Explicit Mathematics with Positive Existential Stratified Comprehension, Join and Uniform Monotone Inductive Definitions

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Der Wirklichkeit ist mit Logik nur zum Teil beizukommen.

Friedrich Dürrenmatt

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Samuel Bucheli

1 Introduction

The aim of this thesis is a proof-theoretic analysis of certain weak systems of Explicit Mathematics. More information about Explicit Mathematics can be found in the following section. This thesis is a follow-up to Krähenbühl's [Krä06], wherein it was shown that

$$\Sigma^+\mathsf{ET} + \mathsf{T\text{-}I_N} \equiv \mathsf{PRA} \equiv \Sigma^+\mathsf{ET} + \mathsf{T\text{-}I_N} + \mathsf{J} + \forall\mathsf{N} + \Sigma^+\mathsf{S\text{-}C} + \mathsf{Pow}^-$$

and

$$\Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \equiv \mathsf{PA} \equiv \Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} + \mathsf{J} + \forall\mathsf{N} + \Sigma^+\mathsf{S\text{-}C} + \mathsf{Pow}^-.$$

where $\Sigma^+\mathsf{ET}$ denotes the explicit type axioms, including basic operations and numbers BON^-, representations axioms $\mathcal{A}_{\mathrm{Rep}}$ and generator axioms $\mathcal{A}_{\Sigma^+\mathsf{E}}$ for types, $\mathsf{T\text{-}I_N}$ and $\mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N}$ are type and full formula induction respectively, J is the principle of disjoint union, $\forall\mathsf{N}$ states that all individuals are natural numbers, $\Sigma^+\mathsf{S\text{-}C}$ is positive existential stratified comprehension and Pow^- denotes the weak power type principle. By a positive existential stratified formula, we mean a positive formula wherein neither the naming relation $\mathfrak{R}$ nor the universal quantifier $\forall$ (for individuals and types) does appear, but existential quantifiers $\exists$ (for individuals and types) are allowed.

In this thesis we add the principle of uniform monotone inductive definitions UMID, which basically states that each monotone operation on types has a fixed point which can be uniformly found from the operation itself. Using an idea of [Tak89] we prove first the consistency of our theory (using the Myhill-Shepherdson theorem) and we will afterwards compare its proof-theoretic strength with well-known systems of Arithmetic. They are introduced in the second section of this thesis.

First questions about the proof-theoretic strength of UMID and related principles can be found in [Fef82], where the principle was also introduced. The results in the following years were mainly concerned about much stronger systems of Explicit Mathematics than the one we will be using. It turned out that UMID is a rather powerful principle amidst these strong systems of Explicit Mathematics. You can find a precise account of this in the introduction to the section about proof theory.

Despite the knowledge that UMID proved to be very strong in other settings, we initially conjectured that it loses all of its strength in our weak systems of Explicit Mathematics, i.e.

$$\mathsf{PRA}^+ \equiv T + \mathsf{T\text{-}I_N}$$

and

$$\mathsf{PA} \equiv T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N}$$

(where $T = \Sigma^+\mathsf{ET} + \Sigma^+\mathsf{S\text{-}C} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}$) of which we were only able to prove the latter, whereas we got the following upper and lower bounds for $T + \mathsf{T\text{-}I_N}$

$$\mathsf{PRA}^+ \leq T + \mathsf{T\text{-}I_N} \leq \mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind}.$$

So the question about the exact proof-theoretic strength of $T + \mathsf{T\text{-}I_N}$ remains open. We still conjecture $\mathsf{PRA}^+ \equiv T + \mathsf{T\text{-}I_N}$, but would like to remind you to allow a slight possibility of $\mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind} \equiv T + \mathsf{T\text{-}I_N}$, since UMID has proved to be prone to surprises, as both Feferman in [Fef82] and Takahashi in [Tak89] admit.

1.1 A Word about Inductive Definitions

Inductive definitions are used in mathematics every day, you find them in this thesis for e.g. terms or formulas (as well as in almost any other text about mathematical logic), but also outside the foundations of mathematics, e.g. in abstract algebra to define the ideal generated by a subset of a ring.

Usually inductive definitions begin something like "the set X is inductively defined as follows" or "the set X is the least set closed under the following conditions" followed by something like

(1) All objects having property φ are in X

(2) If $x_1, \dots, x_n$, then $\psi(x_1, \dots, x_n)$ are in X[1]

(3) ...

(4) More conditions in the style of 2

(5) ...

If you like more specific examples, we could e.g. define the set

$$X = \{a, b, aaa, aba, bab, bbb, aaaaa, aabaa, abbba, ababa, babab, \dots\}$$

inductively as follows

(1) a and b are in X.

(2) If $x \in X$, then $axa \in X$ and $bxb \in X$.

It can easily be seen, that we can assign a formula $\varphi(x, S)$ (in an appropriate second-order language) to the inductive definition which looks something like

$$x = a \lor x = b \lor (\exists y)(y \in S \land (x = aya \lor x = byb))$$

With some further thinking you may convince yourself that such a formula can be found for all inductive definitions (or at least for all inductive definitions that deserve that name). Furthermore you can assign to such a formula $\varphi(x, S)$ an operator Γ_φ (mapping sets to sets, but this depends on the actual framework you are working in, so we will not go into details here) defined by $\Gamma_\varphi(S) = \{x \;;\; \varphi(x, S) \text{ holds }\}$.

As it turns out, most (or even almost all) of the inductive definitions used in everyday's mathematics define a monotone operator Γ_φ, i.e. if $S \subseteq S'$ then $\Gamma_\varphi(S) \subseteq$

[1] where ψ is some kind of transformation or mapping of the objects $x_1, \dots, x_n$

$\Gamma_\varphi(S')$. This actually depends on the fact that in such inductive definitions the (relation) variable S only occurs positively in φ. Hence such inductive definitions are called monotone inductive definitions. Having the monotonicity of Γ_φ at hand, we can use the classical set-theoretic results to show that Γ_φ has a least fixed point, which is the set we wanted to define inductively.

Thus, talking about (monotone) inductive definitions is actually talking about (monotone) operators and their fixed points, which is—as you will see in 2.3.15—the way this is done in the framework of Explicit Mathematics. For more information about monotone inductive definitions see Moschovakis' [Mos74a] and for further information about their non-monotone cousins see [Mos74b].

2 Explicit Mathematics

In this section some systems of Explicit Mathematics will be presented. The terminology and most notations are equal to Krähenbühl's [Krä06].

Systems of Explicit Mathematics were introduced by Feferman in [Fef75] as a formal logical framework for Bishop style constructivism, as is explained in [Fef79]. We will follow the approach of Feferman and Jäger in [FJ93], [FJ96] and [Jäg88]. This means we are using a two-sorted version of Beeson's logic of partial terms (see [Bee85]) as our underlying logic, in order to directly use the application dot $\cdot$ instead of the application relation App. Furthermore a naming relation $\mathfrak{R}(x, X)$ replaces the older $x = X$ and we have generator axioms and comprehension terms instead of a comprehension axiom scheme.

In Explicit Mathematics we have two sorts of objects, namely individuals and types. Types can be considered as some kind of set, i.e. a collection of individuals. Each type has a name, a concept which will become perfectly clear once we have built a recursion-theoretic model of our theory. There the types will be recursively enumerable sets and the names will be the indices of the sets. Explicit Mathematics does also contain combinatorial logic (and hence also lambda calculus) as well as means for talking about natural numbers.

Different systems of Explicit Mathematics are mainly determined by what kind of types can be constructed and the amount of induction. The types that can be constructed are determined by the amount of comprehension you allow and by other principles such as e.g. inductive generation, join and inductive definitions. In our approach we will only consider systems with positive existential stratified comprehension, which means in particular that complements of types can not be constructed in general. These systems are much weaker than Feferman's T_0 which contained full elementary comprehension and an additional axiom for inductive generation (see [Fef75]). In [Fef75] we can also find the construction of a recursion-theoretic model as well as a set-theoretic interpretation of T_0. In [Fef79] the system EM_0 is introduced as a subsystem of T_0. In our notation EM_0 is $\mathsf{EET}^- + \mathsf{E\text{-}C} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I}_{\mathsf{N}}$. Also a more general model construction can be found in [Fef79], namely the construction of a model of T_0 over any model of APP (which is BON^- in our notation), as well as an investigation of role of the join axiom J and power type axioms Pow.

The principles of monotone inductive definition MID and UMID appear in [Fef82]. As mentioned before, they basically state the existence of fixed points for monotone type operations. While MID only asserts the existence of fixed points, UMID is more in the spirit of Explicit Mathematics as it gives an operator lfp which assigns to an operation its fixed point (which is what is meant by "uniformly"). Weaker versions of MID and UMID are MID_A and UMID_A where you restrict inductive definitions to subtypes of the type A. The most prominent example of this might be $\mathsf{UMID}_{\mathsf{N}}$ which gives fixed points only for operations from subsets of the natural numbers to subsets of the natural numbers.

2.1 Syntax

2.1.1 Language, Terms and Formulas

Definition 2.1.1 (Language of Explicit Mathematics)
The language of Explicit Mathematics is given by

(1) A countable set $\mathcal{V}_I$ of individual variables. The individual variables will usually be denoted by lower-case letters $a, b, c,\ f, g, h$ or u, v, w, x, y, z (possibly with subscript).

(2) A countable set $\mathcal{V}_T$ of type variables. The type variables will usually be denoted by upper-case letters $A, B, C,\ U, V, W$ or X, Y, Z (also possibly with subscripts).

(3) The set $\mathcal{C}^{\mathsf{E}} = \{\mathsf{k}, \mathsf{s}, \mathsf{p}, \mathsf{p}_0, \mathsf{p}_1, 0, \mathsf{s_N}, \mathsf{p_N}, \mathsf{d_N}, \mathsf{nat}, \mathsf{id}, \mathsf{neg}, \mathsf{con}, \mathsf{dis}, \mathsf{dom}, \mathsf{inv}, \mathsf{j}, \mathsf{lfp}\}$ of constant symbols.

(4) The function symbol $\cdot$ (centered dot).

(5) The relation symbols $\downarrow, \mathsf{N}, \in, =, \mathfrak{R}$.

(6) The logical symbols $\neg, \vee, \wedge, \exists, \forall$.

(7) The auxiliary symbols (,), , (left bracket, right bracket, comma).

Definition 2.1.2 (Individual Terms $\mathcal{T}^{\mathsf{E}}$)
The set of individual terms $\mathcal{T}^{\mathsf{E}}$ of Explicit Mathematics is defined inductively as follows:

(1) Each individual variable $x \in \mathcal{V}_I$ and each constant symbol in $\mathcal{C}^{\mathsf{E}}$ is a term.

(2) If s, t are terms, then $\cdot(s, t)$ is a term.

Remark 2.1.3 Since we will not define type terms, we will call individual terms simply terms.

Notation 2.1.4
The function symbol $\cdot$ is usually written in infix notation or even omitted, i.e. instead of $\cdot(s, t)$ we write $s \cdot t$ or st.

Furthermore, by using the convention of association to the left, we will write $s_0 \cdot s_1 \cdot \ldots \cdot s_n$ or $s_0 s_1 \ldots s_n$ instead of $(\ldots(s_0 \cdot s_1) \cdot \ldots \cdot s_n)$.

Definition 2.1.5 (Atomic Formulas $\mathcal{F}^{\mathsf{E}}_0$, Formulas $\mathcal{F}^{\mathsf{E}}$, Theories)
The set of formulas $\mathcal{F}^{\mathsf{E}}$ of Explicit Mathematics is defined inductively as follows

(1) If $s, t \in \mathcal{T}^{\mathsf{E}}$ are terms and $X \in \mathcal{V}_T$ is a type variable, then

- $=(s, t)$

- $t\downarrow$
- $\mathsf{N}(t)$
- $\in(t, X)$
- $\mathfrak{R}(t, X)$

are formulas.

(2) If φ, ψ are formulas, then

- $\neg\varphi$
- $(\varphi \vee \psi)$
- $(\varphi \wedge \psi)$

are formulas.

(3) If φ is a formula, $x \in \mathcal{V}_I$ an individual variable and $X \in \mathcal{V}_T$ a type variable, then

- $(\exists x)\varphi$
- $(\forall x)\varphi$
- $(\exists X)\varphi$
- $(\forall X)\varphi$

are formulas.

The set of atomic formulas $\mathcal{F}_0^{\mathsf{E}}$ of Explicit Mathematics is defined as the set of all formulas satisfying only the first clause of the inductive definition above.

Sets of formulas $T \subseteq \mathcal{F}^{\mathsf{E}}$ of Explicit Mathematics are called theories of Explicit Mathematics.

Notation 2.1.6
The atomic formulas $=(s, t), \mathsf{N}(t), \in(t, X)$ are usually written as $s = t, t \in \mathsf{N}, t \in X$ respectively.

As usual, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ will be used as abbreviation for $(\neg\varphi \vee \psi)$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively. Brackets will also be omitted by the usual conventions.

We will also write $(\exists\vec{x})\varphi$ and $(\exists\vec{X})\varphi$ for $(\exists x_0)\ldots(\exists x_n)\varphi$ and $(\exists X_0)\ldots(\exists X_n)\varphi$, respectively if the number of variables is clear from the context. The same holds for $\forall$.

Definition 2.1.7 (Free Individual and Type Variables FV_I, FV_T)
The set of free individual variables $\mathrm{FV}_I(t)$ is inductively defined as follows

(1) if t is $c \in \mathcal{C}^{\mathsf{E}}$, then $\mathrm{FV}_I(t) = \emptyset$

(2) if t is $x \in \mathcal{V}_I$, then $\mathrm{FV}_I(t) = \{x\}$

(3) if t is $r \cdot s$ and $r, s \in \mathcal{T}^{\mathsf{E}}$ are terms, then $\mathrm{FV}_I(t) = \mathrm{FV}_I(r) \cup \mathrm{FV}_I(s)$

for all terms $t \in \mathcal{T}^{\mathsf{E}}$.

The set of free individual variables $\mathrm{FV}_I(\varphi)$ and the set of free type variables $\mathrm{FV}_T(\varphi)$ are inductively defined as follows

(1) if φ is $s = t$ and $s, t \in \mathcal{T}^{\mathsf{E}}$ are terms, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(s) \cup \mathrm{FV}_I(t)$ and $\mathrm{FV}_T(\varphi) = \emptyset$

(2) if φ is $t\downarrow$ or $t \in \mathsf{N}$ and $t \in \mathcal{T}^{\mathsf{E}}$ is a term, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(t)$ and $\mathrm{FV}_T(\varphi) = \emptyset$

(3) if φ is $t \in X$ or $\mathfrak{R}(t, X)$ and $t \in \mathcal{T}^{\mathsf{E}}$ is a term and $X \in \mathcal{V}_T$ is a type variable, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(t)$ and $\mathrm{FV}_T = \{X\}$

(4) if φ is $\neg\psi$ and $\psi \in \mathcal{F}^{\mathsf{E}}$ is a formula, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(\psi)$ and $\mathrm{FV}_T(\varphi) = \mathrm{FV}_T(\psi)$

(5) if φ is $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$ and $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{E}}$ are formulas, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(\psi_1) \cup \mathrm{FV}_I(\psi_2)$ and $\mathrm{FV}_T(\varphi) = \mathrm{FV}_T(\psi_1) \cup \mathrm{FV}_T(\psi_2)$

(6) if φ is $(\forall x)(\psi)$ or $(\exists x)(\psi)$ and $\psi \in \mathcal{F}^{\mathsf{E}}$ is a formula and $x \in \mathcal{V}_I$ is an individual variable, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(\psi) \setminus \{x\}$ and $\mathrm{FV}_T(\varphi) = \mathrm{FV}_T(\psi)$

(7) if φ is $(\forall X)(\psi)$ or $(\exists X)(\psi)$ and $\psi \in \mathcal{F}^{\mathsf{E}}$ is a formula and $X \in \mathcal{V}_T$ is an type variable, then $\mathrm{FV}_I(\varphi) = \mathrm{FV}_I(\psi)$ and $\mathrm{FV}_T(\varphi) = \mathrm{FV}_T(\psi) \setminus \{X\}$

for all formulas $\varphi \in \mathcal{F}^{\mathsf{E}}$.

Definition 2.1.8 (Closed Terms)
Let $t \in \mathcal{T}^{\mathsf{E}}$ be a term. t is called a closed term if and only if $\mathrm{FV}_I(t) = \emptyset$.

Definition 2.1.9 (Term Substitution, Type Variable Substitution)
Let $x_0, \ldots, x_n \in \mathcal{V}_I$ be individual variables, $X_0, \ldots, X_n, Y_0, \ldots, Y_n \in \mathcal{V}_T$ type variables, $s, t_0, \ldots, t_n \in \mathcal{T}^{\mathsf{E}}$ terms and $\varphi \in \mathcal{F}^{\mathsf{E}}$ a formula of Explicit Mathematics. Substitutions are defined in the usual way (involving a quite unspectacular but long inductive definition) and are denoted by

$$s[t_0/x_0, \ldots, t_n/x_n] \text{ or } s[\vec{t}/\vec{x}],$$

$$\varphi[t_0/x_0, \ldots, t_n/x_n] \text{ or } \varphi[\vec{t}/\vec{x}],$$

$$s[Y_0/X_0, \ldots, Y_n/X_n] \text{ or } s[\vec{Y}/\vec{X}]$$

and

$$\varphi[Y_0/X_0, \ldots, Y_n/X_n] \text{ or } \varphi[\vec{Y}/\vec{X}].$$

Definition 2.1.10 (Substitutable Terms and Type Variables FT_T, FT_I)
Let $x \in \mathcal{V}_I$ be an individual variable, $X \in \mathcal{V}_T$ a type variable and $\varphi \in \mathcal{F}^{\mathsf{E}}$ a formula of Explicit Mathematics. The sets $\mathrm{FT}_I(x, \varphi)$ and $\mathrm{FT}_T(X, \varphi)$ of substitutable terms and substitutable type variables are defined inductively as follows

(1) If $\varphi \in \mathcal{F}_0^{\mathsf{E}}$, then $\mathrm{FT}_I(x, \varphi) = \mathcal{T}^{\mathsf{E}}$ and $\mathrm{FT}_T(X, \varphi) = \mathcal{V}_T$.

(2) If φ is $\neg\psi$, then $\mathrm{FT}_I(x, \varphi) = \mathrm{FT}_I(x, \psi)$ and $\mathrm{FT}_T(X, \varphi) = \mathrm{FT}_T(X, \psi)$.

(3) If φ is $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$, then $\mathrm{FT}_I(x, \varphi) = \mathrm{FT}_I(x, \psi_1) \cap \mathrm{FT}_I(x, \psi_2)$ and $\mathrm{FT}_T(X, \varphi) = \mathrm{FT}_T(X, \psi_1) \cap \mathrm{FT}_T(X, \psi_2)$.

(4) If φ is $(\exists y)\psi$ or $(\forall y)(\psi)$, then

$$\mathrm{FT}_I(x, \varphi) = \begin{cases} \mathcal{T}^{\mathsf{E}} & \text{if } y = x \\ \{t \in \mathcal{T}^{\mathsf{E}} \; ; \; y \notin \mathrm{FV}_I(t)\} \cap \mathrm{FT}_I(x, \psi) & \text{if } y \neq x \end{cases}$$

and $\mathrm{FT}_T(X, \varphi) = \mathrm{FT}_T(X, \psi)$.

(5) If φ is $(\exists Y)\psi$ or $(\forall Y)(\psi$, then $\mathrm{FT}_I(x, \varphi) = \mathrm{FT}_I(x, \psi)$ and

$$\mathrm{FT}_I(X, \varphi) = \begin{cases} \mathcal{V}_T & \text{if } Y = X \\ \mathrm{FT}_T(X, \psi) \setminus \{Y\} & \text{if } Y \neq X \end{cases} .$$

Remark 2.1.11 The idea of the definition above is of course that for each formula $\varphi \in \mathcal{F}^{\mathsf{E}}$, all individual variables $\vec{x} \in \mathcal{V}_I$, all type variables $\vec{X}, \vec{Y} \in \mathcal{V}_T$ and all terms $\vec{t} \in \mathcal{T}^{\mathsf{E}}$ we will use $\varphi[\vec{x}/\vec{t}]$ and $\varphi[\vec{X}/\vec{Y}]$ if and only if $t_i \in \mathrm{FT}_I(x_i, \varphi)$ and $Y_i \in \mathrm{FT}_I(X_i, \varphi)$, respectively. If you find a formula with a substitution but can not see this condition near it, then it will most likely have been forgotten and we beg you to forgive us this error.

