu(t), \nu(X)) \\
\mathcal{M}_\nu(\neg\phi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{f} \\
\mathcal{M}_\nu(\phi \vee \psi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{t} \vee \mathcal{M}_\nu(\psi) = \mathbf{t} \\
\mathcal{M}_\nu(\phi \wedge \psi) = \mathbf{t} & :\Leftrightarrow & \mathcal{M}_\nu(\phi) = \mathbf{t} \wedge \mathcal{M}_\nu(\psi) = \mathbf{t} \\
\mathcal{M}_\nu((\exists x)\phi) = \mathbf{t} & :\Leftrightarrow & (\exists m \in |\mathcal{M}|^I)(\mathcal{M}_{\nu[x:m]}(\phi) = \mathbf{t}) \\
\mathcal{M}_\nu((\forall x)\phi) = \mathbf{t} & :\Leftrightarrow & (\forall m \in |\mathcal{M}|^I)(\mathcal{M}_{\nu[x:m]}(\phi) = \mathbf{t}) \\
\mathcal{M}_\nu((\exists X)\phi) = \mathbf{t} & :\Leftrightarrow & (\exists S \in |\mathcal{M}|^T)(\mathcal{M}_{\nu[X:S]}(\phi) = \mathbf{t}) \\
\mathcal{M}_\nu((\forall X)\phi) = \mathbf{t} & :\Leftrightarrow & (\forall S \in |\mathcal{M}|^T)(\mathcal{M}_{\nu[X:S]}(\phi) = \mathbf{t})
\end{array}
$$

Definition 4.33. For $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$, $\phi \in \mathcal{F}^{\mathsf{E}}$ and $T, S \subset \mathcal{F}^{\mathsf{E}}$ we define the following relations:

$$
\begin{array}{lll}
\mathcal{M} \models \phi & :\Leftrightarrow & (\forall \nu \in \mathbb{V}^{\mathcal{M}})(\mathcal{M}_\nu(\phi) = \mathbf{t}) \\
\mathcal{M} \models T & :\Leftrightarrow & (\forall \phi \in T)(\mathcal{M} \models \phi) \\
T \models \phi & :\Leftrightarrow & (\forall \mathcal{M} \in \mathbb{M}^{\mathsf{E}})(\mathcal{M} \models T \Rightarrow \mathcal{M} \models \phi) \\
T \models S & :\Leftrightarrow & (\forall \phi \in S)(T \models \phi)
\end{array}
$$

If $\mathcal{M} \models \phi$ holds then we say ϕ is valid in $\mathcal{M}$, and if we have $\mathcal{M} \models T$ then we say $\mathcal{M}$ is a model of T. Finally if we have $T \models \phi$ (or $T \models S$) then we say ϕ (or S) is a logical consequence of T.

Theorem 4.34 (Adequacy)**.** *Let $\phi \in \mathcal{F}^{\mathsf{E}}$ then we have*

$$T \models \phi \quad \Leftrightarrow \quad T \vdash \phi$$

Example 4.35. We use the following functions defined in (7.1):

$$
\begin{array}{rl}
[\![Ins]\!] : \mathbb{N}^2 \to \mathbb{N} & \quad (3.10) \\
[\![\pi]\!] : \mathbb{N}^2 \to \mathbb{N} & \quad (3.9) \\
[\![Com]\!] : \mathbb{N}^2 \to \mathbb{N} & \quad (6.13)
\end{array}
$$

And for $x_0, \ldots, x_n \in \mathbb{N}$ we use analogous to 3.10 the shorthand notation:

$$\langle x_0, \ldots, x_n \rangle := [\![Ins]\!](\ldots [\![Ins]\!]([\![\pi]\!](0,0), x_0) \ldots, x_n)$$

Now we are ready to define $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ to be the structure consisting of:

(1) $|\mathcal{M}|^I := \mathbb{N}$ and $\infty \notin \mathbb{N}$ (e.g. $\infty = \mathbb{N}$).

(2) For $n \in \mathbb{N}$ we define $T_n := \{x \mid (\exists m)([\![Com]\!](\langle n, x\rangle, m) = 0)\}$ and let $|\mathcal{M}|^T := \{T_n \mid n \in \mathbb{N}\}$.

(3) $\mathsf{0}^{\mathcal{M}} := 0$ and $c^{\mathcal{M}} := [\![\pi]\!](n_c, 0)$ for $c \in \mathcal{C}^{\mathsf{E}} \backslash \{\mathsf{0}\}$ (where n_c is the number assigned to c in 6.10). $c^{\mathcal{M}}$ needs to have this clumsy definition because the definition of Com refers to (6.10).

(4)

$$x \cdot^{\mathcal{M}} y := \begin{cases} n & (\exists m)([\![Com]\!](\langle x, y, n\rangle, m) = 0) \\ \infty & \text{otherwise} \end{cases}$$

The function $\cdot^{\mathcal{M}}$ is well-defined by (6.18), (2.23) and example (2.24).

(5)

$$\mathsf{N}^{\mathcal{M}} := \mathbb{N}$$

$$\Re^{\mathcal{M}} := \{(m, T_n) \mid T_m = T_n\}$$

For the structure $\mathcal{M}$ we have by the embedding theorem (6.28) (i.e. by the particular embedding 6.22 we used for proving it) that: (see 5.11 for $\Sigma^{+}\mathsf{S}$)

$$\mathcal{M} \models \Sigma^{+}\mathsf{ET}+(\mathsf{J})+(\forall\Re)+(\forall\mathsf{N})+(\Sigma^{+}\mathsf{S}\text{-}\mathsf{C})+(\mathsf{Pow}^{-})+(\mathcal{F}\text{-}\mathsf{I}_{\mathsf{N}})$$

The structure $\mathcal{M}$ can be seen as an extension of the structure in the example 2.24. The construction of $\mathcal{M}$ is generic, i.e. any structure $\mathcal{N} \in \mathbb{M}^{\mathsf{A}}$ such that $\mathcal{N} \models \mathsf{PA}$ can be extended similar to the way we described above.

5 Some Aspects of Explicit Mathematics

Having defined the language of Explicit Mathematics and the concept of a formal proof, we now give some useful examples of provable statements. In this way we are building a collection of tools we use in the sequel.

5.1 Applicative Theory

We define λ-abstraction on terms and show that it has the desired properties. We then use λ-abstraction to prove the existence of a fixed point operator in the combinatory algebra. Further we use this fixed point operator to prove some basic facts, e.g. primitive recursion in the theory $\mathsf{BON}^- + (\mathsf{V}\text{-}\mathsf{I}_\mathsf{N})$.

Definition 5.1 ($\lambda : \mathcal{V}_I \times \mathcal{T}^\mathsf{E} \to \mathcal{T}^\mathsf{E}$)**.**

$$\lambda(x,t) := \begin{cases} \mathsf{skk} & t = x \\ \mathsf{s}\lambda(x,t_1)\lambda(x,t_2) & t = t_1 t_2 \\ \mathsf{k}t & \text{otherwise} \end{cases}$$

We usually write $\lambda x.t$ for $\lambda(x,t)$ and $\lambda x_0 \ldots x_n.t$ or $\lambda \vec{x}.t$ for $\lambda x_0.(\ldots \lambda x_n.(t)\ldots)$.

Lemma 5.2.

$$FV_I(\lambda x.t) = FV_I(t) \backslash \{x\}$$

Theorem 5.3 (λ-Abstraction)**.**

(1) $\mathsf{BON}^- \vdash \lambda x.t\downarrow$

(2) $\mathsf{BON}^- \vdash s\downarrow \to (\lambda x.t)s \simeq t[s/x]$

Proof.

(1) By induction on the term $t \in \mathcal{T}^\mathsf{E}$.

(2) We first prove the statement for $s = x$ by induction on the term t, then for arbitrary s we use an instance of $\mathcal{A}^\mathsf{E}_{Quant}$.

□

Lemma 5.4 (Substitution)**.**

$$x \neq y \;\Rightarrow\; \mathsf{BON}^- \vdash (\lambda x.t)[s/y]x \simeq t[s/y]$$

Proof. By induction on t. □

Theorem 5.5 (Fixed Point)**.** *There exists a closed term* $\mathsf{fix} \in \mathcal{T}^{\mathsf{E}}$ *such that*

$$\mathsf{BON}^{-} \vdash \mathsf{fix}f\downarrow \wedge\, f(\mathsf{fix}f)x \simeq (\mathsf{fix}f)x$$

Proof. Let $t := (\lambda yx.f(yy)x)$ and $\mathsf{fix} := (\lambda f.tt)$. □

Lemma 5.6 (Primitive Recursion)**.** *There exists a closed term* $\mathsf{rec} \in \mathcal{T}^{\mathsf{E}}$ *such that*

(1) $\mathsf{BON}^{-} \vdash f{\in}(\mathsf{N}^2{\to}\mathsf{N}) \wedge a{\in}\mathsf{N} \wedge b{\in}\mathsf{N} \rightarrow$
$$(\mathsf{rec}fa)\mathsf{0} = a \wedge (\mathsf{rec}fa)(\mathsf{s_N}b) = fb((\mathsf{rec}fa)b)$$

(2) $\mathsf{BON}^{-}{+}(\mathsf{V}\text{-}\mathsf{I_N}) \vdash f{\in}(\mathsf{N}^2{\to}\mathsf{N}) \wedge a{\in}\mathsf{N} \rightarrow (\mathsf{rec}fa){\in}(\mathsf{N}{\to}\mathsf{N})$

Proof.

(1) Let $t := \lambda hx.(\mathsf{d_N}(\lambda z.a)(\lambda z.f(\mathsf{p_N}x)(h(\mathsf{p_N}x)))x\mathsf{0})\mathsf{0}$ and $\mathsf{rec} := (\lambda fa.\mathsf{fix}t)$.

(2) By (1) and using value induction ($\mathsf{V}\text{-}\mathsf{I_N}$).

□

Lemma 5.7. *There exists a closed term* $\mathsf{not_N} \in \mathcal{T}^{\mathsf{E}}$ *such that*

$$\mathsf{BON}^{-} \vdash \neg(\mathsf{not_N}{\in}\mathsf{N})$$

Proof. Let $t := (\lambda xy.\mathsf{d_N}(\mathsf{s_N}\mathsf{0})\mathsf{0}(xy)\mathsf{0})$ and $\mathsf{not_N} := \mathsf{fix}t\mathsf{0}$ □

Lemma 5.8.

$$\mathsf{BON}^{-}{+}(\mathsf{V}\text{-}\mathsf{I_N}) \vdash \mathsf{S}\text{-}\mathsf{I_N}$$

Proof. Let $s := (\lambda y.\mathsf{d_N}(\lambda z.\mathsf{0})(\lambda z.\mathsf{not_N})y\mathsf{0})$ and $t := \lambda fx.s(fx)$ then we have

$$\mathsf{BON}^{-} \vdash y = 0 \leftrightarrow y{\in}\mathsf{N} \wedge sy{\in}\mathsf{N}$$
$$\mathsf{BON}^{-} \vdash (\forall x{\in}\mathsf{N})(fx = \mathsf{0} \leftrightarrow (tf)x{\in}\mathsf{N})$$

hence we are able to reduce ($\mathsf{S}\text{-}\mathsf{I_N}$) to ($\mathsf{V}\text{-}\mathsf{I_N}$). □

5.2 Explicit Type Theory

First we show how to reduce value induction ($\mathsf{V}\text{-}\mathsf{I_N}$) to type induction ($\mathsf{T}\text{-}\mathsf{I_N}$). Next we use the generator axioms to get type comprehension for a whole class of formulas. Using disjoint union (J) under certain conditions we even get comprehension for formulas containing type quantifiers. This section also contains two inconsistency results about ontological principles. Finally we state the refutation of strong power types by Cantini and Minari [2] in the context of our positive result about weak power types in the theory $\Sigma^{+}\mathsf{ET}{+}(\forall\Re)$.

5.2.1 Induction

Lemma 5.9.

$$\Sigma^{+}\mathsf{ET} \vdash \Re(a, X) \to (x{\in}X \leftrightarrow x\dot{\in}a)$$

Proof. We have $(\Re(a, X) \wedge x{\in}X \to x\dot{\in}a) \in \mathcal{A}^{\mathsf{E}}_{Quant}$. For the other direction we have $(\Re(a, X) \wedge \Re(a, Y) \to X{=}Y) \in \mathcal{A}_{Rep}$ hence we are able to deduce $\Sigma^{+}\mathsf{ET} \vdash \Re(a, X) \wedge (\Re(a, Y) \wedge x{\in}Y) \to x{\in}X$. By an inference rule in $\mathcal{R}^{\mathsf{E}}$ we finally get $\Sigma^{+}\mathsf{ET} \vdash \Re(a, X) \wedge x\dot{\in}a \to x{\in}X$. □

Lemma 5.10.

$$\Sigma^{+}\mathsf{ET}{+}(\mathsf{T\text{-}I_N}) \vdash \mathsf{V\text{-}I_N}$$

Proof. Let $t := \mathsf{inv}\langle\mathsf{nat}, f\rangle$, then from $\mathcal{A}_{\Sigma^{+}\mathsf{E}}$ we get $\Sigma^{+}\mathsf{ET} \vdash fx{\in}\mathsf{N} \leftrightarrow fx\dot{\in}\mathsf{nat}$ and $\Sigma^{+}\mathsf{ET} \vdash \Re(t) \wedge fx\dot{\in}\mathsf{nat} \leftrightarrow x\dot{\in}t$ hence we deduce $\Sigma^{+}\mathsf{ET} \vdash fx{\in}\mathsf{N} \leftrightarrow x\dot{\in}t$. Let

$$\begin{aligned}
\phi_0 &:= f0{\in}\mathsf{N} \wedge (\forall x{\in}\mathsf{N})(fx{\in}\mathsf{N} \to f(\mathsf{s_N}x){\in}\mathsf{N}) \to f{\in}(\mathsf{N}{\to}\mathsf{N})\\
\phi_1 &:= 0\dot{\in}t \wedge (\forall x{\in}\mathsf{N})(x\dot{\in}t \to \mathsf{s_N}x\dot{\in}t) \to (\forall x{\in}\mathsf{N})x\dot{\in}t\\
\phi_2 &:= 0{\in}X \wedge (\forall x{\in}\mathsf{N})(x{\in}X \to \mathsf{s_N}x{\in}X) \to (\forall x{\in}\mathsf{N})x{\in}X
\end{aligned}$$

Now we have $\Sigma^{+}\mathsf{ET} \vdash \phi_1 \leftrightarrow \phi_0$ and we get $\Sigma^{+}\mathsf{ET} \vdash \Re(t, X) \to (\phi_2 \leftrightarrow \phi_1)$ by (5.9), but $\phi_2 \in \mathsf{T\text{-}I_N}$ hence $\Sigma^{+}\mathsf{ET}{+}(\mathsf{T\text{-}I_N}) \vdash \Re(t, X) \to \phi_0$ and by an inference rule $\Sigma^{+}\mathsf{ET}{+}(\mathsf{T\text{-}I_N}) \vdash \Re(t) \to \phi_0$, hence $\Sigma^{+}\mathsf{ET}{+}(\mathsf{T\text{-}I_N}) \vdash \phi_0$. □

5.2.2 Comprehension

Definition 5.11 (Formulas E, $\Sigma^{+}\mathsf{E}$, S, $\Sigma^{+}\mathsf{S}$)**.**

(1) For all $s, t \in \mathcal{T}^{\mathsf{E}}$, $X \in \mathcal{V}_T$ the formulas $s{=}t$, $t\downarrow$, $t{\in}\mathsf{N}$, $t{\in}X$ are in $\Sigma^{+}\mathsf{E}$, $\Sigma^{+}\mathsf{S}$, E, S (but $\Re(t, X)$ is not).

