Belief Change Functions for Multi-Agent Systems

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
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vorgelegt von
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Abstract

Belief change theory is a big area of research in theoretical computer science. Among other things, it is related to updates in database theory. A multi-agent system is a collection of rational agents. These can be computers in a network, processors in a computer, or processes in an operating system. A rational agent is the concept of some object (or person) that can do reasoning based on its beliefs. This thesis deals with the question how to change some rational agents' beliefs. It is divided into two parts.

The first part of this thesis is about belief change in classical propositional logic. That is, facts are represented by propositional formulas, and we only deal with the beliefs of one agent. A belief state of the agent is represented by a set of models, which are usually called possible worlds. There are many types of belief change functions, but we concentrate on four of them. Revision is at the centre of our considerations, and we also investigate the related belief change types: expansion, contraction, and update. We give a survey on traditional belief revision and update theory and we translate specifications and some results to the notion of model sets, our way of representing belief states. In addition, we give an answer to the question how to deal with belief change functions in the context of consistent beliefs. Furthermore, we suggest a way of translating a given revision function to an update function and vice versa. Finally, we introduce new revision and update functions, which also give rise to new contraction functions via our translations.

The second part of this thesis deals with belief expansion in several systems of multi-agent modal logic, also called epistemic or doxastic logic. The beliefs of the agents contain both propositional facts and agents' beliefs. In some systems, the beliefs have to be consistent. There are also some systems where the beliefs have to be true; in this case we talk about knowledge instead of belief. The beliefs can be changed by an announcement of new information encoded by an arbitrary formula. The announcement can come from the environment or can be made by an agent. There is the concept of group announcements, which can be made to arbitrary groups of agents, even to a single agent. Hereby, we distinguish two possible behaviours of the agents: trustful and sceptical. Trustful agents learn every announced formula, whereas sceptical agents reject the information if it contradicts their beliefs. A special case are public announcements, where the new information is always given to the whole community of agents. We give a survey of the well-known truthful public announcements, which are partial, and we provide new syntactical proofs for many known results. We then introduce a new semantics that allows for announcing formulas that are not necessarily true. These announcements are called total public announcements and we use this approach to incorporate public announcements into the logic of knowledge and belief.

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Part I

Propositional belief change

Introduction to Part I

The first part of this thesis is about three types of belief change functions in classical propositional logic: revision, contraction, and update. The contribution of Alchourrón, Gärdenfors, and Makinson [2] can be seen as the foundation of belief revision theory. Revision is the process of consistently adding new information to some representation of a given belief state. There is no problem if the new information is consistent with our beliefs. But if the new information contradicts our beliefs, then we have to retract some of them. There is no unique answer to the question which beliefs should be retracted. It has often been argued that we should retract as few beliefs as possible, which is known as the requirement of minimal change, cf. Gärdenfors [27]. We discuss the properties of revision functions and work towards a definition of a new revision function. In some cases we are going to depart from the idea of minimal change and we will give reasons for that. The alternative we are proposing is called minimax change, described by the minimax revision function.

The revision process is defined to take place in a static world. That is, the facts have not changed, we have just got new information. So if the new information contradicts our beliefs, then we have to retract some of the original beliefs because they are false. Contraction is another type of belief change function that takes place in a static world. It is the process of removing some beliefs from a given belief state representation. This can happen if we learn that it has been wrong to add some information. Contraction is closely related to revision, cf. [2, 27], and the problems with revision translate to the case of contraction. That is, it is not generally clear how to perform a contraction. Similar to revision, the requirement of minimal change demands that we should not retract too many beliefs by performing a contraction. We will investigate the properties of contraction functions and explore the relationships between revision and contraction. These relationships and the new revision function will lead to the definition of the minimax contraction function.

Update is another type of belief change function that is closely related to revision. It is also defined to be the process of consistently adding new information to a given belief state representation. The only difference from revision is that the update process takes place in a dynamic world. That is, the beliefs may need to be changed because of the change of some facts. Katsuno and Mendelzon [47] have argued that update functions should therefore satisfy properties different from those that revision functions are supposed to have. But they agree on the requirement of minimal change. We will analyse the properties of update functions and provide new relationships between revision and update. Moreover, we will define new update functions and investigate how they translate to revision according to the new relationships. Finally, we will define the minimax update function, which corresponds to our new revision function.

Outline

In Chapter 1 we will give the formal definitions and notions that we are going to use in Part I. We will start with the syntax and semantics of classical propositional logic and state some results for later use. Moreover, we will define the two operators Mod and Th, which will be very important for the results in Chapter 2 and Chapter 3. The operator Mod maps a set T of formulas to a set that contains all models satisfying all formulas in T. The operator Th maps a set S of models to the set that contains all formulas satisfied in all models from S.

The main task of Chapter 1 is to give three different definitions of propositional databases. A propositional database is the mathematical formalisation of a belief state. That is, it contains collected information represented by propositional formulas. First, a database can be a set S of models, which we will call a model set. The formulas stored in S are given by the set $\mathrm{Th}(S)$. Model sets are our preferred kind of database, and we will argue for our choice. Second, from traditional belief revision theory, a belief state can be represented by a deductively closed set T of formulas, which is called a belief set, cf. [27]. This set itself is the collection of all facts that we believe to be true. Third, from traditional update theory, a database can be a formula φ, which is called a belief base, cf. [39]. The formulas stored in this kind of database are all the formulas that are logically entailed by φ.

We will show how we can store new information in a database. This operation is called expansion, and we will define the expansion function for each kind of belief state representation. Furthermore, we will show how to translate a database of a given type to a database of another type so that they both contain the same information. The operators Mod and Th will be used for the definition of these translations.

We will then present the famous AGM postulates for revision and contraction from [2] in Chapter 2. They have been formulated in order to capture the notion of minimal change. It has turned out that these two sets of postulates do not lead to the definition of a unique revision and contraction function, cf. [2]. Furthermore, there are examples of revision and contraction functions that satisfy all of the respective postulates, yet it is commonly agreed that these functions are not acceptable for practical use, cf. [2, 27]. The AGM postulates have been formulated in the context of belief sets. We will give the corresponding postulates in the context of model sets and show that they are equivalent to the original ones. For this purpose, the operators Mod and Th from Chapter 1 will again be very useful.

The KM postulates for update from [47] are the corresponding formalisation of minimal change for update functions. They do not uniquely define an update function either, e.g. two different functions satisfying all of the postulates can be found in [39]. In addition, the change performed by an update function is in general considered to be too minimal, cf. [39]. The KM postulates have been stated in the context of belief bases and for a finite set of propositions. We will translate them to the notion of model sets in such a way that we can also consider infinite sets of propositions. We will show that our reformulation of the update postulates is equivalent to the original one.

Although not all of the original postulates are acceptable for practical use, we believe that most of the postulates make sense and should be required. The main goal of our translation to model sets is to explore the relationships between revision and update in Chapter 3. Another goal is to provide alternative postulates for revision, contraction, and update in Chapter 4.

It is the aim of Chapter 3 to provide translations from belief change functions of a given type to functions of the two other types. First, we will define the concept of consistent databases. Consistency is the requirement that a model set must never be empty. We will explain why it is not possible to get consistency of model sets by using integrity constraints, cf. [38, 46, 48, 58]. We will then suggest a consistency preserving contraction function that always results in a non empty model set. Moreover, we will modify postulates for revision, contraction, and update in the context of consistent model sets. We will provide translations that transform a belief change function on model sets to a belief change function on consistent model sets and vice versa. These translations have the following property: if a function satisfies all of the postulates, then its translation also does.

It has been mentioned by Levi [51] that revising with a formula α is equivalent to first contracting with $\neg\alpha$ and then expanding with α. Harper [35] has proposed that contracting with a formula α should be the same as taking the original beliefs that remain after the revision with $\neg\alpha$. It has been proved that these translations from revision to contraction and vice versa preserve the AGM postulates, cf. [27]. We will present the same result for the revision and contraction postulates in the context of model sets, as well as a similar result for the modified revision and contraction postulates in the context of consistent model sets.

Our main contribution in Chapter 3 is the definition of translations from revision to update and vice versa. We will prove that if a function satisfies all revision postulates, then its translation satisfies all update postulates. On the other hand, if a function satisfies all update postulates, then its translation satisfies all but one revision postulates. This is not a big disadvantage since, as we will argue in Chapter 4, this revision postulate may be omitted. Furthermore, we will prove similar results for the translations of revision and update functions on consistent model sets. At the end of Chapter 3 we will argue for some common behaviour for revision and update functions. We will argue that in some cases revision and update functions should perform the same belief change.

We will work towards a new revision function in Chapter 4. For this purpose, we will first define new update functions. We will also get new revision functions by using the translation from update to revision from Chapter 3. The new update functions we are going to define do not satisfy all update postulates. Instead, we will minimise the amount of change by minimising the symmetric difference between the original and the updated model set. One of the new update functions, the cautious standard update, performs minimal change in many cases, but a rather substantial change in some few cases. This function is a good example for our new notion of minimax change. In Appendix A we will compare this new update function with the possible models approach, an update function that satisfies all of the original KM postulates. It turns out that on average, the cautious standard update function actually performs less change.

Finally, we will commit ourselves to new sets of postulates for revision, contraction, and update functions. For revision and contraction, we will just drop two problematic postulates and add a new one. For update, we will reject three postulates, add a new one, and modify a postulate that is too strong. These new sets of postulates are compatible with our new idea of minimax change. We will then adapt the results from Chapter 3 to the new postulates, that is we will define new translations from belief change functions of a given type to functions of another type. In addition, we will give examples of revision, contraction, and update functions that satisfy the new sets of postulates and conform to minimax change.

Chapter 1

Basic definitions

This is a preparatory chapter in order to introduce the notions that we are going to use in the first part of this thesis. In Section 1.1 we will define the syntax and semantics of *classical propositional logic*. In addition, we will provide some useful abbreviations as well as some first properties. We will then define belief states and present three different belief state representations in Section 1.2. We will show how these definitions are related and give arguments for our choice. By defining the belief expansion function for each notion of belief state, we will illustrate how beliefs can be changed, however they are represented. Expansion is the simplest known belief change operation and there is a common agreement among researchers how expansion has to be implemented. One can therefore say that the definition of the expansion function is uniquely determined and beyond controversy.

1.1 Syntax and semantics

In this thesis, we will always deal with a countable set $\mathcal{P} \neq \emptyset$ of *propositions* whose existence we presuppose. Propositions stand for statements that cannot be divided into smaller units, so we also call them *atoms*.

Definition 1.1.1. The language $\mathcal{L}_0$ of classical propositional logic is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$),

$$\alpha \ ::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha).$$

The propositional constants $\bot$ (*falsum*) and $\top$ (*verum*) can be defined by the use of some fixed $p_0 \in \mathcal{P}$, that is by

$$\bot := (p_0 \wedge \neg p_0),$$
$$\top := \neg\bot.$$

Disjunction, implication, and equivalence are defined as usual,

$$\begin{aligned}(\alpha \vee \beta) &:= \neg(\neg\alpha \wedge \neg\beta),\\ (\alpha \to \beta) &:= (\neg\alpha \vee \beta),\\ (\alpha \leftrightarrow \beta) &:= ((\alpha \to \beta) \wedge (\beta \to \alpha)).\end{aligned}$$

If no confusion arises, we will omit brackets with the convention that the connectives $\wedge$ and $\vee$ bind more strongly than $\to$ and $\leftrightarrow$. As usual, a *literal* is an atom or a negated atom, and a disjunction of literals is called a *clause*. Given a finite totally ordered set $A_k = \{\alpha_1, \ldots, \alpha_k\}$, the conjunction and disjunction over A_k are inductively defined as follows,

$$\bigwedge A_0 := \top, \qquad \bigvee A_0 := \bot,$$
$$\bigwedge A_{k+1} := \left(\bigwedge A_k\right) \wedge \alpha_{k+1}, \qquad \bigvee A_{k+1} := \left(\bigvee A_k\right) \vee \alpha_{k+1}.$$

If we deal with an arbitrary finite set of formulas, the conjunction and disjunction over this set can always be made determined. We only have to define a total order on $\mathcal{P} \cup \{\neg, \wedge, (,)\}$. The *lexicographic order* will then induce a total order on the alphabet $(\mathcal{P} \cup \{\neg, \wedge, (,)\})^*$, hence on the set $\mathcal{L}_0$ of all formulas. We refer to the book by Baader and Nipkow [3] for a proof.

Definition 1.1.2. We use the notation $|\alpha|$ for the *length* of a formula $\alpha \in \mathcal{L}_0$, which is defined by induction on the structure of α as follows,

$$\begin{aligned}|p| &:= 1,\\ |\neg\beta| &:= |\beta| + 1,\\ |\beta \wedge \gamma| &:= |\beta| + |\gamma| + 1.\end{aligned}$$

For a formula $\alpha \in \mathcal{L}_0$ we will write “by induction on α” as an abbreviation for “by induction on the length of α”. The notion of *subformula* is such an inductive definition.

Definition 1.1.3. The set $\mathrm{sub}(\alpha)$ of subformulas of a formula $\alpha \in \mathcal{L}_0$ is defined by induction on α as follows,

$$\begin{aligned}\mathrm{sub}(p) &:= \{p\},\\ \mathrm{sub}(\neg\beta) &:= \{\neg\beta\} \cup \mathrm{sub}(\beta),\\ \mathrm{sub}(\beta \wedge \gamma) &:= \{\beta \wedge \gamma\} \cup \mathrm{sub}(\beta) \cup \mathrm{sub}(\gamma).\end{aligned}$$

Substitution is another purely syntactical operation on formulas. It is the process of replacing in a formula all occurrences of another formula by a third one.

Definition 1.1.4. Given $\alpha, \varphi, \psi \in \mathcal{L}_0$, the substitution of φ for ψ in α, denoted by $\alpha[\varphi/\psi]$, is defined by induction on α as follows,

$$p[\varphi/\psi] := \begin{cases} \varphi, & \text{if } \psi = p, \\ p & \text{otherwise,} \end{cases}$$

$$(\neg\beta)[\varphi/\psi] := \begin{cases} \varphi & \text{if } \psi = \neg\beta, \\ \neg(\beta[\varphi/\psi]) & \text{otherwise,} \end{cases}$$

$$(\beta \wedge \gamma)[\varphi/\psi] := \begin{cases} \varphi & \text{if } \psi = \beta \wedge \gamma, \\ (\beta[\varphi/\psi]) \wedge (\gamma[\varphi/\psi]) & \text{otherwise.} \end{cases}$$

A *model* is just a set $w \subseteq \mathcal{P}$ of propositions. The idea is that the elements of w are considered to be the true propositions, the other atoms are all supposed to be false.

Definition 1.1.5. The notion of a formula $\alpha \in \mathcal{L}_0$ being *satisfied* in a model $w \subseteq \mathcal{P}$, denoted by $w \vDash \alpha$, is defined by induction on α as follows,

$$\begin{aligned} w \vDash p &:\Leftrightarrow p \in w, \\ w \vDash \neg\beta &:\Leftrightarrow w \nvDash \beta, \\ w \vDash \beta \wedge \gamma &:\Leftrightarrow w \vDash \beta \text{ and } w \vDash \gamma. \end{aligned}$$

If $w \vDash \alpha$, we also say that α holds in w, or that w is a model of α. A formula α is called *valid*, denoted by $\vDash \alpha$, if it holds in all models. Dually, we call a formula *satisfiable*, if there exists a model satisfying it. Two formulas α and β are called *equivalent*, if the formula $\alpha \leftrightarrow \beta$ is valid, that is if they have exactly the same models.

The following lemma states that classical propositional logic has the *substitution property*.

Lemma 1.1.6. *For all $\alpha, \beta, \varphi \in \mathcal{L}_0$ and all $p \in \mathcal{P}$ we have*

$$\vDash \alpha \leftrightarrow \beta \ \Rightarrow \ \vDash \alpha[\varphi/p] \leftrightarrow \beta[\varphi/p].$$

Proof. Let $w \subseteq \mathcal{P}$ be an arbitrary model. Then we define the model w' by

$$w' := \begin{cases} w \cup \{p\} & \text{if } w \vDash \varphi, \\ w \setminus \{p\} & \text{otherwise.} \end{cases}$$

It is now easy to prove that for all $\gamma \in \mathcal{L}_0$ we have $w \vDash \gamma[\varphi/p]$ if and only if $w' \vDash \gamma$ by induction on gamma. Therefore, we get

$$w \vDash \alpha[\varphi/p] \ \Leftrightarrow \ w' \vDash \alpha \ \Leftrightarrow \ w' \vDash \beta \ \Leftrightarrow \ w \vDash \beta[\varphi/p].$$

Since w has been arbitrarily given, we are done. □

Now, we are going to define the set of atoms occurring in a formula, as well as the set of atoms that really influence the validity of a formula. The latter has been introduced by Herzig and Rifi [39], for instance.