2.1.2 Logical Axioms and Proofs

Definition 2.1.12 (Propositional Axioms, $\mathcal{A}^{\mathsf{E}}_{\mathrm{Prop}}$)
The set of propositional axioms $\mathcal{A}^{\mathsf{E}}_{\mathrm{Prop}}$ is defined as the set consisting of

(1) $(\alpha \wedge \beta) \rightarrow \alpha$

(2) $(\alpha \wedge \beta) \rightarrow \beta$

(3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$

(4) $\alpha \rightarrow (\alpha \vee \beta)$

(5) $\beta \rightarrow (\alpha \vee \beta)$

(6) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$

(7) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(8) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(9) $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

(10) $\neg\neg\alpha \to \alpha$

for all formulas $\alpha, \beta, \gamma \in \mathcal{F}^{\mathsf{E}}$.

Definition 2.1.13 (Equality Axioms, $\mathcal{A}^{\mathsf{E}}_{\mathbf{Eq}}$)
The set of equality axioms $\mathcal{A}^{\mathsf{E}}_{\mathrm{Eq}}$ is defined as the set consisting of

(1) $z = z$

(2) $x_0 = y_0 \land \ldots \land x_n = y_n \land \varphi \to \varphi[\vec{y}/\vec{x}]$

for all individual variables $z, x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{V}_I$ and atomic formulas $\varphi \in \mathcal{F}^{\mathsf{E}}_0$.

Definition 2.1.14 (Quantifier Axioms, $\mathcal{A}^{\mathsf{E}}_{\mathbf{Quant}}$)
The set of quantifier axioms $\mathcal{A}^{\mathsf{E}}_{\mathrm{Quant}}$ is defined as the set consisting of

(1) $\varphi[t/x] \land t\downarrow \to (\exists x)\varphi$

(2) $(\forall x)\varphi \land t\downarrow \to \varphi[t/x]$

(3) $\psi[Y/X] \to (\exists X)\psi$

(4) $(\forall X)\psi \to \psi[Y/X]$

for all formulas $\varphi, \psi \in \mathcal{F}^{\mathsf{E}}$, for all individual variables $x \in \mathcal{V}_I$, for all terms $t \in \mathrm{FT}(x, \varphi)$, for all type variables $X \in \mathcal{V}_T$ and for all type variables $Y \in \mathrm{FT}(X, \psi)$.

Definition 2.1.15 (Definedness Axioms, $\mathcal{A}^{\mathsf{E}}_{\mathbf{Def}}$)
The set of definedness axioms $\mathcal{A}^{\mathsf{E}}_{\mathrm{Def}}$ is defined as the set consisting of

(1) $r\downarrow$

(2) $(s \cdot t)\downarrow \to s\downarrow \land t\downarrow$

(3) $\varphi[\vec{t}/\vec{x}] \to t_0\downarrow \land \ldots \land t_n\downarrow$

for all individual variables and constants $r \in \mathcal{V}_I \cup \mathcal{C}^{\mathsf{E}}$, for all atomic formulas $\varphi \in \mathcal{F}^{\mathsf{E}}_0$, for all individual variables $x_0, \ldots, x_i \in \mathcal{V}_I$ with $x_i \in \mathrm{FV}_I(\varphi)$ and for all terms $s, t, t_0, \ldots, t_n \in \mathcal{T}^{\mathsf{E}}$.

Definition 2.1.16 (Rules of Inference, $\mathcal{R}^{\mathsf{E}}$)
The set of rules of inference $\mathcal{R}^{\mathsf{E}}$ is the set consisting of

(1) $\dfrac{\varphi[y/x] \to \psi}{(\exists x)\varphi \to \psi}$

(2) $\dfrac{\psi \to \varphi[y/x]}{\psi \to (\forall x)\varphi}$

(3) $\dfrac{\varphi[Y/X] \to \psi}{(\exists X)\varphi \to \psi}$

(4) $\dfrac{\psi \to \varphi[Y/X]}{\psi \to (\forall X)\varphi}$

(5) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

for all formulas $\varphi, \psi \in \mathcal{F}^{\mathsf{E}}$, for all individual variables $x, y \in \mathcal{V}_I$ with $y \in \mathrm{FT}(x, \varphi) \setminus \mathrm{FV}_I(\psi)$ and for all type variables $X, Y \in \mathcal{V}_T$ with $Y \in \mathrm{FT}(X, \varphi) \setminus \mathrm{FV}_T(\psi)$.

Notation 2.1.17 ($\mathbf{Prem}(R), \mathbf{Conc}(R)$)
Let

$$R = \frac{\varphi_1 \quad \dots \quad \varphi_n}{\psi} \in \mathcal{R}^{\mathsf{E}}$$

be a rule of inference. The formulas $\varphi_1, \dots, \varphi_n$ will be called premises, the formula ψ is called the conclusion. We will denote the set of premises by $\mathrm{Prem}(R) = \{\varphi_1, \dots, \varphi_n\}$ and the conclusion by $\mathrm{Conc}(R) = \psi$.

Definition 2.1.18 (T-Proof, $T \vdash \varphi$)
Let $T \subseteq \mathcal{F}^{\mathsf{E}}$ be a set of formulas and $\varphi_0, \dots, \varphi_n \in \mathcal{F}^{\mathsf{E}}$ formulas. The sequence of formulas $(\varphi_0, \dots, \varphi_n)$ is called a T-proof, if each φ_i satisfies one of the following conditions

(1) $\varphi_i \in T$,

(2) $\varphi_i \in \mathcal{A}^{\mathsf{E}}_{\mathrm{Prop}} \cup \mathcal{A}^{\mathsf{E}}_{\mathrm{Eq}} \cup \mathcal{A}^{\mathsf{E}}_{\mathrm{Quant}} \cup \mathcal{A}^{\mathsf{E}}_{\mathrm{Def}}$,

(3) there is a $R \in \mathcal{R}^{\mathsf{E}}$ such that $\varphi = \mathrm{Conc}(R)$ and $Prem(R) \subseteq \{\varphi_0, \dots, \varphi_{i-1}\}$.

The proof relation $\vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{E}}) \times \mathcal{F}^{\mathsf{E}}$ is defined as follows:

$T \vdash \varphi$ for all sets of formulas $T \subseteq \mathcal{F}^{\mathsf{E}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{E}}$ if and only if there is a T-proof $(\varphi_0, \dots, \varphi_n)$ such that $\varphi_n = \varphi$.

Definition 2.1.19 (Consistency)
Let $T \subseteq \mathcal{F}^{\mathsf{E}}$ be a set of formulas. T is called consistent if and only if there is a formula $\varphi \in \mathcal{F}^{\mathsf{E}}$ such that there is no T-proof of φ.

2.2 Semantics

2.2.1 Structures

Definition 2.2.1 (Structure of Explicit Mathematics)
A structure of Explicit Mathematics $\mathcal{M}$ consists of

(1) A set M called domain of individuals.

(2) An object $\infty \notin M$ called extra individual.

(3) A set $T \subseteq \mathcal{P}(M)$ of subsets of M called domain of types.

(4) For every constant $c \in \mathcal{C}^{\mathsf{E}}$ an element $c^{\mathcal{M}} \in M$.

(5) A binary operation $\cdot^{\mathcal{M}} : M^\infty \times M^\infty \to M^\infty$ such that for all $x \in M^\infty$

$$\infty \cdot^{\mathcal{M}} x = x \cdot^{\mathcal{M}} \infty = \infty$$

holds, where $M^\infty := M \cup \{\infty\}$.

(6) An unary relation $\mathsf{N}^{\mathcal{M}} \subseteq M$.

(7) A binary relation $\mathfrak{R}^{\mathcal{M}} \subseteq M \times T$.

The sets M, T and M^∞ will usually be denoted as $|\mathcal{M}|^I$, $|\mathcal{M}|^T$ and $|\mathcal{M}|^\infty$, respectively.

The class of all structures of Explicit Mathematics will be denoted $\mathbb{M}^{\mathsf{E}}$.

Definition 2.2.2 (Valuations $\nu \in \mathbb{V}^{\mathcal{M}}$)
Let $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ be a structure of Explicit Mathematics. A valuation ν for the structure $\mathcal{M}$ is a mapping

$$\nu : \mathcal{V}_I \cup \mathcal{V}_T \to |\mathcal{M}|^I \cup |\mathcal{M}|^T$$

satisfying

(1) if $x \in \mathcal{V}_I$, then $\nu(x) \in |\mathcal{M}|^I$,

(2) if $X \in \mathcal{V}_T$, then $\nu(X) \in |\mathcal{M}|^T$.

The set of all valuations for the structure $\mathcal{M}$ is denoted by $\mathbb{V}^{\mathcal{M}}$.

Let $\nu \in \mathbb{V}^{\mathcal{M}}$, $u \in \mathcal{V}_I$ (or $u \in \mathcal{V}_T$) and $m \in |\mathcal{M}|^I$ (or $m \in |\mathcal{M}|^T$, respectively). $\nu[u : m]$ denotes the following valuation for $\mathcal{M}$

$$\nu[u : m](v) := \begin{cases} m, & \text{if } v = u, \\ \nu(v), & \text{otherwise.} \end{cases}$$

Definition 2.2.3 (Interpretations $\mathcal{M}_\nu$)
Let $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ be a structure of Explicit Mathematics and $\nu \in \mathbb{V}^{\mathcal{M}}$ a valuation for $\mathcal{M}$. An interpretation $\mathcal{M}_\nu$ for $\mathcal{M}$ and ν is a mapping

$$\mathcal{M}_\nu : \mathcal{T}^{\mathsf{E}} \to |\mathcal{M}|^\infty$$

satisfying

$$\mathcal{M}_\nu := \begin{cases} t^{\mathcal{M}}, & \text{if } t \in \mathcal{C}^{\mathsf{E}}, \\ \nu(t), & \text{if } t \in \mathcal{V}_I, \\ \mathcal{M}_\nu(r) \cdot^{\mathcal{M}} \mathcal{M}_\nu(s), & \text{if } t = (r \cdot s). \end{cases}$$

An interpretation $\mathcal{M}_\nu$ induces a mapping (which will be given the same name as the interpretation)

$$\mathcal{M}_\nu : \mathcal{F}^{\mathsf{E}} \to \{\mathsf{t}, \mathsf{f}\}$$

which is inductively defined by

$$\begin{aligned}
\mathcal{M}_\nu(s = t) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(s) = \mathcal{M}_\nu(t) \neq \infty \\
\mathcal{M}_\nu(t\downarrow) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(t) \neq \infty \\
\mathcal{M}_\nu(t \in \mathsf{N}) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(t) \in \mathsf{N}^{\mathcal{M}} \\
\mathcal{M}_\nu(t \in X) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(t) \in \nu(X) \\
\mathcal{M}_\nu(\mathfrak{R}(t, X)) = \mathsf{t} &:\Leftrightarrow (\mathcal{M}_\nu(t), \nu(X)) \in \mathfrak{R}^{\mathcal{M}} \\
\mathcal{M}_\nu(\neg\varphi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{f} \\
\mathcal{M}_\nu(\varphi \vee \psi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ or } \mathcal{M}_\nu(\psi) = \mathsf{t} \\
\mathcal{M}_\nu(\varphi \wedge \psi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ and } \mathcal{M}_\nu(\psi) = \mathsf{t} \\
\mathcal{M}_\nu((\exists x)\varphi) = \mathsf{t} &:\Leftrightarrow \text{there is a } m \in |\mathcal{M}|^I \text{ such that } \mathcal{M}_{\nu[x:m]}(\varphi) = \mathsf{t} \\
\mathcal{M}_\nu((\forall x)\varphi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_{\nu[x:m]}(\varphi) = \mathsf{t} \text{ for all } m \in |\mathcal{M}|^I \\
\mathcal{M}_\nu((\exists X)\varphi) = \mathsf{t} &:\Leftrightarrow \text{there is a } S \in |\mathcal{M}|^T \text{ such that } \mathcal{M}_{\nu[X:S]}(\varphi) = \mathsf{t} \\
\mathcal{M}_\nu((\forall X)\varphi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_{\nu[X:S]}(\varphi) = \mathsf{t} \text{ for all } S \in |\mathcal{M}|^T
\end{aligned}$$

for all terms $t, s \in \mathcal{T}^{\mathsf{E}}$, individual variables $x \in \mathcal{V}_I$, type variables $X \in \mathcal{V}_T$ and formulas $\varphi, \psi \in \mathcal{F}^{\mathsf{E}}$.

Definition 2.2.4 (Model relation $\models$)
The model relation $\models \subseteq \mathbb{M}^{\mathsf{E}} \times \mathcal{F}^{\mathsf{E}}$ is defined by

$$\mathcal{M} \models \varphi \quad :\Leftrightarrow \quad \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ for all } \nu \in \mathbb{V}^{\mathcal{M}}$$

for all structures of Explicit Mathematics $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{E}}$.

The model relation can easily be lifted to a relation $\models \subseteq \mathbb{M}^{\mathsf{E}} \times \mathcal{P}(\mathcal{F}^{\mathsf{E}})$ by

$$\mathcal{M} \models T \quad :\Leftrightarrow \quad \mathcal{M} \models \varphi \text{ for all } \varphi \in T$$

for all structures of Explicit Mathematics $\mathcal{M} \in \mathbb{M}^{\mathsf{E}}$ and sets of formulas $T \subseteq \mathcal{F}^{\mathsf{E}}$.

If $\mathcal{M} \models T$ then $\mathcal{M}$ will be called a model of T.

Definition 2.2.5 (Entailment relation, logical consequence $\Vdash$)
The entailment relation $\Vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{E}}) \times \mathcal{F}^{\mathsf{E}}$ is defined by

$$T \Vdash \varphi \quad :\Leftrightarrow \quad \mathcal{M} \models T \text{ implies } \mathcal{M} \models \varphi \text{ for all } \mathcal{M} \in \mathbb{M}^{\mathsf{E}}$$

for all sets of formulas $T \subseteq \mathcal{F}^{\mathsf{E}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{E}}$.

Again the entailment relation can easily be lifted to a relation $\Vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{E}}) \times \mathcal{P}(\mathcal{F}^{\mathsf{E}})$ by

$$T \Vdash S \quad :\Leftrightarrow \quad T \Vdash \varphi \text{ for all } \varphi \in S$$

for all sets of formulas $S, T \subseteq \mathcal{F}^{\mathsf{E}}$.

2.2.2 Adequacy

Theorem 2.2.6 (Adequacy)
Let $\varphi \in \mathcal{F}^{\mathsf{E}}$ and $T \subseteq \mathcal{F}^{\mathsf{E}}$, then

$$T \Vdash \varphi \quad \textit{if and only if} \quad T \vdash \varphi.$$

Proof. See [Bee85]. Correctness is done as usual by induction on the length of the proof. For completeness you may embed the logic of partial terms into classical logic using $n+1$-ary predicates for n-ary partial functions and then use the completeness theorem for classical logic. Another approach using deduction chains can be found in e.g. [Sal01]. □

2.3 Theories and Principles

2.3.1 Basic Operations and Numbers

Notation 2.3.1
Let $x, x_0, \dots, x_k \in \mathcal{V}_I$ be pairwise disjoint individual variables and $s, s_0, \dots, s_n$ as well as $t, t_0, \dots, t_n \in \mathcal{T}^{\mathsf{E}}$ terms. We will use the following abbreviations

$$
\begin{aligned}
s \neq t &\text{ for } s\downarrow \wedge\, t\downarrow \wedge\, \neg(s = t)\\
s \simeq t &\text{ for } (s\downarrow \vee\, t\downarrow) \to s = t\\
\langle s_0, \dots, s_n\rangle &\text{ for } \begin{cases} s_0, & \text{if } n = 0\\ (\mathsf{p}\langle s_0, \dots, s_{n-1}\rangle s_n) & \text{if } n > 0\end{cases}\\
(\exists x \in \mathsf{N})\varphi &\text{ for } (\exists x)(x \in \mathsf{N} \wedge \varphi)\\
(\forall x \in \mathsf{N})\varphi &\text{ for } (\forall x)(x \in \mathsf{N} \to \varphi)\\
t \in (\mathsf{N}^{k+1} \to \mathsf{N}) &\text{ for } (\forall x_0 \in \mathsf{N}) \dots (\forall x_n \in \mathsf{N}) t x_0 \dots x_k \in \mathsf{N}\\
t \in (\mathsf{N} \to \mathsf{N}) &\text{ for } t \in (\mathsf{N}^1 \to \mathsf{N}).
\end{aligned}
$$

Definition 2.3.2 (Basic Operations and Numbers BON^-)
The theory of basic operations and numbers BON^- is defined as the set consisting of

(1) Partial combinatory algebra

  (a) $(\mathsf{k}x)y = x$

  (b) $\mathsf{s}xy\downarrow \wedge\, (\mathsf{s}xy)z \simeq (xz)(yz)$

(2) Pairing and projection

  (a) $\mathsf{p}_0\langle x, y\rangle = x \wedge \mathsf{p}_1\langle x, y\rangle = y$

  (b) $\mathsf{0} \in \mathsf{N} \wedge \mathsf{s_N} \in (\mathsf{N} \to \mathsf{N})$

(3) Natural numbers

  (a) $(\forall x \in \mathsf{N})(\mathsf{s_N}x \neq \mathsf{0} \wedge \mathsf{p_N}(\mathsf{s_N}x) = x)$

  (b) $(\forall x \in \mathsf{N})(x \neq \mathsf{0} \to \mathsf{p_N}x \in \mathsf{N} \wedge \mathsf{s_N}(\mathsf{p_N}x) = x)$

(4) Definition by numerical cases

  (a) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x = y \to (\mathsf{d_N}uv)xy = u$

  (b) $x \in \mathsf{N} \wedge y \in \mathsf{N} \wedge x \neq y \to (\mathsf{d_N}uv)xy = v$

for all individual variables $u, v, x, y, z \in \mathcal{V}_I$.

Remark 2.3.3 The theory BON is BON^- with an additional constant rec and the additional axioms

(5) Primitive recursion

(a) $f \in (\mathsf{N}^2 \to \mathsf{N}) \land a \in \mathsf{N} \land b \in \mathsf{N} \to (\mathsf{rec} f a)\mathsf{0} = a \land (\mathsf{rec} f a)(\mathsf{s_N} b) = fb((\mathsf{rec} f a)b)$

(b) $f \in (\mathsf{N}^2 \to \mathsf{N}) \land a \in \mathsf{N} \to (\mathsf{rec} f a) \in (\mathsf{N} \to \mathsf{N})$

for all individual variables $a, b, f \in \mathcal{V}_I$.

We will see in 2.5.4, that we can prove the first part of the axioms above in BON^- and to prove second part we need some additional axioms, namely $\mathsf{V\text{-}I_N}$.

2.3.2 Explicit Types

Notation 2.3.4
Let $s, t, t_0, \ldots, t_n \in \mathcal{T}^{\mathsf{E}}$ be terms, $x \in \mathcal{V}_I$ an individual variable and $X, X_0, \ldots, X_n \in \mathcal{V}_T$ type variables. We will use the following abbreviations

$$\begin{aligned}
X \subseteq Y &\text{ for } (\forall x)(x \in X \to x \in Y)\\
X = Y &\text{ for } X \subseteq Y \land Y \subseteq X\\
\mathfrak{R}(\vec{t}, \vec{X}) &\text{ for } \mathfrak{R}(t_0, X_0) \land \ldots \land \mathfrak{R}(t_n, X_n)\\
\mathfrak{R}(s) &\text{ for } (\exists X)(\mathfrak{R}(s, X))\\
s \dot{\in} t &\text{ for } (\exists X)(\mathfrak{R}(t, X) \land s \in X)\\
s \dot{\subseteq} t &\text{ for } (\forall x)(x \dot{\in} s \to x \dot{\in} t)\\
s \doteq t &\text{ for } s \dot{\subseteq} t \land t \dot{\subseteq} s
\end{aligned}$$

Definition 2.3.5 (Representation Axioms $\mathcal{A}_{\mathbf{Rep}}$)
The set of representation axioms $\mathcal{A}_{\text{Rep}}$ is defined as the set consisting of

(1) $(\exists x)\mathfrak{R}(x, X)$

(2) $\mathfrak{R}(x, X) \land \mathfrak{R}(x, Y) \to X = Y$

(3) $X = Y \land \mathfrak{R}(x, X) \to \mathfrak{R}(x, Y)$

for all individual variables $x \in \mathcal{V}_I$ and for all type variables $X, Y \in \mathcal{V}_T$.

