Justifying induction on modal μ-formulae

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Abstract

We define a rank function rk, mapping modal μ-formulae on ordinals less than ω^ω, such that $\mathsf{rk}(\mu x.\varphi) > \mathsf{rk}(\varphi^n(x))$ for all approximants $\varphi^n(x) \equiv \varphi[x/\varphi^{n-1}(x)]$, and $\mathsf{rk}(\varphi) > \mathsf{rk}(\psi)$ for all proper subformulae ψ of φ. The corresponding structural induction on formulae, which additionally uses that the approximants $\varphi^n(x)$ are less complex than the fixpoint $\mu x.\varphi$ itself, is thus justified by transfinite induction on ω^ω. We show that ω^ω is the least such ordinal. We further give an algorithm to compute $\mathsf{rk}(\varphi)$ by primitive recursion, and we show how to get formulae φ of any rank $\mathsf{rk}(\varphi)$ in a uniform way.

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1 Introduction

The modal μ-calculus, introduced by Kozen [13], is an extension of modal logic (e.g. Hughes and Cresswell [9]) with least and greatest fixpoint constructors and therefore allows us to study fixpoints on an abstract level. Indeed, the modal μ-calculus is a powerful logic of programs subsuming dynamic and temporal logics like PDL (Fischer and Ladner [8]), PLTL (Pnueli [14]), CTL (Clarke and Emerson [6]) and CTL^* (Emerson and Halpern [7]).

The standard semantics of the modal μ-calculus is based on transition systems. Given a transition system $\mathcal{T}$ with states S, a formula φ is interpreted as the set of states $\|\varphi\|_{\mathcal{T}} \subseteq S$ where the property expressed by the formula φ holds. For x positive in φ, the formula $\mu x.\varphi$ denotes the least fixpoint $\|\mu x.\varphi\|_{\mathcal{T}}$ of the monotone operator $S' \mapsto \|\varphi\|_{\mathcal{T}[x \mapsto S']}$, which by Knaster-Tarski Theorem (cf. [12],[15]) exists and can be computed by iterating this operator, i.e. if we define $L^0_{\varphi,x} = \emptyset$, $L^{\xi+1}_{\varphi,x} = \|\varphi\|_{\mathcal{T}[x \mapsto L^{\xi}_{\varphi,x}]}$ and $L^{\lambda}_{\varphi,x} = \bigcup_{\xi<\lambda} L^{\xi}_{\varphi,x}$ for limit ordinals λ, then for some ordinal ξ with $\xi < |S|^+$, that is $|\xi| \leq |S|$, we get that $L^{\xi}_{\varphi,x} = \|\mu x.\varphi\|_{\mathcal{T}}$.

For any finite transition system $\mathcal{T}$, the least fixpoint $L^{\xi}_{\varphi,x}$ is reached after finitely many steps $\xi = |\xi| \leq |S| < \omega$, and if the formula φ is safe for iteration in x and $\mathcal{T}$, that is if $L^n_{\varphi,x} = \|\varphi^n(x)[x/\bot]\|_{\mathcal{T}}$ for all natural numbers n, where $\varphi^{n+1}(x) \equiv \varphi[x/\varphi^n(x)]$, then the least fixpoint can be written as

$$\|\mu x.\varphi\|_{\mathcal{T}} = \bigcup_{n<\omega} \|\varphi^n(x)[x/\bot]\|_{\mathcal{T}}.$$

The analog for the greatest fixpoint is $\|\nu x.\varphi\|_{\mathcal{T}} = \bigcap_{n<\omega} \|\varphi^n(x)[x/\top]\|_{\mathcal{T}}$. Because of these equalities and having in mind the collection of all (finite) transition systems, from a semantical point of view, the approximants $\varphi^n(x)[x/\bot]$ and $\varphi^n(x)[x/\top]$ can be considered less complex than $\mu x.\varphi$ and $\nu x.\varphi$ respectively.

If we want to study this complexity on the syntactic level, then first of all we need to replace the semantic notion of a formula φ to be safe for iteration in x and $\mathcal{T}$; Therefore, in the following, a formula φ is called *safe*, if all free variables of φ are distinct from all bound variables of φ, hence φ is safe for iteration in arbitrary x and $\mathcal{T}$. It is not obvious at all, how the approximants $\varphi^n(x)$ can be seen to be less complex than $\mu x.\varphi$ and $\nu x.\varphi$ in a purely syntactic way, i.e. how to find a rank function f, mapping formulae to ordinals such that

- $f(\psi) < f(\varphi)$ if ψ is a proper subformula of φ,
- $f(\varphi^n(x)) < f(\sigma x.\varphi)$ for all natural numbers n and $\sigma \in \{\mu, \nu\}$, if φ is safe.

A first attempt to find such an f might result in mapping formulae to the lexicographic ordering of $\omega\times\omega$ by $\varphi \mapsto (n(\varphi), l(\varphi))$, i.e. $f(\varphi) = \omega\cdot n(\varphi)+l(\varphi)$, where $l(\varphi)$ is the ordinary length of the formula and $n(\varphi)$ measures the nesting of μ and ν. But as we are going to see, this must fail because the range of f, $ran(f) = \omega^2$, is to small.

In this thesis we explicitly define a rank function rk^e by primitive recursion, meeting the two requirements and being optimal in the following sense: The quality of rank functions is measured by comparing their range, that is, by comparing the ordinal $\bigcup ran(f)$, and the range of rk^e, that is $ran(\mathsf{rk}^e) = \omega^\omega$, is shown to be as small as possible.

For applications, where this kind of rank functions f are used in proofs by induction on the formula rank $f(\varphi)$, we refer to [2],[3],[4],[5],[10]. In contrast to the formula rank defined in [10], we here define the rank function rk^e without using the syntactic constructs $\sigma^n x.\varphi$ for approximants.

Besides the rank functions f, we even want to consider arbitrary well-founded binary relations $\lhd$ on the set of formulae, fulfilling two analogous requirements, i.e.

- $\psi \lhd \varphi$ if ψ is a proper subformula of φ,
- $\varphi^n(x) \lhd \sigma x.\varphi$ for all natural numbers n and $\sigma \in \{\mu, \nu\}$, if φ is safe.

But given such a relation $\lhd$, by transfinite recursion we can always find a corresponding rank function $f_\lhd$, that is $f_\lhd(\varphi) = \bigcup\{f_\lhd(\psi)+1 \mid \psi \lhd \varphi\}$, such that the range of $f_\lhd$ is equal to the order type of $\lhd$ (Thm. 43).

In this thesis we show that the range of rk^e is ω^ω by determining upper and lower bounds. We further get that ω^ω is minimal, that is, the range of any such rank function is at least ω^ω, and hence any well-founded relation $\lhd$ satisfying the requirements above has order type at least ω^ω.

After introducing the preliminaries in the next section, we show the existence of a rank function rk in Section 3, and we prove the upper bound ω^ω for its range. rk^e is introduced in Section 4, where its equivalence to rk is shown. In Section 5 the lower bound ω^ω for the range of rk and rk^e is provided. And finally in Section 6 we show how to generate formulae of arbitrary comlexity, with respect to this rank functions.

2 Preliminaries

The language $\mathcal{L}_\mu$ of the modal μ-calculus results by adding greatest and least fixpoint operators to propositional modal logic. More precisely, given a set of countably many *propositional variables*, Var , and the set $\mathsf{Cst} = \{\bot, \top\}$ of *propositional constants*, we define the *atoms*, $\mathsf{Atm} = \mathsf{Var} \cup \mathsf{Cst}$, the *literals*, $\mathsf{Lit} = \mathsf{Atm} \cup \{\sim p \mid p \in \mathsf{Var}\}$, and we inductively define the collection $\mathcal{L}_\mu^+$ of *modal μ-formulae* (or simply *formulae*), such that $\mathsf{Lit} \subset \mathcal{L}_\mu^+$ and whenever $\varphi, \psi \in \mathcal{L}_\mu^+$ and $x \in \mathsf{Var}$ then the following are also modal μ-formulae

$$(\varphi \wedge \psi)\ ,\ (\varphi \vee \psi)\ ,\ \neg\varphi\ ,\ \Diamond\varphi\ ,\ \Box\varphi\ ,\ \mu x.\varphi\ ,\ \nu x.\varphi.$$

Remark 1. $\mathcal{L}_\mu^+$ has two kinds of negation symbols, $\sim$ and $\neg$. Usually only one of the symbols is present in the modal μ-calculus, but in the following we can easily cover both cases.

The language $\mathcal{L}_\mu$ of the *modal μ-calculus* is some subset of $\mathcal{L}_\mu^+$, such that only one of the symbols $\sim$ or $\neg$ is present in $\mathcal{L}_\mu$, and occurrences of the variable x (Def. 30) in a formula $\mu x.\varphi$ or $\nu x.\varphi$ must be *positive*, that is, any free occurence of x in φ is not preceded by the symbol $\sim$, or it is preceded by an even number of symbols $\neg$ (Def. 33, Rem. 34), for example the formula $\mu x.\neg(\mu x.x \wedge \neg x)$ is fine.

The *length* $l(\varphi)$ of a formula φ is defined such that

$$l(\varphi) = \begin{cases} 0 & \varphi \in \mathsf{Lit}, \\ l(\alpha) + 1 & \varphi \equiv \neg\alpha, \Diamond\alpha, \Box\alpha, \mu x.\alpha, \nu x.\alpha, \\ l(\alpha) + l(\beta) + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta. \end{cases}$$

The fixpoint operators μx, νx bind the variable x in a way similar to the quantifiers in predicate logic. Therefore we use standard terminology as for quantifiers; $\mathsf{free}(\varphi)$ denotes the set of all propositional variables occurring free in φ (Def. 32), and $\mathsf{bound}(\varphi)$ denotes the set of those variables x occurring in the form μx or νx in φ. We further define the set of variables in φ, $\mathsf{var}(\varphi) = \mathsf{free}(\varphi) \cup \mathsf{bound}(\varphi) \subset \mathsf{Var}$. We write $\psi \leq \varphi$, if ψ is a subformula of φ (Def. 35), and $\psi < \varphi$ for the proper subformulae. Our notion of subformula is such that literals do not have proper subformulae, hence $x \not< \sim x$. $\mathsf{sub}(\varphi)$ is the set of all subformulae of φ, and we further define the atoms of a formula $\mathsf{atm}(\varphi) = \mathsf{var}(\varphi) \cup (\mathsf{sub}(\varphi) \cap \mathsf{Cst}) \subset \mathsf{Atm}$.

Remark 2. If $\varphi \equiv \mu y.\sim x$ then $x, y \not\leq \varphi$, hence $x, y \notin \mathsf{sub}(\varphi)$, but we have $x, y \in \mathsf{var}(\varphi)$.

A variable $x \in \mathsf{bound}(\varphi)$ is called *well-bound in* φ if $x \notin \mathsf{free}(\varphi)$ and if there is only one single occurrence of either μx or νx in φ. A formula φ is called *well-bound* if all $x \in \mathsf{bound}(\varphi)$ are so.

Substitution and *complementation* of formulae are defined simultanously: Let x be a propositional variable and let ψ, φ be formulae, then $\varphi[x/\psi]$, the formula where all free occurrences of x are replaced by ψ, is defined by induction on the structure of φ. For literals $\varphi \in \mathsf{Lit}$ we define

$$\varphi[x/\psi] = \begin{cases} \psi & \text{if } \varphi \equiv x, \\ \overline{\psi} & \text{if } \varphi \equiv {\sim}x, \\ \varphi & \text{otherwise,} \end{cases}$$

substitution distributes over boolean and modal connectives,

$$\varphi[x/\psi] = \begin{cases} \circ(\alpha[x/\psi]) & \text{if } \varphi \equiv \circ\alpha \text{ and } \circ \in \{\neg, \Diamond, \Box\}, \\ \alpha[x/\psi] \circ \beta[x/\psi] & \text{if } \varphi \equiv \alpha \circ \beta \text{ and } \circ \in \{\wedge, \vee\}, \end{cases}$$

and for the fixpoints $\sigma \in \{\mu, \nu\}$ we define that

$$(\sigma y.\alpha)[x/\psi] = \begin{cases} \sigma y.\alpha & \text{if } y \equiv x, \\ \sigma y.(\alpha[x/\psi]) & \text{otherwise.} \end{cases}$$

The complement $\overline{\varphi}$ of the formula φ is defined inductively, such that $\overline{x} = {\sim}x$, $\overline{{\sim}x} = x$, $\overline{\bot} = \top$, $\overline{\top} = \bot$ and $\overline{\neg\alpha} = \neg\overline{\alpha}$, by using de Morgan dualities for the boolean connectives, $\overline{\alpha \wedge \beta} = \overline{\alpha} \vee \overline{\beta}$ and $\overline{\alpha \vee \beta} = \overline{\alpha} \wedge \overline{\beta}$, the usual modal dualities, $\overline{\Diamond\alpha} = \Box\overline{\alpha}$ and $\overline{\Box\alpha} = \Diamond\overline{\alpha}$, and for μx, νx we define that

$$\overline{\mu x.\alpha} = \nu x.(\overline{\alpha}[x/{\sim}x]) \text{ and } \overline{\nu x.\alpha} = \mu x.(\overline{\alpha}[x/{\sim}x]).$$

We further need the *iterated substitution* $\varphi^n(x)$ of a formula φ for $x \in \mathsf{Var}$ and $n \geq 0$, that is $\varphi^0(x) = x$ and $\varphi^{n+1}(x) = \varphi[x/\varphi^n(x)]$.

Remark 3. If $\varphi, \psi \in \mathcal{L}_\mu \subset \mathcal{L}_\mu^+$ then $\varphi[x/\psi]$ and $\varphi^n(x)$ need not be $\mathcal{L}_\mu$-formulae, e.g. if $\varphi \equiv \mu y.x$ and $\psi \equiv {\sim}y$ then $\varphi[x/\psi] \equiv \mu y.{\sim}y \notin \mathcal{L}_\mu$. Or if $\varphi \equiv {\sim}y \wedge \mu y.x$ then $\varphi^2(x) \equiv {\sim}y \wedge \mu y.({\sim}y \wedge \mu y.x) \notin \mathcal{L}_\mu$.

Two formulae φ, ψ are equal up to *renaming* of a bound variable, $\varphi \sim_1 \psi$, if there exist two formulae $\alpha(z')$, $\beta(z'')$ and variables $x, y \notin \mathsf{var}(\alpha)$ such that $\varphi \equiv \beta[z''/\sigma x.\alpha[z'/x]]$ and $\psi \equiv \beta[z''/\sigma y.\alpha[z'/y]]$. The relation $\sim_\infty$ is the transitive closure of $\sim_1$, such that $\varphi \sim_\infty \psi$ holds for formulae that are equal up to renaming of bound variables.

Remark 4. If $\varphi \equiv \mu x.(x \wedge \mu x.x)$ and $\psi \equiv \mu y.(y \wedge \mu x.x)$ then $\varphi \not\sim_1 \psi$, but $\varphi \sim_\infty \psi$ because of $\varphi \sim_1 \mu x.(x \wedge \mu z.z) \sim_1 \mu y.(y \wedge \mu z.z) \sim_1 \psi$.