(2) E (Elementary Formulas)

$$\phi, \psi \in \mathsf{E},\ x \in \mathcal{V}_I \ \Rightarrow\ \neg\phi,\ (\phi \vee \psi),\ (\phi \wedge \psi),\ (\exists x)\phi,\ (\forall x)\phi \in \mathsf{E}$$

(3) $\Sigma^{+}\mathsf{E}$ (Positive Existential Elementary Formulas)

$$\phi, \psi \in \Sigma^{+}\mathsf{E},\ x \in \mathcal{V}_I \ \Rightarrow\ (\phi \vee \psi),\ (\phi \wedge \psi),\ (\exists x)\phi \in \Sigma^{+}\mathsf{E}$$

(4) S (Stratified Formulas)

$$\begin{aligned}
&\phi, \psi \in \mathsf{S}, x \in \mathcal{V}_I,\ X \in \mathcal{V}_T \ \Rightarrow\\
&\qquad \neg\phi,\ (\phi \vee \psi),\ (\phi \wedge \psi),\ (\exists x)\phi,\ (\forall x)\phi,\ (\exists X)\phi,\ (\forall X)\phi \in \mathsf{S}
\end{aligned}$$

(5) $\Sigma^{+}\mathsf{S}$ (Positive Existential Stratified Formulas)

$$\phi, \psi \in \Sigma^{+}\mathsf{S},\ x \in \mathcal{V}_I,\ X \in \mathcal{V}_T \ \Rightarrow\ (\phi \vee \psi),\ (\phi \wedge \psi),\ (\exists x)\phi,\ (\exists X)\phi \in \Sigma^{+}\mathsf{S}$$

Remark 5.12. We have $\Sigma^{+}\mathsf{E} \subset \mathsf{E}$ and $\Sigma^{+}\mathsf{E} \subset \Sigma^{+}\mathsf{S} \subset \mathsf{S}$.

Definition 5.13 (Comprehension Variables)**.** For every formula $\phi \in \mathcal{F}^{\mathsf{E}}$ we define a mapping $\eta_\phi : \mathcal{V}_T \to \mathcal{V}_I$ such that for all X, $Y \in \mathcal{V}_T$ we have

(1) $\eta_\phi(X)$ doesn't occur in ϕ (neither free nor bound).

(2) $\eta_\phi(X) = \eta_\phi(Y) \Rightarrow X = Y$

Definition 5.14 (Comprehension Terms)**.** For every formula $\phi \in \mathsf{E}$ we define a mapping $\tau_\phi : \mathsf{E} \times \mathcal{V}_I \to \mathcal{T}^{\mathsf{E}}$ such that

$$\begin{aligned}
\tau_\phi(s = t, x) &:= \mathsf{inv}\langle \mathsf{id}, \lambda x.\langle s, t\rangle\rangle \\
\tau_\phi(t\downarrow, x) &:= \mathsf{inv}\langle \mathsf{id}, \lambda x.\langle t, t\rangle\rangle \\
\tau_\phi(t\in\mathsf{N}, x) &:= \mathsf{inv}\langle \mathsf{nat}, \lambda x.t\rangle \\
\tau_\phi(t\in X, x) &:= \mathsf{inv}\langle \eta_\phi(X), \lambda x.t\rangle \\
\tau_\phi(\neg\psi, x) &:= \mathsf{neg}\langle \tau_\phi(\psi, x)\rangle \\
\tau_\phi(\psi \vee \xi, x) &:= \mathsf{dis}\langle \tau_\phi(\psi, x), \tau_\phi(\xi, x)\rangle \\
\tau_\phi(\psi \wedge \xi, x) &:= \mathsf{con}\langle \tau_\phi(\psi, x), \tau_\phi(\xi, x)\rangle \\
\tau_\phi((\exists z)\psi, x) &:= \begin{cases} \mathsf{dom}\langle \tau_\phi(\psi[\mathsf{p}_1 x/z], x)\rangle & x = z \\ \mathsf{dom}\langle \tau_\phi(\psi[\mathsf{p}_0 x/x, \mathsf{p}_1 x/z], x)\rangle & x \neq z \end{cases} \\
\tau_\phi((\forall z)\psi, x) &:= \tau_\phi(\neg(\exists z)(\neg\psi), x)
\end{aligned}$$

We usually write $\tau_{\phi,x}$ for $\tau_\phi(\phi, x)$.

Lemma 5.15. *Let $\phi \in \mathcal{F}^{\mathsf{E}}$ such that $FV_T(\phi) = \{X_0, \ldots, X_n\}$ then for $z_i := \eta_\phi(X_i)$ we have*

(1) $FV_I(\tau_{\phi,x}) = \{z_0, \ldots, z_n\} \cup (FV_I(\phi)\backslash\{x\})$

(2) $\phi \in \Sigma^{+}\mathsf{E} \ \Rightarrow\ \Sigma^{+}\mathsf{ET} \vdash \Re(\vec{z}, \vec{X}) \to \Re(\tau_{\phi,x}) \wedge (\forall x)(x \dot{\in} \tau_{\phi,x} \leftrightarrow \phi)$

(3) $\phi \in \mathsf{E} \ \Rightarrow\ \mathsf{EET}^{-} \vdash \Re(\vec{z}, \vec{X}) \to \Re(\tau_{\phi,x}) \wedge (\forall x)(x \dot{\in} \tau_{\phi,x} \leftrightarrow \phi)$

Proof. By induction on ϕ. □

Theorem 5.16 (Elementary Comprehension)**.**

(1) $\Sigma^{+}\mathsf{ET} \vdash \Sigma^{+}\mathsf{E}\text{-}\mathsf{C}$

(2) $\mathsf{EET}^{-} \vdash \mathsf{E\text{-}C}$

Proof. Using λ-abstraction (5.3) on the term $\tau_{\phi,x}$ in lemma (5.15). $\square$

Definition 5.17 (Abbreviations)**.**

$$\begin{aligned} \mathsf{v} &:= \mathsf{inv}\langle \mathsf{id}, \lambda x.\langle x, x\rangle\rangle \\ \mathsf{e} &:= \mathsf{j}\langle \mathsf{v}, \lambda x.x\rangle \end{aligned}$$

Lemma 5.18.

(1) $\Sigma^{+}\mathsf{ET} \vdash \Re(\mathsf{v}) \wedge (\forall x)(x \dot{\in} \mathsf{v})$

(2) $\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \Re(\mathsf{e}) \wedge (\forall x)(x \dot{\in} y \leftrightarrow \langle y, x\rangle \dot{\in} \mathsf{e})$

Definition 5.19 (Elimination of Type Variables)**.** For every formula $\phi \in \mathsf{S}$ we define a mapping $\epsilon_\phi : \mathsf{S} \times \mathcal{V}_T \to \mathsf{E}$ such that

$$\begin{aligned} \epsilon_\phi(s = t, X) &:= s = t \\ \epsilon_\phi(t\downarrow, X) &:= t\downarrow \\ \epsilon_\phi(t \in \mathsf{N}, X) &:= t \in \mathsf{N} \\ \epsilon_\phi(t \in Z, X) &:= \langle \eta_\phi(Z), t\rangle \in X \\ \epsilon_\phi(\neg\psi, X) &:= \neg\epsilon_\phi(\psi, X) \\ \epsilon_\phi(\psi \vee \xi, X) &:= \epsilon_\phi(\psi, X) \vee \epsilon_\phi(\xi, X) \\ \epsilon_\phi(\psi \wedge \xi, X) &:= \epsilon_\phi(\psi, X) \wedge \epsilon_\phi(\xi, X) \\ \epsilon_\phi((\exists x)\psi, X) &:= (\exists x)\epsilon_\phi(\psi, X) \\ \epsilon_\phi((\forall x)\psi, X) &:= (\forall x)\epsilon_\phi(\psi, X) \\ \epsilon_\phi((\exists Z)\psi, X) &:= (\exists \eta_\phi(Z))\epsilon_\phi(\psi, X) \\ \epsilon_\phi((\forall Z)\psi, X) &:= (\forall \eta_\phi(Z))\epsilon_\phi(\psi, X) \end{aligned}$$

We usually write $\epsilon(\phi, X)$ for $\epsilon_\phi(\phi, X)$.

Lemma 5.20. *Let $\phi \in \Sigma^{+}\mathsf{S}$ such that $FV_T(\phi) = \{X_0, \ldots, X_n\}$ then for $z_i := \eta_\phi(X_i)$ and $\psi := \epsilon(\phi, Y)$ and $t := \tau_{\psi,x}[\mathsf{e}/\eta_\psi(Y)]$ we have*

(1) $\psi \in \Sigma^{+}\mathsf{E}$

(2) $\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \Re(\vec{z}, \vec{X}) \wedge \Re(\mathsf{e}, Y) \to (\psi \leftrightarrow \phi)$

(3) $\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \Re(\vec{z}, \vec{X}) \to \Re(t) \wedge (\forall x)(x \dot{\in} t \leftrightarrow \phi)$

Proof.

(1) Immediate, by definition.

(2) By induction on ϕ using that $(\langle z_i, x\rangle \in Y \leftrightarrow x \in X_i)$ by (5.18) and (5.9).

(3) A direct consequence of (5.15) and (1),(2).

$\square$

Theorem 5.21 (Positive Existential Stratified Comprehension)**.**

$$\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \Sigma^{+}\mathsf{S\text{-}C}$$

Proof. Using λ-abstraction (5.3) on the term t in lemma (5.20). $\square$

Theorem 5.22. $\mathsf{EET}^{-}+(\mathsf{j})+(\forall\Re)$ *is inconsistent, i.e. there is a formula* $\phi \in \mathcal{F}^{\mathsf{E}}$ *such that*

$$\mathsf{EET}^{-}+(\mathsf{j})+(\forall\Re) \vdash \phi \leftrightarrow \neg\phi$$

Proof. Let $\psi := (\neg\langle x, x\rangle \in X)$ and $r := \tau_{\psi,x}[\mathsf{e}/\eta_\psi(X)]$ then the formula $\phi := r\dot{\in}r$ does the job. We have

$$\mathsf{EET}^{-} \vdash \Re(e, X) \to (\forall x)(x\dot{\in}r \leftrightarrow \psi) \qquad \text{by (5.15)}$$

$$\Sigma^{+}\mathsf{ET} \vdash \Re(e, X) \to (\psi \leftrightarrow \neg\langle x, x\rangle\dot{\in}\mathsf{e}) \qquad \text{by (5.9)}$$

$$\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \neg x\dot{\in}x \leftrightarrow \neg\langle x, x\rangle\dot{\in}\mathsf{e} \qquad \text{by (5.18)}$$

hence we deduce $\mathsf{EET}^{-}+(\mathsf{j})+(\forall\Re) \vdash \Re(e, X) \to (\forall x)(x\dot{\in}r \leftrightarrow \neg x\dot{\in}x)$. But we also have $\Sigma^{+}\mathsf{ET}+(\mathsf{j})+(\forall\Re) \vdash \Re(e)$ and finally get $\mathsf{EET}^{-}+(\mathsf{j})+(\forall\Re) \vdash \phi \leftrightarrow \neg\phi$ by rules of inference and an instance of $\mathcal{A}^{\mathsf{E}}_{Quant}$. $\square$

5.2.3 Power Types

Theorem 5.23.

(1) $\Sigma^{+}\mathsf{ET}+(\forall\Re) \vdash \mathsf{Pow}^{-}$

(2) $\Sigma^{+}\mathsf{ET} \vdash \neg(\forall Y)(\exists X)Pow^{+}(X, Y)$ *(i.e.* $\Sigma^{+}\mathsf{ET}+\mathsf{Pow}^{+}$ *is inconsistent.)*

Proof.

(1) Let $\phi := ((\exists z)x = \mathsf{con}\langle y, z\rangle)$ then we prove:

$$\Sigma^{+}\mathsf{ET}+(\forall\Re) \vdash \Re(y, Y) \wedge \Re(\tau_{\phi,x}, X) \to Pow^{-}(X, Y)$$

Let $t := \mathsf{con}\langle y, z\rangle$ then we have

$$\Sigma^{+}\mathsf{ET} \vdash \Re(y,Y) \wedge \Re(z,Z) \wedge \Re(\tau_{\phi,x},X) \rightarrow t \in X$$
$$\Sigma^{+}\mathsf{ET} \vdash \Re(y,Y) \wedge \Re(z,Z) \wedge Z \subset Y \rightarrow \Re(t,Z)$$

hence we deduce that $\Sigma^{+}\mathsf{ET}$ proves

$$\Re(y,Y) \wedge \Re(z,Z) \wedge \Re(\tau_{\phi,x},X) \rightarrow (Z \subset Y \rightarrow (t \in X \wedge \Re(t,Z)))$$

and $\Sigma^{+}\mathsf{ET}{+}(\forall\Re)$ proves

$$\Re(y,Y) \wedge \Re(\tau_{\phi,x},X) \rightarrow (\forall Z)(Z \subset Y \rightarrow (\exists a)(a \in X \wedge \Re(a,Z)))$$

For the second part of $Pow^{-}(X,Y)$ we have that $\Sigma^{+}\mathsf{ET}$ proves

$$\Re(y,Y) \wedge \Re(\tau_{\phi,x},X) \wedge \Re(a,Z) \wedge a \in X \rightarrow Z \subset Y$$

hence we get that $\Sigma^{+}\mathsf{ET}{+}(\forall\Re)$ proves

$$\Re(y,Y) \wedge \Re(\tau_{\phi,x},X) \rightarrow (\forall a)(a \in X \rightarrow (\exists Z)(Z \subset Y \wedge \Re(a,Z)))$$

(2) This is a consequence of $\Sigma^{+}\mathsf{ET} \vdash Pow^{+}(X,Y) \rightarrow \Re(\mathsf{v},Y)$. See [2] for a proof.

