Full operational set theory with unbounded existential quantification and power set

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Abstract

We study the extension $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ of Feferman's operational set theory OST provided by adding operational versions of unbounded existential quantification and power set and determine its proof-theoretic strength in terms of a suitable theory of sets and classes.

Keywords: Operational set theory, proof theory, theories of sets and classes.

1 Introduction

This paper is a direct sequel of Jäger [15] and answers the question about the exact consistency-theoretic strength of Feferman's operational set theory extended by operational versions of unbounded existential quantification and power set. We will show that this system, called $\mathsf{OST}(\mathbf{E}, \mathbb{P})$, is equiconsistent with a natural theory of sets and classes $\mathsf{NBG}_{<E_0}$ which is obtained from usual von Neumann-Bernays-Gödel theory NBG by adding $\in$-induction for arbitrary formulas and a specific form of iteration of elementary class comprehension.

Up to a certain degree, operational set theory may be regarded as a set-theoretic variant of explicit mathematics; see Feferman [7]. In its current form, OST has been introduced in Feferman [8] and is also presented in the more recent Feferman [9]. A series of proof-theoretic results about OST and some of its most interesting extensions is established in Jäger [15], and we refer the reader to these three articles for motivation and all sorts of background information about operational set theory. Cantini and Crosilla [6] is a further interesting approach to operational set theory and devoted to its relationship to constructive set theory à la Aczel [1, 2, 3].

As mentioned above, this paper deals with a specific problem left open so far. Familiarity with Feferman [8, 9] and/or Jäger [15] is highly desirable in order to be able to appreciate the general setting and some of the technical approaches which can only be sketched in the following.

We begin with a brief introduction of the theory $\mathsf{OST}(\mathbf{E},\mathbb{P})$, following more or less directly Jäger [15]. Afterwards we turn to von Neumann-Bernays-Gödel set theory NBG and its extension $\mathsf{NBG}_{<E_0}$, obtained by iterating elementary class comprehension sufficiently. The next section deals with interpreting $\mathsf{NBG}_{<E_0}$ into $\mathsf{OST}(\mathbf{E},\mathbb{P})$, before we reduce $\mathsf{OST}(\mathbf{E},\mathbb{P})$ to $\mathsf{NBG}_{<E_0}$ – thus establishing their proof-theoretic equivalence – via an intermediate theory of inductive definitions over set theory.

2 The theory $\mathsf{OST}(\mathbf{E},\mathbb{P})$

Let $\mathcal{L}_1$ be a typical language of admissible or classical set theory with a symbol for the element relation as its only relation symbol and countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \ldots$ (possibly with subscripts). The formulas of $\mathcal{L}_1$ are defined as usual.

$\mathcal{L}^\circ$, the language of $\mathsf{OST}(\mathbf{E},\mathbb{P})$, augments $\mathcal{L}_1$ by the binary function symbol $\circ$ for partial term application, the unary relation symbol $\downarrow$ (defined) and the following constants: (i) the combinators k and s; (ii) $\top$, $\bot$, **el**, **non**, **dis**, **e** and $\mathbf{E}$ for logical operations; (iii) $\mathbb{S}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{P}$ for set-theoretic operations. The meaning of these constants follows from the axioms below.

The *terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. The variables and constants of $\mathcal{L}^\circ$ are terms of $\mathcal{L}^\circ$.
2. If s and t are terms of $\mathcal{L}^\circ$, then so is $\circ(s,t)$.

In the following we often abbreviate $\circ(s,t)$ as $(s \circ t)$, (st) or simply as st. We also adopt the convention of association to the left so that $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2)\ldots s_n)$. In addition, we often write $s(t_1,\ldots,t_n)$ for $st_1\ldots t_n$ if this seems more intuitive. Moreover, we frequently make use of the vector notation $\vec{s}$ as shorthand for a finite string $s_1,\ldots,s_n$ of $\mathcal{L}^\circ$ terms whose length is either not important or evident from the context.

Self-application is possible and meaningful, but it is not necessarily total, and there may be terms which do not denote an object. We make use of the definedness predicate $\downarrow$ to single out those which do, and $(t\downarrow)$ is read "t is defined" or "t has a value".

The formulas $(A, B, C, D, A_1, B_1, C_1, D_1, \ldots)$ of $\mathcal{L}^\circ$ are inductively generated as follows:

1. All expressions of the form $(s \in t)$ and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$; the so-called *atomic* formulas.

2. If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$, $(A \vee B)$ and $(A \wedge B)$.

3. If A is a formula of $\mathcal{L}^\circ$, then so are $\exists x A$ and $\forall x A$.

Since we will be working within classical logic, the remaining logical connectives can be defined as usual. We will often omit parentheses and brackets whenever there is no danger of confusion. The free variables of t and A are defined in the conventional way; the closed $\mathcal{L}^\circ$ terms and closed $\mathcal{L}^\circ$ formulas, also called $\mathcal{L}^\circ$ sentences, are those which do not contain free variables. Equality of sets is introduced by

$$(s = t) \;:=\; (s\downarrow) \wedge (t\downarrow) \wedge \forall x(x \in s \leftrightarrow x \in t).$$

Suppose now that $\vec{u} = u_1, \ldots, u_n$ and $\vec{s} = s_1, \ldots, s_n$. Then $A[\vec{s}/\vec{u}]$ is the $\mathcal{L}^\circ$ formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the $\mathcal{L}^\circ$ terms $\vec{s}$; in order to avoid collision of variables, a renaming of bound variables may be necessary. If the $\mathcal{L}^\circ$ formula A is written as $B[\vec{u}]$, then we often simply write $B[\vec{s}]$ instead of $B[\vec{s}/\vec{u}]$. Further variants of this notation will be obvious.

The logic of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ is the classical *logic of partial terms* due to Beeson [4, 5], including the common equality axioms. Partial equality of terms is introduced by

$$(s \simeq t) \;:=\; (s\downarrow \vee\, t\downarrow \;\rightarrow\; s = t)$$

and says that if either s or t denotes anything, then they both denote the same object.

The non-logical axioms of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ comprise axioms about the applicative structure of the universe, some basic set-theoretic properties, the representation of elementary logical connectives as operations and operational set existence axioms. They divide into four groups.

I. Applicative axioms.

(1) $\mathsf{k} \neq \mathsf{s}$,

(2) $\mathsf{k}xy = x$,

(3) $\mathsf{s}xy\downarrow \,\wedge\, \mathsf{s}xyz \simeq (xz)(yz)$.

Thus the universe is a partial combinatory algebra. We have λ-abstraction and thus can introduce for each $\mathcal{L}^\circ$ term t a term $(\lambda x.t)$ whose variables are those of t other than x such that

$$(\lambda x.t){\downarrow} \;\wedge\; (\lambda x.t)y \simeq t[y/x].$$

As usual we can generalize λ abstraction to several arguments by simply iterating abstraction for one argument. Accordingly, we set for all $\mathcal{L}^\circ$ terms t and all variables $x_1, \ldots, x_n$,

$$(\lambda x_1 \ldots x_n.t) := (\lambda x_1.(\ldots(\lambda x_n.t)\ldots)).$$

Often the term $(\lambda x_1 \ldots x_n.t)$ is also simply written as $\lambda x_1 \ldots x_n.t$. If $\vec{x}$ is the sequence $x_1, \ldots, x_n$, then $\lambda\vec{x}.t$ stands for $\lambda x_1 \ldots x_n.t$ and $t\vec{x}$ for $tx_1 \ldots x_n$.

Furthermore, there exists a closed $\mathcal{L}^\circ$ term fix, a so-called fixed point operator, with

$$\mathsf{fix}(f){\downarrow} \;\wedge\; (\mathsf{fix}(f) = g \;\rightarrow\; gx \simeq f(g, x)).$$

II. Basic set-theoretic axioms. They comprise: (i) the existence of the empty set; (ii) pair, union and infinity; (iii) $\in$-induction is available for arbitrary formulas $A[x]$ of $\mathcal{L}^\circ$,

$$(\mathcal{L}^\circ\text{-I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \;\rightarrow\; \forall x A[x].$$

To increase readability, we will freely use standard set-theoretic terminology. Also, if $A[x]$ is an $\mathcal{L}^\circ$ formula, then $\{x : A[x]\}$ denotes the collection of all sets satisfying A; it may be (extensionally equal to) a set, but this is not necessarily the case. In particular, we set

$$\mathbb{B} \;:=\; \{x : x = \top \vee x = \bot\} \quad \text{and} \quad \mathbb{V} \;:=\; \{x : x{\downarrow}\}$$

so that $\mathbb{B}$ stands for the unordered pair consisting of the truth values $\top$ and $\bot$, which is a set by the previous axioms. $\mathbb{V}$ is the collection of all sets but not a set itself. The following shorthand notations, for n an arbitrary natural number,

$$\begin{aligned}(f : a \rightarrow b) \;&:=\; (\forall x \in a)(fx \in b),\\ (f : a^{n+1} \rightarrow b) \;&:=\; (\forall x_1, \ldots, x_{n+1} \in a)(f(x_1, \ldots, x_{n+1}) \in b)\end{aligned}$$

express that f, in the operational sense, is a unary and $(n{+}1)$-ary mapping from a to b, respectively. They do not say, however, that f is a unary or $(n{+}1)$-ary function in the set-theoretic sense.

In the previous definition the set variables a and/or b may be replaced by $\mathbb{V}$ and/or $\mathbb{B}$. So, for example, $(f : a \to \mathbb{V})$ means that f is an operation which is total on a, and $(f : \mathbb{V} \to b)$ means that f maps all sets into b. If we have $(f : a \to \mathbb{B})$, we may regard f as a *definite predicate* on a. The n-ary *Boolean operations* are those f for which $(f : \mathbb{B}^n \to \mathbb{B})$.

III. Logical operations axioms.

(L1) $\top \neq \bot$.

(L2) $(\mathbf{el} : \mathbb{V}^2 \to \mathbb{B}) \ \wedge \ \forall x \forall y (\mathbf{el}(x, y) = \top \ \leftrightarrow \ x \in y)$.

(L3) $(\mathbf{non} : \mathbb{B} \to \mathbb{B}) \ \wedge \ (\forall x \in \mathbb{B})(\mathbf{non}(x) = \top \ \leftrightarrow \ x = \bot)$.

(L4) $(\mathbf{dis} : \mathbb{B}^2 \to \mathbb{B}) \ \wedge \ (\forall x, y \in \mathbb{B})(\mathbf{dis}(x, y) = \top \ \leftrightarrow \ (x = \top \ \vee \ y = \top))$.

(L5) $(f : a \to \mathbb{B}) \ \to \ (\mathbf{e}(f, a) \in \mathbb{B} \ \wedge \ (\mathbf{e}(f, a) = \top \ \leftrightarrow \ (\exists x \in a)(fx = \top)))$.

(L6) $(f : \mathbb{V} \to \mathbb{B}) \ \to \ (\mathbf{E}(f) \in \mathbb{B} \wedge (\mathbf{E}(f) = \top \ \leftrightarrow \ \exists x(fx = \top)))$.

Axiom (6) provides for unbounded existential quantification. It is not available in OST and gives us more power in representing formulas by terms; see the following lemma.

The *pure formulas* of $\mathcal{L}^\circ$ are those $\mathcal{L}^\circ$ formulas which do not contain the function symbol $\circ$ or the relation symbol $\downarrow$. Hence they are the usual formulas of set theory, possibly containing additional constants. The logical operations make it possible to represent all such formulas by constant $\mathcal{L}^\circ$ terms.

Lemma 1 *Let $\vec{u}$ be the sequence of variables $u_1, \ldots, u_n$. For every pure formula $A[\vec{u}]$ of $\mathcal{L}^\circ$ with at most the variables $\vec{u}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that the axioms introduced so far yield*

$$t_A\downarrow \ \wedge \ (t_A : \mathbb{V}^n \to \mathbb{B}) \ \wedge \ \forall \vec{x}(A[\vec{x}] \ \leftrightarrow \ t_A(\vec{x}) = \top).$$

In Feferman [8, 9] and Jäger [15] we have a corresponding result for OST and Δ_0 formulas. It should be obvious how, by making use of (6) to deal with unbounded quantifiers, it can be lifted to pure formulas of $\mathcal{L}^\circ$.

IV. Operational set-theoretic axioms.

(S1) Separation for definite operations:

$$(f : a \to \mathbb{B}) \ \to \ (\mathbb{S}(f, a)\downarrow \ \wedge \ \forall x(x \in \mathbb{S}(f, a) \ \leftrightarrow \ (x \in a \ \wedge \ fx = \top))).$$

(S2) Replacement:

$$(f : a \to \mathbb{V}) \;\to\; (\mathbb{R}(f,a)\downarrow \;\wedge\; \forall x(x \in \mathbb{R}(f,a) \;\leftrightarrow\; (\exists y \in a)(x = fy))).$$

(S3) Choice:

$$\exists x(fx = \top) \;\to\; (\mathbb{C}f\downarrow \;\wedge\; f(\mathbb{C}f) = \top).$$

(S4) Power set:

$$(\mathbb{P} : \mathbb{V} \to \mathbb{V}) \;\wedge\; \forall x \forall y(x \in \mathbb{P}y \;\leftrightarrow\; x \subset y).$$

This finishes the description of the non-logical axioms of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$. Recall that OST is the subsystem of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ in which the axioms (L6) and (S4) are omitted.

From Feferman [8] and Jäger [15] we know that, provably in the systems OST and $\mathsf{OST}(\mathbf{E}, \mathbb{P})$, there exist closed $\mathcal{L}^\circ$ terms $\emptyset$ for the empty set, **uopa** for forming unordered pairs, **un** for forming unions, **p** for forming ordered pairs (Kuratowski pairs) and **prod** for forming Cartesian products. In addition, there are closed $\mathcal{L}^\circ$ terms $\mathbf{p}_L$ and $\mathbf{p}_R$ which act as projections with respect to **p**, i.e.

$$\mathbf{p}_L(\mathbf{p}(a,b)) = a \quad \text{and} \quad \mathbf{p}_R(\mathbf{p}(a,b)) = b.$$

To comply with the set-theoretic conventions, we generally write $\{a, b\}$ instead of $\mathbf{uopa}(a,b)$, $\cup a$ instead of $\mathbf{un}(a)$, $\langle a, b\rangle$ instead of $\mathbf{p}(a,b)$ and $a \times b$ instead of $\mathbf{prod}(a,b)$.

We end this section with remarks about a form of definition by cases and global choice in OST and $\mathsf{OST}(\mathbf{E}, \mathbb{P})$. This will be relevant later, when we interpret an extension of von Neumann-Bernays-Gödel set theory into $\mathsf{OST}(\mathbf{E}, \mathbb{P})$.