The standard semantics for the modal μ-calculus is given by transition systems. A *transition system* $\mathcal{T}$ is a triple $(\mathsf{S}, \rightarrow, \lambda)$ consisting of a nonempty set S of *states*, a binary *transition relation* $\rightarrow$ on S, and a *valuation* $\lambda : \mathsf{Var} \rightarrow \wp(\mathsf{S})$ ($\wp(\mathsf{S})$ the powerset of S), assigning to each variable x a set $\lambda(x) \subseteq \mathsf{S}$. Given a transition system $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$, a subset $\mathsf{S}' \subseteq \mathsf{S}$ and a variable x, we denote by $\mathcal{T}[x \mapsto \mathsf{S}']$ the system $(\mathsf{S}, \rightarrow, \lambda')$, where λ' is

$$\lambda'(y) = \begin{cases} \mathsf{S}' & \text{if } y \equiv x, \\ \lambda(y) & \text{otherwise.} \end{cases}$$

The set of states of a transition system $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$ where φ holds, is called the *denotation of* φ *in* $\mathcal{T}$ and is denoted by $\|\varphi\|_\mathcal{T}$. We inductively define $\|\varphi\|_\mathcal{T}$, simultaneously for all λ (with S, $\rightarrow$ fixed), such that $\|x\|_\mathcal{T} = \lambda(x)$, $\|{\sim}x\|_\mathcal{T} = \mathsf{S} \backslash \lambda(x)$ for $x \in \mathsf{Var}$, $\|\bot\|_\mathcal{T} = \emptyset$, $\|\top\|_\mathcal{T} = \mathsf{S}$, $\|\neg\alpha\|_\mathcal{T} = \mathsf{S} \backslash \|\alpha\|_\mathcal{T}$ and

$$\|\alpha \wedge \beta\|_\mathcal{T} = \|\alpha\|_\mathcal{T} \cap \|\beta\|_\mathcal{T}, \qquad \|\alpha \vee \beta\|_\mathcal{T} = \|\alpha\|_\mathcal{T} \cup \|\beta\|_\mathcal{T},$$

$$\|\Box\alpha\|_\mathcal{T} = \{a \in \mathsf{S} \mid \forall b((a \rightarrow b) \Rightarrow b \in \|\alpha\|_\mathcal{T})\},$$
$$\|\Diamond\alpha\|_\mathcal{T} = \{a \in \mathsf{S} \mid \exists b((a \rightarrow b) \wedge b \in \|\alpha\|_\mathcal{T})\},$$
$$\|\nu x.\alpha\|_\mathcal{T} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\alpha\|_{\mathcal{T}[x \mapsto \mathsf{S}']}\},$$
$$\|\mu x.\alpha\|_\mathcal{T} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha\|_{\mathcal{T}[x \mapsto \mathsf{S}']} \subseteq \mathsf{S}'\}.$$

Remark 5. For later use we compute some denotations of formulae in the transition system $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda) = (\{a, b\}, \{a \rightarrow b\}, \lambda)$.

$$\|\Diamond y\|_\mathcal{T} = \{a \in \mathsf{S} \mid \exists b((a \rightarrow b) \wedge b \in \lambda(y))\} = \begin{cases} \emptyset & b \notin \lambda(y), \\ \{a\} & b \in \lambda(y), \end{cases}$$
$$\|\nu y.x\|_\mathcal{T} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|x\|_{\mathcal{T}[y \mapsto \mathsf{S}']}\} = \lambda(x),$$
$$\|(\nu y.x) \wedge \Diamond y\|_\mathcal{T} = \begin{cases} \emptyset & b \notin \lambda(y), \\ \{a\} \cap \lambda(x) & b \in \lambda(y), \end{cases}$$
$$\|\nu y.((\nu y.x) \wedge \Diamond y)\|_\mathcal{T} = \emptyset \quad \text{because if } b \in \lambda(y) \text{ then } \lambda(y) \not\subseteq \{a\} \cap \lambda(x),$$
$$\|\nu x.((\nu y.x) \wedge \Diamond y)\|_\mathcal{T} = \begin{cases} \bigcup\{\emptyset\} = \emptyset & b \notin \lambda(y), \\ \bigcup\{\emptyset, \{a\}\} = \{a\} & b \in \lambda(y), \end{cases}$$
$$\|\nu y.\nu x.((\nu y.x) \wedge \Diamond y)\|_\mathcal{T} = \emptyset \quad \text{because if } b \in \lambda(y) \text{ then } \lambda(y) \not\subseteq \{a\}.$$

The operator $I_{\alpha,x}$ with $I_{\alpha,x}(\mathsf{S}') = \|\alpha\|_{\mathcal{T}[x\mapsto\mathsf{S}']}$ can be shown to be monotone for variables x occurring only positive in α (Thm. 39), hence by Knaster-Tarski Theorem (cf. [12],[15]) for such x we have that $\|\mu x.\alpha\|_{\mathcal{T}}$ and $\|\nu x.\alpha\|_{\mathcal{T}}$ are the least and the greatest fixpoints of $I_{\alpha,x}$, respectively (Thm. 40). If we iterate the operator, such that $L^0_{\alpha,x} = \emptyset$, $G^0_{\alpha,x} = \mathsf{S}$, $L^{\xi+1}_{\alpha,x} = I_{\alpha,x}(L^{\xi}_{\alpha,x})$, $G^{\xi+1}_{\alpha,x} = I_{\alpha,x}(G^{\xi}_{\alpha,x})$ and $L^{\rho}_{\alpha,x} = \bigcup_{\xi<\rho} L^{\xi}_{\alpha,x}$, $G^{\rho}_{\alpha,x} = \bigcap_{\xi<\rho} G^{\xi}_{\alpha,x}$ for limit ordinals ρ, then for some ordinal ξ with $|\xi| \leq |\mathsf{S}|$ we get (Thm. 41)

$$\|\mu x.\alpha\|_{\mathcal{T}} = L^{\xi}_{\alpha,x} \quad \text{and} \quad \|\nu x.\alpha\|_{\mathcal{T}} = G^{\xi}_{\alpha,x}.$$

Remark 6. (cf. Rem. 5) For $\alpha \equiv (\nu y.x) \wedge \Diamond y$ and for the transition system $\mathcal{T} = (\{a,b\}, \{a \to b\}, \lambda)$ with $\lambda(x) = \lambda(y) = \{a,b\}$, we find the fixpoint $G^1_{\alpha,x} = \|\nu x.\alpha\|_{\mathcal{T}} = \{a\}$ such that $I_{\alpha,x}(\{a\}) = \{a\}$. But the fixpoint property is not reflected in the denotation, i.e. $\|\nu x.\alpha\|_{\mathcal{T}} \neq \|\alpha[x/\nu x.\alpha]\|_{\mathcal{T}} = \emptyset$. Notice that $\mathsf{bound}(\alpha) \cap \mathsf{free}(\alpha) \neq \emptyset$.

If two formulae φ, ψ are such that $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset$, then it can be shown that for all transition systems $\mathcal{T}$ we have $\|\varphi[x/\psi]\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}$ (Thm. 42). By using this fact for formulae α with $\mathsf{bound}(\alpha) \cap \mathsf{free}(\alpha) = \emptyset$, we get

$$L^n_{\alpha,x} = \|\alpha^n(x)[x/\bot]\|_{\mathcal{T}} \quad \text{and} \quad G^n_{\alpha,x} = \|\alpha^n(x)[x/\top]\|_{\mathcal{T}}.$$

Remark 7. (cf. Rem. 5) For the formula $\alpha \equiv (\nu y.x) \wedge \Diamond y$ and for the transition system $\mathcal{T} = (\{a,b\}, \{a \to b\}, \lambda)$ with $\lambda(x) = \lambda(y) = \{a,b\}$, we have $G^2_{\alpha,x} = \{a\} \neq \emptyset = \|\alpha^2(x)[x/\top]\|_{\mathcal{T}}$.

If in addition to $\mathsf{bound}(\alpha) \cap \mathsf{free}(\alpha) = \emptyset$, we have that x is positive in α, that is $I_{\alpha,x}$ is monotone, then for finite transition systems $\mathcal{T}$ with $|\mathsf{S}| \leq n$ we get that

$$\|\nu x.\alpha(x)\|_{\mathcal{T}} = \|\alpha^n(x)[x/\top]\|_{\mathcal{T}} \quad \text{and} \quad \|\mu x.\alpha(x)\|_{\mathcal{T}} = \|\alpha^n(x)[x/\bot]\|_{\mathcal{T}}.$$

In the following a formulae φ is called *safe*, if $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$.

Remark 8. Well-bound formulae are safe, and for any formula φ we can find a well-bound formula φ^* such that $\varphi \sim_\infty \varphi^*$. Notice further, that subformulae of well-bound formulae are well-bound, but subformulae of safe formulae need not be safe, e.g. if $\varphi \equiv x \wedge \mu x.x$ then $\mu x.\varphi$ is safe, but φ is not.

By Ω we denote the first uncountable ordinal (hence the union of a countable subset of Ω is in Ω, and Ω is closed under addition). For any set X there is the set Ω^X of all functions $f : X \to \Omega$, i.e. the set of all sequences of ordinals indexed by elements of X. $\mathbf{0} \in \Omega^X$ is the function which maps every argument to 0.

A subset $\mathcal{F} \subseteq \mathcal{L}_\mu^+$ is *closed under subformulae* if $\psi \in \mathcal{F}$ whenever $\psi < \varphi$ and $\varphi \in \mathcal{F}$. And $\mathcal{F}$ is *closed under approximation* if $\varphi^n(x) \in \mathcal{F}$ for all natural numbers n, if $\mu x.\varphi \in \mathcal{F}$ or $\nu x.\varphi \in \mathcal{F}$ and φ is safe. Further we call $\mathcal{F}$ an *initial segment* of formulae, if it is closed under subformulae and approximation.

Remark 9. Observe that $\mathcal{L}_\mu$ is an initial segment of formulae, i.e. it is closed under approximation. For $\varphi^n(x)$ to be in $\mathcal{L}_\mu$ it is important that φ is safe, see Remark 3.

For initial segments $\mathcal{F} \subseteq \mathcal{L}_\mu^+$, a *$\mu$-rank on $\mathcal{F}$* is any mapping $|.| : \mathcal{F} \to \Omega$ such that

- $|\psi| < |\varphi|$ if $\psi < \varphi$,
- $|\varphi^n(x)| < |\sigma x.\varphi|$ for all natural numbers n and $\sigma \in \{\mu, \nu\}$, if φ is safe.

3 Minimal μ-ranks with range ω^ω

In order to define a μ-rank on $\mathcal{L}_\mu^+$ we first define the mapping $[\![\varphi]\!] : \Omega^{\mathsf{Var}} \to \Omega$ for any formula φ. We need the following: Given a sequence $s \in \Omega^{\mathsf{Var}}$, a variable x and $\xi \in \Omega$, the sequence $s[x{:}\xi] \in \Omega^{\mathsf{Var}}$ is defined such that

$$s[x{:}\xi](y) = \begin{cases} \xi & x \equiv y, \\ s(y) & \text{otherwise.} \end{cases}$$

The *composition* in x of $f, g : \Omega^{\mathsf{Var}} \to \Omega$ is $(f \circ_x g)(s) = f(s[x{:}g(s)])$, and the *iteration* of f in x is $f_x^{n+1} = f \circ_x (f_x^n)$ with $f_x^0 = \mathbf{0}$.

The first part of the following definition is by Afshari and Leigh [1].

Definition 10.

(1) For every $\varphi \in \mathcal{L}_\mu^+$ we define the function $[\![\varphi]\!] : \Omega^{\mathsf{Var}} \to \Omega$ such that

$$[\![\varphi]\!](s) = \begin{cases} 0 & \varphi \in \mathsf{Cst}, \\ s(x) & \varphi \equiv x, {\sim}x, \\ [\![\alpha]\!](s) + 1 & \varphi \equiv \neg\alpha, \Diamond\alpha, \Box\alpha, \\ \max\{[\![\alpha]\!](s), [\![\beta]\!](s)\} + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta, \\ \sup_{n<\omega}\{[\![\alpha]\!]_x^n(s) + 1\} & \varphi \equiv \mu x.\alpha, \nu x.\alpha. \end{cases}$$

(2) For every $\varphi \in \mathcal{L}_\mu^+$ we define the function $[\![\varphi]\!]' : \Omega^{\mathsf{Var}} \to \Omega$ such that

$$[\![\varphi]\!]'(s) = \begin{cases} 0 & \varphi \in \mathsf{Cst}, \\ s(x) & \varphi \equiv x, {\sim}x, \\ [\![\alpha]\!]'(s) + 1 & \varphi \equiv \neg\alpha, \Diamond\alpha, \Box\alpha, \\ \max\{[\![\alpha]\!]'(s), [\![\beta]\!]'(s)\} + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta, \\ \sup_{n<\omega}\{[\![\alpha]\!]'^n_x(s) + 1\} & \varphi \equiv \mu x.\alpha, \nu x.\alpha,\ \alpha \text{ safe}, \\ [\![\alpha[x/\bot]]\!]'(s) + 1 & \varphi \equiv \mu x.\alpha, \nu x.\alpha,\ \alpha \text{ not safe}. \end{cases}$$

(3) The functions $\mathsf{rk}, \mathsf{rk}' : \mathcal{L}_\mu^+ \to \Omega$ are defined such that

$$\mathsf{rk}(\varphi) = [\![\varphi]\!](\mathbf{0}) \quad \text{and} \quad \mathsf{rk}'(\varphi) = [\![\varphi]\!]'(\mathbf{0}).$$

In the remaining of this section we show that rk and rk' are μ-ranks on $\mathcal{L}_\mu^+$ with range ω^ω. We show that $\mathsf{rk}'(\varphi)$ is minimal with respect to any other μ-rank, and that rk is minimal for well-bound formulae, i.e $\mathsf{rk}(\varphi) = \mathsf{rk}'(\varphi)$ for well-bound formulae φ. In order to do this we need the following technical lemma.

Lemma 11.

For all $\varphi, \psi \in \mathcal{L}_\mu^+$, $x, y \in \mathsf{Var}$, $\xi \in \Omega$ and natural numbers n we have

(1) $[\![\varphi]\!] = [\![\varphi[x/{\sim}x]]\!]$ and $[\![\varphi[x/\bot]]\!] = [\![\varphi[x/\top]]\!]$ and $[\![\varphi]\!] = [\![\overline{\varphi}]\!]$

(2) $x \notin \mathsf{free}(\varphi) \quad \Rightarrow \quad [\![\varphi]\!](s[x{:}\xi]) = [\![\varphi]\!](s)$

(3) $x \not\equiv y,\ y \notin \mathsf{free}(\psi) \quad \Rightarrow \quad ([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n = [\![\varphi]\!]_y^n \circ_x [\![\psi]\!]$

(4) $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset \quad \Rightarrow \quad [\![\varphi[x/\psi]]\!] = [\![\varphi]\!] \circ_x [\![\psi]\!]$

(5) φ safe $\quad \Rightarrow \quad [\![\varphi]\!]_x^n = [\![\varphi^n(x)[x/\bot]]\!]$

(6) Statements (1)$-$(5) still hold if $[\![.]\!]$ is replaced by $[\![.]\!]'$

(7) φ well-bound $\quad \Rightarrow \quad [\![\varphi]\!]' = [\![\varphi]\!]$

Proof. Part 1 is proved by induction on $l(\varphi)$ and is left to the reader.