□

6 Proof-Theoretic Analysis

In this last section we are going to relate systems of first order arithmetic to systems of Explicit Mathematics. The relation we exhibit is based on the formal concept of proofs, that is we compare the proofs of the two systems. We take the following two standard approaches:

(1) Arithmetic statements (formulas) are translated into the systems of Explicit Mathematics. The translation is such that the statements still have the same arithmetic meaning but now in the language of Explicit Mathematics. We are then able to compare the two systems by comparing the sets of provable arithmetic statements.

(2) We first define what it means for a function $f : \mathbb{N} \to \mathbb{N}$ to be definable in the two systems. Now we can ask if a definable function is recognized as being total within the system, i.e. if the system is able to prove totality of the function. The two systems are then compared by their sets of provably total functions.

6.1 Embeddings

A common way to find the provable arithmetic statements of a system, is by embedding it into an appropriate system of arithmetic and vice versa.

6.1.1 Embedding Arithmetic into Explicit Mathematics

We follow the construction (5.1) in [5].

Definition 6.1 (Numerals $\overline{n} \in \mathcal{T}^{\mathsf{E}}$)**.** We use the following notation:

$$\overline{n+1} := (\mathsf{s_N} \cdot \overline{n}) \qquad [\overline{0} := \mathsf{0}]$$

Definition 6.2 ($.^{N} : Prim \to \mathcal{T}^{\mathsf{E}}$)**.**

$$f^N := \begin{cases} \mathsf{s_N} & f = S \\ \lambda x_1 \dots x_n.\overline{i} & f = Cs_i^n \\ \lambda x_1 \dots x_n.x_{i+1} & f = Pr_i^n \\ \lambda x_1 \dots x_n.h^N(g_1{}^N x_1 \dots x_n) \dots (g_m{}^N x_1 \dots x_n) & f = Comp^n(h, g_1, \dots, g_m) \\ \lambda x_1 \dots x_n.\mathsf{rec}(g^N x_1 \dots x_n)(h^N x_1 \dots x_n) & f = Rec^{n+1}(h, g) \end{cases}$$

Definition 6.3 ($.^{N} : \mathcal{T}^{\mathsf{A}} \to \mathcal{T}^{\mathsf{E}}$)**.**

(1) $\mathsf{0}^N := \mathsf{0}$

(2) $x^N \in \mathcal{V}_I$ such that $x^N = y^N \Rightarrow x = y$

(3) $f(t_0, \ldots, t_n)^N := f^N {t_0}^N \ldots {t_n}^N$

Definition 6.4 ($.^\circ : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$)**.**

(1) $(s = t)^\circ := (s^N = t^N)$

(2) $(\neg\phi)^\circ := \neg(\phi^\circ)$

(3) $(\phi \vee \psi)^\circ := \phi^\circ \vee \psi^\circ$

(4) $(\phi \wedge \psi)^\circ := \phi^\circ \wedge \psi^\circ$

(5) $((\exists x)\phi)^\circ := (\exists x^N \in \mathsf{N})\phi^\circ$

(6) $((\forall x)\phi)^\circ := (\forall x^N \in \mathsf{N})\phi^\circ$

Lemma 6.5. *Let $\phi \in \mathcal{F}^{\mathsf{A}}$ and $x \in \mathcal{V}_A$ then*

$$x \in FV_A(\phi) \Leftrightarrow x^N \in FV_I(\phi^\circ)$$

Definition 6.6 ($.^N : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$)**.**

$$\phi^N := \begin{cases} \phi^\circ & FV_A(\phi) = \{\} \\ {x_0}^N \in \mathsf{N} \wedge \ldots \wedge {x_n}^N \in \mathsf{N} \to \phi^\circ & FV_A(\phi) = \{x_0, \ldots, x_n\} \end{cases}$$

Remark: There is an ambiguity in the ordering of $x_0, \ldots, x_n$ in the formula ϕ^N.

Lemma 6.7. *Let $\phi \in \mathcal{A}^{\mathsf{A}}_{Prim}$ be a defining equation, $f \in Prim_{k+1}$ and $n_0, \ldots, n_k \in \mathbb{N}$ then we have*

(1) $\mathsf{BON}^- \vdash \phi^N$

(2) $\mathsf{BON}^- \vdash f^N \overline{n_0} \ldots \overline{n_k} = \overline{[\![f]\!](n_0, \ldots, n_k)}$

(3) $\mathsf{BON}^- + (\mathsf{V}\text{-}\mathsf{I}_\mathsf{N}) \vdash f^N \in (\mathsf{N}^{k+1} \to \mathsf{N})$

Proof.

(1) By the definition of f^N.

(2) We prove

$$f \in Prim_{k+1} \Rightarrow \ (\forall n_0, \ldots, n_k)(\mathsf{BON}^- \vdash f^N \overline{n_0} \ldots \overline{n_k} = \overline{[\![f]\!](n_0, \ldots, n_k)})$$

by induction on the function symbols $f \in Prim$.

(3) We prove $f \in Prim_{k+1} \Rightarrow \mathsf{BON}^{-}+(\mathsf{V}\text{-}\mathsf{I_N}) \vdash f^N \in (\mathsf{N}^{k+1} \to \mathsf{N})$ by induction on the function symbols $f \in Prim$. We need value induction ($\mathsf{V}\text{-}\mathsf{I_N}$) in case $f = Rec^{k+1}(g, h)$.

□

Lemma 6.8. *For every quantifier free formula $\phi \in QF$ and $\{x_1, \ldots, x_k\} \supset FV_I(\phi^\circ)$ there is a closed term $t \in \mathcal{T}^{\mathsf{E}}$ such that*

(1) $\mathsf{BON}^{-}+(\mathsf{V}\text{-}\mathsf{I_N}) \vdash (\forall x_1 \in \mathsf{N}) \ldots (\forall x_k \in \mathsf{N})(\phi^\circ \leftrightarrow t x_1 \ldots x_k = \mathsf{0})$

(2) $\mathsf{BON}^{-}+(\mathsf{V}\text{-}\mathsf{I_N}) \vdash t \in (\mathsf{N}^k \to \mathsf{N})$

Proof. We can inductively define terms $t := t_\phi$ (on the complexity of ϕ) such that (1) is fulfilled, and we especially choose t_ϕ such that (2) holds. The construction of t_ϕ is similar to the usual construction of the function symbol f_ϕ such that $\mathsf{PRA} \vdash \phi \leftrightarrow f_\phi(x_1, \ldots, x_k) = \mathsf{0}$. □

Theorem 6.9 (Embedding Theorem I)**.** *For all $\phi \in \mathcal{F}^{\mathsf{A}}$ we have:*

(1) $\mathsf{PRA} \vdash \phi \quad \Rightarrow \quad \mathsf{BON}^{-}+(\mathsf{V}\text{-}\mathsf{I_N}) \vdash \phi^N$

(2) $\mathsf{PA} \vdash \phi \quad \Rightarrow \quad \mathsf{BON}^{-}+(\mathcal{F}\text{-}\mathsf{I_N}) \vdash \phi^N$

Proof. By induction on the length k of the proof $T \vdash^k \phi$. In case $\phi \in QF\text{-}\mathsf{Ind}$ we use (6.8) and an instance of $\mathsf{V}\text{-}\mathsf{I_N}$ to prove $\mathsf{BON}^{-}+(\mathsf{V}\text{-}\mathsf{I_N}) \vdash \phi^N$. □

6.1.2 Computation Sequences

For the embedding of Explicit Mathematics into arithmetic we need to translate the application operation ($x \cdot y = z$) and the element relation ($x \in Y$). Inspired by the operator form in [5] we will find a primitive recursive function leading to a suitable notion of computation sequence.

Definition 6.10 ($\hat{\cdot} : \mathcal{C}^{\mathsf{E}} \cup \mathcal{V}_I \cup \mathcal{V}_T \to \mathcal{T}^{\mathsf{A}}$)**.** For every $r \in \mathcal{C}^{\mathsf{E}} \backslash \{\mathsf{0}\}$ we fix a number $n_r > 4$ and we define $\hat{r} := \pi(\overline{n_r}, \mathsf{0})$ and $\hat{\mathsf{0}} := \mathsf{0}$. For every $r \in \mathcal{V}_I \cup \mathcal{V}_T$ we fix a variable $x_r \in \mathcal{V}_A$ and we define $\hat{r} := x_r$. The choice of numbers and variables is such that $\hat{r} = \hat{s} \Rightarrow r = s$ holds and for $\hat{\mathcal{V}}_A := \{\hat{r} \mid r \in \mathcal{V}_I \cup \mathcal{V}_T\}$ we want that $\mathcal{V}_A \backslash \hat{\mathcal{V}}_A$ contains countably many variables.

Definition 6.11 (Computation Sequence I)**.** Let $F_1, \ldots, F_{22} \in Prim_1$ be function symbols such that PRA proves:

(1) $F_1(x) = x \doteq \langle \hat{\mathsf{k}}, (x)_1, \langle \hat{\mathsf{k}}, (x)_1 \rangle \rangle$

(2) $F_2(x) = x \doteq \langle\langle \hat{\mathsf{k}}, (x)_2\rangle, (x)_1, (x)_2\rangle$

(3) $F_3(x) = x \doteq \langle \hat{\mathsf{s}}, (x)_1, \langle \hat{\mathsf{s}}, (x)_1\rangle\rangle$

(4) $F_4(x) = x \doteq \langle\langle \hat{\mathsf{s}}, (x)_{0,1}\rangle, (x)_1, \langle \hat{\mathsf{s}}, (x)_{0,1}, (x)_1\rangle\rangle$

(5) $F_5(x) = x \doteq \langle \hat{\mathsf{p}}, (x)_1, \langle \hat{\mathsf{p}}, (x)_1\rangle\rangle$

(6) $F_6(x) = x \doteq \langle\langle \hat{\mathsf{p}}, (x)_{0,1}\rangle, (x)_1, \langle (x)_{0,1}, (x)_1\rangle\rangle$

(7) $F_7(x) = x \doteq \langle \widehat{\mathsf{p}_0}, \langle (x)_2, (x)_{1,1}\rangle, (x)_2\rangle$

(8) $F_8(x) = x \doteq \langle \widehat{\mathsf{p}_1}, \langle (x)_{1,0}, (x)_2\rangle, (x)_2\rangle$

(9) $F_9(x) = x \doteq \langle \widehat{\mathsf{s}_{\mathsf{N}}}, (x)_1, S((x)_1)\rangle$

(10) $F_{10}(x) = x \doteq \langle \widehat{\mathsf{p}_{\mathsf{N}}}, S((x)_2), (x)_2\rangle$

(11) $F_{11}(x) = x \doteq \langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_1, \langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_1\rangle\rangle$

(12) $F_{12}(x) = x \doteq \langle\langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}\rangle, (x)_1, \langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}, (x)_1\rangle\rangle$

(13) $F_{13}(x) = x \doteq \langle\langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}, (x)_{0,2}\rangle, (x)_1, \langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}, (x)_{0,2}, (x)_1\rangle\rangle$

(14) $F_{14}(x) = (x \doteq \langle\langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}, (x)_{0,2}, (x)_{0,3}\rangle, (x)_1, (x)_{0,1}\rangle) \mathbin{\dot\wedge} ((x)_{0,3} \doteq (x)_1)$

(15) $F_{15}(x) = (x \doteq \langle\langle \widehat{\mathsf{d}_{\mathsf{N}}}, (x)_{0,1}, (x)_{0,2}, (x)_{0,3}\rangle, (x)_1, (x)_{0,2}\rangle) \mathbin{\dot\wedge} \dot\neg((x)_{0,3} \doteq (x)_1)$

(16) $F_{16}(x) = x \doteq \langle \widehat{\mathsf{nat}}, (x)_1\rangle$

(17) $F_{17}(x) = x \doteq \langle \widehat{\mathsf{id}}, \langle (x)_{1,0}, (x)_{1,0}\rangle\rangle$

(18) $F_{18}(x) = x \doteq \langle \widehat{\mathsf{con}}, \langle (x)_{1,0}, (x)_{1,1}\rangle, \langle \widehat{\mathsf{con}}, (x)_{1,0}, (x)_{1,1}\rangle\rangle$

(19) $F_{19}(x) = x \doteq \langle \widehat{\mathsf{dis}}, \langle (x)_{1,0}, (x)_{1,1}\rangle, \langle \widehat{\mathsf{dis}}, (x)_{1,0}, (x)_{1,1}\rangle\rangle$

(20) $F_{20}(x) = x \doteq \langle \widehat{\mathsf{dom}}, (x)_1, \langle \widehat{\mathsf{dom}}, (x)_1\rangle\rangle$

(21) $F_{21}(x) = x \doteq \langle \widehat{\mathsf{inv}}, \langle (x)_{1,0}, (x)_{1,1}\rangle, \langle \widehat{\mathsf{inv}}, (x)_{1,0}, (x)_{1,1}\rangle\rangle$

(22) $F_{22}(x) = x \doteq \langle \hat{\mathsf{j}}, \langle (x)_{1,0}, (x)_{1,1}\rangle, \langle \hat{\mathsf{j}}, (x)_{1,0}, (x)_{1,1}\rangle\rangle$

Definition 6.12 (Computation Sequence II)**.**

The interpretation of the functions $G_n(x,y,z)$ defined below is the following: x is the element we want to compute, z is a computation sequence and y is a witnessing sequence, i.e. $(y)_i = \mathsf{0}$ is a witness that $(z)_i$ is computable. The computation sequence z can also contain elements $(z)_j$ not computable or not yet computed, indicated by the fact that $(y)_j \neq \mathsf{0}$. The meaning of $G_n(x,y,z) = \mathsf{0}$ then is, that based on the intermediate computations $(z)_i$ (such that $(y)_i = \mathsf{0}$) we can compute x in only one step.