Lemma 2 *There exist closed $\mathcal{L}^\circ$ terms $\mathsf{d}_=$, $\mathsf{d}_\emptyset$ and $\mathsf{d}_\mathbb{B}$ such that OST proves:*

1. $(u = v \;\wedge\; \mathsf{d}_=(a,b,u,v) = a) \;\vee\; (u \neq v \;\wedge\; \mathsf{d}_=(a,b,u,v) = b)$.
2. $(u = \emptyset \;\wedge\; \mathsf{d}_\emptyset(a,b,u) = a) \;\vee\; (u \neq \emptyset \;\wedge\; \mathsf{d}_\emptyset(a,b,u) = b)$.
3. $\mathsf{d}_\mathbb{B}(a,b,\top) = a \;\wedge\; \mathsf{d}_\mathbb{B}(a,b,\bot) = b$.

Proof. A lemma in Feferman [8, 9] and Jäger [15] about the representation of Δ_0 formulas implies that there exists a closed term t so that OST proves $(t : \mathbb{V}^5 \to \mathbb{B})$ and

$$t(a,b,c,u,v) = \top \;\leftrightarrow\; ((u = v \;\wedge\; c = a) \;\vee\; (u \neq v \;\wedge\; c = b)).$$

Now simply set $\mathsf{d}_= := \lambda abuv.\mathbb{C}(\lambda c.t(a,b,c,u,v))$ and verify that it has the required property. The terms $\mathsf{d}_\emptyset$ and $\mathsf{d}_\mathbb{B}$ are easily defined from $\mathsf{d}_=$. $\square$

Theorem 3 *There exists a closed $\mathcal{L}^\circ$ term* choice *such that* OST *proves*

$$(\mathsf{choice} : \mathbb{V} \to \mathbb{V}) \;\wedge\; \forall x(x \neq \emptyset \;\to\; \mathsf{choice}(x) \in x) \;\wedge\; \mathsf{choice}(\emptyset) = \top.$$

PROOF. We recall that any λ term is defined and pick choice to be the closed term defined by

$$\mathsf{choice} \;:=\; \lambda x.\mathsf{d}_\emptyset(\lambda y.\top, \lambda y.\mathbb{C}(\lambda z.\mathbf{el}(z,y)), x)x.$$

The assertion of our theorem follows from the axioms for **el** and $\mathbb{C}$ and the previous lemma. □

3 The theory $\mathsf{NBG}_{<E_0}$

A well-established theory of sets and classes is the so-called von Neumann-Bernays-Gödel set theory NBG. It is presented in full detail, for example, in Levy [19] and Mendelson [20]; here we confine ourselves on those facts which will be essential for what follows.

NBG is a theory of sets and classes conservative over the system ZFC of Zermelo-Fraenkel set theory with the axiom of choice. NBG is known to be finitely axiomatizable although the version we are going to present below permits axiom schemas and as such is an infinite axiomatization.

$\mathcal{L}_2$, the language of NBG, augments $\mathcal{L}_1$ by a second sort of countably many variables $U, V, W, X, Y, Z, \ldots$ (possibly with subscripts) for classes. The set terms of $\mathcal{L}_2$ are the terms of $\mathcal{L}_1$, as class terms we simply have the class variables.

The *formulas* $(A, B, C, D, A_1, B_1, C_1, D_1, \ldots)$ of $\mathcal{L}_2$ are inductively generated as follows:

1. If s and t are set terms of $\mathcal{L}_2$ and U is a class variable, then all expressions of the form $(s \in t)$ and $(s \in U)$ are (atomic) formulas of $\mathcal{L}_2$.

2. If A and B are formulas of $\mathcal{L}_2$, then so are are $\neg A$, $(A \vee B)$ and $(A \wedge B)$.

3. If A is a formula and t a set term of $\mathcal{L}_2$ which does not contain x, then $\exists x A$, $\forall x A$, $\exists X A$ and $\forall X A$ are formulas of $\mathcal{L}_2$.

As before, the remaining logical connectives are introduced as abbreviations, and we will often omit parentheses and brackets whenever there is no danger of confusion. Equalities between sets/sets, sets/classes, classes/sets and classes/classes are not atomic formulas of $\mathcal{L}_2$ but defined as

$$(\mathit{Var}_1 = \mathit{Var}_2) \;:=\; \forall x(x \in \mathit{Var}_1 \;\leftrightarrow\; x \in \mathit{Var}_2)$$

where Var_1 and Var_2 denote set or class variables. A formula of $\mathcal{L}_2$ is called *elementary* or a Π^1_0 formula if it does not contain bound class variables; free class variables, however, are permitted. The Σ^1_1 formulas of $\mathcal{L}_2$ are those of the form $\exists X A$ with elementary A.

The logic of NBG is classical two-sorted logic with equality for the first sort. The non-logical axioms of NBG are given in six groups. To increase readability, we freely use standard set-theoretic terminology.

I. Elementary comprehension For any elementary formula $A[u]$ of $\mathcal{L}_2$:

(ECA) $$\exists X \forall y (y \in X \leftrightarrow A[y]).$$

Hence every elementary NBG formula $A[u]$ defines a class, which is typically written as $\{x : A[x]\}$. It may be (extensionally equal to) a set, but this is not necessarily the case. The intersection of a class with a set, however, is always supposed to produce a set by the following principle of Aussonderung.

II. Aussonderung

(AUS) $$\forall X \forall y \exists z (z = X \cap y).$$

From logical reasons, (ECA) and (AUS) we conclude that there is a unique set which has no members; it is denoted by $\emptyset$.

III. Basic set existence

(Pair) $$\forall x \forall y \exists z (z = \{x, y\}),$$

(Union) $$\forall x \exists y (y = \cup x),$$

(Powerset) $$\forall x \exists y \forall z (z \in y \leftrightarrow z \subset x),$$

(Infinity) $$\exists x (\emptyset \in x \wedge (\forall y \in x)(y \cup \{y\} \in x)).$$

As in $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ we write $\langle a, b \rangle$ for the ordered pair of the sets a and b à la Kuratowski. Class relations are classes which consist of ordered pairs only, and class functions are class relations which assign to every set exactly one set; i.e. for all U we set:

$$\begin{aligned} Rel[U] &:= \forall x (x \in U \rightarrow \exists y \exists z (x = \langle y, z \rangle)), \\ Fun[U] &:= Rel[U] \wedge \forall x \exists ! y (\langle x, y \rangle \in U). \end{aligned}$$

If U is a function we write $U(x)$ for the uniquely determined y associated to x by U. Replacement states that the range of a set under a function is a set.

IV. Replacement

(REP) $\forall X(Fun[X] \to \forall y \exists z(z = \{X(u) : u \in y\}))$.

Global choice is a very uniform principle of choice which claims the existence of a class function which picks an element of any non-empty set.

V. Global choice

(GC) $\exists X(Fun[X] \wedge \forall y(y \neq \emptyset \to X(y) \in y))$.

To complete the list of axioms of NBG, we add foundation. In NBG it is claimed that the element relation is well-founded with respect to classes.

VI. Class foundation

(Class-$\mathrm{I}_\in$) $\forall X(X \neq \emptyset \to \exists y(y \in X \wedge X \cap y = \emptyset))$.

A set a is called an *ordinal* if a itself and all its elements are transitive, On stands for the class of all ordinals; i.e.

$$On := \{x : Tran[x] \wedge (\forall y \in x)\, Tran[y]\}.$$

The axioms (Infinity) and (Class-$\mathrm{I}_\in$) imply that there exists a least infinite ordinal, which we denote by ω, as usual. The elements of ω are identified with the natural numbers in the sense that $0 := \emptyset,\ 1 := \{0\},\ 2 := 1 \cup \{1\}$ and so on. In the following the first small Greek letters $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts) are supposed to range over On.

According to a well-known result, NBG is a conservative extension of ZFC. A proof of this fact can be found, for example, in Levy [19].

Theorem 4 *A sentence of the language $\mathcal{L}_1$ is provable in* NBG *if and only if it is provable in* ZFC.

In order to characterize $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ in terms of a theory of sets and classes we introduce the extension $\mathsf{NBG}_{<E_0}$ of the system NBG: we add to NBG the schema of $\in$-induction for arbitrary $\mathcal{L}_2$ formulas $A[u]$,

($\mathcal{L}_2$-$\mathrm{I}_\in$) $\forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x]$,

plus axioms (It-ECA) about specific iterations of elementary comprehension, to be described below. Before formulating them, we have to say a few words about the notation system $(E_0, \lhd)$.

The basic idea is very simple: $(E_0, \lhd)$ provides notations for all order types which we obtain from the ordinals together with the order type of the class of all ordinals by closing those under addition and ω-exponentiation. As such, $(E_0, \lhd)$ can be considered as the canonical blowing up of $(\varepsilon_0, <)$ triggered by replacing the natural numbers by the ordinals. In particular:

(i) E_0 is an elementarily definable class, and $\lhd$ is an elementarily definable strict linear ordering on E_0.

(ii) For any ordinal α the code $\overline{\alpha} := \langle 0, \alpha \rangle$ of α belongs to E_0; for any ordinals α and β, we have $\overline{\alpha} \lhd \overline{\beta}$ if and only if $\alpha < \beta$.

(iii) E_0 contains an element Ω such that $(\Omega, \lhd)$ is an isomorphic copy of $(On, <)$.

(iv) There are a binary class function $\oplus$ and a unary class function Exp_ω, both elementary, such that E_0 is closed under $\oplus$ and Exp_ω. These two functions are for the addition and ω-exponentiation of elements of E_0 in the expected sense.

In the following we write $(a + b)$ – or often simply $a + b$ – for $\oplus(a, b)$ and ω^a for $Exp_\omega(a)$. For all natural numbers n, the ordinal terms Ω_n are inductively defined by

$$\Omega_0 \;:=\; \omega^{\Omega+1} \qquad \text{and} \qquad \Omega_{n+1} \;:=\; \omega^{\Omega_n}.$$

All additional relevant details concerning $(E_0, \lhd)$ are worked out in detail in Jäger and Krähenbühl [16]. In particular, it is shown there that, for any standard natural number k, the theory $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathrm{I}_\in)$ proves transfinite induction along $\lhd$ up to Ω_k. To be precise, given an $\mathcal{L}_2$ formula $A[u]$, we set

$$TI_\lhd[v, A] \;:=\; \forall x((\forall y \lhd x)A[y] \to A[x]) \;\to\; (\forall x \lhd v)A[x],$$

formulating transfinite induction with respect to the formula $A[u]$ along the relation $\lhd$ up to v. See Jäger and Krähenbühl [16] for the following.

Lemma 5 *For any standard natural number k and for any formula A of the language $\mathcal{L}_2$ we have*

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathrm{I}_\in) \;\vdash\; TI_\lhd[\Omega_k, A].$$

Hierarchies of classes are coded in NBG by working with projections of classes. For this purpose, we set

$$(U)_a \;:=\; \{x : \langle a, x \rangle \in U\} \quad \text{and} \quad \Sigma(U, a) \;:=\; \{\langle b, x \rangle \in U : b \lhd a\}.$$

Therefore, $\Sigma(U, a)$ stands for the disjoint union of the projections of U, collected along $\lhd$ up to a.

Definition 6 *Let $A[U, V, u, v]$ be an elementary $\mathcal{L}_2$ formula with at most the variables U, V, u, v free. Then we write $Hier_A[a, U, V]$ for the elementary $\mathcal{L}_2$ formula*

$$(\forall b \lhd a)((V)_b = \{x : A[U, \Sigma(V, b), b, x]\}).$$

$\mathsf{NBG}_{<E_0}$ is the theory of sets and classes which extends $\mathsf{NBG} + (\mathcal{L}_2\text{-I}_\in)$ by claiming the existence of such hierarchies along each initial segment of E_0: the axioms of $\mathsf{NBG}_{<E_0}$ comprise the axioms of NBG, the schema $(\mathcal{L}_2\text{-I}_\in)$ plus the schema

(It-ECA) $$\forall X \exists Y \, \mathit{Hier}_A[\Omega_k, X, Y]$$

for all standard natural numbers k and all elementary formulas $A[U, V, u, v]$ of $\mathcal{L}_2$ with at most the variables U, V, u, v free.

4 Embedding $\mathsf{NBG}_{<E_0}$ into $\mathsf{OST}(\mathbf{E}, \mathbb{P})$

Our next aim is to show that $\mathsf{NBG}_{<E_0}$ can be embedded into $\mathsf{OST}(\mathbf{E}, \mathbb{P})$. To do so, we begin with selecting a translation $\check{}$ which maps the set variables u and class variables U of $\mathcal{L}_2$ to set variables $\check{u}$ and $\check{U}$ of $\mathcal{L}_1$ so that no conflicts arise. The basic idea is that the $\mathcal{L}_2$ set variables will be interpreted as ranging over the sets of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ and the $\mathcal{L}_2$ class variables as ranging over $\mathbb{T}$, the total operations from $\mathbb{V}$ to $\mathbb{B}$;

$$(f \in \mathbb{T}) \;:=\; \forall x(fx = \top \,\vee\, fx = \bot).$$

Consequently, the atomic formulas $(s \in U)$ and $(s \notin U)$ are interpreted as $(\check{U}\check{s} = \top)$ and $(\check{U}\check{s} = \bot)$, respectively.

Definition 7 *The translations A^+ and A^- of an $\mathcal{L}_2$ formula A are inductively defined as follows:*

1. *If A is a formula $(u \in v)$, then $A^+ := (\check{u} \in \check{v})$ and $A^- := (\check{u} \notin \check{v})$.*
2. *If A is a formula $(u \in V)$, then*
$$A^+ := (\check{V}\check{u} = \top) \quad \text{and} \quad A^- := (\check{V}\check{u} = \bot).$$
3. *If A is a formula $\neg B$, then $A^+ := B^-$ and $A^- := B^+$; if A is a formula $(B \vee C)$, then $A^+ := (B^+ \vee C^+)$ and $A^- := (B^- \wedge C^-)$; if A is a formula $(B \wedge C)$, then $A^+ := (B^+ \wedge C^+)$ and $A^- := (B^- \vee C^-)$.*
4. *If A is a formula $\exists x B[x]$, then $A^+ := \exists \check{x} B^+[\check{x}]$ and $A^- := \forall \check{x} B^-[\check{x}]$; if A is a formula $\forall x B[x]$, then $A^+ := \forall \check{x} B^+[\check{x}]$ and $A^- := \exists \check{x} B^-[\check{x}]$.*
5. *If A is a formula $\exists X B[X]$, then*
$$A^+ \;:=\; (\exists \check{X} \in \mathbb{T}) B^+[\check{X}] \quad \text{and} \quad A^- \;:=\; (\forall \check{X} \in \mathbb{T}) B^-[\check{X}];$$
if A is a formula $\forall X B[X]$, then
$$A^+ \;:=\; (\forall \check{X} \in \mathbb{T}) B^+[\check{X}] \quad \text{and} \quad A^- \;:=\; (\exists \check{X} \in \mathbb{T}) B^-[\check{X}].$$

This translation is so that A^- is equivalent to the negation of A^+ and vice versa provided that all class parameters in A are interpreted by operations from $\mathbb{T}$. This is spelled out in detail in the following lemma which can be proved by routine induction on A.