Definition 1.1.7. We write $\mathrm{atm}(\alpha)$ for the set of propositions occurring in a formula $\alpha \in \mathcal{L}_0$. It is defined by induction on α as follows,

$$\begin{aligned} \mathrm{atm}(p) &:= \{p\}, \\ \mathrm{atm}(\neg\beta) &:= \mathrm{atm}(\beta), \\ \mathrm{atm}(\beta \wedge \gamma) &:= \mathrm{atm}(\beta) \cup \mathrm{atm}(\gamma). \end{aligned}$$

A proposition $p \in \mathcal{P}$ is called *relevant* for a formula $\alpha \in \mathcal{L}_0$, if for every $\beta \in \mathcal{L}_0$ that is equivalent to α we have $p \in \mathrm{atm}(\beta)$. That is,

$$p \text{ is relevant for } \alpha \;:\Leftrightarrow\; p \in \bigcap\{\mathrm{atm}(\beta) : \vDash \beta \leftrightarrow \alpha\}.$$

The set of propositions that are relevant for α is denoted by $\mathrm{atm}^\sharp(\alpha)$. A proposition $p \in \mathrm{atm}(\alpha)$ that is not relevant for α is called *redundant* in α.

As an immediate consequence of Definition 1.1.7, we get that the redundant propositions of α are given by the set $\mathrm{atm}(\alpha) \setminus \mathrm{atm}^\sharp(\alpha)$. For instance, p and q are both relevant for $p \vee q$, whereas p is redundant in both formulas $p \to p$ and $q \wedge (p \vee q)$. The following lemma has been mentioned by Herzig and Rifi [38] in a slightly different way.

Lemma 1.1.8. *For all* $\alpha \in \mathcal{L}_0$ *we have*

$$\mathrm{atm}^\sharp(\alpha) = \{p \in \mathcal{P} : \nvDash \alpha[\top/p] \leftrightarrow \alpha[\bot/p]\}.$$

Proof. We will show that for all $p \in \mathcal{P}$ we have $p \notin \mathrm{atm}^\sharp(\alpha)$ if and only if $\vDash \alpha[\top/p] \leftrightarrow \alpha[\bot/p]$. For the direction from left to right, assume $p \notin \mathrm{atm}^\sharp(\alpha)$. Then for some $\beta \in \mathcal{L}_0$ we have $\vDash \beta \leftrightarrow \alpha$ and $p \notin \mathrm{atm}(\beta)$. We obviously get $\vDash \beta[\top/p] \leftrightarrow \beta[\bot/p]$. Therefore, we immediately get $\vDash \alpha[\top/p] \leftrightarrow \alpha[\bot/p]$ by Lemma 1.1.6. For the converse direction, assume $\vDash \alpha[\top/p] \leftrightarrow \alpha[\bot/p]$ and let $w \subseteq \mathcal{P}$ be arbitrarily given. First, if $w \vDash p$, then we have $w \vDash \alpha \leftrightarrow \alpha[\top/p]$. On the other hand, if $w \vDash \neg p$, then we have $w \vDash \alpha \leftrightarrow \alpha[\bot/p]$, and we get $w \vDash \alpha \leftrightarrow \alpha[\top/p]$ by assumption. Since w has been arbitrarily given, we get $\vDash \alpha \leftrightarrow \alpha[\top/p]$. Of course, we have $p \notin \mathrm{atm}(\alpha[\top/p])$, and we get $p \notin \mathrm{atm}^\sharp(\alpha)$ by definition. □

A *theory* is a set $T \subseteq \mathcal{L}_0$ of formulas. We write $w \vDash T$ to express that for all $\alpha \in T$ we have $w \vDash \alpha$. A theory T is called *consistent*, if there is a model w such that $w \vDash T$. If $S \subseteq \mathrm{Pow}(\mathcal{P})$ is a set of models, we write $S \vDash \alpha$ to say that for every $w \in S$ we have $w \vDash \alpha$.

Definition 1.1.9. The operators Th and Mod are defined by

$$\begin{aligned} \mathrm{Th}\colon \mathrm{Pow}(\mathrm{Pow}(\mathcal{P})) &\to \mathrm{Pow}(\mathcal{L}_0), & \mathrm{Th}(S) &:= \{\alpha \in \mathcal{L}_0 : S \models \alpha\}, \\ \mathrm{Mod}\colon \mathrm{Pow}(\mathcal{L}_0) &\to \mathrm{Pow}(\mathrm{Pow}(\mathcal{P})), & \mathrm{Mod}(T) &:= \{w \subseteq \mathcal{P} : w \models T\}. \end{aligned}$$

Clearly, a theory T is consistent, if and only if $\mathrm{Mod}(T) \neq \emptyset$. Accordingly, the non empty sets of models are called *consistent*, as well. Moreover, a formula $\alpha \in \mathcal{L}_0$ is called *consistent with* a set S of models, if $S \not\models \neg\alpha$.

We will use the abbreviation $\|\alpha\|$ for the expression $\mathrm{Mod}(\{\alpha\})$ in the following. In addition, we will make use of the facts that for all $\alpha, \beta \in \mathcal{L}_0$ and all $S \subseteq \mathrm{Pow}(\mathcal{P})$ we have

$$\begin{aligned} \|\neg\alpha\| &= \mathrm{Pow}(\mathcal{P}) \setminus \|\alpha\|, & \models \alpha \to \beta \;&\Leftrightarrow\; \|\alpha\| \subseteq \|\beta\|, \\ \|\alpha \wedge \beta\| &= \|\alpha\| \cap \|\beta\|, & S \models \alpha \;&\Leftrightarrow\; S \subseteq \|\alpha\|. \end{aligned}$$

As an immediate consequence of these facts, we get the following lemma.

Lemma 1.1.10. *For all $S \subseteq \mathrm{Pow}(\mathcal{P})$ and all $\alpha, \beta \in \mathcal{L}_0$ we have*

$$S \models \alpha \to \beta \;\Leftrightarrow\; S \cap \|\alpha\| \models \beta.$$

We will now prove some properties of the operators Mod and Th, which will be important in what follows.

Lemma 1.1.11. *For all $\mathcal{I} \subseteq \mathrm{Pow}(\mathrm{Pow}(\mathcal{P}))$ and all $\mathcal{J} \subseteq \mathrm{Pow}(\mathcal{L}_0)$ we have*

$$\begin{aligned} \bigcap_{S \in \mathcal{I}} \mathrm{Th}(S) &= \mathrm{Th}\Big(\bigcup \mathcal{I}\Big), & \bigcap_{T \in \mathcal{J}} \mathrm{Mod}(T) &= \mathrm{Mod}\Big(\bigcup \mathcal{J}\Big), \\ \bigcup_{S \in \mathcal{I}} \mathrm{Th}(S) &\subseteq \mathrm{Th}\Big(\bigcap \mathcal{I}\Big), & \bigcup_{T \in \mathcal{J}} \mathrm{Mod}(T) &\subseteq \mathrm{Mod}\Big(\bigcap \mathcal{J}\Big). \end{aligned}$$

Proof. Given a formula $\alpha \in \mathcal{L}_0$, we directly get

$$\begin{aligned} \alpha \in \bigcap_{S \in \mathcal{I}} \mathrm{Th}(S) \;&\Leftrightarrow\; \text{for all } S \in \mathcal{I},\ S \subseteq \|\alpha\| \\ &\Leftrightarrow\; \bigcup \mathcal{I} \subseteq \|\alpha\| \;\Leftrightarrow\; \alpha \in \mathrm{Th}\Big(\bigcup \mathcal{I}\Big), \end{aligned}$$

$$\begin{aligned} \alpha \in \bigcup_{S \in \mathcal{I}} \mathrm{Th}(S) \;&\Leftrightarrow\; \text{for some } S \in \mathcal{I},\ S \subseteq \|\alpha\| \\ &\Rightarrow\; \bigcap \mathcal{I} \subseteq \|\alpha\| \;\Leftrightarrow\; \alpha \in \mathrm{Th}\Big(\bigcap \mathcal{I}\Big). \end{aligned}$$

The assertions concerning the operator Mod can similarly be proved. □

In order to see that the two inclusions of Lemma 1.1.11 can be strict, we give the following counterexamples.

Example 1.1.12. Let $\mathcal{P} := \{p, q\}$. First, we define $S := \|p\|$, $S' := \|q\|$, and $\alpha := p \wedge q$. Then we immediately get $\alpha \in \mathrm{Th}(S \cap S') = \mathrm{Th}(\|p \wedge q\|)$, but $\alpha \notin \mathrm{Th}(S) \cup \mathrm{Th}(S') = \mathrm{Th}(\|p\|) \cup \mathrm{Th}(\|q\|)$. For the second example, we define $T := \{\neg p, p \wedge q\}$, $T' := \{\neg q, p \wedge q\}$, and $w := \{p, q\}$. Then we have $w \in \mathrm{Mod}(T \cap T') = \|p \wedge q\|$, but $w \notin \mathrm{Mod}(T) \cup \mathrm{Mod}(T') = \emptyset$.

The following lemma states that the operators Mod and Th form a *Galois connection*[1]. We use the same definition of Galois connections as Birkhoff does in [14].

Lemma 1.1.13. *For all* $S \subseteq \mathrm{Pow}(\mathcal{P})$ *and all* $T \subseteq \mathcal{L}_0$ *we have*

$$S \subseteq \mathrm{Mod}(T) \;\Leftrightarrow\; T \subseteq \mathrm{Th}(S).$$

Proof. For the direction from left to right, assume $S \subseteq \mathrm{Mod}(T)$. Then we have

$$\begin{aligned} \alpha \in T \;&\Rightarrow\; \mathrm{Mod}(T) \models \alpha \\ &\Rightarrow\; S \models \alpha \qquad\qquad \text{by assumption} \\ &\Leftrightarrow\; \alpha \in \mathrm{Th}(S). \end{aligned}$$

The direction from right to left is similar. □

The next lemma lists a few properties that every two operators forming a Galois connection satisfy. This result can also be found in [14].

Lemma 1.1.14. *For all* $S, S' \subseteq \mathrm{Pow}(\mathcal{P})$ *and all* $T, T' \subseteq \mathcal{L}_0$ *we have*

$$\begin{aligned} &S \subseteq S' \;\Rightarrow\; \mathrm{Th}(S) \supseteq \mathrm{Th}(S'), && T \subseteq T' \;\Rightarrow\; \mathrm{Mod}(T) \supseteq \mathrm{Mod}(T'), \\ &S \subseteq \mathrm{Mod}(\mathrm{Th}(S)), && T \subseteq \mathrm{Th}(\mathrm{Mod}(T)), \\ &\mathrm{Th}(S) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(S))), && \mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{Th}(\mathrm{Mod}(T))). \end{aligned}$$

Because we will often use the operators $(\mathrm{Mod} \circ \mathrm{Th})$ and $(\mathrm{Th} \circ \mathrm{Mod})$, we will now introduce convenient abbreviations for these operators.

Definition 1.1.15. For all $S \subseteq \mathrm{Pow}(\mathcal{P})$ and all $T \subseteq \mathcal{L}_0$ we define

$$\overline{S} := \mathrm{Mod}(\mathrm{Th}(S)), \qquad\qquad \overline{T} := \mathrm{Th}(\mathrm{Mod}(T)).$$

[1] Like Galois theory, Galois connections are named after the French mathematician Évariste Galois (1811-1832).

For instance, some of the properties of Lemma 1.1.14 can be abbreviated as follows,

$$S \subseteq \overline{S}, \qquad\qquad T \subseteq \overline{T},$$
$$\mathrm{Th}(S) = \overline{\mathrm{Th}(S)} = \mathrm{Th}(\overline{S}), \qquad \mathrm{Mod}(T) = \overline{\mathrm{Mod}(T)} = \mathrm{Mod}(\overline{T}).$$

Of course, the abbreviations from Definition 1.1.15 have been chosen because the operators (Mod ∘ Th) and (Th ∘ Mod) are both *closure operators*, as is defined in the book by Davey and Priestley [16]. This actuality is directly implied by the fact that Mod and Th form a Galois connection, see [14, 16] for a proof. Therefore, we have the following additional properties.

Lemma 1.1.16. *For all* $S, S' \subseteq \mathrm{Pow}(\mathcal{P})$ *and all* $T, T' \subseteq \mathcal{L}_0$ *we have*

$$S \subseteq S' \;\Rightarrow\; \overline{S} \subseteq \overline{S'}, \qquad T \subseteq T' \;\Rightarrow\; \overline{T} \subseteq \overline{T'},$$
$$\overline{\overline{S}} = \overline{S}, \qquad \overline{\overline{T}} = \overline{T},$$
$$S \subseteq \overline{S'} \;\Leftrightarrow\; \overline{S} \subseteq \overline{S'}, \qquad T \subseteq \overline{T'} \;\Leftrightarrow\; \overline{T} \subseteq \overline{T'}.$$

The operator (Mod ∘ Th) even is a *topological closure operator*, as we state in the following lemma. A proof has been given by Parikh [56].

Lemma 1.1.17. *For all* $S, S' \subseteq \mathrm{Pow}(\mathcal{P})$ *we have*

$$S = \emptyset \;\Rightarrow\; \overline{S} = \emptyset, \qquad \overline{S \cup S'} = \overline{S} \cup \overline{S'}.$$

It is not hard to find a theory T that is a proper subset of $\overline{T}$. For instance, for every finite $T \subseteq \mathcal{L}_0$, we have that $\overline{T}$ is infinite, hence a proper superset of T. On the other hand, it is not obvious that there are sets S of models such that $S \neq \overline{S}$. We give such a set of models in the following example.

Example 1.1.18. Let the set $\mathcal{P} = \{p_i : i \in \mathbb{N}\}$ be infinite and define $S := \|p_0\| \setminus \{\{p_0\}\}$. We will now show that $\overline{S} = \|p_0\|$. Since we have $S \subseteq \|p_0\|$ and $\|p_0\| = \overline{\|p_0\|}$ by Lemma 1.1.14, we get $\overline{S} \subseteq \|p_0\|$ by Lemma 1.1.16. For the other inclusion, we only have to show that $\{p_0\} \in \overline{S}$, because we have $S = \|p_0\| \setminus \{\{p_0\}\} \subseteq \overline{S}$. By Lemma 1.1.13, it is enough to show that $\mathrm{Th}(S)$ is a subset of $\mathrm{Th}(\{\{p_0\}\})$. For this purpose, let $\alpha \in \mathcal{L}_0$ be given and assume $\alpha \notin \mathrm{Th}(\{\{p_0\}\})$. Then we get $\{\{p_0\}\} \not\models \alpha$, that is $\{p_0\} \not\models \alpha$. Now, if we define $w := \{p_0\} \cup \{p_i : p \notin \mathrm{atm}(\alpha)\}$, then we get $w \neq \{p_0\}$ and $w \in \|p_0\|$, so we obviously get $w \in S$ and $w \not\models \alpha$. Therefore, we have $S \not\models \alpha$, which is equivalent to $\alpha \notin \mathrm{Th}(S)$, and we are done.

If a set of models contains only one model, then it is closed, as we state in the following lemma.

Lemma 1.1.19. *For all $w \subseteq \mathcal{P}$ we have*

$$\overline{\{w\}} = \{w\}.$$

Proof. By Lemma 1.1.14, it will suffice to show $\overline{\{w\}} \subseteq \{w\}$. So let $v \in \overline{\{w\}}$, that is $v \models \mathrm{Th}(\{w\})$. Since we have $w \cup \{\neg p : p \in \mathcal{P} \setminus w\} \subseteq \mathrm{Th}(\{w\})$, we get $v \models w \cup \{\neg p : p \in \mathcal{P} \setminus w\}$. But this implies $v = w$, and we are done. □

Given a theory T and a formula α, we will sometimes write $T \models \alpha$ to express that α is a *logical consequence* of T. This means that for all models w we have that $w \models T$ implies $w \models \alpha$. That is, we have $T \models \alpha$ if and only if $\mathrm{Mod}(T) \models \alpha$. Therefore, we can say that a theory T is consistent if and only if $T \not\models \bot$, which is equivalent to $\overline{T} \neq \mathcal{L}_0$.