Let $G_1, \ldots, G_7 \in \mathit{Prim}_3$ and $H_1, \ldots, H_7 \in \mathit{Prim}$ be function symbols such that PRA proves:

(1)

$$\begin{aligned} G_1(x,y,z) &= \dot{\exists}_0 H_1(|z|, x, y, z) \\ H_1(i,x,y,z) &= (y)_i \mathbin{\dot{\wedge}} ((z)_i \doteq x) \end{aligned}$$

(2)

$$\begin{aligned} G_2(x,y,z) &= (x \doteq \langle\langle \hat{\mathsf{s}}, (x)_{0,1}, (x)_{0,2}\rangle, (x)_1, (x)_2\rangle) \mathbin{\dot{\wedge}} \\ &\qquad \dot{\exists}_0 \dot{\exists}_1 \dot{\exists}_2 H_2(|z|, |z|, |z|, x, y, z) \\ H_2(i,j,k,x,y,z) &= H_1(i, \langle (x)_{0,1}, (x)_1, (z)_{i,2}\rangle, y, z) \mathbin{\dot{\wedge}} \\ &\qquad H_1(j, \langle (x)_{0,2}, (x)_1, (z)_{j,2}\rangle, y, z) \mathbin{\dot{\wedge}} \\ &\qquad H_1(k, \langle (z)_{i,2}, (z)_{j,2}, (x)_2\rangle, y, z) \end{aligned}$$

(3)

$$\begin{aligned} G_3(x,y,z) &= (x \doteq \langle\langle \hat{\mathsf{con}}, (x)_{0,1}, (x)_{0,2}\rangle, (x)_1\rangle) \mathbin{\dot{\wedge}} \\ &\qquad \dot{\exists}_0 \dot{\exists}_1 H_3(|z|, |z|, x, y, z) \\ H_3(i,j,x,y,z) &= H_1(i, \langle (x)_{0,1}, (x)_1\rangle, y, z) \mathbin{\dot{\wedge}} \\ &\qquad H_1(j, \langle (x)_{0,2}, (x)_1\rangle, y, z) \end{aligned}$$

(4)

$$\begin{aligned} G_4(x,y,z) &= (x \doteq \langle\langle \hat{\mathsf{dis}}, (x)_{0,1}, (x)_{0,2}\rangle, (x)_1\rangle) \mathbin{\dot{\wedge}} \\ &\qquad \dot{\exists}_0 H_4(|z|, x, y, z) \\ H_4(i,x,y,z) &= H_1(i, \langle (x)_{0,1}, (x)_1\rangle, y, z) \mathbin{\dot{\vee}} \\ &\qquad H_1(i, \langle (x)_{0,2}, (x)_1\rangle, y, z) \end{aligned}$$

(5)

$$\begin{aligned} G_5(x,y,z) = {} & (x \doteq \langle\langle \hat{\mathsf{dom}}, (x)_{0,1}\rangle, (x)_1\rangle) \mathbin{\dot\wedge} \\ & \dot\exists_0 H_5(|z|, x, y, z) \\ H_5(i,x,y,z) = {} & H_1(i, \langle (x)_{0,1}, \langle (x)_1, (z)_{i,1,1}\rangle\rangle, y, z) \end{aligned}$$

(6)

$$\begin{aligned} G_6(x,y,z) = {} & (x \doteq \langle\langle \hat{\mathsf{inv}}, (x)_{0,1}, (x)_{0,2}\rangle, (x)_1\rangle) \mathbin{\dot\wedge} \\ & \dot\exists_0 \dot\exists_1 H_6(|z|, |z|, x, y, z) \\ H_6(i,j,x,y,z) = {} & H_1(i, \langle (x)_{0,2}, (x)_1, (z)_{i,2}\rangle, y, z) \mathbin{\dot\wedge} \\ & H_1(j, \langle (x)_{0,1}, (z)_{i,2}\rangle, y, z) \end{aligned}$$

(7)

$$\begin{aligned} G_7(x,y,z) = {} & (x \doteq \langle\langle \hat{\mathsf{j}}, (x)_{0,1}, (x)_{0,2}\rangle, \langle (x)_{1,0}, (x)_{1,1}\rangle\rangle) \mathbin{\dot\wedge} \\ & \dot\exists_0 \dot\exists_1 \dot\exists_2 H_7(|z|, |z|, |z|, x, y, z) \\ H_7(i,j,k,x,y,z) = {} & H_1(i, \langle (x)_{0,1}, (x)_{1,0}\rangle, y, z) \mathbin{\dot\wedge} \\ & H_1(j, \langle (x)_{0,2}, (x)_{1,0}, (z)_{j,2}\rangle, y, z) \mathbin{\dot\wedge} \\ & H_1(k, \langle (z)_{j,2}, (x)_{1,1}\rangle, y, z) \end{aligned}$$

Definition 6.13 (Computation Sequence III)**.**

The main part of this definition is the function $H(n,z)$. This function assigns to every computation sequence z its witnessing sequence $H(n,z)$, i.e. we have $(H(n,z))_i = \mathsf{0}$ only if $(z)_i$ is computable from z in n steps (iterations).

Let Com, F, $H \in Prim_2$ and $G \in Prim_3$ be function symbols such that PRA proves:

(1) $F(i,z) = F_1((z)_i) \mathbin{\dot\vee} \ldots \mathbin{\dot\vee} F_{22}((z)_i)$

(2) $G(i,y,z) = F(i,z) \mathbin{\dot\vee} G_2((z)_i, y, z) \mathbin{\dot\vee} \ldots \mathbin{\dot\vee} G_7((z)_i, y, z)$

(3) $H(0,z) = F^*(|z|, z)$

(4) $H(S(u), z) = G^*(|z|, H(u,z), z)$

(5) $Com(x,z) = G_1(x, H(|z|, z), z)$

Lemma 6.14.

(1) $\mathsf{PRA} \vdash H_1(i, x, H(u, z), z) = \mathsf{0} \leftrightarrow [(H(u, z))_i = \mathsf{0} \wedge (z)_i = x]$

(2) $\mathsf{PRA} \vdash [(H(u, z))_i = \mathsf{0} \wedge (H(u, z))_j = \mathsf{0} \wedge (z)_{i,0} = (z)_{j,0} \wedge (z)_{i,1} = (z)_{j,1}] \rightarrow (z)_{i,2} = (z)_{j,2}$

Proof.

(1) By definition of H_1.

(2) (sketchy, informal) Let f, $g \in \mathit{Prim}$ be function symbols such that PRA proves:

$$g(i, j, z) = \dot{\neg}\, [(z)_{i,0} \doteq (z)_{j,0})\ \dot{\wedge}\ ((z)_{i,1} \doteq (z)_{j,1})]\ \dot{\vee}\ ((z)_{i,2} \doteq (z)_{j,2})$$
$$f(i, j, u, z) = \dot{\neg}\, [(H(u, z))_i\ \dot{\wedge}\ (H(u, z))_j]\ \dot{\vee}\ g(i, j, z)$$

We can prove $\mathsf{PRA} \vdash \dot{\forall}_0\, \dot{\forall}_1\, f(|z|, |z|, u, z) = \mathsf{0}$ by induction on u (and by long and tedious argumentation about F and G). Just mentioning two peculiar cases (same informal reasoning also holds for j instead of i): In case $|(z)_i| = \overline{2}$ and $(z)_{i,0} = (z)_{j,0}$ we get $|(z)_i| = |(z)_j|$ by definition of F and G, hence $(z)_{i,2} = (z)_{j,2} = \overline{2}$ by (3.12). In case $i \geq |z|$ and $(H(u, z))_i = 0$ we have $|z| = |H(u, z)| = (H(u, z))_i = 0$ by (3.16), hence $j \geq |z|$ and $(z)_i = |z| = (z)_j$ by (3.12).

□

Lemma 6.15.

$$\mathsf{PRA} \vdash \mathit{Com}(\langle u, x, y\rangle, w) = \mathsf{0} \wedge \mathit{Com}(\langle u, x, z\rangle, w) = \mathsf{0} \rightarrow y = z$$

Proof. Because of (3.8) we have

$$\mathsf{PRA} \vdash \mathit{Com}(\langle u, x, v\rangle, w) = \mathsf{0} \leftrightarrow (\exists k)(k < |w| \wedge H_1(k, \langle u, x, v\rangle, H(|w|, w), w) = \mathsf{0})$$

From (6.14) we get

$$\mathsf{PRA} \vdash H_1(i, \langle u, x, y\rangle, H(|w|, w), w) = \mathsf{0} \wedge H_1(j, \langle u, x, z\rangle, H(|w|, w), w) = \mathsf{0} \rightarrow y = z$$

Extending the left side of this implication with $(i < |w| \wedge j < |w|)$ and applying twice the $\exists$-rule of inference (in $\mathcal{R}^{\mathsf{A}}$) we get the desired proof. □

Lemma 6.16.

$$\mathsf{PRA} \vdash Com(x, z) = \mathsf{0} \to Com(x, y * z) = \mathsf{0} \wedge Com(x, z * y) = \mathsf{0}$$

Proof. (sketchy) First we prove

$$\mathsf{PRA} \vdash u \geq v \wedge G_1(x, H(v, z), z) = \mathsf{0} \to G_1(x, H(u, z), z) = \mathsf{0}$$

Next we show by induction on u using (3.15) that

$$\mathsf{PRA} \vdash G_1(x, H(u, z), z) = \mathsf{0} \to G_1(x, H(u, y * z), y * z) = \mathsf{0}$$

Putting the pieces together knowing by (3.15) that $|y * z| \geq |z|$ we get

$$\mathsf{PRA} \vdash Com(x, z) = \mathsf{0} \to Com(x, y * z) = \mathsf{0}$$

Analogous for $z * y$ instead of $y * z$. □

6.1.3 Embedding Explicit Mathematics into Arithmetic

Definition 6.17 ($\alpha : \mathcal{V}_A{}^3 \to \mathcal{F}^{\mathsf{A}}, \beta : \mathcal{V}_A{}^2 \to \mathcal{F}^{\mathsf{A}}$)**.**

(1) $\alpha(u, x, y) := (\exists z) Com(\langle u, x, y\rangle, z) = \mathsf{0}$

(2) $\beta(x, y) := (\exists z) Com(\langle y, x\rangle, z) = \mathsf{0}$

Where the variable $z \in \mathcal{V}_A \backslash \hat{\mathcal{V}}_A$ is different from x, y, $u \in \mathcal{V}_A$.

Lemma 6.18.

$$\mathsf{PRA} \vdash \alpha(u, x, y) \wedge \alpha(u, x, z) \to y = z$$

Proof. From (6.16) we get

$$\begin{aligned} \mathsf{PRA} \vdash & Com(\langle u, x, y\rangle, v) = \mathsf{0} \wedge Com(\langle u, x, z\rangle, w) = \mathsf{0} \to \\ & Com(\langle u, x, y\rangle, v * w) = \mathsf{0} \wedge Com(\langle u, x, z\rangle, v * w) = \mathsf{0} \end{aligned}$$

Using this in conjunction with (6.15) we get

$$\mathsf{PRA} \vdash Com(\langle u, x, y\rangle, v) = \mathsf{0} \wedge Com(\langle u, x, z\rangle, w) = \mathsf{0} \to y = z$$

Applying twice the $\exists$-rule of inference (in $\mathcal{R}^{\mathsf{A}}$) we get the desired proof. □

Definition 6.19 ($\gamma : \mathcal{T}^{\mathsf{E}} \times \mathcal{V}_A \to \mathcal{F}^{\mathsf{A}}$)**.**

(1) $t \in \mathcal{C}^{\mathsf{E}} \cup \mathcal{V}_I \Rightarrow \gamma(t, x) := \hat{t} = x$

(2) $\gamma(t_1 \cdot t_2, x) := (\exists z_1)(\exists z_2)[\gamma(t_1, z_1) \wedge \gamma(t_2, z_2) \wedge \alpha(z_1, z_2, x)]$

Where z_1, $z_2 \in \mathcal{V}_A \backslash \hat{\mathcal{V}}_A$ are different variables and z_1, z_2 are different from x.

Lemma 6.20.

(1) $\mathsf{PRA} \vdash \gamma(t, x) \wedge \gamma(t, y) \to x = y$

(2) $\mathsf{PRA} \vdash \gamma(\overline{n}, x) \to x = \overline{n}$

Proof. (1) By induction on t and (2) by induction on n. □

Definition 6.21 ($.^\star : \mathcal{F}_0^{\mathsf{E}} \to \mathcal{F}^{\mathsf{A}}$).

(1) $(s = t)^\star := (\exists x)(\gamma(s, x) \wedge \gamma(t, x))$

(2) $t{\downarrow}^\star := t{\in}\mathsf{N}^\star := (\exists x)\gamma(t, x)$

(3) $t{\in}X^\star := (\exists x)(\gamma(t, x) \wedge \beta(x, \hat{X}))$

(4) $\Re(t, X)^\star := (\exists x)[\gamma(t, x) \wedge (\forall y)(\beta(y, x) \leftrightarrow \beta(y, \hat{X}))]$

Where x, $y \in \mathcal{V}_A \backslash \hat{\mathcal{V}}_A$ are different variables.

Definition 6.22 ($.^\star : \mathcal{F}^{\mathsf{E}} \to \mathcal{F}^{\mathsf{A}}$).

(1) For $\phi \in \mathcal{F}_0^{\mathsf{E}}$ see (6.21).