Lemma 8 *For all formulas $A[U_1, \ldots, U_n]$ of $\mathcal{L}_2$ with at most the indicated class variables free and possibly additional set parameters,* OST *proves:*

$$\bigwedge_{i=1}^{n}(f_i \in \mathbb{T}) \;\rightarrow\; (\neg A^+[f_1, \ldots, f_n] \;\leftrightarrow\; A^-[f_1, \ldots, f_n]).$$

Given this interpretation of class variables, Lemma 1 can be lifted in a suitable sense from pure formulas of $\mathcal{L}^\circ$ to the translations of elementary formulas of $\mathcal{L}_2$. For the proof of the following lemma proceed by induction on buildup of the formula and simply follow the corresponding proofs in Feferman [8, 9], taking into account what we remarked subsequent to Lemma 1; the previous lemma helps to treat negation.

Lemma 9 *For every elementary formula $A[\vec{U}, \vec{v}]$ of $\mathcal{L}_2$ with at most the class variables $\vec{U}$ and set variables $\vec{v}$ free, there exists a closed $\mathcal{L}^\circ$ term t_A such that* $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ *proves*

$$t_A{\downarrow} \;\wedge\; (\forall \vec{x} \in \mathbb{T})(t_A(\vec{x}) : \mathbb{V}^n \to \mathbb{B}) \;\wedge\; (\forall \vec{x} \in \mathbb{T})\forall \vec{y}(A^+[\vec{x}, \vec{y}] \;\leftrightarrow\; t_A(\vec{x}, \vec{y}) = \top).$$

Here we assume that the length of the vector $\vec{U}$ agrees with that of $\vec{x}$ and that both vectors $\vec{v}$ and $\vec{y}$ have length n.

This lemma provides for the translation of elementary comprehension; see Theorem 11 below. For dealing with replacement, we abbreviate

$$\begin{aligned} \mathit{Rel}_C[f] &\;:=\; f \in \mathbb{T} \;\wedge\; \forall x(fx = \top \;\rightarrow\; \exists y \exists z(x = \langle y, z\rangle)), \\ \mathit{Fun}_C[f] &\;:=\; \mathit{Rel}_C[f] \;\wedge\; \forall x \exists! y(f\langle x, y\rangle = \top), \end{aligned}$$

expressing that f is a code of a class relation and class function, respectively. The translation of replacement is a consequence of the following lemma.

Lemma 10 *For any a and f we can prove in* OST *that*

$$\mathit{Fun}_C[f] \;\rightarrow\; \exists x(x = \{y : (\exists z \in a)(f\langle z, y\rangle = \top)\}).$$

Proof. Depending on f, first let $s := \lambda z.\mathbb{C}(\lambda y.f\langle z, y\rangle)$. Hence s is defined and, provided that $\mathit{Fun}_C[f]$, axiom (S3) about operational choice implies $(s : \mathbb{V} \to \mathbb{V})$ and

(*) $$\forall z(f\langle z, sz\rangle = \top).$$

Then consider $\mathbb{R}(s,a)$. In view of axiom (S2) about operational replacement we conclude that $\mathbb{R}(s,a)$ is the set for which

$$\forall y(y \in \mathbb{R}(s,a) \leftrightarrow (\exists z \in a)(y = sz)).$$

Because of $Fun_C[f]$ and (*) this implies

$$\mathbb{R}(s,a) = \{y : (\exists z \in a)(f\langle z,y\rangle = \top)\},$$

thus finishing the proof of our assertion. □

After this preparatory work, the embedding of NBG into $\mathsf{OST}(\mathbf{E},\mathbb{P})$ is easily achieved.

Theorem 11 *The theory $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathrm{I}_\in)$ can be embedded into $\mathsf{OST}(\mathbf{E},\mathbb{P})$; i.e. for all closed formulas A of $\mathcal{L}_2$ we have*

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathrm{I}_\in) \vdash A \quad \Longrightarrow \quad \mathsf{OST}(\mathbf{E},\mathbb{P}) \vdash A^+.$$

Proof. This assertion is established once we have shown, by induction on the length of the derivation in NBG, that

$$\mathsf{NBG} \vdash A[U_1,\dots,U_n] \quad \Longrightarrow \quad \mathsf{OST}(\mathbf{E},\mathbb{P}) \vdash \bigwedge_{i=1}^{n}(\check{U}_i \in \mathbb{T}) \to A^+[\check{U}_1,\dots,\check{U}_n]$$

for all formulas $A[U_1,\dots,U_n]$ of $\mathcal{L}_2$ with at most the indicated free class variables and possibly additional set parameters. If $A[U_1,\dots,U_n]$ is a logical axiom or the conclusion of an inference, the assertion follows, possibly using the induction hypothesis, by simple reasoning within $\mathsf{OST}(\mathbf{E},\mathbb{P})$. Hence it only remains to treat the non-logical axioms of NBG:

1. The translation of Aussonderung is easily proved in $\mathsf{OST}(\mathbf{E},\mathbb{P})$ by means of separation for definite operations; all basic set existence axioms of NBG are basic set-theoretic axioms of OST or a consequence of operational power set (S4); the translation of class foundation and the translation of any instance of $(\mathcal{L}_2\text{-}\mathrm{I}_\in)$ are directly implied by $(\mathcal{L}^\circ\text{-}\mathrm{I}_\in)$.

2. $A[U_1,\dots,U_n]$ is an instance of elementary comprehension. Then there exists an elementary $\mathcal{L}_2$ formula $B[U_1,\dots,U_n,\vec{v},w]$ with at most the indicated free variables such that $A[U_1,\dots,U_n]$ is

$$\exists X\forall y(y \in X \leftrightarrow B[U_1,\dots,U_n,\vec{v},y]).$$

Now we select a closed $\mathcal{L}^\circ$ term t_B as provided by Lemma 9 and define $s := \lambda y.t_B(\check{U}_1,\dots,\check{U}_n,\vec{v},y)$. Hence, in view of Lemma 9, $\mathsf{OST}(\mathbf{E},\mathbb{P})$ proves

$$\bigwedge_{i=1}^{n}(\check{U}_i \in \mathbb{T}) \to (s \in \mathbb{T} \wedge \forall y(sy = \top \leftrightarrow B^+[\check{U}_1,\dots,\check{U}_n,\vec{v},y])).$$

This means that s is a suitable code for the witness claimed to exist by elementary comprehension.

3. $A[U_1, \ldots, U_n]$ is replacement. Then it is the axiom

$$\forall X(Fun[X] \to \forall y \exists z(z = \{X(u) : u \in y\})),$$

the translation of which is equivalent to

$$(\forall \check{X} \in \mathbb{T})(Fun_C[\check{X}] \to \forall y \exists z(z = \{v : (\exists u \in y)(\check{X}\langle u, v\rangle = \top)\})).$$

According to Lemma 10, this assertion is provable in $\mathsf{OST}(\mathbf{E}, \mathbb{P})$.

4. $A[U_1, \ldots, U_n]$ is global choice. Then it is the axiom

$$\exists X(Fun[X] \wedge \forall y(y \neq \emptyset \to X(y) \in y)).$$

Recalling Theorem 3, we know that global choice is available in $\mathsf{OST}(\mathbf{E}, \mathbb{P})$. All we have to do is to rewrite it so that it validates the translation of the axiom (GC) of NBG. For this purpose we first pick a closed $\mathcal{L}^\circ$ term $\mathbf{eq}$ for the characteristic function of equality of sets, which exists according to Lemma 1 or the corresponding lemmas of Feferman [8, 9], and define

$$s := \lambda xy.\mathbf{eq}(x, \langle y, \mathsf{choice}(y)\rangle) \quad \text{and} \quad t := \lambda x.\mathbf{E}(sx).$$

For any a, we then have $(sa : \mathbb{V} \to \mathbb{B})$ so that, by axiom (L6), $\mathbf{E}(sa) \in \mathbb{B}$. Consequently, $t \in \mathbb{T}$. Together with axiom (L6), the definitions of s and t also yield

$$\forall x(tx = \top \leftrightarrow \exists y(x = \langle y, \mathsf{choice}(y)\rangle)).$$

Since $(\mathsf{choice} : \mathbb{V} \to \mathbb{V})$ and $\mathsf{choice}(a) \in a$ for all non-empty a, we conclude $Fun_C[t]$ and

$$\forall y \forall z(y \neq \emptyset \wedge t\langle y, z\rangle = \top \to z \in y),$$

and thus t is a suitable witness for the translation of (GC). This finishes our proof since now (the translations of) all non-logical axioms of NBG have been proved in $\mathsf{OST}(\mathbf{E}, \mathbb{P})$. □

We are left with the iteration axioms (It-ECA) of $\mathsf{NBG}_{<E_0}$. They will be handled by combining a fixed point construction on operations with verifying, by transfinite induction along initial segments of $(E_0, \lhd)$, that we obtain a family of operations belonging to $\mathbb{T}$.

In order to speak about transfinite induction along the elementary $(E_0, \lhd)$ within the framework of $\mathsf{OST}(\mathbf{E}, \mathbb{P})$, we first fix a closed $\mathcal{L}^\circ$ term $\ell_\lhd$ which codes the ordering $\lhd$ on E_0 in the sense of Lemma 9. In the context of $\mathcal{L}^\circ$

we write $(a \lhd b)$ for $(\ell_\lhd(a,b) = \top)$ and $(a \not\lhd b)$ for $(\ell_\lhd(a,b) = \bot)$. Then as before (but now within the language $\mathcal{L}^\circ$) transfinite induction along $\lhd$ up to an element v is canonically defined, for an $\mathcal{L}^\circ$ formula $A[u]$, by

$$TI_\lhd[v,A] \;:=\; \forall x((\forall y \lhd x)A[y] \to A[x]) \;\to\; (\forall x \lhd v)A[x].$$

As shown above, NBG is contained in $\mathsf{OST}(\mathbf{E},\mathbb{P})$. Therefore the following lemma is proved by a direct adaptation of the corresponding result in Jäger and Krähenbühl [16], and there is no need to reproduce any details here.

Lemma 12 *For any standard natural number k and for any formula $A[u]$ of $\mathcal{L}^\circ$ we have*

$$\mathsf{OST}(\mathbf{E},\mathbb{P}) \vdash TI_\lhd[\Omega_k, A].$$

As mentioned above, this lemma will play a crucial role in the proof of the subsequent Theorem 14. However, it is convenient to afore supply the operational version of a specific form of disjoint union.

Lemma 13 *There exists a closed $\mathcal{L}^\circ$ term jn such that $\mathsf{OST}(\mathbf{E},\mathbb{P})$ proves:*

1. $(\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\to\; \forall x(\mathsf{jn}(f,a,x) \in \mathbb{B})$.
2. $(\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\to\; \forall x(\mathsf{jn}(f,a,x) = \top \;\leftrightarrow\; J(f,a,x))$.

Here $J(f,a,x)$ expresses that x is an element of the disjoint union of the classes coded by fb with $b \lhd a$, i.e.

$$J(f,a,x) \;:=\; (\exists y \lhd a)\exists z(x = \langle y,z\rangle \;\wedge\; f(y,z) = \top).$$

Proof. As an auxiliary term, we introduce

$$s \;:=\; \lambda fabc.\mathsf{d}_\mathbb{B}(\lambda y.f(b,y), \lambda y.\top, \ell_\lhd(b,a))c.$$

This term s is closed and defined and satisfies

$$(1) \qquad (\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\wedge\; u \lhd a \;\to\; s(f,a,u,v) = f(u,v),$$

$$(2) \qquad u \not\lhd a \;\to\; s(f,a,u,v) = \top.$$

In particular, we can conclude from (1) and (2) that

$$(3) \qquad (\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\to\; \forall y\forall z(s(f,a,y,z) \in \mathbb{B}).$$

Now consider the formula $J(\lambda yz.s(f,a,y,z),a,x)$, i.e.

$$(\exists y \lhd a)\exists z(x = \langle y,z\rangle \;\wedge\; s(f,a,y,z) = \top).$$

Because of (3) and by means of the logical operation axioms we can find a closed $\mathcal{L}^\circ$ term jn such that

$$(\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\rightarrow\; \forall x(\mathsf{jn}(f,a,x) \in \mathbb{B}),$$

$$(\forall b \lhd a)\forall x(f(b,x) \in \mathbb{B}) \;\rightarrow\; \forall x(\mathsf{jn}(f,a,x) = \top \;\leftrightarrow\; J(\lambda yz.s(f,a,y,z),a,x)).$$

Together with assertion (1) this shows that the term jn has the required property. $\square$

Theorem 14 *Let k be a standard natural number and $A[U,V,u,v]$ an elementary $\mathcal{L}_2$ formula with at most the variables U,V,u,v free. Then there exists a closed $\mathcal{L}^\circ$ term it_A for which we can prove in $\mathsf{OST}(\mathbf{E},\mathbb{P})$:*

1. $f \in \mathbb{T} \wedge a \lhd \Omega_k \;\rightarrow\; \forall x(\mathsf{it}_A(f,a,x) \in \mathbb{B})$.
2. $f \in \mathbb{T} \wedge a \lhd \Omega_k \;\rightarrow$
$$\forall x(\mathsf{it}_A(f,a,x) = \top \;\leftrightarrow\; A^+[f,\lambda y.\mathsf{jn}(\mathsf{it}_A f,a,y),a,x]).$$

Proof. We fix an element f of $\mathbb{T}$ and proceed in three steps. Firstly, Lemma 9 shows

$$(1) \qquad (\forall g \in \mathbb{T})\forall x\forall y(t_A(f,g,x,y) \in \mathbb{B}),$$

$$(2) \qquad (\forall g \in \mathbb{T})\forall x\forall y(t_A(f,g,x,y) = \top \;\leftrightarrow\; A^+[f,g,x,y])$$

for an appropriately selected closed $\mathcal{L}^\circ$ term t_A. Secondly, we make use of the fixed point operator (see Section 2) to provide a closed $\mathcal{L}^\circ$ term it_A fulfilling the partial equality

$$(3) \qquad \mathsf{it}_A(f,a,x) \;\simeq\; t_A(f,\lambda y.\mathsf{jn}(\mathsf{it}_A f,a,y),a,x)$$

for any a and x. Thirdly, we establish $\forall x(\mathsf{it}_A(f,a,x) \in \mathbb{B})$ for all $a \lhd \Omega_k$ by transfinite induction along $\lhd$. The induction hypothesis gives us

$$(\forall b \lhd a)\forall x(\mathsf{it}_A(f,b,x) \in \mathbb{B}).$$

Therefore, by Lemma 13, we also have $\forall x(\mathsf{jn}(\mathsf{it}_A f,a,x) \in \mathbb{B})$. This means $\lambda y.\mathsf{jn}(\mathsf{it}_A f,a,y) \in \mathbb{T}$ which, by (1), yields

$$t_A(f,\lambda y.(\mathsf{jn}(\mathsf{it}_A f,a,y),a,u) \in \mathbb{B}$$

for every u. Combined with (3), we conclude $\forall x(\mathsf{it}_A(f,a,x) \in \mathbb{B})$, and our first assertion is proved.