Part 2 is proved by induction on $l(\varphi)$; We only consider the case $\varphi \equiv \mu y.\psi$ and we are going to show $[\![\psi]\!]_y^n(s[x{:}\xi]) = [\![\psi]\!]_y^n(s)$ by induction on n, hence $[\![\varphi]\!](s[x{:}\xi]) = [\![\varphi]\!](s)$. Assuming $x \notin \mathsf{free}(\varphi)$ we either have that $x \equiv y$ or $x \notin \mathsf{free}(\psi)$. If $n = 0$ then $[\![\psi]\!]_y^n = \mathbf{0}$ by definition. For $n + 1 > 0$ and $x \not\equiv y$ (similar for $x \equiv y$) we have that

$$\begin{aligned}
[\![\psi]\!]_y^{n+1}(s[x{:}\xi]) &= [\![\psi]\!] \circ_y [\![\psi]\!]_y^n(s[x{:}\xi]) = [\![\psi]\!]((s[x{:}\xi])[y{:}[\![\psi]\!]_y^n(s[x{:}\xi])]) \\
&= [\![\psi]\!]((s[x{:}\xi])[y{:}[\![\psi]\!]_y^n(s)]) \quad \text{by i.h. for } n \\
&= [\![\psi]\!]((s[y{:}[\![\psi]\!]_y^n(s)])[x{:}\xi]) \quad \text{because } x \not\equiv y \\
&= [\![\psi]\!](s[y{:}[\![\psi]\!]_y^n(s)]) = [\![\psi]\!]_y^{n+1}(s). \quad \text{by i.h. for } l(\psi)
\end{aligned}$$

Part 3 is proved by induction on n. For $n = 0$ we have

$$([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n = \mathbf{0} = \mathbf{0} \circ_x [\![\psi]\!] = [\![\varphi]\!]_y^n \circ_x [\![\psi]\!].$$

For the induction step $n + 1 > 0$ we have the following

$$\begin{aligned}
([\![\varphi]\!] \circ_x [\![\psi]\!])_y^{n+1}(s) &= ([\![\varphi]\!] \circ_x [\![\psi]\!]) \circ_y ([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n(s) \\
&= ([\![\varphi]\!] \circ_x [\![\psi]\!]) \circ_y ([\![\varphi]\!]_y^n \circ_x [\![\psi]\!])(s) \quad \text{by i.h. for } n \\
&= ([\![\varphi]\!] \circ_x [\![\psi]\!])(s[y{:}\xi]) \quad \text{with } \xi = ([\![\varphi]\!]_y^n \circ_x [\![\psi]\!])(s) \\
&= [\![\varphi]\!]((s[y{:}\xi])[x{:}[\![\psi]\!](s[y{:}\xi])]) \\
&= [\![\varphi]\!]((s[y{:}\xi])[x{:}[\![\psi]\!](s)]) \quad \text{by Part 2, } y \notin \mathsf{free}(\psi) \\
&= [\![\varphi]\!]((s[x{:}[\![\psi]\!](s)])[y{:}\xi]) \quad \text{because } x \not\equiv y \\
&= ([\![\varphi]\!] \circ_y [\![\varphi]\!]_y^n)(s[x{:}[\![\psi]\!](s)]) \quad \text{because } \xi = [\![\varphi]\!]_y^n(s[x{:}[\![\psi]\!](s)]) \\
&= [\![\varphi]\!]_y^{n+1} \circ_x [\![\psi]\!](s).
\end{aligned}$$

Part 4 goes by induction on $l(\varphi)$. In case $\varphi \equiv \sim x$, we use Part 1. But here we only consider the case $\varphi \equiv \mu y.\alpha$. If $x \not\equiv y$ then

$$\begin{aligned}
[\![\varphi[x/\psi]]\!](s) &= \sup_{n<\omega}\{[\![\alpha[x/\psi]]\!]_y^n(s) + 1\} \\
&= \sup_{n<\omega}\{([\![\alpha]\!]\circ_x[\![\psi]\!])_y^n(s) + 1\} \quad \text{by i.h. for } l(\alpha) \\
&= \sup_{n<\omega}\{[\![\alpha]\!]_y^n\circ_x[\![\psi]\!](s) + 1\} \quad \text{by Part 3, } x \not\equiv y,\ y \notin \mathsf{free}(\psi) \\
&= \sup_{n<\omega}\{[\![\alpha]\!]_y^n(s[x{:}[\![\psi]\!](s)]) + 1\} \\
&= [\![\varphi]\!](s[x{:}[\![\psi]\!](s)]) = [\![\varphi]\!]\circ_x[\![\psi]\!](s),
\end{aligned}$$

else if $x \equiv y$ then $x \notin \mathsf{free}(\varphi)$, hence $[\![\varphi[x/\psi]]\!](s) = [\![\varphi]\!](s) = [\![\varphi]\!]\circ_x[\![\psi]\!](s)$ by Part 2.

For Part 5 we assume $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$ and we are going to show $[\![\varphi]\!]_x^n = [\![\varphi^n(x)[x/\bot]]\!]$ by induction on n. For $n = 0$ this is $\mathbf{0} = [\![\bot]\!]$, and for $n+1 > 0$ we get

$$\begin{aligned}
[\![\varphi]\!]_x^{n+1} &= [\![\varphi]\!]\circ_x[\![\varphi]\!]_x^n = [\![\varphi]\!]\circ_x[\![\varphi^n(x)[x/\bot]]\!] \quad \text{by i.h. for } n \\
&= [\![\varphi[x/\varphi^n(x)[x/\bot]]]\!] \quad \text{by Part 4, } \mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi^n(x)) = \emptyset \\
&= [\![(\varphi[x/\varphi^n(x)])[x/\bot]]\!] = [\![\varphi^{n+1}(x)[x/\bot]]\!].
\end{aligned}$$

Part 6 is proved analogous to (1)−(5). We only need to consider the new case in the proofs by induction on $l(\varphi)$ for (1),(2) and (4), that is the case $\varphi \equiv \mu y.\alpha$ where α is not safe. For Part 6/(1), e.g. because of $l(\alpha[y/\bot]) = l(\alpha)$ and $\overline{\alpha}[y/\sim y][y/\bot] \equiv \overline{\alpha}[y/\top] \equiv \overline{\alpha[y/\bot]}$, by using the i.h. we get that

$$\begin{aligned}
[\![\overline{\mu y.\alpha}]\!]'(s) &= [\![\nu y.\overline{\alpha}[y/\sim y]]\!]'(s) = [\![\overline{\alpha}[y/\sim y][y/\bot]]\!]'(s) + 1 \\
&= [\![\overline{\alpha[y/\bot]}]\!]'(s) + 1 = [\![\alpha[y/\bot]]\!]'(s) + 1 = [\![\mu y.\alpha]\!]'(s).
\end{aligned}$$

For Part 6/(2), with $x \notin \mathsf{free}(\varphi)$ we have $x \notin \mathsf{free}(\alpha[y/\bot])$, hence by i.h.

$$[\![\mu y.\alpha]\!]'(s[x{:}\xi]) = [\![\alpha[y/\bot]]\!]'(s[x{:}\xi]) + 1 = [\![\alpha[y/\bot]]\!]'(s) + 1 = [\![\mu y.\alpha]\!]'(s).$$

For Part 6/(4), if $x \not\equiv y$ then by i.h. and because $y \notin \mathsf{free}(\psi)$ we get that

$$\begin{aligned}
[\![\mu y.\alpha[x/\psi]]\!]'(s) &= [\![\alpha[x/\psi][y/\bot]]\!]'(s) + 1 = [\![\alpha[y/\bot][x/\psi]]\!]'(s) + 1 \\
&= [\![\alpha[y/\bot]]\!]'\circ_x[\![\psi]\!]'(s) + 1 = [\![\alpha[y/\bot]]\!]'(s[x{:}[\![\psi]\!]'(s)]) + 1 \\
&= [\![\mu y.\alpha]\!]'(s[x{:}[\![\psi]\!]'(s)]) = [\![\mu y.\alpha]\!]'\circ_x[\![\psi]\!]'(s).
\end{aligned}$$

Part 7 is proved by induction on $l(\varphi)$, by using that subformulae of well-bound formulae are well-bound and safe. $\square$

The following theorem shows that rk and rk' are μ-ranks on $\mathcal{L}_\mu^+$, and that $\mathsf{rk}'(\varphi)$ is minimal with respect to any other μ-rank.

Theorem 12. (Minimal μ-ranks)

(1) rk and rk' are μ-ranks on $\mathcal{L}_\mu^+$.

(2) If $|.| : \mathcal{F} \to \Omega$ is any μ-rank on an initial segment $\mathcal{F} \subseteq \mathcal{L}_\mu^+$, then we have that

(i) $\mathsf{rk}'(\varphi) \leq |\varphi|$ for all $\varphi \in \mathcal{F}$,

(ii) $\mathsf{rk}(\varphi) \leq |\varphi|$ for all well-bound $\varphi \in \mathcal{F}$.

Proof. Part 1 is proved by induction on $l(\varphi)$. The only non trivial case is where φ is $\mu x.\alpha$ or $\nu x.\alpha$, but this case follows by Lemma 11.5/11.6 and because of $[\![\alpha]\!]_x^1(\mathbf{0}) = \mathsf{rk}(\alpha)$ and $[\![\alpha]\!]_x^n(\mathbf{0}) = [\![\alpha^n(x)[x/\bot]]\!](\mathbf{0}) = \mathsf{rk}(\alpha^n(x))$ for safe α (the same for $[\![.]\!]'$ and rk').

Part 2(i) is proved by induction on the μ-rank $\mathsf{rk}'(\varphi)$. We only consider the case where $\varphi \equiv \mu x.\alpha$. If α is safe, then

$$\begin{aligned}\mathsf{rk}'(\mu x.\alpha) = [\![\mu x.\alpha]\!]'(\mathbf{0}) &= \sup_{n<\omega}\{([\![\alpha(x)]\!]'^{n}_{x}(\mathbf{0}) + 1\} \\ &= \sup_{n<\omega}\{([\![\alpha^n(x)[x/\bot]]\!]'(\mathbf{0}) + 1\} \quad \text{by 11.6} \\ &\leq \sup_{n<\omega}\{|\alpha^n(x)| + 1\} \quad \text{by i.h. for } \alpha^n(x) \\ &\leq |\mu x.\alpha|. \quad \text{by definition of } \mu\text{-rank}\end{aligned}$$

Otherwise if α is not safe, then

$$\mathsf{rk}'(\mu x.\alpha) = [\![\mu x.\alpha]\!]'(\mathbf{0}) = [\![\alpha[x/\bot]]\!]'(\mathbf{0}) + 1 \leq |\alpha| + 1 \leq |\mu x.\alpha|.$$

Part 2(ii) follows by Lemma 11.7 and Part 2(i). □

The next lemma shows that rk' and rk are surjective functions onto the same ordinal.

Lemma 13. (Surjectivity)

For all $\varphi, \psi \in \mathcal{L}_\mu^+$, $\xi \in \Omega$ and $\mathcal{F} \in \{\mathcal{L}_\mu, \mathcal{L}_\mu^+\}$ we have that

(1) $\varphi \sim_\infty \psi \quad \Rightarrow \quad [\![\varphi]\!] = [\![\psi]\!]$

(2) $\xi \in \mathsf{rk}[\mathcal{F}] \quad \Rightarrow \quad \xi \subset \mathsf{rk}[\mathcal{F}]$

(3) $\mathsf{rk}'[\mathcal{F}] = \mathsf{rk}[\mathcal{F}]$

Proof. For Part 1 we first show $([\![\alpha]\!]\circ_z[\![x]\!])_x^n = [\![\alpha]\!]_z^n$ for $x \notin \mathsf{free}(\alpha)$ by induction on n.

For $n = 0$ this is $\mathbf{0} = \mathbf{0}$, and for $n + 1 > 0$ we have

$$
\begin{aligned}
(\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)_x^{n+1}(s) &= (\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)\circ_x(\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)_x^n(s) \\
&= (\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)\circ_x\llbracket\alpha\rrbracket_z^n(s) \quad \text{by i.h. for } n \\
&= (\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)(s[x{:}\xi]) \\
&= \llbracket\alpha\rrbracket((s[x{:}\xi])[z{:}\llbracket x\rrbracket(s[x{:}\xi])]) \quad \text{let } \xi = \llbracket\alpha\rrbracket_z^n(s) \\
&= \llbracket\alpha\rrbracket((s[x{:}\xi])[z{:}\xi]) \\
&= \llbracket\alpha\rrbracket((s[z{:}\xi])[x{:}\xi]) = \llbracket\alpha\rrbracket(s[z{:}\xi]) \;\; \text{by 11.2, } x \notin \mathsf{free}(\alpha) \\
&= \llbracket\alpha\rrbracket\circ_z\llbracket\alpha\rrbracket_z^n(s) = \llbracket\alpha\rrbracket_z^{n+1}(s).
\end{aligned}
$$

From this we get $\llbracket\mu x.\alpha[z/x]\rrbracket = \llbracket\mu z.\alpha\rrbracket$ for $x \notin \mathsf{var}(\alpha)$, because

$$
\begin{aligned}
\llbracket\mu x.\alpha[z/x]\rrbracket &= \sup_{n<\omega}\{\llbracket\alpha[z/x]\rrbracket_x^n(s) + 1\} \\
&= \sup_{n<\omega}\{(\llbracket\alpha\rrbracket\circ_z\llbracket x\rrbracket)_x^n(s) + 1\} \quad \text{by 11.4, } x \notin \mathsf{bound}(\alpha) \\
&= \sup_{n<\omega}\{\llbracket\alpha\rrbracket_z^n(s) + 1\} = \llbracket\mu z.\alpha\rrbracket. \quad \text{because } x \notin \mathsf{free}(\alpha)
\end{aligned}
$$

For formulae $\varphi \sim_1 \psi$ such that $\varphi \equiv \beta[z'/\mu x.\alpha[z/x]]$ and $\psi \equiv \beta[z'/\mu y.\alpha[z/y]]$ and $x, y \notin \mathsf{var}(\alpha)$, we are now able to prove $\llbracket\varphi\rrbracket = \llbracket\psi\rrbracket$ by induction on $l(\beta)$. The claim follows because $\sim_\infty$ is the transitive closure of $\sim_1$.