(2) $(\neg\phi)^\star := \neg(\phi^\star)$

(3) $(\phi \vee \psi)^\star := \phi^\star \vee \psi^\star$

(4) $(\phi \wedge \psi)^\star := \phi^\star \wedge \psi^\star$

(5) $((\exists x)\phi)^\star := (\exists \hat{x})\phi^\star$

(6) $((\forall x)\phi)^\star := (\forall \hat{x})\phi^\star$

(7) $((\exists X)\phi)^\star := (\exists \hat{X})\phi^\star$

(8) $((\forall X)\phi)^\star := (\forall \hat{X})\phi^\star$

Lemma 6.23. *For all variables u, v, $x_0, \ldots, x_n$, $y_0, \ldots, y_n \in \mathcal{V}_I$ and all formulas $\phi \in \mathcal{F}^{\mathsf{E}}$ such that $y_i \in FT(x_i, \phi)$ we have*

(1) $u \in FV_I(\phi) \Leftrightarrow \hat{u} \in FV_A(\phi^\star)$

(2) $u \in FT_I(v, \phi) \Leftrightarrow \hat{u} \in FT_A(\hat{v}, \phi^\star)$

(3) $\vdash \phi[\vec{y}/\vec{x}]^\star \leftrightarrow \phi^\star[\hat{\vec{y}}/\hat{\vec{x}}]$ $\qquad$ ($[\hat{\vec{y}}/\hat{\vec{x}}]$ *stands for* $[\hat{y_0}/\hat{x_0}, \ldots, \hat{y_n}/\hat{x_n}]$)

Proof. By induction on ϕ. $\square$

Lemma 6.24. *For all variables U, V, $X_0, \ldots, X_n$, $Y_0, \ldots, Y_n \in \mathcal{V}_T$ and all formulas $\phi \in \mathcal{F}^{\mathsf{E}}$ such that $Y_i \in FT(X_i, \phi)$ we have that all statements analogous to (6.23) hold.*

Lemma 6.25. *For all terms s, $t \in \mathcal{T}^{\mathsf{E}}$ such that $y \in FT_A(\hat{x}, \gamma(s, z))$ we have*

$$z \neq \hat{x} \quad \Rightarrow \quad \mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\gamma(s[t/x], z) \leftrightarrow \gamma(s, z)[y/\hat{x}])$$

Proof. By induction on s. $\square$

Lemma 6.26. *For all formulas $\phi \in \mathcal{F}^{\mathsf{E}}$ such that $y \in FT_A(\hat{x}, \phi^\star)$ we have*

$$\mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\phi[t/x]^\star \leftrightarrow \phi^\star[y/\hat{x}])$$

Proof. By induction on ϕ. $\square$

Lemma 6.27.

(1) $\mathsf{PRA} \vdash (x{\in}\mathsf{N})^\star$

(2) $\mathsf{PRA} \vdash ((\forall x{\in}\mathsf{N})\phi)^\star \leftrightarrow (\forall \hat{x})\phi^\star$

(3) $\mathsf{PRA} \vdash \Re(x)^\star$

(4) $\mathsf{PRA} \vdash (x \dot{\in} y)^\star \leftrightarrow \beta(\hat{x}, \hat{y})$

(5) $\mathsf{PRA} \vdash (x{\cdot}y = z)^\star \leftrightarrow \alpha(\hat{x}, \hat{y}, \hat{z})$

Proof.

(1) $(x{\in}\mathsf{N})^\star = (\exists y)\gamma(x, y) = (\exists y)\hat{x} = y$.

(2) $((\forall x{\in}\mathsf{N})\phi)^\star = (\forall \hat{x})((x{\in}\mathsf{N})^\star \rightarrow \phi^\star)$ and (1).

(3) $\Re(x)^\star = ((\exists X)\Re(x, X))^\star = (\exists \hat{X})(\exists z)(\hat{x} = z \wedge (\forall y)(\beta(y, z) \leftrightarrow \beta(y, \hat{X})))$ but we surely have that $\mathsf{PRA} \vdash \hat{x} = \hat{x} \wedge (\forall y)(\beta(y, \hat{x}) \leftrightarrow \beta(y, \hat{x}))$.

(4) $(x \dot{\in} y)^\star =$ $(\exists \hat{X})[(\exists u)(\hat{y} = u \wedge (\forall w)(\beta(w, u) \leftrightarrow \beta(w, \hat{X}))) \wedge (\exists v)(\hat{x} = v \wedge \beta(v, \hat{X}))]$ hence $(x \dot{\in} y)^\star \leftrightarrow (\exists \hat{X})[(\forall w)(\beta(w, \hat{y}) \leftrightarrow \beta(w, \hat{X})) \wedge \beta(\hat{x}, \hat{X})]$ is provable in PRA and finally $\mathsf{PRA} \vdash (x \dot{\in} y)^\star \leftrightarrow \beta(\hat{x}, \hat{y})$.

(5) $(x{\cdot}y = z)^\star = (\exists u)[(\exists v_1)(\exists v_2)(\hat{x} = v_1 \wedge \hat{y} = v_2 \wedge \alpha(v_1, v_2, u)) \wedge \hat{z} = u)]$

$\square$

Theorem 6.28 (Embedding Theorem II)**.**

For all $\phi \in \mathcal{F}^{\mathsf{E}}$ *and for* $T = \Sigma^{+}\mathsf{ET}{+}(\mathsf{J}){+}(\forall\Re){+}(\forall\mathsf{N}){+}(\Sigma^{+}\mathsf{S}\text{-}\mathsf{C}){+}(\mathsf{Pow}^{-})$ *we have:*

(1) $T{+}(\mathcal{F}\text{-}\mathsf{I_N}) \vdash \phi \quad \Rightarrow \quad \mathsf{PA} \vdash \phi^{\star}$

(2) $T{+}(\mathsf{T}\text{-}\mathsf{I_N}) \vdash \phi \quad \Rightarrow \quad \mathsf{PRA}^{+} \vdash \phi^{\star}$

Proof. We have $T \vdash \phi \;\Rightarrow\; \Sigma^{+}\mathsf{ET}{+}(\mathsf{J}){+}(\forall\Re){+}(\forall\mathsf{N}) \vdash \phi$ by (5.21) and (5.23), hence it is enough to prove the statement for $T = \Sigma^{+}\mathsf{ET}{+}(\mathsf{J}){+}(\forall\Re){+}(\forall\mathsf{N})$. The proof is by induction on the length k of the proof $T' \vdash^{k} \phi$.

(1) $\phi \in \Sigma^{+}\mathsf{ET}$

(a) $\phi \in \mathcal{A}^{\mathsf{E}}_{Prop} \Rightarrow \phi^{\star} \in \mathcal{A}^{\mathsf{A}}_{Prop}$

(b) $\phi \in \mathcal{A}^{\mathsf{E}}_{Equal}$

- $\phi = (x = x)$.
  $\phi^{\star} = (\exists y)(\hat{x} = y \wedge \hat{x} = y)$ hence $\vdash \phi^{\star}$.
- $\phi = (x_0 = y_0 \wedge \ldots \wedge x_n = y_n \wedge \psi \rightarrow \psi[\vec{y}/\vec{x}])$ with $\psi \in \mathcal{F}^{\mathsf{E}}_{0}$.
  Let $\xi = (x_0 = y_0 \wedge \ldots \wedge x_n = y_n)$ then
  $\vdash \xi^{\star} \rightarrow (\hat{x_0} = \hat{y_0} \wedge \ldots \wedge \hat{x_n} = \hat{y_n})$. We have
  $\vdash \hat{x_0} = \hat{y_0} \wedge \ldots \wedge \hat{x_n} = \hat{y_n} \wedge \psi^{\star} \rightarrow \psi^{\star}[\hat{\vec{y}}/\hat{\vec{x}}]$ by (3.18), hence
  $\vdash \hat{x_0} = \hat{y_0} \wedge \ldots \wedge \hat{x_n} = \hat{y_n} \wedge \psi^{\star} \rightarrow \psi[\vec{y}/\vec{x}]^{\star}$ by (6.23). Finally we get $\vdash \xi^{\star} \rightarrow (\psi^{\star} \rightarrow \psi[\vec{y}/\vec{x}]^{\star})$ and hence $\vdash \phi^{\star}$.

(c) $\phi \in \mathcal{A}^{\mathsf{E}}_{Quant}$

- $\phi = (\psi[t/x] \wedge t\!\downarrow \rightarrow (\exists x)\psi)$.
  Let $y \in \mathcal{V}_A$ be such that $y \notin FV_A((\psi[t/x] \rightarrow (\exists x)\psi)^{\star})$ and $y \in FT_A(\hat{x}, \psi^{\star})$ then by (6.26) we have
  $\mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\psi[t/x]^{\star} \rightarrow \psi^{\star}[y/\hat{x}])$.
  But $(\psi^{\star}[y/\hat{x}] \rightarrow (\exists \hat{x})\psi^{\star}) \in \mathcal{A}^{\mathsf{A}}_{Quant}$ hence
  $\mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\psi[t/x]^{\star} \rightarrow (\exists \hat{x})\psi^{\star})$. Now we can apply the $\exists$-rule in $\mathcal{R}^{\mathsf{A}}$ to get $\mathsf{PRA} \vdash t\!\downarrow^{\star} \rightarrow (\psi[t/x]^{\star} \rightarrow (\exists \hat{x})\psi^{\star})$ and finally $\mathsf{PRA} \vdash \phi^{\star}$.
- $\phi = ((\forall x)\psi \wedge t\!\downarrow \rightarrow \psi[t/x])$, similar to the previous case.
- $(\psi[Y/X] \rightarrow (\exists X)\psi)^{\star} \in \mathcal{A}^{\mathsf{A}}_{Quant}$
- $((\forall X)\psi \rightarrow \psi[Y/X])^{\star} \in \mathcal{A}^{\mathsf{A}}_{Quant}$

(d) $\phi \in \mathcal{A}^{\mathsf{E}}_{Def}$

- $\phi = r\!\downarrow$ with $r \in \mathcal{V}_I \cup \mathcal{C}^{\mathsf{E}}$, then $\phi^{\star} = ((\exists x)\hat{r} = x)$ hence $\vdash \phi^{\star}$

- $\phi = ((s{\cdot}t){\downarrow} \to s{\downarrow} \wedge t{\downarrow})$ Suppose that x, y, z are different variables in $\mathcal{V}_A \backslash \hat{\mathcal{V}}_A$. We have that $\gamma(s,x) \to (\exists x)\gamma(s,x)$ and $\gamma(t,y) \to (\exists y)\gamma(t,y)$ are in $\mathcal{A}^{\mathsf{A}}_{Quant}$, hence we can prove $\vdash \gamma(s,x) \wedge \gamma(t,y) \wedge \alpha(x,y,z) \to s{\downarrow}^\star \wedge t{\downarrow}^\star$. Threefold application of the $\exists$-rule in $\mathcal{R}^{\mathsf{A}}$ yields $\vdash \phi^\star$.
- $\phi = (\psi[\vec{t}/\vec{x}] \to t_0{\downarrow} \wedge \ldots \wedge t_n{\downarrow})$ with $\psi \in \mathcal{F}^{\mathsf{E}}_0$, similar to the previous case.

(e) $\phi \in \mathsf{BON}^-$

- $\phi = ((\mathsf{k}x)y = x)$. For $t = \langle \hat{\mathsf{k}}, \hat{x}, \langle \hat{\mathsf{k}}, \hat{x} \rangle \rangle$ and $t = \langle \langle \hat{\mathsf{k}}, \hat{x} \rangle, \hat{y}, \hat{x} \rangle$ we have $\mathsf{PRA} \vdash Com(t, \langle t \rangle)$, hence we can prove
  $\mathsf{PRA} \vdash (\gamma(\mathsf{k}, z_1) \wedge \gamma(x, z_2) \wedge \alpha(z_1, z_2, z_3))[\hat{\mathsf{k}}/z_1, \hat{x}/z_2, \langle \hat{\mathsf{k}}, \hat{x} \rangle/z_3]$,
  using $\mathcal{A}^{\mathsf{A}}_{Quant}$ we get $\mathsf{PRA} \vdash \gamma(\mathsf{k}x, z_3)[\widehat{\langle \mathsf{k}, \hat{x} \rangle}/z_3]$, then
  $\mathsf{PRA} \vdash (\gamma(\mathsf{k}x, z_3) \wedge \gamma(y, z_4) \wedge \alpha(z_3, z_4, z_5))[\langle \hat{\mathsf{k}}, \hat{x} \rangle/z_3, \hat{y}/z_4, \hat{x}/z_5]$
  and $\mathsf{PRA} \vdash \gamma((\mathsf{k}x)y, z_3)[\hat{x}/z_3]$ and finally
  $\mathsf{PRA} \vdash (\gamma((\mathsf{k}x)y, z_3) \wedge \gamma(x, z_3))[\hat{x}/z_3]$ hence $\mathsf{PRA} \vdash \phi^\star$.
- $\phi = (\mathsf{s}xy{\downarrow} \wedge (\mathsf{s}xy)z \simeq (xz)(yz))$. We can prove $\mathsf{PRA} \vdash (\mathsf{s}xy{\downarrow})^\star$ similar to the previous case. We are going to show that $\mathsf{PRA} \vdash \gamma((\mathsf{s}xy)z, v_3) \leftrightarrow \gamma((xz)(yz), v_3)$ holds, and by this we get $\mathsf{PRA} \vdash ((\mathsf{s}xy)z{\downarrow} \vee (xz)(yz){\downarrow} \to (\mathsf{s}xy)z = (xz)(yz))^\star$, hence $\mathsf{PRA} \vdash \phi^\star$.
  1. $\mathsf{PRA} \vdash \gamma((\mathsf{s}xy)z, v_3) \to \gamma((xz)(yz), v_3)$.
     Let $t := \langle \langle \hat{\mathsf{s}}, \hat{x}, \hat{y} \rangle, \hat{z}, v_3 \rangle$ then we have
     $\mathsf{PRA} \vdash \gamma((\mathsf{s}xy)z, v_3) \to (\exists w) G_1(t, H(|w|, w), w)$,
     $\mathsf{PRA} \vdash G_1(t, H(u, w), w) \to G_2(t, H(u, w), w)$ and
     $$\begin{aligned}\mathsf{PRA} \vdash G_2(t, w_0, w_1) \to (\exists v_1)(\exists v_2)[&G_1(\langle \hat{x}, \hat{z}, v_1 \rangle, w_0, w_1) \wedge \\ &G_1(\langle \hat{y}, \hat{z}, v_2 \rangle, w_0, w_1) \wedge \\ &G_1(\langle v_1, v_2, v_3 \rangle, w_0, w_1)].\end{aligned}$$
     We have that $\mathsf{PRA} \vdash G_1(\langle \hat{x}, \hat{z}, v_1 \rangle, w_0, w_1) \to \gamma(xz, v_1)$ and $\mathsf{PRA} \vdash G_1(\langle \hat{y}, \hat{z}, v_2 \rangle, w_0, w_1) \to \gamma(yz, v_2)$, hence we get
     $$\mathsf{PRA} \vdash G_2(t, w_0, w_1) \to (\exists v_1)(\exists v_2)\gamma(xz, v_1) \wedge \gamma(yz, v_2) \wedge \alpha(v_1, v_2, v_3),$$ that is $\mathsf{PRA} \vdash G_2(t, w_0, w_1) \to \gamma((xz)(yz), v_3)$. Putting all together yields $\mathsf{PRA} \vdash \gamma((\mathsf{s}xy)z, v_3) \to \gamma((xz)(yz), v_3)$.
  2. $\mathsf{PRA} \vdash \gamma((\mathsf{s}xy)z, v_3) \leftarrow \gamma((xz)(yz), v_3)$. We have
     $$\begin{aligned}\mathsf{PRA} \vdash Com(u_1, w_1) \wedge Com(u_2, w_2) \wedge Com(u_3, w_3) \to\ &Com(u_1, (w_1 * w_2) * w_3) \wedge \\ &Com(u_2, (w_1 * w_2) * w_3) \wedge \\ &Com(u_3, (w_1 * w_2) * w_3)\end{aligned}$$
     and for $t := \langle \langle \hat{\mathsf{s}}, \hat{x}, \hat{y} \rangle, \hat{z}, v_3 \rangle$ we have