The second assertion easily follows from the first since, given $a \lhd \Omega_k$, it implies that $\lambda y.\mathsf{jn}(\mathsf{it}_A f,a,y) \in \mathbb{T}$. Hence (2) and (3) yield what we want. $\square$

Corollary 15 *The theory* $\mathsf{NBG}_{<E_0}$ *can be embedded into* $\mathsf{OST}(\mathbf{E},\mathbb{P})$*; i.e. for all closed formulas* A *of* $\mathcal{L}_2$ *we have*

$$\mathsf{NBG}_{<E_0} \vdash A \quad \Longrightarrow \quad \mathsf{OST}(\mathbf{E},\mathbb{P}) \vdash A^+.$$

Proof. Keeping the proof of Theorem 11 in mind, we just have to interpret the iteration axioms (It-ECA). Hence let k be a standard natural number and $A[U,V,u,v]$ an elementary $\mathcal{L}_2$ formula with at most the variables U,V,u,v free. We have to show that the translation of $\forall X \exists Y \mathit{Hier}_A[\Omega_k, X, Y]$, i.e.

$$(\forall f \in \mathbb{T})(\exists g \in \mathbb{T})\mathit{Hier}_A^+[\Omega_k, f, g],$$

is provable in $\mathsf{OST}(\mathbf{E},\mathbb{P})$. Pick the closed $\mathcal{L}^\circ$ term it_A of the previous theorem and, given $f \in \mathbb{T}$, set

$$s \;:=\; \lambda x.\mathsf{jn}(\lambda ay.\mathsf{it}_A(f,a,y), \Omega_k, x).$$

Lemma 13 and the previous theorem provide all we need to verify that s is a suitable witness for g. $\Box$

This corollary determines a lower bound of the consistency strength of our operational set theory $\mathsf{OST}(\mathbf{E},\mathbb{P})$. Our goal of the next sections is to show that this bound is sharp.

5 An inductive extension of ZF

Similar to Feferman and Jäger [10], Jäger and Studer [18] or Jäger and Strahm [17] we will utilize an inductive model constructions. However, this model is not constructed within $\mathsf{NBG}_{<E_0}$ directly but within a new system $\mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-I}_\in)$. In the next section $\mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-I}_\in)$ will be reduced to $\mathsf{NBG}_{<E_0}$.

When building up the inductive model of $\mathsf{OST}(\mathbf{E},\mathbb{P})$, we have to handle the choice axiom (S3). For this end it is convenient to have a global well-ordering of the set-theoretic universe at our disposal. Therefore, let $\mathcal{L}_1(\mathcal{W})$ be the extension of $\mathcal{L}_1$ by the fresh binary relation symbol $\mathcal{W}$ and let ZFW be the extension of ZF which comprises all axioms of ZF – formulated, of course, with respect to the new language $\mathcal{L}_1(\mathcal{W})$ – plus the following global well-ordering axiom

(GWO) $\qquad \forall x \exists! \alpha \mathcal{W}(x,\alpha) \;\wedge\; \forall x \forall y \forall \alpha (\mathcal{W}(x,\alpha) \wedge \mathcal{W}(y,\alpha) \;\rightarrow\; x = y).$

From axiom (GWO) the desired well-ordering of the universe of sets is canonically obtained if we set

$$(<_{\mathcal{W}}) \qquad a <_{\mathcal{W}} b \;:=\; \exists\alpha\exists\beta(\mathcal{W}(a,\alpha) \wedge \mathcal{W}(b,\beta) \wedge \alpha < \beta).$$

Now we pick an n-ary relation symbol R which does not belong to the language $\mathcal{L}_1(\mathcal{W})$ and write $\mathcal{L}_1(\mathcal{W}, R)$ for the extension of $\mathcal{L}_1(\mathcal{W})$ by R. An $\mathcal{L}_1(\mathcal{W}, R)$ formula which contains at most $a_1, \ldots, a_n$ free is called an n-ary operator form, and we let $\mathfrak{A}[R, a_1, \ldots, a_n]$ range over such forms.

Based on a model $\mathcal{M}$ of ZFW with universe $|\mathcal{M}|$, any n-ary operator form $\mathfrak{A}[R, \vec{a}]$ gives rise to subsets $I^{\zeta}_{\mathfrak{A}}$ of $|\mathcal{M}|^n$ generated inductively for all ordinals ζ (not only those belonging to $|\mathcal{M}|$) by

$$I^{<\zeta}_{\mathfrak{A}} \;:=\; \bigcup_{\eta<\zeta} I^{\eta}_{\mathfrak{A}} \quad \text{and} \quad I^{\zeta}_{\mathfrak{A}} \;:=\; \{\langle\vec{x}\rangle \in |\mathcal{M}|^n : \mathcal{M} \models \mathfrak{A}[I^{<\zeta}_{\mathfrak{A}}, \vec{x}]\}.$$

These sets $I^{\zeta}_{\mathfrak{A}}$ are the *stages* of the inductive definition induced by $\mathfrak{A}[R, \vec{a}]$, relative to $\mathcal{M}$; for many models $\mathcal{M}$, operator forms $\mathfrak{A}[R, \vec{a}]$ and ordinals ζ the $I^{\zeta}_{\mathfrak{A}}$ are not elements of $|\mathcal{M}|$. We now enrich ZFW so that we can speak about such stages.

Given an n-ary operator form $\mathfrak{A}[R, a_1, \ldots, a_n]$, the theory $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW})$ is formulated in the language $\mathcal{L}_{\mathfrak{A}}$ which extends $\mathcal{L}_1(\mathcal{W})$ by adding a new sort of so called stage variables $\rho, \sigma, \tau, \ldots$ (possibly with subscripts) as well as new binary relation symbols $\prec$ and $\equiv$ for the less and equality relation for stage variables, respectively, plus an $(n+1)$-ary relation symbol $Q_{\mathfrak{A}}$.

The atomic formulas of $\mathcal{L}_{\mathfrak{A}}$ are the atomic formulas of $\mathcal{L}_1(\mathcal{W})$ as well as all expressions $(\sigma \prec \tau)$, $(\sigma \equiv \tau)$ and $Q_{\mathfrak{A}}(\sigma, \vec{s})$. Usually we write $Q^{\sigma}_{\mathfrak{A}}(\vec{s})$ instead of $Q_{\mathfrak{A}}(\sigma, \vec{s})$.

The formulas $(A, B, C, A_1, B_1, C_1, \ldots)$ of $\mathcal{L}_{\mathfrak{A}}$ are generated from these atoms by closure under negation, conjunction and disjunction, bounded and unbounded quantification over sets, bounded stage quantification $(\exists\sigma \prec \tau)$ and $(\forall\sigma \prec \tau)$ as well as unbounded stage quantification $\exists\sigma$ and $\forall\sigma$. The $\Delta^S_0(\mathfrak{A})$ formulas are those $\mathcal{L}_{\mathfrak{A}}$ formulas that do not contain unbounded stage quantifiers. An $\mathcal{L}_{\mathfrak{A}}$ formula A is is called $\Sigma^S(\mathfrak{A})$ if all positive occurrences of unbounded stage quantifiers in A are existential and all negative occurrences of unbounded stage quantifiers in A are universal; it is called $\Pi^S(\mathfrak{A})$ if all positive occurrences of unbounded stage quantifiers in A are universal and all negative occurrences of unbounded stage quantifiers in A are existential.

Further, given an $\mathcal{L}_{\mathfrak{A}}$ formula A and a stage variable σ not occurring in A, we write A^{σ} to denote the $\mathcal{L}_{\mathfrak{A}}$ formula which is obtained from A by replacing all unbounded stage quantifiers $Q\tau$ in A by bounded stage quantifiers $(Q\tau \prec \sigma)$. Additional abbreviations are

$$Q_{\mathfrak{A}}^{\prec\sigma}(\vec{s}) \ := \ (\exists\tau \prec \sigma)Q_{\mathfrak{A}}^{\tau}(\vec{s}) \qquad \text{and} \qquad Q_{\mathfrak{A}}(\vec{s}) \ := \ \exists\sigma Q_{\mathfrak{A}}^{\sigma}(\vec{s}).$$

Clearly, any formula of $\mathcal{L}_1(\mathcal{W})$ is a $\Delta_0^S(\mathfrak{A})$ formula, and A^σ is $\Delta_0^S(\mathfrak{A})$ for any $\mathcal{L}_{\mathfrak{A}}$ formula A.

The theory $\mathsf{E}_{\mathfrak{A}}^r(\mathsf{ZFW})$ is formulated in classical two-sorted predicate logic with equality in both sorts; in addition, it contains as non-logical axioms all ZFW-axioms of the language $\mathcal{L}_1(\mathcal{W})$, some axioms about stage variables and operator forms, reflection for $\Sigma^S(\mathfrak{A})$ formulas, separation and replacement for $\Delta_0^S(\mathfrak{A})$ formulas plus induction along $\in$ and $\prec$ for $\Delta_0^S(\mathfrak{A})$ formulas.

I. ZFW-axioms. All axioms of the theory ZFW formulated in the language $\mathcal{L}_1(\mathcal{W})$; they do not refer to stage variables or relation symbols associated to operator forms.

II. Linearity axioms. For all stage variables ρ, σ and τ:

$$\sigma \not\prec \sigma \ \wedge \ (\rho \prec \sigma \wedge \sigma \prec \tau \ \rightarrow \ \rho \prec \tau) \ \wedge \ (\sigma \prec \tau \ \vee \ \sigma \equiv \tau \ \vee \ \tau \prec \sigma).$$

III. Operator axioms. For all $\vec{u}$:

$$Q_{\mathfrak{A}}^{\sigma}(\vec{u}) \ \leftrightarrow \ \mathfrak{A}[Q_{\mathfrak{A}}^{\prec\sigma}, \vec{u}].$$

IV. Δ_0^S Separation. For all $\Delta_0^S(\mathfrak{A})$ formulas $A[u]$ and all a:

$(\Delta_0^S\text{-Sep})$ $$\exists x(x = \{y \in a : A[y]\}).$$

V. Δ_0^S Replacement. For all $\Delta_0^S(\mathfrak{A})$ formulas $A[u, v]$ and all a:

$(\Delta_0^S\text{-Rep})$ $$(\forall x \in a)\exists! y A[x, y] \ \rightarrow \ \exists z \forall y(y \in z \ \leftrightarrow \ (\exists x \in a)A[x, y]).$$

VI. Σ^S reflection. For all $\Sigma^S(\mathfrak{A})$ formulas A:

$(\Sigma^S\text{-Ref})$ $$A \ \rightarrow \ \exists\sigma A^\sigma.$$

VII. Δ_0^S induction along $\in$ and $\prec$. For all $\Delta_0^S(\mathfrak{A})$ formulas $A[u]$:

$(\Delta_0^S\text{-I}_\in)$ $$\forall x((\forall y \in x)A[y] \rightarrow A[x]) \ \rightarrow \ \forall x A[x],$$

$(\Delta_0^S\text{-I}_\prec)$ $$\forall\sigma((\forall\tau \prec \sigma)A[\tau] \rightarrow A[\sigma]) \ \rightarrow \ \forall\sigma A[\sigma].$$

It is important to observe that the stage variables do not belong to the collection of sets; they constitute a different entity which is used to "enumerate" the stages of the inductive definition associated to each operator form. However, in the form of $\Delta_0^S(\mathfrak{A})$ separation and $\Delta_0^S(\mathfrak{A})$ replacement they can nevertheless help to constitute new sets in a carefully restricted way.

The theory $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW})$ is a restricted system (hence the superscript "r") in the sense that the axioms in groups IV, V and VII are restricted to $\Delta_0^S(\mathfrak{A})$ formulas. By $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_\in)$ is meant $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW})$ extended by the schema of $\in$-induction for arbitrary $\mathcal{L}_{\mathfrak{A}}$ formulas $A[u]$,

$$(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x].$$

It remains to follow the pattern of the embedding of $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ into a similar theory ZFL^r_Ω, as carried through in Jäger [15]. For any natural number n greater than 0 we select (i) a Δ_0 formula $Tup_n(a)$ formalizing that a is an ordered n-tuple and (ii) a Δ_0 formula $(a)_n = b$ formalizing that b the projection of a on its nth component so that

$$Tup_n(a) \wedge (a)_1 = b_1 \wedge \ldots \wedge (a)_n = b_n \rightarrow a = \langle b_1, \ldots, b_n \rangle.$$

Then we fix pairwise different sets $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\top}$, $\widehat{\bot}$, $\widehat{\mathbf{el}}$, $\widehat{\mathbf{non}}$, $\widehat{\mathbf{dis}}$, $\widehat{\mathbf{e}}$, $\widehat{\mathbf{E}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$, $\widehat{\mathbb{C}}$ and $\widehat{\mathbb{P}}$ which all do not belong to the collection of ordered pairs and triples; they will later act as the codes of the corresponding constants of $\mathcal{L}^\circ$. We are going to code the $\mathcal{L}^\circ$ terms $\mathsf{k}x$, $\mathsf{s}x$, $\mathsf{s}xy$, ... by the ordered tuples $\langle \widehat{\mathsf{k}}, x\rangle$, $\langle \widehat{\mathsf{s}}, x\rangle$, $\langle \widehat{\mathsf{s}}, x, y\rangle$, ... of the corresponding form. For example, to satisfy $\mathsf{k}xy = x$ we interpret $\mathsf{k}x$ as $\langle \widehat{\mathsf{k}}, x\rangle$, and "$\langle \widehat{\mathsf{k}}, x\rangle$ applied to y" is taken to be x.

For finding the required interpretation of the application operation of the theory $\mathsf{OST}(\mathbf{E},\mathbb{P})$ we introduce a specific ternary operator form $\mathfrak{F}[R,a,b,c]$, with R being a fresh ternary relation symbol.