Part 2 is proved by using the minimality of rk. First we observe that $\mathcal{F}$ is an initial segment of formulae. We define $|.| : \mathcal{F} \to \Omega$ by transfinite recursion such that

$$|\varphi| = \bigcup\{|\alpha| + 1 \mid \mathsf{rk}(\alpha) < \mathsf{rk}(\varphi), \alpha \in \mathcal{F}\}.$$

$|.|$ is a μ-rank because $\mathsf{rk}(\psi) < \mathsf{rk}(\varphi)$ implies $|\varphi| \geq |\psi| + 1$. By induction on $\mathsf{rk}(\varphi)$ we get $|\varphi| \leq \mathsf{rk}(\varphi)$, and this yields $|\varphi| = \mathsf{rk}(\varphi)$ for well-bound φ by Theorem 12.2(ii). For $\varphi \sim_\infty \psi$ we get $|\varphi| = |\psi|$ by the definition of $|.|$ and by Part 1, and for any $\varphi \in \mathcal{F}$ there is a well-bound $\varphi^* \in \mathcal{F}$ such that $\varphi \sim_\infty \varphi^*$, hence $|\varphi| = |\varphi^*| = \mathsf{rk}(\varphi^*) = \mathsf{rk}(\varphi)$, that is $|\varphi| = \mathsf{rk}(\varphi)$ for all $\varphi \in \mathcal{F}$. Now we assume that there is some $\zeta < \xi$ with $\zeta \notin \mathsf{rk}[\mathcal{F}]$. Given such ζ we define the set $Z = \{\mathsf{rk}(\varphi) \mid \mathsf{rk}(\varphi) > \zeta, \varphi \in \mathcal{F}\}$. We have $Z \neq \emptyset$ because $\xi \in \mathsf{rk}[\mathcal{F}]$, hence there is some $\varphi_0 \in \mathcal{F}$ with $\mathsf{rk}(\varphi_0) = \min Z$, and $\mathsf{rk}(\alpha) < \mathsf{rk}(\varphi_0)$ implies $\mathsf{rk}(\alpha) < \zeta$ for $\alpha \in \mathcal{F}$. But now we get

$$\mathsf{rk}(\varphi_0) = |\varphi_0| = \bigcup\{\mathsf{rk}(\alpha) + 1 \mid \mathsf{rk}(\alpha) < \zeta, \alpha \in \mathcal{F}\} \leq \zeta,$$

in contradiction to $\mathsf{rk}(\varphi_0) \in Z$. Hence there is no such ζ, and $\xi \subset \mathsf{rk}[\mathcal{F}]$.

For Part 3 we show both inclusions. Let $\mathcal{W} \subset \mathcal{F}$ be the set of all well-bound formulae in $\mathcal{F}$. We have that $\mathsf{rk}[\mathcal{F}] = \mathsf{rk}[\mathcal{W}]$ by Part 1, hence $\mathsf{rk}[\mathcal{F}] = \mathsf{rk}'[\mathcal{W}] \subseteq \mathsf{rk}'[\mathcal{F}]$ by Lemma 11.7. On the other hand for any ordinal $\xi = \mathsf{rk}'(\varphi) \in \mathsf{rk}'[\mathcal{F}]$ we have $\xi \leq \mathsf{rk}(\varphi)$ by Theorem 12.2(i), hence either $\xi = \mathsf{rk}(\varphi) \in \mathsf{rk}[\mathcal{F}]$ or $\xi \in \mathsf{rk}(\varphi) \subset \mathsf{rk}[\mathcal{F}]$ by Part 2, that is $\mathsf{rk}'[\mathcal{F}] \subseteq \mathsf{rk}[\mathcal{F}]$. $\square$

Theorem 14. (Upper bound)

For all $\varphi, \psi \in \mathcal{L}_\mu^+$, $x \in \mathsf{Var}$ and natural numbers n we have that

(1) $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset,\ x \notin \mathsf{free}(\psi) \Rightarrow \quad [\![\varphi[x/\psi]]\!](s) \leq [\![\psi]\!](s) + [\![\varphi]\!](s)$

(2) $[\![\varphi]\!]_x^n(s) \leq [\![\varphi]\!](s) \cdot n$

(3) $\mathsf{rk}(\varphi) < \omega^\omega$ (i.e. $\mathsf{rk}'[\mathcal{L}_\mu^+] = \mathsf{rk}[\mathcal{L}_\mu^+] \subseteq \omega^\omega$)

Proof. Part 1 goes by induction on the μ-rank $\mathsf{rk}(\varphi)$. We only consider the case $\varphi \equiv \mu y.\alpha$ and $x \not\equiv y$. To show this, we need a further case distinction: If φ is well-bound, then α is safe and we have

$$\begin{aligned}
[\![\varphi[x/\psi]]\!](s) &= \sup_{n<\omega}\{[\![\alpha[x/\psi]]\!]_y^n(s) + 1\} \\
&= \sup_{n<\omega}\{([\![\alpha]\!]\circ_x[\![\psi]\!])_y^n(s) + 1\} \text{ by 11.4, } \mathsf{bound}(\alpha) \cap \mathsf{free}(\psi) = \emptyset \\
&= \sup_{n<\omega}\{([\![\alpha]\!]_y^n\circ_x[\![\psi]\!])(s) + 1\} \quad \text{by 11.3, } x \neq y,\ x \notin \mathsf{free}(\psi) \\
&= \sup_{n<\omega}\{([\![\alpha^n(y)[y/\bot]]\!]\circ_x[\![\psi]\!])(s) + 1\} \quad \text{by 11.5, } \alpha \text{ safe} \\
&= \sup_{n<\omega}\{[\![\alpha^n(y)[y/\bot][x/\psi]]\!](s) + 1\} \quad \text{by 11.4} \\
&\leq \sup_{n<\omega}\{[\![\psi]\!](s) + [\![\alpha^n(y)[y/\bot]]\!](s) + 1\} \text{ i.h. for } \mathsf{rk}(\alpha^n(y)[y/\bot]) \\
&= [\![\psi]\!](s) + \sup_{n<\omega}\{[\![\alpha]\!]_y^n(s) + 1\} \quad \text{by 11.5, } \alpha \text{ safe} \\
&= [\![\psi]\!](s) + [\![\varphi]\!](s).
\end{aligned}$$

Otherwise if φ is not well-bound, then we can find a well-bound formula φ^* with $\varphi^* \sim_\infty \varphi$ and $\mathsf{bound}(\varphi^*) \cap \mathsf{free}(\psi) = \emptyset$, hence $\varphi^*[x/\psi] \sim_\infty \varphi[x/\psi]$. By Lemma 13.1, because of $\mathsf{rk}(\varphi^*) = \mathsf{rk}(\varphi)$ and by the previous case we get

$$[\![\varphi[x/\psi]]\!](s) = [\![\varphi^*[x/\psi]]\!](s) \leq [\![\psi]\!](s) + [\![\varphi^*]\!](s) = [\![\psi]\!](s) + [\![\varphi]\!](s).$$

Part 2 is first proved for well-bound formulae φ by induction on n. For $n = 0$ this is $\mathbf{0}(s) \leq 0$ and for $n + 1 > 0$ we get

$$\begin{aligned}
[\![\varphi]\!]_x^{n+1}(s) &= [\![\varphi^{n+1}(x)[x/\bot]]\!](s) = [\![\varphi[x/\varphi^n(x)[x/\bot]]]\!](s) \quad \text{by 11.5} \\
&\leq [\![\varphi^n(x)[x/\bot]]\!](s) + [\![\varphi]\!](s) \quad \begin{array}{l}\text{by Part 1, } x \notin \mathsf{free}(\varphi^n(x)[x/\bot]), \\ \mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi^n(x)) = \emptyset\end{array} \\
&= [\![\varphi]\!]_x^n(s) + [\![\varphi]\!](s) \leq [\![\varphi]\!](s) \cdot (n+1). \quad \text{by i.h for } n
\end{aligned}$$

For any formula φ there is a well-bound formula φ^* such that $\varphi^* \sim_\infty \varphi$, hence the full claim now follows because of $[\![\varphi^*]\!] = [\![\varphi]\!]$ by Lemma 13.1.

Part 3 is proved by induction on $l(\varphi)$. We only consider $\varphi \equiv \mu x.\alpha$, but for such φ we have $\mathsf{rk}(\mu x.\alpha) = \sup_{n<\omega}\{[\![\alpha]\!]_x^n(\mathbf{0}) + 1\} \leq \mathsf{rk}(\alpha) \cdot \omega + 1 < \omega^\omega$ by i.h. and Part 2. $\square$

4 Effective computation of the μ-rank

In this section we provide an alternative definition of rk, namely rk^e, which is defined by primitive recursion and hence is effectively computable, and we prove the equivalence of the two definitions.

Definition 15.

(1) For every $\varphi \in \mathcal{L}_\mu^+$ we define $\langle\varphi\rangle \in \Omega^{\mathsf{Atm}}$ such that $\langle\varphi\rangle_u = 0$ if $u \notin \mathsf{atm}(\varphi)$ and otherwise

$$\langle\varphi\rangle_u = \begin{cases} 0 & \varphi \in \mathsf{Lit}, \\ \langle\alpha\rangle_u + 1 & \varphi \equiv \neg\alpha, \Box\alpha, \Diamond\alpha, \\ \max\{\langle\alpha\rangle_u, \langle\beta\rangle_u\} + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta, \\ \langle\alpha\rangle_u + 1 + \langle\alpha\rangle_x \cdot \omega & \varphi \equiv \mu x.\alpha, \nu x.\alpha. \end{cases}$$

(2) We fix a mapping $\varphi \mapsto \varphi^*$ on $\mathcal{L}_\mu^+$ such that φ^* is well-bound and $\varphi^* \sim_\infty \varphi$, and such that $\varphi^* \equiv \varphi$ if φ is well-bound. The mappings $\mathsf{f}^e, \mathsf{rk}^e : \mathcal{L}_\mu^+ \to \Omega$ are now defined such that

$$\mathsf{f}^e(\varphi) = \max_{u \in \mathsf{Atm}}\{\langle\varphi\rangle_u\} \quad \text{and} \quad \mathsf{rk}^e(\varphi) = \mathsf{f}^e(\varphi^*).$$

Remark 16. $\mathsf{f}^e(\varphi) = \max\limits_{u \in \mathsf{atm}(\varphi)}\{\langle\varphi\rangle_u\}$ because $\langle\varphi\rangle_u = 0$ for $u \notin \mathsf{atm}(\varphi)$.

The following technical lemmas are used to show the equivalence of rk and rk^e.

Lemma 17.

Let φ be well-bound and $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi) = \emptyset$, then we have that

$$\langle\varphi[x/\psi]\rangle_u = \begin{cases} \langle\varphi\rangle_u & x \notin \mathsf{free}(\varphi) \text{ or } u \notin \mathsf{atm}(\psi), \\ \max\{\langle\varphi\rangle_u, \langle\psi\rangle_u + \langle\varphi\rangle_x\} & x \in \mathsf{free}(\varphi) \text{ and } u \in \mathsf{atm}(\psi). \end{cases}$$

Proof. We distinguish three cases. In the first case we assume $x \notin \mathsf{free}(\varphi)$. But then we have $\varphi[x/\psi] \equiv \varphi$ hence $\langle\varphi[x/\psi]\rangle_u = \langle\varphi\rangle_u$.

In the second case we assume $x \in \mathsf{free}(\varphi)$ and $u \notin \mathsf{atm}(\psi)$. We show this case by induction on $l(\varphi)$. Further, we assume $u \in \mathsf{atm}(\varphi)$, because if $u \notin \mathsf{atm}(\varphi) \cup \mathsf{atm}(\psi)$ then $\langle\varphi[x/\psi]\rangle_u = \langle\varphi\rangle_u = 0$ by definition. If $\varphi \equiv x$ then $\langle\varphi[x/\psi]\rangle_u = \langle\psi\rangle_u = 0 = \langle\varphi\rangle_u$. For $l(\varphi) > 0$, here we only consider

$\varphi \equiv \mu y.\alpha$. In this case we have $y \notin \mathsf{atm}(\psi)$ because of the assumption $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi) = \emptyset$, hence

$$\begin{aligned}\langle \varphi[x/\psi] \rangle_u = \langle \mu y.\alpha[x/\psi] \rangle_u &= \langle \alpha[x/\psi] \rangle_u + 1 + \langle \alpha[x/\psi] \rangle_y \cdot \omega \\ &= \langle \alpha \rangle_u + 1 + \langle \alpha \rangle_y \cdot \omega = \langle \varphi \rangle_u. \quad \text{by i.h.}\end{aligned}$$

In the third case we assume $x \in \mathsf{free}(\varphi)$ and $u \in \mathsf{atm}(\psi)$. In this case we prove $\langle \varphi[x/\psi] \rangle_u = \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}$ by induction on $l(\varphi)$. If $\varphi \equiv x$, then we have

$$\langle \varphi[x/\psi] \rangle_u = \langle \psi \rangle_u = \max\{0, \langle \psi \rangle_u + 0\} = \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}.$$

For $l(\varphi) > 0$, here we only consider the cases $\varphi \equiv \alpha \wedge \beta$ and $\varphi \equiv \mu y.\alpha$. If $\varphi \equiv \alpha \wedge \beta$ we further distinguish the following two cases:
(i) In case $x \in \mathsf{free}(\alpha) \cap \mathsf{free}(\beta)$, by using the i.h. we have the following

$$\begin{aligned}\langle \varphi[x/\psi] \rangle_u = \langle (\alpha \wedge \beta)[x/\psi] \rangle_u &= \max\{\langle \alpha[x/\psi] \rangle_u, \langle \beta[x/\psi] \rangle_u\} + 1 \\ &= \max\{\max\{\langle \alpha \rangle_u, \langle \psi \rangle_u + \langle \alpha \rangle_x\}, \max\{\langle \beta \rangle_u, \langle \psi \rangle_u + \langle \beta \rangle_x\}\} + 1 \\ &= \max\{\max\{\langle \alpha \rangle_u, \langle \beta \rangle_u\}, \max\{\langle \psi \rangle_u + \langle \alpha \rangle_x, \langle \psi \rangle_u + \langle \beta \rangle_x\}\} + 1 \\ &= \max\{\max\{\langle \alpha \rangle_u, \langle \beta \rangle_u\} + 1, \langle \psi \rangle_u + \max\{\langle \alpha \rangle_x, \langle \beta \rangle_x\} + 1\} \\ &= \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}.\end{aligned}$$

(ii) For $x \notin \mathsf{free}(\alpha) \cap \mathsf{free}(\beta)$ we consider the case $x \notin \mathsf{free}(\alpha)$ and $x \in \mathsf{free}(\beta)$. Because φ is well-bound, we have $x \notin \mathsf{bound}(\varphi)$, hence $x \notin \mathsf{atm}(\alpha)$ and $\langle \alpha \rangle_x = 0$, and from this we get

$$\begin{aligned}\langle \varphi[x/\psi] \rangle_u &= \max\{\langle \alpha[x/\psi] \rangle_u, \langle \beta[x/\psi] \rangle_u\} + 1 \\ &= \max\{\langle \alpha \rangle_u, \max\{\langle \beta \rangle_u, \langle \psi \rangle_u + \langle \beta \rangle_x\}\} + 1 \quad \text{by i.h. and case two} \\ &= \max\{\max\{\langle \alpha \rangle_u, \langle \beta \rangle_u\}, \langle \psi \rangle_u + \langle \beta \rangle_x\} + 1 \\ &= \max\{\max\{\langle \alpha \rangle_u, \langle \beta \rangle_u\} + 1, \langle \psi \rangle_u + \max\{\langle \alpha \rangle_x, \langle \beta \rangle_x\} + 1\} \\ &= \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}.\end{aligned}$$

If $\varphi \equiv \mu y.\alpha$ then $y \notin \mathsf{var}(\psi)$ because of $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi) = \emptyset$, hence $\langle \alpha[x/\psi] \rangle_y = \langle \alpha \rangle_y$ by case two, and finally we have

$$\begin{aligned}\langle \varphi[x/\psi] \rangle_u &= \langle \alpha[x/\psi] \rangle_u + 1 + \langle \alpha[x/\psi] \rangle_y \cdot \omega = \langle \alpha[x/\psi] \rangle_u + 1 + \langle \alpha \rangle_y \cdot \omega \\ &= \max\{\langle \alpha \rangle_u, \langle \psi \rangle_u + \langle \alpha \rangle_x\} + 1 + \langle \alpha \rangle_y \cdot \omega \quad \text{by i.h.} \\ &= \max\{\langle \alpha \rangle_u + 1 + \langle \alpha \rangle_y \cdot \omega, \langle \psi \rangle_u + \langle \alpha \rangle_x + 1 + \langle \alpha \rangle_y \cdot \omega\} \\ &= \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}.\end{aligned}$$

□

Lemma 18.