Definition 2.3.6 (Generator Axioms $\mathcal{A}_{\mathbf{\Sigma+E}}$)
The set of generator axioms $\mathcal{A}_{\mathsf{\Sigma+E}}$ is defined as the set consisting of

(1) $\mathfrak{R}(\mathsf{nat}) \land (\forall x)(x \dot{\in} \mathsf{nat} \leftrightarrow x \in \mathsf{N})$

(2) $\mathfrak{R}(\mathsf{id}) \land (\forall x)(x \dot{\in} \mathsf{id} \leftrightarrow (\exists y)(x = \langle y, y\rangle))$

(3) $\mathfrak{R}(a) \land \mathfrak{R}(b) \to \mathfrak{R}(\mathsf{con}\langle a, b\rangle) \land (\forall x)(x \dot{\in} \mathsf{con}\langle a, b\rangle \leftrightarrow x \dot{\in} a \land x \dot{\in} b)$

(4) $\mathfrak{R}(a) \land \mathfrak{R}(b) \to \mathfrak{R}(\mathsf{dis}\langle a, b\rangle) \land (\forall x)(x \dot{\in} \mathsf{dis}\langle a, b\rangle \leftrightarrow x \dot{\in} a \lor x \dot{\in} b)$

(5) $\mathfrak{R}(a) \to \mathfrak{R}(\mathsf{dom}\langle a\rangle) \land (\forall x)(x \dot{\in} \mathsf{dom}\langle a\rangle \leftrightarrow (\exists y)(\langle x, y\rangle \dot{\in} a))$

(6) $\mathfrak{R}(a) \rightarrow \mathfrak{R}(\mathsf{inv}\langle a, f\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{inv}\langle a, f\rangle \leftrightarrow fx\dot{\in}a)$

for all individual variables $a, b, f, x, y \in \mathcal{V}_I$.

Definition 2.3.7 (Generator Axioms $\mathcal{A}_{\mathsf{E}}$)
The set of generator axioms $\mathcal{A}_{\mathsf{E}}$ is defined as the set consisting of

(1) all axioms of $\mathcal{A}_{\Sigma^+\mathsf{E}}$

(2) $\mathfrak{R}(a) \rightarrow \mathfrak{R}(\mathsf{neg}\langle a\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{neg}\langle a\rangle \leftrightarrow \neg x\dot{\in}a)$

for all individual variables $a, x \in \mathcal{V}_I$.

Definition 2.3.8 (Explicit Types $\Sigma^+\mathsf{ET}$, EET^-)
The sets of explicit type axioms $\Sigma^+\mathsf{ET}, \mathsf{EET}^-$ are defined as

(1) $\Sigma^+\mathsf{ET} := \mathsf{BON}^- \cup \mathcal{A}_{\mathrm{Rep}} \cup \mathcal{A}_{\Sigma^+\mathsf{E}}$.

(2) $\mathsf{EET}^- := \mathsf{BON}^- \cup \mathcal{A}_{\mathrm{Rep}} \cup \mathcal{A}_{\mathsf{E}}$.

Definition 2.3.9 (Disjoint Union J)
The set of disjoint union axioms J is defined as the set consisting of

$$\mathfrak{R}(a)\wedge(\forall x\dot{\in}a)\mathfrak{R}(fx) \rightarrow \mathfrak{R}(\mathsf{j}\langle a, f\rangle) \wedge (\forall x)(x\dot{\in}\mathsf{j}\langle a, f\rangle \leftrightarrow (\exists y)(\exists z)(x = \langle y, z\rangle \wedge y\dot{\in}a \wedge z\dot{\in}fy)).$$

2.3.3 Ontological Principles

Definition 2.3.10 (All Individuals are Names of Types $\forall\mathfrak{R}$)
The set $\forall\mathfrak{R}$ is defined as the set consisting of

$$(\forall x)\mathfrak{R}(x)$$

for all individual variables $x \in \mathcal{V}_I$.

Definition 2.3.11 (All Individuals are Numbers $\forall\mathsf{N}$)
The set $\forall\mathsf{N}$ is defined as the set consisting of

$$(\forall x)\mathsf{N}(x)$$

for all individual variables $x \in \mathcal{V}_I$.

Definition 2.3.12 (Uniform Comprehension $\mathcal{X}$-C)
Let $\mathcal{X} \subseteq \mathcal{F}^{\mathsf{E}}$. The set $\mathcal{X}$-C is defined as the set consisting of

$$(\exists f)(\forall \vec{x}, \vec{y}, \vec{X})[\mathfrak{R}(\vec{x}, \vec{X}) \rightarrow \mathfrak{R}(fx0 \dots x_n y_0 \dots y_m)\wedge(\forall z)(z\dot{\in}fx_0 \dots x_n y_0 \dots y_m \leftrightarrow \varphi)]$$

for all formulas $\varphi \in \mathcal{X}$ where $\mathrm{FV}_T(\varphi) = \{X_0, \dots, X_n\}$ and $\{y_0, \dots, y_m\} = \mathrm{FV}_I(\varphi)\setminus \{z\}$ and all individual variables $x_0, \dots x_n \in \mathcal{V}_I$.

2.3.4 Uniform Monotone Inductive Definition

Notation 2.3.13 (Monotone Operation Mon, Least Fixed Point Lfp)
Let $f \in \mathcal{T}^{\mathsf{E}}$ be a term and $x, y \in \mathcal{V}_I$ individual variables. We will use the following abbreviations

$$\begin{aligned} f \in (\mathfrak{R} \to \mathfrak{R}) &\text{ for } (\forall x)(\mathfrak{R}(x) \to \mathfrak{R}(fx)) \\ \mathrm{Mon}(f) &\text{ for } (\forall x, y)(\mathfrak{R}(x) \land \mathfrak{R}(y) \land x \dot{\subseteq} y \to fx \dot{\subseteq} fy) \\ \mathrm{Lfp}(y, f) &\text{ for } \mathfrak{R}(y) \land fy \dot{\subseteq} y \land (\forall x)(\mathfrak{R}(x) \land fx \dot{\subseteq} x \to y \dot{\subseteq} x) \end{aligned}$$

Remark 2.3.14 Usually, one would require a type operation $f : \mathfrak{R} \to \mathfrak{R}$ to be extensional, i.e. to satisfy $\mathrm{Ext}(f)$, which is an abbreviation for

$$(\forall x)(\forall y)(\mathfrak{R}(x) \land \mathfrak{R}(y) \land x \dot{=} y \to fx \dot{=} fy).$$

But obviously $\mathrm{Mon}(f) \to \mathrm{Ext}(f)$, i.e. monotonicity implies extensionality. Since we will always have extensionality or monotonicity as additional assumption, we will not require it in the definition of type operations.

Definition 2.3.15 (Uniform Monotone Inductive Definition, UMID)
Let $f \in \mathcal{V}_I$ be an individual variable. The principle of uniform monotone inductive definition UMID is defined as

$$(\forall f)(f \in (\mathfrak{R} \to \mathfrak{R}) \land \mathrm{Mon}(f) \to \mathrm{Lfp}(\mathsf{lfp}\langle f \rangle, f)).$$

Remark 2.3.16 (Monotone Inductive Definition, MID*)* The principle of monotone inductive definition is defined by

$$(\forall f)(f \in (\mathfrak{R} \to \mathfrak{R}) \land \mathrm{Mon}(f) \to (\exists x)\mathrm{Lfp}(x, f)),$$

i.e. only the existence of least fixed point is asserted but there are no means to get them uniformly from the type operation f.

2.3.5 Induction Principles

Definition 2.3.17 (Set Induction $\mathsf{S\text{-}I_N}$)
The set of set induction axioms $\mathsf{S\text{-}I_N}$ is defined as the set consisting of

$$f \in (\mathsf{N} \to \mathsf{N}) \land f\mathsf{0} = \mathsf{0} \land (\forall x \in \mathsf{N})(fx = \mathsf{0} \to f(\mathsf{s_N}x) = \mathsf{0}) \to (\forall x \in \mathsf{N})(fx = \mathsf{0})$$

for all individual variables $x, f \in \mathcal{V}_I$.

Definition 2.3.18 (Value Induction $\mathsf{V\text{-}I_N}$)
The set of value induction axioms $\mathsf{V\text{-}I_N}$ is defined as the set consisting of

$$f\mathsf{0} \in \mathsf{N} \land (\forall x \in \mathsf{N})(fx \in \mathsf{N} \to f(\mathsf{s_N}x) \in \mathsf{N}) \to f \in (\mathsf{N} \to \mathsf{N})$$

for all individual variables $x, f \in \mathcal{V}_I$.

Definition 2.3.19 (Type Induction $\mathsf{T}\text{-}\mathsf{I}_\mathsf{N}$)
The set of type induction axioms $\mathsf{T}\text{-}\mathsf{I}_\mathsf{N}$ is defined as the set consisting of

$$\mathsf{0} \in X \wedge (\forall x \in \mathsf{N})(x \in X \rightarrow \mathsf{s_N} x \in X) \rightarrow (\forall x \in \mathsf{N}) x \in X$$

for all individual variables $x \in \mathcal{V}_I$ and type variables $X \in \mathcal{V}_T$.

Definition 2.3.20 (Formula Induction $\mathcal{F}^\mathsf{E}\text{-}\mathsf{I}_\mathsf{N}$)
The set of formula induction axioms $\mathcal{F}^\mathsf{E}\text{-}\mathsf{I}_\mathsf{N}$ is defined as the set consisting of

$$\varphi[\mathsf{0}/x] \wedge (\forall x \in \mathsf{N})(\varphi \rightarrow \varphi[\mathsf{s_N} x/x]) \rightarrow (\forall x \in \mathsf{N})\varphi$$

for all individual variables $x \in \mathcal{V}_I$ and formulas $\varphi \in \mathcal{F}^\mathsf{E}$.

2.4 Consistency Results

The aim of this subsection is to establish the consistency of the theory $\Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}$ by building a model of this theory. The model construction follows the idea of [Tak89], it is basically the classical recursion-theoretic model with some extended thinking about why UMID holds. Takahashi showed in [Tak89] the consistency of the much stronger $\mathsf{T_0}+\mathsf{MID}$ and $\mathsf{T_0}+\mathsf{UMID}$ and pointed out that the Myhill-Shepherdson theorem (see [MS55]) is the key to proving the consistency of $\mathsf{APP} + \Sigma^+\text{-}\mathsf{ECA} + \mathsf{UMID}$ which is $\mathsf{BON}^- + \Sigma^+\mathsf{S\text{-}C} + \mathsf{UMID}$ in our notation.

We will assume some basic recursion-theoretic knowledge, which can be found in any textbook such as [Cut80] or [Rog87]. The notations and proofs which are used in the following are mostly from [Cut80]. Most of the following results are well-known in a slightly different version, as they are usually stated for partial-recursive functions, while we are stating them for recursively enumerable sets. In most textbooks their destiny is to remain hidden in the exercise section until an eager student may find them, but we will pay them a little bit more attention, as they will play a crucial role in the consistency proof as well as in the proof-theoretic embedding.

2.4.1 Recursion-Theoretic Preliminaries

Notation 2.4.1
We will use the following recursion theoretic notations

(1) Let $f : A \to B$ be a partial function, $\mathrm{dom}(f)$ denotes the domain of f, i.e. $\mathrm{dom}(f) = \{x \in A \ ; \ f \text{ defined on } x\}$, $\mathrm{ran}(f)$ denotes the range of f, i.e. $\mathrm{ran}(f) = \{y \in B \ ; \text{ there is a } x \in \mathrm{dom}(f) \text{ such that } f(x) = y)\}$

(2) Let $f : A \to B$ be a partial function and $i \in \mathbb{N}$ a natural number. f^i is defined by
$$f^i(x) := \begin{cases} x & \text{if } i = 0 \\ f(f^{i-1}(x)) & \text{else} \end{cases}$$
for all $x \in \mathbb{N}$, i.e. f^i is the i-th iteration of f.

(3) $\mathcal{F}$ denotes the set of all partial functions f with $\mathrm{dom}(f) \subseteq \mathbb{N}$ and $\mathrm{ran}(f) \subseteq \mathbb{N}$.

(4) Let $n \in \mathbb{N}$, then $\{n\}$ denotes the partial recursive function with index n. $\{\ \}$ is called the Kleene bracket.

(5) $\mathcal{C}$ denotes the set of all partial recursive functions, i.e. $\mathcal{C} = \{\{n\} \ ; \ n \in \mathbb{N}\}$.

(6) $\mathcal{C}_n$ denotes the set of all n-ary partial recursive functions.

(7) $W_n := \mathrm{dom}(\{n\})$ for $n \in \mathbb{N}$.

(8) $\mathcal{S}$ denotes the set of all recursively enumerable sets, i.e. $\mathcal{S} = \{W_n \ ; \ n \in \mathbb{N}\}$.

(9) If $X \subseteq \mathbb{N}^n$, then $\overline{X}$ denotes the complement of X, i.e. $\overline{X} = \mathbb{N}^n \setminus X$.

(10) K denotes the halting problem, i.e. $K = \{x \,;\, x \in W_x\}$.

Theorem 2.4.2 (*s-m-n* Theorem)
Let $f = \{i\} \in \mathcal{C}_{m+n}$ be a $(m+n)$-ary partial recursive function. Then there is a total recursive $(m+1)$-ary function $s_n^m \in \mathcal{C}_{m+1}$ such that

$$f(\vec{x}, \vec{y}) = \{i\}(\vec{x}, \vec{y}) = \{s_n^m(i, \vec{x})\}(\vec{y})$$

for all $\vec{x} \in \mathbb{N}^m$ and $\vec{y} \in \mathbb{N}^n$.

Proof. See [Cut80] or [Rog87]. □

Theorem 2.4.3 (Kleene's Second Recursion Theorem)
Let $f \in \mathcal{C}_1$ be a total recursive function. There is an index $e \in \mathbb{N}$ such that

$$\{e\}(x, \vec{y}) \simeq \{f(e)\}(x, \vec{y})$$

for all $x, \vec{y} \in \mathbb{N}$.

Proof. See [Cut80] or [Rog87]. □

Corollary 2.4.4
Let $n \in \mathbb{N}$ be a natural number and $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_{n+2}$ partial recursive functions. There is a partial recursive function $f \in \mathcal{C}_{n+1}$ such that

$$\begin{aligned} f(0, \vec{y}) &= g(\vec{y}) \\ f(x+1, \vec{y}) &\simeq h(f(x, \vec{y}), x, \vec{y}) \end{aligned}$$

for all $x, y_1, \ldots, y_n \in \mathbb{N}$.

Proof. Let $r \in \mathcal{C}$ be a total recursive function that assigns to each natural number $n \in \mathbb{N}$ an index $r(n)$ such that

$$\begin{aligned} \{r(n)\}(0, \vec{y}) &= g(\vec{y}) \\ \{r(n)\}(x+1, \vec{y}) &\simeq h(\{n\}(x, \vec{y}), x, \vec{y}). \end{aligned}$$

The existence of r is given by an appeal to Church's thesis (or a long and tedious construction).

By 2.4.3 there is an index $e \in \mathbb{N}$ such that

$$\{e\}(x, \vec{y}) \simeq \{r(e)\}(x, \vec{y})$$

for all $x, y_1, \ldots, y_n \in \mathbb{N}$.

You may now easily convince yourself that we can take $\{e\}$ as the function f desired above. □

Theorem 2.4.5 (Rice-Shapiro-Theorem, set version)
Let $\mathcal{T} \subseteq \mathcal{S}$ be a set of recursively enumerable sets such that the set $\{n\ ;\ W_n \in \mathcal{T}\}$ is recursively enumerable.

Then $X \in \mathcal{T}$ if and only if there is a finite $Y \subseteq X$ such that $Y \in \mathcal{T}$.

Proof. Define $T := \{n\ ;\ W_n \in \mathcal{T}\}$. T is recursively enumerable by assumption.

Now assume (towards a contradiction) that there is an $X \in \mathcal{T}$ such that $Y \notin \mathcal{T}$ for all finite $Y \subseteq X$. Since X is recursively enumerable, X has an index i, i.e. $X = W_i$. Furthermore let k be an index of K. Now define the partial recursive function f by

$$f(x, y) = \begin{cases} \text{undefined} & \text{if } \{k\}(x) \text{ stops in } y \text{ or less steps} \\ \{i\}(y) & \text{else} \end{cases}$$

By 2.4.2 there is a total recursive function s such that $f(x, y) \simeq \{s(x)\}(y)$. Obviously we have $\{s(x)\} \subseteq \{k\}$ for all $x \in \mathbb{N}$ by construction. We can now easily check the following implications

$$\begin{aligned} x \in K \Rightarrow\ & \text{there is a minimal } y_0 \text{ such that } \{k\}(x) \text{ stops in } y_0 \text{ steps} \\ \Rightarrow\ & \{s(x)\}(y) \text{ defined implies } y < y_0 \\ \Rightarrow\ & \{s(x)\} \text{ finite} \\ \Rightarrow\ & W_{s(x)} \notin \mathcal{T} \\ x \notin K \Rightarrow\ & \{k\}(x) \text{ is undefined} \\ \Rightarrow\ & \{s(x)\}(y) \simeq \{i\}(y) \text{ for all } y \in \mathbb{N} \\ \Rightarrow\ & W_{s(x)} = W_i = X \in \mathcal{T} \end{aligned}$$

So we have

$$x \in \overline{K} \quad \Leftrightarrow \quad s(x) \in T.$$

Since T is recursively enumerable and s is a total recursive function, this would imply the recursive enumerability of $\overline{K}$. But we know $\overline{K}$ to be not recursively enumerable and so we arrive at a contradiction.

For the other direction assume (again towards a contradiction) that there is a recursively enumerable set $X = W_j$ such that there is a finite $Y \subseteq W_j$ with $Y \in \mathcal{T}$ but $X \notin \mathcal{T}$. Now define the partial recursive function g by

$$g(x, y) = \begin{cases} \{j\}(y) & x \in K \text{ or } y \in Y \\ \text{undefined} & \text{else} \end{cases}$$

By 2.4.2 there is a total recursive function t such that $g(x, y) \simeq \{t(x)\}(y)$. Again we easily check two simple implications

$$\begin{aligned} x \in K \Rightarrow\ & \{t(x)\}(y) \simeq \{j\}(y) \\ \Rightarrow\ & W_{t(x)} = W_j \notin \mathcal{T} \\ x \notin K \Rightarrow\ & \{t(x)\}(y) \text{ is defined if and only if } y \in Y \\ \Rightarrow\ & W_{t(x)} = Y \in \mathcal{T} \end{aligned}$$

So we have again

$$x \in \overline{K} \quad \Leftrightarrow \quad t(x) \in T$$

which leads us to the same contradiction. □

Definition 2.4.6 (continuous)
A mapping $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is called continuous if and only if

$$x \in \Psi(X) \quad \Leftrightarrow \quad \text{there is a finite } Y \subseteq X \text{ such that } x \in \Psi(Y)$$

holds for all $X \subseteq \mathbb{N}$.

Definition 2.4.7 (monotone on indices, extensional on indices)
Let $f \in \mathcal{C}_1$ be a partial recursive function.

(1) f is called monotone on indices if and only if

$$W_m \subseteq W_n \Rightarrow W_{f(m)} \subseteq W_{f(n)}$$

(2) f is called extensional on indices if and only if

$$W_m = W_n \Rightarrow W_{f(m)} = W_{f(n)}$$

Remark 2.4.8 If f is monotone on indices, then f is extensional on indices.

Theorem 2.4.9 (Myhill-Shepherdson, poor man's version)
Let $h \in \mathcal{C}_1$ be a total recursive function which is extensional on indices. Then there is a continuous mapping $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $\Psi(W_n) = W_{h(n)}$ for all $n \in \mathbb{N}$.