Definition 1.1.20. Given $T \subseteq \mathcal{L}_0$, the set $\mathrm{Cn}(T)$ of all logical consequences of T is defined by

$$\mathrm{Cn}(T) := \{\alpha \in \mathcal{L}_0 : T \models \alpha\}.$$

Clearly, we have $\mathrm{Cn}(T) = \{\alpha \in \mathcal{L}_0 : \mathrm{Mod}(T) \models \alpha\}$. A theory $T \subseteq \mathcal{L}_0$ is called *closed* (under consequences), if $\mathrm{Cn}(T) = T$. It is immediate that Cn is the same operator as $(\mathrm{Th} \circ \mathrm{Mod})$, which we state in the following lemma.

Lemma 1.1.21. *For all $T \subseteq \mathcal{L}_0$ we have*

$$\mathrm{Cn}(T) = \overline{T}.$$

Proof. We have $\overline{T} = \mathrm{Th}(\mathrm{Mod}(T)) = \{\alpha \in \mathcal{L}_0 : \mathrm{Mod}(T) \models \alpha\} = \mathrm{Cn}(T)$. □

Due to Lemma 1.1.21, a theory $T \subseteq \mathcal{L}_0$ is closed if and only if $\overline{T} = T$. Accordingly, a set $S \subseteq \mathrm{Pow}(\mathcal{P})$ is called *closed*, if $\overline{S} = S$. If the set $\mathcal{P}$ is finite and $S \subseteq \mathrm{Pow}(\mathcal{P})$ is a set of models, then we can define a formula whose models are exactly the elements of S.

Definition 1.1.22. If $\mathcal{P}$ is finite, then for all $S \subseteq \mathrm{Pow}(\mathcal{P})$ we define

$$\mathrm{fml}(S) := \bigvee_{w \in S} \bigwedge (w \cup \{\neg p : p \in \mathcal{P} \setminus w\}).$$

Although we need a total order on $\mathcal{L}_0$ to make the formula $\mathrm{fml}(S)$ uniquely defined, it does not matter which order we take. We can prove that the models of the formula $\mathrm{fml}(S)$ are the same for all $S \subseteq \mathrm{Pow}(\mathcal{P})$.

Lemma 1.1.23. *If $\mathcal{P}$ is finite, then for all $S \subseteq \mathrm{Pow}(\mathcal{P})$ we have*

$$\|\mathrm{fml}(S)\| = S.$$

Proof. For all $w \subseteq \mathcal{P}$ we have

$$\begin{aligned} w \in \|\mathrm{fml}(S)\| &\Leftrightarrow w \models \mathrm{fml}(S) \\ &\Leftrightarrow \text{for some } v \in S,\ w \models \bigwedge(v \cup \{\neg p : p \in \mathcal{P} \setminus v\}) \\ &\Leftrightarrow \text{for some } v \in S,\ w = v, \end{aligned}$$

which is equivalent to saying that w is an element of S. □

As a consequence of Lemma 1.1.14 and Lemma 1.1.23, we immediately get the following result.

Corollary 1.1.24. *If $\mathcal{P}$ is finite, then for all $S, S' \subseteq \mathrm{Pow}(\mathcal{P})$ we have*

$$\overline{S} = S, \qquad\qquad S \subseteq S' \ \Leftrightarrow\ \mathrm{Th}(S) \supseteq \mathrm{Th}(S').$$

The following lemma states two more properties of the operator fml, which will be useful in what follows.

Lemma 1.1.25. *If $\mathcal{P}$ is finite, then for all $\alpha \in \mathcal{L}_0$ and all $T \subseteq \mathcal{L}_0$ we have*

$$\models \mathrm{fml}(\|\alpha\|) \leftrightarrow \alpha, \qquad\qquad \overline{\{\mathrm{fml}(\mathrm{Mod}(T))\}} = \overline{T}.$$

Proof. Both assertions are a direct consequence of Lemma 1.1.23. □

We are now going to define the notion of a complete formula. The idea is that a complete formula is either unsatisfiable or has exactly one model.

Definition 1.1.26. A formula $\alpha \in \mathcal{L}_0$ is called *complete*, if for all $\beta \in \mathcal{L}_0$ we have $\models \alpha \to \beta$ or $\models \alpha \to \neg\beta$.

Observe that if $\mathcal{P}$ is infinite, then the only complete formulas are the unsatisfiable ones. It is not hard to show that a formula α is complete if and only if for all $p \in \mathcal{P}$ we have that at least one of the formulas $\alpha \to p$ or $\alpha \to \neg p$ is valid.

Definition 1.1.27. Given two sets (or models) w and v, the *symmetric difference* of w and v is defined by

$$w \,\Delta\, v := (w \setminus v) \cup (v \setminus w).$$

The symmetric difference of two sets w and v is also called the *distance* between w and v. It is easy to see that for all sets w and v we have

$$w \,\Delta\, v = v \,\Delta\, w = (w \cup v) \setminus (w \cap v).$$

1.2 Propositional databases

A *propositional database* is an object that has to satisfy at least the following two properties. First, it must be possible to *store* new information in the database. Such information is usually given by a propositional formula. Second, there must be a way to find out whether or not a given formula *holds* in the database. This means that we can check if a formula α is in the closure $\overline{T}$ of the set T of all formulas that have been stored in the database. It is not necessary that we can find out which formulas have been stored. But when storing a formula, the database has to be modified in such a way that this formula will hold afterwards. Throughout this thesis, we will follow the *open world assumption*, that is to say it can happen for some formula α, that neither α nor $\neg\alpha$ holds in a given database.

A propositional database can be seen as the representation of a *belief state*. Since our beliefs can change with time, a belief state is a momentary description of our beliefs. A formula α has to hold in a given database, if and only if we believe that the facts represented by α are true. We are now ready to define three different types of belief state representations. All of them have been widely discussed in literature, see Gärdenfors [27, 28] for an overview.

Definition 1.2.1. We have the following kinds of propositional databases.

1. A *model set* is a set of models. We say that a formula α *holds* in the model set S, if $S \vDash \alpha$. A model set S is called *consistent*, if $S \neq \emptyset$. $\mathcal{M} := \mathrm{Pow}(\mathrm{Pow}(\mathcal{P}))$ denotes the set of all model sets.
2. A *belief set* is a closed theory. A formula α *holds* in a belief set T, if $\alpha \in T$. A belief set T is called *consistent*, if $T \neq \mathcal{L}_0$. The set of all belief sets is denoted by $\mathcal{B} := \{T \subseteq \mathcal{L}_0 : \overline{T} = T\}$.
3. A *belief base* is a formula. A formula α *holds* in a belief base φ, if the implication $\varphi \to \alpha$ is valid. A belief base φ is called *consistent*, if it is satisfiable. Clearly, $\mathcal{L}_0$ is the set of all belief bases.

In the context of belief change, belief states have been defined to be model sets by Harper [35] and Grove [31], for example. The original motivation for model sets is the possible worlds approach often used in philosophy, see Hintikka [40]. We focus on the notion of model sets, because we will define belief change functions in modal logic in Part II. There, the beliefs of the agents are represented by sets of possible worlds. A possible world is a model augmented with some additional information. Belief change functions defined on models sets can be transferred to functions on sets of possible worlds. There are two technical advantages of the notion on model sets.

First, if the set $\mathcal{P}$ is finite, then every model set is also finite. Second, model sets can be arbitrarily given, they need not be closed sets of models. By Lemma 1.1.14, we immediately get that for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\overline{S} \models \alpha \;\Leftrightarrow\; S \models \alpha.$$

Belief sets have been used in traditional belief revision theory, in particular in one of the first contributions by Alchourrón, Gärdenfors, and Makinson [2]. They have been considered as “the simplest way of modelling a belief state”, see Gärdenfors [28]. But note that even if the number of propositions is finite, every non empty belief set is an infinite set.

Belief bases are used in update theory, which is still a very relevant subject in computer science. A belief base can be seen as the conjunction over a finite set of formulas consisting of the collected facts we believe to be true. That is where the name belief base comes from. Hence one can distinguish between the collected information and the derivable one. Depending on the application, this property can be an advantage.

We are now going to show how the different notions of databases are related. Every database of a certain type can be translated to a belief state representation of another type, such that exactly the same formulas hold in both databases. Given a model set S, the corresponding belief set is given by $\mathrm{Th}(S)$. On the other hand, the model set $\mathrm{Mod}(T)$ corresponds to any belief set T. If φ is a belief base, then we have that $\|\varphi\|$ and $\overline{\{\varphi\}}$ are the respective model set and belief set. If the set $\mathcal{P}$ is finite, the belief base $\mathrm{fml}(S)$ corresponds to a given model set S and a given belief set T translates to the belief base $\mathrm{fml}(\mathrm{Mod}(T))$. The next lemma shows that these translations lead to equivalent belief state representations.

Lemma 1.2.2. *Let $S \in \mathcal{M}$, $T \in \mathcal{B}$, and $\varphi \in \mathcal{L}_0$ be arbitrary belief state representations. Then for all $\alpha \in \mathcal{L}_0$ we have*

$$S \models \alpha \;\Leftrightarrow\; \alpha \in \mathrm{Th}(S), \qquad \models \varphi \to \alpha \;\Leftrightarrow\; \|\varphi\| \models \alpha,$$
$$\alpha \in T \;\Leftrightarrow\; \mathrm{Mod}(T) \models \alpha, \qquad \models \varphi \to \alpha \;\Leftrightarrow\; \alpha \in \overline{\{\varphi\}}.$$

Moreover, if the set $\mathcal{P}$ is finite, we have

$$S \models \alpha \;\Leftrightarrow\; \models \mathrm{fml}(S) \to \alpha,$$
$$\alpha \in T \;\Leftrightarrow\; \models \mathrm{fml}(\mathrm{Mod}(T)) \to \alpha.$$

Proof. We will prove three assertions. First, we have $S \models \alpha \;\Leftrightarrow\; \alpha \in \mathrm{Th}(S)$ directly by Definition 1.1.9. Second, we have $\alpha \in T$ implies $\mathrm{Mod}(T) \models \alpha$ by

Definition 1.1.9, as well. For the converse direction, we have that $\mathrm{Mod}(T) \models \alpha$ implies $\alpha \in \overline{T} = T$, because $T \in \mathcal{B}$. Third, we have $\models \varphi \to \alpha$ if and only if $\|\varphi\| \subseteq \|\alpha\|$, which is equivalent to $\|\varphi\| \models \alpha$. The other three assertions have similar proofs. □

In Definition 1.2.1 we have only defined when a formula is supposed to hold in a database. We have not yet given the functions that store new information in a belief state representation. This kind of belief change function is called *expansion*. Expansion is the process of adding new information to a given belief state without checking for consistency, thus it is rather simple to implement. For the notion of belief sets, the *Gärdenfors postulates* for expansion from [27] define the behaviour of expansion functions.

Definition 1.2.3. A function $+\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is an *expansion function*, if for all $T, T' \in \mathcal{B}$ and all $\alpha \in \mathcal{L}_0$ we have

(E1$_\mathcal{B}$) $\overline{T + \alpha} = T + \alpha$,

(E2$_\mathcal{B}$) $\alpha \in T + \alpha$,

(E3$_\mathcal{B}$) $T \subseteq T + \alpha$,

(E4$_\mathcal{B}$) $\alpha \in T \;\Rightarrow\; T + \alpha = T$,

(E5$_\mathcal{B}$) $T \subseteq T' \;\Rightarrow\; T + \alpha \subseteq T' + \alpha$,

(E6$_\mathcal{B}$) For all $T \in \mathcal{B}$ and all $\alpha \in \mathcal{L}_0$, $T + \alpha$ is the smallest belief set that satisfies (E1$_\mathcal{B}$)–(E5$_\mathcal{B}$).

For a discussion on these postulates, we refer to the book by Gärdenfors [27]. It has turned out that the Gärdenfors postulates for expansion determine a unique expansion function. A proof of the following theorem can be found in [27].

Theorem 1.2.4. *A function* $+\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ *is an expansion function, if and only if for all* $T \in \mathcal{B}$ *and all* $\alpha \in \mathcal{L}_0$ *we have*

$$T + \alpha = \overline{T \cup \{\alpha\}}.$$

From now on, we will exclusively use the symbol + to denote the uniquely defined expansion function from Theorem 1.2.4.

Before we are going to define the expansion function for the notion of model sets, we will mention an important property that belief change functions on model sets should satisfy. Given a belief change function $\circledast\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, we require that for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(\overline{S} \circledast \alpha) = \mathrm{Th}(S \circledast \alpha). \tag{1.1}$$

This ensures that given two model sets S, S' with $\mathrm{Th}(S) = \mathrm{Th}(S')$, we get $\mathrm{Th}(S \circledast \alpha) = \mathrm{Th}(S' \circledast \alpha)$. Throughout this thesis, we will require property (1.1) for all belief change functions. If we deal with a given belief change function $\star\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$, then we want that $\models (\varphi \star \alpha) \leftrightarrow (\psi \star \alpha)$ whenever $\models \varphi \leftrightarrow \psi$ for the same reason. Finally, if $*\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is our belief change function, then we require another important property, because different belief sets always contain different formulas. But we have to ensure that $\overline{T * \alpha} = T * \alpha$, which is the same as to specify $T * \alpha \in \mathcal{B}$.

Since there exists a unique expansion function on belief sets, we do not translate the expansion postulates to the notion of model sets. We just give the appropriate definition and show that it corresponds to the function $+$.

Definition 1.2.5. The expansion function $\oplus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$S \oplus \alpha := S \cap \|\alpha\|.$$

First of all, we are going to show that the expansion function $\oplus$ indeed satisfies property (1.1).

Lemma 1.2.6. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$\mathrm{Th}(\overline{S} \oplus \alpha) = \mathrm{Th}(S \oplus \alpha).$$

Proof. For all $\beta \in \mathcal{L}_0$ we have $\beta \in \mathrm{Th}(\overline{S} \oplus \alpha)$ if and only if $\alpha \to \beta \in \mathrm{Th}(\overline{S})$ by Lemma 1.1.10. Since $\mathrm{Th}(\overline{S}) = \mathrm{Th}(S)$, we get the desired result by again applying Lemma 1.1.10. □

Belief bases can be expanded by the conjunction $\wedge\colon (\varphi, \alpha) \mapsto \varphi \wedge \alpha$. It is not hard to see that the functions $\oplus$ and $\wedge$ are the adequate reformulations of $+$ in the context of model sets and belief bases respectively.

Lemma 1.2.7. *Let $S \in \mathcal{M}$, $T \in \mathcal{B}$, and $\varphi \in \mathcal{L}_0$ be arbitrary belief state representations. Then for all $\alpha, \beta \in \mathcal{L}_0$ we have*

$$\begin{aligned} S \oplus \alpha \models \beta \;&\Leftrightarrow\; \beta \in \mathrm{Th}(S) + \alpha, \\ \beta \in T + \alpha \;&\Leftrightarrow\; \mathrm{Mod}(T) \oplus \alpha \models \beta, \\ \models \varphi \wedge \alpha \to \beta \;&\Leftrightarrow\; \|\varphi\| \oplus \alpha \models \beta, \\ \models \varphi \wedge \alpha \to \beta \;&\Leftrightarrow\; \beta \in \overline{\{\varphi\}} + \alpha. \end{aligned}$$

Furthermore, if the set $\mathcal{P}$ is finite, we have

$$\begin{aligned} S \oplus \alpha \models \beta \;&\Leftrightarrow\; \models \mathrm{fml}(S) \wedge \alpha \to \beta, \\ \beta \in T + \alpha \;&\Leftrightarrow\; \models \mathrm{fml}(\mathrm{Mod}(T)) \wedge \alpha \to \beta. \end{aligned}$$

Proof. We show how to prove the first three assertions. First, we have

$$\begin{aligned} \mathrm{Th}(S \oplus \alpha) &= \mathrm{Th}(\overline{S} \oplus \alpha) && \text{by Lemma 1.2.6} \\ &= \mathrm{Th}(\overline{S} \cap \|\alpha\|) \\ &= \overline{\mathrm{Th}(S) \cup \{\alpha\}} && \text{by Lemma 1.1.11} \\ &= \mathrm{Th}(S) + \alpha. \end{aligned}$$

For the second assertion, we have

$$\begin{aligned} T + \alpha &= \overline{T \cup \{\alpha\}} \\ &= \mathrm{Th}(\mathrm{Mod}(T) \cap \|\alpha\|) && \text{by Lemma 1.1.11} \\ &= \mathrm{Th}(\mathrm{Mod}(T) \oplus \alpha). \end{aligned}$$

The third assertion can be proved as follows. We have $\models \varphi \wedge \alpha \to \beta$ if and only if $\|\varphi \wedge \alpha\| \subseteq \|\beta\|$, which is equivalent to $\|\varphi\| \oplus \alpha \models \beta$. The other three equivalences can similarly be proved. □

We have now seen that the same belief state can be represented in three different ways. Accordingly, we have also defined three expansion functions, one for each type of belief state representation. These expansion functions have exactly the same impact on each kind of database. The expansion function is just adding new information without retracting any beliefs. There is no problem as long as the incoming information is consistent with our belief state. Otherwise, the expanded database is inconsistent and our collected information is lost.