For any variables x, y, z, z' and any formula φ we have that

(1) $x \not\equiv y,\ x \not\equiv z,\ y, z \notin \mathsf{bound}(\varphi) \quad \Rightarrow \quad \langle \varphi[z'/y] \rangle_x = \langle \varphi[z'/z] \rangle_x$

(2) $x \notin \mathsf{bound}(\varphi),\ y, z \notin \mathsf{var}(\varphi) \quad \Rightarrow \quad \langle \varphi[x/y] \rangle_y = \langle \varphi[x/z] \rangle_z$

(3) If φ, ψ are well-bound formulae with $\varphi \sim_\infty \psi$ and $x \in \mathsf{free}(\varphi)$, then we have that $\langle \varphi \rangle_x = \langle \psi \rangle_x$.

Proof. For Part 1, we first observe that $x \in \mathsf{atm}(\varphi[z'/y])$ iff $x \in \mathsf{atm}(\varphi[z'/z])$, because of $x \not\equiv y$ and $x \not\equiv z$. Hence if $x \notin \mathsf{atm}(\varphi[z'/y])$ then we have $\langle \varphi[z'/y] \rangle_x = 0 = \langle \varphi[z'/z] \rangle_x$. The other case is proved by induction on $l(\varphi)$. If $\varphi \in \mathsf{Lit}$ then $\varphi[z'/y] \in \mathsf{Lit}$ and $\varphi[z'/z] \in \mathsf{Lit}$ hence $\langle \varphi[z'/y] \rangle_x = 0 = \langle \varphi[z'/z] \rangle_x$. Using the induction hypothesis we get for $\circ \in \{\neg, \Box, \Diamond\}$ and $\varphi \equiv \circ\alpha$ that

$$\langle \varphi[z'/y] \rangle_x = \langle \alpha[z'/y] \rangle_x + 1 = \langle \alpha[z'/z] \rangle_x + 1 = \langle \varphi[z'/z] \rangle_x.$$

Analogous for $\circ \in \{\wedge, \vee\}$ and $\varphi \equiv \alpha \circ \beta$. If $\sigma \in \{\mu, \nu\}$ and $\varphi \equiv \sigma u.\alpha$, and in case of $u \equiv z'$, we have $\varphi[z'/y] \equiv \varphi[z'/z]$. Otherwise, if $u \not\equiv z'$ then because of i.h. and $y, z \notin \mathsf{bound}(\varphi)$, that is $u \not\equiv y$ and $u \not\equiv z$, we have

$$\begin{aligned}
\langle \varphi[z'/y] \rangle_x &= \langle \sigma u.(\alpha[z'/y]) \rangle_x \\
&= \langle \alpha[z'/y] \rangle_x + 1 + \langle \alpha[z'/y] \rangle_u \cdot \omega \\
&= \langle \alpha[z'/z] \rangle_x + 1 + \langle \alpha[z'/z] \rangle_u \cdot \omega = \langle \sigma u.(\alpha[z'/z]) \rangle_x = \langle \varphi[z'/z] \rangle_x.
\end{aligned}$$

For Part 2, we first observe that $y \in \mathsf{atm}(\varphi[x/y])$ iff $z \in \mathsf{atm}(\varphi[x/z])$, because of $y, z \notin \mathsf{var}(\varphi)$. Hence if $y \notin \mathsf{atm}(\varphi[x/y])$ then we have that $\langle \varphi[x/y] \rangle_y = 0 = \langle \varphi[x/z] \rangle_z$. The other case is proved by induction on $l(\varphi)$. If $\varphi \in \mathsf{Lit}$ then $\varphi[x/y] \in \mathsf{Lit}$ and $\varphi[x/z] \in \mathsf{Lit}$ hence $\langle \varphi[x/y] \rangle_y = 0 = \langle \varphi[x/z] \rangle_z$. Using the induction hypothesis we easily get the claim for φ built from the connectives $\neg, \Box, \Diamond, \wedge, \vee$. If $\sigma \in \{\mu, \nu\}$ and $\varphi \equiv \sigma u.\alpha$ then because of $x \notin \mathsf{bound}(\varphi)$ and $y, z \notin \mathsf{var}(\varphi)$ we have $u \not\equiv x$, $u \not\equiv y$, $u \not\equiv z$, and because of i.h. and Part 1 we get

$$\begin{aligned}
\langle \varphi[x/y] \rangle_y &= \langle \sigma u.(\alpha[x/y]) \rangle_y \\
&= \langle \alpha[x/y] \rangle_y + 1 + \langle \alpha[x/y] \rangle_u \cdot \omega \\
&= \langle \alpha[x/z] \rangle_z + 1 + \langle \alpha[x/z] \rangle_u \cdot \omega = \langle \sigma u.(\alpha[x/z]) \rangle_z = \langle \varphi[x/z] \rangle_z.
\end{aligned}$$

For Part 3, we first observe that there is a finite sequence of formulae

$$\varphi \equiv \varphi_0 \sim_1 \varphi_1 \sim_1 \ldots \sim_1 \varphi_n \equiv \psi$$

such that φ_i is well-bound and $x \in \mathsf{free}(\varphi_i)$ for $i \leq n$, hence it is enough to prove the claim for $\varphi \sim_1 \psi$ instead of $\varphi \sim_\infty \psi$. Assuming $\varphi \sim_1 \psi$, we have

$$\varphi \equiv \beta[z''/\sigma y.\alpha[z'/y]] \quad \text{ and } \quad \psi \equiv \beta[z''/\sigma z.\alpha[z'/z]],$$

for some $y, z \notin \mathsf{var}(\alpha)$ and w.l.o.g. $z' \notin \mathsf{bound}(\alpha)$. We further have $x \not\equiv y$ and $x \not\equiv z$ because of $x \in \mathsf{free}(\varphi) \cap \mathsf{free}(\psi)$, and we have $y, z \notin \mathsf{bound}(\alpha)$ because φ and ψ are well-bound. Now by using Part 1 and Part 2 we get

$$\begin{aligned} \langle \sigma y.\alpha[z'/y] \rangle_x &= \langle \alpha[z'/y] \rangle_x + 1 + \langle \alpha[z'/y] \rangle_y \cdot \omega \\ &= \langle \alpha[z'/z] \rangle_x + 1 + \langle \alpha[z'/z] \rangle_z \cdot \omega = \langle \sigma z.\alpha[z'/z] \rangle_x. \end{aligned}$$

Using this and Lemma 17 we finally get $\langle \varphi \rangle_x = \langle \psi \rangle_x$. □

Lemma 19.

Let φ be well-bound and $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi) = \emptyset$, then we have that

$$x \in \mathsf{free}(\varphi) \quad \Rightarrow \quad \mathsf{f}^e(\varphi[x/\psi]) = \max\{\mathsf{f}^e(\varphi), \mathsf{f}^e(\psi) + \langle \varphi \rangle_x\}.$$

Proof. If $x \in \mathsf{free}(\varphi)$ then by Lemma 17 we have that $\langle \varphi[\psi/x] \rangle_u = \langle \varphi \rangle_u$ if $u \notin \mathsf{atm}(\psi)$ and that $\langle \varphi[\psi/x] \rangle_u = \max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}$ if $u \in \mathsf{atm}(\psi)$. Therefore, we get

$$\begin{aligned} \mathsf{f}^e(\varphi[x/\psi]) &= \max_{u \in \mathsf{Atm}} \{\langle \varphi[x/\psi] \rangle_u\} = \max \begin{cases} \max\limits_{u \notin \mathsf{atm}(\psi)} \{\langle \varphi \rangle_u\} \\ \max\limits_{u \in \mathsf{atm}(\psi)} \{\max\{\langle \varphi \rangle_u, \langle \psi \rangle_u + \langle \varphi \rangle_x\}\} \end{cases} \\ &= \max \begin{cases} \max\limits_{u \notin \mathsf{atm}(\psi)} \{\langle \varphi \rangle_u\} \\ \max\{\max\limits_{u \in \mathsf{atm}(\psi)} \{\langle \varphi \rangle_u\}, \max\limits_{u \in \mathsf{atm}(\psi)} \{\langle \psi \rangle_u\} + \langle \varphi \rangle_x\} \end{cases} \\ &= \max\{\max_{u \in \mathsf{Atm}} \{\langle \varphi \rangle_u\}, \max_{u \in \mathsf{atm}(\psi)} \{\langle \psi \rangle_u\} + \langle \varphi \rangle_x\} \\ &= \max\{\mathsf{f}^e(\varphi), \mathsf{f}^e(\psi) + \langle \varphi \rangle_x\}. \end{aligned}$$

□

The next lemma is some kind of a generalisation of the previous two lemmas.

Lemma 20.

Let $x_0, \ldots, x_n \in \mathsf{free}(\varphi)$ be pairwise distinct variables.

(1) If φ is well-bound, $y \notin \mathsf{bound}(\varphi)$ and $x_i \neq y$ for $i \leq n$ then

$$\langle \varphi[x_0/y] \dots [x_n/y] \rangle_y = \max\{\langle \varphi \rangle_y, \max_{i \leq n}\{\langle \varphi \rangle_{x_i}\}\}.$$

(2) If $\varphi[x_0/\psi_0] \dots [x_n/\psi_n]$ is well-bound, $x_j \notin \mathsf{var}(\psi_i)$ for $i < j \leq n$ and $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi_i) = \mathsf{bound}(\psi_i) \cap \mathsf{var}(\psi_j) = \emptyset$ for $i < j \leq n$, then we have that

$$\mathsf{f}^e(\varphi[x_0/\psi_0] \dots [x_n/\psi_n]) = \max\{\mathsf{f}^e(\varphi), \max_{i \leq n}\{\mathsf{f}^e(\psi_i) + \langle \varphi \rangle_{x_i}\}\}.$$

Proof. Part 1 is proved by induction on n by using Lemma 17. We further use that $\alpha_i \equiv \varphi[x_0/y] \dots [x_{i-1}/y]$ is well-bound, $y \notin \mathsf{bound}(\alpha_i)$ and $\langle \alpha_i \rangle_{x_i} = \langle \varphi \rangle_{x_i}$ for all $i \leq n$. Part 2 is proved by induction on n by using Lemma 19. Notice that $\beta_i \equiv \varphi[x_0/\psi_0] \dots [x_{i-1}/\psi_{i-1}]$ is well-bound, $\mathsf{bound}(\beta_i) \cap \mathsf{var}(\psi_i) = \emptyset$, $x_i \in \mathsf{free}(\beta_i)$ and $\langle \beta_i \rangle_{x_i} = \langle \varphi \rangle_{x_i}$ for all $i \leq n$. □

The next theorem shows the equivalence of rk and rk^e. Therefore, it provides a method to compute the μ-rank rk by primitive recursion.

Theorem 21. (Effective computation)

$$\text{For all } \varphi \in \mathcal{L}_\mu^+ \text{ we have that } \mathsf{rk}(\varphi) = \mathsf{rk}^e(\varphi)$$

Proof. Let $^* : \mathcal{L}_\mu^+ \to \Omega$ be the mapping used in Definition 15.2. By induction on $\mathsf{rk}(\varphi)$ we show that $\mathsf{rk}(\varphi) = \mathsf{f}^e(\varphi)$ for all well-bound formulae φ. The full claim of the theorem follows by Lemma 13.1 because for any φ we then have that

$$\mathsf{rk}(\varphi) = \mathsf{rk}(\varphi^*) = \mathsf{f}^e(\varphi^*) = \mathsf{rk}^e(\varphi).$$

In the induction we only consider the case $\varphi \equiv \mu x.\alpha$. By Lemma 11.5 and because α is well-bound we get

$$\mathsf{rk}(\varphi) = \sup_{n<\omega}\{[\![\alpha]\!]_x^n(\mathbf{0}) + 1\} = \sup_{n<\omega}\{\mathsf{rk}(\alpha^n(x)) + 1\}.$$

For each natural number n we have $\alpha^n(x)^* \sim_\infty \alpha^n(x)$ with $\alpha^n(x)^*$ well-bound. Hence by Lemma 13.1 and i.h. we get

$$\mathsf{rk}(\varphi) = \sup_{n<\omega}\{\mathsf{rk}(\alpha^n(x)^*) + 1\} = \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n(x)^*) + 1\}.$$

Now we compute $\mathsf{f}^e(\alpha^n(x)^*)$ by distinguishing two cases. In the first case we assume that $\langle \alpha \rangle_x = 0$ or $x \notin \mathsf{free}(\alpha)$, but then $\alpha^n(x) \equiv \alpha$ for $n > 0$ and

$$\begin{aligned} \mathsf{rk}(\varphi) &= \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n(x)^*) + 1\} = \mathsf{f}^e(\alpha^*) + 1 = \mathsf{f}^e(\alpha) + 1 \quad \text{because } \alpha^* \equiv \alpha \\ &= \max_{u \in \mathsf{Atm}}\{\langle \alpha \rangle_u\} + 1 = \max_{u \in \mathsf{Atm}}\{\langle \alpha \rangle_u + 1 + \langle \alpha \rangle_x \cdot \omega\} = \mathsf{f}^e(\varphi). \end{aligned}$$

In the second case we assume that $\langle\alpha\rangle_x > 0$ and $x \in \mathsf{free}(\alpha)$. For $n > 0$ we first show by induction on n that

$$\mathsf{f}^e(\alpha^n(x)^*) = \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-1).$$

For $n = 1$ we have $\alpha^n(x)^* \equiv \alpha^* \equiv \alpha$ and $n - 1 = 0$. For $n > 1$ we have $\alpha^n(x) \equiv \alpha[x/\alpha^{n-1}(x)]$ and there are distinct variables $x_0, \dots, x_k$ and well-bound formulae $\hat{\alpha}$ and $\psi_0, \dots, \psi_k$ such that

(1) $\alpha \sim_\infty \hat{\alpha}[x_0/x]\dots[x_k/x]$ and $\hat{\alpha}[x_0/x]\dots[x_k/x]$ is well-bound,

(2) $\alpha^{n-1}(x)^* \sim_\infty \psi_i$ for $i \leq k$,

(3) $\alpha^n(x)^* \sim_\infty \hat{\alpha}[x_0/\psi_0]\dots[x_k/\psi_k]$ and $\hat{\alpha}[x_0/\psi_0]\dots[x_k/\psi_k]$ is well-bound,

(4) $x_i \in \mathsf{free}(\hat{\alpha})$ and $x_j \notin \mathsf{var}(\psi_i)$ and $x_i \not\equiv x$ for $i < j \leq k$.

