A Syntactical Treatment of Simultaneous Fixpoints in the Modal μ-Calculus

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Abstract

We provide a purely syntactical treatment of simultaneous fixpoints in the modal μ-calculus by proving directly in Kozen's axiomatisation their properties as greatest and least fixpoints, that is, the fixpoint axiom and the induction rule. Further, we apply our result in order to get a completeness result for characteristic formulae of finite pointed transition systems.
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1 Introduction

Modal μ-calculus is an extension of modal logic with least and greatest fixpoint constructors and therefore allows us to study fixpoints, which play an important role as extensions for many modal logics, on a sufficiently abstract level.

The expression 'μ-calculus' combined with the idea to introduce fixpoint constructors to monotonic functions on complete lattices was first introduced by Scott and De Bakker in [7]. The book of Arnod and Niwinski [2] provides a good overview over this general notion of μ-calculus. Modal μ-calculus can be seen as a special case where we restrict ourselves to the complete lattice given by the powerset of states of a transition system. It was introduced by Kozen in his seminal work [6]. There, also the axiomatisation KOZ is introduced which is basically the extension of minimal modal logic K with the so-called Park fixpoint induction principles. Kozen himself could prove completeness for the aconjunctive fragment but failed for the full language. Full completeness was established by Walukiewicz in [11], the proof is very involved and strongly relies on methods from automata theory and infinite games.

Induction principles in a modal context represent a big challenge for proof theorists. Namely fixpoint extensions of modal logic are very difficult to handle in a pure syntactical manner and, therefore, proof theoretical research on the modal μ-calculus has concentrated on, mainly infinitary, systems different from KOZ (see e.g. Jäger, Kretz and Studer in [5] and Dam and Sprenger [4]). One task of this paper is getting a better proof theoretical understandig of KOZ by working syntactically exclusively in that deductive system.

Simultaneous fixpoints are introduced by vectors of formulae: On a transition system with states S a modal μ-formula $\varphi(x_1, \ldots, x_n)$ defines a function from $\mathbb{P}(\mathsf{S})^n$ to $\mathbb{P}(\mathsf{S})$, therefore, n formulae $\varphi_i(x_1, \ldots, x_n)$, each of them monotone in all x_j, define a monotone function from $\mathbb{P}(\mathsf{S})^n$ to $\mathbb{P}(\mathsf{S})^n$ which by Tarski-Knaster [9] has a greatest and least (simultaneous) fixpoint. The question arises whether modal μ-calculus, which was introduced to deal with normal fixpoints, is strong enough to express simultaneous fixpoints. In fact, by using a result of Bekić (see [3, 2]) from the theory of complete lattices one gets that there exists a n-vector of modal μ-formulae expressing the greatest (resp. least) simultaneous fixpoint. Obviously, simultaneous fixpoints satisfy the fixpoint axiom and the induction rule.

In the main part of our work we show that Kozen's axiomatisation proves the fixpoint axiom and the induction rule for simultaneous fixpoints by working exclusively syntactically in KOZ, that is, without taking a detour via semantics by using completeness.

A straightforward application of our syntactical analysis is the completeness proof for the fragment of characteristic formulae. Given a finite pointed transition system $(\mathcal{S}, s_0)$ there is a formula $\chi_{(\mathcal{S},s_0)}$ characterizing it, that is, we have for all φ that φ is satisfied in $(\mathcal{S}, s_0)$ if and only if the implication $\chi_{(\mathcal{S},s_0)} \to \varphi$ is valid. With characteristic formulae we can translate the model checking question to the question of validity in all transition systems. Characteristic formulae are constructed by, first, introducing simultaneous greatest fixpoints of $\mathcal{L}_\mu$-formulae which express bisimilarity to the transition system $(\mathcal{S}, s_0)$ and, second, by applying the aforementioned Bekić result. We prove, obviously without using the full completeness of Walukiewicz, completeness for any formula of the form $\chi_{(\mathcal{S},s_0)} \to \varphi$.

In the next section we introduce the basic notions and results and extend them to a syntax allowing vectors of $\mathcal{L}_\mu$-formulae; a significant part is devoted to a exact definition of substitution. In Section 3 we introduce simultaneous fixpoints semantically and syntactically and prove the basic fixpoint properties in KOZ. This result is then applied in Section 4 in order to get the partial completeness for characteristic formulae.

2 Preliminaries

2.1 Syntax

We define the set of formulae of the modal μ-calculus starting from a set of *propositional variables* $\mathsf{P} = \{p, q, \ldots x, y, z, \ldots\}$ and the symbols $\top, \bot, \wedge, \vee, \neg, \Box, \Diamond, \mu$ and ν. The *class of all* $\mathcal{L}_\mu$*-formulae*, $\mathcal{L}_\mu$, is the smallest set with $\mathsf{P} \cup \{\top, \bot\} \subset \mathcal{L}_\mu$ and such that if

$$\varphi, \psi \in \mathcal{L}_\mu \text{ then } (\varphi \wedge \psi), (\varphi \vee \psi), \neg\varphi, \Box\varphi, \Diamond\varphi, \mu x.\varphi, \nu x.\varphi \in \mathcal{L}_\mu.$$

As usual, for formulae of the form $\mu x.\varphi, \nu x.\varphi$ we require that each occurrence of x in φ is positive, that is, in the scope of an even number of negations. If we do not require this syntactical restriction in the inductive definition above then we define the class of all $\mathcal{L}_{\mathsf{free}}$-formulae, $\mathcal{L}_{\mathsf{free}}$. Both $\mathcal{L}_\mu$-formulae and $\mathcal{L}_{\mathsf{free}}$-formulae will be denoted by small Greek letters, $\alpha, \beta, \varphi, \psi$.... We omit the parentheses if there is no danger of confusion and we sometimes abbreviate $\neg\alpha \vee \beta$ by $\alpha \rightarrow \beta$. Given a $\mathcal{L}_\mu$-formula (resp. $\mathcal{L}_{\mathsf{free}}$-formula) of the form $\sigma x.\varphi$, where $\sigma \in \{\mu, \nu\}$, we say that x is *bound by* $\sigma x.\varphi$. A variable x is *bound in* φ if there is a subformula binding x, otherwise it is a *free variable*. $\mathsf{Free}(\varphi)$ denotes all free variables of φ and $\mathsf{Bound}(\varphi)$ all bound variables. We write $\varphi(x_1, \ldots, x_n)$ if all occurrences of x_i are free in φ and all x_i are pairwise distinct. For a syntactical treatment of simultaneous fixpoints we need to extend our notation such that we permit also vectors of formulae: Let $\varphi_1, \ldots, \varphi_n$ be $\mathcal{L}_\mu$-formulae (resp. $\mathcal{L}_{\mathsf{free}}$-formulae). A vector of formulae $(\varphi_1, \ldots, \varphi_n)$ is denoted as $\vec{\varphi}$. Let $x_1, \ldots, x_m$ be propositional variables, we sometimes write $(\varphi_1, \ldots, \varphi_n)(x_1, \ldots, x_m)$ or $\vec{\varphi}(x_1, \ldots, x_m)$ or simply $\vec{\varphi}(\vec{x})$ if all appearances of x_i are pairwise distinct and free in all φ_j. For any vector $\vec{\varphi} \equiv (\varphi_1, \ldots, \varphi_n)$ (of formulae for example) by $\vec{\varphi}^{-i}$ we denote the vector $(\varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n)$. Simultaneous substitution is defined for $\mathcal{L}_{\mathsf{free}}$-formulae as follows: Let $\vec{x} \equiv (x_1, \ldots, x_n)$ and $\vec{\psi} \equiv (\psi_1, \ldots, \psi_n)$ be vectors of pairwise distinct propositonal variables and $\mathcal{L}_{\mathsf{free}}$-formulae and let φ be a $\mathcal{L}_{\mathsf{free}}$-formula. The $\mathcal{L}_{\mathsf{free}}$-formula $\varphi[\vec{x}/\vec{\psi}]$, the $\mathcal{L}_{\mathsf{free}}$-formula φ where all free occurrences of x_i are substituted simultaneously by ψ_i, is defined recursively as follows:

- $x[\vec{x}/\vec{\psi}] \equiv \begin{cases} \psi_i & \text{if } x \equiv x_i \\ x & \text{else.} \end{cases}$
- For $\star \in \{\wedge, \vee\}$: $\alpha \star \beta[\vec{x}/\vec{\psi}] \equiv \alpha[\vec{x}/\vec{\psi}] \star \beta[\vec{x}/\vec{\psi}]$.
- For $\star \in \{\Box, \Diamond, \neg\}$: $\star\alpha[\vec{x}/\vec{\psi}] \equiv \star(\alpha[\vec{x}/\vec{\psi}])$.
- For $\sigma \in \{\mu, \nu\}$: $\sigma x.\alpha[\vec{x}/\vec{\psi}] \equiv \begin{cases} \sigma x.(\alpha[\vec{x}^{-i}/\vec{\psi}^{-i}]) & \text{if } x \equiv x_i \\ \sigma x.(\alpha[\vec{x}/\vec{\psi}]) & \text{else.} \end{cases}$

We sometimes write $\varphi[x_1/\psi_1, \ldots, x_n/\psi_n]$ for $\varphi[\vec{x}/\vec{\psi}]$ and we write $\varphi[x'/\psi', \vec{x}/\vec{\psi}]$ for $\varphi[x'/\psi', x_1/\psi_1, \ldots, x_n/\psi_n]$. If $\varphi(x_1, \ldots, x_n)$ is a $\mathcal{L}_{\mathsf{free}}$-formula then by $\varphi(\psi_1, \ldots, \psi_n)$ we mean $\varphi[x_1/\psi_1, \ldots, x_n/\psi_n]$. For a vector of fomulae $\vec{\varphi} \equiv (\varphi_1, \ldots, \varphi_n)$, a formula ψ and a variable x we write $\vec{\varphi}[x/\psi]$ for $(\varphi_1[x/\psi], \ldots, \varphi_n[x/\psi])$, analogously for $\vec{\varphi}[\vec{x}/\vec{\psi}]$. For $\mathcal{L}_\mu$-formulae $\varphi, \psi_1, \ldots, \psi_n \in \mathcal{L}_\mu$ if $\varphi[x_1/\psi_1, \ldots, x_n/\psi_n]$ is a $\mathcal{L}_\mu$-formula, too, then we have an admissible substitution.

Remark 2.1. Note, that if $\varphi, \psi \in \mathcal{L}_\mu$ then $\varphi[x/\psi]$ need not be a $\mathcal{L}_\mu$-formula, for example, if we set $\varphi \equiv \mu y.x$ and $\psi \equiv \neg y$ then we have $\varphi[x/\psi] \equiv \mu y.\neg y \notin \mathcal{L}_\mu$. Therefore, in order to formally define substitution we had to intoduce the class of $\mathcal{L}_{\mathsf{free}}$-formulae. In the sequel we will concentrate on $\mathcal{L}_\mu$-formulae mainly and, if nothing is mentioned, by formulae we implicitly mean $\mathcal{L}_\mu$-formulae.

For serial substitution we write $\varphi[x_1/\psi_1][x_2/\psi_2]$ for $(\varphi[x_1/\psi_1])[x_2/\psi_2]$; similarly for $\varphi[x_1/\psi_1]\dots[x_n/\psi_n]$. The next lemma is proven by induction on φ.

Lemma 2.2. Let φ, ψ be $\mathcal{L}_{\mathsf{free}}$-formulae, y, z variables, $(\alpha_1, \dots, \alpha_n)$ a vector of $\mathcal{L}_{\mathsf{free}}$-formulae and $(x_1, \dots, x_n)$ a vector of propositional variables such that $x_1, \dots, x_n, y$ are pairwise distinct. We have:

(1) If $x_1, \dots, x_n \notin \mathsf{Free}(\psi)$ then $\varphi[y/\psi][\vec{x}/\vec{\alpha}] \equiv \varphi[y/\psi, \vec{x}/\vec{\alpha}]$.

(2) If $y \notin \mathsf{Free}(\alpha_i)$ for all i then $\varphi[\vec{x}/\vec{\alpha}][y/\psi] \equiv \varphi[y/\psi, \vec{x}/\vec{\alpha}]$.

(3) If $y \notin \mathsf{Free}(\psi)$ then $\varphi[y/\psi, \vec{x}/\vec{\alpha}][y/\psi] \equiv \varphi[\vec{x}/\vec{\alpha}][y/\psi]$.

(4) If $y \notin \mathsf{Bound}(\varphi)$ then $\varphi[y/\psi, \vec{x}/(\vec{\alpha}[y/\psi])] \equiv \varphi[\vec{x}/\vec{\alpha}][y/\psi]$.

(5) Let $\vec{y} \equiv (y_1, \dots, y_n), \vec{z} \equiv (z_1, \dots, z_n)$ be vectors of pairwise distinct variables and $\vec{\psi} \equiv (\psi_1, \dots, \psi_n)$ a vector of formulae. If $z_j \notin (\mathsf{Bound}(\varphi) \cup \mathsf{Free}(\varphi))$ and if $z_j \notin \mathsf{Free}(\alpha_i)$ for all i and j then $\varphi[\vec{x}/\vec{\alpha}, \vec{y}/\vec{z}][\vec{z}/\vec{\psi}] \equiv \varphi[\vec{x}/\vec{\alpha}, \vec{y}/\vec{\psi}]$.

The class of formulae where all free variables are among a subset of propositional variables $\mathsf{L} \subseteq \mathsf{P}$ is denoted by $\mathcal{L}_\mu(\mathsf{L})$, that is,

$$\mathcal{L}_\mu(\mathsf{L}) = \{\varphi \in \mathcal{L}_\mu \mid \mathsf{Free}(\varphi) \subseteq \mathsf{L}\}.$$

Clearly, a formula containing no fixpoint operator is a *modal formula*. The set of all modal formulae is denoted by $\mathcal{L}_{\mathsf{mod}}$. A formula φ can be transformed in *negation normal form*, $\mathsf{nnf}(\varphi)$, by shifting all negations inside the formula by using the dualities of the connectives defined. For any formulae $\alpha(x)$ and natural number $n \in \mathbb{N}$ we define recursively $\alpha^n(x)$, such that $\alpha^1(x) \equiv \alpha(x)$ and $\alpha^{n+1}(x) \equiv \alpha[x/\alpha^n(x)]$. *Kozen's Axiomatisation*, KOZ, is a Hilbert-Style axiomatisation and consists of the following axioms and rules.

Axioms:
KOZ contains all axioms of the classical propositional calculus, the *distribution axiom*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

the *fixpoint axiom*

$$\nu x.\varphi \leftrightarrow \varphi(\nu x.\varphi)$$

and the *duality axioms*

$$\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi \quad \text{and} \quad \neg\nu x.\neg\varphi[x/\neg x] \leftrightarrow \mu x.\varphi$$

which are necessary since we did not introduce $\Diamond$ as $\neg\Box\neg$ and μ and ν similarly.

Inference Rules:
In addition to the classical *Modus Ponens* [MP] we have the *Necessitation Rule* [Nec] from Modal Logic.

$$[\mathsf{MP}]: \quad \frac{\varphi \quad \varphi \to \psi}{\psi} \qquad [\mathsf{Nec}]: \quad \frac{\varphi}{\Box\varphi}$$

Further, for any formula $\varphi(x)$ such that x appears only positively we have the *Induction Rule* [Ind] to handle fixpoints.

$$[\mathsf{Ind}]: \quad \frac{\psi \to \varphi(\psi)}{\psi \to \nu x.\varphi}.$$

If there is a proof for a formula φ we write $\vdash \varphi$. The proof of the next lemma is left to the reader.

Lemma 2.3. Let φ, ψ be $\mathcal{L}_\mu$-formulae and let $\alpha(x)$ be a $\mathcal{L}_\mu$-formula with x appearing only positively . The following holds:

(1) $\vdash \varphi \leftrightarrow \mathsf{nnf}(\varphi)$.