Lemma 1.2.8. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \models \neg\alpha \;\Leftrightarrow\; S \oplus \alpha = \emptyset.$$

Proof. We have $S \models \neg\alpha \;\Leftrightarrow\; S \subseteq \|\neg\alpha\| \;\Leftrightarrow\; S \oplus \alpha = S \cap \|\alpha\| = \emptyset$. □

In the following chapter, we will introduce more sophisticated belief change functions in order to avoid such unsatisfactory situations.

Chapter 2

Belief change functions

It is the aim of this chapter to introduce the concepts of traditional belief revision and update theory. The types of belief change functions of our interest are expansion, revision, contraction, and update. We have already defined expansion in Section 1.2. Revision stands in the middle of our considerations, whereas contraction and update are closely related to it. There are several other kinds of belief change: merging, consolidation, deletion/erasure, and so on. But they are not directly related to revision and are therefore beyond the scope of this thesis. Section 2.1, Section 2.2, and Section 2.3 deal with the principles of revision, contraction, and update functions respectively. We will show how these principles translate into the notion of model sets, our preferred representation of belief states.

2.1 Revision

Revision is the process of adding new information to a belief state while attending to consistency. Moreover, the revision process is defined to take place in a *static world*. This means that the original belief state and the new information both refer to the same situation. An inconsistency between the beliefs and the incoming information is explained by the possibility of having false beliefs. Thus the incoming information is always regarded as the most credible one and a *revision function*

$$\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$$

modifies a model set S in a way that a new information α holds in the revised model set $S \otimes \alpha$.

It has been argued by many authors that it is not possible to define a revision function that can be used for all situations, see Alchourrón, Gärdenfors, and

Makinson [2], Gärdenfors [28], as well as Friedman and Halpern [26]. The following example should illustrate this statement.

Example 2.1.1. Let $\mathcal{P} = \{p, q\}$ and $S = \{\{p\}\}$, so we believe that p is true and q is false. Assume we learn that q holds. Then we have the following possibilities to revise our beliefs.

1. If p means "lions are mammals" and q means "lions are carnivores", we might want to get $S \otimes q = \{\{p, q\}\}$, because we consider p and q to be independent.
2. If p means "lions are herbivores" and q means "lions eat zebra", we might want to get $S \otimes q = \{\{q\}\}$, because we believe that q implies $\neg p$.
3. If p means "lions eat plants" and q means "lions eat zebra", we might want to get $S \otimes q = \{\{q\}, \{p, q\}\}$, because we are not sure about p being true anymore.

There are several proposals in literature how to deal with the above mentioned problem, all of them adding some properties to the logic. We confine ourselves to mentioning two different approaches.

Epistemic entrenchment is a relation $\leq$ on the formulas and leads to a degree of how much a formula is epistemically entrenched. The meaning of $\alpha \leq \beta$ is that we have to retract α as soon as we give up β, see Gärdenfors [28].

Another similar approach is the definition of—possibly infinitely many—*degrees of belief*. A *preference relation* on the models leads to the different degrees of belief, see van Ditmarsch [20]. A special feature of this solution is, that a revision operation also changes the degrees of belief, including the preference relation, whereas an epistemic entrenchment is defined to be constant.

However, the use of an epistemic entrenchment relation or a preference relation both lead to a unique revision function, but there exist infinitely many entrenchment and preference relations. That is, the problem is only shifted to another level, but is not really solved.

As a consequence of all these difficulties, researchers have started to discuss several properties that have to be satisfied by revision functions. One point that almost everybody agrees with is the term of *minimal change*. That is, revising a belief state should change it as little as possible. Since it is not clear how the amount of change can be measured, researchers have been stating several properties a revision function has to satisfy. The pioneering work by Alchourrón, Gärdenfors, and Makinson [2] provides the famous *AGM postulates for revision*, which are supposed to describe rational revision functions.

The following version of the postulates by Gärdenfors [27] has been most commonly used in contributions to belief revision.

Definition 2.1.2. A function $\dot{+}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is an *AGM revision* (on belief sets), if for all $T \in \mathcal{B}$ and all $\alpha, \beta \in \mathcal{L}_0$ we have

($\mathsf{R1}_\mathcal{B}$) $\overline{T \dot{+} \alpha} = T \dot{+} \alpha$,

($\mathsf{R2}_\mathcal{B}$) $\alpha \in T \dot{+} \alpha$,

($\mathsf{R3}_\mathcal{B}$) $T \dot{+} \alpha \subseteq T + \alpha$,

($\mathsf{R4}_\mathcal{B}$) $\neg\alpha \notin T \ \Rightarrow\ T + \alpha \subseteq T \dot{+} \alpha$,

($\mathsf{R5}_\mathcal{B}$) $T \dot{+} \alpha = \mathcal{L}_0 \ \Leftrightarrow\ \models \neg\alpha$,

($\mathsf{R6}_\mathcal{B}$) $\models \alpha \leftrightarrow \beta \ \Rightarrow\ T \dot{+} \alpha = T \dot{+} \beta$,

($\mathsf{R7}_\mathcal{B}$) $T \dot{+} (\alpha \wedge \beta) \subseteq (T \dot{+} \alpha) + \beta$,

($\mathsf{R8}_\mathcal{B}$) $\neg\beta \notin T \dot{+} \alpha \ \Rightarrow\ (T \dot{+} \alpha) + \beta \subseteq T \dot{+} (\alpha \wedge \beta)$.

Observe that we could drop ($\mathsf{R1}_\mathcal{B}$) by specifying $\dot{+}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathcal{B}$. But we state the postulates in the original way for historical reasons. We want to mention that the other postulates are formulated in such a way that they only make sense if ($\mathsf{R1}_\mathcal{B}$) holds.

($\mathsf{R2}_\mathcal{B}$) requires that the new information has to hold in the revised belief set. The postulates ($\mathsf{R3}_\mathcal{B}$) and ($\mathsf{R4}_\mathcal{B}$) are stating that revision should have the same effect as expansion, whenever the new information is consistent with the original beliefs. ($\mathsf{R5}_\mathcal{B}$) is directly implied by the general definition of the revision process. It makes sure that the revised belief set is consistent as long as the new information is satisfiable. ($\mathsf{R6}_\mathcal{B}$) specifies that the revision process has to be *syntax independent*. The postulates ($\mathsf{R1}_\mathcal{B}$)–($\mathsf{R6}_\mathcal{B}$) are also called the *basic AGM postulates for revision*. The postulates ($\mathsf{R7}_\mathcal{B}$) and ($\mathsf{R8}_\mathcal{B}$) describe the behaviour of iterated revision, see Katsuno and Mendelzon [46, 48] or Halpern [33] for a detailed discussion.

Remark 2.1.3. The AGM postulates have also been reformulated in order to fit the notion of belief bases, see Katsuno and Mendelzon [46, 48]. In this context, a revision function maps from $\mathcal{L}_0 \times \mathcal{L}_0$ to $\mathcal{L}_0$.

It has been mentioned by Gärdenfors [28] that the AGM postulates for revision do not uniquely determine a revision function. He believes "it would be a mistake to expect that only logical properties are sufficient to characterise the revision process". However, in the context of classical propositional logic, we think that it should be possible to define one or two revision functions that one can use for most applications. In Chapter 4 we will suggest two potential candidates and explore their properties.

We will now give an example that illustrates that not every revision function satisfying $(\mathsf{R1}_{\mathcal{B}})$–$(\mathsf{R8}_{\mathcal{B}})$ behaves as we would expect. The presented function has been introduced by Alchourrón, Gärdenfors, and Makinson [2].

Example 2.1.4. The *full meet revision* function $\dot{+}_{\mathsf{fm}} \colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is defined by

$$T \dot{+}_{\mathsf{fm}} \alpha := \begin{cases} T + \alpha & \text{if } \neg\alpha \notin T, \\ \overline{\{\alpha\}} & \text{otherwise.} \end{cases}$$

The function $\dot{+}_{\mathsf{fm}}$ is an AGM revision on belief sets, a proof can easily be derived from the results in [2]. This function has often been mentioned in literature, cf. Nebel [53, 54]. Alchourrón, Gärdenfors, and Makinson argue in [2] that in the case $\neg\alpha \in T$, the belief set $\overline{\{\alpha\}}$ "is far too small in general to represent the result of an intuitive process of revision of T so as to bring in α". The case $\neg\alpha \notin T$ is beyond controversy.

We are now going to give the appropriate translation of the *AGM postulates for revision* to the notion of model sets. Since model sets need not be closed sets of models, we do not require a revised model set to be closed either. So our first postulate is property (1.1) instead. The other seven postulates are an exact reformulation of $(\mathsf{R2}_{\mathcal{B}})$–$(\mathsf{R8}_{\mathcal{B}})$.

Definition 2.1.5. A function $\otimes \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an *AGM revision* (on model sets), if for all $S \in \mathcal{M}$ and all $\alpha, \beta \in \mathcal{L}_0$ we have

$(\mathsf{R1}_{\mathcal{M}})$ $\mathrm{Th}(\overline{S} \otimes \alpha) = \mathrm{Th}(S \otimes \alpha)$,

$(\mathsf{R2}_{\mathcal{M}})$ $S \otimes \alpha \vDash \alpha$,

$(\mathsf{R3}_{\mathcal{M}})$ $\mathrm{Th}(S \otimes \alpha) \subseteq \mathrm{Th}(S \oplus \alpha)$,

$(\mathsf{R4}_{\mathcal{M}})$ $S \nvDash \neg\alpha \;\Rightarrow\; \mathrm{Th}(S \oplus \alpha) \subseteq \mathrm{Th}(S \otimes \alpha)$,

$(\mathsf{R5}_{\mathcal{M}})$ $S \otimes \alpha = \emptyset \;\Leftrightarrow\; \vDash \neg\alpha$,

$(\mathsf{R6}_{\mathcal{M}})$ $\vDash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S \otimes \beta)$,

$(\mathsf{R7}_{\mathcal{M}})$ $\mathrm{Th}(S \otimes (\alpha \wedge \beta)) \subseteq \mathrm{Th}((S \otimes \alpha) \oplus \beta)$,

$(\mathsf{R8}_{\mathcal{M}})$ $S \otimes \alpha \nvDash \neg\beta \;\Rightarrow\; \mathrm{Th}((S \otimes \alpha) \oplus \beta) \subseteq \mathrm{Th}(S \otimes (\alpha \wedge \beta))$.

According to the definition of the full meet revision function $\dot{+}_{\mathsf{fm}}$, we define a similar revision function on model sets, which satisfies all of the translated postulates for revision.

Example 2.1.6. The function $\otimes_{\mathsf{fm}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S \otimes_{\mathsf{fm}} \alpha := \begin{cases} S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ \|\alpha\| & \text{otherwise,} \end{cases}$$

is an AGM revision on model sets. We show that $\otimes_{\mathsf{fm}}$ satisfies ($\mathsf{R7}_{\mathcal{M}}$). Let $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given. If $S \not\models \neg(\alpha \wedge \beta)$, then we also have $S \not\models \neg\alpha$ and thus, we get $\mathrm{Th}(S \otimes_{\mathsf{fm}} (\alpha \wedge \beta)) = \mathrm{Th}(S \cap \|\alpha \wedge \beta\|) = \mathrm{Th}((S \cap \|\alpha\|) \cap \|\beta\|) = \mathrm{Th}((S \otimes_{\mathsf{fm}} \alpha) \oplus \beta)$. If $S \models \neg(\alpha \wedge \beta)$, we distinguish the following two cases. If $S \models \neg\alpha$, then we get $\mathrm{Th}(S \otimes_{\mathsf{fm}} (\alpha \wedge \beta)) = \mathrm{Th}(\|\alpha \wedge \beta\|) = \mathrm{Th}(\|\alpha\| \cap \|\beta\|) = \mathrm{Th}((S \otimes_{\mathsf{fm}} \alpha) \oplus \beta)$. If $S \not\models \neg\alpha$, then we have

$$\begin{aligned} \mathrm{Th}(S \otimes_{\mathsf{fm}} (\alpha \wedge \beta)) &= \mathrm{Th}(\|\alpha \wedge \beta\|) = \mathrm{Th}(\|\alpha\| \cap \|\beta\|) \\ &\subseteq \mathrm{Th}((S \cap \|\alpha\|) \cap \|\beta\|) = \mathrm{Th}((S \otimes_{\mathsf{fm}} \alpha) \oplus \beta). \end{aligned}$$

It is immediate how to translate a function defined on belief sets into a function operating on models sets and vice versa. We use the translations from belief sets to model sets and backwards.

Definition 2.1.7. Given a function $\dotplus \colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$, the function

$$\otimes_{\dotplus} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \mathrm{Mod}(\mathrm{Th}(S) \dotplus \alpha),$$

is the corresponding function on models sets. On the other hand, given a function $\otimes \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\dotplus_{\otimes} \colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0), \qquad (T, \alpha) \mapsto \mathrm{Th}(\mathrm{Mod}(T) \otimes \alpha),$$

is the corresponding function on belief sets.

Definition 2.1.7 makes sure that the functions $\otimes_{\dotplus}$ and $\dotplus_{\otimes}$ satisfy ($\mathsf{R1}_{\mathcal{M}}$) and ($\mathsf{R1}_{\mathcal{B}}$) respectively. Moreover, we can prove that the postulates ($\mathsf{R2}_{\mathcal{M}}$)–($\mathsf{R8}_{\mathcal{M}}$) are equivalent to ($\mathsf{R2}_{\mathcal{B}}$)–($\mathsf{R8}_{\mathcal{B}}$) with respect to the above defined translations.

Lemma 2.1.8. *We have the following correspondences between the two sets of revision postulates.*

1. *Let $\dotplus \colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ be given. Then the function $\otimes_{\dotplus}$ satisfies* ($\mathsf{R1}_{\mathcal{M}}$). *If $\dotplus$ satisfes* ($\mathsf{R1}_{\mathcal{B}}$)*, then we have that $\dotplus$ satisfies* ($\mathsf{R2}_{\mathcal{B}}$)–($\mathsf{R8}_{\mathcal{B}}$) *if and only if $\otimes_{\dotplus}$ satisfies* ($\mathsf{R2}_{\mathcal{M}}$)–($\mathsf{R8}_{\mathcal{M}}$).
2. *Let $\otimes \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then the function $\dotplus_{\otimes}$ satisfies* ($\mathsf{R1}_{\mathcal{B}}$)*. If $\otimes$ satisfes* ($\mathsf{R1}_{\mathcal{M}}$)*, then we have that $\otimes$ satisfies* ($\mathsf{R2}_{\mathcal{M}}$)–($\mathsf{R8}_{\mathcal{M}}$) *if and only if $\dotplus_{\otimes}$ satisfies* ($\mathsf{R2}_{\mathcal{B}}$)–($\mathsf{R8}_{\mathcal{B}}$).

Proof. It is obvious that the functions $\otimes_{\dot{+}}$ and $\dot{+}_{\otimes}$ both satisfy the first postulate. We will show the equivalence of one postulate for both assertions.