Hence we have $x \notin \mathsf{var}(\hat{\alpha})$ and $\mathsf{bound}(\hat{\alpha}) \cap \mathsf{var}(\psi_i) = \mathsf{bound}(\psi_i) \cap \mathsf{var}(\psi_j) = \emptyset$ for $i < j \leq k$, and we have the following two facts:
(i)

$$\begin{aligned}
\mathsf{f}^e(\alpha) &= \mathsf{f}^e(\hat{\alpha}[x_0/x]\dots[x_k/x]) \quad \text{by i.h. for } \mathsf{rk}(\alpha) \text{ and 13.1} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i\leq k}\{\mathsf{f}^e(x) + \langle\hat{\alpha}\rangle_{x_i}\}\} \text{ by 20.2} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i\leq k}\{\langle\hat{\alpha}\rangle_{x_i}\}\} = \mathsf{f}^e(\hat{\alpha}).
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathsf{f}^e(\alpha^n(x)^*) &= \mathsf{f}^e(\hat{\alpha}[x_0/\psi_0]\dots[x_k/\psi_k]) \quad \text{by i.h. for } \mathsf{rk}(\alpha^n(x)^*) \text{ and 13.1} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i\leq k}\{\mathsf{f}^e(\psi_i) + \langle\hat{\alpha}\rangle_{x_i}\}\} \quad \text{by 20.2} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}(x)^*) + \max_{i\leq k}\{\langle\hat{\alpha}\rangle_{x_i}\}\} \text{ i.h. for } \mathsf{rk}(\alpha^{n-1}(x)) \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}(x)^*) + \langle\hat{\alpha}[x_0/x]\dots[x_k/x]\rangle_x\} \quad \text{by 20.1} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}(x)^*) + \langle\alpha\rangle_x\} \quad \text{by 18.3} \\
&= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-2) + \langle\alpha\rangle_x\} \quad \text{by i.h. for } n-1 \\
&= \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-1). \quad \text{by i.h. and (i)}
\end{aligned}$$

For $n > 1$ we have $\mathsf{f}^e(\alpha^n(x)^*) + 1 \leq \mathsf{f}^e(\alpha^{n+1}(x)^*)$ because of $\langle\alpha\rangle_x > 0$ hence

$$\begin{aligned}
\mathsf{rk}(\varphi) &= \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n(x)^*) + 1\} = \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n(x)^*)\} \\
&= \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot \omega = \mathsf{f}^e(\alpha) + 1 + \langle\alpha\rangle_x \cdot \omega = \mathsf{f}^e(\varphi).
\end{aligned}$$

□

5 Minimum range ω^ω for μ-ranks

Having established the equivalence of rk and rk^e, this now allows us to provide the sharp lower bound ω^ω for $\mathsf{rk}[\mathcal{L}_\mu] = \mathsf{rk}'[\mathcal{L}_\mu]$ in the remaining of this section. The lower bound is given by computing the μ-rank $\mathsf{rk}^e(\varphi)$ of a special class of formulae φ.

Definition 22.

We fix an infinite sequence of propositional variables $p_0, p_1, \ldots$ such that $p_i \neq p_j$ for $i \neq j$, and we define the formulae Ψ_n^k and Φ_n^k such that

$$\Psi_n^k \equiv (p_{n+k} \wedge \ldots \wedge (p_n \wedge p_0)),$$
$$\Phi_n^k \equiv \begin{cases} \bot \wedge p_0 & k = 0, \\ \mu p_{(n+k-1)} \ldots \mu p_n . \Psi_n^{k-1} & k > 0. \end{cases}$$

Theorem 23. (Lower bound)

For all natural numbers n and k we have that

(1) $u \in \mathsf{atm}(\Phi_n^k) \quad \Rightarrow \quad \langle \Phi_n^k \rangle_u = \omega^k$

(2) $\omega^\omega \subseteq \mathsf{rk}[\mathcal{L}_\mu] = \mathsf{rk}'[\mathcal{L}_\mu]$

Proof. Part 1 is proved by induction on k. If $k = 0$ and $u \in \mathsf{atm}(\Phi_n^k)$ we have that $\langle \Phi_n^k \rangle_u = \langle \bot \wedge p_0 \rangle_u = 1 = \omega^0$. If $k > i \geq 0$ then define $\varphi_i = \mu p_{n+i} \ldots \mu p_n . \Psi_n^{k-1}$. We show $u \in \mathsf{atm}(\Phi_n^k) \Rightarrow \langle \varphi_i \rangle_u = \omega^{i+1}$ by induction on i.

- If $i = 0$ then
$$\langle \varphi_0 \rangle_u = \langle \Psi_n^{k-1} \rangle_u + 1 + \langle \Psi_n^{k-1} \rangle_{p_n} \cdot \omega = \omega$$
because of $0 < \langle \Psi_n^{k-1} \rangle_u \leq \langle \Psi_n^{k-1} \rangle_{p_n} < \omega$.

- For $k > i > 0$ we have $\langle \varphi_{i-1} \rangle_u = \langle \varphi_{i-1} \rangle_{p_{n+i}} = \omega^i$ by i.h. hence
$$\begin{aligned} \langle \varphi_i \rangle_u = \langle \mu p_{n+i} . \varphi_{i-1} \rangle_u &= \langle \varphi_{i-1} \rangle_u + 1 + \langle \varphi_{i-1} \rangle_{p_{n+i}} \cdot \omega \\ &= \omega^i + 1 + \omega^i \cdot \omega = \omega^{i+1}. \end{aligned}$$

Since $\langle \Phi_n^k \rangle_u = \langle \varphi_{k-1} \rangle_u = \omega^k$ the first part is proved.

Part 2 holds because we have $\omega^k = \mathsf{rk}(\Phi_n^k) \in \mathsf{rk}[\mathcal{L}_\mu]$ for all natural numbers k by Part 1 and Theorem 21, hence $\omega^k \subset \mathsf{rk}[\mathcal{L}_\mu]$ by Lemma 13.2, that is $\omega^\omega \subseteq \mathsf{rk}[\mathcal{L}_\mu]$. $\square$

Corollary 24.

$$\mathsf{rk}[\mathcal{L}_\mu] = \mathsf{rk}'[\mathcal{L}_\mu] = \mathsf{rk}[\mathcal{L}_\mu^+] = \mathsf{rk}'[\mathcal{L}_\mu^+] = \omega^\omega$$

6 Generating formulae of any rank

In this section we show how to generate formulae of arbitrary rank in a uniform way.

Definition 25.

For ordinals ξ with $0 < \xi < \omega^\omega$ there is a unique representation in *cantor normal form* (see cf. [11]), that is

$$\xi =_{\mathit{CNF}} \omega^{k_0} + \ldots + \omega^{k_n} \quad \text{with} \quad \omega > k_0 \geq \ldots \geq k_n \geq 0.$$

Based on the cantor normal form we define the mapping $\Theta : \omega^\omega \to \mathcal{L}_\mu$ such that

$$\Theta_\xi = \begin{cases} \bot & \xi = 0, \\ \Phi_1^k[p_0/\Theta_0] & \xi =_{\mathit{CNF}} \omega^k, \\ \Phi_{1+k_0+\ldots+k_{n-1}}^{k_n}[p_0/\Theta_{\omega^{k_0}+\ldots+\omega^{k_{n-1}}}] & \xi =_{\mathit{CNF}} \omega^{k_0} + \ldots + \omega^{k_n}. \end{cases}$$

Remark 26. Some examples to illustrate the structure of the formula Θ_ξ.

$$\begin{aligned} \Theta_{\omega^2} &= \Phi_1^2[p_0/\bot] = \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)), \\ \Theta_{\omega^2\cdot 2} &= \Phi_3^2[p_0/\Theta_{\omega^2}] = \mu p_4 \mu p_3 (p_4 \wedge (p_3 \wedge \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)))), \\ \Theta_{\omega^2\cdot 2+\omega+2} &= \bot \wedge (\bot \wedge \mu p_5 (p_5 \wedge \mu p_4 \mu p_3 (p_4 \wedge (p_3 \wedge \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)))))). \end{aligned}$$

Theorem 27.

For all $\xi < \omega^\omega$ we have that

$$\mathsf{rk}'(\Theta_\xi) = \mathsf{rk}(\Theta_\xi) = \mathsf{rk}^e(\Theta_\xi) = \xi$$

Proof. This is proved by induction on ξ. We simultaneously show the following:

(i) $\mathsf{atm}(\Theta_\xi) = \{\bot, p_0, \ldots, p_{k_0+\ldots+k_n}\} \backslash \{p_0\}$ for $\xi =_{\mathit{CNF}} \omega^{k_0} + \ldots + \omega^{k_n}$, $\mathsf{atm}(\Theta_0) = \{\bot\}$,

(ii) Θ_ξ is well-bound,

(iii) $\mathsf{rk}^e(\Theta_\xi) = \xi$.

If $\xi = 0$ then $\Theta_0 = \bot$ is well-bound, $\mathsf{atm}(\bot) = \{\bot\}$, $\mathsf{rk}^e(\bot) = \max_{u \in \mathsf{Atm}} \{0\} = 0$.
If $\xi =_{\mathit{CNF}} \omega^{k_0} + \ldots + \omega^{k_n}$ and $\zeta = \omega^{k_0} + \ldots + \omega^{k_{n-1}} < \xi$ and $s = k_0 + \ldots + k_{n-1}$ (for $n = 0$ let $\zeta = 0$ and $s = 0$) then $\Theta_\xi = \Phi^{k_n}_{1+s}[p_0/\Theta_\zeta]$. By the definition of $\Phi^{k_n}_{1+s}$ we have that $\Phi^{k_n}_{1+s}$ is well-bound and

$$\mathsf{bound}(\Phi^{k_n}_{1+s}) = \mathsf{atm}(\Phi^{k_n}_{1+s}) \backslash \{\bot, p_0\} = \{p_{1+s}, \ldots, p_{s+k_n}\}.$$

By i.h. we get that Θ_ζ is well-bound, and that $\mathsf{atm}(\Theta_\zeta) = \{\bot, p_1, \ldots, p_s\}$. Thus, because there is only one occurrence of p_0 in $\Phi^{k_n}_{1+s}$ and because of $\mathsf{bound}(\Phi^{k_n}_{1+s}) \cap \mathsf{var}(\Theta_\zeta) = \emptyset$, we have that

$$\mathsf{atm}(\Theta_\xi) = \{\bot, p_1, \ldots, p_{s+k_n}\} \text{ and } \Theta_\xi \text{ is well-bound.}$$

Now because Θ_ξ, Θ_ζ and $\Phi^{k_n}_{1+s}$ are well-bound and because $p_0 \in \mathsf{free}(\Phi^{k_n}_{1+s})$ and $\mathsf{bound}(\Phi^{k_n}_{1+s}) \cap \mathsf{var}(\Theta_\zeta) = \emptyset$ the following holds by Lemma 19:

$$\begin{aligned}
\mathsf{rk}^e(\Theta_\xi) = \mathsf{rk}^e(\Phi^{k_n}_{1+s}[p_0/\Theta_\zeta]) &= \max\{\mathsf{rk}^e(\Phi^{k_n}_{1+s}), \mathsf{rk}^e(\Theta_\zeta) + \langle \Phi^{k_n}_{1+s} \rangle_{p_0}\} \\
&= \max\{\omega^{k_n}, \mathsf{rk}^e(\Theta_\zeta) + \omega^{k_n}\} = \mathsf{rk}^e(\Theta_\zeta) + \omega^{k_n} \qquad \text{by 23.1} \\
&= \zeta + \omega^{k_n} = \xi. \qquad \text{by i.h.}
\end{aligned}$$

Since Θ_ξ is well-bound, we get $\mathsf{rk}'(\Theta_\xi) = \mathsf{rk}(\Theta_\xi) = \mathsf{rk}^e(\Theta_\xi) = \xi$ for $\xi < \omega^\omega$ by Lemma 11.7 and Theorem 21. $\square$

7 Appendix

In this appendix we give some thorough definitions of notions that are used in the preliminaries, and we prove some important propositions stated in the preliminaries. In the following $\mathbb{N}$ stands for the set of natural numbers.

Definition 28.

For any natural number k we inductively define the set of numbers $U_k \subseteq \mathbb{N}$ such that $k \in U_k$, and if $n \in U_k$ then $2n \in U_k$ and $2n+1 \in U_k$.

The next lemma is the key to some definitions below, because it helps to assign unique numbers to the nodes of any binary tree in a uniform way.

Lemma 29.

For any natural number $k > 0$ we have that

(1) $n \in U_k \quad \Leftrightarrow \quad U_n \subseteq U_k$

(2) $U_{2k} \cup U_{2k+1} = U_k \setminus \{k\}$

(3) $U_k \cap U_{k+1} = \emptyset$

Proof. For Part 1 we have $\Leftarrow$ because $n \in U_n \subseteq U_k$. For $\Rightarrow$ we show $U_n \subseteq U_k$ by induction on $m \in U_n$. If $m = n$ then $m \in U_k$. If $m > n$ then $m = 2r$ or $m = 2r+1$ for some $r \in U_n$, and by i.h. $r \in U_k$ hence $m \in U_k$.

For Part 2 we get $U_{2k} \cup U_{2k+1} \subseteq U_k \setminus \{k\}$ by Part 1. And we show $U_k \setminus \{k\} \subseteq U_{2k} \cup U_{2k+1}$ by induction on $m \in U_k \setminus \{k\}$. We observe that $\{2k, 2k+1\} \subset U_{2k} \cup U_{2k+1}$, and for $2k+1 < m \in U_k \setminus \{k\}$ we have $m = 2n$ or $m = 2n+1$ for some $n \in U_k \setminus \{k\}$, and by i.h. we get $n \in U_{2k} \cup U_{2k+1}$ hence $m \in U_{2k} \cup U_{2k+1}$.