Definition 16 *The operator form $\mathfrak{F}[R,a,b,c]$ is defined to be the disjunction of the following clauses:*

(1) $a = \widehat{\mathsf{k}} \wedge c = \langle \widehat{\mathsf{k}}, b\rangle$,

(2) $Tup_2(a) \wedge (a)_1 = \widehat{\mathsf{k}} \wedge (a)_2 = c$,

(3) $a = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, b\rangle$,

(4) $Tup_2(a) \wedge (a)_1 = \widehat{\mathsf{s}} \wedge c = \langle \widehat{\mathsf{s}}, (a)_2, b\rangle$,

(5) $Tup_3(a) \wedge (a)_1 = \widehat{\mathsf{s}} \wedge \exists x \exists y (R((a)_2, b, x) \wedge R((a)_3, b, y) \wedge R(x, y, c))$,

(6) $a = \widehat{\mathbf{el}} \ \wedge \ c = \langle \widehat{\mathbf{el}}, b \rangle$,

(7) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{el}} \ \wedge \ (a)_2 \in b \ \wedge \ c = \widehat{\top}$,

(8) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{el}} \ \wedge \ (a)_2 \notin b \ \wedge \ c = \widehat{\bot}$,

(9) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\bot}$,

(10) $a = \widehat{\mathbf{non}} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\top}$,

(11) $a = \widehat{\mathbf{dis}} \ \wedge \ c = \langle \widehat{\mathbf{dis}}, b \rangle$,

(12) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(13) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\bot} \ \wedge \ b = \widehat{\top} \ \wedge \ c = \widehat{\top}$,

(14) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{dis}} \ \wedge \ (a)_2 = \widehat{\bot} \ \wedge \ b = \widehat{\bot} \ \wedge \ c = \widehat{\bot}$,

(15) $a = \widehat{\mathbf{e}} \ \wedge \ c = \langle \widehat{\mathbf{e}}, b \rangle$,

(16) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{e}} \ \wedge \ (\exists x \in b) R((a)_2, x, \widehat{\top}) \ \wedge \ c = \widehat{\top}$,

(17) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbf{e}} \ \wedge \ (\forall x \in b) R((a)_2, x, \widehat{\bot}) \ \wedge \ c = \widehat{\bot}$,

(18) $a = \widehat{\mathbb{S}} \ \wedge \ c = \langle \widehat{\mathbb{S}}, b \rangle$,

(19) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbb{S}} \ \wedge \ (\forall x \in b)(R((a)_2, x, \widehat{\top}) \ \vee \ R((a)_2, x, \widehat{\bot})) \ \wedge$
$\forall x (x \in c \ \leftrightarrow \ x \in b \wedge R((a)_2, x, \widehat{\top}))$,

(20) $a = \widehat{\mathbb{R}} \ \wedge \ c = \langle \widehat{\mathbb{R}}, b \rangle$,

(21) $Tup_2(a) \ \wedge \ (a)_1 = \widehat{\mathbb{R}} \ \wedge \ (\forall x \in b)(\exists y \in c) R((a)_2, x, y) \ \wedge$
$(\forall y \in c)(\exists x \in b) R((a)_2, x, y)$,

(22) $a = \widehat{\mathbb{C}} \ \wedge \ R(b, c, \widehat{\top}) \ \wedge \ \forall x (x <_{\mathcal{W}} c \ \rightarrow \ \neg R(b, x, \widehat{\top})) \ \wedge \ \forall x \neg R(\widehat{\mathbb{C}}, b, x)$,

(23) $a = \widehat{\mathbb{P}} \ \wedge \ \forall x (x \in c \ \leftrightarrow \ x \subset b)$,

(24) $a = \widehat{\mathbf{E}} \ \wedge \ \exists x R(b, x, \widehat{\top}) \ \wedge \ c = \widehat{\top}$,

(25) $a = \widehat{\mathbf{E}} \ \wedge \ \forall x R(b, x, \widehat{\bot}) \ \wedge \ c = \widehat{\bot}$.

This definition differs from the corresponding definition in Jäger [15] only in the global well-ordering $<_L$ being replaced by the global well-ordering $<_{\mathcal{W}}$, a change without any consequences for the considerations leading to Theorem 19.

As in Jäger [15] it is easily shown that $Q_{\mathfrak{F}}(a,b,c)$ is functional in its third argument and, therefore, suitable for translating the operational application of $\mathsf{OST}(\mathbf{E},\mathbb{P})$.

Definition 17 *For each $\mathcal{L}^\circ$ term t we introduce an $\mathcal{L}_{\mathfrak{F}}$ formula $[\![t]\!]_{\mathfrak{F}}(u)$, with u not occurring in t, which is inductively defined as follows:*

1. *If t is a set variable, then $[\![t]\!]_{\mathfrak{F}}(u)$ is the formula $(t = u)$.*
2. *If t is a constant, then $[\![t]\!]_{\mathfrak{F}}(u)$ is the formula $(\widehat{t} = u)$.*
3. *If t is the term (rs), then we set*

$$[\![t]\!]_{\mathfrak{F}}(u) \;:=\; \exists x \exists y([\![r]\!]_{\mathfrak{F}}(x) \,\wedge\, [\![s]\!]_{\mathfrak{F}}(y) \,\wedge\, Q_{\mathfrak{F}}(x,y,u)).$$

For any $\mathcal{L}^\circ$ term t, the formula $[\![t]\!]_{\mathfrak{F}}(u)$ expresses that u is the value of t under the interpretation of the operational application via the formula $Q_{\mathfrak{F}}(a,b,c)$. By this treatment of the terms of $\mathcal{L}^\circ$, the translation of arbitrary formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}_{\mathfrak{F}}$ is predetermined.

Definition 18 *The translation of an $\mathcal{L}^\circ$ formula A into the $\mathcal{L}_{\mathfrak{F}}$ formula A^* is inductively defined as follows:*

1. *For the atomic formulas of $\mathcal{L}^\circ$ we stipulate*

$$\begin{aligned} (t{\downarrow})^* &:= \exists x [\![t]\!]_{\mathfrak{F}}(x), \\ (s \in t)^* &:= \exists x \exists y([\![s]\!]_{\mathfrak{F}}(x) \,\wedge\, [\![t]\!]_{\mathfrak{F}}(y) \,\wedge\, x \in y). \end{aligned}$$

2. *If A is a formula $\neg B$, then A^* is $\neg B^*$.*
3. *If A is a formula $(B \diamond C)$ for $\diamond$ being the binary junctor $\vee$ or $\wedge$, then A^* is $(B^* \diamond C^*)$.*
4. *If A is a formula $QxB[x]$ for a quantifier Q, then A^* is $QxB^*[x]$.*

The proof of the first part of the following theorem can be directly taken form Jäger [15]. Its second part is a direct consequence from the first because every instance of ($\mathcal{L}^\circ$-$\mathrm{I}_\in$) translates into an instance of ($\mathcal{L}_{\mathfrak{F}}$-$\mathrm{I}_\in$).

Theorem 19 *The theories* $\mathsf{OST}^r(\mathbf{E},\mathbb{P})$ *and* $\mathsf{OST}(\mathbf{E},\mathbb{P})$ *are interpretable in* $\mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW})$ *and* $\mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-}\mathrm{I}_\in)$*, respectively; i.e. for all formulas* A *of* $\mathcal{L}^\circ$ *we have:*

1. $\mathsf{OST}^r(\mathbf{E},\mathbb{P}) \vdash A \quad \Longrightarrow \quad \mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) \vdash A^*$.
2. $\mathsf{OST}(\mathbf{E},\mathbb{P}) \vdash A \quad \Longrightarrow \quad \mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-}\mathrm{I}_\in) \vdash A^*$.

In combination with Corollary 15 this result implies the proof-theoretic equivalence of the systems $\mathsf{OST}(\mathbf{E},\mathbb{P})$ and $\mathsf{NBG}_{<E_0}$ as soon as the reduction of the theory $\mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-}\mathrm{I}_\in)$ to $\mathsf{NBG}_{<E_0}$ is established. This is the content on the next section.

6 Reducing $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_\in)$ to $\mathsf{NBG}_{<E_0}$

It is notationally convenient to restrict ourselves from now to a unary operator form $\mathfrak{A}[R,a]$. It is obvious, however, that and how all results of this section can be generalized to operator forms of arbitrary arities. We begin our reduction process with embedding $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_\in)$ into the auxiliary system $\mathsf{G}^\infty_{\mathfrak{A}}$, which is a Gentzen-style reformulation of $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW})$ with an additional infinitary rule branching over all ordinals. Afterwards, we carry through a partial cut elimination argument before an asymmetric interpretation in $\mathsf{NBG}_{<E_0}$ is performed.

In the following we develop, within $\mathsf{NBG}_{<E_0}$, the infinitary system $\mathsf{G}^\infty_{\mathfrak{A}}$. For this purpose we code the set variables as the pairs $\langle 0,n\rangle$, the stage variables as the pairs $\langle 1,n\rangle$, n always a natural number. For every set a we have the set constant $\langle 2,a\rangle$ and for every $b \in E_0$ the stage constant $\langle 3,b\rangle$. For natural numbers n, sets a and elements b of E_0 we set

$$\begin{aligned} e_n &:= \langle 0,n\rangle, & \xi_n &:= \langle 1,n\rangle, \\ \mathsf{p}_a &:= \langle 2,a\rangle, & \mathsf{q}_b &:= \langle 3,b\rangle. \end{aligned}$$

We also fix several elementary class functions defined, for arbitrary sets a,b,c, by (some are written in infix or another mnemonically convenient notation):

$$\begin{aligned} (a \mathrel{\dot\in} b) &:= \langle 4,a,b\rangle, & \dot{\mathcal{W}}(a,b) &:= \langle 5,a,b\rangle, \\ (a \mathrel{\dot\prec} b) &:= \langle 6,a,b\rangle, & (a \mathrel{\dot\equiv} b) &:= \langle 7,a,b\rangle, \\ \dot{Q}_{\mathfrak{A}}(a,b) &:= \langle 8,a,b\rangle, & \dot\neg\, a &:= \langle 9,a\rangle, \\ (a \mathrel{\dot\vee} b) &:= \langle 10,a,b\rangle, & (a \mathrel{\dot\wedge} b) &:= \langle 11,a,b\rangle, \end{aligned}$$

$$\dot{\exists}\, a\, b \;:=\; \langle 12, a, b\rangle, \qquad \dot{\forall}\, a\, b \;:=\; \langle 13, a, b\rangle,$$

$$(\dot{\exists}\, a \mathrel{\dot{\prec}} b)\, c \;:=\; \langle 14, a, b, c\rangle, \qquad (\dot{\forall}\, a \mathrel{\dot{\prec}} b)\, c \;:=\; \langle 15, a, b, c\rangle.$$

To proceed with our development of $\mathsf{G}^{\infty}_{\mathfrak{A}}$ within $\mathsf{NBG}_{<E_0}$, all formulas of $\mathsf{G}^{\infty}_{\mathfrak{A}}$ are presented as sets, mimicking the built up of the formulas of $\mathcal{L}_{\mathfrak{A}}$ with additional sets and stage constants.

Definition 20 *The class $F^{\infty}_{\mathfrak{A}}$ is defined to be the smallest class which satisfies the following closure properties:*

(1) For all natural numbers m, n and all sets a, b the class $F^{\infty}_{\mathfrak{A}}$ contains

$$(e_m \mathrel{\dot{\in}} e_n), \quad (e_m \mathrel{\dot{\in}} \mathsf{p}_a), \quad (\mathsf{p}_a \mathrel{\dot{\in}} e_m), \quad (\mathsf{p}_a \mathrel{\dot{\in}} \mathsf{p}_b),$$
$$\dot{\mathcal{W}}(e_m, e_n), \quad \dot{\mathcal{W}}(e_m, \mathsf{p}_a), \quad \dot{\mathcal{W}}(\mathsf{p}_a, e_m), \quad \dot{\mathcal{W}}(\mathsf{p}_a, \mathsf{p}_b).$$

(2) For all natural numbers m, n and all elements a, b of E_0, the class $F^{\infty}_{\mathfrak{A}}$ contains

$$(\xi_m \mathrel{\dot{\prec}} \xi_n), \quad (\xi_m \mathrel{\dot{\prec}} \mathsf{q}_a), \quad (\mathsf{q}_a \mathrel{\dot{\prec}} \xi_m), \quad (\mathsf{q}_a \mathrel{\dot{\prec}} \mathsf{q}_b),$$
$$(\xi_m \mathrel{\dot{\equiv}} \xi_n), \quad (\xi_m \mathrel{\dot{\equiv}} \mathsf{q}_a), \quad (\mathsf{q}_a \mathrel{\dot{\equiv}} \xi_m), \quad (\mathsf{q}_a \mathrel{\dot{\equiv}} \mathsf{q}_b).$$

(3) For all natural numbers m, n, all sets a and all elements b of E_0, the class $F^{\infty}_{\mathfrak{A}}$ contains

$$\dot{Q}_{\mathfrak{A}}(\xi_m, e_n), \quad \dot{Q}_{\mathfrak{A}}(\xi_m, \mathsf{p}_a), \quad \dot{Q}_{\mathfrak{A}}(\mathsf{q}_b, e_n), \quad \dot{Q}_{\mathfrak{A}}(\mathsf{q}_b, \mathsf{p}_a).$$

(4) For all $x, y \in F^{\infty}_{\mathfrak{A}}$, the class $F^{\infty}_{\mathfrak{A}}$ also contains

$$\dot{\neg}\, x, \quad (x \mathrel{\dot{\vee}} y), \quad (x \mathrel{\dot{\wedge}} y).$$

(5) For all $x \in F^{\infty}_{\mathfrak{A}}$ and all natural numbers n, the class $F^{\infty}_{\mathfrak{A}}$ also contains

$$\dot{\exists}\, e_n\, x, \quad \dot{\forall}\, e_n\, x, \quad \dot{\exists}\, \xi_n\, x, \quad \dot{\forall}\, \xi_n\, x.$$

(6) For all $x \in F^{\infty}_{\mathfrak{A}}$, all natural numbers m, n and all elements a of E_0, the class $F^{\infty}_{\mathfrak{A}}$ also contains

$$(\dot{\exists}\, \xi_m \mathrel{\dot{\prec}} \xi_n)\, x, \quad (\dot{\exists}\, \xi_m \mathrel{\dot{\prec}} \mathsf{q}_a)\, x \quad (\dot{\forall}\, \xi_m \mathrel{\dot{\prec}} \xi_n)\, x, \quad (\dot{\forall}\, \xi_m \mathrel{\dot{\prec}} \mathsf{q}_a)\, x.$$

This definition could be reformulated as an explicit elementary formula, for the prize of being less perspicuous. We are not going to work out the details, only formulate the corresponding assertion.

Lemma 21 *$F_{\mathfrak{A}}^{\infty}$ is an elementarily definable class of* $\mathsf{NBG}_{<E_0}$.

It is also elementarily decidable whether elements of $F_{\mathfrak{A}}^{\infty}$ contain set or stage constants or whether a set or stage variable occurs freely (in the usual sense) within an element of $F_{\mathfrak{A}}^{\infty}$. Moreover, there is an elementary class function Sub mapping any set constant p_a, set variable e_m and element x of $F_{\mathfrak{A}}^{\infty}$ onto that element $Sub(\mathsf{p}_a, e_m, x)$ of $F_{\mathfrak{A}}^{\infty}$ which is obtained from x by replacing all free occurrences of e_m by p_a. The simultaneous replacements

$$Sub(\langle \mathsf{p}_{a_1}, \ldots, \mathsf{p}_{a_m}, \mathsf{q}_{b_1}, \ldots, \mathsf{q}_{b_n} \rangle, \langle e_{i_1}, \ldots, e_{i_m}, \xi_{j_1}, \ldots, \xi_{j_n} \rangle, x)$$

of free occurrences of set and stage variables within an element x of $F_{\mathfrak{A}}^{\infty}$ are dealt with accordingly.

Clearly, $(a \mathbin{\dot{\rightarrow}} b)$, for any sets a and b, stands for $(\dot{\neg}\, a \mathbin{\dot{\vee}} b)$, and other abbreviations of this sort are used as expected. To give an example, if an element φ of $F_{\mathfrak{A}}^{\infty}$ is given as $\psi[e_1, \ldots, e_m]$, then $\psi[\mathsf{p}_{a_1}, \ldots, \mathsf{p}_{a_m}]$ is often written instead of $Sub(\langle \mathsf{p}_{a_1}, \ldots, \mathsf{p}_{a_m} \rangle, \langle e_1, \ldots, e_m \rangle, \varphi)$.