1. We prove that ($\mathsf{R4}_{\mathcal{B}}$) and ($\mathsf{R4}_{\mathcal{M}}$) are equivalent. For the direction from left to right, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. From $S \not\models \neg\alpha$ we can directly conclude $\neg\alpha \notin \mathrm{Th}(S)$. Then we have

$$\begin{aligned} \mathrm{Th}(S \oplus \alpha) &= \mathrm{Th}(S) + \alpha && \text{by Lemma 1.2.7} \\ &\subseteq \mathrm{Th}(S) \dot{+} \alpha && \text{by } (\mathsf{R4}_{\mathcal{B}}) \\ &= \overline{\mathrm{Th}(S) \dot{+} \alpha} && \text{by } (\mathsf{R1}_{\mathcal{B}}) \\ &= \mathrm{Th}(S \otimes_{\dot{+}} \alpha). \end{aligned}$$

   For the direction from right to left, let $T \in \mathcal{B}$ and $\alpha \in \mathcal{L}_0$ be given and assume $\neg\alpha \notin T$. Since $T = \overline{T}$, we immediately get that $\mathrm{Mod}(T) \not\models \neg\alpha$ and we have

$$\begin{aligned} T + \alpha &= \mathrm{Th}(\mathrm{Mod}(T) \oplus \alpha) && \text{by Lemma 1.2.7} \\ &\subseteq \mathrm{Th}(\mathrm{Mod}(T) \otimes_{\dot{+}} \alpha) && \text{by } (\mathsf{R4}_{\mathcal{M}}) \\ &= \overline{\overline{T} \dot{+} \alpha} \\ &= \overline{T \dot{+} \alpha} \\ &= T \dot{+} \alpha && \text{by } (\mathsf{R1}_{\mathcal{B}}). \end{aligned}$$

2. We give the proof for the equivalence of the postulates ($\mathsf{R3}_{\mathcal{M}}$) and ($\mathsf{R3}_{\mathcal{B}}$). For the direction from left to right, let $T \in \mathcal{B}$ and $\alpha \in \mathcal{L}_0$ be given. Then we have

$$\begin{aligned} T \dot{+}_{\otimes} \alpha &= \mathrm{Th}(\mathrm{Mod}(T) \otimes \alpha) \\ &\subseteq \mathrm{Th}(\mathrm{Mod}(T) \oplus \alpha) && \text{by } (\mathsf{R3}_{\mathcal{M}}) \\ &= T + \alpha && \text{by Lemma 1.2.7.} \end{aligned}$$

   For the other direction, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. Then we have

$$\begin{aligned} \mathrm{Th}(S \otimes \alpha) &= \mathrm{Th}(\overline{S} \otimes \alpha) && \text{by } (\mathsf{R1}_{\mathcal{M}}) \\ &= \mathrm{Th}(S) \dot{+}_{\otimes} \alpha \\ &\subseteq \mathrm{Th}(S) + \alpha && \text{by } (\mathsf{R3}_{\mathcal{B}}) \\ &= \mathrm{Th}(S \oplus \alpha) && \text{by Lemma 1.2.7.} \end{aligned}$$

The equivalence of the other postulates can be proved the same way. □

The proof of Lemma 2.1.8 shows that the postulates ($\mathsf{R1}_{\mathcal{B}}$) and ($\mathsf{R1}_{\mathcal{M}}$) are important. They are also a necessary condition for the following result.

Lemma 2.1.9. *Composing* $\otimes_{\dotplus}$ *and* $\dotplus_{\otimes}$ *from Definition 2.1.7 results in equivalent belief state representations.*

1. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{R1}_{\mathcal{M}})$*, then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes_{(\dotplus_{\otimes})} \alpha) = \mathrm{Th}(S \otimes \alpha)$*.*
2. *If a function* $\dotplus\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ *satisfies* $(\mathsf{R1}_{\mathcal{B}})$*, then for all* $T \in \mathcal{B}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $T \dotplus_{(\otimes_{\dotplus})} \alpha = T \dotplus \alpha$*.*

Proof. For the first assertion, we have $\mathrm{Th}(S \otimes_{(\dotplus_{\otimes})} \alpha) = \mathrm{Th}(\mathrm{Mod}(\mathrm{Th}(\overline{S} \otimes \alpha)))$ by definition, which is the same as $\mathrm{Th}(S \otimes \alpha)$ by Lemma 1.1.14 and $(\mathsf{R1}_{\mathcal{M}})$. For the second assertion, we have $T \dotplus_{(\otimes_{\dotplus})} \alpha = \mathrm{Th}(\mathrm{Mod}(\overline{T} \dotplus \alpha))$ by definition, which equals $T \dotplus \alpha$ by $(\mathsf{R1}_{\mathcal{B}})$ and the fact that $\overline{T} = T$. $\square$

In the next section we will introduce contraction, which is closely related to revision, and we will establish similar results for contraction functions.

2.2 Contraction

Contraction is the process of removing some information from a belief state without adding any new beliefs. Like revision, contraction is an action that takes place in a *static world*. The request to remove some belief from a given belief state can occur, if we learn that it has been wrong to add it. The new awareness that one has to remove some belief is always regarded as the most reliable information. Thus, a *contraction function*

$$\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$$

modifies a set of possible worlds S in a way that the retracted belief α does not hold in the contracted model set $S \ominus \alpha$, except in case α is valid. Clearly, this task is different from (consistently) adding $\neg\alpha$. Nevertheless, contraction is closely related to revision, as we will see in Section 3.2.

We will now give an example to illustrate why it is difficult to agree on how a contraction function should behave. Observe that we will have to add models to a given model set S, if we want to reduce the set of beliefs holding in S.

Example 2.2.1. Let $\mathcal{P} = \{p, q\}$ and $S = \{\{p\}, \{p, q\}\}$ be given and assume we learn that it was wrong to add p, that is we do not have any reason to believe wether p is true or not. Then the contracted belief state can be one of the following model sets.

1. If p means “lions eat plants” and q means “lions are herbivores”, we might want to get $S \ominus p = \{\{p\}, \{p, q\}, \emptyset\}$, because we believe that q implies p.

2. If p means "lions eat plants" and q means "lions eat meat", we might want to get $S \ominus p = \{\{p\}, \{p, q\}, \{q\}\}$, because we believe that $p \vee q$ is true.
3. If p means "lions eat plants" and q means "lions are mammals", we might want to get $S \ominus p = \{\{p\}, \{p, q\}, \emptyset, \{q\}\}$, because we consider p and q to be independent.

One could argue that there are several other model sets that have to be considered as a possible contracted belief state. But if we took away one of the models in S, then we would automatically add some new beliefs[1]. This is something we cannot accept because of the definition of the contraction process.

Similar to revision, there have been a lot of discussions whether or not it is possible to define one contraction function with the most rational behaviour. Again, the only agreement has turned out to be the notion of minimal change. The eight *AGM postulates for contraction* were presented by Alchourrón, Gärdenfors, and Makinson [2] for the first time, we present the version by Gärdenfors from [27].

Definition 2.2.2. A function $\dot{-} : \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is an *AGM contraction* (on belief sets), if for all $T \in \mathcal{B}$ and all $\alpha, \beta \in \mathcal{L}_0$ we have

($\mathsf{C1}_\mathcal{B}$) $\overline{T \dot{-} \alpha} = T \dot{-} \alpha$,

($\mathsf{C2}_\mathcal{B}$) $T \dot{-} \alpha \subseteq T$,

($\mathsf{C3}_\mathcal{B}$) $\alpha \notin T \;\Rightarrow\; T \dot{-} \alpha = T$,

($\mathsf{C4}_\mathcal{B}$) $\not\models \alpha \;\Rightarrow\; \alpha \notin T \dot{-} \alpha$,

($\mathsf{C5}_\mathcal{B}$) $\alpha \in T \;\Rightarrow\; T \subseteq (T \dot{-} \alpha) + \alpha$,

($\mathsf{C6}_\mathcal{B}$) $\models \alpha \leftrightarrow \beta \;\Rightarrow\; T \dot{-} \alpha = T \dot{-} \beta$,

($\mathsf{C7}_\mathcal{B}$) $(T \dot{-} \alpha) \cap (T \dot{-} \beta) \subseteq T \dot{-} (\alpha \wedge \beta)$,

($\mathsf{C8}_\mathcal{B}$) $\alpha \notin T \dot{-} (\alpha \wedge \beta) \;\Rightarrow\; T \dot{-} (\alpha \wedge \beta) \subseteq T \dot{-} \alpha$.

Again, all of the postulates are formulated in such a way that they only make sense if ($\mathsf{C1}_\mathcal{B}$) holds. Similar to revision, the postulates ($\mathsf{C1}_\mathcal{B}$)–($\mathsf{C6}_\mathcal{B}$) are called the *basic AGM postulates for contraction.*

($\mathsf{C2}_\mathcal{B}$) makes sure that no new beliefs can occur in a contracted belief base. ($\mathsf{C3}_\mathcal{B}$) states that if the formula we are supposed to contract with is not part of our beliefs, then we must not retract any belief. ($\mathsf{C4}_\mathcal{B}$) is an immediate

[1] If $\mathcal{P}$ was infinite, then this would not necessarily be the case.

consequence of the definition of the contraction process. It requires that the formula we contract with does not hold in the contracted belief set, whenever the formula is not valid. ($\mathsf{C5}_\mathcal{B}$) states that we must be able to recover the old beliefs by expanding the contracted belief state with the same formula. ($\mathsf{C6}_\mathcal{B}$) is the specification of *syntax independence*. Finally, the postulates ($\mathsf{C7}_\mathcal{B}$) and ($\mathsf{C8}_\mathcal{B}$) are the technical counterparts to the revision postulates ($\mathsf{R7}_\mathcal{B}$) and ($\mathsf{R8}_\mathcal{B}$). For a detailed discussion on the postulates for both revision and contraction, we refer to Chapter 3 of the book by Gärdenfors [27].

In the following example we will present an AGM contraction.

Example 2.2.3. The *full meet contraction* function defined by

$$T \mathbin{\dot{-}}_{\mathsf{fm}} \alpha := \begin{cases} T \cap \overline{\{\neg\alpha\}} & \text{if } \alpha \in T, \\ T & \text{otherwise,} \end{cases}$$

is an AGM contraction on belief sets. A proof and some further discussion can be found in [2]. It has been argued in [2] that in the case $\alpha \in T$, the set $T \cap \overline{\{\neg\alpha\}}$ "is in general far too small".

The following postulates are the adequate translation of the AGM postulates for contraction to the notion of model sets. Again, we have replaced the first postulate by property (1.1).

Definition 2.2.4. A function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an *AGM contraction* (on model sets), if for all $S \in \mathcal{M}$ and all $\alpha, \beta \in \mathcal{L}_0$ we have

($\mathsf{C1}_\mathcal{M}$) $\mathrm{Th}(\overline{S} \ominus \alpha) = \mathrm{Th}(S \ominus \alpha)$,

($\mathsf{C2}_\mathcal{M}$) $\mathrm{Th}(S \ominus \alpha) \subseteq \mathrm{Th}(S)$,

($\mathsf{C3}_\mathcal{M}$) $S \not\models \alpha \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) = \mathrm{Th}(S)$,

($\mathsf{C4}_\mathcal{M}$) $\not\models \alpha \;\Rightarrow\; S \ominus \alpha \not\models \alpha$,

($\mathsf{C5}_\mathcal{M}$) $S \models \alpha \;\Rightarrow\; \mathrm{Th}(S) \subseteq \mathrm{Th}((S \ominus \alpha) \oplus \alpha)$,

($\mathsf{C6}_\mathcal{M}$) $\models \alpha \leftrightarrow \beta \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) = \mathrm{Th}(S \ominus \beta)$,

($\mathsf{C7}_\mathcal{M}$) $\mathrm{Th}(S \ominus \alpha) \cap \mathrm{Th}(S \ominus \beta) \subseteq \mathrm{Th}(S \ominus (\alpha \wedge \beta))$,

($\mathsf{C8}_\mathcal{M}$) $S \ominus (\alpha \wedge \beta) \not\models \alpha \;\Rightarrow\; \mathrm{Th}(S \ominus (\alpha \wedge \beta)) \subseteq \mathrm{Th}(S \ominus \alpha)$.

Inspired by the full meet contraction function, we will now give an example of an AGM contraction on model sets.

Example 2.2.5. The function $\ominus_{\mathsf{fm}}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S \ominus_{\mathsf{fm}} \alpha := \begin{cases} S \cup \|\neg\alpha\| & \text{if } S \models \alpha, \\ S & \text{otherwise,} \end{cases}$$

is an AGM contraction on model sets. We show how to prove that $\ominus_{\mathsf{fm}}$ satisfies $(\mathsf{C5}_{\mathcal{M}})$. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given and assume that $S \vDash \alpha$. Then we have

$$\mathrm{Th}(S) \subseteq \mathrm{Th}(S \cap \|\alpha\|) = \mathrm{Th}((S \cup \|\neg\alpha\|) \cap \|\alpha\|) = \mathrm{Th}((S \ominus_{\mathsf{fm}} \alpha) \oplus \alpha).$$

Of course, we can also use the translations from Definition 2.1.7 for contraction functions. Given a function $\dot{-}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$, the function

$$\ominus_{\dot{-}}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \mathrm{Mod}(\mathrm{Th}(S) \dot{-} \alpha),$$

is the adequate translation that operates on models sets. If the function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is given, then the function

$$\dot{-}_{\ominus}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0), \qquad (T, \alpha) \mapsto \mathrm{Th}(\mathrm{Mod}(T) \ominus \alpha),$$

is the corresponding function on belief sets. Again, the functions $\ominus_{\dot{-}}$ and $\dot{-}_{\ominus}$ obviously satisfy $(\mathsf{C1}_{\mathcal{M}})$ and $(\mathsf{C1}_{\mathcal{B}})$ respectively. In addition, the reformulation of the AGM postulates for contraction ensures that $(\mathsf{C2}_{\mathcal{M}})$–$(\mathsf{C8}_{\mathcal{M}})$ are equivalent to $(\mathsf{C2}_{\mathcal{B}})$–$(\mathsf{C8}_{\mathcal{B}})$ with respect to the above defined translations.

Lemma 2.2.6. *We have the following correspondences between the two sets of contraction postulates.*

1. *Let* $\dot{-}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ *be given. Then the function* $\ominus_{\dot{-}}$ *satisfies* $(\mathsf{C1}_{\mathcal{M}})$. *If* $\dot{-}$ *satisfies* $(\mathsf{C1}_{\mathcal{B}})$, *then we have that* $\dot{-}$ *satisfies* $(\mathsf{C2}_{\mathcal{B}})$–$(\mathsf{C8}_{\mathcal{B}})$ *if and only if* $\ominus_{\dot{-}}$ *satisfies* $(\mathsf{C2}_{\mathcal{M}})$–$(\mathsf{C8}_{\mathcal{M}})$.
2. *Let* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *be given. Then the function* $\dot{-}_{\ominus}$ *satisfies* $(\mathsf{C1}_{\mathcal{B}})$. *If* $\ominus$ *satisfies* $(\mathsf{C1}_{\mathcal{M}})$, *then we have that* $\ominus$ *satisfies* $(\mathsf{C2}_{\mathcal{M}})$–$(\mathsf{C8}_{\mathcal{M}})$ *if and only if* $\dot{-}_{\ominus}$ *satisfies* $(\mathsf{C2}_{\mathcal{B}})$–$(\mathsf{C8}_{\mathcal{B}})$.

Proof. It is easy to see that the functions $\ominus_{\dot{-}}$ and $\dot{-}_{\ominus}$ both satisfy the first postulate. For both assertions, we show the equivalence of one postulate.

1. We prove that $(\mathsf{C4}_{\mathcal{B}})$ and $(\mathsf{C4}_{\mathcal{M}})$ are equivalent. For the direction from left to right, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given, and assume that α is not valid. Then, by $(\mathsf{C1}_{\mathcal{B}})$ and $(\mathsf{C4}_{\mathcal{B}})$, we have that $\alpha \notin \overline{\mathrm{Th}(S) \dot{-} \alpha}$, and we immediately get

$$S \ominus_{\dot{-}} \alpha = \mathrm{Mod}(\mathrm{Th}(S) \dot{-} \alpha) \nsubseteq \|\alpha\|.$$

For the other direction, let $T \in \mathcal{B}$ and $\alpha \in \mathcal{L}_0$ be given and assume that α is not valid. By $(\mathsf{C4}_{\mathcal{M}})$ we have that $\mathrm{Mod}(T) \ominus_{\dot{-}} \alpha \nsubseteq \|\alpha\|$, and we can directly conclude

$$\alpha \notin \mathrm{Th}(\mathrm{Mod}(T) \ominus_{\dot{-}} \alpha) = \overline{T \dot{-} \alpha},$$

and therefore, $\alpha \notin T \dot{-} \alpha$.