$\mathsf{PRA} \vdash Com(\langle \hat{x}, \hat{z}, v_1\rangle, w) \wedge Com(\langle \hat{y}, \hat{z}, v_2\rangle, w) \wedge$
$Com(\langle v_1, v_2, v_3\rangle, w) \rightarrow Com(t, w * \langle t\rangle)$ hence
$\mathsf{PRA} \vdash Com(\langle \hat{x}, \hat{z}, v_1\rangle, w_1) \wedge Com(\langle \hat{y}, \hat{z}, v_2\rangle, w_2) \wedge$
$Com(\langle v_1, v_2, v_3\rangle, w_3) \rightarrow \alpha(\langle \hat{\mathsf{s}}, \hat{x}, \hat{y}\rangle, \hat{z}, v_3)$. We have
$\mathsf{PRA} \vdash \gamma((xz)(yz), v_3) \rightarrow$
$(\exists v_1)(\exists v_2)[(\exists w_1)Com(\langle \hat{x}, \hat{z}, v_1\rangle, w_1) \wedge$
$(\exists w_2)Com(\langle \hat{y}, \hat{z}, v_2\rangle, w_2) \wedge$
$(\exists w_3)Com(\langle v_1, v_2, v_3\rangle, w_3)]$ hence
$\mathsf{PRA} \vdash \gamma((xz)(yz), v_3) \rightarrow \alpha(\langle \hat{\mathsf{s}}, \hat{x}, \hat{y}\rangle, \hat{z}, v_3)$. Now because
$\mathsf{PRA} \vdash (\gamma(\mathsf{s}xy, v_1) \wedge \gamma(z, v_2))[\langle \hat{\mathsf{s}}, \hat{x}, \hat{y}\rangle/v_1, \hat{z}/v_2]$ we get
$\mathsf{PRA} \vdash \gamma((xz)(yz), v_3) \rightarrow \gamma((\mathsf{s}xy)z, v_3)$.

- The next few cases are omitted, but can be proved similar to the previous ones:
  $\phi = (\mathsf{p_0}\langle x, y\rangle = x \wedge \mathsf{p_1}\langle x, y\rangle = y)$
  $\phi = (0{\in}\mathsf{N} \wedge \mathsf{s_N}{\in}(\mathsf{N}{\rightarrow}\mathsf{N}))$
  $\phi = ((\forall x{\in}\mathsf{N})(\mathsf{s_N}x \neq 0 \wedge \mathsf{p_N}(\mathsf{s_N}x) = x))$
  $\phi = ((\forall x{\in}\mathsf{N})(x \neq 0 \rightarrow \mathsf{p_N}x{\in}\mathsf{N} \wedge \mathsf{s_N}(\mathsf{p_N}x) = x))$
  $\phi = (x{\in}\mathsf{N} \wedge y{\in}\mathsf{N} \wedge x = y \rightarrow (\mathsf{d_N}uv)xy = u)$
  $\phi = (x{\in}\mathsf{N} \wedge y{\in}\mathsf{N} \wedge x \neq y \rightarrow (\mathsf{d_N}uv)xy = v)$

(f) $\phi \in \mathcal{A}_{Rep}$

- $\phi = (\exists x)\Re(x, X)$.
  Let $\psi := \gamma(x, z) \wedge (\forall y)(\beta(y, z) \leftrightarrow \beta(y, \hat{X}))$ then
  $(\psi[\hat{X}/\hat{x}])[\hat{X}/z] \rightarrow (\exists z)(\psi[\hat{X}/\hat{x}])$ and
  $((\exists z)\psi)[\hat{X}/\hat{x}] \rightarrow (\exists \hat{x})(\exists z)\psi$ are in $\mathcal{A}^{\mathsf{A}}_{Quant}$, hence from
  $\vdash \hat{X} = \hat{X} \wedge (\forall y)(\beta(y, \hat{X}) \leftrightarrow \beta(y, \hat{X}))$ we get $\vdash \phi^\star$.
- $\phi = (\Re(x, X) \wedge \Re(x, Y) \rightarrow X = Y)$.
  We have $\vdash \Re(x, U)^\star \leftrightarrow (\forall y)(\beta(y, \hat{x}) \leftrightarrow \beta(y, \hat{U}))$ and
  $\vdash (X = Y)^\star \leftrightarrow (\forall y)(\beta(y, \hat{X}) \leftrightarrow \beta(y, \hat{Y}))$, hence from
  $\vdash (\forall y)(\beta(y, \hat{x}) \leftrightarrow \beta(y, \hat{X})) \wedge (\forall y)(\beta(y, \hat{x}) \leftrightarrow \beta(y, \hat{Y})) \rightarrow$
  $(\forall y)(\beta(y, \hat{X}) \leftrightarrow \beta(y, \hat{Y}))$
  we get $\vdash \phi^\star$.
- $X = Y \wedge \Re(t, X) \rightarrow \Re(t, Y)$, similar to the previous case.

(g) $\phi \in \mathcal{A}_{\Sigma^+\mathsf{E}}$
An exemplary proof similar to the proofs of the following cases is in (2) ($\phi \in \mathsf{J}$).

- $\phi = (\Re(\mathsf{nat}) \wedge (\forall x)(x \dot{\in} \mathsf{nat} \leftrightarrow x{\in}\mathsf{N}))$
- $\phi = (\Re(\mathsf{id}) \wedge (\forall x)(x \dot{\in} \mathsf{id} \leftrightarrow (\exists y)x = \langle y, y\rangle))$

- $\phi = (\Re(a) \wedge \Re(b) \rightarrow$
  $\Re(\mathsf{con}\langle a, b\rangle) \wedge (\forall x)(x \dot{\in} \mathsf{con}\langle a, b\rangle \leftrightarrow x \dot{\in} a \wedge x \dot{\in} b))$
- $\phi = (\Re(a) \wedge \Re(b) \rightarrow$
  $\Re(\mathsf{dis}\langle a, b\rangle) \wedge (\forall x)(x \dot{\in} \mathsf{dis}\langle a, b\rangle \leftrightarrow x \dot{\in} a \vee x \dot{\in} b))$
- $\phi = (\Re(a) \rightarrow \Re(\mathsf{dom}\langle a\rangle) \wedge (\forall x)(x \dot{\in} \mathsf{dom}\langle a\rangle \leftrightarrow (\exists y)\langle x, y\rangle \dot{\in} a))$
- $\phi = (\Re(a) \rightarrow \Re(\mathsf{inv}\langle a, f\rangle) \wedge (\forall x)(x \dot{\in} \mathsf{inv}\langle a, f\rangle \leftrightarrow f x \dot{\in} a))$

(2) $\phi \in \mathsf{J}$

- $\phi = (\Re(a) \wedge (\forall x \dot{\in} a)\Re(fx) \rightarrow \Re(\mathsf{j}\langle a, f\rangle) \wedge$
  $(\forall x)[x \dot{\in} \mathsf{j}\langle a, f\rangle \leftrightarrow (\exists y)(\exists z)(x = \langle y, z\rangle \wedge y \dot{\in} a \wedge z \dot{\in} f y)])$
  For $t := \mathsf{j}\langle a, f\rangle$ we have $\mathsf{PRA} \vdash \gamma(t, y)[\langle \hat{\mathsf{j}}, \hat{a}, \hat{f}\rangle / y]$ hence $\mathsf{PRA} \vdash (t\downarrow)^\star$.
  We also have $\mathsf{PRA} \vdash ((\forall x)\Re(x))^\star$ by (6.27), hence because of (1)(c) we get $\mathsf{PRA} \vdash \Re(\mathsf{j}\langle a, f\rangle)^\star$. Now let
  $\psi := x \dot{\in} \mathsf{j}\langle a, f\rangle \leftrightarrow (\exists y)(\exists z)(x = \langle y, z\rangle \wedge y \dot{\in} a \wedge z \dot{\in} f y)$ and
  $\xi := \beta(\hat{x}, v_2) \leftrightarrow (\hat{x} = \langle v_0, v_1\rangle \wedge \beta(v_0, \hat{a}) \wedge (\exists z)(\alpha(\hat{f}, v_0, z) \wedge \beta(v_1, z)))$
  then we have that $\mathsf{PRA} \vdash \xi[(\hat{x})_0/v_0, (\hat{x})_1/v_1, \langle \hat{\mathsf{j}}, \hat{a}, \hat{f}\rangle/v_2] \leftrightarrow \psi^\star$.
  Similar to the second case in (1)(e), arguing about computation sequences we can prove $\mathsf{PRA} \vdash \xi[(\hat{x})_0/v_0, (\hat{x})_1/v_1, \langle \hat{\mathsf{j}}, \hat{a}, \hat{f}\rangle/v_2]$ hence we get $\mathsf{PRA} \vdash \phi^\star$.

(3) $\phi \in \forall\Re \Rightarrow \mathsf{PRA} \vdash \phi^\star$ by (6.27).

(4) $\phi \in \forall\mathsf{N} \Rightarrow \mathsf{PRA} \vdash \phi^\star$ by (6.27).

(5) $\phi \in \mathcal{F}\text{-}\mathsf{I}_\mathsf{N}$

- $\phi = (\psi[0/x] \wedge (\forall x \in \mathsf{N})(\psi \rightarrow \psi[\mathsf{s_N} x/x]) \rightarrow (\forall x \in \mathsf{N})\psi)$.
  Suppose $y \in \mathcal{V}_A$ is such that $y \notin (\psi[\mathsf{s_N} x/x])^\star$ and $y \in FT_A(\hat{x}, \psi^\star)$ then by (6.26) we have
  $\mathsf{PRA} \vdash (\gamma(\mathsf{s_N} x, y) \rightarrow (\psi[\mathsf{s_N} x/x]^\star \leftrightarrow \psi^\star[y/\hat{x}]))[S(\hat{x})/y]$
  and because of $\mathsf{PRA} \vdash \gamma(\mathsf{s_N} x, y)[S(\hat{x})/y]$ we get
  $\mathsf{PRA} \vdash \psi[\mathsf{s_N} x/x]^\star \leftrightarrow \psi^\star[S(\hat{x})/\hat{x}]$. Similar reasoning leads to
  $\mathsf{PRA} \vdash \psi[0/x]^\star \leftrightarrow \psi^\star[0/\hat{x}]$. Using this and (6.27) yields for
  $\xi := \psi^\star[0/\hat{x}] \wedge (\forall \hat{x})(\psi^\star \rightarrow \psi^\star[S(\hat{x})/\hat{x}]) \rightarrow (\forall \hat{x})\psi^\star$
  that we have $\mathsf{PRA} \vdash \xi \leftrightarrow \phi^\star$. But $\xi \in \mathcal{F}\text{-}\mathsf{Ind}$ hence $\mathsf{PA} \vdash \phi^\star$.

(6) $\phi \in \mathsf{T}\text{-}\mathsf{I}_\mathsf{N}$

- $\phi = (0 \in X \wedge (\forall x \in \mathsf{N})(x \in X \rightarrow \mathsf{s_N} x \in X) \rightarrow (\forall x \in \mathsf{N}) x \in X)$.
  Analogous to the previous case with $\psi := (x \in X)$. Knowing that $\mathsf{PRA} \vdash \psi^\star \leftrightarrow \beta(\hat{x}, \hat{X})$ we get $\mathsf{PRA} \vdash \xi \leftrightarrow \phi^\star$ for
  $\xi := \beta(\hat{x}, \hat{X})[0/\hat{x}] \wedge (\forall \hat{x})(\beta(\hat{x}, \hat{X}) \rightarrow \beta(\hat{x}, \hat{X})[S(\hat{x})/\hat{x}]) \rightarrow (\forall \hat{x})\, \beta(\hat{x}, \hat{X})$.
  But $\xi \in \Sigma_1\text{-}\mathsf{Ind}$ hence $\mathsf{PRA}^+ \vdash \phi^\star$.

(7) $\mathcal{R}^{\mathsf{E}}$, Translation of the rules of inference:

- $\dfrac{\phi^\star \quad (\phi \to \psi)^\star}{\psi^\star}$ follows trivially from $(\phi \to \psi)^\star = \phi^\star \to \psi^\star$ and the corresponding rule (modus ponens) in $\mathcal{R}^{\mathsf{A}}$.
- $\dfrac{(\phi[y/x] \to \psi)^\star}{((\exists x)\phi \to \psi)^\star}$ is achieved in the following way:

$$\begin{aligned}
\text{We have}\quad & (\phi[y/x] \to \psi)^\star = \phi[y/x]^\star \to \psi^\star \\
\text{and}\quad & \vdash \phi[y/x]^\star \leftrightarrow \phi^\star[\hat{y}/\hat{x}]. \\
\text{Hence from}\quad & T' \vdash (\phi[y/x] \to \psi)^\star \\
\text{and}\quad & \vdash (\phi[y/x]^\star \to \psi^\star) \to (\phi^\star[\hat{y}/\hat{x}] \to \psi^\star) \\
\text{we get}\quad & T' \vdash \phi^\star[\hat{y}/\hat{x}] \to \psi^\star.
\end{aligned}$$

  Applying the corresponding rule in $\mathcal{R}^{\mathsf{A}}$ we get $T' \vdash (\exists\hat{x})\phi^\star \to \psi^\star$ hence $T' \vdash ((\exists x)\phi \to \psi)^\star$.
- The other rules of inference in $\mathcal{R}^{\mathsf{E}}$ are translated analogous to the previous case.