(2) If $\vdash \varphi \to \psi$ then $\vdash \alpha(\varphi) \to \alpha(\psi)$

(3) For all $n \in \mathbb{N}$ we have $\vdash \alpha^n(x)[x/\bot] \to \mu x.\alpha(x)$.

(4) If $y \notin \mathsf{Free}(\varphi)$ we have $\vdash \sigma x.\varphi \leftrightarrow \sigma y.(\varphi[x/y])$ where $\sigma \in \{\mu, \nu\}$.

(5) If $y \notin \mathsf{Free}(\varphi)$ we have $\vdash \psi(\sigma x.\varphi) \leftrightarrow \psi(\sigma y.(\varphi[x/y]))$ where $\sigma \in \{\mu, \nu\}$.

Remark 2.4. Having in mind Lemma 2.3 (v), in the sequel we assume formulae to be *well-named*, that is, bound and free variables are distinct and for two distinct subformulae $\sigma x.\alpha$ and $\sigma' y.\beta$ we have $x \not\equiv y$ (where $\sigma, \sigma' \in \{\mu, \nu\}$).

2.2 Semantics

The standard semantics for modal μ-calculus is given by transition systems. A *transition system* $\mathcal{S}$ is a triple $(\mathsf{S}, \to_\mathcal{S}, \lambda)$ consisting of

- a set S of *states*,
- a binary relation $\to_\mathcal{S} \subseteq \mathsf{S} \times \mathsf{S}$ called *transition relation*,
- the *valuation* $\lambda : \mathsf{P} \to \mathbb{P}(\mathsf{S})$ assigning to each propositional variable p a subset $\lambda(p)$ of S.

We write $s \to_\mathcal{S} t$ for $(s,t) \in \to_\mathcal{S}$. Let λ be a valuation on $\mathbb{P}(\mathsf{S})$, p a propositional variable and S' an element of $\mathbb{P}(\mathsf{S})$; we set for all propositional variables p'

$$\lambda[p \mapsto \mathsf{S}'](p') = \begin{cases} \mathsf{S}' & \text{if } p' = p, \\ \lambda(p') & \text{otherwise.} \end{cases}$$

Given a transition system $\mathcal{S} = (\mathsf{S}, \to_\mathcal{S}, \lambda)$, then $\mathcal{S}[p \mapsto \mathsf{S}']$ denotes the transition system $(\mathsf{S}, \to_\mathcal{S}, \lambda[p \mapsto \mathsf{S}'])$. For any transition system $\mathcal{S}$ and state $s_0 \in \mathsf{S}$ we define the *pointed transition system* as the tuple $(\mathcal{S}, s_0)$. Let φ be a $\mathcal{L}_\mu$-formula and $\mathcal{S}$ a transition system, the set of states where φ holds, denoted by $\|\varphi\|_\mathcal{S}$, is called the *denotation of φ in $\mathcal{S}$*. The definition of $\|\varphi\|_\mathcal{S}$ is by induction on the complexity of φ. Simultaneously for all transition systems $\mathcal{S}$ we set:

- $\|p\|_{\mathcal{S}} = \lambda(p)$ for all $p \in \mathsf{P}$
- $\|\neg\alpha\|_{\mathcal{S}} = \mathsf{S} - \|\alpha\|_{\mathcal{S}}$
- $\|\alpha \wedge \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cap \|\beta\|_{\mathcal{S}}$
- $\|\alpha \vee \beta\|_{\mathcal{S}} = \|\alpha\|_{\mathcal{S}} \cup \|\beta\|_{\mathcal{S}}$
- $\|\Box\alpha\|_{\mathcal{S}} = \{s \in \mathsf{S} \mid \forall t((s \rightarrow_{\mathcal{S}} t) \Rightarrow t \in \|\alpha\|_{\mathcal{S}})\}$
- $\|\Diamond\alpha\|_{\mathcal{S}} = \{s \in \mathsf{S} \mid \exists t((s \rightarrow_{\mathcal{S}} t) \wedge t \in \|\alpha\|_{\mathcal{S}})\}$
- $\|\nu x.\alpha\|_{\mathcal{S}} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\alpha(x)\|_{\mathcal{S}[x \mapsto \mathsf{S}']}\}$
- $\|\mu x.\alpha\|_{\mathcal{S}} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha(x)\|_{\mathcal{S}[x \mapsto \mathsf{S}']} \subseteq \mathsf{S}'\}$

Given a formula $\varphi(x)$ and set of states $\mathsf{S}' \subseteq \mathsf{S}$ we sometimes write $\|\varphi(\mathsf{S}')\|_{\mathcal{S}}$ instead of $\|\varphi(x)\|_{\mathcal{S}[x \mapsto \mathsf{S}']}$, analogously, if $\mathsf{S}_1, \ldots, \mathsf{S}_n \subseteq \mathsf{S}$, for $\|\varphi(\mathsf{S}_1, \ldots, \mathsf{S}_n)\|_{\mathcal{S}}$. When clear from the context we use $\|\varphi(x)\|_{\mathcal{S}}$ for the function

$$\|\varphi(x)\|_{\mathcal{S}} : \begin{cases} \mathbb{P}(\mathsf{S}) \rightarrow \mathbb{P}(\mathsf{S}) \\ \mathsf{S}' \mapsto \|\varphi(\mathsf{S}')\|_{\mathcal{S}}. \end{cases}$$

Analogously for $\|\varphi(x_1, \ldots, x_n)\|_{\mathcal{S}}$.

By Tarski-Knaster Theorem, c.f. [9], $\|\nu x.\alpha(x)\|_{\mathcal{S}}$ is the greatest and $\|\mu x.\alpha(x)\|_{\mathcal{S}}$ the least fixpoint of the operator $\|\alpha(x)\|_{\mathcal{S}}$, we have

$$\|\nu x.\alpha(x)\|_{\mathcal{S}} = \mathsf{GFP}(\|\alpha(x)\|_{\mathcal{S}}) \quad \text{and} \quad \|\mu x.\alpha(x)\|_{\mathcal{S}} = \mathsf{LFP}(\|\alpha(x)\|_{\mathcal{S}})$$

and, moreover, for any finite transition system $\mathcal{S}$ of cardinality n we have

$$\|\nu x.\alpha(x)\|_{\mathcal{S}} = \|\alpha^n(\top)\|_{\mathcal{S}} \quad \text{and} \quad \|\mu x.\alpha(x)\|_{\mathcal{S}} = \|\alpha^n(\bot)\|_{\mathcal{S}}.$$

Tarski-Knaster Theorem does not restrict the second claim to finite domains. Since for this work this already suffices and, therefore, transfinite iterations are not necessarily needed we cite this less general variant.

If $s \in \|\varphi\|_{\mathcal{S}}$ we say that φ *is valid in* s. If φ is valid in all states s of $\mathcal{S}$ then φ is said to be *valid in* $\mathcal{S}$ and we write $\mathcal{S} \models \varphi$. For pointed transition systems we write $(\mathcal{S}, s_0) \models \varphi$ if $s_0 \in \|\varphi\|_{\mathcal{S}}$. φ is valid in all transition systems $\mathcal{S}$ then φ is *valid* and we write $\models \varphi$, obviously this is exactly the case when it is valid in all pointed transition systems. Correctness of KOZ is proven by a straightforward induction on the proof length.

Proposition 2.5 (Correctness). For all $\mathcal{L}_\mu$-formulae φ we have

$$\vdash \varphi \quad \Rightarrow \quad \models \varphi.$$

We extend the notion of denotation to vectors of formulae $(\varphi_1, \dots, \varphi_n) \equiv \vec{\varphi}$ whereby for all transition systems $\mathcal{S}$ $\|\vec{\varphi}\|_{\mathcal{S}}$ denotes the vector $(\|\varphi_1\|_{\mathcal{S}}, \dots, \|\varphi_n\|_{\mathcal{S}}) \subseteq \mathbb{P}(\mathsf{S})^n$. For any vector of sets of states $\vec{\mathsf{S}'} = (\mathsf{S}'_1, \dots, \mathsf{S}'_n)$ and any vector of propositional variables $\vec{x} = (x_1, \dots, x_n)$ by $\mathcal{S}[\vec{x} \mapsto \vec{\mathsf{S}'}]$ we denote the transition system $\mathcal{S}[x_1 \mapsto \mathsf{S}'_1] \dots [x_n \mapsto \mathsf{S}'_n]$ (note that, the order does not matter here). The next lemma is proven by induction on φ.