For Part 3 we assume $U_k \cap U_{k+1} \neq \emptyset$ and $m = \min(U_k \cap U_{k+1})$. We have $m > k+1$ because $k \notin U_{k+1}$ and $k+1 \notin U_k$. Hence $m = 2n$ or $m = 2n+1$ with $n \in U_k \cap U_{k+1}$, in contradiction to the minimality of m. $\square$

Definition 30. (Occurrences of symbols)

(1) The set of *symbols* $\mathcal{S}_\mu^+$ of the language $\mathcal{L}_\mu^+$ is defined such that

$$\mathcal{S}_\mu^+ = \mathsf{Atm} \cup \{\sim, \neg, \wedge, \vee, \Diamond, \Box\} \cup \{\mu x \mid x \in \mathsf{Var}\} \cup \{\nu x \mid x \in \mathsf{Var}\}.$$

(2) The set of *occurrences of symbols* in a formula φ, $\mathsf{Symb}(\varphi) \subset \mathcal{S}_\mu^+ \times \mathbb{N}$ is defined by $\mathsf{Symb}(\varphi) = f(\varphi, 1)$ with $f : \mathcal{L}_\mu^+ \times \mathbb{N} \to \wp(\mathcal{S}_\mu^+ \times \mathbb{N})$ such that

$$f(\varphi, n) = \begin{cases} \{(\varphi, n)\} & \varphi \in \mathsf{Atm}, \\ f(\alpha, 2n) \cup \{(\circ, n)\} & \varphi \equiv \circ\alpha, \circ \in \{\sim, \neg, \Diamond, \Box\}, \\ f(\alpha, 2n) \cup f(\beta, 2n{+}1) \cup \{(\circ, n)\} & \varphi \equiv \alpha \circ \beta,\ \circ \in \{\wedge, \vee\}, \\ f(\alpha, 2n) \cup \{(\sigma x, n)\} & \varphi \equiv \sigma x.\alpha,\ \sigma \in \{\mu, \nu\}. \end{cases}$$

(3) The set of *occurrences of free variables* in a formula φ, $\mathsf{Free}(\varphi) \subset \mathsf{Var} \times \mathbb{N}$ is defined by $\mathsf{Free}(\varphi) = g(\varphi, 1)$ with $g : \mathcal{L}_\mu^+ \times \mathbb{N} \to \wp(\mathsf{Var} \times \mathbb{N})$ such that

$$g(\varphi, n) = \begin{cases} \emptyset & \varphi \in \mathsf{Cst} \\ \{(\varphi, n)\} & \varphi \in \mathsf{Var}, \\ g(\alpha, 2n) & \varphi \equiv \circ\alpha, \ \circ \in \{\sim, \neg, \Diamond, \Box\}, \\ g(\alpha, 2n) \cup g(\beta, 2n{+}1) & \varphi \equiv \alpha \circ \beta, \ \circ \in \{\wedge, \vee\}, \\ g(\alpha, 2n) \setminus (\{x\} \times \mathbb{N}) & \varphi \equiv \sigma x.\alpha, \ \sigma \in \{\mu, \nu\}. \end{cases}$$

(4) The set of *occurrences of bound variables* in φ, $\mathsf{Bound}(\varphi) \subset \mathsf{Var} \times \mathbb{N}$ is defined by $\mathsf{Bound}(\varphi) = h(\varphi, 1)$ with $h : \mathcal{L}_\mu^+ \times \mathbb{N} \to \wp(\mathsf{Var} \times \mathbb{N})$ such that

$$h(\varphi, n) = \begin{cases} \emptyset & \varphi \in \mathsf{Atm} \\ h(\alpha, 2n) & \varphi \equiv \circ\alpha, \ \circ \in \{\sim, \neg, \Diamond, \Box\}, \\ h(\alpha, 2n) \cup h(\beta, 2n{+}1) & \varphi \equiv \alpha \circ \beta, \ \circ \in \{\wedge, \vee\}, \\ h(\alpha, 2n) \cup \{(\sigma x, n)\} & \varphi \equiv \sigma x.\alpha, \ \sigma \in \{\mu, \nu\}. \end{cases}$$

Remark 31. Observe that $\mathsf{Free}(\varphi) \cup \mathsf{Bound}(\varphi) \subseteq \mathsf{Symb}(\varphi)$.

Having at hand the sets of occurrences $\mathsf{Free}(\varphi)$ and $\mathsf{Bound}(\varphi)$, it is easy to define the sets of free and bound variables of φ.

Definition 32. (Free and bound variables)

The set of *free variables* $\mathsf{free}(\varphi) \subset \mathsf{Var}$ and the set of *bound variables* $\mathsf{bound}(\varphi) \subset \mathsf{Var}$ of a formula φ are defined such that

$$\mathsf{free}(\varphi) = \{x \mid (x, n) \in \mathsf{Free}(\varphi)\},$$
$$\mathsf{bound}(\varphi) = \{x \mid (\sigma x, n) \in \mathsf{Bound}(\varphi).\}$$

Definition 33. (Preceding symbols)

The set of *preceding symbols* $\mathsf{pre}(\varphi, n) \subset \mathsf{Symb}(\varphi)$ of an occurrence $(s, n) \in \mathsf{Symb}(\varphi)$ of the symbol $s \in \mathcal{S}_\mu^+$ in the formula φ is defined such that

$$\mathsf{pre}(\varphi, n) = \{(t, k) \in \mathsf{Symb}(\varphi) \mid k < n, n \in U_k\}.$$

Remark 34. The number of negation symbols $\neg$ preceding an occurrence $(x, n) \in \mathsf{Symb}(\varphi)$ of a variable x in the formula φ is equal to the cardinality of the set $\{k \mid (\neg, k) \in \mathsf{pre}(\varphi, n)\}$.

Definition 35. (Subformulae)

The *subformulae* $\mathsf{sub}(\varphi) \subset \mathcal{L}_\mu^+$ of a formula φ are defined such that

$$\mathsf{sub}(\varphi) = \begin{cases} \{\varphi\} & \varphi \in \mathsf{Lit} \\ \{\varphi\} \cup \mathsf{sub}(\alpha) & \varphi \equiv \circ\alpha, \circ \in \{\neg, \Diamond, \Box\}, \\ \{\varphi\} \cup \mathsf{sub}(\alpha) \cup \mathsf{sub}(\beta) & \varphi \equiv \alpha \circ \beta, \circ \in \{\wedge, \vee\}, \\ \{\varphi\} \cup \mathsf{sub}(\alpha) & \varphi \equiv \sigma x.\alpha, \sigma \in \{\mu, \nu\}. \end{cases}$$

In the following we prove some important properties of the semantics of the modal μ-calculus.

Lemma 36. (Complementation)

For any transition system $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$ and formula $\varphi \in \mathcal{L}_\mu^+$ we have

$$\|\overline{\varphi}\|_\mathcal{T} = \|\neg\varphi\|_\mathcal{T}.$$

Proof. We show $\|\overline{\varphi}\|_\mathcal{T} = \mathsf{S} \setminus \|\varphi\|_\mathcal{T}$ by induction on $l(\varphi)$. We have that

$$\begin{aligned} \|\overline{\bot}\|_\mathcal{T} &= \|\top\|_\mathcal{T} = \mathsf{S} = \mathsf{S} \setminus \emptyset = \mathsf{S} \setminus \|\bot\|_\mathcal{T}, \\ \|\overline{\top}\|_\mathcal{T} &= \|\bot\|_\mathcal{T} = \emptyset = \mathsf{S} \setminus \mathsf{S} = \mathsf{S} \setminus \|\top\|_\mathcal{T}, \\ \|\overline{x}\|_\mathcal{T} &= \|{\sim}x\|_\mathcal{T} = \mathsf{S} \setminus \|x\|_\mathcal{T}, \\ \|\overline{{\sim}x}\|_\mathcal{T} &= \|x\|_\mathcal{T} = \mathsf{S} \setminus (\mathsf{S} \setminus \|x\|_\mathcal{T}) = \mathsf{S} \setminus \|{\sim}x\|_\mathcal{T}. \end{aligned}$$

Further by using the induction hypothesis we get that

$$\begin{aligned} \|\overline{\neg\alpha}\|_\mathcal{T} &= \|\neg\overline{\alpha}\|_\mathcal{T} = \mathsf{S} \setminus \|\overline{\alpha}\|_\mathcal{T} = \mathsf{S} \setminus \|\neg\alpha\|_\mathcal{T}, \\ \|\overline{\alpha \wedge \beta}\|_\mathcal{T} &= \|\overline{\alpha} \vee \overline{\beta}\|_\mathcal{T} = \|\overline{\alpha}\|_\mathcal{T} \cup \|\overline{\beta}\|_\mathcal{T} = (\mathsf{S} \setminus \|\alpha\|_\mathcal{T}) \cup (\mathsf{S} \setminus \|\beta\|_\mathcal{T}) \\ &= \mathsf{S} \setminus (\|\alpha\|_\mathcal{T} \cap \|\beta\|_\mathcal{T}) = \mathsf{S} \setminus (\|\alpha \wedge \beta\|_\mathcal{T}), \\ \|\overline{\alpha \vee \beta}\|_\mathcal{T} &= \|\overline{\alpha} \wedge \overline{\beta}\|_\mathcal{T} = \|\overline{\alpha}\|_\mathcal{T} \cap \|\overline{\beta}\|_\mathcal{T} = (\mathsf{S} \setminus \|\alpha\|_\mathcal{T}) \cap (\mathsf{S} \setminus \|\beta\|_\mathcal{T}) \\ &= \mathsf{S} \setminus (\|\alpha\|_\mathcal{T} \cup \|\beta\|_\mathcal{T}) = \mathsf{S} \setminus (\|\alpha \vee \beta\|_\mathcal{T}), \\ \|\overline{\Box\alpha}\|_\mathcal{T} &= \|\Diamond\overline{\alpha}\|_\mathcal{T} = \{a \in \mathsf{S} \mid \exists b((a \rightarrow b) \wedge b \in \|\overline{\alpha}\|_\mathcal{T})\} \\ &= \mathsf{S} \setminus \{a \in \mathsf{S} \mid \forall b((a \rightarrow b) \Rightarrow b \in \|\alpha\|_\mathcal{T})\} = \mathsf{S} \setminus \|\Box\alpha\|_\mathcal{T}, \\ \|\overline{\Diamond\alpha}\|_\mathcal{T} &= \|\Box\overline{\alpha}\|_\mathcal{T} = \{a \in \mathsf{S} \mid \forall b((a \rightarrow b) \Rightarrow b \in \|\overline{\alpha}\|_\mathcal{T})\} \\ &= \mathsf{S} \setminus \{a \in \mathsf{S} \mid \exists b((a \rightarrow b) \wedge b \in \|\alpha\|_\mathcal{T})\} = \mathsf{S} \setminus \|\Diamond\alpha\|_\mathcal{T}, \end{aligned}$$

$$
\begin{aligned}
\|\overline{\nu x.\alpha}\|_{\mathcal{T}} &= \|\mu x.(\overline{\alpha}[x/{\sim}x])\|_{\mathcal{T}} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\overline{\alpha}[x/{\sim}x]\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\overline{\alpha}\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]} \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid (\mathsf{S} \setminus \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]}) \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid (\mathsf{S} \setminus \mathsf{S}') \subseteq \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]}\} \\
&= \mathsf{S} \setminus \bigcup\{(\mathsf{S} \setminus \mathsf{S}') \subseteq \mathsf{S} \mid (\mathsf{S} \setminus \mathsf{S}') \subseteq \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]}\} \\
&= \mathsf{S} \setminus \|\nu x.\alpha\|_{\mathcal{T}}, \\
\|\overline{\mu x.\alpha}\|_{\mathcal{T}} &= \|\nu x.(\overline{\alpha}[x/{\sim}x])\|_{\mathcal{T}} = \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\overline{\alpha}[x/{\sim}x]\|_{\mathcal{T}[x\mapsto \mathsf{S}']}\} \\
&= \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq \|\overline{\alpha}\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]}\} \\
&= \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq (\mathsf{S} \setminus \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]})\} \\
&= \bigcup\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]} \subseteq (\mathsf{S} \setminus \mathsf{S}')\} \\
&= \mathsf{S} \setminus \bigcap\{(\mathsf{S} \setminus \mathsf{S}') \subseteq \mathsf{S} \mid \|\alpha\|_{\mathcal{T}[x\mapsto (\mathsf{S}\setminus\mathsf{S}')]} \subseteq (\mathsf{S} \setminus \mathsf{S}')\} \\
&= \mathsf{S} \setminus \|\mu x.\alpha\|_{\mathcal{T}}. \qquad \square
\end{aligned}
$$

The following definition is used to prove Theorem 39.

Definition 37. (Negation normal form)

For any formula $\varphi \in \mathcal{L}_\mu^+$ we define its *negation normal form* $\widetilde{\varphi} \in \mathcal{L}_\mu^+$ such that

$$
\widetilde{\varphi} = \begin{cases} \varphi & \varphi \in \mathsf{Lit} \\ \overline{\alpha} & \varphi \equiv \neg\alpha, \\ \circ(\widetilde{\alpha}) & \varphi \equiv \circ\alpha, \circ \in \{\Diamond, \Box\}, \\ \widetilde{\alpha} \circ \widetilde{\beta} & \varphi \equiv \alpha \circ \beta, \circ \in \{\wedge, \vee\}, \\ \sigma x.\widetilde{\alpha} & \varphi \equiv \sigma x.\alpha, \sigma \in \{\mu, \nu\}. \end{cases}
$$

The next lemma shows that with respect to the semantics, we can restrict ourselfs to formulae in negation normal form.

Lemma 38. (Normal form)

For any formula $\varphi \in \mathcal{L}_\mu^+$ we have that

(1) The negation symbol $\neg$ does not occur in $\widetilde{\varphi}$.

(2) $\widetilde{\widetilde{\varphi}} \equiv \widetilde{\varphi}$

(3) $\|\widetilde{\varphi}\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}}$ for all transition systems $\mathcal{T}$.

Proof. Part 1, 2 and 3 are proved by induction on $l(\varphi)$. Part 1 holds by the definition of $\widetilde{\varphi}$. We use Part 1 to prove Part 2, and we use Lemma 36 to prove Part 3. $\square$

Theorem 39. (Monotonicity)

For variables x occurring positive in the formula $\varphi \in \mathcal{L}_\mu^+$ we have for any transition system $\mathcal{T} = (\mathsf{S}, \to, \lambda)$ that

$$\mathsf{S}' \subseteq \mathsf{S}'' \subseteq \mathsf{S} \quad \Rightarrow \quad \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \subseteq \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}$$

Proof. W.l.o.g. we assume that the negation symbol $\neg$ does not occur in φ, because otherwise we use Lemma 38 and consider $\|\widetilde{\varphi}\|_{\mathcal{T}[x\mapsto \mathsf{S}']} = \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']}$ (observe that x occurs positive in $\widetilde{\varphi}$ because it does so in φ).