□

6.2 Provable Arithmetic Sentences

Arithmetic statements are translated into systems of Explicit Mathematics by the mapping $(.)^N$ from (6.6). The translation is such that the statements still have the same arithmetic meaning but now expressed in the language of Explicit Mathematics. We use this translation to compare systems of Explicit Mathematics with systems of arithmetic by comparing their sets of provable arithmetic statements.

Lemma 6.29. *For terms $t \in \mathcal{T}^{\mathsf{A}}$ with $FV_A(t) = \{x_0, \dots, x_n\}$ we have*

$$\mathsf{PRA} \vdash \gamma(t^N, y) \leftrightarrow t[\hat{x_0}^N/x_0, \dots, \hat{x_n}^N/x_n] = y$$

Proof. First we prove $\mathsf{PRA} \vdash \gamma(f(x_0, \dots, x_m)^N, y) \leftrightarrow f(\hat{x_0}^N, \dots, \hat{x_n}^N) = y$ by induction on the function symbols f. Next we prove the statement for general terms by induction on t. □

Lemma 6.30. *For formulas $\phi \in \mathcal{F}^{\mathsf{A}}$ with $FV_A(\phi) = \{x_0, \dots, x_n\}$ we have*

$$\mathsf{PRA} \vdash (\phi^N)^\star \leftrightarrow (\phi[\hat{x_0}^N/x_0, \dots, \hat{x_n}^N/x_n])$$

Proof. By induction on the formula ϕ. $\square$

Theorem 6.31 (Proof-Theoretic Equivalence).

Let $T \subset \mathcal{F}^{\mathsf{E}}$ *such that* $\Sigma^{+}\mathsf{ET} \subset T \subset \Sigma^{+}\mathsf{ET}+(\mathsf{J})+(\forall\Re)+(\forall\mathsf{N})+(\Sigma^{+}\mathsf{S}\text{-}\mathsf{C})+(\mathsf{Pow}^{-})$ *then the following holds:*

(1) *The provable arithmetic sentences of* $T+(\mathcal{F}\text{-}\mathsf{I}_{\mathsf{N}})$ *and* PA *are exactly the same, that is for closed* $\phi \in \mathcal{F}^{\mathsf{A}}$ *we have*

$$\mathsf{PA} \vdash \phi \quad \Leftrightarrow \quad T+(\mathcal{F}\text{-}\mathsf{I}_{\mathsf{N}}) \vdash \phi^{N}$$

(2) *The provable arithmetic* Π_2*-sentences of* $T+(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}})$ *and* PRA *are exactly the same, that is for closed* $\phi \in \Pi_2$ *we have*

$$\mathsf{PRA} \vdash \phi \quad \Leftrightarrow \quad T+(\mathsf{T}\text{-}\mathsf{I}_{\mathsf{N}}) \vdash \phi^{N}$$

Proof.

(1) We have $\Rightarrow$ by (6.9) and (5.10), and we get the other direction by (6.28) and (6.30).

(2) Similar to (1), but we additionally use that $\mathsf{PRA}^{+} \vdash \phi \Rightarrow \mathsf{PRA} \vdash \phi$ by (3.27).

$\square$

6.3 Provably Total Functions

First we define what it means for a function $f : \mathbb{N} \to \mathbb{N}$ to be provably total in systems of arithmetic and in systems of Explicit Mathematics, after this we determine the class of provably total functions of some specific systems.

Definition 6.32 (Provably total functions of $T \subset \mathcal{F}^{\mathsf{A}}$). A function $F : \mathbb{N} \to \mathbb{N}$ is provably total in $T \subset \mathcal{F}^{\mathsf{A}}$ if there exists a formula $\phi \in \Sigma_1$ such that $\{x, y\} = FV_A(\phi)$ (x, y different variables) and the following holds:

(1) $T \vdash (\forall x)(\exists y)\phi$

(2) $T \vdash \phi \wedge \phi[z/y] \to y = z \qquad$ (z different from x, y)

(3) $(\forall n \in \mathbb{N}) \quad T \vdash \phi[\overline{n}/x, \overline{F(n)}/y]$

Theorem 6.33. *The provably total functions of* PRA *are exactly the functions in* $PRIM_1$.

Proof. First we show that all $F \in PRIM_1$ are provably total in PRA. If $F \in PRIM_1$ then there is an $f \in Prim_1$ such that $F = [\![f]\!]$. We define $\phi := (f(x) = y)$ and show for this $\phi \in \Sigma_1$ the properties in (6.32):

(1) We have $\vdash \phi[f(x)/y]$ by (3.20). From $(\phi[f(x)/y] \rightarrow (\exists y)\phi) \in \mathcal{A}^{\mathsf{A}}_{Quant}$ we get $\vdash (\exists y)\phi$ and we finally arrive at $\vdash (\forall x)(\exists y)\phi$ by (3.19) .

(2) We can derive $\vdash (f(x) = y \wedge f(x) = z \wedge f(x) = f(x) \rightarrow y = z)$ from $\mathcal{A}^{\mathsf{A}}_{Equal}$, hence from $\vdash f(x) = f(x) \rightarrow (f(x) = y \wedge f(x) = z \rightarrow y = z)$ and $\vdash f(x) = f(x)$ we deduce $\vdash \phi \wedge \phi[z/y] \rightarrow y = z$.

(3) For arbitrary $n \in \mathbb{N}$ we have $\mathsf{PRA} \vdash f(\overline{n}) = \overline{[\![f]\!](n)}$ by (3.21), and because $[\![f]\!](n) = F(n)$ this is the same as $\mathsf{PRA} \vdash \phi[\overline{n}/x, \overline{F(n)}/y]$.

Suppose we have $F : \mathbb{N} \rightarrow \mathbb{N}$ and $\phi = (\exists z)\psi \in \Sigma_1$ satisfying (6.32). Now we need to show $F \in PRIM_1$. Because of $\mathsf{PRA} \vdash (\forall x)(\exists y)(\exists z)\psi$ we also have that $\mathsf{PRA} \vdash (\exists y)(\exists z)\psi$, hence by (3.26) there is a term $t \in \mathcal{T}^{\mathsf{A}}$ such that $\mathsf{PRA} \vdash \psi[(t)_0/y, (t)_1/z]$. By (3.23) there is a function symbol $f \in Prim_1$ such that $\mathsf{PRA} \vdash (t)_0 = f(x)$, hence $\mathsf{PRA} \vdash \phi[f(x)/y]$ by (3.18). For arbitrary $n \in \mathbb{N}$ we have $\mathsf{PRA} \vdash \phi[f(x)/y][\overline{n}/x]$, hence $\mathsf{PRA} \vdash \phi[\overline{n}/x, f(\overline{n})/y]$. By (3.21) and (3.18) we get $\mathsf{PRA} \vdash \phi[\overline{n}/x, \overline{[\![f]\!](n)}/y]$, and using (6.32)(3)+(2) yields $\mathsf{PRA} \vdash \overline{[\![f]\!](n)} = \overline{F(n)}$, hence $(\forall n \in \mathbb{N})\ [\![f]\!](n) = F(n)$ and finally $F = [\![f]\!] \in PRIM_1$. □

Definition 6.34 (Provably total functions of $T \subset \mathcal{F}^{\mathsf{E}}$)**.** A function $F : \mathbb{N} \rightarrow \mathbb{N}$ is provably total in $T \subset \mathcal{F}^{\mathsf{E}}$ if there exists a closed term $t \in \mathcal{T}^{\mathsf{E}}$ such that the following holds:

(1) $T \vdash t \in (\mathsf{N} \rightarrow \mathsf{N})$

(2) $(\forall n \in \mathbb{N}) \quad T \vdash t \cdot \overline{n} = \overline{F(n)}$

Theorem 6.35.
Let $T \subset \mathcal{F}^{\mathsf{E}}$ such that $\Sigma^{+}\mathsf{ET} \subset T \subset \Sigma^{+}\mathsf{ET} + (\mathsf{J}) + (\forall\Re) + (\forall\mathsf{N}) + (\Sigma^{+}\mathsf{S\text{-}C}) + (\mathsf{Pow}^{-})$ then the provably total functions of $T + (\mathsf{T\text{-}I_N})$ and PRA are exactly the same.

Proof. Because of (6.7) we have that all functions in $PRIM_1$ are provably total in $\mathsf{BON}^{-} + (\mathsf{V\text{-}I_N})$, hence provably total in $T + (\mathsf{T\text{-}I_N})$ by (5.10). Next we want to show that every provably total function in $T + (\mathsf{T\text{-}I_N})$ is also provably total in PRA. Suppose we have $F : \mathbb{N} \rightarrow \mathbb{N}$ and $t \in \mathcal{T}^{\mathsf{E}}$ satisfying (6.34). We need to show that there is a formula $\phi \in \Sigma_1$ such that ϕ and F satisfy definition (6.32). For $\psi := (\exists z)(\gamma(t, z) \wedge \alpha(z, x, y))$ we have

(1) $\mathsf{PRA}^{+} \vdash (\forall x)(\exists y)\psi$.
Because of (6.28) we have $\mathsf{PRA}^{+} \vdash (t{\in}(\mathsf{N}{\to}\mathsf{N}))^{\star}$ and by *-translation we get $\mathsf{PRA}^{+} \vdash (\exists y)\gamma(x,y) \;\to\; (\exists y)\gamma(t{\cdot}x,y)$. We have $\mathsf{PRA}^{+} \vdash (\exists y)\gamma(x,y)$ and hence get $\mathsf{PRA}^{+} \vdash (\exists y)\gamma(t{\cdot}x,y)$ by modus ponens. Now using that $\mathsf{PRA}^{+} \vdash \gamma(t{\cdot}x,y) \leftrightarrow \psi$ yields $\mathsf{PRA}^{+} \vdash (\exists y)\psi$ and finally $\mathsf{PRA}^{+} \vdash (\forall x)(\exists y)\psi$.

(2) $\mathsf{PRA}^{+} \vdash \psi \wedge \psi[z/y] \to y = z$.
Because of (6.18) and (6.20).

(3) $\mathsf{PRA}^{+} \vdash \psi[\overline{n}/x, \overline{F(n)}/y]$.
Because of (6.28) we have $\mathsf{PRA}^{+} \vdash (t{\cdot}\overline{n} = \overline{F(n)})^{\star}$ and by *-translation we get $\mathsf{PRA}^{+} \vdash (\exists y)[(\exists z)(\exists x)(\gamma(t,z) \wedge \gamma(\overline{n},x) \wedge \alpha(z,x,y)) \wedge \gamma(\overline{F(n)},y)]$ hence $\mathsf{PRA}^{+} \vdash (\exists x)(\exists y)[x = \overline{n} \wedge y = \overline{F(n)} \wedge (\exists z)(\gamma(t,z) \wedge \alpha(z,x,y))]$ because of (6.20) and finally $\mathsf{PRA}^{+} \vdash \psi[\overline{n}/x, \overline{F(n)}/y]$.

By (3.27) we get the same for PRA instead of PRA^{+}, that is

(1) $\mathsf{PRA} \vdash (\forall x)(\exists y)\psi$

(2) $\mathsf{PRA} \vdash \psi \wedge \psi[z/y] \to y = z$

(3) $\mathsf{PRA} \vdash \psi[\overline{n}/x, \overline{F(n)}/y]$

Finally there is a formula $\phi \in \Sigma_1$ such that $\mathsf{PRA} \vdash \phi \leftrightarrow \psi$, hence F is provably total in PRA. $\square$

7 Appendix

7.1 Primitive Recursive Functions

Definition 7.1 (Primitive Recursive Functions $PRIM$)**.** For every function symbol $f \in Prim_n$ we define the primitive recursive function $[\![f]\!] : \mathbb{N}^n \to \mathbb{N}$:

(1) $[\![S]\!](x) := x + 1$

(2) $[\![Cs_i^n]\!](x_1, \ldots, x_n) := i$

(3) $[\![Pr_i^n]\!](x_1, \ldots, x_n) := x_{i+1}$

(4) $[\![Comp^n(f, g_1, \ldots, g_m)]\!](x_1, \ldots, x_n) :=$
$[\![f]\!]([\![g_1]\!](x_1, \ldots, x_n), \ldots, [\![g_m]\!](x_1, \ldots, x_n))$

(5) $[\![Rec^{n+1}(f, g)]\!](x_1, \ldots, x_n, 0) := [\![f]\!](x_1, \ldots, x_n)$

(6) $[\![Rec^{n+1}(f, g)]\!](x_1, \ldots, x_n, x + 1) :=$
$[\![g]\!](x_1, \ldots, x_n, x, [\![Rec^{n+1}(f, g)]\!](x_1, \ldots, x_n, x))$

The set of all primitive recursive functions is $PRIM := \{[\![f]\!] \mid f \in Prim\}$. We write $PRIM_n$ for the set of primitive recursive functions of arity n, that is $PRIM_n := \{[\![f]\!] \mid f \in Prim_n\}$.