Lemma 2.6. For all μ formulae φ, ψ and vector of formulae $\vec{\psi}$ and transition systems $\mathcal{S}$ we have

(1) $\|\varphi[x/\psi]\|_{\mathcal{S}} = \|\varphi\|_{\mathcal{S}[x \mapsto \|\psi\|_{\mathcal{S}}]}$ and

(2) $\|\varphi[\vec{x}/\vec{\psi}]\|_{\mathcal{S}} = \|\varphi\|_{\mathcal{S}[\vec{x} \mapsto \|\vec{\psi}\|_{\mathcal{S}}]}$.

Let $\alpha_i(x_1, \dots, x_n)$ be formulae for all $i \in \{1, \dots, n\}$ with x_i appearing only positively. We can define a new functional $\|(\alpha_1, \dots, \alpha_n)(x_1, \dots, x_n)\|_{\mathcal{S}}$ as follows ($\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}$ abbreviates $\|(\alpha_1, \dots, \alpha_n)(x_1, \dots, x_n)\|_{\mathcal{S}}$).

$$\begin{array}{llcl} \|\vec{\alpha}(\vec{x})\|_{\mathcal{S}} : & \mathbb{P}(\mathsf{S})^n & \to & \mathbb{P}(\mathsf{S})^n \\ & (\mathsf{S}_1, \dots, \mathsf{S}_n) & \mapsto & (\|\alpha_1(\mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}, \dots, \|\alpha_n(\mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}) \end{array}$$

By Tarski-Knaster $\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}$ has a greatest fixpoint $\mathsf{GFP}(\vec{\alpha})$ and a least fixpoint $\mathsf{LFP}(\vec{\alpha})$ on $\mathbb{P}(\mathsf{S})^n$, where the order relation $\subseteq_n$ is defined such that $(\mathsf{S}_1, \dots, \mathsf{S}_n) \subseteq_n (\mathsf{S}'_1, \dots, \mathsf{S}'_n)$ if and only if $\mathsf{S}_i \subseteq \mathsf{S}'_i$ for all $i \in \{1, \dots, n\}$. In the following if there is no danger of confusion we write $\subseteq$ instead of $\subseteq_n$.

2.3 Bisimulation

Bisimulation is used to formalize the notion of observational equivalence and defines and a central concept in operational semantics. Let $\mathcal{S} = (\mathsf{S}, \to_{\mathcal{S}}, \lambda_{\mathcal{S}})$ and $\mathcal{S}' = (\mathsf{S}, \to_{\mathcal{S}'}, \lambda_{\mathcal{S}'})$ be two transition systems and let $\mathsf{L} \subseteq \mathsf{P}$ be a subset of propositional variables. A relation $R \subseteq \mathsf{S} \times \mathsf{S}'$ is a L-*bisimulation* if the following hold:

(1) if $(s, s') \in R$ and $s \to_{\mathcal{S}} t$ then there is a $t' \in \mathsf{S}'$ such that $s' \to_{\mathcal{S}'} t'$ and $(t, t') \in R$,

(2) if $(s, s') \in R$ and $s' \to_{\mathcal{S}'} t'$ then there is a $t \in \mathsf{S}$ such that $s \to_{\mathcal{S}} t$ and $(t, t') \in R$, and

(3) if $(s, s') \in R$ then for all propositional variables $p \in \mathsf{L}$ we have $s \in \lambda_{\mathcal{S}}(p)$ if and only if $s' \in \lambda_{\mathcal{S}'}(p)$.

Two pointed transition systems $(\mathcal{S}, s)$ and $(\mathcal{S}', s')$ are L-*bisimilar* if there is a L-bisimulation relation $R \subseteq \mathsf{S} \times \mathsf{S}'$ such that $(s, s') \in R$; we write $(\mathcal{S}, s) \sim_{\mathsf{L}} (\mathcal{S}', s')$. $\|(\mathcal{S}, s)\|_{\sim_{\mathsf{L}}}$ denotes the (proper) class of all pointed transition systems which are L-bisimilar to $(\mathcal{S}, s)$. The next folklore lemma states that two bisimilar states fulfill exactly the same formulae with free variables in L. The proof is by unwinding the definitions of bisimilarity and denotation.

Lemma 2.7. Let $\mathcal{S} = (\mathsf{S}, \to_{\mathcal{S}}, \lambda_{\mathcal{S}})$ and $\mathcal{S}' = (\mathsf{S}', \to_{\mathcal{S}'}, \lambda_{\mathcal{S}'})$ be two transition systems and let $s \in \mathsf{S}$ and $s' \in \mathsf{S}'$. We have

$$(\mathcal{S}, s) \sim_{\mathsf{L}} (\mathcal{S}', s') \quad \Rightarrow \quad (\text{all } \varphi \in \mathcal{L}_\mu(\mathsf{L}) \quad s \in \|\varphi\|_{\mathcal{S}} \text{ if and only if } s' \in \|\varphi\|_{\mathcal{S}'}).$$

3 Simultaneous Fixpoints

In the first subsection we motivate simultaneous fixpoints by showing that they can be used to characterize bisimlarity. In the second section we show that in modal μ-calculus simultaneous fixpoints can be seen as tuples of $\mathcal{L}_\mu$-formulae and in the last subsection we prove that the basic properties are derivable in KOZ.

3.1 Bisimulation Classes of finite transition systems

Let L be a finite subset of propositional variables and let $\mathcal{S}$ be the finite transition system of the form $(\{s_1, \dots, s_n\}, \to, \lambda)$. For any state s_i we define a modal formula $\phi^{\mathsf{L}}_{s_i} \in \mathcal{L}_{\mathsf{mod}}(\mathsf{L} \cup \{s_1 \dots, s_n\})$ characterizing it locally modulo L-bisimulation, the *local characteristic* L*-formula*, as

$$\phi^{\mathsf{L}}_{s_i}(s_1, \dots, s_n) \equiv \Box(\bigvee_{\{s_j | s_i \to s_j\}} s_j) \wedge \bigwedge_{\{s_j | s_i \to s_j\}} \Diamond s_j \wedge \bigwedge_{\substack{s_i \in \|p\|_{\mathcal{S}} \\ p \in \mathsf{L}}} p \wedge \bigwedge_{\substack{s_i \in \|\neg p\|_{\mathcal{S}} \\ p \in \mathsf{L}}} \neg p.$$

(Note, that $\bigvee \emptyset \equiv \bot$ and $\bigwedge \emptyset \equiv \top$.) For any transition system $\mathcal{S}'$ with states S' starting for the (finitely many) local characteristic L-formulae we can define the following function:

$$\begin{array}{rrcl} \|\vec{\phi}(\vec{x})\|_{\mathcal{S}'} : & \mathbb{P}(\mathsf{S}')^n & \to & \mathbb{P}(\mathsf{S}')^n \\ & (\mathsf{S}'_1, \dots, \mathsf{S}'_n) & \mapsto & (\|\phi^{\mathsf{L}}_{s_1}(\mathsf{S}'_1, \dots, \mathsf{S}'_n)\|_{\mathcal{S}'}, \dots, \|\phi^{\mathsf{L}}_{s_n}(\mathsf{S}'_1, \dots, \mathsf{S}'_n)\|_{\mathcal{S}'}) \end{array}$$

$\|\vec{\phi}\|_{\mathcal{S}'}$ is a monotone operator on $\mathbb{P}(\mathsf{S}')^n$ ordered by $\subseteq_n$ and therefore, by Tarski-Knaster [9], it follows that $\mathsf{GFP}(\|\vec{\phi}\|_{\mathcal{S}'})$ exists. The following theorem shows that bisimulation equivalence classes of finite transition systems can be seen as simultaneous greatest fixpoints. It has first been established in the context of Hennessy-Milner logic (see Aceto and Ingolfsdottir [1]).