2. We give the proof for the equivalence of the postulates ($\mathsf{C8}_{\mathcal{M}}$) and ($\mathsf{C8}_{\mathcal{B}}$). For the direction from left to right, let $T \in \mathcal{B}$ and $\alpha \in \mathcal{L}_0$ be given and assume $\alpha \notin T \dot{-}_{\ominus} (\alpha \wedge \beta) = \mathrm{Th}(\mathrm{Mod}(T) \ominus (\alpha \wedge \beta))$. Then we immediately get $\mathrm{Mod}(T) \ominus (\alpha \wedge \beta) \not\models \alpha$, thus we have

$$\begin{aligned} T \dot{-}_{\ominus} (\alpha \wedge \beta) &= \mathrm{Th}(\mathrm{Mod}(T) \ominus (\alpha \wedge \beta)) \\ &\subseteq \mathrm{Th}(\mathrm{Mod}(T) \ominus \alpha) && \text{by } (\mathsf{C8}_{\mathcal{M}}) \\ &= T \dot{-}_{\ominus} \alpha. \end{aligned}$$

For the direction from right to left, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given and assume $S \ominus (\alpha \wedge \beta) \not\models \alpha$. Then we get $\alpha \notin \mathrm{Th}(S \ominus (\alpha \wedge \beta))$, which is equivalent to $\alpha \notin \mathrm{Th}(S) \dot{-}_{\ominus} (\alpha \wedge \beta)$ by ($\mathsf{C1}_{\mathcal{M}}$), and we get

$$\begin{aligned} \mathrm{Th}(S \ominus (\alpha \wedge \beta)) &= \mathrm{Th}(\overline{S} \ominus (\alpha \wedge \beta)) && \text{by } (\mathsf{C1}_{\mathcal{M}}) \\ &= \mathrm{Th}(S) \dot{-}_{\ominus} (\alpha \wedge \beta) \\ &\subseteq \mathrm{Th}(S) \dot{-}_{\ominus} \alpha && \text{by } (\mathsf{C8}_{\mathcal{B}}) \\ &= \mathrm{Th}(\overline{S} \ominus \alpha) \\ &= \mathrm{Th}(S \ominus \alpha) && \text{by } (\mathsf{C1}_{\mathcal{M}}). \end{aligned}$$

The equivalence of the other postulates can similarly be shown. □

As we can see in the proof of Lemma 2.2.6, the respective first contraction postulate is important for both kinds of database. The postulates ($\mathsf{C1}_{\mathcal{B}}$) and ($\mathsf{C1}_{\mathcal{M}}$) are also necessary for the following result.

Lemma 2.2.7. *Composing $\ominus_{\dot{-}}$ and $\dot{-}_{\ominus}$ from Definition 2.1.7 results in equivalent belief state representations.*

1. *If a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{C1}_{\mathcal{M}}$), then for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S \ominus_{(\dot{-}_{\ominus})} \alpha) = \mathrm{Th}(S \ominus \alpha)$.*
2. *If a function $\dot{-}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ satisfies ($\mathsf{C1}_{\mathcal{B}}$), then for all $T \in \mathcal{B}$ and all $\alpha \in \mathcal{L}_0$ we have $T \dot{-}_{(\ominus_{\dot{-}})} \alpha = T \dot{-} \alpha$.*

Proof. The proof is identical to the proof of Lemma 2.1.9 for the following reasons. First, $\ominus_{\dot{-}}$ and $\dot{-}_{\ominus}$ are the same functions as $\otimes_{\dot{+}}$ and $\dot{+}_{\otimes}$ respectively. Second, the postulates ($\mathsf{C1}_{\mathcal{M}}$) and ($\mathsf{C1}_{\mathcal{B}}$) are identical to ($\mathsf{R1}_{\mathcal{M}}$) and ($\mathsf{R1}_{\mathcal{B}}$) respectively. □

In the next section we will introduce update, which is also related to revision, and we will establish similar results concerning update functions.

2.3 Update

Update is the process of incorporating new information into a knowledge base with regard to consistency. Unlike revision, update has to be performed because of some real change in a *dynamic world*. The general assumption is that our beliefs are true. Thus, only a change in the real world can bring us to change the database. An inconsistency between the beliefs and the incoming information is explained by the change that has taken place. The incoming information is always regarded as the most current one and an *update function*

$$\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$$

modifies a model set S in a such way that a new information α holds in the updated model set $S \odot \alpha$.

Similar to revision, it is in general not clear how to update a database, see Herzig and Rifi [39] for a discussion on many different update functions. The following example shows that it can depend on the situation how the updated knowledge base has to be defined.

Example 2.3.1. Let $\mathcal{P} = \{p, q\}$ and $S = \{\emptyset\}$ be given, that is we believe that p and q are both false. Assume we learn that $p \vee q$ is true. Then the updated belief state can be one of the following model sets.

1. If p means "the road is wet" and q means "it is raining", we might want to get $S \odot (p \vee q) = \{\{p\}, \{p, q\}\}$, because we believe that q implies p.
2. If p means "Switzerland has won the championship" and q means "All Swiss people are sad", we might want to get $S \odot (p \vee q) = \{\{p\}, \{q\}\}$, because we believe that p and q cannot both be true at the same time.
3. If p means "Switzerland has won the championship" and q means "it is raining", we might want to get $S \odot (p \vee q) = \{\{p\}, \{q\}, \{p, q\}\}$, because we consider p and q to be independent.

Herzig has illustrated in [37] how we can formalise interrelations between formulas using *dependence functions*. A dependence function maps a proposition p to a set $\text{dep}(p)$ of propositions, which means that the propositions in $\text{dep}(p)$ are dependent from p. For a formula α, $\text{dep}(\alpha)$ is defined to be the union of $\text{dep}(p)$ over all $p \in \text{atm}(\alpha)$. Dependence functions are an additional tool to improve the behaviour of update functions. However, they do not prevent us from carefully defining reasonable update functions.

The following postulates have been presented by Katsuno and Mendelzon [47]. They can be seen as an answer to the AGM postulates for revision.

Katsuno and Mendelzon argue that the update process is different from revision and that it therefore needs a different set of postulates. They have presented eight postulates, where the first five *KM postulates for update* are the counterpart of the first seven AGM postulates for revision. We have to mention here that we have split the fourth KM postulate from the original version into $(\mathsf{U0}_{\mathcal{L}_0})$ and $(\mathsf{U4}_{\mathcal{L}_0})$, because $(\mathsf{U0}_{\mathcal{L}_0})$ is the exact reformulation of property (1.1).

Definition 2.3.2. A function $\diamond\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ is a *KM update* (on belief bases), if for all (belief bases) $\varphi, \psi \in \mathcal{L}_0$ and all (formulas) $\alpha, \beta \in \mathcal{L}_0$ we have

$(\mathsf{U0}_{\mathcal{L}_0})$ $\models \varphi \leftrightarrow \psi \;\Rightarrow\; \models (\varphi \diamond \alpha) \leftrightarrow (\psi \diamond \alpha)$,

$(\mathsf{U1}_{\mathcal{L}_0})$ $\models (\varphi \diamond \alpha) \to \alpha$,

$(\mathsf{U2}_{\mathcal{L}_0})$ $\models \varphi \to \alpha \;\Rightarrow\; \models (\varphi \diamond \alpha) \leftrightarrow \varphi$,

$(\mathsf{U3}_{\mathcal{L}_0})$ $\not\models \neg\varphi$ and $\not\models \neg\alpha \;\Rightarrow\; \not\models \neg(\varphi \diamond \alpha)$,

$(\mathsf{U4}_{\mathcal{L}_0})$ $\models \alpha \leftrightarrow \beta \;\Rightarrow\; \models (\varphi \diamond \alpha) \leftrightarrow (\varphi \diamond \beta)$,

$(\mathsf{U5}_{\mathcal{L}_0})$ $\models (\varphi \diamond \alpha) \wedge \beta \to (\varphi \diamond (\alpha \wedge \beta))$,

$(\mathsf{U6}_{\mathcal{L}_0})$ $\models (\varphi \diamond \alpha) \to \beta$ and $\models (\varphi \diamond \beta) \to \alpha \;\Rightarrow\; \models (\varphi \diamond \alpha) \leftrightarrow (\varphi \diamond \beta)$,

$(\mathsf{U7}_{\mathcal{L}_0})$ φ is complete $\;\Rightarrow\; \models (\varphi \diamond \alpha) \wedge (\varphi \diamond \beta) \to (\varphi \diamond (\alpha \vee \beta))$,

$(\mathsf{U8}_{\mathcal{L}_0})$ $\models ((\varphi \vee \psi) \diamond \alpha) \leftrightarrow (\varphi \diamond \alpha) \vee (\psi \diamond \alpha)$.

The postulates $(\mathsf{U0}_{\mathcal{L}_0})$ and $(\mathsf{U4}_{\mathcal{L}_0})$ are the requirement for *syntax independence*. $(\mathsf{U0}_{\mathcal{L}_0})$ states that equivalent belief bases must be updated the same way, whereas $(\mathsf{U4}_{\mathcal{L}_0})$ states that two updates with equivalent formulas must result in the same belief base. $(\mathsf{U1}_{\mathcal{L}_0})$ requires that the new information always holds in the updated belief base. $(\mathsf{U2}_{\mathcal{L}_0})$ states that if the new information holds in the original belief base, then the update must not change it. $(\mathsf{U3}_{\mathcal{L}_0})$ makes sure that the updated belief base is consistent whenever the original belief base is consistent and the new information is satisfiable. $(\mathsf{U5}_{\mathcal{L}_0})$ is the translation of $(\mathsf{R7}_{\mathcal{M}})$ to the notion of belief base. $(\mathsf{U6}_{\mathcal{L}_0})$ states that if two formulas are equivalent under a given belief base, then they have the same effect on this belief base. $(\mathsf{U7}_{\mathcal{L}_0})$ is stated for complete belief bases, exclusively. Remember that a complete belief base is a formula that is either unsatisfiable or has exactly one model. While the former case is not that interesting, the latter can only happen if the set $\mathcal{P}$ is finite. With model sets we do not need this restriction. A complete model set is just a singleton or the empty set. Since we learn from $(\mathsf{U8}_{\mathcal{L}_0})$ that every model of the belief base can be updated separately, we interpret $(\mathsf{U7}_{\mathcal{L}_0})$ in the following way. It defines how the update of every single model has to be performed with disjunctive input.

We are now giving an example for a KM update. The following function has been used to present the KM postulates in [47], where we can also find a proof that it satisfies ($\mathsf{U0}_{\mathcal{L}_0}$)–($\mathsf{U8}_{\mathcal{L}_0}$).

Example 2.3.3. Let $\mathcal{P}$ be finite. Then the function $\diamond_{\mathsf{pma}} \colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ called *possible models approach* is defined as follows,

$$\|\varphi \diamond_{\mathsf{pma}} \alpha\| := \bigcup_{w \,\in\, \|\varphi\|} \{v \in \|\alpha\| : \text{for all } u \in \|\alpha\|,\ w \,\Delta\, u \not\subset w \,\Delta\, v\}.$$

As usual, the resulting belief base is defined by its models, cf. [39]. The function $\diamond_{\mathsf{pma}}$ has been introduced by Winslett [64] and follows the idea that only the "closest models" (with respect to the symmetric distance) are possible models for the updated belief base. Herzig and Rifi [39] have shown that the function $\diamond_{\mathsf{pma}}$ is in general too restrictive in the following sense. Given a belief base $\varphi \in \mathcal{L}_0$ and propositions $p, q \in \mathcal{P}$ satisfying $\models (\varphi \diamond_{\mathsf{pma}} p) \to \neg q$ and $\models (\varphi \diamond_{\mathsf{pma}} q) \to \neg p$, we get that the formula $(p \wedge \neg q) \vee (\neg p \wedge q)$ always holds in the updated belief base $\varphi \diamond_{\mathsf{pma}} (p \vee q)$. This behaviour is known under the term "the problem of *disjunctive input*" and should be avoided, see [39] for a detailed discussion.

The following update postulates are an exact translation of the KM postulates to the notion of model sets. Observe that postulate ($\mathsf{U0}_{\mathcal{L}_0}$) translates to property (1.1).

Definition 2.3.4. A function $\odot \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a *KM update* (on model sets), if for all $S, S' \in \mathcal{M}$ and all $\alpha, \beta \in \mathcal{L}_0$, we have

($\mathsf{U0}_{\mathcal{M}}$) $\mathrm{Th}(\overline{S} \odot \alpha) = \mathrm{Th}(S \odot \alpha)$,

($\mathsf{U1}_{\mathcal{M}}$) $S \odot \alpha \models \alpha$,

($\mathsf{U2}_{\mathcal{M}}$) $S \models \alpha \ \Rightarrow\ \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S)$,

($\mathsf{U3}_{\mathcal{M}}$) $S \neq \emptyset$ and $\not\models \neg\alpha \ \Rightarrow\ S \odot \alpha \neq \emptyset$,

($\mathsf{U4}_{\mathcal{M}}$) $\models \alpha \leftrightarrow \beta \ \Rightarrow\ \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S \odot \beta)$,

($\mathsf{U5}_{\mathcal{M}}$) $\mathrm{Th}(S \odot (\alpha \wedge \beta)) \subseteq \mathrm{Th}((S \odot \alpha) \oplus \beta)$,

($\mathsf{U6}_{\mathcal{M}}$) $S \odot \alpha \models \beta$ and $S \odot \beta \models \alpha \ \Rightarrow\ \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S \odot \beta)$,

($\mathsf{U7}_{\mathcal{M}}$) $\mathrm{Card}(S) \leq 1 \ \Rightarrow\ \mathrm{Th}(S \odot (\alpha \vee \beta)) \subseteq \mathrm{Th}((S \odot \alpha) \cap (S \odot \beta))$,

($\mathsf{U8}_{\mathcal{M}}$) $\mathrm{Th}((S \cup S') \odot \alpha) = \mathrm{Th}((S \odot \alpha) \cup (S' \odot \alpha))$.

The original KM postulates for update have been stated for finite sets $\mathcal{P}$ of propositions. The postulates ($\mathsf{U0}_{\mathcal{M}}$)–($\mathsf{U8}_{\mathcal{M}}$) are formulated in such a way that they also make sense if $\mathcal{P}$ is infinite. One could ask why we have not formulated the eighth update postulate as follows,

$$\mathrm{Th}\Big(\big(\bigcup \mathcal{I}\big) \odot \alpha\Big) = \mathrm{Th}\Big(\bigcup_{S \,\in\, \mathcal{I}} (S \odot \alpha)\Big)$$

for every set $\mathcal{I}$ of model sets. However, this version would have been much stronger and there would not have been any advantage for our purposes.

Inspired by the possible models approach $\diamond_{\mathsf{pma}}$ on belief sets, we are now going to define a KM update on model sets. Since the definition of $\diamond_{\mathsf{pma}}$ works with models instead of formulas, it is not very hard to define the corresponding function on model sets. Nevertheless, there is a technical detail that is necessary to make $(\mathsf{U0}_{\mathcal{M}})$ being satisfied. The reason is that the set $\mathcal{P}$ does not need to be finite.

Example 2.3.5. The function $\odot_{\mathsf{pma}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S \odot_{\mathsf{pma}} \alpha := \bigcup_{w \,\in\, \overline{S}} \{v \in \|\alpha\| : \text{for all } u \in \|\alpha\|,\ w \mathbin{\Delta} u \not\subset w \mathbin{\Delta} v\}$$

is a KM update. By taking the union over all $w \in \overline{S}$, we make sure that $\odot_{\mathsf{pma}}$ satisfies $(\mathsf{U0}_{\mathcal{M}})$. Observe that Lemma 1.1.17 directly implies that $(\mathsf{U8}_{\mathcal{M}})$ is satisfied. The other postulates can be proved the same way as the respective postulates for the function $\diamond_{\mathsf{pma}}$.

Due to the existence of the function fml, we are able to translate functions on belief bases to functions on model sets and vice versa. Observe that both translations make use of fml, hence $\mathcal{P}$ must be finite.

Definition 2.3.6. Let $\mathcal{P}$ be finite. Given a function $\diamond\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$, then the function

$$\odot_{\diamond} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \|\mathrm{fml}(S) \diamond \alpha\|,$$

is the respective function defined on model sets. On the other hand, given a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\diamond_{\odot} \colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0, \qquad (\varphi, \alpha) \mapsto \mathrm{fml}(\|\varphi\| \odot \alpha),$$

is the corresponding function operating on belief bases.

If $\mathcal{P}$ is finite, we can apply Lemma 1.1.23 and Corollary 1.1.24, and we get that for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$S \odot_{(\diamond_{\mathsf{pma}})} \alpha = \bigcup_{w \,\in\, S} \{v \in \|\alpha\| : \text{for all } u \in \|\alpha\|,\ w \mathbin{\Delta} u \not\subset w \mathbin{\Delta} v\} = S \odot_{\mathsf{pma}} \alpha.$$

It is easy to see that the function $\odot_\diamond$ satisfies $(\mathsf{U0}_\mathcal{M})$ because the set $\mathcal{P}$ is finite, and the function $\diamond_\odot$ satisfies $(\mathsf{U0}_{\mathcal{L}_0})$ by definition. Furthermore, we can prove that the postulates $(\mathsf{U1}_\mathcal{M})$–$(\mathsf{U8}_\mathcal{M})$ are an exact translation of the postulates $(\mathsf{U1}_{\mathcal{L}_0})$–$(\mathsf{U8}_{\mathcal{L}_0})$.