7.2 Free Variables and Substitution

7.2.1 Arithmetic

Definition 7.2 (Free Variables $FV_A : \mathcal{T}^{\mathsf{A}} \cup \mathcal{F}^{\mathsf{A}} \to \mathbf{P}(\mathcal{V}_A)$)**.**

(1) $FV_A(t) := \begin{cases} \{\} & t = \mathsf{0} \\ \{t\} & t \in \mathcal{V}_A \\ \bigcup_{i \leq n} FV_A(s_i) & t = f(s_0, \ldots, s_n) \end{cases}$

(2) $FV_A(s = t) := FV_A(s) \cup FV_A(t)$

(3) $FV_A(\neg\phi) := FV_A(\phi)$

(4) $FV_A(\phi \vee \psi) := FV_A(\phi \wedge \psi) := FV_A(\phi) \cup FV_A(\psi)$

(5) $FV_A((\exists x)\phi) := FV_A((\forall x)\phi) := FV_A(\phi) \backslash \{x\}$

Definition 7.3 (Term Substitution θ_n^{A}).

$$\theta_n^{\mathsf{A}} : (\mathcal{T}^{\mathsf{A}} \cup \mathcal{F}^{\mathsf{A}}) \times (\mathcal{T}^{\mathsf{A}})^{n+1} \times \mathcal{V}_A{}^{[n+1]} \to (\mathcal{T}^{\mathsf{A}} \cup \mathcal{F}^{\mathsf{A}})$$
$$\text{where } \mathcal{V}_A{}^{[n+1]} := \{(x_0, \dots, x_n) \in \mathcal{V}_A{}^{n+1} \mid (\forall i < j) x_i \neq x_j\}$$

For $\theta_n^{\mathsf{A}}(s, t_0, \dots, t_n, x_0, \dots, x_n)$ we write $s[t_0/x_0, \dots, t_n/x_n]$ or $s[\vec{t}/\vec{x}]$.

(1) $r \in \mathcal{V}_A \cup \{\mathsf{0}\} \;\Rightarrow\; r[\vec{t}/\vec{x}] := \begin{cases} t_i & r = x_i \\ r & (\forall i) r \neq x_i \end{cases}$

(2) $f(s_0, \dots, s_n)[\vec{t}/\vec{x}] := f(s_0[\vec{t}/\vec{x}], \dots, s_n[\vec{t}/\vec{x}])$

(3) $(r = s)[\vec{t}/\vec{x}] := (r[\vec{t}/\vec{x}] = s[\vec{t}/\vec{x}])$

(4) $(\neg\phi)[\vec{t}/\vec{x}] := \neg(\phi[\vec{t}/\vec{x}])$

(5) $(\phi \vee \psi)[\vec{t}/\vec{x}] := (\phi[\vec{t}/\vec{x}] \vee \psi[\vec{t}/\vec{x}])$

(6) $(\phi \wedge \psi)[\vec{t}/\vec{x}] := (\phi[\vec{t}/\vec{x}] \wedge \psi[\vec{t}/\vec{x}])$

(7) $((\exists y)\phi)[\vec{t}/\vec{x}] := \begin{cases} (\exists y)(\phi[t_0/x_0, \dots, x_i/x_i, \dots, t_n/s_n]) & y = x_i \\ (\exists y)(\phi[\vec{t}/\vec{x}]) & (\forall i) y \neq x_i \end{cases}$

(8) $((\forall y)\phi)[\vec{t}/\vec{x}] := \begin{cases} (\forall y)(\phi[t_0/x_0, \dots, x_i/x_i, \dots, t_n/s_n]) & y = x_i \\ (\forall y)(\phi[\vec{t}/\vec{x}]) & (\forall i) y \neq x_i \end{cases}$

Definition 7.4 (Substitutable Terms $FT_A : \mathcal{V}_A \times \mathcal{F}^{\mathsf{A}} \to \mathbf{P}(\mathcal{T}^{\mathsf{A}})$).

(1) $FT_A(x, s = t) := \mathcal{T}^{\mathsf{A}}$

(2) $FT_A(x, \neg\phi) := FT_A(x, \phi)$

(3) $FT_A(x, \phi \vee \psi) := FT_A(x, \phi \wedge \psi) := FT_A(x, \phi) \cap FT_A(x, \psi)$

(4) $FT_A(x, (\exists y)\phi) := \begin{cases} \mathcal{T}^{\mathsf{A}} & y = x \\ \{t \in \mathcal{T}^{\mathsf{A}} \mid y \notin FV_A(t)\} \cap FT_A(x, \phi) & y \neq x \end{cases}$

(5) $FT_A(x, (\forall y)\phi) := FT_A(x, (\exists y)\phi)$

7.2.2 Explicit Mathematics

Definition 7.5 (Free Individual Variables $FV_I : \mathcal{T}^{\mathsf{E}} \cup \mathcal{F}^{\mathsf{E}} \to \mathbf{P}(\mathcal{V}_I)$)**.**

(1) $r \in \mathcal{C}^{\mathsf{E}} \;\Rightarrow\; FV_I(r) := \{\}$

(2) $x \in \mathcal{V}_I \;\Rightarrow\; FV_I(x) := \{x\}$

(3) $S \in \{\cdot, \downarrow, \mathsf{N}, =\} \;\Rightarrow\; FV_I(S(t_0, \ldots, t_n)) := \bigcup_{i \leq n} FV_I(t_i)$

(4) $S \in \{\in, \Re\}, \;\Rightarrow\; FV_I(S(t, X)) := FV_I(t)$

(5) $FV_I(\neg\phi) := FV_I(\phi)$

(6) $FV_I(\phi \vee \psi) := FV_I(\phi \wedge \psi) := FV_I(\phi) \cup FV_I(\psi)$

(7) $FV_I((\exists x)\phi) := FV_I((\forall x)\phi) := FV_I(\phi)\backslash\{x\}$

(8) $FV_I((\exists X)\phi) := FV_I((\forall X)\phi) := FV_I(\phi)$

Definition 7.6 (Free Type Variables $FV_T : \mathcal{F}^{\mathsf{E}} \to \mathbf{P}(\mathcal{V}_T)$)**.**

(1) $S \in \{\downarrow, \mathsf{N}, =\} \;\Rightarrow\; FV_T(S(t_0, \ldots, t_n)) := \{\}$

(2) $S \in \{\in, \Re\}, \;\Rightarrow\; FV_T(S(t, X)) := \{X\}$

(3) $FV_T(\neg\phi) := FV_T(\phi)$

(4) $FV_T(\phi \vee \psi) := FV_T(\phi \wedge \psi) := FV_T(\phi) \cup FV_T(\psi)$

(5) $FV_T((\exists x)\phi) := FV_T((\forall x)\phi) := FV_T(\phi)$

(6) $FV_T((\exists X)\phi) := FV_T((\forall X)\phi) := FV_T(\phi)\backslash\{X\}$

Definition 7.7 (Term Substitution θ_n^{E})**.**

$$\theta_n^{\mathsf{E}} : (\mathcal{T}^{\mathsf{E}} \cup \mathcal{F}^{\mathsf{E}}) \times (\mathcal{T}^{\mathsf{E}})^{n+1} \times {\mathcal{V}_I}^{[n+1]} \to (\mathcal{T}^{\mathsf{E}} \cup \mathcal{F}^{\mathsf{E}})$$
$$\text{where } {\mathcal{V}_I}^{[n+1]} := \{(x_0, \ldots, x_n) \in {\mathcal{V}_I}^{n+1} \mid (\forall i < j) x_i \neq x_j\}$$

For $\theta_n^{\mathsf{E}}(s, t_0, \ldots, t_n, x_0, \ldots, x_n)$ we write $s[t_0/x_0, \ldots, t_n/x_n]$ or $s[\vec{t}/\vec{x}]$.

(1) $r \in \mathcal{V}_I \cup \mathcal{C}^{\mathsf{E}} \;\Rightarrow\; r[\vec{t}/\vec{x}] := \begin{cases} t_i & r = x_i \\ r & (\forall i) r \neq x_i \end{cases}$

(2) $S \in \{\cdot, \downarrow, \mathsf{N}, =\} \;\Rightarrow\; S(s_0, \ldots, s_k)[\vec{t}/\vec{x}] := S(s_0[\vec{t}/\vec{x}], \ldots, s_k[\vec{t}/\vec{x}])$

(3) $S \in \{\in, \Re\} \;\Rightarrow\; S(s, X)[\vec{t}/\vec{x}] := S(s[\vec{t}/\vec{x}], X)$

(4) $(\neg\phi)[\vec{t}/\vec{x}] := \neg(\phi[\vec{t}/\vec{x}])$

(5) $(\phi \vee \psi)[\vec{t}/\vec{x}] := (\phi[\vec{t}/\vec{x}] \vee \psi[\vec{t}/\vec{x}])$

(6) $(\phi \wedge \psi)[\vec{t}/\vec{x}] := (\phi[\vec{t}/\vec{x}] \wedge \psi[\vec{t}/\vec{x}])$

(7) $((\exists y)\phi)[\vec{t}/\vec{x}] := \begin{cases} (\exists y)(\phi[t_0/x_0, \ldots, x_i/x_i, \ldots, t_n/s_n]) & y = x_i \\ (\exists y)(\phi[\vec{t}/\vec{x}]) & (\forall i) y \neq x_i \end{cases}$

(8) $((\forall y)\phi)[\vec{t}/\vec{x}] := \begin{cases} (\forall y)(\phi[t_0/x_0, \ldots, x_i/x_i, \ldots, t_n/s_n]) & y = x_i \\ (\forall y)(\phi[\vec{t}/\vec{x}]) & (\forall i) y \neq x_i \end{cases}$

(9) $((\exists X)\phi)[\vec{t}/\vec{x}] := (\exists X)(\phi[\vec{t}/\vec{x}])$

(10) $((\forall X)\phi)[\vec{t}/\vec{x}] := (\forall X)(\phi[\vec{t}/\vec{x}])$

Definition 7.8 (Type Variable Substitution Θ_n)**.**

$$\begin{gathered} \Theta_n : \mathcal{F}^{\mathsf{E}} \times {\mathcal{V}_T}^{n+1} \times {\mathcal{V}_T}^{[n+1]} \to \mathcal{F}^{\mathsf{E}} \\ \text{where } {\mathcal{V}_T}^{[n+1]} := \{(X_0, \ldots, X_n) \in {\mathcal{V}_T}^{n+1} \mid (\forall i < j) X_i \neq X_j\} \end{gathered}$$

For $\Theta_n(\phi, Y_0, \ldots, Y_n, X_0, \ldots, X_n)$ we write $\phi[Y_0/X_0, \ldots, Y_n/X_n]$ or $\phi[\vec{Y}/\vec{X}]$.

(1) $S \in \{\downarrow, \mathsf{N}, =\} \;\Rightarrow\; S(t_0, \ldots, t_k)[\vec{Y}/\vec{X}] := S(t_0, \ldots, t_k)$

(2) $S \in \{\in, \Re\} \;\Rightarrow\; S(t, Z)[\vec{Y}/\vec{X}] := \begin{cases} S(t, Y_i) & Z = X_i \\ S(t, Z) & (\forall i) Z \neq X_i \end{cases}$

(3) $(\neg\phi)[\vec{Y}/\vec{X}] := \neg(\phi[\vec{Y}/\vec{X}])$

(4) $(\phi \vee \psi)[\vec{Y}/\vec{X}] := (\phi[\vec{Y}/\vec{X}] \wedge \psi[\vec{Y}/\vec{X}])$

(5) $(\phi \wedge \psi)[\vec{Y}/\vec{X}] := (\phi[\vec{Y}/\vec{X}] \vee \psi[\vec{Y}/\vec{X}])$

(6) $((\exists x)\phi)[\vec{Y}/\vec{X}] := (\exists x)(\phi[\vec{Y}/\vec{X}])$

(7) $((\forall x)\phi)[\vec{Y}/\vec{X}] := (\forall x)(\phi[\vec{Y}/\vec{X}])$

(8) $((\exists Z)\phi)[\vec{Y}/\vec{X}] := \begin{cases} (\exists Z)(\phi[Y_0/X_0, \ldots, X_i/X_i, \ldots, Y_n/X_n]) & Z = X_i \\ (\exists Z)(\phi[\vec{Y}/\vec{X}]) & (\forall i) Z \neq X_i \end{cases}$

(9) $((\forall Z)\phi)[\vec{Y}/\vec{X}] := \begin{cases} (\forall Z)(\phi[Y_0/X_0, \ldots, X_i/X_i, \ldots, Y_n/X_n]) & Z = X_i \\ (\forall Z)(\phi[\vec{Y}/\vec{X}]) & (\forall i) Z \neq X_i \end{cases}$

Definition 7.9 (Substitutable Terms $FT_I : \mathcal{V}_I \times \mathcal{F}^{\mathsf{E}} \to \mathbf{P}(\mathcal{T}^{\mathsf{E}})$).

(1) $\phi \in \mathcal{F}_0^{\mathsf{E}} \ \Rightarrow \ FT_I(x, \phi) := \mathcal{T}^{\mathsf{E}}$

(2) $FT_I(x, \neg\phi) := FT_I(x, \phi)$

(3) $FT_I(x, \phi \vee \psi) := FT_I(x, \phi \wedge \psi) := FT_I(x, \phi) \cap FT_I(x, \psi)$

(4) $FT_I(x, (\exists y)\phi) := \begin{cases} \mathcal{T}^{\mathsf{E}} & y = x \\ \{t \in \mathcal{T}^{\mathsf{E}} \mid y \notin FV_I(t)\} \cap FT_I(x, \phi) & y \neq x \end{cases}$

(5) $FT_I(x, (\forall y)\phi) := FT_I(x, (\exists y)\phi)$

(6) $FT_I(x, (\exists X)\phi) := FT_I(x, (\forall X)\phi) := FT_I(x, \phi)$

We usually write $FT(x, \phi)$ for $FT_I(x, \phi)$.

Definition 7.10 (Substitutable Type Variables $FT_T : \mathcal{V}_T \times \mathcal{F}^{\mathsf{E}} \to \mathbf{P}(\mathcal{V}_T)$).

(1) $\phi \in \mathcal{F}_0^{\mathsf{E}} \ \Rightarrow \ FT_T(X, \phi) := \mathcal{V}_T$

(2) $FT_T(X, \neg\phi) := FT_T(X, \phi)$

(3) $FT_T(X, \phi \vee \psi) := FT_T(X, \phi \wedge \psi) := FT_T(X, \phi) \cap FT_T(X, \psi)$

(4) $FT_T(X, (\exists x)\phi) := FT_T(X, (\forall x)\phi) := FT_T(X, \phi)$

(5) $FT_T(X, (\exists Y)\phi) := FT_T(X, (\forall Y)\phi) := \begin{cases} \mathcal{V}_T & Y = X \\ FT_T(X, \phi) \backslash \{Y\} & Y \neq X \end{cases}$

We usually write $FT(X, \phi)$ for $FT_T(X, \phi)$.

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