Theorem 3.1. *Let* L *be a finite set of propositional variables and let* $\mathcal{S}$ *be a finite transition system states with* $\{s_1, \dots, s_n\}$ *and with corresponding local characteristic* L*-formulae* $\phi_{s_1}, \dots, \phi_{s_n}$*. For all transition systems* $\mathcal{S}'$ *and* n*-tuples of states* $(s'_1, \dots, s'_n)$ *we have*

$$(s'_1, \dots, s'_n) \in \mathsf{GFP}(\|\vec{\phi}\|_{\mathcal{S}'}) \quad \textit{if and only if} \quad (\mathcal{S}', s'_i) \sim_{\mathsf{L}} (\mathcal{S}, s_i) \ \textit{all } i.$$

3.2 Simultaneous Fixpoints as $\mathcal{L}_\mu$-formulae

Definition 3.2 (Syntactical Simultaneous Fixpoints). Let n be a natural number and let $\vec{x} \equiv (x_1, \ldots, x_n)$ be a vector of variables. For any vector of formulae $(\alpha_1(\vec{x}), \ldots, \alpha_n(\vec{x}))$ where each x_j appears only positively in each α_i we define recursively the *greatest simultaneous fixpoint* and the *projections*. The greatest simultaneous fixpoint is an n-tuple of $\mathcal{L}_\mu$-formulae and is denoted by $\nu(x_1, \ldots, x_n).\vec{\alpha}$ or shorter $\nu\vec{x}.\vec{\alpha}$. For all $j \leq n$ the j-th projection of $\nu\vec{x}.\vec{\alpha}$ is the $\mathcal{L}_\mu$-formula denoted by $\pi_j^n(\nu\vec{x}.\vec{\alpha})$. The defining recursion is the following:

$$\begin{array}{rrcl} n = 1: & \pi_1^1(\nu\vec{x}.\vec{\alpha}) & \equiv & \nu x_1.\alpha_1(x_1) \\ & \nu\vec{x}.\vec{\alpha} & \equiv & \pi_1^1(\nu\vec{x}.\vec{\alpha}) \\ n > 1: & \pi_i^n(\nu\vec{x}.\vec{\alpha}) & \equiv & \nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}] \\ & \nu\vec{x}.\vec{\alpha} & \equiv & (\pi_1^n(\nu\vec{x}.\vec{\alpha}), \ldots, \pi_n^n(\nu\vec{x}.\vec{\alpha})) \end{array}$$

The simultaneous least fixpoint $\mu\vec{x}.\vec{\alpha}$ is defined analogously.

Remark 3.3. The restriction to variables x_i which appear only positively guarantees that the substitutions applied in the definition above are admissible and, therefore, that simultaneous fixpoints of $\mathcal{L}_\mu$-formulae are $\mathcal{L}_\mu$-formulae, too.

Example 3.4. Let $\alpha(x, y), \beta(x, y) \in \mathcal{L}_\mu$ such that x, y appear only positively. Then $\nu(x, y).(\alpha(x, y), \beta(x, y)) \equiv (\nu x.\alpha(x, \nu y.\beta(x, y)), \nu y.\beta(\nu x.\alpha(x, y), y))$.

The next theorem shows that syntactical simultaneous fixpoints correspond to the semantical ones. It is an adaptation of the fixpoint result of Bekić [3] to our framework. A similar proof can also be found in Arnold and Niwinski [2].

Theorem 3.5. *Let $\alpha_i(x_1, \ldots, x_n)$ be formulae for all $i \in \{1, \ldots, n\}$ with all x_j appearing only positively. For all transition systems $\mathcal{S}$ we have*

$$\mathsf{GFP}(\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}) = \|\nu\vec{x}.\vec{\alpha}\|_{\mathcal{S}} \quad \textit{and} \quad \mathsf{LFP}(\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}) = \|\mu\vec{x}.\vec{\alpha}\|_{\mathcal{S}}.$$

Proof. By induction on n, simultaneously for all formulae α_i and transition systems $\mathcal{S}$. We show the arguments for the greatest fixpoints, the least fixpoints are left to the reader. The case where $n = 1$ is trivial. For $n > 1$ let us fix an arbitrary transition system $\mathcal{S}$ with states S and define $\vec{A}, \vec{B} \in \mathbb{P}(\mathsf{S})^n$ whereby

$$\vec{A} = (A_1, \ldots, A_n) = \mathsf{GFP}(\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}) \text{ and } \vec{B} = (B_1, \ldots, B_n) = \|\nu\vec{x}.\vec{\alpha}\|_{\mathcal{S}}.$$

We have to show $\vec{A} = \vec{B}$. Let us fist show $\vec{A} \subseteq \vec{B}$. Since $(A_1, \ldots, A_n)$ is a fixpoint of $\|\vec{\alpha}(\vec{x})\|_{\mathcal{S}}$ for all i we have $A_i = \|\alpha_i(A_1, \ldots, A_n)\|_{\mathcal{S}}$. Thus, for all i and all $j \neq i$ we have

$$A_j = \|\alpha_j(A_1, \ldots, A_{i-1}, x_i, A_{i+1}, \ldots, A_n)\|_{\mathcal{S}[x_i \mapsto A_i]}.$$

It is clear from definition of $\vec{A}$ that the greatest fixpoint of the $(n-1)$-ary functional $\|\vec{\alpha}^{-i}(\vec{x}^{-i})\|_{\mathcal{S}[x_i \mapsto A_i]}$ must be $\vec{A}^{-i}$ and by applying the induction hypothesis on the transition system $\mathcal{S}[x_i \mapsto A_i]$ to $\vec{\alpha}^{-i}$ we get $\vec{A}^{-i} \subseteq \|\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)\|_{\mathcal{S}[x_i \mapsto A_i]}$ and by monotonicity of α_i in all components we can infer

$$\|\alpha_i(A_1, \ldots, A_n)\|_{\mathcal{S}[x_i \mapsto A_i]} \subseteq \|\alpha_i(\vec{x})\|_{\mathcal{S}[x_i \mapsto A_i][\vec{x}^{-i} \mapsto \|\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)\|_{\mathcal{S}[x_i \mapsto A_i]}]}$$

and since $A_i = \|\alpha_i(A_1, \ldots, A_n)\|_{\mathcal{S}[x_i \mapsto A_i]}$

$$A_i \subseteq \|\alpha_i(\vec{x})\|_{\mathcal{S}[x_i \mapsto A_i][\vec{x}^{-i} \mapsto \|\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)\|_{\mathcal{S}[x_i \mapsto A_i]}]}.$$

Therefore, A_i is a pre-fixpoint and we can infer

$$A_i \subseteq \|\nu x_i.\alpha_i(\vec{x})\|_{\mathcal{S}[x_i \mapsto A_i][\vec{x}^{-i} \mapsto \|\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)\|_{\mathcal{S}[x_i \mapsto A_i]}]}$$

and by Lemma 2.6 we get

$$A_i \subseteq \|\nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)]\|_{\mathcal{S}[x_i \mapsto A_i]}.$$

It follows from the fact that x_i does not appear free in the inequality above that

$$A_i \subseteq \|\nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(x_i)]\|_{\mathcal{S}}$$

and, therefore, $A_i \subseteq \|\pi_i^n(\vec{x})\|_{\mathcal{S}}$. Since i was chosen arbitrarily we have shown $\vec{A} \subseteq \vec{B}$. For the other inclusion, first observe that for all B_i since they are defined equal to $\|\pi_i^n(\nu\vec{x}.\vec{\alpha})\|_{\mathcal{S}} = \|\nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}]\|_{\mathcal{S}}$ we have

$$B_i = \|\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}]\|_{\mathcal{S}[x_i \mapsto B_i]}.$$

Define $\vec{C}^{-i}$ as $\|\nu\vec{x}^{-i}.\vec{\alpha}^{-i}(\vec{x})\|_{\mathcal{S}[x_i \mapsto B_i]}$, with the equation above and Lemma 2.6 we infer

$$B_i = \|\alpha_i(\vec{x})\|_{\mathcal{S}[\vec{x}^{-i} \mapsto \vec{C}^{-i}][x_i \mapsto B_i]}. \tag{1}$$

Further, by induction hypothesis $\vec{C}^{-i}$ is a fixpoint of $\|\vec{\alpha}^{-i}(\vec{x}^{-i})\|_{\mathcal{S}[x_i \mapsto B_i]}$ (in fact, the greatest fixpoint) and, thus, we also have

$$\vec{C}^{-i} = \|\vec{\alpha}^{-i}(\vec{x})\|_{\mathcal{S}[\vec{x}^{-i} \mapsto \vec{C}^{-i}][x_i \mapsto B_i]}. \tag{2}$$

Combining equation 1 and 2 we have that the vector of subsets of states

$$(C_1, \ldots, C_{i-1}, B_i, C_{i+1}, \ldots, C_n)$$

is a fixpoint of $\vec{\alpha}(\vec{x})$. And therefore we have for all i

$$(C_1, \ldots, C_{i-1}, B_i, C_{i+1}, \ldots, C_n) \subseteq \vec{A}$$

and, since i was chosen arbitrary, $\vec{B} \subseteq \vec{A}$.