Proof. Define a mapping $\Psi_0 : \mathcal{S} \to \mathcal{S}$ by $\Psi_0(W_n) := W_{h(n)}$ for all $n \in \mathbb{N}$. Ψ_0 is well-defined since h is extensional on indices. For $x \in \mathbb{N}$ define the set $\mathcal{T}_x := \{X \in \mathcal{S} \; ; \; x \in \Psi_0(X)\}$. The set $T_x := \{n \; ; \; W_n \in \mathcal{T}_x\} = \{n \; ; \; x \in W_{h(n)}\}$ is obviously recursively enumerable. So we can apply 2.4.5 which yields

$$X \in \mathcal{T}_x \quad \Leftrightarrow \quad \text{there is a finite } Y \subseteq X \text{ such that } Y \in \mathcal{T}_x$$

for a recursively enumerable set $X \in \mathcal{S}$. So we have

$$x \in \Psi_0(X) \quad \Leftrightarrow \quad \text{there is a finite } Y \subseteq X \text{ such that } x \in \Psi_0(Y).$$

Now we can define $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Psi(Z) := \{z \in \mathbb{N} \; ; \; \text{there is a finite } Y \subset Z \text{ such that } z \in \Psi_0(Y)\}$$

for all $Z \subseteq \mathbb{N}$.

Clearly Ψ extends Ψ_0 and is continuous by construction. So we have a mapping with the desired properties. □

Definition 2.4.10 (least fixed point)
Let $f \in \mathcal{F}$ be a total function. A recursively enumerable set W_n is called the least fixed point of f if and only if

(1) $W_{f(n)} = W_n$,

(2) if $W_{f(m)} = W_m$ then $W_n \subseteq W_m$ for all $m \in \mathbb{N}$.

Remark 2.4.11 It makes sense speaking of *the* least fixed point of f, since—as can be easily checked—it is unique.

Theorem 2.4.12
Let $f \in \mathcal{C}_1$ be a total recursive function which is monotone on indices. Then f has a least fixed point W_n.

Proof. Surely the empty set $\emptyset$ is recursively enumerable, so there is an $e \in \mathbb{N}$ such that $W_e = \emptyset$. An easy induction shows, that $W_{f^i(e)} \subseteq W_{f^{i+1}(e)}$ for all $i \in \mathbb{N}$ since $\emptyset = W_e \subseteq W_{f(e)}$ and f is monotone on indices. Note please that the existence of f^i is guaranteed by 2.4.4.

The set $\bigcup_{i\in\mathbb{N}} W_{f^i(e)}$ can be defined by $(\exists i \in \mathbb{N})(x \in W_{f^i(e)})$ and is hence recursively enumerable. Let n be an index of $\bigcup_{i\in\mathbb{N}} W_{f^i(e)}$, i.e. $W_n = \bigcup_{i\in\mathbb{N}} W_{f^i(e)}$.

We will now show, that W_n has the desired properties.

(1) Show $W_{f(n)} = W_n$:

- Show $W_{f(n)} \supseteq W_n$: Let $x \in W_n$. Then there is an $i \in \mathbb{N}$ such that $x \in W_{f^i(e)}$. By definition $W_{f^i(e)} \subseteq W_n$ and by the monotonicity of f $W_{f^{i+1}(e)} \subseteq W_{f(n)}$. But we also have $W_{f^i(e)} \subseteq W_{f^{i+1}(e)}$ and so $x \in W_{f^i(e)} \subseteq W_{f^{i+1}(e)} \subseteq W_{f(n)}$.
- Show $W_{f(n)} \subseteq W_n$: Let $x \in W_{f(n)}$. Then $x \in \mathrm{dom}(\{f(n)\})$. By 2.4.9 f defines a continuous mapping $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ with $\Psi(W_n) = W_{f(n)}$. Since $x \in W_{f(n)}$ and Ψ is continuous there is a finite $Y \subseteq W_n$ such that $x \in \Psi(Y)$.

  Since $W_n = \bigcup_{i\in\mathbb{N}} W_{f^i(e)}$ and $W_{f^i(e)} \subseteq W_{f^{i+1}(e)}$ there must be an $i_0 \in \mathbb{N}$ such that $Y \subseteq W_{f^{i_0}(e)}$ (else Y would not be finite). Let j be an index of Y, i.e. $Y = W_j$. We have $x \in \Psi(Y) = \Psi(W_j) = W_{f(j)}$ and from $Y = W_j \subseteq W_{f^{i_0}(e)}$ and the monotonicity on indices of f we get $W_{f(j)} \subseteq W_{f^{i_0+1}(e)}$. A combination of this yields finally $x \in \Psi(Y) = \Psi(W_j) = W_{f(j)} \subseteq W_{f^{i_0+1}(e)} \subseteq \bigcup_{i\in\mathbb{N}} W_{f^i(e)} = W_n$.

(2) Let W_m be such that $W_{f(m)} = W_m$. By induction on $i \in \mathbb{N}$ we will show that $W_{f^i(e)} \subseteq W_m$ holds for all i. Clearly we have $\emptyset = W_a \subseteq W_m$. Now let $W_{f^i(e)} \subseteq W_m$. By the monotonicity on indices of f we have $W_{f^{i+1}(e)} \subseteq W_{f(m)} = W_m$. So we have $W_{f^i(e)} \subseteq W_m$ for all $i \in \mathbb{N}$ and hence also $W_n = \bigcup_{i\in\mathbb{N}} W_{f^i(e)} \subseteq W_m$.

2.4.2 The Model $\mathcal{M}_{\mathbb{N}}$

Lemma 2.4.13 ($\hat{\mathsf{k}}$)
There is a $\hat{\mathsf{k}} \in \mathbb{N}$ *such that*

$$\{\{\hat{\mathsf{k}}\}(x)\}(y) = x$$

for all $x, y \in \mathbb{N}$.

Proof. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $f(x, y) := x$. f is obviously partial recursive. So we can fix an index $n \in \mathbb{N}$ such that $\{n\} = f$.

By 2.4.2 there is a recursive $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\{n\}(x, y) = \{s(n, x)\}(y)$. A second application of 2.4.2 yields a recursive $t : \mathbb{N} \to \mathbb{N}$ such that $\{t(n)\}(x) = s(n, x)$.

Define $\hat{\mathsf{k}}$ as $t(n)$.

We have

$$\begin{aligned}
\{\{\hat{\mathsf{k}}\}(x)\}(y) &= \{\{t(n)\}(x)\}(y) \\
&= \{s(n, x)\}(y) \\
&= \{n\}(x, y) \\
&= f(x, y) \\
&= x.
\end{aligned}$$ □

Lemma 2.4.14 ($\hat{\mathsf{s}}$)
There is a $\hat{\mathsf{s}} \in \mathbb{N}$ *such that*

$$dom(\{\{\hat{\mathsf{s}}\}(x)\}) = \mathbb{N}$$

and

$$\{\{\hat{\mathsf{s}}\}(x)\}(y) \simeq \{\{x\}(z)\}(\{y\}(z))$$

for all $x, y, z \in \mathbb{N}$.

Proof. Similar to the proof above, but slightly more involved. □

Lemma 2.4.15 ($\hat{\mathsf{p}}, \hat{\mathsf{p}}_0, \hat{\mathsf{p}}_1$)
There are $\hat{\mathsf{p}}, \hat{\mathsf{p}}_0, \hat{\mathsf{p}}_1 \in \mathbb{N}$ *such that*

$$\{p_0\}(\{\{p\}(x)\}(y)) = x$$

and

$$\{p_1\}(\{\{p\}(x)\}(y)) = y$$

Proof. Using a recursive pairing function and 2.4.2 one easily finds indices of recursive functions satisfying the desired properties. □

Notation 2.4.16
We will use the following abbreviation

$$\langle x_0, \dots, x_n \rangle := \begin{cases} x_0, & \text{if } n = 0 \\ \{\{\hat{\mathsf{p}}\}(\langle s_0, \dots, s_{n-1} \rangle)\}(s_n) & \text{if } n > 0 \end{cases}$$

for all natural number $n \in \mathbb{N}$ and $x_0, \dots x_n \in \mathbb{N}$.

Lemma 2.4.17 ($\hat{\mathsf{0}}, \hat{\mathsf{s_N}}, \hat{\mathsf{p_N}}$)
There are $\hat{\mathsf{0}}, \hat{\mathsf{s_N}}, \hat{\mathsf{p_N}} \in \mathbb{N}$ *such that*

$$\{\hat{\mathsf{s_N}}\}(x) \neq \hat{\mathsf{0}}$$

and

$$\{\hat{\mathsf{p_N}}\}(\{\hat{\mathsf{s_N}}\}(x)) = x$$

for all $x \in \mathbb{N}$. *Furthermore for all* $0 \neq x \in \mathbb{N}$

$$\{\hat{\mathsf{s_N}}\}(\{\hat{\mathsf{p_N}}\}(x)) = x$$

holds.

Proof. Take $\hat{\mathsf{0}} := 0 \in \mathbb{N}$ and let $\hat{\mathsf{s_N}}$ and $\hat{\mathsf{p_N}}$ be indices of $\varphi(x) := x + 1$ and $\psi(x) := x - 1$, respectively. □

Lemma 2.4.18 ($\hat{\mathsf{d_N}}$)
There is a $\hat{\mathsf{d_N}} \in \mathbb{N}$ *such that*

$$\{\{\{\{\hat{\mathsf{d_N}}\}(u)\}(v)\}(x)\}(y) = \begin{cases} u & \text{if } x = y \\ v & \text{if } x \neq y \end{cases}$$

for all $x, y, u, v \in \mathbb{N}$.

Proof. As above. □

Lemma 2.4.19 ($\hat{\mathsf{nat}}, \hat{\mathsf{id}}, \hat{\mathsf{con}}, \hat{\mathsf{dis}}, \hat{\mathsf{dom}}, \hat{\mathsf{inv}}, \hat{\mathsf{j}}$)
There are $\hat{\mathsf{nat}}, \hat{\mathsf{id}}, \hat{\mathsf{con}}, \hat{\mathsf{dis}}, \hat{\mathsf{dom}}, \hat{\mathsf{inv}}, \hat{\mathsf{j}} \in \mathbb{N}$ *such that*

$$\begin{aligned}
W_{\hat{\mathsf{nat}}} &= \mathbb{N}, \\
W_{\hat{\mathsf{id}}} &= \{\langle x, x \rangle \; ; \; x \in \mathbb{N}\}, \\
W_{\{\hat{\mathsf{con}}\}\langle a,b \rangle} &= W_a \cap W_b, \\
W_{\{\hat{\mathsf{dis}}\}\langle a,b \rangle} &= W_a \cup W_b, \\
W_{\{\hat{\mathsf{dom}}\}(a)} &= \{x \in \mathbb{N} \; ; \; \textit{there is a } y \textit{ such that } \langle x, y \rangle \in W_a\}, \\
W_{\{\hat{\mathsf{inv}}\}\langle a,f \rangle} &= \{x \in \mathbb{N} \; ; \; \{f\}(x) \in W_a\}, \\
W_{\{\hat{\mathsf{j}}\}\langle a,f \rangle} &= \{x \in \mathbb{N} \; ; \; \textit{ there are } y, z \in \mathbb{N} \\
&\qquad \textit{such that } x = \langle y, z \rangle, \, y \in W_a \textit{ and } z \in W_{\{f\}(y)}\}
\end{aligned}$$

for all $a, b, f \in \mathbb{N}$.

Proof. Obvious by the usual recursion theoretic methods. □

Lemma 2.4.20 ($\hat{\mathsf{lfp}}$)
There is a $\hat{\mathsf{lfp}} \in \mathbb{N}$ *such that, if* $\{f\}$ *is a total function which is monotone on indices, then* $W_{\{\hat{\mathsf{lfp}}\}(f)}$ *is the least fixed point of* $\{f\}$ *for all* $f \in \mathbb{N}$.

Proof. A close inspection of the proof of 2.4.12 and a firm belief in Church's thesis reveals that given an index of a total function which is monotone on indices, the index of its least fixed point can be obtained uniformly, i.e. there is a recursive function taking the index of the function as input and giving the index of the least fixed point as output. Let $\hat{\mathsf{lfp}}$ be the index of such a function.[2] □

Definition 2.4.21 (Model $\mathcal{M}_{\mathbb{N}}$)
The model $\mathcal{M}_{\mathbb{N}}$ is given by

(1) the domain of individuals $|\mathcal{M}_{\mathbb{N}}|^I := \mathbb{N}$

(2) an individual $\infty \notin \mathbb{N}$.

(3) the domain of types $|\mathcal{M}_{\mathbb{N}}|^T := \{W_n \ ; \ n \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N})$

(4) the constants $\mathsf{k}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{k}}$, $\mathsf{s}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{s}}$, $\mathsf{p}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{p}}$, $\mathsf{p}_0^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{p}}_0$, $\mathsf{p}_1^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{p}}_1$, $0^{\mathcal{M}_{\mathbb{N}}} := \hat{0}$, $\mathsf{s}_{\mathsf{N}}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{s}_{\mathsf{N}}}$, $\mathsf{p}_{\mathsf{N}}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{p}_{\mathsf{N}}}$, $\mathsf{d}_{\mathsf{N}}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{d}_{\mathsf{N}}}$, $\mathsf{nat}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{nat}}$, $\mathsf{id}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{id}}$, $\mathsf{neg}^{\mathcal{M}_{\mathbb{N}}} := 17$, $\mathsf{con}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{con}}$, $\mathsf{dis}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{dom}}$, $\mathsf{inv}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{inv}}$, $\mathsf{j}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{j}}$ and $\mathsf{lfp}^{\mathcal{M}_{\mathbb{N}}} := \hat{\mathsf{lfp}}$, as defined previously.

(5) the binary operation $\cdot^{\mathcal{M}_{\mathbb{N}}} : (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \to (\mathbb{N} \cup \{\infty\})$ defined by

$$t \cdot^{\mathcal{M}_{\mathbb{N}}} s := \begin{cases} \{t\}(s) & \text{if } t \neq \infty,\ s \neq \infty \text{ and } s \in \mathrm{dom}(\{t\}) \\ \infty & \text{else} \end{cases}$$

(6) the unary relation $\mathsf{N}^{\mathcal{M}_{\mathbb{N}}} := \mathbb{N}$

(7) the binary relation $\mathfrak{R}^{\mathcal{M}_{\mathbb{N}}} := \{\langle n, W_n \rangle \ ; \ n \in \mathbb{N}\}$.

Theorem 2.4.22
$\mathcal{M}_{\mathbb{N}}$ *is a model of* $\Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I}_{\mathsf{N}} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}$.

Proof. Immediate by the lemmas above. □

Corollary 2.4.23
$\Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I}_{\mathsf{N}} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}$ *is consistent.*

[2] If you are not absolutely convinced by this argument, we recommend a look at 3.5.12.

2.5 Aspects of Explicit Mathematics

Having the consistency of our theory at hand, we will now take a look what our theory is able to prove. Most of this results can be found in this or a similar form in many texts about Explicit Mathematics and therefore we will present them in a very abridged version. For more information about combinatory algebra and lambda calculus we refer to the classical textbooks as e.g. [Bar84] but remind you of the fact that these mostly deal with total versions whereas ours is partial.

2.5.1 Lambda Calculus

Definition 2.5.1 ($\boldsymbol{\lambda}$)
The mapping $\lambda : \mathcal{V}_I \times \mathcal{T}^{\mathsf{E}} \to \mathcal{T}^{\mathsf{E}}$ is defined inductively by

$$\lambda(x,t) := \begin{cases} \mathsf{skk}, & \text{if } t = x, \\ \mathsf{s}\lambda(x,t_1)\lambda(x,t_2), & \text{if } t = t_1t_2, \\ \mathsf{k}t, & \text{else,} \end{cases}$$

for each term $t \in \mathcal{T}^{\mathsf{E}}$ and individual variable $x \in \mathcal{V}_I$.

Notation 2.5.2
$\lambda x.t$ is usually written for $\lambda(x,t)$, $\lambda x_0.(\dots\lambda x_n.(t)\dots)$ is abbreviated by $\lambda x_0 \dots x_n.t$ or $\lambda\vec{x}.t$.

Lemma 2.5.3
Let $t, s \in \mathcal{T}^{\mathsf{E}}$ be terms, $x, y \in \mathcal{V}_I$ individual variables. Then

(1) $FV_I(\lambda x.t) = FV_I(t) \setminus \{x\}$.

(2) $\mathsf{BON}^- \vdash \lambda x.t\downarrow$.

(3) $\mathsf{BON}^- \vdash s\downarrow \to (\lambda x.t)s \simeq t[s/x]$.

(4) $x \neq y \Rightarrow \mathsf{BON}^- \vdash (\lambda x.t)[s/y]x \simeq t[s/y]$.

Proof. Straightforward. □

Theorem 2.5.4 (Fixed Point, Primitive Recursion, $\mathsf{not_N}$)

(1) There is a closed term $\mathsf{fix} \in \mathcal{T}^{\mathsf{E}}$ *such that*

$$\mathsf{BON}^- \vdash \mathsf{fix}f\downarrow \wedge f(\mathsf{fix}f)x \simeq (\mathsf{fix}f)x$$

for all variables $f, x \in \mathcal{V}_I$.

(2) There is a closed term $\mathsf{rec} \in \mathcal{T}^{\mathsf{E}}$ *such that*

(a) $\mathsf{BON}^- \vdash f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \wedge b \in \mathsf{N} \to (\mathsf{rec}fa)\mathsf{0} \simeq a \wedge (\mathsf{rec}fa)(\mathsf{s_N}b) \simeq fb((\mathsf{rec}fa)b)$

(b) $\mathsf{BON}^- + (\mathsf{V}\text{-}\mathsf{I_N}) \vdash f \in (\mathsf{N}^2 \to \mathsf{N}) \wedge a \in \mathsf{N} \to (\mathsf{rec}fa) \in (\mathsf{N} \to \mathsf{N})$

for all variables $a, b, f \in \mathcal{V}_I$.

(3) There is a closed term $\mathsf{not_N} \in \mathcal{T}^{\mathsf{E}}$ *such that* $\mathsf{BON}^- \vdash \neg(\mathsf{not_N} \in \mathsf{N})$.

Proof. (1) Choose fix as $\lambda f.tt$ where t is $\lambda yx.f(yy)x$.

(2) Choose rec as $\lambda fa.\mathsf{fix}t$ where t is $\lambda hx.(\mathsf{d_N}(\lambda z.a)(\lambda z.f(\mathsf{p_N}x)(h(\mathsf{p_N}x)))x\mathsf{0})\mathsf{0}$.

(3) Choose $\mathsf{not_N}$ as $\mathsf{fix}t\mathsf{0}$ where t is $\lambda xy.\mathsf{d_N}(\mathsf{s_N}\mathsf{0})\mathsf{0}(xy)\mathsf{0}$. □

2.5.2 Induction

Theorem 2.5.5

(1) $\mathsf{BON}^- + \mathsf{V\text{-}I_N} \vdash \mathsf{S\text{-}I_N}$.

(2) $\Sigma^+\mathsf{ET} + \mathsf{T\text{-}I_N} \vdash \mathsf{V\text{-}I_N}$.

(3) $\Sigma^+\mathsf{ET} + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \vdash \mathsf{T\text{-}I_N}$.

Proof. See e.g. [Krä06]. □

2.5.3 Comprehension

Definition 2.5.6
The sets $\Sigma^+\mathsf{E}, \mathsf{E}, \Sigma^+\mathsf{S}, \mathsf{S}$ of positive existential elementary, elementary, positive existential stratified and stratified formulas are defined by

(1) for all terms $s, t \in \mathcal{T}^{\mathsf{E}}$ and type variables $X \in \mathcal{V}_T$ the formulas

- $s = t$
- $t \downarrow$
- $t \in \mathsf{N}$
- $t \in X$

are in $\Sigma^+\mathsf{E}, \mathsf{E}, \Sigma^+\mathsf{S}$ and S.

(2) if $\varphi, \psi \in \Sigma^+\mathsf{E}$ are positive existential elementary formulas and $x \in \mathcal{V}_I$ is an individual variable, then

- $\varphi \vee \psi$ and $\varphi \wedge \psi$
- $(\exists x)\varphi$

are also in $\Sigma^+\mathsf{E}$.