The previous definition is so that Gödel numbers, all belonging to $F_{\mathfrak{A}}^{\infty}$, can be canonically assigned to the formulas of $\mathcal{L}_{\mathfrak{A}}$. For this purpose we begin with fixing a mapping $\natural$ which assigns natural numbers to all set and stage variables, making sure that different variables are mapped onto different natural numbers.

If u, v are set variables and σ, τ stage variables of $\mathcal{L}_{\mathfrak{A}}$, we define

$$\begin{aligned} \ulcorner (u \in v) \urcorner &:= (e_{\natural(u)} \mathbin{\dot{\in}} e_{\natural(v)}), & \ulcorner \mathcal{W}(u,v) \urcorner &:= \dot{\mathcal{W}}(e_{\natural(u)}, e_{\natural(v)}), \\ \ulcorner (\sigma \prec \tau) \urcorner &:= (\xi_{\natural(\sigma)} \mathbin{\dot{\prec}} \xi_{\natural(\tau)}), & \ulcorner (\sigma \equiv \tau) \urcorner &:= (\xi_{\natural(\sigma)} \mathbin{\dot{\equiv}} \xi_{\natural(\tau)}), \\ \ulcorner Q_{\mathfrak{A}}^{\sigma}(u) \urcorner &:= \dot{Q}_{\mathfrak{A}}(\xi_{\natural(\sigma)}, e_{\natural(u)}). \end{aligned}$$

The Gödel numbers of the non-atomic formulas of $\mathcal{L}_{\mathfrak{A}}$ are inductively calculated in compliance with the equations

$$\begin{aligned} \ulcorner \neg A \urcorner &:= \dot{\neg}\, \ulcorner A \urcorner, \\ \ulcorner (A \vee B) \urcorner &:= (\ulcorner A \urcorner \mathbin{\dot{\vee}} \ulcorner B \urcorner), \\ \ulcorner (A \wedge B) \urcorner &:= (\ulcorner A \urcorner \mathbin{\dot{\wedge}} \ulcorner B \urcorner), \\ \ulcorner \exists x A \urcorner &:= \dot{\exists}\, e_{\natural(x)} \ulcorner A \urcorner, \\ \ulcorner \forall x A \urcorner &:= \dot{\forall}\, e_{\natural(x)} \ulcorner A \urcorner, \\ \ulcorner \exists \sigma A \urcorner &:= \dot{\exists}\, \xi_{\natural(\sigma)} \ulcorner A \urcorner, \end{aligned}$$

$$\ulcorner \forall \sigma A \urcorner \;:=\; \dot{\forall}\, \xi_{\natural(\sigma)} \ulcorner A \urcorner,$$

$$\ulcorner (\exists \sigma \prec \tau) A \urcorner \;:=\; (\dot{\exists}\, \xi_{\natural(\sigma)} \mathrel{\dot{\prec}} \xi_{\natural(\tau)}) \ulcorner A \urcorner,$$

$$\ulcorner (\forall \sigma \prec \tau) A \urcorner \;:=\; (\dot{\forall}\, \xi_{\natural(\sigma)} \mathrel{\dot{\prec}} \xi_{\natural(\tau)}) \ulcorner A \urcorner.$$

The elements of $F^{\infty}_{\mathfrak{A}}$ are called $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas and will be denoted by the small Greek letters θ, φ, χ and ψ (possibly with subscripts). To increase the readability we often omit the dots when it is clear from the context that we speak about elements of $F^{\infty}_{\mathfrak{A}}$.

The *set-closed* formulas of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ are those $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas which do not contain free set variables and stage constants (but they may contain free stage variables and set constants); the closed formulas of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ are those $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas which contain neither free set variables nor free stage variables. We collect the set-closed formulas of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ in the class $SCF^{\infty}_{\mathfrak{A}}$ and the closed formulas of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ in the class $CF^{\infty}_{\mathfrak{A}}$; both classes are elementary definable.

The capital Greek letters $\Theta, \Phi, \Psi, \ldots$ (possibly with subscripts) denote finite sequences of set-closed $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas. If Φ is the sequence of set-closed $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas $\varphi_1, \ldots, \varphi_m$ and Ψ the sequence of set-closed $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas $\psi_1, \ldots, \psi_n$, then

$$\langle 16, m, n, \varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \rangle$$

is the sequent with antecedent Φ and succedent Ψ; typically, it will be written as $(\Phi \supset \Psi)$ or simply as $\Phi \supset \Psi$.

The $\Delta^S_0(\mathfrak{A})$, $\Sigma^S(\mathfrak{A})$ and $\Pi^S(\mathfrak{A})$ formulas of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ are defined analogously to the corresponding classes of $\mathcal{L}_{\mathfrak{A}}$ formulas; set and stage constants are now, of course, permitted as parameters. Furthermore, a formula of $\mathcal{L}^{\infty}_{\mathfrak{A}}$ is $\Sigma^S_1(\mathfrak{A})$ if it is $\Delta^S_0(\mathfrak{A})$ or of the form $\exists \xi_n \varphi[\xi_n]$ where $\varphi[\xi_n]$ is a $\Delta^S_0(\mathfrak{A})$ formula of $\mathcal{L}^{\infty}_{\mathfrak{A}}$; it is $\Pi^S_1(\mathfrak{A})$ if it is $\Delta^S_0(\mathfrak{A})$ or of the form $\forall \xi_n \varphi[\xi_n]$ where $\varphi[\xi_n]$ is a $\Delta^S_0(\mathfrak{A})$ formula of $\mathcal{L}^{\infty}_{\mathfrak{A}}$.

Looking at the axioms of the groups (I)–(V) of the theory $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW})$, we can convince ourselves that corresponding axioms can be formulated within the language $\mathcal{L}^{\infty}_{\mathfrak{A}}$, all belonging to $\Delta^S_0(\mathfrak{A})$. We replace all free occurrences of set variables within these axioms by set constants and collect the resulting set-closed formulas in the class $AX_{\mathfrak{A}}$.

Definition 22 *The* degree $dg(\varphi)$ *of a set-closed $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formula φ is inductively defined as follows:*

1. *If φ is a set-closed $\Sigma^S_1(\mathfrak{A})$ formula of $\mathcal{L}^{\infty}_{\mathfrak{A}}$, then $dg(\varphi) := 0$.*

2. For all set-closed $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formulas not belonging to $\Sigma_1^S(\mathfrak{A})$ we set

$$\begin{aligned}
dg(\neg\psi) &:= dg(\psi)+1,\\
dg(\psi_1 \vee \psi_2) &:= \max(dg(\psi_1), dg(\psi_2))+1,\\
dg(\psi_1 \wedge \psi_2) &:= \max(dg(\psi_1), dg(\psi_2))+1,\\
dg(\exists e_n \psi[e_n]) &:= dg(\psi[\mathsf{p}_\emptyset])+1,\\
dg(\forall e_n \psi[e_n]) &:= dg(\psi[\mathsf{p}_\emptyset])+1,\\
dg(\exists \xi_n \psi[\xi_n]) &:= dg(\psi[\xi_n])+1,\\
dg(\forall \xi_n \psi[\xi_n]) &:= dg(\psi[\xi_n])+1,\\
dg((\exists \xi_n \prec \xi_m)\psi[\xi_n]) &:= dg(\psi[\xi_n])+2,\\
dg((\forall \xi_n \prec \xi_m)\psi[\xi_n]) &:= dg(\psi[\xi_n])+2.
\end{aligned}$$

$\mathsf{G}_{\mathfrak{A}}^{\infty}$ is an extension of the classical Gentzen sequent calculus LK (cf., e.g., Girard [11] or Takeuti [22]) by additional axioms and rules of inference which take care of the non-logical axioms of $\mathsf{E}_{\mathfrak{A}}^{r}(\mathsf{ZFW})$. Universal set quantification in the succedent and the corresponding existential set quantification in the anticedent are infinitary rules branching over the collection of all sets. The axioms and rules of $\mathsf{G}_{\mathfrak{A}}^{\infty}$ can be grouped as follows.

I. Axioms. For all set-closed $\Delta_0^S(\mathfrak{A})$ formulas φ of $\mathcal{L}_{\mathfrak{A}}^{\infty}$, all elements ψ of $AX_{\mathfrak{A}}$ and all sets a, b:

(A1) $\varphi \supset \varphi$,

(A2) $\supset \psi$,

(A3) $\supset (\mathsf{p}_a \in \mathsf{p}_b)$ if $a \in b$,

(A4) $\supset (\mathsf{p}_a \notin \mathsf{p}_b)$ if $a \notin b$.

II. Structural rules. The structural rules of $\mathsf{G}_{\mathfrak{A}}^{\infty}$ consist of the usual weakening, exchange and contraction rules.

III. Propositional rules. The propositional rules of $\mathsf{G}_{\mathfrak{A}}^{\infty}$ consist of the usual rules for introducing the propositional connectives on the left and right hand sides of sequents.

IV. Quantifier rules for sets. Formulated only for succedents; there are also corresponding rules for the anticedents. For all set variables e_n, all set constants p_a and all set-closed formulas $\varphi[\mathsf{p}_\emptyset]$ of $\mathcal{L}_{\mathfrak{A}}^\infty$:

$$\frac{\Phi \;\supset\; \Psi,\, \varphi[\mathsf{p}_a]}{\Phi \;\supset\; \Psi,\, \exists e_n \varphi[e_n]}\,, \quad \frac{\Phi \;\supset\; \Psi,\, \varphi[\mathsf{p}_b] \quad \text{for all sets b}}{\Phi \;\supset\; \Psi,\, \forall e_n \varphi[e_n]}\,.$$

V. Quantifier rules for stages. Formulated only for succedents; there are also corresponding rules for the anticedents. By $(\star)$ we mark those rules where the designated free variables are not to occur in the conclusion. For all stage variables ξ_k, ξ_m, ξ_n and all set-closed formulas $\varphi[\xi_n]$ of $\mathcal{L}_{\mathfrak{A}}^\infty$:

$$\frac{\Phi \;\supset\; \Psi,\, \varphi[\xi_m]}{\Phi \;\supset\; \Psi,\, \exists \xi_n \varphi[\xi_n]}\,, \quad \frac{\Phi \;\supset\; \Psi,\, \varphi[\xi_m]}{\Phi \;\supset\; \Psi,\, \forall \xi_n \varphi[\xi_n]}\,(\star),$$

$$\frac{\Phi \;\supset\; \Psi,\, \xi_m \prec \xi_k \,\wedge\, \varphi[\xi_m]}{\Phi \;\supset\; \Psi,\, (\exists \xi_n \prec \xi_k) A[\xi_n]}\,, \quad \frac{\Phi \;\supset\; \Psi,\, \xi_m \prec \xi_k \,\rightarrow\, \varphi[\xi_m]}{\Phi \;\supset\; \Psi,\, (\forall \xi_n \prec \xi_k) \varphi[\xi_n]}\,(\star).$$

VI. Σ^S reflection rules. For all set-closed $\Sigma^S(\mathfrak{A})$ formulas φ of $\mathcal{L}_{\mathfrak{A}}^\infty$ and all stage variables ξ_n which are not free in φ:

$$\frac{\Phi \;\supset\; \Psi,\, \varphi}{\Phi \;\supset\; \Psi,\, \exists \xi_n \varphi^{\xi_n}}\,.$$

VII. Δ_0^S induction rules along $\prec$. For all stage variables ξ_k, ξ_m, ξ_n and all set-closed $\Delta_0^S(\mathfrak{A})$ formulas $\varphi[\xi_n]$ of $\mathcal{L}_{\mathfrak{A}}^\infty$:

$$\frac{\Phi \;\supset\; \Psi,\, \forall \xi_k ((\forall \xi_n \prec \xi_k) \varphi[\xi_n] \rightarrow \varphi[\xi_k])}{\Phi \;\supset\; \Psi,\, \varphi[\xi_m]}\,.$$

VIII. Cuts. For all set-closed formulas φ of $\mathcal{L}_{\mathfrak{A}}^\infty$:

$$\frac{\Phi \;\supset\; \Psi,\, \varphi \qquad \Phi,\, \varphi \;\supset\; \Psi}{\Phi \;\supset\; \Psi}\,.$$

The formula φ is called the cut formula of this cut; the degree of a cut is the degree of its cut formula.

Since $\mathsf{G}_{\mathfrak{A}}^\infty$ has inference rules which branch over all sets, namely the rules for introducing universal quantification over sets in the succedents and existential quantification over sets in the anticedents, infinite proof trees may occur. We confine ourselves to those whose depths are bounded by initial segments of E_0.

Definition 23 *Let k be an arbitrary standard natural number. For any notation $a \lhd \Omega_k$, any $n < \omega$ and any sequent $\Phi \supset \Psi$, we define $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_n \Phi \supset \Psi$ by induction on a.*

1. *If $\Phi \supset \Psi$ is an axiom of $\mathsf{G}^{\infty}_{\mathfrak{A}}$, then we have $\mathsf{G}^{\infty}_{(\mathfrak{A},k)}) \vdash^a_n \Phi \supset \Psi$ for all $a \lhd \Omega_k$ and $n < \omega$.*

2. *If $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a_x}_n \Phi_x \supset \Psi_x$ and $a_x \lhd a$ for every premise of a rule which is not a cut, then we have $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_n \Phi \supset \Psi$ for the conclusion $\Phi \supset \Psi$ of this rule.*

3. *If $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a_i}_n \Phi_i \supset \Psi_i$ and $a_i \lhd a$ for the two premises $\Phi_i \supset \Psi_i$ of a cut $(i = 1, 2)$ whose degree is less than n, then we have $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_n \Phi \supset \Psi$ for the conclusion $\Phi \supset \Psi$ of this cut.*

To be precise, given a standard natural number k, we employ axiom (It-ECA) to introduce a class U such that, for any $a \lhd \Omega_k$, the projection $(U)_a$ consists of all pairs $(\Phi \supset \Psi, n)$ for which we have $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_n \Phi \supset \Psi$.

$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_0 \Phi \supset \Psi$ says that there exists a cut-free proof in $\mathsf{G}^{\infty}_{\mathfrak{A}}$ whose depth is bounded by the notation a and $a \lhd \Omega_k$. If we have $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_1 \Phi \supset \Psi$, then only $\Sigma^S_1(\mathfrak{A})$ formulas are permitted as cut formulas.

The axioms (A2)–(A3), the $\Sigma^S(\mathfrak{A})$ reflection rules and the $\Delta^S_0(\mathfrak{A})$ induction rules along $\prec$ block total cut elimination. But since the main formulas of these axioms and rules belong to $\Sigma^S_1(\mathfrak{A})$, partial cut elimination – eliminating all those cuts whose cut formula is not from $\Sigma^S_1(\mathfrak{A})$ – can be proved by standard techniques as presented, for example, in Schütte [21].