3.3 Provability of the fixpoint properties in KOZ

Before we show that KOZ proves the induction rule and the fixpoint axiom for simultaneous fixpoints we prove two technical lemmas dealing with substitution.

Lemma 3.6. Let $\alpha_1(x_1, \ldots, x_n, y), \ldots, \alpha_n(x_1, \ldots, x_n, y)$ be $\mathcal{L}_\mu$-formulae such that all x_i appear only positively and let ψ be a formula such that $x_1, \ldots, x_n, y \notin \mathsf{Free}(\psi)$. If we define $\hat{\alpha}_i \equiv \alpha_i[y/\psi]$ then we have

$$\pi_i^n(\nu\vec{x}.\vec{\hat{\alpha}}) \equiv \pi_i^n(\nu\vec{x}.\vec{\alpha})[y/\psi] \quad \text{and} \quad \nu\vec{x}.\vec{\hat{\alpha}} \equiv \nu\vec{x}.\vec{\alpha}[y/\psi].$$

Proof. By induction on n we simultaneously prove both equivalences. The case where $n = 1$ is clear. If $n > 1$ then we can establish the following equivalences

$$\begin{array}{rll} \pi_i^n(\nu\vec{x}.\vec{\hat{\alpha}}) \equiv & \nu x_i.\hat{\alpha}_i(x)[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\hat{\alpha}}^{-i}] & \\ \equiv & \nu x_i.\hat{\alpha}_i(x)[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}[y/\psi]] & \text{Ind. hyp.} \\ \equiv & \nu x_i.\alpha_i(x)[y/\psi][\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}[y/\psi]] & \text{Definiton of } \hat{\alpha}_i \\ \equiv & \nu x_i.\alpha_i(x)[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}][y/\psi] & \text{Lemma 2.2 (iv)} \\ \equiv & \pi_i^n(\nu\vec{x}.\vec{\alpha})[y/\psi]. & \end{array}$$

The induction step which shows $\nu\vec{x}.\vec{\hat{\alpha}} \equiv \nu\vec{x}.\vec{\alpha}[y/\psi]$ follows from the previous one straightforwardly.

Lemma 3.7. Let $\vec{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ be a vector of formulae and $\vec{x} \equiv (x_1, \ldots, x_n), \vec{y} \equiv (y_1, \ldots, y_n)$ pairwise distinct vectors of variables such that all x_i appear only free and positively in all α_j and such that $y_1, \ldots, y_n \notin (\mathsf{Free}(\alpha_j) \cup \mathsf{Bound}(\alpha_j))$. We have

$$\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \leftrightarrow \pi_i^n(\nu\vec{y}.(\vec{\alpha}[\vec{x}/\vec{y}])).$$

Proof. By induction on n. If $n = 1$ the claim follows from Lemma 2.3 (iv). If $n > 1$ then by definition we have

$$\pi_i^n(\nu\vec{x}.\vec{\alpha}) \equiv \nu x_i.\alpha_i[x_1/\pi_1^{n-1}, x_{i-1}/\pi_{i-1}^{n-1}, x_{i+1}/\pi_i^{n-1}, \ldots, x_n/\pi_{n-1}^{n-1}] \tag{3}$$

where $\pi_j^{n-1} \equiv \pi_j^{n-1}(\nu\vec{x}^{-i}.\vec{\alpha}^{-i})$. By induction hypothesis for all $j \neq i$ we have

$$\vdash \pi_j^{n-1}(\nu\vec{x}^{-i}.\vec{\alpha}^{-i}) \rightarrow \pi_j^{n-1}(\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}])). \tag{4}$$

Applying Lemma 2.3 (ii) to equations 3 and 4 we get

$$\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \rightarrow \nu x_i.\alpha_i[\vec{x}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}])]$$

and with Lemma 2.3 (iv) we get

$$\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \rightarrow \nu y_i.\alpha_i[\vec{x}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}])][x_i/y_i].$$

The following equivalences complete the proof of $\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \rightarrow \pi_i^n(\nu\vec{y}.(\vec{\alpha}[\vec{x}/\vec{y}]))$.

$$\begin{array}{rcll}
\pi_i^n(\nu\vec{y}.(\vec{\alpha}[\vec{x}/\vec{y}])) & \equiv & \nu y_i.(\alpha_i[\vec{x}/\vec{y}])[\vec{y}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}/\vec{y}])] & \\
& \equiv & \nu y_i.(\alpha_i[\vec{x}/\vec{y}][\vec{y}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}/\vec{y}])]) & \text{Def. Subst.} \\
& \equiv & \nu y_i.(\alpha_i[x_i/y_i, \vec{x}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}/\vec{y}])]) & \text{Lemma 2.2 (v)} \\
& \equiv & \nu y_i.(\alpha_i[x_i/y_i, \vec{x}^{-i}/\nu\vec{y}^{-i}.\vec{\alpha}^{-i}[\vec{x}/\vec{y}]]) & \text{Def. Subst.} \\
& \equiv & \nu y_i.(\alpha_i[x_i/y_i, \vec{x}^{-i}/\nu\vec{y}^{-i}.\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}][x_i/y_i]]) & \text{Lemma 2.2 (ii)} \\
& \equiv & \nu y_i.\alpha_i[\vec{x}^{-i}/\nu\vec{y}^{-i}.\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}]][x_i/y_i] & \text{Lemma 2.2 (iv)} \\
& \equiv & \nu y_i.\alpha_i[\vec{x}^{-i}/\nu\vec{y}^{-i}.(\vec{\alpha}^{-i}[\vec{x}^{-i}/\vec{y}^{-i}])][x_i/y_i] & \text{Def. Subst.}
\end{array}$$

The implication $\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \leftarrow \pi_i^n(\nu\vec{y}.(\vec{\alpha}[\vec{x}/\vec{y}]))$ follows from the other implication and from the fact that $\nu\vec{x}.((\vec{\alpha}[\vec{x}/\vec{y}])[\vec{y}/\vec{x}]) \equiv \nu\vec{x}.\vec{\alpha}$.