(3) if $\varphi, \psi \in \mathsf{E}$ are elementary formulas and $x \in \mathcal{V}_I$ is an individual variable, then

- $\neg\varphi$

– $\varphi \vee \psi$ and $\varphi \wedge \psi$
– $(\exists x)\varphi$ and $(\forall x)\varphi$

are also in E.

(4) if $\varphi, \psi \in \Sigma^+\mathsf{S}$ are positive existential stratified formulas $x \in \mathcal{V}_I$ is an individual variable and $X \in \mathcal{V}_T$ is a type variable, then

– $\varphi \vee \psi$ and $\varphi \wedge \psi$
– $(\exists x)\varphi$
– $(\exists X)\varphi$

are also in $\Sigma^+\mathsf{S}$.

(5) $\varphi, \psi \in \mathsf{S}$ are stratified formulas, $x \in \mathcal{V}_I$ is an individual variable and $X \in \mathcal{V}_T$ is a type variable, then

– $\neg\varphi$
– $\varphi \vee \psi$ and $\varphi \wedge \psi$
– $(\exists x)\varphi$ and $(\forall x)\varphi$
– $(\exists X)\varphi$ and $(\forall X)\varphi$

are also in S.

Remark 2.5.7 The sets $\Sigma^+\mathsf{E}, \mathsf{E}, \Sigma^+\mathsf{S}, \mathsf{S}$ have in common, that they do not contain any formula wherein the naming relation $\mathfrak{R}$ does appear. In addition to this condition we have:

– $\Sigma^+\mathsf{E}$ contains only positive formulas (formulas without negations) without quantified type variables and without universal quantifiers $\forall$, but existential quantifiers for type variables are allowed,

– E contains only formulas without quantified type variables, but both kinds of quantifiers for individual variables are allowed

– and $\Sigma^+\mathsf{S}$ contains only positive formulas. Universal quantifiers $\forall$ are not allowed (neither for types nor for individuals), but existential quantifiers $\exists$ may be used for types as well as for individuals.

Theorem 2.5.8
(1) $\Sigma^+\mathsf{ET} \vdash \Sigma^+\mathsf{E}\text{-}\mathsf{C}$.

(2) $\mathsf{EET}^- \vdash \mathsf{E}\text{-}\mathsf{C}$.

(3) $\Sigma^+\mathsf{ET} + \mathsf{J} + \forall\mathfrak{R} \vdash \Sigma^+\mathsf{S}\text{-}\mathsf{C}$.

(4) $\mathsf{EET}^- + \mathsf{J} + \forall\mathfrak{R}$ *is inconsistent.*

Proof. See e.g. [Krä06]. □

3 Arithmetic

In this section we will present the well-known systems of arithmetic. As before, the terminology and most notations are similar to [Krä06]. We use classical first-order logic with equality as our logical framework, which can be found in almost each textbook about mathematical logic, e.g. Shoenfield's [Sho67].

The most famous system of arithmetic is the Peano Arithmetic, which is a first-order version of the second-order Peano Axioms. Giuseppe Peano introduced his Axioms as early as 1889, a translation of the original work can be found in [Pea77]. In contrast to the classical definitions of systems of arithmetic, we will include function symbols for each primitive-recursive function and defining equations for these function symbols.

We will get various systems of arithmetic by allowing different amounts of inductions, ranging from the primitive-recursive arithmetic PRA, where only induction for quantifier-free formulas is allowed to the Peano Arithmetic PA, where induction for all formulas is allowed. By classifying formulas (in prenex normal form) by the arithmetical hierarchy, i.e. by counting alternations of quantifiers we get a whole hierarchy of systems inbetween PRA and PA. For more information about systems of arithmetic and their proof-theoretic strength, we refer to Sieg's [Sie85] and Hajek and Pudlak's [HP93].

In contrast to the section about Explicit Mathematics we will not prove the consistency of our systems of arithmetic. The consistency of Peano Arithmetic can be shown rather easily in any standard set theory, such as e.g. Zermelo-Fraenkel set theory. Such proofs can be found in almost any textbook about set theory such as e.g. Moschovakis' elaborate account in [Mos06]. There are also more proof-theoretically flavoured consistency proofs, such as Gentzen's ordinal analysis which can be found in [Gen36] and [Gen38] and translated in [Sza69]. While these results are about the consistency of full Peano Arithmetic, there are also interesting results about the consistency of fragments of Arithmetic, one of which we mention in the section about some aspects of arithmetic, but as it needs some theory that lies beyond the scope of this thesis, we refer again to [Sie85] and [HP93] for a full account. While using the heavy machinery of set theory to prove the consistency of Peano Arithmetic (or a fragment thereof) is mostly quite a simple affair, the proof-theoretic consistency proofs tend to be more sophisticated. They usually yield a lot more of information about the inner structure of Peano Arithmetic and were also an important (and somehow unexpected) step after the failure of Hilbert's program, as they—in view of Gödel's incompleteness theorems—are as close as possible to the original idea of Hilbert.

Besides the many (from the viewpoint of formal logic) informal texts about arithmetic and recursion theory as e.g. the already mentioned [Cut80] from Cutland and [Rog87] from Rogers we will also occasionally refer to more formal treatments of the subject such as Hinman's [Hin78] and Troelstra's and van Dalen's [TvD88].

As systems of arithmetic were in the focus of the logical community since their beginning there are a lot more results about it than could ever fit these pages. So our account of arithmetic will remain essentially incomplete, as we will mostly only mention the results we are actually using for our proof-theoretic analysis.

One main goal of this section is to show that we can formalise all the necessary results for the consistency proof. This will, as we will see in the following section, immediately give us the upper bound for the proof-theoretic strength of our systems of Explicit Mathematics. So we will keep an eye on the complexity of the formulas we use. As before in the consistency result, the Myhill-Shepherdson theorem (see [MS55]) will be playing a central role. On this occasion we also would like to apologise for the lot of new notations that is introduced and the sometimes not quite standard namings of the variables (which was chosen in order to keep as close as possible to the informal results).

3.1 Syntax

3.1.1 Language, Terms and Formulas

Definition 3.1.1 (Primitive Recursive Functions)
The sets Prim_n of n-ary primitve recursive functions is inductively defined as follows

(1) S is an unary primitive recursive function, where $\mathrm{S} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathrm{S}(x) := x + 1$$

for all $x \in \mathbb{N}$.

(2) Cs_i^n is a n-ary primitive recursive function, where $\mathrm{Cs}_i^n : \mathbb{N}^n \to \mathbb{N}$ is defined by

$$\mathrm{Cs}_i^n(x_1, \ldots, x_n) := i$$

for all $i, n, \vec{x} \in \mathbb{N}$.

(3) Pr_i^n is a n-ary primitive recursive function, where $\mathrm{Pr}_i^n : \mathbb{N}^n \to \mathbb{N}$ is defined by

$$\mathrm{Pr}_i^n(x_1, \ldots, x_n) := x_{i+1}$$

for all $i, n, \vec{x} \in \mathbb{N}$ with $i < n$.

(4) if f is a m-ary primitive recursive function and $g_1, \ldots, g_m$ are n-ary primitive recursive functions, then $\mathrm{Comp}^n(f, g_1, \ldots, g_m)$ is a n-ary primitive recursive function, where $\mathrm{Comp}^n(f, g_1, \ldots, g_m) : \mathbb{N}^n \to \mathbb{N}$ is defined by

$$\mathrm{Comp}^n(f, g_1, \ldots, g_m)(x_1, \ldots, x_n) := f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$$

for all $n, \vec{x} \in \mathbb{N}$.

(5) if f is n-ary primitive recursive function and g is a $(n+2)$-ary primitive recursive function, then $\mathrm{Rec}^{n+1}(f, g)$ is a $(n+1)$-ary primitive recursive function, where $\mathrm{Rec}^{n+1}(f, g) : \mathbb{N}^{n+1} \to \mathbb{N}$ is defined by

$$\mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, 0) := f(x_1, \ldots, x_n)$$

and

$$\mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, \mathrm{S}(x)) := g(x_1, \ldots, x_n, x, \mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, x))$$

for all $n, x, \vec{x} \in \mathbb{N}$.

for all $n \in \mathbb{N}$.

The set Prim of all primitive recursive functions is defined as

$$\mathrm{Prim} := \bigcup_{n \in \mathbb{N}} \mathrm{Prim}_n.$$

Definition 3.1.2 (Language of Arithmetic)
The language of Arithmetic is given by

(1) A countable set $\mathcal{V}_A$ of variables. The variables will usually be denoted by lower-case letters $a, b, c,\ f, g, h$ or u, v, w, x, y, z (possibly with subscript).

(2) The constant symbol $\mathsf{0}$.

(3) A function symbol for each primitive recursive function. The function symbols will usually be denoted by lower-case letters f, g, h (possibly with subscripts) or by the same symbol that is used to name the intended function, e.g. Cs_i^n.

(4) The relation symbol $=$.

(5) The logical symbols $\neg, \vee, \wedge, \exists, \forall$.

(6) The auxiliary symbols $($, $)$, $,$ (left bracket, right bracket, comma).

Definition 3.1.3 (Terms $\mathcal{T}^{\mathsf{A}}$)
The set of terms $\mathcal{T}^{\mathsf{A}}$ of Arithmetic is inductively defined as follows

(1) Each variable $x \in \mathcal{V}_A$ is a term.

(2) The constant symbol $\mathsf{0}$ is a term.

(3) If $t_1, \ldots, t_n$ are terms and f is function symbol for a n-ary primitive recursive function, then $f(t_1, \ldots, t_n)$ is a term.

Definition 3.1.4 (Formulas $\mathcal{F}^{\mathsf{A}}$, Quantifier Free Formulas QF)
The set of formulas $\mathcal{F}^{\mathsf{A}}$ of Arithmetic is inductively defined as follows

(1) If $s, t \in \mathcal{T}^{\mathsf{A}}$ are terms, then $(s = t)$ is a formula.

(2) If φ and ψ are formulas, then

- $\neg\varphi$
- $(\varphi \vee \psi)$
- $(\varphi \wedge \psi)$

are formulas.

(3) If φ is a formula and $x \in \mathcal{V}_A$ is a variable, then

- $(\exists x)\varphi$
- $(\forall x)\varphi$

are formulas.

The set QF of quantifier free formulas of Arithmetic is defined as the set of all formulas satisfying only the first two clauses of the inductive definition above, i.e. the set of all formulas in which no quantifier occurs.

Definition 3.1.5 (Arithmetical Hierarchy, Σ^0_n, Π^0_n)
The sets Σ^0_n, Π^0_n are inductively defined as follows

(1) $\Sigma^0_0 := \Pi^0_0 := \mathrm{QF}$,

(2) $\Sigma^0_{n+1} :=$
$\{\varphi \in \mathcal{F}^{\mathsf{A}}$; there is a $\psi \in \Pi^0_n$ and $x_1, \ldots, x_m \in \mathcal{V}_A$ such that φ is $(\exists\vec{x})\psi\}$,

(3) $\Pi^0_{n+1} :=$
$\{\varphi \in \mathcal{F}^{\mathsf{A}}$; there is a $\psi \in \Sigma^0_n$ and $x_1, \ldots, x_m \in \mathcal{V}_A$ such that φ is $(\forall\vec{x})\psi\}$,

for all $n \in \mathbb{N}$.

Notation 3.1.6
As usual $s \neq t$, $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ will be used as abbreviations for $\neg(s = t)$, $(\neg\varphi \vee \psi)$ and $(\varphi \to \psi) \wedge (\psi \to \varphi)$, respectively, where $s, t \in \mathcal{T}^{\mathsf{A}}$ and $\varphi, \psi \in \mathcal{F}^{\mathsf{A}}$. As before, brackets may be omitted by the usual conventions.

Definition 3.1.7 (Free Variables, FV_A)
The set of free individual variables $\mathrm{FV}_A(t)$ is inductively defined as follows

(1) if t is $\mathsf{0}$, then $\mathrm{FV}_A(t) = \emptyset$

(2) if t is $x \in \mathcal{V}_A$, then $\mathrm{FV}_A(t) = \{x\}$

(3) if t is $f(t_1, \ldots, t_n)$ and f is a function symbol for a n-ary primitive recursive function and $t_1, \ldots, t_n \in \mathcal{T}^{\mathsf{A}}$ are terms, then $\mathrm{FV}_A(t) = \mathrm{FV}_A(t_1) \cup \cdots \cup \mathrm{FV}_A(t_n)$

for all terms $t \in \mathcal{T}^{\mathsf{A}}$.

The set of free individual variables $\mathrm{FV}_A(\varphi)$ is inductively defined as follows

(1) if φ is $s = t$ and $s, t \in \mathcal{T}^{\mathsf{A}}$ are terms, then $\mathrm{FV}_A(\varphi) = \mathrm{FV}_A(s) \cup \mathrm{FV}_A(t)$

(2) if φ is $\neg\psi$ and $\psi \in \mathcal{F}^{\mathsf{A}}$ is a formula, then $\mathrm{FV}_A(\varphi) = \mathrm{FV}_A(\psi)$

(3) if φ is $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$ and $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{A}}$ are formulas, then $\mathrm{FV}_A(\varphi) = \mathrm{FV}_A(\psi_1) \cup \mathrm{FV}_A(\psi_2)$

(4) if φ is $(\exists x)\psi$ or $(\forall x)\psi$ and $\psi \in \mathcal{F}^{\mathsf{A}}$ is a formula and $x \in \mathcal{V}_A$ is a variable, then $\mathrm{FV}_A(\varphi) = \mathrm{FV}_A(\psi) \setminus \{x\}$

for all formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$.

Definition 3.1.8 (Closed Terms, Sentences)
Let $t \in \mathcal{T}^{\mathsf{A}}$ be a term and $\varphi \in \mathcal{F}^{\mathsf{A}}$ be a formula of Arithmetic. t is called a closed term if and only if $\mathrm{FV}_A(t) = \emptyset$ and φ is called a sentence if and only if $\mathrm{FV}_A(\varphi) = \emptyset$.

Definition 3.1.9 (Substitution)
Let $x_1, \ldots, x_n \in \mathcal{V}_A$ be variables, $t_1, \ldots, t_n \in \mathcal{T}^{\mathsf{A}}$ terms and $\varphi \in \mathcal{F}^{\mathsf{A}}$ a formula of Arithmetic. Term substitution is defined as usual and is denoted by $\varphi[\vec{t}/\vec{x}]$.

Definition 3.1.10 (Substitutable Terms $\mathbf{FT}_A$)
Let $x \in \mathcal{V}_A$ be a variable and $\varphi \in \mathcal{F}^{\mathsf{A}}$ a formula of Arithmetic. The set $\mathrm{FT}_A(x, \varphi)$ of substitutable terms is defined inductively as follows

(1) If φ is $s = t$, then $\mathrm{FT}_A(x, \varphi) = \mathcal{T}^{\mathsf{A}}$.

(2) If φ is $\neg\psi$, then $\mathrm{FT}_A(x, \varphi) = \mathrm{FT}_A(x, \psi)$.

(3) If φ is $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$, then $\mathrm{FT}_A(x, \varphi) = \mathrm{FT}_A(x, \psi_1) \cap \mathrm{FT}_A(x, \psi_2)$.

(4) If φ is $(\exists y)\psi$ or $(\forall y)\psi$, then

$$\mathrm{FT}_A(x, \varphi) = \begin{cases} \mathcal{T}^{\mathsf{A}}, & \text{if } y = x, \\ \{t \in \mathcal{T}^{\mathsf{A}} \ ; \ y \notin \mathrm{FV}_A(t)\} \cap \mathrm{FT}_A(x, \psi), & \text{if } y \neq x. \end{cases}$$

Remark 3.1.11 Remark 2.1.11 (in an appropriately modified form) applies here as well.

3.1.2 Logical Axioms and Proofs

Definition 3.1.12 (Propositional Axioms, $\mathcal{A}^{\mathsf{A}}_{\mathbf{Prop}}$)
The set of propositional axioms $\mathcal{A}^{\mathsf{A}}_{\mathrm{Prop}}$ is defined as the set consisting of

(1) $(\alpha \wedge \beta) \to \alpha$

(2) $(\alpha \wedge \beta) \to \beta$

(3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(4) $\alpha \to (\alpha \vee \beta)$

(5) $\beta \to (\alpha \vee \beta)$

(6) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(7) $\alpha \to (\beta \to \alpha)$

(8) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(9) $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

(10) $\neg\neg\alpha \to \alpha$

for all formulas $\alpha, \beta, \gamma \in \mathcal{F}^{\mathsf{A}}$.

Definition 3.1.13 (Equality Axioms, $\mathcal{A}^{\mathsf{A}}_{\mathrm{Eq}}$)
The set of equality axioms $\mathcal{A}^{\mathsf{A}}_{\mathrm{Eq}}$ is defined as the set consisting of

(1) $x = x$

(2) $x_0 = y_0 \wedge \ldots \wedge x_n = y_n \wedge s = t \rightarrow (s = t)[\vec{y}/\vec{x}]$

for all variables $x, x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathcal{V}_A$ and terms $t, s \in \mathcal{T}^{\mathsf{A}}$.

Definition 3.1.14 (Quantifier Axioms, $\mathcal{A}^{\mathsf{A}}_{\mathrm{Quant}}$)
The set of Quantifier axioms $\mathcal{A}^{\mathsf{A}}_{\mathrm{Quant}}$ is defined as the set consisting of

(1) $\varphi[t/x] \rightarrow (\exists x)\varphi$

(2) $(\forall x)\varphi \rightarrow \varphi[t/x]$

for all formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$, variables $x \in \mathcal{V}_A$ and terms $t \in \mathcal{T}^{\mathsf{A}}$.

Definition 3.1.15 (Rules of Inference, $\mathcal{R}^{\mathsf{A}}$)
The set of rules of inference $\mathcal{R}^{\mathsf{A}}$ is the set consisting of

(1) $\dfrac{\varphi[y/x] \rightarrow \psi}{(\exists x)\varphi \rightarrow \psi}$

(2) $\dfrac{\psi \rightarrow \varphi[y/x]}{\psi \rightarrow (\forall x)\varphi}$

(3) $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

for all formulas $\varphi, \psi \in \mathcal{F}^{\mathsf{A}}$ and variables $x, y \in \mathcal{V}_A$ with $y \in \mathrm{FT}_A(x, \varphi) \setminus \mathrm{FV}_A(\psi)$.

Notation 3.1.16 ($\mathbf{Prem}(\boldsymbol{R}), \mathbf{Conc}(\boldsymbol{R})$)
Let

$$R = \frac{\varphi_1 \quad \ldots \varphi_n}{\psi} \in \mathcal{R}^{\mathsf{A}}$$

be a rule of inference. The formulas $\varphi_1, \ldots, \varphi_n$ will be called premises and the formula ψ is called the conclusion. We will denote the set of premises by $\mathrm{Prem}(R) = \{\varphi_1, \ldots, \varphi_n\}$ and the conclusion by $\mathrm{Conc}(R) = \psi$.

Definition 3.1.17 ($\boldsymbol{T}$-Proof, $\boldsymbol{T} \vdash \boldsymbol{\varphi}$)
Let $T \subseteq \mathcal{F}^{\mathsf{A}}$ be a set of formulas and $\varphi_0, \ldots, \varphi_n \in \mathcal{F}^{\mathsf{A}}$ formulas. The sequence of formulas $(\varphi_0, \ldots, \varphi_n)$ is called a T-proof, if each φ_i satisfies one of the following conditions

(1) $\varphi_i \in T$,

(2) $\varphi_i \in \mathcal{A}^{\mathsf{A}}_{\mathrm{Prop}} \cup \mathcal{A}^{\mathsf{A}}_{\mathrm{Eq}} \cup \mathcal{A}^{\mathsf{A}}_{\mathrm{Quant}}$,

(3) there is a $R \in \mathcal{R}^{\mathsf{A}}$ such that $\varphi_i = \mathrm{Conc}(R)$ and $\mathrm{Prem}(R) \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$.

The proof relation $\vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{A}}) \times \mathcal{F}^{\mathsf{A}}$ is defined as follows:

$T \vdash \varphi$ for all sets of formulas $T \subseteq \mathcal{F}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$ if and only if there is a T-proof $(\varphi_0, \ldots, \varphi_n)$ such that $\varphi_n = \varphi$.