Lemma 2.3.7. *Let $\mathcal{P}$ be finite. Then we have the following correspondences between the two sets of update postulates.*

1. *Let $\diamond\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ be given. Then the function $\odot_\diamond$ satisfies $(\mathsf{U0}_\mathcal{M})$. If $\diamond$ satisfies $(\mathsf{U0}_{\mathcal{L}_0})$, then we have that $\diamond$ satisfies $(\mathsf{U1}_{\mathcal{L}_0})$–$(\mathsf{U8}_{\mathcal{L}_0})$ if and only if $\odot_\diamond$ satisfies $(\mathsf{U1}_\mathcal{M})$–$(\mathsf{U8}_\mathcal{M})$.*
2. *Let $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then the function $\diamond_\odot$ satisfies $(\mathsf{U0}_{\mathcal{L}_0})$. In addition, we have that $\odot$ satisfies $(\mathsf{U1}_\mathcal{M})$–$(\mathsf{U8}_\mathcal{M})$ if and only if $\diamond_\odot$ satisfies $(\mathsf{U1}_{\mathcal{L}_0})$–$(\mathsf{U8}_{\mathcal{L}_0})$.*

Proof. It is obvious that $\odot_\diamond$ and $\diamond_\odot$ satisfy $(\mathsf{U0}_\mathcal{M})$ and $(\mathsf{U0}_{\mathcal{L}_0})$ respectively. We will prove the equivalence of one postulate for each assertion.

1. We show that $(\mathsf{U8}_{\mathcal{L}_0})$ and $(\mathsf{U8}_\mathcal{M})$ are equivalent. For the direction from left to right, let $S, S' \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. We have

$$\begin{aligned}
(S \cup S') \odot_\diamond \alpha &= \|\mathrm{fml}(S \cup S') \diamond \alpha\| \\
&= \|(\mathrm{fml}(S) \vee \mathrm{fml}(S')) \diamond \alpha\| && \text{by } (\mathsf{U0}_{\mathcal{L}_0}) \text{ and Lemma 1.1.23} \\
&= \|(\mathrm{fml}(S) \diamond \alpha) \vee (\mathrm{fml}(S') \diamond \alpha)\| && \text{by } (\mathsf{U8}_{\mathcal{L}_0}) \\
&= \|\mathrm{fml}(S) \diamond \alpha\| \cup \|\mathrm{fml}(S') \diamond \alpha\| \\
&= (S \odot_\diamond \alpha) \cup (S' \odot_\diamond \alpha).
\end{aligned}$$

For the converse direction, let (the belief bases) $\varphi, \psi \in \mathcal{L}_0$ and (the formula) $\alpha \in \mathcal{L}_0$ be given. We get

$$\begin{aligned}
\|(\varphi \vee \psi) \diamond \alpha\| &= \|\mathrm{fml}(\|\varphi \vee \psi\|) \diamond \alpha\| && \text{by } (\mathsf{U0}_{\mathcal{L}_0}) \text{ and Lemma 1.1.25} \\
&= \|\varphi \vee \psi\| \odot_\diamond \alpha \\
&= (\|\varphi\| \cup \|\psi\|) \odot_\diamond \alpha \\
&= (\|\varphi\| \odot_\diamond \alpha) \cup (\|\psi\| \odot_\diamond \alpha) && \text{by } (\mathsf{U8}_\mathcal{M}) \\
&= \|\mathrm{fml}(\|\varphi\|) \diamond \alpha\| \cup \|\mathrm{fml}(\|\psi\|) \diamond \alpha\| \\
&= \|(\mathrm{fml}(\|\varphi\|) \diamond \alpha) \vee (\mathrm{fml}(\|\psi\|) \diamond \alpha)\| \\
&= \|(\varphi \diamond \alpha) \vee (\psi \diamond \alpha)\| && \text{by } (\mathsf{U0}_{\mathcal{L}_0}).
\end{aligned}$$

2. Now, we show the equivalence of the postulates ($\mathsf{U5}_{\mathcal{M}}$) and ($\mathsf{U5}_{\mathcal{L}_0}$). First, let (the belief base) $\varphi \in \mathcal{L}_0$ as well as (the formulas) $\alpha, \beta \in \mathcal{L}_0$ be given. Then we have

$$\begin{aligned}
\|(\varphi \diamond_{\odot} \alpha) \wedge \beta\| &= \|\mathrm{fml}(\|\varphi\| \odot \alpha) \wedge \beta\| \\
&= \|\mathrm{fml}(\|\varphi\| \odot \alpha)\| \cap \|\beta\| \\
&= (\|\varphi\| \odot \alpha) \cap \|\beta\| && \text{by Lemma 1.1.23} \\
&= (\|\varphi\| \odot \alpha) \oplus \beta \\
&\subseteq \|\varphi\| \odot (\alpha \wedge \beta) && \text{by } (\mathsf{U5}_{\mathcal{M}}) \text{ and} \\
& && \text{Corollary 1.1.24} \\
&= \|\mathrm{fml}(\|\varphi\| \odot (\alpha \wedge \beta))\| \\
&= \|\varphi \diamond_{\odot} (\alpha \wedge \beta)\|.
\end{aligned}$$

For the direction from right to left, let $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given. We get

$$\begin{aligned}
(S \odot \alpha) \oplus \beta &= (S \odot \alpha) \cap \|\beta\| \\
&= \|\mathrm{fml}(\|\mathrm{fml}(S)\| \odot \alpha)\| \cap \|\beta\| && \text{by Lemma 1.1.23} \\
&= \|\mathrm{fml}(S) \diamond_{\odot} \alpha\| \cap \|\beta\| \\
&= \|(\mathrm{fml}(S) \diamond_{\odot} \alpha) \wedge \beta\| \\
&\subseteq \|\mathrm{fml}(S) \diamond_{\odot} (\alpha \wedge \beta)\| && \text{by } (\mathsf{U5}_{\mathcal{L}_0}) \\
&= \|\mathrm{fml}(\|\mathrm{fml}(S)\| \odot (\alpha \wedge \beta))\| \\
&= S \odot (\alpha \wedge \beta) && \text{by Lemma 1.1.23.}
\end{aligned}$$

The proof of the other equivalences is similar. □

Note that ($\mathsf{U0}_{\mathcal{L}_0}$) is important for the proof of Lemma 2.3.7. By Corollary 1.1.24, we have that ($\mathsf{U0}_{\mathcal{M}}$) is always satisfied if $\mathcal{P}$ is finite, hence it is not an assumption in Lemma 2.3.7. ($\mathsf{U0}_{\mathcal{L}_0}$) is also essential for the following result.

Lemma 2.3.8. *Let $\mathcal{P}$ be finite. Then we have that composing $\odot_{\diamond}$ and $\diamond_{\odot}$ from Definition 2.3.6 results in equivalent belief state representations.*

1. *Let $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have $S \odot_{(\diamond_{\odot})} \alpha = S \odot \alpha$.*
2. *If a function $\diamond\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ satisfies ($\mathsf{U0}_{\mathcal{L}_0}$), then for all (belief bases) $\varphi \in \mathcal{L}_0$ and all (formulas) $\alpha \in \mathcal{L}_0$ we have $\vDash (\varphi \diamond_{(\odot_{\diamond})} \alpha) \leftrightarrow (\varphi \diamond \alpha)$.*

Proof. For the first assertion, we have $S \odot_{(\diamond_\odot)} \alpha = \|\mathrm{fml}(\|\mathrm{fml}(S)\| \odot \alpha)\|$ by definition, which is the same as $\|\mathrm{fml}(S)\| \odot \alpha$ by Lemma 1.1.23. Applying the same lemma again results in $S \odot \alpha$. For the second assertion, we have that $\varphi \diamond_{(\odot_\diamond)} \alpha = \mathrm{fml}(\|\mathrm{fml}(\|\varphi\|) \diamond \alpha\|)$ by definition, which is equivalent to $\mathrm{fml}(\|\varphi\|) \diamond \alpha$ by Lemma 1.1.25. This is equivalent to $\varphi \diamond \alpha$ by ($\mathsf{U0}_{\mathcal{L}_0}$) and again Lemma 1.1.25. □

We conclude this section by defining the *standard update* function, which will be relevant in the following. There is a syntax dependent version by Winslett [65], who has called this function the standard update function. Additionally, there are three syntax independent definitions by Doherty et al. [24], Hegner [36], and Herzig et al. [38, 39], which have all turned out to be equivalent, see [39] for a proof. We present a slightly modified version of the syntax independent standard update function that fits the notion of model sets. We call this function $\odot_{\mathsf{su}}$ for historical reasons.

Definition 2.3.9. The standard update function $\odot_{\mathsf{su}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$S \odot_{\mathsf{su}} \alpha := \bigcup_{w \in \overline{S}} \{v \in \|\alpha\| : w \,\Delta\, v \subseteq \mathrm{atm}^\sharp(\alpha)\}.$$

The standard update function is not the best representation of minimal change for the following reason. If a model set S satisfies a formula α, then it can happen that the updated model set $S \odot_{\mathsf{su}} \alpha$ is a proper superset of S, as we can see in the proof of the next lemma. However, if S satisfies $\neg\alpha$, then the updated model set is suitable, as we will argue in Section 3.3 and Section 4.2. The following lemma states that the function $\odot_{\mathsf{su}}$ only satisfies five out of nine postulates. Although this result is not new (cf. Herzig and Rifi [39]), we will illustrate how to prove it in the context of model sets.

Lemma 2.3.10. *The standard update function* $\odot_{\mathsf{su}}$ *only satisfies* ($\mathsf{U0}_{\mathcal{M}}$), ($\mathsf{U1}_{\mathcal{M}}$), ($\mathsf{U3}_{\mathcal{M}}$), ($\mathsf{U4}_{\mathcal{M}}$), *and* ($\mathsf{U8}_{\mathcal{M}}$).

Proof. Let $S, S' \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given. ($\mathsf{U0}_{\mathcal{M}}$) is obviously satisfied because we have $\overline{\overline{S}} = \overline{S}$ by Lemma 1.1.16. ($\mathsf{U1}_{\mathcal{M}}$) trivially holds by definition. We get that ($\mathsf{U4}_{\mathcal{M}}$) is satisfied because $\vDash \alpha \leftrightarrow \beta$ implies $\|\alpha\| = \|\beta\|$ and $\mathrm{atm}^\sharp(\alpha) = \mathrm{atm}^\sharp(\beta)$ by Lemma 1.1.6 and Lemma 1.1.8. By Lemma 1.1.17 we have $\overline{S \cup S} = \overline{S} \cup \overline{S'}$, hence ($\mathsf{U8}_{\mathcal{M}}$) is also satisfied. Now, we will show that ($\mathsf{U3}_{\mathcal{M}}$) holds. Suppose that $S \neq \emptyset$ and $\|\alpha\| \neq \emptyset$, and take an arbitrary $w \in \overline{S}$. If for any $v \in \|\alpha\|$ we have $w \,\Delta\, v \subseteq \mathrm{atm}^\sharp(\alpha)$, then $v \in S \odot_{\mathsf{su}} \alpha$, and we are done. If $w \,\Delta\, v \nsubseteq \mathrm{atm}^\sharp(\alpha)$, then we construct a model v' by changing in v the values of the propositions in $(w \,\Delta\, v) \setminus \mathrm{atm}^\sharp(\alpha)$. Obviously, we have $w \,\Delta\, v' \subseteq \mathrm{atm}^\sharp(\alpha)$ and $v' \vDash \alpha$, thus $v' \in S \odot_{\mathsf{su}} \alpha$. In both cases we have $S \odot_{\mathsf{su}} \alpha \neq \emptyset$. We conclude this proof by giving counterexamples for the remaining postulates.

- Let $\mathcal{P} = \{p, q\}$, $S = \{\{p\}\}$, and $\alpha = p \vee q$. Then we have $S \vDash \alpha$ and we get $S \odot_{\mathsf{su}} \alpha = \{\{p\}, \{q\}, \{p, q\}\}$. ($\mathsf{U2}_{\mathcal{M}}$) is not satisfied because we have $p \in \mathrm{Th}(S)$ but $p \notin \mathrm{Th}(S \odot_{\mathsf{su}} \alpha)$.
- Let $\mathcal{P} = \{p, q\}$, $S = \{\emptyset\}$, $\alpha = p \vee q$, and $\beta = p$. Then we have $\mathrm{atm}^\sharp(\alpha \wedge \beta) = \{p\}$ and we immediately get $S \odot_{\mathsf{su}} (\alpha \wedge \beta) = \{\{p\}\}$ and $(S \odot_{\mathsf{su}} \alpha) \oplus \beta = \{\{p\}, \{p, q\}\}$. ($\mathsf{U5}_{\mathcal{M}}$) is not satisfied because we have $\neg q \in \mathrm{Th}(S \odot_{\mathsf{su}} (\alpha \wedge \beta))$ but $\neg q \notin \mathrm{Th}((S \odot_{\mathsf{su}} \alpha) \oplus \beta)$.
- Let $\mathcal{P} = \{p, q, r\}$, $S = \{\{q\}\}$, $\alpha = p$, and $\beta = p \wedge (q \vee r)$. Then we get $S \odot_{\mathsf{su}} \alpha = \{\{p, q\}\} \vDash \beta$ and $S \odot_{\mathsf{su}} \beta = \{\{p, q\}, \{p, r\}, \{p, q, r\}\} \vDash \alpha$. Postulate ($\mathsf{U6}_{\mathcal{M}}$) is not satisfied because we have $\neg r \in \mathrm{Th}(S \odot_{\mathsf{su}} \alpha)$ but $\neg r \notin \mathrm{Th}(S \odot_{\mathsf{su}} \beta)$.
- Let $\mathcal{P} = \{p, q, r\}$, $S = \{\emptyset\}$, $\alpha = p \wedge (q \vee r)$, and $\beta = p \wedge (\neg q \vee r)$. Then we have $\mathrm{Card}(S) = 1$ and we get $S \odot_{\mathsf{su}} (\alpha \vee \beta) = \{\{p\}\}$ and $(S \odot_{\mathsf{su}} \alpha) \cap (S \odot_{\mathsf{su}} \beta) = \{\{p, r\}, \{p, q, r\}\}$. ($\mathsf{U7}_{\mathcal{M}}$) is not satisfied because we have $\neg q \in \mathrm{Th}(S \odot_{\mathsf{su}} (\alpha \vee \beta))$ but $\neg q \notin \mathrm{Th}((S \odot_{\mathsf{su}} \alpha) \cap (S \odot_{\mathsf{su}} \beta))$.

These counterexamples will also be used in Section 4.1. □

As we will argue in Section 4.1, we think that it is a big disadvantage of the standard update function not satisfying ($\mathsf{U2}_{\mathcal{M}}$). At least, we can show that the worlds already satisfying the new information do not get lost by performing the standard update.

Lemma 2.3.11. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \oplus \alpha \subseteq S \odot_{\mathsf{su}} \alpha.$$

Proof. Let $w \in S \oplus \alpha$. Then we obviously have $w \in S$, $w \in \|\alpha\|$, and $w \,\Delta\, w = \emptyset \subseteq \mathrm{atm}^\sharp(\alpha)$. Hence, we get $w \in S \odot_{\mathsf{su}} \alpha$. □

By Lemma 1.1.14, we can therefore say that the standard update function $\odot_{\mathsf{su}}$ satisfies the revision postulate ($\mathsf{R3}_{\mathcal{M}}$). The following result is a direct consequence of the fact that $S \oplus \alpha = (S \oplus \alpha) \oplus \alpha$ and Lemma 2.3.11.

Corollary 2.3.12. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \oplus \alpha \subseteq (S \oplus \alpha) \odot_{\mathsf{su}} \alpha.$$

In the following chapter we will consider belief change functions in the context of consistent model sets, and we will examine the relationships of belief revision to the contraction and update processes.

Chapter 3

Relationships

In this chapter we will show how given belief change functions can be transformed into other functions. From now on, we will focus on belief change functions operating on model sets. In Section 3.1 we will investigate how the different belief change functions can be modified in order to fit the framework of consistent databases. Section 3.2 deals with the well-known identities by Levi [51] and Harper [35]. The Levi identity allows to define a revision function from a given contraction function, the Harper identity translates a given revision function to its corresponding contraction function. We will show that these identities can both be translated to the notion of model sets, where they satisfy the same properties as before. In addition, we will formulate similar identities for revision and contraction functions in the context of consistent model sets. Finally, we will introduce new identities between revision and update functions in Section 3.3. We cannot get such strong results as with revision and contraction, because the postulates for updates have been stated from a different perspective. We will end the section by postulating some common behaviour for revision and update functions.