Proposition 3.8 (Simultaneous Induction Rule). Let $\alpha_i(x_1, \ldots, x_n)$ be formulae for all $i \in \{1, \ldots, n\}$ with all x_j appearing only positively and let $\psi_1, \ldots, \psi_n$ be $\mathcal{L}_\mu$-formulae. If we have

$$\vdash \psi_i \rightarrow \alpha_i(\psi_1, \ldots, \psi_n) \text{ for all } i \in \{1, \ldots, n\}$$

then we also have

$$\vdash \psi_i \rightarrow \pi_i^n(\nu\vec{x}.\vec{\alpha}) \text{ for all } i \in \{1, \ldots, n\}.$$

Proof. We first show the lemma assuming that $x_1, \ldots, x_n \notin \mathsf{Free}(\psi_i)$ for all i. The proof goes by induction on n simultaneously for all formulae. The case where $n = 1$ is easily verified by applying the induction rule [Ind]. If $n > 1$ for any $i \in \{1, \ldots, n\}$ we have

$$\vdash \psi_i \rightarrow \alpha_i(\psi_1, \ldots, \psi_n). \tag{5}$$

We fix an arbitrary i and for all $j \neq i$ we define $\hat{\alpha}_j \equiv \alpha_j[x_i/\psi_i]$. By applying the induction hypothesis to the vector of formulae $(\hat{\alpha}_1, \ldots, \hat{\alpha}_{i-1}, \hat{\alpha}_{i+1}, \ldots, \hat{\alpha}_n)$ for all $j \neq i$ we get $\vdash \psi_j \rightarrow \pi_j^n(\nu\vec{x}^{-i}.\vec{\hat{\alpha}}^{-i})$ and with Lemma 3.6, since we assumed that $x_1, \ldots, x_n \notin \mathsf{Free}(\psi_i)$ for all i, we get

$$\vdash \psi_j \rightarrow \pi_j^n(\nu\vec{x}^{-i}.\vec{\alpha}^{-i})[x_i/\psi_i] \quad i \neq j. \tag{6}$$

Since $\vdash \psi_i \rightarrow \alpha_i(\psi_1, \ldots, \psi_n)$, by equation 6 and by Lemma 2.3 (ii) we can deduce

$$\begin{array}{ll}
\vdash \psi_i \rightarrow \alpha_i(\vec{x}) & [x_1/\pi_1^n(\nu\vec{x}^{-i}.\vec{\alpha}^{-i})[x_i/\psi_i] \\
& , \ldots, x_i/\psi_i, \ldots, \\
& x_n/\pi_n^n(\nu\vec{x}^{-i}.\vec{\alpha}^{-i})[x_i/\psi_i]]
\end{array}$$

and by applying Lemma 2.2 (iv) we get $\vdash \psi_i \rightarrow \alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}][x_i/\psi_i]$. An application of the induction rule [Ind] leads to $\vdash \psi_i \rightarrow \nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}]$. By definition we have $\pi_i^n(\nu\vec{x}.\vec{\alpha}) \equiv \nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}]$. Since i was chosen arbitrarily we have completed the induction step.

In order to complete the proof we have to show that we can drop the assumption that $x_1, \ldots, x_n \notin \mathsf{Free}(\psi_i)$ for all i. First, define for all i the formula $\hat{\alpha}_i(y_1, \ldots, y_n) \equiv \alpha_i(x_1, \ldots, x_n)[x_1/y_1, \ldots, x_n/y_n]$, such that all y_j are new variables. Trivially, if

$$\vdash \psi_i \to \alpha_i(\psi_1, \ldots, \psi_n) \text{ for all } i \in \{1, \ldots, n\}$$

then, since $\hat{\alpha}_i(\psi_1, \ldots, \psi_n) \equiv \alpha_i(\psi_1, \ldots, \psi_n)$, we have

$$\vdash \psi_i \to \hat{\alpha}_i(\psi_1, \ldots, \psi_n) \text{ for all } i \in \{1, \ldots, n\}.$$

Since $y_1, \ldots, y_n \notin \mathsf{Free}(\psi_i)$ for all i we can apply the claim of this lemma and get

$$\vdash \psi_i \to \pi_i^n(\nu\vec{y}.\vec{\hat{\alpha}}) \text{ for all } i \in \{1, \ldots, n\}.$$

By Lemma 3.7 we have $\vdash \pi_i^n(\nu\vec{y}.\vec{\hat{\alpha}}) \to \pi_i^n(\nu\vec{x}.\vec{\alpha})$ and, therefore,

$$\vdash \psi_i \to \pi_i^n(\nu\vec{x}.\vec{\alpha}) \text{ for all } i \in \{1, \ldots, n\}.$$

Proposition 3.9 (Simultaneous Fixpoint Axiom 1). Let $\alpha_i(x_1, \ldots, x_n)$ be formulae for all $i \in \{1, \ldots, n\}$ with all x_j appearing only positively. For all i we have

$$\vdash \pi_i^n(\nu\vec{x}.\vec{\alpha}) \to \alpha_i(\pi_1^n(\nu\vec{x}.\vec{\alpha}), \ldots, \pi_n^n(\nu\vec{x}.\vec{\alpha})).$$

Proof. By induction on n. The case where $n = 1$ follows from definition of $\pi_1^1(\nu\vec{x}.\vec{\alpha})$ and fixpoint axiom. For the case where $n > 1$ for all i we abbreviate $\pi_i^n(\nu\vec{x}.\vec{\alpha}) \equiv \nu x_i.\alpha_i(\vec{x})[\vec{x}^{-i}/\nu\vec{x}^{-i}.\vec{\alpha}^{-i}]$ by π_i^n. For a given and arbitrary i we define a vector of formulae $\vec{\psi}^{-i}$ to be $\nu\vec{x}^{-i}.\vec{\alpha}^{-i}[x_i/\pi_i^n]$. Note, that for all $\psi_i \in \vec{\psi}^{-i}$ we have $\vec{x} \notin \mathsf{Free}(\psi_i)$. Since $\vec{x} \notin \mathsf{Free}(\pi_i^n)$ by Lemma 3.6 we have for all $j \neq i$ that $\nu\vec{x}^{-i}.\vec{\alpha}^{-i}[x_i/\pi_i^n] \equiv \nu\vec{x}^{-i}.\vec{\hat{\alpha}}^{-i}$ where $\hat{\alpha}_j \equiv \alpha_j[x_i/\pi_i^n]$. By induction hypothesis for $\nu\vec{x}^{-i}.\vec{\hat{\alpha}}^{-i}$ for all $\psi_j \in \vec{\psi}^{-i}$ we get $\vdash \psi_j \to \hat{\alpha}_j[\vec{x}^{-i}/\vec{\psi}^{-i}]$ and unwinding the definition of $\hat{\alpha}_j$ we get $\vdash \psi_j \to \alpha_j[x_i/\pi_i^n][\vec{x}^{-i}/\vec{\psi}^{-i}]$. Since $\vec{x} \notin \mathsf{Free}(\pi_i^n)$ we can apply Lemma 2.2 (i) and get $\alpha_j[x_i/\pi_i^n][\vec{x}^{-i}/\vec{\psi}^{-i}] \equiv \alpha_j[x_i/\pi_i^n, \vec{x}^{-i}/\vec{\psi}^{-i}]$. Therefore, the implication above can be reformulated as

$$\vdash \psi_j \to \alpha_j[x_i/\pi_i^n, \vec{x}^{-i}/\vec{\psi}^{-i}]. \tag{7}$$

Further, by applying the fixpoint axiom to π_i^n we have

$$\vdash \pi_i^n \to \alpha_i[x_i/\pi_i^n, \vec{x}^{-i}/\vec{\psi}^{-i}]. \tag{8}$$

Equations 7 and 8 fulfill the requirements of Proposition 3.8 and therefore we can apply it and get $\vdash \psi_j \to \pi_j^n$ for all $j \neq i$. By monotonicity of all α_j with Lemma 2.3 (ii) we can infer from equation 8 the following implication

$$\vdash \pi_i^n \to \alpha_i(\pi_1^n, \ldots, \pi_n^n).$$

Since i was chosen arbitrary we have completed the induction step and the proof.

Proposition 3.10 (Simultaneous Fixpoint Axiom 2). Let $\alpha_i(x_1, \ldots, x_n)$ be formulae for all $i \in \{1, \ldots, n\}$ with all x_j appearing only positively. For all i we have

$$\vdash \alpha_i(\pi_1^n(\nu\vec{x}.\vec{\alpha}), \ldots, \pi_n^n(\nu\vec{x}.\vec{\alpha})) \to \pi_i^n(\nu\vec{x}.\vec{\alpha})$$

Proof. We abbreviate $\pi_i^n(\nu\vec{x}.\vec{\alpha})$ by π_1^n. By Proposition 3.9 for all i we have $\vdash \pi_i^n \to \alpha_i(\pi_1^n, \ldots, \pi_n^n)$. Applying Lemma 2.3 (ii) for all i we can infer

$$\vdash \alpha_i(\pi_1^n, \ldots, \pi_n^n) \to \alpha_i(\alpha_1(\pi_1^n, \ldots, \pi_n^n), \ldots, \alpha_n(\pi_1^n, \ldots, \pi_n^n)).$$

Applying Proposition 3.8 gives us the desired result.