3.2 Semantics

3.2.1 Structures

Definition 3.2.1 (Structure of Arithmetic)
A structure $\mathcal{M}$ of Arithmetic consists of

(1) A set M called the domain of numbers.

(2) An element $\mathsf{0}^{\mathcal{M}} \in M$.

(3) For every $n \in \mathbb{N}$ and every function symbol $f \in \mathrm{Prim}_n$ a function

$$f^{\mathcal{M}} : M^n \to M.$$

The set M will usually be denoted as $\mid \mathcal{M} \mid$, the class of all structures of Arithmetic is denoted by $\mathbb{M}^{\mathsf{A}}$.

Definition 3.2.2 (Valuations $\nu \in \mathbb{V}^{\mathcal{M}}$)
Let $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ be a structure of Arithmetic. A valuation ν for the structure $\mathcal{M}$ is a mapping

$$\nu : \mathcal{V}_A \to \mid \mathcal{M} \mid .$$

The set of all valuations for the structure $\mathcal{M}$ is denoted by $\mathbb{V}^{\mathcal{M}}$.

Let $\nu \in \mathbb{V}^{\mathcal{M}}$, $u \in \mathcal{V}_A$ and $m \in \mid \mathcal{M} \mid$. $\nu[u : m]$ denotes the following valuation for $\mathcal{M}$

$$\nu[u : m](v) := \begin{cases} m, & \text{if } v = u \\ \nu(v), & \text{otherwise.} \end{cases}$$

Definition 3.2.3 (Interpretations)
Let $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ be a structure of Arithmetic and $\nu \in \mathbb{V}^{\mathcal{M}}$ a valuation for $\mathcal{M}$. An interpretation $\mathcal{M}_\nu$ for $\mathcal{M}$ and ν is a mapping

$$\mathcal{M}_\nu : \mathcal{T}^{\mathsf{A}} \to \mid \mathcal{M} \mid$$

satisfying

$$\mathcal{M}_\nu := \begin{cases} \mathsf{0}^{\mathcal{M}}, & \text{if } t = \mathsf{0} \\ \nu(t), & \text{if } t \in \mathcal{V}_A \\ f^{\mathcal{M}}(\mathcal{M}_\nu(s_0), \ldots, \mathcal{M}_\nu(s_n)), & \text{if } t = f(s_0, \ldots, s_n). \end{cases}$$

An interpretation $\mathcal{M}_\nu$ induces a mapping (which we will be given the same name as the interpretation)

$$\mathcal{M}_\nu : \mathcal{F}^{\mathsf{A}} \to \{\mathsf{t}, \mathsf{f}\}$$

which is inductively defined by

$$\begin{aligned}
\mathcal{M}_\nu(s = t) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(s) = \mathcal{M}_\nu(t)\\
\mathcal{M}_\nu(\neg\varphi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{f}\\
\mathcal{M}_\nu(\varphi \vee \psi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ or } \mathcal{M}_\nu(\psi) = \mathsf{t}\\
\mathcal{M}_\nu(\varphi \wedge \psi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ and } \mathcal{M}_\nu(\psi) = \mathsf{t}\\
\mathcal{M}_\nu((\exists x)\varphi) = \mathsf{t} &:\Leftrightarrow \text{there is a } m \in \mid\mathcal{M}\mid^I \text{ such that } \mathcal{M}_{\nu[x:m]}(\varphi) = \mathsf{t}\\
\mathcal{M}_\nu((\forall x)\varphi) = \mathsf{t} &:\Leftrightarrow \mathcal{M}_{\nu[x:m]}(\varphi) = \mathsf{t} \text{ for all } m \in \mid\mathcal{M}\mid^I
\end{aligned}$$

for all terms $t, s \in \mathcal{T}^{\mathsf{A}}$, variables $x \in \mathcal{V}_A$ and formulas $\varphi, \psi \in \mathcal{F}^{\mathsf{A}}$.

Definition 3.2.4 (Model relation $\models$)
The model relation $\models \subseteq \mathbb{M}^{\mathsf{A}} \times \mathcal{F}^{\mathsf{A}}$ is defined by

$$\mathcal{M} \models \varphi \quad :\Leftrightarrow \quad \mathcal{M}_\nu(\varphi) = \mathsf{t} \text{ for all } \nu \in \mathbb{V}^{\mathcal{M}}$$

for all structures of Arithmetic $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$.

The model relation can easily be lifted to a relation $\models \subseteq \mathbb{M}^{\mathsf{A}} \times \mathcal{P}(\mathcal{F}^{\mathsf{A}})$ by

$$\mathcal{M} \models T \quad :\Leftrightarrow \quad \mathcal{M} \models \varphi \text{ for all } \varphi \in T$$

for all structures of Arithmetic $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ and sets of formulas $T \subseteq \mathcal{F}^{\mathsf{A}}$.

If $\mathcal{M} \models T$ then $\mathcal{M}$ will be called a model of T.

Definition 3.2.5 (Entailment relation, logical consequence $\Vdash$)
The entailment relation $\Vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{A}}) \times \mathcal{F}^{\mathsf{A}}$ is defined by

$$T \Vdash \varphi \quad :\Leftrightarrow \quad \mathcal{M} \models T \text{ implies } \mathcal{M} \models \varphi \text{ for all } \mathcal{M} \in \mathbb{M}^{\mathsf{A}}$$

for all sets of formulas $T \subseteq \mathcal{F}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$.

Again the entailment relation can easily be lifted to a relation $\Vdash \subseteq \mathcal{P}(\mathcal{F}^{\mathsf{A}}) \times \mathcal{P}(\mathcal{F}^{\mathsf{A}})$ by

$$T \Vdash S \quad :\Leftrightarrow \quad T \Vdash \varphi \text{ for all } \varphi \in S$$

for all sets of formulas $S, T \subseteq \mathcal{F}^{\mathsf{A}}$.

3.2.2 Adequacy

Theorem 3.2.6 (Correctness)
For all theories $T \subseteq \mathcal{F}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$

$$\textit{If } T \vdash \varphi \quad \textit{then} \quad T \Vdash \varphi.$$

Proof. See e.g. [Sho67]. □

Theorem 3.2.7 (Henkin's Theorem)
A theory $T \subseteq \mathcal{F}^{\mathsf{A}}$ is consistent, if and only if there is a model $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$ of T, i.e. $\mathcal{M} \models T$.

Proof. See e.g. [Sho67]. □

Corollary 3.2.8 (Completeness)
For all theories $T \subseteq \mathcal{F}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$

$$\textit{If } T \Vdash \varphi \quad \textit{then} \quad T \vdash \varphi.$$

Proof. We will prove the contraposition. Without loss of generality we may assume that $\mathrm{FV}_A(\varphi) = \emptyset$. Let $T \nvdash \varphi$.

Assume (towards a contradiction) that $T \cup \{\neg\varphi\}$ is inconsistent. Then $T \cup \{\neg\varphi\} \vdash \varphi$ since inconsistent theories prove every formula. By some further thinking, this yields $T \vdash \varphi$ which contradicts our assumption $T \nvdash \varphi$.

So $T \cup \{\neg\varphi\}$ is consistent and has hence by 3.2.7 a model, say $\mathcal{M} \in \mathbb{M}^{\mathsf{A}}$. Since $\mathcal{M} \models T \cup \{\neg\varphi\}$ we get especially $\mathcal{M} \models T$ and $\mathcal{M} \models \neg\varphi$, i.e. $\mathcal{M} \models T$ but $\mathcal{M} \not\models \varphi$ and so $T \nVdash \varphi$. □

Theorem 3.2.9 (Adequacy)
For all theories $T \subseteq \mathcal{F}^{\mathsf{A}}$ and formulas $\varphi \in \mathcal{F}^{\mathsf{A}}$

$$T \Vdash \varphi \quad \textit{if and only if} \quad T \vdash \varphi.$$

Proof. Follows immediately by combining 3.2.6 and 3.2.8. □

3.3 Theories and Principles

3.3.1 Primitive Recursive Arithmetic

Definition 3.3.1 (Numerals $\overline{n} \in \mathcal{T}^{\mathsf{A}}$)
For each $n \in \mathbb{N}$ the term $\overline{n} \in \mathcal{T}^{\mathsf{A}}$ is defined inductively as follows

(1) $\overline{0} := \mathsf{0}$,

(2) $\overline{n+1} := \mathrm{S}(\overline{n})$.

Definition 3.3.2 (Defining Equations $\mathcal{A}^{\mathsf{A}}_{\mathbf{Prim}}$)
The set of defining equations $\mathcal{A}^{\mathsf{A}}_{\mathrm{Prim}}$ is defined as the set consisting of

(1) $\neg\mathrm{S}(x) = \mathsf{0}$

(2) $\mathrm{Cs}^n_i(x_1, \ldots, x_n) = \overline{i}$

(3) $\mathrm{Pr}^n_i(x_1, \ldots, x_n) = x_{i+1}$

(4) $\mathrm{Comp}^n(f, g_1, \ldots, g_m)(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$

(5) $\mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, \mathsf{0}) = f(x_1, \ldots, x_n)$

(6) $\mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, \mathrm{S}(x)) = g(x_1, \ldots, x_n, x, \mathrm{Rec}^{n+1}(f, g)(x_1, \ldots, x_n, x))$

for all $i, n \in \mathbb{N}$ with $n > 0$, function symbols Cs^n_i, Pr^n_i, $\mathrm{Comp}^n(f, g_1, \ldots, g_m)$, $\mathrm{Rec}^{n+1}(f, g)$ and variables $x, x_1, \ldots, x_n \in \mathcal{V}_A$.

3.3.2 Induction Axioms

Definition 3.3.3 (Induction Axioms $\mathcal{X}$-Ind)
Let $\mathcal{X} \subseteq \mathcal{F}^{\mathsf{A}}$ be a set of formulas. The set $\mathcal{X}\text{-}\mathsf{Ind} \subseteq \mathcal{F}^{\mathsf{A}}$ is defined as the set consisting of

$$\varphi[\mathsf{0}/x] \land (\forall x)(\varphi \to \varphi[\mathrm{S}(x)/x]) \to (\forall x)\varphi$$

for all $\varphi \in \mathcal{X}$ and $x \in \mathcal{V}_A$.

Remark 3.3.4 Often we will use $\mathcal{X}\text{-}\mathsf{Ind}$ with $\mathcal{X} = \mathrm{QF}, \Sigma^0_1, \Pi^0_2$ or $\mathcal{F}^{\mathsf{A}}$.

3.3.3 Further Principles

Notation 3.3.5
Let f be a function symbol for the primitive-recursive characteristic function of the order relation $<$, i.e. $f(y, x) = 0$ if and only if $y < x$, $\varphi \in \mathcal{F}^{\mathsf{A}}$ a formula and $x, y \in \mathcal{V}_A$ variables. We will use $(\forall y < x)\varphi$ as an abbreviation for $(\forall y)(f(y, x) = 0 \to \varphi)$.

Definition 3.3.6 (Least Element Principle $\mathcal{X}$-LEP)
Let $\mathcal{X} \subseteq \mathcal{F}^{\mathsf{A}}$ be a set of formulas. The set $\mathcal{X}$-LEP $\subseteq \mathcal{F}^{\mathsf{A}}$ is defined as the set consisting of

$$(\exists x)\varphi \to (\exists x)(\varphi \wedge (\forall y < x)\neg\varphi[y/x])$$

for all $\varphi \in \mathcal{X}$ and $x, y \in \mathcal{V}_A$.

Remark 3.3.7 The least element principle states, that if we can find an element satisfying some property, then we can find a least element satisfying that property. As we will see in 3.4.3, this concept is (not surprisingly) closely related to induction.

Definition 3.3.8 (Collection Principle $\mathcal{X}$-CP)
Let $\mathcal{X} \subseteq \mathcal{F}^{\mathsf{A}}$ be a set of formulas. The set $\mathcal{X}$-CP $\subseteq \mathcal{F}^{\mathsf{A}}$ is defined as the set consisting of

$$(\forall x < t)(\exists y)\varphi \to (\exists z)(\forall x < t)(\exists y < z)\varphi$$

for all $\varphi \in \mathcal{X}$, $t \in \mathcal{T}^{\mathsf{A}}$ and $x, y, z \in \mathcal{V}_A$.

Remark 3.3.9 The collection principle is a so-called number theoretic choice principle. Information about similar weak choice principles and also about stronger (second-order) choice principles in the context of arithmetic can be found in e.g. [Sie85].

3.3.4 PRA, PRA$^+$ and PA

Definition 3.3.10 (Primitive Recursive Arithmetic PRA)
The theory of primitive recursive arithmetic PRA is defined as

$$\mathsf{PRA} := \mathcal{A}^{\mathsf{A}}_{\mathrm{Prim}} \cup \mathrm{QF}\text{-}\mathsf{Ind}.$$

Definition 3.3.11 (PRA$^+$)
The theory PRA$^+$ is defined as

$$\mathsf{PRA}^+ := \mathsf{PRA} \cup \Sigma^0_1\text{-}\mathsf{Ind}.$$

Definition 3.3.12 (Peano Arithmetic PA)
The theory of Peano Arithmetic PA is defined as

$$\mathsf{PA} := \mathsf{PRA} \cup \mathcal{F}^{\mathsf{A}}\text{-}\mathsf{Ind}.$$

3.4 Aspects of Arithmetic

Here we will present some of the interesting and sometimes also surprising results about systems of Arithmetic. Not all of them will be used in the following parts.

Theorem 3.4.1 (Parsons' Theorem)
Let $\varphi \in \Pi^0_2$ be a formula of Arithmetic. Then

$$\mathsf{PRA}^+ \vdash \varphi \quad \textit{if and only if} \quad \mathsf{PRA} \vdash \varphi,$$

i.e. PRA^+ is conservative over PRA *for Π^0_2-formulas.*

Proof. See e.g. [Sie91]. □

Theorem 3.4.2 (Equivalence of Σ^0_n-Ind and Π^0_n-Ind)
Π^0_n-Ind *is equivalent to* Σ^0_n-Ind *over* PRA *for all $n \in \mathbb{N}$.*

Proof. Let $\varphi \in \Pi^0_n$ be a formula of arithmetic and $x, y \in \mathcal{V}_A$ variables such that y is different from x and $x \dot{-} y \in \mathrm{FT}_A(x, \varphi)$. Then

$$\neg\varphi[x \dot{-} y/x]$$

is an element of Σ^0_n, where $\dot{-}$ denotes the (primitive recursive) limited subtraction. Using the Σ^0_n induction axiom for this formula and some further thinking will yield a prove of the corresponding Π^0_n induction axiom for φ. The reverse direction is established in the same way. See e.g. [Sie85] for a full account of this proof. □

Theorem 3.4.3
For all $n \in \mathbb{N}$

(1) Σ^0_{n+1}-CP *is contained in* Σ^0_{n+1}-Ind *over* PRA*.*

(2) Π^0_n-LEP *is equivalent to* Σ^0_n-Ind *over* PRA*.*

(3) Σ^0_n-LEP *is equivalent to* Π^0_n-Ind *over* PRA*.*

Proof. (1) By induction on n and the induction axiom for the formula

$$b \leq a \to (\exists z)(\forall x < b)(\exists y < z)\varphi,$$

where φ is the formula you want to prove the collection principle for. See e.g. [Sie85] for details.

(2) For the left-to-right direction assume that the least element principle holds but the induction axiom fails. Using a bit of arithmetic machinery will give a contradiction. For the reverse direction, if you assume $(\exists x)\varphi$, let ψ be

$$(\forall y < z)\varphi[y/x].$$

Then the induction axiom for ψ and the initial assumption $(\exists x)\varphi$ can be used to get

$$\neg(\forall x)(\psi[x/z] \rightarrow \psi[\mathrm{S}(x)/z]$$

which gives the consequence of the least element principle. For details we refer again to e.g. [Sie85].

Another way to prove this is by comparing the least element principle for a formula φ to the contraposition of the induction axiom for $\neg\varphi$.

(3) Parallel to the prove above. See e.g. [Sie85]. □

Corollary 3.4.4
For all $n \in \mathbb{N}$

(1) if $\varphi \in \Sigma^0_n$-LEP*, then* $\mathsf{PRA} + \Sigma^0_n$-$\mathsf{Ind} \vdash \varphi$*.*

(2) if $\varphi \in \Sigma^0_{n+1}$-$\mathsf{CP}$*, then* $\mathsf{PRA} + \Pi^0_{n+1}$-$\mathsf{Ind} \vdash \varphi$*.*

Theorem 3.4.5
$\mathsf{PRA} + \Sigma^0_{n+1}$-$\mathsf{IR}$ *proves the consistency of* $\mathsf{PRA} + \Pi^0_n$-Ind *for all $n \in \mathbb{N}$.*

Proof. See e.g. [Sie85]. □

Remark 3.4.6 Let $\mathcal{X} \subseteq \mathcal{F}^{\mathsf{A}}$. Then $\mathcal{X}$-IR denotes the induction rule for formulas in $\mathcal{X}$, i.e. the rule

$$\frac{\varphi[\mathsf{0}/x] \qquad (\forall x)(\varphi \rightarrow \varphi[\mathrm{S}(x)/x])}{(\forall x)\varphi} \quad \varphi \in \mathcal{X}.$$

The induction rule Σ^0_n-IR is weaker than the corresponding induction axioms Σ^0_n-Ind, for details see e.g. [Sie85].

3.5 Recursion Theory Formalised

In the following we will show that the recursion theoretic results we used to prove the consistency of our systems of Explicit Mathematics also hold in the formalised settings of arithmetic. As we will see later, these results are the key to the proof-theoretic analysis of our systems of Explicit Mathematics and so we pay some special attention to the complexity of formulas. The first step of formalization is introducing a formal version of the Kleene bracket, which we do using a version of Kleene's T-Predicate. Afterwards we are working informally in PRA^+ and $\mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind}$, respectively. The main problem one encounters when trying to formalise the recursion theoretic results is the lack of second order objects and the fact that many implicit dependencies become explicit. We will also occasionally refer to classical (informal) recursion-theoretic theorems from the section about the consistency of our theory, but we actually mean formalised counterparts of these theorems.

3.5.1 Notations and Definitions

Definition 3.5.1
Let T be the function symbol for a formalised version of the (primitive-recursive) characteristic function of Kleene's T-Predicate, i.e. $\{x\}(y) \simeq z$ in u or less steps if and only if $T(u, x, y, z) = 0$.

Furthermore let S be the function symbol for the function defined by $\{x\}(y)\downarrow$ in u or less steps if and only if $S(u, x, y) = 0$.

Remark 3.5.2 To be more precisely, we could take the characteristic functions of the primitive-recursive predicates S_1 and H_1 of Corollary 5-1.3 of [Cut80] for T and S.

A more formal construction of (functions similar to our) T can be found in many textbooks, such as [TvD88] or [Hin78].

Remark 3.5.3 In the following we will use characteristic functions of predicates as if they were the predicates, i.e. we will write $P(x, y, z)$ instead of $P(x, y, z) = 0$.

Furthermore we will use primitive recursive relations like $\leq$ in infix notation.

Definition 3.5.4 (Kleene terms $\mathcal{T}^{\mathsf{A}}_K$)
The Kleene terms are defined inductively as follows

(1) each variable is a Kleene term,

(2) if $t_1, \ldots, t_n$ are Kleene terms and f a n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a Kleene term,

(3) if t and $s_1, \ldots, s_n$ are Kleene terms, then $\{t\}(s_1, \ldots, s_n)$ is a Kleene term.

We will denote the set of Kleene terms by $\mathcal{T}^{\mathsf{A}}_K$. Please note that $\mathcal{T}^{\mathsf{A}} \subseteq \mathcal{T}^{\mathsf{A}}_K$.

Remark 3.5.5 Kleene terms are not part of the language of Arithmetic. They will be used as abbreviations, as will be seen in the following definition.