The lemma is proved by induction on $l(\varphi)$. For $\varphi \in \mathsf{Lit}$ we have $\varphi \not\equiv {\sim}x$ because x occurs positive in φ, and if $\varphi \equiv x$ then we have

$$\|x\|_{\mathcal{T}[x\mapsto \mathsf{S}']} = \mathsf{S}' \subseteq \mathsf{S}'' = \|x\|_{\mathcal{T}[x\mapsto \mathsf{S}'']},$$

and in all other cases of literals φ, $\|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']} = \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}$. By using the induction hypothesis we further have

$$\begin{aligned}
\|\alpha \wedge \beta\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \cap \|\beta\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \\
&\subseteq \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']} \cap \|\beta\|_{\mathcal{T}[x\mapsto \mathsf{S}'']} = \|\alpha \wedge \beta\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}, \\
\|\alpha \vee \beta\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \cup \|\beta\|_{\mathcal{T}[x\mapsto \mathsf{S}']} \\
&\subseteq \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']} \cup \|\beta\|_{\mathcal{T}[x\mapsto \mathsf{S}'']} = \|\alpha \vee \beta\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}, \\
\|\Box\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \{a \in \mathsf{S} \mid \forall b((a \to b) \Rightarrow b \in \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']})\} \\
&\subseteq \{a \in \mathsf{S} \mid \forall b((a \to b) \Rightarrow b \in \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']})\} = \|\Box\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}, \\
\|\Diamond\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \{a \in \mathsf{S} \mid \exists b((a \to b) \wedge b \in \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']})\} \\
&\subseteq \{a \in \mathsf{S} \mid \exists b((a \to b) \wedge b \in \|\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']})\} = \|\Diamond\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}.
\end{aligned}$$

If $\varphi \equiv \sigma y.\alpha$ and $y \equiv x$ then clearly $\|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']} = \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}$. Otherwise if $y \not\equiv x$ we get that

$$\begin{aligned}
\|\nu y.\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \bigcup\{\mathsf{Q} \subseteq \mathsf{S} \mid \mathsf{Q} \subseteq \|\alpha\|_{(\mathcal{T}[x\mapsto \mathsf{S}'])[y\mapsto \mathsf{Q}]}\} \\
&= \bigcup\{\mathsf{Q} \subseteq \mathsf{S} \mid \mathsf{Q} \subseteq \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{Q}])[x\mapsto \mathsf{S}']}\} \\
&\subseteq \bigcup\{\mathsf{Q} \subseteq \mathsf{S} \mid \mathsf{Q} \subseteq \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{Q}])[x\mapsto \mathsf{S}'']}\} = \|\nu y.\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}, \\
\|\mu y.\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}']} &= \bigcap\{\mathsf{Q} \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[x\mapsto \mathsf{S}'])[y\mapsto \mathsf{Q}]} \subseteq \mathsf{Q}\} \\
&= \bigcap\{\mathsf{Q} \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{Q}])[x\mapsto \mathsf{S}']} \subseteq \mathsf{Q}\} \\
&\subseteq \bigcap\{\mathsf{Q} \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{Q}])[x\mapsto \mathsf{S}'']} \subseteq \mathsf{Q}\} = \|\mu y.\alpha\|_{\mathcal{T}[x\mapsto \mathsf{S}'']}.
\end{aligned}$$

$\square$

Theorem 40. (Fixpoint theorem, Knaster [12])

> If the variable x is occurring positive in the formula φ then for all transition systems $\mathcal{T}$ we have that $\|\mu x.\varphi\|_{\mathcal{T}}$ and $\|\nu x.\varphi\|_{\mathcal{T}}$ are the least and the greatest fixpoint of the mapping $\mathsf{S}' \mapsto \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']}$, respectively.

Proof. Let $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$ be any transition system. We define the shortcuts $I(\mathsf{S}') = \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']}$, $L = \|\mu x.\varphi\|_{\mathcal{T}}$ and $G = \|\nu x.\varphi\|_{\mathcal{T}}$. By Theorem 39 and by definition we have that

$$\begin{aligned} \mathsf{S}' \subseteq \mathsf{S}'' \quad &\Rightarrow \quad I(\mathsf{S}') \subseteq I(\mathsf{S}''), \\ L = \bigcap &\{\mathsf{S}' \subseteq \mathsf{S} \mid I(\mathsf{S}') \subseteq \mathsf{S}'\}, \\ G = \bigcup &\{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq I(\mathsf{S}')\}. \end{aligned}$$

We show that G is the greatest fixpoint. Let $H_G = \{\mathsf{S}' \subseteq \mathsf{S} \mid \mathsf{S}' \subseteq I(\mathsf{S}')\}$. For any $\mathsf{S}' \in H_G$ we have $\mathsf{S}' \subseteq G$, hence by the monotonicity of I we get $I(\mathsf{S}') \subseteq I(G)$, that is $\mathsf{S}' \subseteq I(\mathsf{S}') \subseteq I(G)$. Because of $\mathsf{S}' \subseteq I(G)$ for any $\mathsf{S}' \in H_G$ we now get $\bigcup H_G \subseteq I(G)$, that is $G \subseteq I(G)$. Again by monotonicity we get $I(G) \subseteq I(I(G))$, hence $I(G) \in H_G$ and $I(G) \subseteq \bigcup H_G = G$. We have $G = I(G)$. For any fixpoint $\mathsf{S}' = I(\mathsf{S}')$ we have $\mathsf{S}' \in H_G$, that is $\mathsf{S}' \subseteq G$.

The least fixpoint L is proved analogous. $\square$

Theorem 41. (Fixpoint by iteration)

> Let $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$ be a transition system, x a variable occurring positive in the formula φ and
>
> $$\begin{aligned} L^0 = \emptyset,\ L^{\xi+1} = \|\varphi\|_{\mathcal{T}[x\mapsto L^\xi]} \text{ and } L^\rho = \textstyle\bigcup_{\xi<\rho} L^\xi \text{ for limit ordinals } \rho, \\ G^0 = \mathsf{S},\ G^{\xi+1} = \|\varphi\|_{\mathcal{T}[x\mapsto G^\xi]} \text{ and } G^\rho = \textstyle\bigcap_{\xi<\rho} G^\xi \text{ for limit ordinals } \rho, \end{aligned}$$
>
> then there is an ordinal $|\xi| \leq |\mathsf{S}|$ such that
>
> $$L^\xi = L^{\xi+1} = \|\mu x.\varphi\|_{\mathcal{T}} \quad \text{and} \quad G^\xi = G^{\xi+1} = \|\nu x.\varphi\|_{\mathcal{T}}.$$

Proof. Let $I(\mathsf{S}') = \|\varphi\|_{\mathcal{T}[x\mapsto \mathsf{S}']}$. By induction on ξ we have $L^\xi \subseteq L^{\xi+1}$, because of $L^0 = \emptyset \subseteq I(\emptyset) = L^1$, and $L^{\xi+1} = I(L^\xi) \subseteq I(L^{\xi+1}) = L^{(\xi+1)+1}$ by i.h. and the monotonicity of I, Theorem 39, and for limit ordinals ξ we have for any $\zeta < \xi$ that $I(L^\zeta) \subseteq I(\bigcup_{\zeta<\xi} L^\zeta) = I(L^\xi) = L^{\xi+1}$ by the monotonicity of I, hence by i.h. we get $L^\xi = \bigcup_{\zeta<\xi} L^\zeta \subseteq \bigcup_{\zeta<\xi} L^{\zeta+1} = \bigcup_{\zeta<\xi} I(L^\zeta) \subseteq L^{\xi+1}$.

Now let γ be the least ordinal such that $|\gamma| > |\mathsf{S}|$ and assume $L^\xi \neq L^{\xi+1}$ for all $\xi < \gamma$, then there is a one-one mapping $f : \gamma \rightarrow \mathsf{S}$ with $f(\xi) \in L^{\xi+1} \setminus L^\xi$

in contradiction to $|\gamma| > |\mathsf{S}|$. Hence there is some $\xi < \gamma$ with $|\xi| \leq |\mathsf{S}|$ and $L^\xi = L^{\xi+1}$, that is L^ξ is a fixpoint of I. We have $\|\mu x.\varphi\|_{\mathcal{T}} \subseteq L^\xi$ by Theorem 40. By induction on ξ we show $L^\xi \subseteq \|\mu x.\varphi\|_{\mathcal{T}} = \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid I(\mathsf{S}') \subseteq \mathsf{S}'\}$, that is $L^\xi \subseteq \mathsf{S}'$ for any $\mathsf{S}' \subseteq \mathsf{S}$ with $I(\mathsf{S}') \subseteq \mathsf{S}'$. Clearly $L^0 = \emptyset \subset \|\mu x.\varphi\|_{\mathcal{T}}$. Now let S' such that $I(\mathsf{S}') \subseteq \mathsf{S}'$. If $\xi = \zeta + 1$ then $L^\zeta \subseteq \mathsf{S}'$ by i.h., and by the monotonicity of I we get $L^\xi = I(L^\zeta) \subseteq I(\mathsf{S}') \subseteq \mathsf{S}'$. If ξ is a limit ordinal then by i.h. we have $L^\zeta \subset \mathsf{S}'$ for any $\zeta < \xi$, that is $L^\xi = \bigcup_{\zeta<\xi} L^\zeta \subseteq \mathsf{S}'$. We have shown $L^\xi = L^{\xi+1} = \|\mu x.\varphi\|_{\mathcal{T}}$.

The proof for $G^\xi = G^{\xi+1} = \|\nu x.\varphi\|_{\mathcal{T}}$ is analogous. $\square$

Theorem 42. (Substitution)

> If $\varphi, \psi \in \mathcal{L}_\mu^+$ are such that $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset$, then for any transition system $\mathcal{T} = (\mathsf{S}, \rightarrow, \lambda)$ we have that
>
> $$\|\varphi[x/\psi]\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}$$

Proof. This is proved by induction on $l(\varphi)$. For literals $\varphi \in \mathsf{Lit}$ we have

$$\begin{aligned}
\|x[x/\psi]\|_{\mathcal{T}} &= \|\psi\|_{\mathcal{T}} = \|x\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]},\\
\|{\sim}x[x/\psi]\|_{\mathcal{T}} &= \|\overline{\psi}\|_{\mathcal{T}} = \|\neg\psi\|_{\mathcal{T}} = \mathsf{S} \setminus \|\psi\|_{\mathcal{T}} = \|{\sim}x\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}
\end{aligned}$$

by using Lemma 36. In all other base cases we have $x \notin \mathsf{free}(\varphi)$ hence

$$\|\varphi[x/\psi]\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}.$$

By using the induction hypothesis we further get

$$\begin{aligned}
\|(\neg\alpha)[x/\psi]\|_{\mathcal{T}} &= \|\neg(\alpha[x/\psi])\|_{\mathcal{T}} = \mathsf{S} \setminus \|\alpha[x/\psi]\|_{\mathcal{T}}\\
&= \mathsf{S} \setminus \|\alpha\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]} = \|\neg\alpha\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]},\\
\|(\alpha \wedge \beta)[x/\psi]\|_{\mathcal{T}} &= \|\alpha[x/\psi]\|_{\mathcal{T}} \cap \|\beta[x/\psi]\|_{\mathcal{T}}\\
&= \|\alpha[x/\psi]\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]} \cap \|\beta[x/\psi]\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}\\
&= \|(\alpha \wedge \beta)[x/\psi]\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]},\\
\|(\Box\alpha)[x/\psi]\|_{\mathcal{T}} &= \|\Box(\alpha[x/\psi])\|_{\mathcal{T}}\\
&= \{a \in \mathsf{S} \mid \forall b((a \rightarrow b) \Rightarrow b \in \|\alpha[x/\psi]\|_{\mathcal{T}})\}\\
&= \{a \in \mathsf{S} \mid \forall b((a \rightarrow b) \Rightarrow b \in \|\alpha\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]})\}\\
&= \|\Box\alpha\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]},
\end{aligned}$$

similar for $\varphi \equiv \alpha \vee \beta, \Diamond\alpha$. If $\varphi \equiv \mu y.\alpha$ and $x \equiv y$ then $x \notin \mathsf{free}(\varphi)$ and $\|\varphi[x/\psi]\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}} = \|\varphi\|_{\mathcal{T}[x\mapsto\|\psi\|_{\mathcal{T}}]}$, otherwise if $x \not\equiv y$ then because of $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset$, hence $y \notin \mathsf{free}(\psi)$, we have that

$$\begin{aligned}
\|(\mu y.\alpha)[x/\psi]\|_{\mathcal{T}} &= \|\mu y.(\alpha[x/\psi])\|_{\mathcal{T}} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha[x/\psi]\|_{\mathcal{T}[y\mapsto \mathsf{S}']} \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{S}'])[x\mapsto \|\psi\|_{\mathcal{T}[y\mapsto \mathsf{S}']}]} \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[y\mapsto \mathsf{S}'])[x\mapsto \|\psi\|_{\mathcal{T}}]} \subseteq \mathsf{S}'\} \\
&= \bigcap\{\mathsf{S}' \subseteq \mathsf{S} \mid \|\alpha\|_{(\mathcal{T}[x\mapsto \|\psi\|_{\mathcal{T}}])[y\mapsto \mathsf{S}']} \subseteq \mathsf{S}'\} = \|\mu y.\alpha\|_{\mathcal{T}[x\mapsto \|\psi\|_{\mathcal{T}}]}.
\end{aligned}$$

The analogous argument for $\varphi \equiv \nu y.\alpha$ finishes the proof. □

The next theorem is some general statement about well-founded binary relations.

Theorem 43. (Rank function)

> Let $\lhd$ ba a well-founded binary relation on the set $\mathcal{S}$, and $\mathcal{F} \subseteq \mathcal{S}$ be a $\lhd$-downward closed set, i.e. if $\varphi \in \mathcal{F}$ and $\psi \lhd \varphi$ then $\psi \in \mathcal{F}$. If $f_\lhd$ is the mapping from $\mathcal{S}$ to the ordinals with $f_\lhd(\varphi) = \bigcup\{f_\lhd(\psi) + 1 \mid \psi \lhd \varphi\}$, then for any ordinal ξ and order preserving map $g : \mathcal{F} \to \xi$, i.e. if $\varphi \in \mathcal{F}$ and $\psi \lhd \varphi$ then $g(\psi) < g(\varphi)$, we have that
>
> $$\bigcup f_\lhd[\mathcal{F}] \leq \xi.$$

Proof. We show $f_\lhd(\varphi) \leq g(\varphi)$ for $\varphi \in \mathcal{F}$ by induction on $\lhd$. If $\varphi \in \mathcal{F}$ is such that $\{\psi \in \mathcal{F} \mid \psi \lhd \varphi\} = \emptyset$ then $\{\psi \mid \psi \lhd \varphi\} = \emptyset$, hence $f_\lhd(\varphi) = 0 \leq g(\varphi)$. If $\varphi \in \mathcal{F}$ and $\{\psi \in \mathcal{F} \mid \psi \lhd \varphi\} \neq \emptyset$ then by i.h. we have that

$$\begin{aligned}
f_\lhd(\varphi) &= \bigcup\{f_\lhd(\psi) + 1 \mid \psi \lhd \varphi\} \\
&= \bigcup\{f_\lhd(\psi) + 1 \mid \psi \in \mathcal{F},\ \psi \lhd \varphi\} \\
&\leq \bigcup\{g(\psi) + 1 \mid \psi \in \mathcal{F},\ \psi \lhd \varphi\} \\
&\leq \bigcup\{g(\varphi) \mid \psi \in \mathcal{F},\ \psi \lhd \varphi\} = g(\varphi).
\end{aligned}$$

This yields $f_\lhd(\varphi) \leq g(\varphi) < \xi$ for any $\varphi \in \mathcal{F}$, hence $\bigcup f_\lhd[\mathcal{F}] \leq \xi$. □

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