4 Completeness for characteristic formulae

The next theorem says that a finite transition system can be characterized modulo bisimulation by his characteristic formula. It is a consequence of the Theorems 3.1 and 3.5 and has first been proven by Steffen in [8].

Theorem 4.1 (Characteristic Formula). *For any finite transition system $\mathcal{S}$, finite set of propositional variables L and any state s there exists a formula $\chi^{\mathsf{L}}_{(\mathcal{S},s)}$ such that for all transition systems $\mathcal{S}'$ and all states s' we have*

$$s' \in \|\chi^{\mathsf{L}}_{(\mathcal{S},s)}\|_{\mathcal{S}'} \quad \textit{if and only if} \quad (\mathcal{S}, s) \sim_{\mathsf{L}} (\mathcal{S}', s').$$

Remember that $\chi_{(\mathcal{S},s)}$ is the greatest simultaneous fixpoint $\nu\vec{x}.\vec{\phi}^{\mathsf{L}}$ of all local characteristic L-formulae defined for every state s_i in $\mathcal{S}$ as

$$\phi^{\mathsf{L}}_{s_i}(s_1, \ldots, s_n) \equiv \Box(\bigvee_{\{s_j | s_i \to s_j\}} s_j) \wedge \bigwedge_{\{s_j | s_i \to s_j\}} \Diamond s_j \wedge \bigwedge_{\substack{s_i \in \|p\|_{\mathcal{S}} \\ p \in \mathsf{L}}} p \wedge \bigwedge_{\substack{s_i \in \|\neg p\|_{\mathcal{S}} \\ p \in \mathsf{L}}} \neg p.$$

By remembering the definitions of $\nu\vec{x}.\vec{\alpha}$ and $\pi_i^n(\nu\vec{x}.\vec{\alpha})$ we can remark the following.

Remark 4.2. Any characteristic formula $\chi_{(\mathcal{S},s_0)}$ only contains greatest fixpoints; that is, it belongs to the first level of the fixpoint hierarchy where only greatest fixpoint constructors are allowed (when formulae are assumed to be in negation normal form).

Further, Lemma 3.9 can be reformulated as follows:

Lemma 4.3. Let $\mathcal{S}$ be a finite transition system with states $\{s_1, \ldots, s_n\}$ and let L be a finite set of propositional variables. For all i we have

$$\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s_i)} \to \phi^{\mathsf{L}}_{s_i}(\chi^{\mathsf{L}}_{(\mathcal{S},s_1)}, \ldots, \chi^{\mathsf{L}}_{(\mathcal{S},s_n)}).$$

Let $\mathsf{S}' \subseteq \mathsf{S}$ be an arbitrary subset of the states of a finite transition system. We define

$$\chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}')} \equiv \bigvee_{s' \in \mathsf{S}'} \chi^{\mathsf{L}}_{(\mathcal{S},s')}$$

and state the following Lemma.

Lemma 4.4. Let $s \in \mathsf{S}$ be an arbitrary state of a finite transition system $\mathcal{S}$ and let $\mathcal{R}(s) = \{s' \in \mathsf{S} \mid s \to_{\mathcal{S}} s'\}$. We have

(1) $\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \Box(\chi^{\mathsf{L}}_{(\mathcal{S},\mathcal{R}(s))})$, and

(2) $\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \Diamond\chi^{\mathsf{L}}_{(\mathcal{S},s')}$ for any $s' \in \mathcal{R}(s)$.

Proof. By Lemma 4.3 we have $\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \phi^{\mathsf{L}}_s(\chi^{\mathsf{L}}_{(\mathcal{S},s_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},s_n)})$. By remembering the definition of ϕ^{L}_s with classical propositional reasoning we can infer both parts of the lemma.

Theorem 4.5 (Model Checking with KOZ). *Let $\mathcal{S} = (\mathsf{S}, \to_{\mathcal{S}}, \lambda)$ be a finite transition system and L a finite set of propositional variables. For all states $s \in \mathsf{S}$, formulae $\varphi(x_1, \dots, x_n) \in \mathcal{L}_\mu(\mathsf{L})$ and set of states $\mathsf{S}_1, \dots, \mathsf{S}_n$ we have*

$$s \in \|\varphi(\mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}} \quad \Leftrightarrow \quad \vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \varphi(\chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)})$$

Proof. The direction from right to left follows from Theorem 4.1 and the correctness of Kozen's Axiomatisation 2.5. The other direction is proved by induction on the complexity of φ. Since we have $\vdash \varphi \leftrightarrow \mathsf{nnf}(\varphi)$ we assume that all formulae are in negation normal form. The cases where φ is of the form $p, \neg p, \alpha \wedge \beta, \alpha \vee \beta$ are left to the reader. The cases where φ is of the form $\Box\alpha, \Diamond\alpha$ use Lemma 4.4 and are left to the reader, too.

$\varphi \equiv \mu x.\alpha(x, x_1, \dots, x_n)$: If $s \in \|\mu x.\alpha(x, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}$ then, since $\mathcal{S}$ is a finite transition system, there is a n such that

$$s \in \|\alpha^n(\bot, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}$$

By induction hypothesis we have

$$\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \alpha^n(\bot, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)})$$

and with Lemma 2.3.3 can derive

$$\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \mu x.\alpha(x, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)}).$$

$\varphi \equiv \nu x.\alpha(x, x_1, \dots, x_n)$: If $s \in \|\nu x.\alpha(x, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}$ then, by the fixpoint properties we have

$$s \in \|\alpha(\|\nu x.\alpha(x, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}.$$

By induction hypothesis we have

$$\vdash \chi_{(\mathcal{S},s)} \to \alpha(\chi^{\mathsf{L}}_{(\mathcal{S},\|\nu x.\alpha(x,\mathsf{S}_1,\dots,\mathsf{S}_n)\|_{\mathcal{S}})}, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)}).$$

Since this is valid for all $s \in \|\nu x.\alpha(x, \mathsf{S}_1, \dots, \mathsf{S}_n)\|_{\mathcal{S}}$ we get

$$\vdash \chi^{\mathsf{L}}_{(\mathcal{S},\|\nu x.\alpha(x,\mathsf{S}_1,\dots,\mathsf{S}_n)\|_{\mathcal{S}})} \to \alpha(\chi^{\mathsf{L}}_{(\mathcal{S},\|\nu x.\alpha(x,\mathsf{S}_1,\dots,\mathsf{S}_n)\|_{\mathcal{S}})}, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)}).$$

With the induction rule [Ind] we get

$$\vdash \chi_{(\mathcal{S},\|\nu x.\alpha(x,\mathsf{S}_1,\dots,\mathsf{S}_n)\|_{\mathcal{S}})} \to \nu x.\alpha(x, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_1)}, \dots, \chi^{\mathsf{L}}_{(\mathcal{S},\mathsf{S}_n)}).$$

and since $\vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \to \chi^{\mathsf{L}}_{\|\nu x.\alpha(x,\mathsf{S}_1,\dots,\mathsf{S}_n)\|_{\mathcal{S}}}$ we get the induction step.

Our partial completeness result follows from Theorems 4.5 and 4.1

Corollary 4.6 (Completeness for Characteristic formulae). Let $\chi^{\mathsf{L}}_{(\mathcal{S},s)}$ be the characteristic formula for an arbitrary state s in a finite transition system $\mathcal{S}$ and let $\varphi \in \mathcal{L}_\mu(\mathsf{L})$. We have

$$\models \chi^{\mathsf{L}}_{(\mathcal{S},s)} \rightarrow \varphi \quad \text{if and only if} \quad \vdash \chi^{\mathsf{L}}_{(\mathcal{S},s)} \rightarrow \varphi.$$

Proof. The "if" direction follows form correctness 2.5. For the "only if" observe that from $\models \chi^{\mathsf{L}}_{(\mathcal{S},s)} \rightarrow \varphi$ we have that $s \in \|\varphi\|_{\mathcal{S}}$. By applying Theorem 4.5 we get the desired result.

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