Definition 3.5.6 ($\gamma : \mathcal{T}_K^{\mathsf{A}} \times \mathcal{V}_A \to \mathcal{F}^{\mathsf{A}}$)
The function $\gamma : \mathcal{T}_K^{\mathsf{A}} \times \mathcal{V}_A \to \mathcal{F}^{\mathsf{A}}$ is defined for each Kleene term $t \in \mathcal{T}_K^{\mathsf{A}}$ and each variable $x \in \mathcal{V}_A$ inductively by

(1) if t is a variable $y \in \mathcal{V}_A$, then $\gamma(t, x)$ is $x = y$.

(2) if t is $f(t_1, \ldots, t_n)$, then $\gamma(t, x)$ is

$$(\exists \vec{y})(\gamma(t_1, y_1) \wedge \ldots \wedge \gamma(t_n, y_n) \wedge f(y_1, \ldots, y_n) = x).$$

(3) if t is $\{s\}(r_1, \ldots, r_n)$, then $\gamma(t, x)$ is

$$(\exists y, \vec{z})(\gamma(s, y) \wedge \gamma(r_1, z_1) \wedge \ldots \wedge \gamma(r_n, z_n) \wedge \exists w(T(w, y, \langle z_1, \ldots, z_n \rangle, x))).$$

Notation 3.5.7
Let $t \in \mathcal{T}_K^{\mathsf{A}}$ be a Kleene term and $\varphi \in \mathcal{F}^{\mathsf{A}}$ be a formula. By $\varphi(t/x)$ we mean the formula φ with all occurrences of x replaced by t. If $t \in \mathcal{T}_K^{\mathsf{A}} \cap \mathcal{T}^{\mathsf{A}}$ then we will interpret $\varphi(t/x)$ as usual. If $t \in \mathcal{T}_K^{\mathsf{A}} \setminus \mathcal{T}^{\mathsf{A}}$, we will interpret $\varphi(t/x)$ as follows

(1) if φ is of the form $x = s$, where $s \in \mathcal{T}^{\mathsf{A}}$ is a term, then $\varphi(t/x)$ is

$$(\exists z)(\gamma(t, z) \wedge z = y[z/x]),$$

where $z \in \mathcal{V}_A$ is a variable that does occur neither in t nor in s.

(2) if φ is non-atomic, then $\varphi(t/x)$ is defined inductively in the obvious way (again being careful no variable clashes occur).

Notation 3.5.8
(1) Let t, s be terms of arithmetic. We use the following abbreviations

(a) $t \overset{\star}{\in} s$ for $(\exists u)(S(u, s, t))$.

(b) $t \overset{\star}{\subseteq} s$ for $(\forall x)(x \overset{\star}{\in} t \to x \overset{\star}{\in} s)$.

(c) $t \overset{\star}{=} s$ for $t \overset{\star}{\subseteq} s \wedge s \overset{\star}{\subseteq} t$.

(d) $\mathrm{Fin}(t)$ for $(\exists z)(\forall y)(y \overset{\star}{\in} t \to y \leq z)$.

(2) Let $t, s \in \mathcal{T}_K^{\mathsf{A}}$ be Kleene terms. We will write $t \simeq s$ as an abbreviation for

$$(\forall x)(\gamma(t, x) \leftrightarrow \gamma(s, x)).$$

(3) $k \in \mathbb{N}$ denotes an index of the halting set K.

(4) $\langle . \rangle$ denotes a primitive-recursive pairing function.

(5) Let f, t be terms. We us the following abbreviations

(a) Ext(f) for $(\forall x, y)(x \overset{\star}{=} y \to \{f\}(x) \overset{\star}{=} \{f\}(y))$.

(b) Mon(f) for $(\forall x, y)(x \overset{\star}{\subseteq} y \to \{f\}(x) \overset{\star}{\subseteq} \{f\}(y))$.

(c) Tot(f) for $(\forall x)(\exists y)(\{f\}(x) = y)$.

(d) Cont(f) for $(\forall x, y)(x \overset{\star}{\in} \{f\}(y) \leftrightarrow \exists z(z \overset{\star}{\subseteq} y \wedge \mathrm{Fin}(z) \wedge x \overset{\star}{\in} \{f\}(z)))$.

(e) Lfp(t, f) for $\{f\}(t) \overset{\star}{\subseteq} t \wedge (\forall s)(\{f\}(s) \overset{\star}{\subseteq} s \to t \overset{\star}{\subseteq} s)$.

(f) ExtClos(t) for $(\forall x, y)(x \overset{\star}{\in} t \wedge y \overset{\star}{=} x \to y \overset{\star}{\in} t)$.

Remark 3.5.9 Please remember that a formula like $t \overset{\star}{\subseteq} s$ is an abbreviation for

$$(\forall x)(x \overset{\star}{\in} t \to x \overset{\star}{\in} s),$$

which is an abbreviation for

$$(\forall x)((\exists w_1)S(w_1, t, x) \to (\exists w_2)S(w_2, s, x)),$$

which is again an abbreviation for

$$(\forall x)((\forall w_1)\neg S(w_1, t, x) \vee (\exists w_2)S(w_2, s, x)).$$

So we easily see that this formula (even if we replace t and s by Kleene terms) is equivalent to a formula in $\Pi^0_2 \cap \Sigma^0_2$.

3.5.2 The Theorems

Theorem 3.5.10 (Rice-Shapiro, formalised)
Let $t \in \mathcal{T}^{\mathsf{A}}$ be a term.

$$\mathsf{PRA}^+ \vdash \mathit{ExtClos}(t) \to (\forall x)(x \overset{\star}{\in} t \leftrightarrow (\exists y)(y \overset{\star}{\subseteq} x \wedge \mathit{Fin}(y) \wedge y \overset{\star}{\in} t)).$$

Proof. We will only prove the left-to-right direction of the equivalence, the proof of the other direction being similar[3].

So assume (towards a contradiction), that we have a x such that

$$x \overset{\star}{\in} t \wedge (\forall y)(y \overset{\star}{\subseteq} x \wedge \mathrm{Fin}(y) \to \neg y \overset{\star}{\in} t).$$

With the usual recursion theoretic methods, we can find an $f \in \mathbb{N}$ such that

$$(\forall u, v)((\exists w)(w \leq v \wedge S(w, \overline{k}, u) \to \neg(\exists w)(S(w, \{\overline{f}\}(x), u, v))))$$

and

$$(\forall u, v)(\neg(\exists w)(w \leq v \wedge S(w, \overline{k}, v) \to (\forall w)(\{\{\overline{f}\}(x)\}(u, v) = w \leftrightarrow \{x\}(u) = w))).$$

[3]and actually only the left-to-right direction is used in the following theorems

Using 2.4.2, we can get a $s \in \mathbb{N}$ such that

$$(\forall u, v, w)(\{\{\overline{f}\}(x)\}(u, v) = w \leftrightarrow \{\{\{\overline{s}\}(x)\}(u)\}(v) = w).$$

Now let $u \overset{\star}{\in} \overline{k}$. Using an instance of Σ^0_1-LEP (which we can by 3.4.4), we can show

$$(\exists v_0)(\forall v)(v < v_0 \leftrightarrow \neg S(v, \overline{k}, u))$$

and from there we easily get

$$\mathrm{Fin}(\{\{\overline{s}\}(x)\}(u)).$$

By our initial assumption this means also

$$\neg\{\{\overline{s}\}(x)\}(u) \overset{\star}{\in} t.$$

On the other hand let now $\neg u \overset{\star}{\in} \overline{k}$, by definition of f and s we get

$$(\forall v, w)(\{\{\{\overline{s}(x)\}(u)\}\}(v) = w \leftrightarrow \{x\}(v) = w)$$

and therefore by ExtClos(t)

$$\{\{\overline{s}\}(x)\}(u) \overset{\star}{\in} t.$$

So we finally get

$$\forall u(\neg u \overset{\star}{\in} \overline{k} \leftrightarrow \{\{\overline{s}\}(x)\}(u) \overset{\star}{\in} t),$$

which can be used to construct a contradiction to the undecidability of the halting problem. But this means our initial assumption was wrong and hence we get the desired result. □

Theorem 3.5.11 (Myhill-Shepherdson, formalised)

$$\mathsf{PRA}^+ \vdash (\forall f)(\mathit{Ext}(f) \wedge \mathit{Tot}(f) \rightarrow \mathit{Cont}(f)).$$

Proof. First, find a $t \in \mathbb{N}$ such that

$$(\forall x, y)(y \overset{\star}{\in} \{\overline{t}\}(f, x) \leftrightarrow x \overset{\star}{\in} \{f\}(y)).$$

From Ext(f) we instantly get ExtClos($\{\overline{t}\}(f, x)$) and so we can use 3.5.10 to get

$$y \overset{\star}{\in} \{\overline{t}\}(f, x) \leftrightarrow (\exists z)(z \overset{\star}{\subseteq} y \wedge \mathrm{Fin}(z) \wedge z \overset{\star}{\in} \{\overline{t}\}(f, x)).$$

So we get immediately

$$x \overset{\star}{\in} \{f\}(y) \leftrightarrow (\exists z)(z \overset{\star}{\subseteq} y \wedge \mathrm{Fin}(z) \wedge x \overset{\star}{\in} \{f\}(z)),$$

which is Cont(f). □

Theorem 3.5.12 (least fixed point, formalised)
There is a $n \in \mathbb{N}$ such that

$$\mathsf{PRA}^{+} + \Pi^0_2\text{-}\mathsf{Ind} \vdash (\forall f)(\mathit{Tot}(f) \wedge \mathit{Mon}(f) \rightarrow (\mathit{Lfp}(\{\overline{n}\}(f), f)).$$

Proof. Let $e \in \mathbb{N}$ be an index of the empty set. By 2.4.4 we can find a $f_{\text{iter}} \in \mathbb{N}$ such that

$$\{\overline{f_{\text{iter}}}\}(f, 0) \simeq \overline{e}$$

and

$$\{\overline{f_{\text{iter}}}\}(f, \mathrm{S}(n)) \simeq \{f\}(\{\overline{f_{\text{iter}}}\}(f, n)).$$

Furthermore we can find a $n \in \mathbb{N}$ such that

$$x \overset{\star}{\in} \{\overline{n}\}(f) \leftrightarrow (\exists i)(x \overset{\star}{\in} \{\overline{f_{\text{iter}}}\}(f, i)).$$

Using an appropriate instance of Π^0_2-Ind (see 3.5.9) and the property Mon(f) we can first show

$$(\forall i)(\{\overline{f_{\text{iter}}}\}(f, i) \overset{\star}{\subseteq} \{\overline{f_{\text{iter}}}\}(f, \mathrm{S}(i))).$$

Since Mon$(f) \rightarrow$ Ext(f) we can use 3.5.11 to get Cont(f). So from $x \overset{\star}{\in} \{f\}(\{\overline{n}\}(f)))$ we get

$$(\exists y)(y \overset{\star}{\subseteq} \{\overline{n}\}(f) \wedge \mathrm{Fin}(y) \wedge x \overset{\star}{\in} \{f\}(y)).$$

From Fin(y) we get

$$(\exists u)(\forall v)(v \overset{\star}{\in} y \rightarrow v \leq u)$$

and from $y \overset{\star}{\subseteq} \{\overline{n}\}(f)$ we get

$$(\forall v)(v \overset{\star}{\in} y \rightarrow (\exists i)(v \overset{\star}{\in} \{\overline{f_{\text{iter}}}\}(f, i))).$$

A combination of this yields

$$(\forall v \leq u)(\exists i)(v \overset{\star}{\in} y \rightarrow v \overset{\star}{\in} \{\overline{f_{\text{iter}}}\}(f, i)).$$

Using an instance of Π^0_2-CP (which we can by 3.4.4 and 3.5.9), we get

$$(\exists i_0)(\forall v \leq u)(\exists i < i_0)(v \overset{\star}{\in} y \rightarrow v \overset{\star}{\in} \{\overline{f_{\text{iter}}}\}(f, i)).$$

And since $v \overset{\star}{\in} y \rightarrow v \leq u$ and $\{\overline{f_{\text{iter}}}\}(f, i) \overset{\star}{\subseteq} \{\overline{f_{\text{iter}}}\}(f, i_0)$ for $i < i_0$ we are able to deduce

$$(\exists i_0)(y \overset{\star}{\subseteq} \{\overline{f_{\text{iter}}}\}(f, i_0))$$

and so we get

$$x \overset{\star}{\in} \{f\}(\{\overline{f_{\text{iter}}}\}(f, i_0)),$$

which, by definition of f_{iter}, leads to

$$x \stackrel{\star}{\in} \{\overline{f_{\text{iter}}}\}(f, \mathrm{S}(i_0))$$

and further, by definition of n, to

$$x \stackrel{\star}{\in} \{\overline{n}\}(f).$$

So we have shown $\{f\}(\overline{n}) \stackrel{\star}{\subseteq} \overline{n}$.

Now finally assume we have a s, such that $\{f\}(s) = s$. Using an instance of Π^0_2-Ind we can show

$$(\forall i)(\{\overline{f_{\text{iter}}}\}(f, i) \stackrel{\star}{\subseteq} s)$$

and so, by definition of n again,

$$\{\overline{n}\}(f) \stackrel{\star}{\subseteq} s,$$

which concludes our proof. □

Remark 3.5.13 A little extra thinking reveals that $\{n\}$ is even a primitive-recursive function, so we could (but will not) use a function symbol from the language of arithmetic to represent it.

4 Proof Theory

In [Fef79] Feferman showed (amongst others) the following proof-theoretic results about T_0 and EM_0

$$\mathsf{EM}_0 + \mathsf{J} \equiv \Sigma^1_1\text{-}\mathsf{AC}$$

$$T_0 = \mathsf{EM}_0 + \mathsf{J} + \mathsf{IG} \leq \Sigma^1_2\text{-}\mathsf{AC} + \mathsf{BI}$$

where IG stands for the axiom of inductive generation, BI for the principle of bar induction and Σ^1_n-AC for the axiom scheme for the axiom of choice for Σ^1_n-formulas. Furthermore Feferman conjectured

$$T_0 \equiv \Sigma^1_2\text{-}\mathsf{AC} + \mathsf{BI},$$

which was proved by Jäger in [Jäg83].

In [Fef82] Feferman raised the question about the strength of $\mathsf{T}_0 + \mathsf{MID}$ and $\mathsf{T}_0 + \mathsf{UMID}$. In [Tak89] Takahashi showed that

$$\mathsf{EM}_0 + \mathsf{J} + \mathsf{MID} \equiv \Pi^1_1\text{-}\mathsf{CA}$$

and also that T_0 is interpretable in Π^1_2-$\mathsf{CA} + \mathsf{BI}$, where Π^1_n-CA stands for the the axiom scheme of comprehension for Π^1_n-formulas.

A survey about some further answers to Feferman's question can be found in [Rat02]. One interesting fact is, that $\mathsf{T}_0{\restriction} + \mathsf{MID}$ proves the consistency of T_0 (where $\mathsf{T}_0{\restriction}$ is a version of T_0 with restricted inductive generation $\mathsf{IG}{\restriction}$ and type induction T-I_N instead of full formula induction $\mathcal{F}^\mathsf{E}$-I_N) and that UMID is even stronger. Rathjen also conjectured

$$\mathsf{T}_0 + \mathsf{UMID}_\mathsf{N} \equiv \mathsf{T}_0 + \mathsf{UMID} \equiv \Pi^1_2\text{-}\mathsf{CA} + \mathsf{BI}.$$

and the following results about UMID could be proved:

$$\mathsf{T}_0{\restriction} + \mathsf{UMID}_\mathsf{N} \equiv \Pi^1_2\text{-}\mathsf{CA}{\restriction},$$

$$\mathsf{T}_0{\restriction} + \mathsf{IND}_\mathbb{N} + \mathsf{UMID}_\mathsf{N} \equiv \Pi^1_2\text{-}\mathsf{CA}$$

and

$$\Pi^1_2\text{-}\mathsf{CA} < \mathsf{T}_0 + \mathsf{UMID}_\mathsf{N} \leq \Pi^1_2\text{-}\mathsf{CA} + \mathsf{BI},$$

where $\mathsf{IND}_\mathbb{N}$ is $\mathcal{F}^\mathsf{E}$-I_N in our notation and Π^1_2-$\mathsf{CA}{\restriction}$ is second order arithmetic with Π^1_2 comprehension schema and induction restricted to Π^1_2 sets.

While a lot of investigation about MID and UMID in the setting of classical logic has been done, the question about these principles in intuitionistic Explicit Mathematics was only recently addressed in [Tup04], where Tupailo, using a result of Möllerfeld from [Möl02], showed that

$$(\mathsf{EETJ}{\restriction} + \mathsf{UMID}_\mathsf{N})^i \equiv (\mathsf{T}_0{\restriction} + \mathsf{UMID}_\mathsf{N})^i \equiv \Pi^1_2\text{-}\mathsf{CA}{\restriction},$$

where $(.)^i$ denotes the intuitionistic version of the respective theory and $\mathsf{EETJ}\!\upharpoonright$ is $\mathsf{EET}^- + \mathsf{J} + \mathsf{T}\text{-}\mathsf{I_N}$ in our notation. Thus, using intuitionistic logic does not change the proof theoretic strength.

The exact strength of $\mathsf{T}_0 + \mathsf{UMID}$ remains an open question in both classical and intuitionistic logic.

In this section we will give lower and upper bounds for the proof-theoretic strength of systems of Explicit Mathematics T with

$$\Sigma^+\mathsf{ET} \;\subseteq\; T \;\subseteq\; \Sigma^+\mathsf{ET} + \Sigma^+\mathsf{S\text{-}C} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}.$$

For the lower bounds we will map the primitive-recursive function symbols to terms of the lambda calculus, for the upper bounds we will basically map terms $s \cdot t$ to $\{s\}(t)$ and use the results from the formalised recursion theory. For systems $T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N}$, i.e. systems with full formula induction, we will get exact results, but for systems $T + \mathsf{T}\text{-}\mathsf{I_N}$, i.e. systems with type induction, the question of determining the exact proof-theoretic strength remains open.

4.1 Lower Bounds

In this part we will embed systems of arithmetic into systems of Explicit Mathematics. This is done in a very classical way by mapping the primitive-recursive functions to their lambda-calculus counterparts. As this method is very familiar we will not go into detail and refer mostly to [Krä06] for a full account of the proofs.

4.1.1 The Embedding

Definition 4.1.1 (Numerals $\overline{n} \in \mathcal{T}^{\mathsf{E}}$)
For each $n \in \mathbb{N}$ the term $\overline{n} \in \mathcal{T}^{\mathsf{E}}$ is defined inductively as follows

(1) $\overline{0} := \mathsf{0}$,

(2) $\overline{n+1} := (\mathsf{s_N} \cdot \overline{n})$.

Definition 4.1.2 (mapping $\hat{.} : \mathcal{V}_A \to \mathcal{V}_I$)
Let $\hat{.} : \mathcal{V}_A \to \mathcal{V}_I$ be a mapping with following properties

– $\hat{.}$ is one-to-one

– $\mathcal{V}_I \setminus \hat{\mathcal{V}}_I$ is (countably) infinite

where $\hat{\mathcal{V}}_I := \{\hat{v} \; ; \; v \in \mathcal{V}_A\}$.

Definition 4.1.3 (mapping $.^\circ : \mathcal{T}^{\mathsf{A}} \to \mathcal{T}^{\mathsf{E}}$)
The mapping $.^\circ : \mathcal{T}^{\mathsf{A}} \to \mathcal{T}^{\mathsf{E}}$ is defined inductively for each term $t \in \mathcal{T}^{\mathsf{A}}$ by

(1) if t is $\mathsf{0}$, then t° is $\mathsf{0}$,

(2) if t is $x \in \mathcal{V}_A$, then t° is $\hat{x}$,

(3) if t is $f(t_0, \dots, t_n)$, then t° is $f^\circ t_0^\circ \dots t_n^\circ$,

where f° is defined inductively by

– if f is S, then f° is $\mathsf{s_N}$,

– if f is Cs_i^n, then f° is $\lambda x_1 \dots x_n.\overline{i}$,

– if f is Pr_i^n, then f° is $\lambda x_1 \dots x_n.x_{i+1}$,

– if f is $\mathrm{Comp}^n(h, g_1, \dots, g_m)$, then f° is

$$\lambda x_1 \dots x_n.h^\circ(g_1^\circ x_1 \dots x_n) \dots (g_m^\circ x_1 \dots x_n),$$

– if f is $\mathrm{Rec}^{n+1}(h, g)$ then f° is

$$\lambda x_1 \dots x_n.\mathsf{rec}(g^\circ x_1 \dots x_n)(h^\circ x_1 \dots x_n),$$

for each primitive recursive function symbol f and for all natural numbers $i, n \in \mathbb{N}$ with the additional condition that $x_1, \ldots x_n \in \mathcal{V}_I \setminus \hat{\mathcal{V}}_I$.