Theorem 24 (Partial cut elimination) *Let k be a standard natural number. Then $\mathsf{NBG}_{<E_0}$ proves for all $n < \omega$, all $a \in E_0$ such that $\omega^a \lhd \Omega_k$ and all sequents $\Phi \supset \Psi$ that*

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^a_{n+2} \Phi \supset \Psi \quad \rightarrow \quad \mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\omega^a}_{n+1} \Phi \supset \Psi.$$

$\mathsf{G}^{\infty}_{\mathfrak{A}}$ is so that apart from $\in$-induction, all axioms of $\mathsf{E}^r_{\mathfrak{A}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{A}}\text{-I}_{\in})$ are directly verified within $\mathsf{G}^{\infty}_{\mathfrak{A}}$. For proving the instances of $(\mathcal{L}_{\mathfrak{A}}\text{-I}_{\in})$ infinite derivations are required in general.

Lemma 25 *Let k be a standard natural number. Then $\mathsf{NBG}_{<E_0}$ proves for all set-closed $\mathcal{L}^{\infty}_{\mathfrak{A}}$ formulas $\varphi[\mathsf{p}_\emptyset]$:*

1. *For all ordinals α, all sets a of set-theoretic rank α and all ordinals β such that $\beta = \omega^\alpha + \omega + 1$,*

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{\beta}}_0 \forall e_m((\forall e_n \in e_m)\varphi[e_n] \rightarrow \varphi[e_m]) \supset \varphi[\mathsf{p}_a].$$

2. $\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\Omega}_{0} \forall e_m((\forall e_n \in e_m)\varphi[e_n] \to \varphi[e_m]) \supset \forall e_m \varphi[e_m]$.

PROOF. We let ψ be the formula $\forall e_m((\forall e_n \in e_m)\varphi[e_n] \to \varphi[e_m])$ and show the first assertion by induction on α. Given a set a of rank α, the induction hypothesis implies for all $b \in a$

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{\gamma}}_{0} \psi \supset \varphi[\mathsf{p}_b] \tag{1}$$

where $\gamma := \omega^{\alpha}$. If $b \notin a$, then according to (A4) and weakening

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{1}}_{0} \psi \supset \mathsf{p}_b \notin \mathsf{p}_a. \tag{2}$$

From (1) and (2) we conclude, for any set b,

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{\gamma+1}}_{0} \psi \supset \mathsf{p}_b \notin \mathsf{p}_a \vee \varphi[\mathsf{p}_b].$$

By universal set quantification we thus have

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{\gamma+2}}_{0} \psi \supset (\forall e_n \in \mathsf{p}_a)\varphi[e_n],$$

and from this, simple manipulations within $\mathsf{G}^{\infty}_{\mathfrak{A}}$ also lead to

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\overline{\gamma+\omega}}_{0} \psi, (\forall e_n \in \mathsf{p}_a)\varphi[e_n] \to \varphi[\mathsf{p}_a] \supset \varphi[\mathsf{p}_a].$$

Universal set quantification within the anticedent therefore finishes the proof of our first assertion. The second assertion follows from the first by a universal set quantification in the succedent. □

It is now routine to verify by induction on the lengths of the proofs in the system $\mathsf{E}^{r}_{\mathfrak{A}}(\mathsf{ZFW})+(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_{\in})$ that every theorem of $\mathsf{E}^{r}_{\mathfrak{A}}(\mathsf{ZFW})+(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_{\in})$ is derivable in $\mathsf{G}^{\infty}_{\mathfrak{A}}$.

Theorem 26 *Let k be a standard natural number and A a formula of $\mathcal{L}_{\mathfrak{A}}$ without free set variables. If A is derivable in the theory $\mathsf{E}^{r}_{\mathfrak{A}}(\mathsf{ZFW})+(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_{\in})$, then there exist standard natural numbers m and n such that $\mathsf{NBG}_{<E_0}$ proves*

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{\Omega+\overline{m}}_{n} \supset \ulcorner A \urcorner.$$

Applying Theorem 24 finitely often we can strengthen this theorem to an interpretation of $\mathsf{E}^{r}_{\mathfrak{A}}(\mathsf{ZFW})+(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_{\in})$ in $\mathsf{G}^{\infty}_{\mathfrak{A}}$ with proofs whose cut formulas are $\Sigma^{S}_{1}(\mathfrak{A})$ formulas and whose depths are bounded by Ω_k for suitable standard natural numbers k.

Corollary 27 *Let A be a formula of $\mathcal{L}_{\mathfrak{A}}$ without free set variables. If A is derivable in $\mathsf{E}^{r}_{\mathfrak{A}}(\mathsf{ZFW})+(\mathcal{L}_{\mathfrak{A}}\text{-}\mathrm{I}_{\in})$, then there exists a standard natural number k such that $\mathsf{NBG}_{<E_0}$ proves that there exists a notation $a \lhd \Omega_k$ such that*

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a}_{1} \supset \ulcorner A \urcorner.$$

We continue by defining what it means that elements of $F_{\mathfrak{A}}^{\infty}$ are true. In the following definition of this notion the set quantifiers range over the universe of sets; the existential and universal stage quantifiers, on the other hand, are interpreted over (not necessarily the same) initial segments of E_0.

Before going into the details of this definition, we have to take care of the well-ordering relation $\mathcal{W}$ of ZFW. This is done by observing that the global choice axiom (GC) of NBG induces a well-ordering of the universe: in NBG we can prove that there exists a class, call it W_{glob}, for which

$$\text{(G-WO)} \qquad\qquad \forall x \exists! \alpha (\langle x, \alpha \rangle \in W_{glob}).$$

Obviously, W_{glob} is the right candidate to interpret $\mathcal{W}$. Also, we write $\widehat{\mathfrak{A}}[U, a]$ for the formula of $\mathcal{L}_2$ which is obtained from our operator form $\mathfrak{A}[R, a]$ by replacing all occurrences of $\mathcal{W}(x, y)$ by $(\langle x, y \rangle \in W_{glob})$ and all occurrences of $R(x)$ by $(x \in U)$. Many of formulas we work with until the end of this section contain the class W_{glob} as parameter, but we forbear from indicating this parameter in general.

Let us write $(U)_{\lhd x}$ for the class $\bigcup \{(U)_y : y \lhd x\}$ and $(U)_{<m}$ for the class $\bigcup \{(U)_n : n < m\}$. Then the iteration axiom (It-ECA) has two special cases: first,

$$\text{(It-1)} \qquad\qquad \exists X (\forall y \lhd \Omega_k)((X)_y = \{x : \widehat{\mathfrak{A}}[(X)_{\lhd y}, x]\}),$$

and second, for any elementary $\mathcal{L}_2$ formula $A[U, V, u, v]$ with at most the variables U, V, a, b free,

$$\text{(It-2)} \qquad\qquad \forall X \exists Y (\forall m < \omega)((Y)_m = \{x : A[X, (Y)_{<m}, x, m]\}).$$

(It-1) and (It-2) play an important role in the next considerations, but before applying them a further (lengthy) definition is needed. In this definition Lh is the elementary class function which assigns to each element a of $F_{\mathfrak{A}}^{\infty}$ the number $Lh(a)$ of occurrences of logical connectives in a.

Definition 28 *Let k be a standard natural number. Then $Sat_k[U, V, a, b]$ is defined to be the elementary formula*

$$a \in CF_{\mathfrak{A}}^{\infty} \ \wedge \ Lh(a) = b \ \wedge \ A_k[U, V, a],$$

where $A_k[U, V, a]$ is the auxiliary formula taken to be the disjunction of the following clauses:

(1) $\exists x \exists y (a = (\mathsf{p}_x \mathrel{\dot{\in}} \mathsf{p}_y) \ \wedge \ x \in y)$,

(2) $\exists x \exists y (a = \dot{\mathcal{W}}(\mathsf{p}_x, \mathsf{p}_y) \wedge \langle x, y\rangle \in W_{glob})$,

(3) $(\exists x, y \lhd \Omega_k)(a = (\mathsf{q}_x \mathrel{\dot{\prec}} \mathsf{q}_y) \wedge x \lhd y)$,

(4) $(\exists x \lhd \Omega_k)(a = (\mathsf{q}_x \mathrel{\dot{\equiv}} \mathsf{q}_x))$,

(5) $\exists x (\exists y \lhd \Omega_k)(a = \dot{Q}_{\mathfrak{A}}(\mathsf{q}_y, \mathsf{p}_x) \wedge \langle y, x\rangle \in U)$,

(6) $\exists x (a = \dot{\neg}\, x \wedge x \notin V)$,

(7) $\exists x \exists y (a = (x \mathrel{\dot{\vee}} y) \wedge (x \in V \vee y \in V))$,

(8) $\exists x \exists y (a = (x \mathrel{\dot{\wedge}} y) \wedge x \in V \wedge y \in V)$,

(9) $\exists x (\exists m < \omega)(a = \dot{\exists}\, e_m\, x \wedge \exists y (Sub(\mathsf{p}_y, e_m, x) \in V))$,

(10) $\exists x (\exists m < \omega)(a = \dot{\forall}\, e_m\, x \wedge \forall y (Sub(\mathsf{p}_y, e_m, x) \in V))$,

(11) $\exists x (\exists m < \omega)(a = \dot{\exists}\, \xi_m\, x \wedge (\exists y \lhd \Omega_k)(Sub(\mathsf{q}_y, \xi_m, x) \in V))$,

(12) $\exists x (\exists m < \omega)(a = \dot{\forall}\, \xi_m\, x \wedge (\forall y \lhd \Omega_k)(Sub(\mathsf{q}_y, \xi_m, x) \in V))$,

(13) $\exists x (\exists y \lhd \Omega_k)(\exists m < \omega)(a = (\dot{\exists}\, \xi_m \mathrel{\dot{\prec}} \mathsf{q}_y)\, x \wedge$
$$(\exists z \lhd y)(Sub(\mathsf{q}_z, \xi_m, x) \in V)),$$

(14) $\exists x (\exists y \lhd \Omega_k)(\exists m < \omega)(a = (\dot{\forall}\, \xi_m \mathrel{\dot{\prec}} \mathsf{q}_y)\, x \wedge$
$$(\forall z \lhd y)(Sub(\mathsf{q}_z, \xi_m, x) \in V)).$$

The next step is now to apply principle (It-2) to this formula $Sat_k[U, V, a, b]$, providing us with a class W such that, for all natural numbers m,

$$(W)_m = \{x \in CF^{\infty}_{\mathfrak{A}} : Lh(x) = m \wedge Sat_k[U, (W)_{<m}, x, m]\}.$$

Therefore $(W)_{<\omega} := \bigcup\{(W)_m : m < \omega\}$ consists of all elements of $CF^{\infty}_{\mathfrak{A}}$ which are true in the intended sense, with the only exception that the relation symbol $Q_{\mathfrak{A}}$ is interpreted by the class U. To correct this deficiency, all we have to do is to replace U by the appropriate interpretation for $Q_{\mathfrak{A}}$. For that we have principle (It-1).

Definition 29 *Let k be a standard natural number. Then we define*

$$Tr_{(\mathfrak{A},k)}[a] := \begin{cases} \exists X \exists Y(\, (\forall z \lhd \Omega_k)((X)_z = \{x : \widehat{\mathfrak{A}}[(X)_{\lhd z}, x]\}) \wedge \\ \qquad (\forall m < \omega)((Y)_m = \{x : Sat_k[X, (Y)_{<m}, x, m]\}) \wedge \\ \qquad (\exists m < \omega)(a \in (Y)_m)\,). \end{cases}$$

Although $Tr_{(\mathfrak{A},k)}[a]$ is not an elementary $\mathcal{L}_2$ formula, $\mathsf{NBG}_{<E_0}$ guarantees that it defines a class, namely, given a standard natural number k, we easily derive from (It-1) that there exists a unique class U with

$$(\forall z \lhd \Omega_k)((U)_z = \{x : \widehat{\mathfrak{A}}[(U)_{\lhd z}, x]\}) \ \wedge \ U = \Sigma(\Omega_k, U);$$

therefore and since $Sat_k[U, V, a, b]$ only refers to segments of U less than Ω_k, the schema (It-2) furnishes us with a unique class V for which

$$(\forall m < \omega)((V)_m = \{x : Sat_k[U, (V)_{<m}, x, m]\}) \ \wedge \ V = \Sigma(\overline{\omega}, V).$$

These observations immediately establish that the $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formulas which are true in the sense of Definition 29 form a class.

Lemma 30 *For every standard natural number k we can prove in $\mathsf{NBG}_{<E_0}$ that*

$$\exists X \forall y (y \in X \ \leftrightarrow \ Tr_{(\mathfrak{A},k)}[y]).$$

It should now be evident that the formulas of $\mathcal{L}_{\mathfrak{A}}$ are directly represented by their Gödel numbers and this truth definition. If $\vec{u}$ is the sequence $u_1, \ldots, u_n$, then we write $\mathsf{p}_{\vec{u}}$ and $e_{\natural(\vec{u})}$ for the sequences $\mathsf{p}_{u_1}, \ldots, \mathsf{p}_{u_n}$ and $e_{\natural(u_1)}, \ldots, e_{\natural(u_n)}$, respectively.

Lemma 31 *For every standard natural number k and for every $\mathcal{L}_1$ formula $A[\vec{u}]$ with at most the variables $\vec{u}$ free the theory $\mathsf{NBG}_{<E_0}$ proves*

$$\forall \vec{x}(A[\vec{x}] \ \leftrightarrow \ Tr_{(\mathfrak{A},k)}[Sub(\langle \mathsf{p}_{\vec{x}} \rangle, \langle e_{\natural(\vec{u})} \rangle, \ulcorner A[\vec{u}] \urcorner)]).$$

The proof of this assertion is by simple induction on the complexity of the $\mathcal{L}_{\mathfrak{A}}$ formula $A[\vec{u}]$, and there is no need to present it in detail. A further property of our truth definition deals with the stages of the inductive definition. Its proof can be omitted as well.

Lemma 32 *For every standard natural number k the theory $\mathsf{NBG}_{<E_0}$ proves that for any class U which satisfies*

$$(\forall y \lhd \Omega_k)((U)_y = \{x : \widehat{\mathfrak{A}}[(U)_{\lhd y}, x]\})$$

we also have, for all sets a and all $b \lhd \Omega_k$:

1. $a \in (U)_b \ \leftrightarrow \ Tr_{(\mathfrak{A},k)}[Sub(\langle \mathsf{p}_a, \mathsf{q}_b \rangle, \langle e_{\natural(u)}, \xi_{\natural(\sigma)} \rangle, \ulcorner Q_{\mathfrak{A}}^{\sigma}(u) \urcorner)]$,
2. $a \in (U)_b \ \leftrightarrow \ Tr_{(\mathfrak{A},k)}[Sub(\langle \mathsf{p}_a, \mathsf{q}_b \rangle, \langle e_{\natural(u)}, \xi_{\natural(\sigma)} \rangle, \ulcorner \mathfrak{A}[Q_{\mathfrak{A}}^{\prec \sigma}, u] \urcorner)]$.

After this brief respite for introducing the truth definitions $Tr_{(\mathfrak{A},k)}$, we return to $\mathsf{G}^\infty_\mathfrak{A}$ and use them to reflect $\mathsf{G}^\infty_\mathfrak{A}$ within $\mathsf{NBG}_{<E_0}$. A first observation concerns some axioms of $\mathsf{G}^\infty_\mathfrak{A}$.