Definition 4.1.4 (mapping $.^\circ : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$)
The mapping $.^\circ : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$ is defined inductively for each formula $\varphi \in \mathcal{F}^{\mathsf{A}}$ by

(1) if φ is $s = t$ where $s, t \in \mathcal{T}^{\mathsf{A}}$ are terms, then φ° is $t^\circ = s^\circ$,

(2) if φ is $\neg\psi$ where $\psi \in \mathcal{F}^{\mathsf{A}}$ is a formula, then φ° is $\neg(\psi^\circ)$,

(3) if φ is $\psi_1 \vee \psi_2$ where $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{A}}$ a formulas, then φ° is $\psi_1^\circ \vee \psi_2^\circ$,

(4) if φ is $\psi_1 \wedge \psi_2$ where $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{A}}$ a formulas, then φ° is $\psi_1^\circ \wedge \psi_2^\circ$,

(5) if φ is $(\exists x)\psi$ where $\psi \in \mathcal{F}^{\mathsf{A}}$ is a formula and $x \in \mathcal{V}_A$ is a variable, then φ° is $(\exists \hat{x} \in \mathsf{N})\psi^\circ$,

(6) if φ is $(\forall x)\psi$ where $\psi \in \mathcal{F}^{\mathsf{A}}$ is a formula and $x \in \mathcal{V}_A$ is a variable, then φ° is $(\forall \hat{x} \in \mathsf{N})\psi^\circ$.

Definition 4.1.5 (embedding $.^N : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$)
The embedding $.^N : \mathcal{F}^{\mathsf{A}} \to \mathcal{F}^{\mathsf{E}}$ is defined by

$$\varphi^N := \begin{cases} \varphi^\circ, & \text{if } \mathrm{FV}_A(\varphi) = \emptyset, \\ \hat{x_0} \in \mathsf{N} \wedge \ldots \wedge \hat{x_n} \in \mathsf{N} \to \varphi^\circ, & \text{if } \mathrm{FV}_A(\varphi) = \{x_0, \ldots, x_n\}, \end{cases}$$

for each formula $\varphi \in \mathcal{F}^{\mathsf{A}}$.

4.1.2 The Embedding Theorems

Theorem 4.1.6 (Embedding Theorem)
For all $\varphi \in \mathcal{F}^{\mathsf{A}}$ we have

(1) $\mathsf{PRA} \vdash \varphi \Rightarrow \mathsf{BON}^- + \mathsf{V}\text{-}\mathsf{I_N} \vdash \varphi^N$.

(2) $\mathsf{PA} \vdash \varphi \Rightarrow \mathsf{BON}^- + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \vdash \varphi^N$.

Proof. See [Krä06]. □

Corollary 4.1.7
Let $T = \Sigma^+\mathsf{ET} + \mathsf{J} + \forall\mathfrak{R} + \forall\mathsf{N} + \mathsf{UMID}$. For all $\varphi \in \mathcal{F}^{\mathsf{A}}$ we have

(1) $\mathsf{PRA} \vdash \varphi \Rightarrow T + \mathsf{V}\text{-}\mathsf{I_N} \vdash \varphi^N$.

(2) $\mathsf{PA} \vdash \varphi \Rightarrow T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \vdash \varphi^N$.

4.2 Upper Bounds

In this section we will embed systems of Explicit Mathematics into systems of Arithmetic. The main idea is to map a term $s \cdot t$ to $\{s\}(t)$, as we have already done in the consistency proof. So it is not a big surprise that we will use the formalised recursion theoretic results in order to proof the translation of UMID. As you will see, we have again chosen the terminology and the notations as similar as possible to the embedding in [Krä06] and we will also see that these embeddings share some important basic properties. So it will be possible to refer to [Krä06] for some results that rely only on this general aspects and focus on the more specific parts of our systems.

4.2.1 The Embedding

Remark 4.2.1 T and S denote the same functions as defined in 3.5.1.

Definition 4.2.2 ($\hat{.}$)
Let $\hat{.} : \mathcal{V}_I \cup \mathcal{V}_T \to \mathcal{V}_A$ be a mapping with following properties

– $\hat{.}$ is one-to-one

– $\mathcal{V}_A \setminus \hat{\mathcal{V}}_A$ is (countably) infinite

where $\hat{\mathcal{V}}_A := \{\hat{v} \ ; \ v \in \mathcal{V}_I \cup \mathcal{V}_T\}$.

Definition 4.2.3 (α, β)
The mappings $\alpha : \mathcal{V}_A^3 \to \mathcal{F}^{\mathsf{A}}$ and $\beta : \mathcal{V}_A^2 \to \mathcal{F}^{\mathsf{A}}$ are defined by

(1) $\alpha(x, y, z) = (\exists u)(T(u, x, y, z) = 0)$,

(2) $\beta(y, x) = (\exists u)(S(u, x, y) = 0)$,

for all variables $x, y, z \in \mathcal{V}_A$ with the additional condition that $z \in \mathcal{V}_A \setminus \hat{\mathcal{V}}_A$ is different from x, y, z.

Definition 4.2.4 (γ)
The mapping $\gamma : \mathcal{T}^{\mathsf{E}} \times \mathcal{V}_A \to \mathcal{F}^{\mathsf{A}}$ is inductively defined for each term $t \in \mathcal{T}^{\mathsf{E}}$ and variable $x \in \mathcal{V}_A$ by

(1) if t is $c \in \mathcal{C}^{\mathsf{E}}$, then $\gamma(t, x)$ is $\bar{\hat{c}} = x$. Here $\hat{c}$ denotes the natural number which is an index for the desired function as in 2.4.21 and 3.5.12,

(2) if t is $y \in \mathcal{V}_I$, then $\gamma(t, x)$ is $\hat{y} = x$,

(3) if t is $t_1 \ldots t_2$ with $t_1, t_2 \in \mathcal{T}^{\mathsf{E}}$, then $\gamma(t, x)$ is $(\exists y_1)(\exists y_2)(\gamma(t_1, y_1) \wedge \gamma(t_2, y_2) \wedge \alpha(y_1, y_2, x)$.

Definition 4.2.5 (the embedding $.^* : \mathcal{F}^{\mathsf{E}} \to \mathcal{F}^{\mathsf{A}}$)
The mapping $.^* : \mathcal{F}^{\mathsf{E}} \to \mathcal{F}^{\mathsf{A}}$ is inductively defined for each formula $\varphi \in \mathcal{F}^{\mathsf{E}}$ of Explicit Mathematics by

(1) if φ is $s = t$ with $s, t \in \mathcal{T}^{\mathsf{E}}$ terms, then φ^* is $(\exists x)(\gamma(s,x) \wedge \gamma(t,x))$,

(2) if φ is $t\downarrow$ with $t \in \mathcal{T}^{\mathsf{E}}$ a term, then φ^* is $(\exists x)(\gamma(t,x))$,

(3) if φ is $t \in \mathsf{N}$ with $t \in \mathcal{T}^{\mathsf{E}}$ a term, then φ^* is $(\exists x)(\gamma(t,x))$,

(4) if φ is $t \in X$ with $t \in \mathcal{T}^{\mathsf{E}}$ a term and $X \in \mathcal{V}_T$ a type variable, then φ^* is $(\exists x)(\gamma(t,x) \wedge \beta(x, \hat{X}))$,

(5) if φ is $\mathfrak{R}(t, X)$ with $t \in \mathcal{T}^{\mathsf{E}}$ a term and $X \in \mathcal{V}_T$ a type variable, then φ^* is $(\exists x)(\gamma(t,x) \wedge (\forall y)(\beta(y,x)) \leftrightarrow \beta(y, \hat{X})))$,

(6) if φ is $\neg\psi$ with $\psi \in \mathcal{F}^{\mathsf{E}}$ a formula, then φ^* is $\neg(\varphi^*)$,

(7) if φ is $\psi_1 \vee \psi_2$ with $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{E}}$ formulas, then φ^* is ${\psi_1}^* \vee {\psi_2}^*$,

(8) if φ is $\psi_1 \wedge \psi_2$ with $\psi_1, \psi_2 \in \mathcal{F}^{\mathsf{E}}$ formulas, then φ^* is ${\psi_1}^* \wedge {\psi_2}^*$,

(9) if φ is $(\exists x)\psi$ with $\psi \in \mathcal{F}^{\mathsf{E}}$ a formula and $x \in \mathcal{V}_I$ an individual variable, then φ^* is $(\exists \hat{x})\psi^*$,

(10) if φ is $(\forall x)\psi$ with $\psi \in \mathcal{F}^{\mathsf{E}}$ a formula and $x \in \mathcal{V}_I$ an individual variable, then φ^* is $(\forall \hat{x})\psi^*$,

(11) if φ is $(\exists X)\psi$ with $\psi \in \mathcal{F}^{\mathsf{E}}$ a formula and $X \in \mathcal{V}_T$ a type variable, then φ^* is $(\exists \hat{X})\psi^*$,

(12) if φ is $(\forall X)\psi$ with $\psi \in \mathcal{F}^{\mathsf{E}}$ a formula and $X \in \mathcal{V}_T$ a type variable, then φ^* is $(\forall \hat{X})\psi^*$,

where $x, y \in \mathcal{V}_A \setminus \hat{\mathcal{V}}_A$ are different variables.

Lemma 4.2.6
Let $\varphi \in \mathcal{F}^{\mathsf{E}}$ be a formula of Explicit Mathematics. Then

(1) $x \in FV_I(\varphi) \quad \Leftrightarrow \quad \hat{x} \in FV_A(\varphi^*)$,

(2) $x \in FT_I(y, \varphi) \quad \Leftrightarrow \quad \hat{x} \in FT_A(\hat{y}, \varphi^*)$,

(3) $\vdash (\varphi[\vec{y}/\vec{x}])^* \leftrightarrow (\varphi^*)[\hat{y_0}/\hat{x_0}, \dots, \hat{y_n}/\hat{x_n}]$,

(4) $X \in FV_T(\varphi) \quad \Leftrightarrow \quad \hat{X} \in FV_A(\varphi^*)$,

(5) $X \in FT_T(Y, \varphi) \quad \Leftrightarrow \quad \hat{X} \in FT_A(\hat{Y}, \varphi^*)$,

(6) $\vdash (\varphi[\vec{Y}/\vec{X}])^* \leftrightarrow (\varphi^*)[\hat{Y}_0/\hat{X}_0, \dots, \hat{Y}_n/\hat{X}_n]$,

holds for all individual variables $x, y, x_0, \dots, x_n, y_0, \dots, y_n \in \mathcal{V}_I$ with $y_i \in FT_I(x_i, \varphi)$ and type variables $X, Y, X_0, \dots, X_n, Y_0, \dots, Y_n \in \mathcal{V}_T$ with $Y_i \in FT_T(X_i, \varphi)$.

Proof. Obvious. □

Lemma 4.2.7

(1) Let $s, t \in \mathcal{T}^{\mathsf{E}}$ be terms, $x \in \mathcal{V}_I$ an individual variable and $y, z \in \mathcal{V}_A$ be variables such that $z \neq \hat{x}$ and $y \in FT_A(\hat{x}, \gamma(s, z))$. Then

$$\mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\gamma(s[t/x], z) \leftrightarrow \gamma(s, z)[y/\hat{x}].$$

(2) Let $\varphi \in \mathcal{F}^{\mathsf{E}}$ be a formula of Explicit Mathematics, $x \in \mathcal{V}_I$ an individual variable and $y \in \mathcal{V}_A$ a variable such that $y \in FT_A(\hat{x}, \varphi^*)$. Then

$$\mathsf{PRA} \vdash \gamma(t, y) \rightarrow (\varphi[t/x]^* \leftrightarrow \varphi^*[y/\hat{x}]).$$

Proof. (1) By induction on the term s.

(2) By induction on the formula φ. □

Lemma 4.2.8

Let $x, y, z \in \mathcal{V}_I$ be individual variables. Then

(1) $\mathsf{PRA} \vdash (x \in \mathsf{N})^*$

(2) $\mathsf{PRA} \vdash ((\forall x \in \mathsf{N})\varphi)^* \leftrightarrow (\forall \hat{x})\varphi^*$

(3) $\mathsf{PRA} \vdash (\mathfrak{R}(x))^*$

(4) $\mathsf{PRA} \vdash (y \dot{\in} x)^* \leftrightarrow \beta(\hat{y}, \hat{x})$

(5) $\mathsf{PRA} \vdash (x \cdot y = z)^* \leftrightarrow \alpha(\hat{x}, \hat{y}, \hat{z})$

Proof. (1) $(x \in \mathsf{N})^*$ is $(\exists y)(\hat{x} = y)$, which is surely provable in PRA.

(2) $((\forall x \in \mathsf{N})\varphi)^*$ is $(\forall \hat{x})((x \in \mathsf{N})^* \rightarrow \varphi^*)$, putting this together with the result above yields the desired statement.

(3) $(\mathfrak{R}(x))^*$ is $(\exists \hat{X})(\exists z)[\hat{x} = z \wedge (\forall y)(\beta(y, z) \leftrightarrow \beta(y, \hat{X}))]$. Since $\mathsf{PRA} \vdash \hat{x} = \hat{x} \wedge (\forall y)[\beta(y, \hat{x}) \leftrightarrow \beta(y, \hat{x})]$ we get the desired result by suitable substitutions.

(4) $(x \dot{\in} y)^*$ is $(\exists \hat{Y})(\exists (u))(\hat{y} = u \wedge (\forall v)(\beta(v, u) \leftrightarrow \beta(v, \hat{Y})) \wedge (\exists w)(\hat{x} = w \wedge \beta(w, \hat{Y})))$, so we have $\mathsf{PRA} \vdash (x \dot{\in} y)^* \leftrightarrow (\exists \hat{Y})[(\forall v)(\beta(v, \hat{y}) \leftrightarrow \beta(v, \hat{Y})) \wedge \beta(\hat{x}, \hat{Y})]$ which immediately leads to the desired results.

(5) $(x \cdot y = z)^*$ is $(\exists u)(\exists v_1)(\exists v_2)[\hat{x} = v_1 \wedge \hat{y} = v_2 \wedge \alpha(v_1, v_2, u) \wedge \hat{z} = u]$. Applying the same reasoning as above will give the desired result. □

4.2.2 The Embedding Theorems

Theorem 4.2.9
Let $T = \mathsf{\Sigma^+ET} + \mathsf{\Sigma^+S\text{-}C} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}$. *For all* $\varphi \in \mathcal{F}^{\mathsf{E}}$

(1) $T + \mathsf{T\text{-}I_N} \vdash \varphi \quad \Rightarrow \quad \mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind} \vdash \varphi^*$.

(2) $T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \vdash \varphi \quad \Rightarrow \quad \mathsf{PA} \vdash \varphi^*$.

Proof. We prove the statement by induction on the length of the T-proof of φ. By 2.5.8 we do not have to care about the axioms in $\mathsf{\Sigma^+S\text{-}C}$.

Since our embedding satisfies the same basic properties as the embedding in [Krä06], the proofs for the logical axioms and rules will be the same ones and are therefore omitted. We only mention that we can prove the translation of instances of the type induction axiom schema by appropriate instances of the Σ^0_1 induction schema because the relation of set membership translates to Σ^0_1 formulas.

The proofs for the non-logical axioms rely on the right choice of constant, i.e. the indices of the corresponding partial-recursive functions. We give an example for the case that φ is $(\mathsf{k}x)y = x$.

φ^* is equivalent to $(\exists z)(\exists u_1)(\exists u_2)(\exists v_1)(\exists v_2)[\hat{x} = z \wedge \hat{y} = u_2 \wedge \hat{\mathsf{k}} = v_1 \wedge \hat{x} = v_2 \wedge \alpha(v_1, v_2, u_1) \wedge \alpha(u_1, u_2, z)]$.

Let f be a function symbol such that

$$\mathsf{PRA}^+ \vdash f(x, y) = x.$$

We find an index $n \in \mathbb{N}$ of f such that

$$\mathsf{PRA}^+ \vdash f(x, y) = z \leftrightarrow \alpha(\overline{n}, \langle x, y\rangle, z).$$

By 2.4.2 we can find a $s \in \mathbb{N}$ such that

$$\mathsf{PRA}^+ \vdash \alpha(\overline{n}, \langle x, y\rangle, z) \leftrightarrow (\alpha(\overline{s}, \langle \overline{n}, x\rangle, u) \wedge \alpha(u, y, z))$$

and by a second application of 2.4.2 we get a $t \in \mathbb{N}$ such that

$$\mathsf{PRA}^+ \vdash \alpha(\overline{s}, \langle \overline{n}, x\rangle, z) \leftrightarrow (\alpha(\overline{t}, \overline{n}, u) \wedge \alpha(u, x, z)).$$

By construction we have that

$$\mathsf{PRA}^+ \vdash \alpha(\overline{t}, \overline{n}, \overline{\hat{\mathsf{k}}}).$$

Putting this all together in the right way and using 4.2.8 will give

$$\mathsf{PRA}^+ \vdash \varphi^*.$$

For the case that φ is an instance of UMID, we refer to 3.5.12 and the fact that our embedding is basically the interpretation of the application operator as in 2.4.21 using Kleene terms as defined in 3.5.1. □

4.3 Results

Lemma 4.3.1

(1) Let $t \in \mathcal{T}^{\mathsf{A}}$ be a term of Arithmetic with $FV_A(t) = \{x_0, \ldots, x_n\}$. Then

$$\mathsf{PRA} \vdash \gamma(t^N, x) \leftrightarrow t[\hat{x_0}^N/x_0, \ldots, \hat{x_n}^N/x_n].$$

(2) Let $\varphi \in \mathcal{F}^{\mathsf{A}}$ be a formula of Arithmetic with $FV_A(\varphi) = \{x_0, \ldots, x_n\}$.

$$\mathsf{PRA} \vdash (\varphi^N)^* \leftrightarrow \varphi[\hat{x_0}^N/x_0, \ldots, \hat{x_n}^N/x_n].$$

Proof. By induction on terms (which requires an induction on function symbols first) and formulas. This result is again basically the same as in [Krä06] since the embedding $.^N$ is the same and the embedding $.^*$ shares the same necessary basic properties. □

Theorem 4.3.2 (Proof-Theoretic Equivalence)
Let $T \subseteq \mathcal{F}^{\mathsf{E}}$ be a set of formulas with

$$\Sigma^+\mathsf{ET} \ \subseteq \ T \ \subseteq \ \Sigma^+\mathsf{ET} + \Sigma^+\mathsf{S\text{-}C} + \mathsf{J} + \forall\mathsf{N} + \forall\mathfrak{R} + \mathsf{UMID}.$$

We have

(1)

$$\mathsf{PA} \equiv T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N}$$

where $\equiv$ stands for proof-theoretic equivalence, which means PA and $T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N}$ have the same provable arithmetic sentences, i.e. for all sentences $\varphi \in \mathcal{F}^{\mathsf{A}}$ we have

$$\mathsf{PA} \vdash \varphi \quad \textit{if and only if} \quad T + \mathcal{F}^{\mathsf{E}}\text{-}\mathsf{I_N} \vdash \varphi^N.$$

(2)

$$\mathsf{PRA}^+ \leq T + \mathsf{T\text{-}I_N} \leq \mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind}$$

where $\leq$ gives the proof-theoretic upper and lower bound for provable arithmetic sentences, which means for all sentences $\varphi \in \mathcal{F}^{\mathsf{A}}$ we have

$$\mathsf{PRA}^+ \vdash \varphi \quad \textit{implies} \quad T + \mathsf{T\text{-}I_N} \vdash \varphi^N$$

and

$$T + \mathsf{T\text{-}I_N} \vdash \varphi^N \quad \textit{implies} \quad \mathsf{PRA}^+ + \Pi^0_2\text{-}\mathsf{Ind} \vdash \varphi.$$

Proof. Combine 4.1.6, 4.1.7, 4.2.9 and 4.3.1. □

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