Lemma 33 *For every standard natural number k and every φ the theory $\mathsf{NBG}_{<E_0}$ proves*

$$\varphi \in AX_\mathfrak{A} \;\rightarrow\; Tr_{(\mathfrak{A},k)}[\forall(\varphi)],$$

where $\forall(\varphi)$ denotes the universal closure of φ with respect to its stage variables.

Proof. We work within $\mathsf{NBG}_{<E_0}$ and let $AX_{(\mathfrak{A},k)}$ be the class of all those closed $\mathcal{L}^\infty_\mathfrak{A}$ formulas which are obtained from the elements of $AX_\mathfrak{A}$ by substituting stage constants q_a with $a \lhd \Omega_k$ for their free stage variables. It is easily seen that our lemma follows from, for any φ,

$$(*) \qquad \varphi \in AX_{(\mathfrak{A},k)} \;\rightarrow\; Tr_{(\mathfrak{A},k)}[\varphi].$$

If φ stems from an $\mathsf{E}^r_\mathfrak{A}(\mathsf{ZFW})$ axiom of group (I), group (II) or group (III), the formula $Tr_{(\mathfrak{A},k)}[\varphi]$ follows directly from the axioms of NBG, the assertion (G-WO), the linearity of the ordering $\lhd$ and Lemma 32.

If φ stems from an $\mathsf{E}^r_\mathfrak{A}(\mathsf{ZFW})$ axiom of group (IV), the formula $Tr_{(\mathfrak{A},k)}[\varphi]$ translates into a statement of the form

$$\exists x(x = \{y \in a : Tr_{(\mathfrak{A},k)}[Sub(\mathsf{p}_y, e_n, \psi)]\})$$

for some $\Delta^S_0(\mathfrak{A})$ formula ψ of $\mathcal{L}^\infty_\mathfrak{A}$ which may contain e_n and several stage variables free, but without any other free set variables. Choosing a class X according to Lemma 30, this can be rewritten as

$$\exists x(x = \{y \in a : Sub(\mathsf{p}_y, e_n, \psi) \in X\})$$

and, therefore, is provable in $\mathsf{NBG}_{<E_0}$.

The last case we have to consider is that φ stems from an $\mathsf{E}^r_\mathfrak{A}(\mathsf{ZFW})$ axiom of group (V). Then $Tr_{(\mathfrak{A},k)}[\varphi]$ translates into a statement of the form

$$(\forall x \in a)\exists! y\, Tr_{(\mathfrak{A},k)}[Sub(\langle \mathsf{p}_x, \mathsf{p}_y\rangle, \langle e_m, e_n\rangle, \psi)] \;\rightarrow$$
$$\exists z \forall y(y \in z \;\leftrightarrow\; (\exists x \in a)\, Tr_{(\mathfrak{A},k)}[Sub(\langle \mathsf{p}_x, \mathsf{p}_y\rangle, \langle e_m, e_n\rangle, \psi)]).$$

for some $\Delta^S_0(\mathfrak{A})$ formula ψ of $\mathcal{L}^\infty_\mathfrak{A}$ which may contain e_m and e_n plus several stage variables free, but without any other free set variables. As in the previous case, we make use of Lemma 30, pick a class X which has the same extension as $Tr_{(\mathfrak{A},k)}$ and transform the previous implication into

$$(\forall x \in a)\exists! y(Sub(\langle \mathsf{p}_x, \mathsf{p}_y\rangle, \langle e_m, e_n\rangle, \psi) \in X) \;\rightarrow$$
$$\exists z \forall y(y \in z \;\leftrightarrow\; (\exists x \in a)(Sub(\langle \mathsf{p}_x, \mathsf{p}_y\rangle, \langle e_m, e_n\rangle, \psi) \in X)).$$

As before, this assertion is a theorem of $\mathsf{NBG}_{<E_0}$. This completes the proof of our auxiliary assertion (*) and thus of our lemma. $\square$

Given the interpretation of the ordering $\prec$ on the stages by our truth definition, transfinite induction along $\lhd$ carries over directly to the truth of induction along $\prec$. Hence the following result is evident.

Lemma 34 *Let k be a standard natural number. Then $\mathsf{NBG}_{<E_0}$ proves, for every closed $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formula $\forall \xi_m \varphi[\xi_m]$ and every $a \lhd \Omega_k$,*

$$Tr_{(\mathfrak{A},k)}[(\forall \xi_m \prec \mathsf{q}_a)((\forall \xi_n \prec \xi_m)\varphi[\xi_n] \rightarrow \varphi[\xi_m])] \;\rightarrow\; Tr_{(\mathfrak{A},k)}[(\forall \xi_m \prec \mathsf{q}_a)\varphi[\xi_m]].$$

Let us now fix, for the rest of this paper, a few useful notations which are required for Theorem 35 below:

(i) If Φ and Ψ are finite sequences of $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formulas, then $(\neg\Phi \vee \Psi)$ is the disjunction whose disjuncts are the negated formulas of Φ and the formulas of Ψ.

(ii) If φ is an $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formula and a an element of E_0, then $\varphi^{(a)}$ is the $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formula obtained from φ by replacing all unrestricted stage quantifiers $Q\xi_n$ by $(Q\xi_n \prec \mathsf{q}_a)$.

(iii) If φ is an $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formula and a an element of E_0, then the class $SI(a, \varphi)$ consists of all substitution instances of φ which are obtained from φ by replacing all occurrences of free stage variables by stage constants from $\{\mathsf{q}_x : x \lhd a\}$.

(iv) Suppose that Φ and Ψ are finite sequence of set-closed formulas of $\mathcal{L}_{\mathfrak{A}}^{\infty}$. If a and b are elements of E_0, then we write $SI(a, b, \Phi \supset \Psi)$ for the class $SI(a, (\neg\Phi \vee \Psi)^{(b)})$. Every element of $SI(a, b, \Phi \supset \Psi)$ is a closed $\mathcal{L}_{\mathfrak{A}}^{\infty}$ formula.

We assume that the reader can carry out all these syntactic transformations in detail and is sufficiently convinced that they can be described by elementary $\mathcal{L}_2$ formulas. Notably, everything can be performed within $\mathsf{NBG}_{<E_0}$.

Now the stage is set for carrying through an asymmetric interpretation of $\mathsf{G}_{(\mathfrak{A},k)}^{\infty}$. The technique of asymmetric interpretations is well-established in proof theory; see, for example, Schütte [21] and Jäger [12, 13]. Systems similar to $\mathsf{G}^\infty_{\mathfrak{A}}$, with explicit stages of inductive definitions, have been treated in Jäger [14, 15] and Jäger and Strahm [17].

Theorem 35 *Let k be a standard natural number. In $\mathsf{NBG}_{<E_0}$ we can prove that, for all $a, b, c \lhd \Omega_k$, all finite sequences Φ of set-closed $\Pi^S(\mathfrak{A})$ formulas, all finite sequences Ψ of set-closed $\Sigma^S(\mathfrak{A})$ formulas and all closed $\mathcal{L}^\infty_{\mathfrak{A}}$ formulas φ,*

$$\mathsf{G}^\infty_{(\mathfrak{A},k)} \vdash^a_1 \Phi \supset \Psi \;\wedge\; b + \omega^a \unlhd c \;\wedge\; \varphi \in SI(b, c, \Phi \supset \Psi) \quad\rightarrow\quad Tr_{(\mathfrak{A},k)}[\varphi].$$

Proof. We show this theorem by induction on a, which is justified in view of Lemma 5, and distinguish the following cases:

1. $\Phi \supset \Psi$ is an axiom or a conclusion of a structural rule, a propositional rule, a quantifier rule for sets, a quantifier rule for stages or a Δ^S_0 induction rule along $\prec$. Then the assertion is trivially satisfied or follows from the induction hypothesis and Lemma 34 (plus some obvious logical transformations).

2. $\Phi \supset \Psi$ is a conclusion of a Σ^S reflection rule. Then the sequence Ψ is of the form $\Psi_0, \exists\xi_n\psi^{\xi_n}$ for some set-closed $\Sigma^S(\mathfrak{A})$ formula ψ of $\mathcal{L}^\infty_{\mathfrak{A}}$, and there exists an $a_0 \lhd a$ such that

$$(1) \qquad \mathsf{G}^\infty_{(\mathfrak{A},k)} \vdash^{a_0}_1 \Phi \;\supset\; \Psi_0,\, \psi.$$

Every $\varphi \in SI(b, c, \Phi \supset \Psi)$ is logically equivalent to a formula

$$\chi^{(c)} \;\vee\; (\exists\xi_n \prec \mathsf{q}_c)\theta^{\xi_n},$$

where $\chi \in SI(b, (\neg\Phi \vee \Psi_0))$ and $\theta \in SI(b, \psi)$. Set $c_0 := b + \omega^{a_0}$; then by the induction hypothesis we obtain from (1) that

$$Tr_{(\mathfrak{A},k)}[\chi^{(c_0)} \vee \theta^{(c_0)}]$$

which actually implies, since $c_0 \lhd c$,

$$Tr_{(\mathfrak{A},k)}[\chi^{(c_0)} \vee (\exists\xi_n \prec \mathsf{q}_c)\theta^{\xi_n}].$$

By an obvious persistency argument, we can lift the bound c_0 in $\chi^{(c_0)}$ to c and conclude that $Tr_{(\mathfrak{A},k)}[\varphi]$.

3. $\Phi \supset \Psi$ is a conclusion of a cut. By assumption, its cut formula has to be a $\Delta^S_0(\mathfrak{A})$ formula or a formula of the form $\exists\xi_n\theta[\xi_n]$, where $\theta[\xi_n]$ is $\Delta^S_0(\mathfrak{A})$. In the remainder we concentrate on the second and more complicated case. Then there exists $a_1, a_2 \lhd a$ such that

$$(2) \qquad \mathsf{G}^\infty_{(\mathfrak{A},k)} \vdash^{a_1}_1 \Phi \;\supset\; \Psi,\, \exists\xi_n\theta[\xi_n],$$

$$(3) \qquad \mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a_2}_{1} \Phi,\ \exists \xi_n \theta[\xi_n] \supset \Psi.$$

Now pick a formula $\varphi \in SI(b, c, \Phi \supset \Psi)$. It can be written as $\psi^{(c)}$, where $\psi \in SI(b, (\neg\Phi \vee \Psi))$. Set $c_1 := b + \omega^{a_1}$, take some $\exists \xi_n \chi[\xi_n] \in SI(b, \exists \xi_n \theta[\xi_n])$ and apply the induction hypothesis to (2). Then we obtain

$$(4) \qquad Tr_{(\mathfrak{A},k)}[(\exists \xi_n \prec \mathsf{q}_{c_1})\chi[\xi_n] \vee \psi^{(c_1)}].$$

Furthermore, by an inversion argument (we did not formulate it explicitly but it can be proved in a straightforward way), assertion (3) also gives

$$(5) \qquad \mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a_2}_{1} \Phi,\ \theta[\xi_m] \supset \Psi$$

for a fresh stage variable ξ_m which does not occur in $\Phi \supset \Psi$ and $\exists \xi_n \theta[\xi_n]$. For $c_2 := c_1 + \omega^{a_2}$ and every q_x such that $x \lhd c_1$ the induction hypothesis applied to (5) – with a, b and c replaced by a_2, c_1 and c_2, respectively – yields

$$Tr_{(\mathfrak{A},k)}[\neg\chi[\mathsf{q}_x] \vee \psi^{(c_2)}]$$

and hence

$$Tr_{(\mathfrak{A},k)}[(\forall \xi_n \prec \mathsf{q}_{c_1})\neg\chi[\xi_n] \vee \psi^{(c_2)}].$$

Together with (4) this implies

$$Tr_{(\mathfrak{A},k)}[\psi^{(c_1)} \vee \psi^{(c_2)}].$$

We recall that $c_2 = c_1 + \omega^{a_2} = b + \omega^{a_1} + \omega^{a_2} \unlhd b + \omega^{a} \unlhd c$ and weaken the previous, by persistency, to $Tr_{(\mathfrak{A},k)}[\psi^{(c)}]$, as desired.

Therefore all possible cases for deriving the sequent $\Phi \supset \Psi$ within $\mathsf{G}^{\infty}_{(\mathfrak{A},k)}$ have been considered, proving our theorem. $\square$

Corollary 36 *Let k be a standard natural number and A a closed formula of the language $\mathcal{L}_1$ of set theory. In $\mathsf{NBG}_{<E_0}$ we can prove that, for all $a \lhd \Omega_k$,*

$$\mathsf{G}^{\infty}_{(\mathfrak{A},k)} \vdash^{a}_{1} \ulcorner A \urcorner \quad \rightarrow \quad Tr_{(\mathfrak{A},k)}[\ulcorner A \urcorner].$$

Now it is time to bring everything together and to reduce operational set theory $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ with unbounded existential quantification and power set to the extension $\mathsf{NBG}_{<E_0}$ of von Neumann-Bernays-Gödel set theory.

Theorem 37 (Reduction) *The theory $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ can be reduced to the system $\mathsf{NBG}_{<E_0}$ with respect to all sentences of the first order language $\mathcal{L}_1$ of set theory; i.e. for all closed formulas A of $\mathcal{L}_1$ we have*

$$\mathsf{OST}(\mathbf{E}, \mathbb{P}) \vdash A \quad \Longrightarrow \quad \mathsf{NBG}_{<E_0} \vdash A.$$

PROOF. Let A be an $\mathcal{L}_1$ sentence provable in $\mathsf{OST}(\mathbf{E},\mathbb{P})$. Therefore, by Theorem 19, we also have

$$(+) \qquad \mathsf{E}^r_{\mathfrak{F}}(\mathsf{ZFW}) + (\mathcal{L}_{\mathfrak{F}}\text{-I}_\in) \vdash A.$$

Now we turn to the infinitary system $\mathsf{G}^\infty_{\mathfrak{F}}$. This is the system analogous to the system $\mathsf{G}^\infty_{\mathfrak{A}}$ treated in Section 6 with the operator form $\mathfrak{A}[R,a]$ replaced by the operator form $\mathfrak{F}[R,a,b,c]$; all results carry over. According to Corollary 27, we deduce from (+) that there exists a standard natural number k such that

$$\mathsf{NBG}_{<E_0} \ \vdash \ (\exists a \lhd \Omega_k)(\mathsf{G}^\infty_{(\mathfrak{F},k)} \vdash^a_1 \ulcorner A \urcorner).$$

Hence the previous corollary yields

$$\mathsf{NBG}_{<E_0} \vdash \mathit{Tr}_{(\mathfrak{F},k)}[\ulcorner A \urcorner],$$

and it only remains to apply the reflection property described in Lemma 31 in order to derive $\mathsf{NBG}_{<E_0} \vdash A$. □

Corollary 38 *The theories* $\mathsf{OST}(\mathbf{E},\mathbb{P})$ *and* $\mathsf{NBG}_{<E_0}$ *are equiconsistent.*

This final result of this article is an immediate consequence of Corollary 15 and the previous reduction theorem.

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