3.1 Consistent databases

We have seen in Chapter 2 that there can be situations where we do not know how to revise, contract, or update a database. Example 2.1.1, Example 2.2.1, and Example 2.3.1 illustrate that we can have a clear picture of how we expect the changed belief state to look like. But if these common beliefs are not represented in a given model set, they will not influence the result of the belief change functions. For this purpose, the use of *integrity constraints* has been proposed by many authors. In the context of revision, contraction, and update functions, we recommend to consult the work of Katsuno and

Mendelzon [48], Katsuno and Mendelzon [46], and Herzig and Rifi [38] respectively. Further discussions on integrity constraints can be found in the contribution of Reiter [58]. Given a finite set $I = \{\mu_1, \ldots, \mu_k\} \subseteq \mathcal{L}_0$ of constraints, we define the integrity constraint $\iota := \bigwedge I$, so we can always deal with one formula. Usually, given a function $\circledast\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ and a set I of constraints with its corresponding formula ι, the belief change function $\circledast$ is modified in the following way. For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we set

$$\begin{aligned} S \circledast_I^r \alpha &:= S \circledast (\iota \wedge \alpha), \\ S \circledast_I^c \alpha &:= S \circledast (\iota \to \alpha), \\ S \circledast_I^u \alpha &:= (S \circledast \alpha) \cap \|\iota\|, \end{aligned}$$

depending on whether $\circledast$ is a revision, contraction, or update function respectively. Integrity constraints are a powerful tool, but using an integrity constraint like $I = \{\neg\bot\}$ does not help in ensuring consistency of model sets, because we have

$$S \circledast_I^r \alpha = S \circledast_I^c \alpha = S \circledast_I^u \alpha = S \circledast \alpha$$

whenever the function $\circledast$ is syntax independent. Therefore, in order to get every changed model set being consistent, we have to define a new requirement for arbitrary belief change functions.

Definition 3.1.1. We write $\mathcal{M}^c := \mathcal{M} \setminus \{\emptyset\}$ for the set of all consistent model sets. For all functions $\circledast\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, all $S \in \mathcal{M}^c$, and all $\alpha \in \mathcal{L}_0$, the consistency requirement is given by

$$S \circledast \alpha \neq \emptyset. \tag{3.1}$$

We will now define an expansion function in the context of consistent model sets. Since we have a unique expansion function $\oplus$, we define one concrete function $\oplus^c\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ that satisfies property (3.1). Our proposal is to reject the new information, whenever it is inconsistent with the original beliefs. This approach will be applied in a modal logic setting in Section 6.2.

Definition 3.1.2. The *consistent expansion function* $\oplus^c\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$S \oplus^c \alpha := \begin{cases} S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ S & \text{otherwise.} \end{cases}$$

The following theorem states that the new expansion function $\oplus^c$ satisfies the properties (1.1) and (3.1), three of the properties that are related to some of the Gärdenfors postulates for expansion (cf. Definition 1.2.3), as well as syntax independence.

Theorem 3.1.3. *For all $S \in \mathcal{M}^c$ and all $\alpha, \beta \in \mathcal{L}_0$ we have*

1. $\mathrm{Th}(\overline{S} \oplus^c \alpha) = \mathrm{Th}(S \oplus^c \alpha)$,
2. $S \oplus^c \alpha \neq \emptyset$,
3. $S \not\models \neg\alpha \;\Rightarrow\; S \oplus^c \alpha \models \alpha$,
4. $\mathrm{Th}(S) \subseteq \mathrm{Th}(S \oplus^c \alpha)$,
5. $S \models \alpha \;\Rightarrow\; \mathrm{Th}(S \oplus^c \alpha) = \mathrm{Th}(S)$,
6. $\models \alpha \leftrightarrow \beta \;\Rightarrow\; \mathrm{Th}(S \oplus^c \alpha) = \mathrm{Th}(S \oplus^c \beta)$.

Proof. We show how to prove the first assertion and distinguish two cases. First, if $S \not\models \neg\alpha$, then we have $\overline{S} \not\models \neg\alpha$ by Lemma 1.1.14, and we get

$$\begin{aligned}\mathrm{Th}(\overline{S} \oplus^c \alpha) &= \mathrm{Th}(\overline{S} \oplus \alpha) \\ &= \mathrm{Th}(S \oplus \alpha) && \text{by Lemma 1.2.6} \\ &= \mathrm{Th}(S \oplus^c \alpha).\end{aligned}$$

Second, if $S \models \neg\alpha$, then we have $\overline{S} \models \neg\alpha$ by again Lemma 1.1.14, and we get

$$\begin{aligned}\mathrm{Th}(\overline{S} \oplus^c \alpha) &= \mathrm{Th}(\overline{S}) \\ &= \mathrm{Th}(S) && \text{by Lemma 1.1.14} \\ &= \mathrm{Th}(S \oplus^c \alpha).\end{aligned}$$

The proofs of the other assertions are similar. □

In order to adapt the requirement of minimal change to the context of consistent model sets, we will now reformulate some of the AGM postulates for revision. We are going to change as few as possible, such that the new set of postulates is compatible with the consistency requirement (3.1).

Definition 3.1.4. A function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent AGM revision*, if it satisfies the revision postulates $(\mathsf{R1}_{\mathcal{M}})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R6}_{\mathcal{M}})$, and $(\mathsf{R7}_{\mathcal{M}})$, as well as the following modified postulates,

$(\mathsf{R2}_{\mathcal{M}^c})$ $\not\models \neg\alpha \;\Rightarrow\; S \otimes \alpha \models \alpha$,

$(\mathsf{R5}_{\mathcal{M}^c})$ $S \otimes \alpha \neq \emptyset$,

$(\mathsf{R8}_{\mathcal{M}^c})$ $\not\models \neg\alpha$ and $S \otimes \alpha \not\models \neg\beta \;\Rightarrow\; \mathrm{Th}((S \otimes \alpha) \oplus \beta) \subseteq \mathrm{Th}(S \otimes (\alpha \wedge \beta))$.

We want to mention that the expansion function $\oplus$ has not been replaced by the consistent expansion function $\oplus^c$ in $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R7}_{\mathcal{M}})$, and $(\mathsf{R8}_{\mathcal{M}^c})$ for technical reasons.

We are now going to define the translation of a revision function on model sets to a revision function on consistent model sets and the other way round. Note that the term “revision function” and the symbol $\otimes$ say nothing about the properties of the function, they are just an indication that the mentioned function is supposed to revise belief states.

Definition 3.1.5. Given a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\otimes^{cr}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S \otimes \alpha & \text{if } \not\models \neg\alpha, \\ S & \text{otherwise} \end{cases}$$

is the corresponding consistency preserving revision function. On the other hand, if the function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is given, then the function

$$\otimes^{ir}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S \otimes \alpha & \text{if } \not\models \neg\alpha \text{ and } S \neq \emptyset, \\ \|\alpha\| & \text{otherwise} \end{cases}$$

is the adequate revision function on possibly inconsistent model sets.

The functions from Definition 3.1.5 are supposed to translate an AGM revision into a consistent AGM revision and vice versa. The following lemma shows that our translations fulfil this requirement.

Lemma 3.1.6. *We have the following correspondences between the two sets of revision postulates.*

1. *If a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an AGM revision, then the function $\otimes^{cr}$ is a consistent AGM revision.*
2. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent AGM revision, then the function $\otimes^{ir}$ is an AGM revision.*

Proof. First, we want to mention that $\otimes^{cr}$ satisfies property (3.1), whenever $\otimes$ satisfies ($\mathsf{R5}_{\mathcal{M}}$). Now, we will prove one postulate for each assertion.

1. We show that the function $\otimes^{cr}$ satisfies ($\mathsf{R8}_{\mathcal{M}^c}$). Let $S \in \mathcal{M}^c$ and $\alpha, \beta \in \mathcal{L}_0$ be given and assume that α is satisfiable and $S \otimes^{cr} \alpha \not\models \neg\beta$. Then we have $\mathrm{Th}((S \otimes^{cr} \alpha) \oplus \beta) = \mathrm{Th}((S \otimes \alpha) \oplus \beta)$ and $S \otimes \alpha \not\models \neg\beta$. Now, suppose that $\alpha \wedge \beta$ is not satisfiable. Then, we get $S \otimes \alpha \models \neg\beta$, because $S \otimes \alpha \models \alpha$ by ($\mathsf{R2}_{\mathcal{M}}$), and we have a contradiction. Hence, the formula $\alpha \wedge \beta$ must be satisfiable. Therefore, $\mathrm{Th}(S \otimes^{cr} (\alpha \wedge \beta))$ is equal to $\mathrm{Th}(S \otimes (\alpha \wedge \beta))$, and the claim follows from the assumption that $\otimes$ satisfies ($\mathsf{R8}_{\mathcal{M}}$).

2. We show that the function $\otimes^{ir}$ satisfies ($\mathsf{R7}_{\mathcal{M}}$). Let $S \in \mathcal{M}$ and the formulas $\alpha, \beta \in \mathcal{L}_0$ be given. If $S = \emptyset$, then we have

$$\begin{aligned}\mathrm{Th}(S \otimes^{ir} (\alpha \wedge \beta)) &= \mathrm{Th}(\|\alpha \wedge \beta\|)\\ &= \mathrm{Th}(\|\alpha\| \cap \|\beta\|)\\ &= \mathrm{Th}((S \otimes^{ir} \alpha) \cap \|\beta\|)\\ &= \mathrm{Th}((S \otimes^{ir} \alpha) \oplus \beta).\end{aligned}$$

If $S \neq \emptyset$, then we distinguish three cases. First, if $\alpha \wedge \beta$ is satisfiable, then so also is α and we get $\mathrm{Th}(S \otimes^{ir} (\alpha \wedge \beta)) = \mathrm{Th}(S \otimes (\alpha \wedge \beta))$ and $\mathrm{Th}(S \otimes^{ir} \alpha) = \mathrm{Th}(S \otimes \alpha)$. So the claim directly follows from the assumption that $\otimes$ satisfies ($\mathsf{R7}_{\mathcal{M}}$). Second, if $\alpha \wedge \beta$ and α are both unsatisfiable, then we have $\mathrm{Th}(S \otimes^{ir} (\alpha \wedge \beta)) = \mathrm{Th}((S \otimes^{ir} \alpha) \oplus \beta) = \mathcal{L}_0$, so the postulate trivially holds. In the last case, if $\alpha \wedge \beta$ is unsatisfiable and α is satisfiable, we get $S \otimes^{ir} \alpha = S \otimes \alpha \models \neg\beta$, because $S \otimes \alpha \models \alpha$ by ($\mathsf{R2}_{\mathcal{M}^c}$). Thus, we have $\mathrm{Th}((S \otimes^{ir} \alpha) \oplus \beta) = \mathrm{Th}(S \otimes^{ir} (\alpha \wedge \beta)) = \mathcal{L}_0$.

The proofs for the other postulates are similar. □

For some later results, we want to mention that Lemma 3.1.6 also holds for certain subsets of the postulates. This fact is an immediate consequence of the proof of Lemma 3.1.6.

Lemma 3.1.7. *We have the following special cases of Lemma 3.1.6.*

1. *If a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R6}_{\mathcal{M}}$), then the function $\otimes^{cr}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), and ($\mathsf{R6}_{\mathcal{M}}$). If $\otimes$ additionally satisfies ($\mathsf{R7}_{\mathcal{M}}$), then so also does the function $\otimes^{cr}$.*
2. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), and ($\mathsf{R6}_{\mathcal{M}}$), then the function $\otimes^{ir}$ satisfies ($\mathsf{R2}_{\mathcal{M}}$), ($\mathsf{R3}_{\mathcal{M}}$), and ($\mathsf{R6}_{\mathcal{M}}$). Moreover, if $\otimes$ additionally satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), and ($\mathsf{R5}_{\mathcal{M}^c}$), then the function $\otimes^{ir}$ also satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), and ($\mathsf{R5}_{\mathcal{M}}$).*

We are now going to define two new postulates, which will be useful for some further results.

Definition 3.1.8. A function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies the following additional revision postulate if for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

($\mathsf{R9}_{\mathcal{M}}$) $\quad S = \emptyset \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(\|\alpha\|)$,

and for $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, $S \in \mathcal{M}^c$, and $\alpha \in \mathcal{L}_0$ we require

($\mathsf{R9}_{\mathcal{M}^c}$) $\quad \models \neg\alpha \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S)$.

For instance, the full meet revision function $\otimes_{\mathsf{fm}}$ from Example 2.1.6 satisfies ($\mathsf{R9}_{\mathcal{M}}$). The motivation for ($\mathsf{R9}_{\mathcal{M}}$) is the following. If $S = \emptyset$, then every formula holds in the model set S. That is, we are in an inconsistent state, which is even worse than having no information at all. So instead of revising the inconsistent state with α, we revise the belief state that contains all models. This results in $\|\alpha\|$ by ($\mathsf{R2}_{\mathcal{M}}$)–($\mathsf{R4}_{\mathcal{M}}$), that is what we require with ($\mathsf{R9}_{\mathcal{M}}$). ($\mathsf{R9}_{\mathcal{M}^c}$) specifies that we must not modify our belief state if it is not possible to consistently integrate the new information. So ($\mathsf{R9}_{\mathcal{M}^c}$) carries on the idea of minimal change. Due to these new postulates, we are able to state the following lemma.

Lemma 3.1.9. *We have the following interchangeability results.*

1. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* ($\mathsf{R9}_{\mathcal{M}}$) *and at least one of the postulates* ($\mathsf{R2}_{\mathcal{M}}$) *and* ($\mathsf{R5}_{\mathcal{M}}$)*, then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\ (\otimes^{cr})^{ir}\ \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* ($\mathsf{R9}_{\mathcal{M}^c}$)*, then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\ (\otimes^{ir})^{cr}\ \alpha) = \mathrm{Th}(S \otimes \alpha)$.

Proof. We show how to prove the first assertion. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. First, if $\not\models \neg\alpha$ and $S \neq \emptyset$, then we have $S\ (\otimes^{cr})^{ir}\ \alpha = S \otimes^{cr} \alpha = S \otimes \alpha$ by definition, and the claim directly follows. Second, if $S = \emptyset$, then we have $\mathrm{Th}(S\ (\otimes^{cr})^{ir}\ \alpha) = \mathrm{Th}(\|\alpha\|)$, which is the same as $\mathrm{Th}(S \otimes \alpha)$ by ($\mathsf{R9}_{\mathcal{M}}$). In the last case, if $\models \neg\alpha$, then we get $S\ (\otimes^{cr})^{ir}\ \alpha = \|\alpha\| = \emptyset$, which is equal to $S \otimes \alpha$ by ($\mathsf{R2}_{\mathcal{M}}$) or ($\mathsf{R5}_{\mathcal{M}}$). The proof of the second assertion is similar. □

The definition of the notion of a consistent AGM contraction does not need any modified contraction postulates, because the contraction of a consistent model set always results in a non empty model set, provided that ($\mathsf{C2}_{\mathcal{M}}$) holds. That is, the second contraction postulate directly implies property (3.1) in the context of consistent model sets.

Definition 3.1.10. A function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent AGM contraction*, if it satisfies ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C8}_{\mathcal{M}}$), that is, if it the restriction of an AGM contraction to consistent model sets.

As a consequence of Definition 3.1.10, the translation of an AGM contraction to a consistent AGM contraction will just be its restriction to consistent model set. The definition of the translation of a consistent AGM contraction to an AGM contraction is a bit more elaborate.

Definition 3.1.11. Given a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\ominus^{cc}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto S \ominus \alpha$$

is the appropriate consistency preserving contraction function. In addition, for all functions $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\ominus^{ic}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S,\alpha) \mapsto \begin{cases} S \ominus \alpha & \text{if } S \neq \emptyset, \\ \|\neg\alpha\| & \text{otherwise} \end{cases}$$

is the corresponding contraction function on possibly inconsistent model sets.

We can prove that the translations from Definition 3.1.11 preserve the AGM postulates for contraction.

Lemma 3.1.12. *We have the following correspondences between the AGM contractions and consistent AGM contractions.*

1. *If a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an AGM contraction, then the function $\ominus^{cc}$ is a consistent AGM contraction.*
2. *If a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent AGM contraction, then the function $\ominus^{ic}$ is an AGM contraction.*

Proof. The first assertion is trivial, since the function $\ominus^{cc}$ is the restriction of $\ominus$ to $\mathcal{M}^c \times \mathcal{L}_0$. For the second assertion, we content ourselves with proving that $\ominus^{ic}$ satisfies ($\mathsf{C7}_{\mathcal{M}}$). Let $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given. If $S \neq \emptyset$, then the claim follows from the assumption that $\ominus$ satisfies ($\mathsf{C7}_{\mathcal{M}}$). If $S = \emptyset$, then we have

$$\begin{aligned} \mathrm{Th}(S \ominus^{ic} \alpha) \cap \mathrm{Th}(S \ominus^{ic} \beta) &= \mathrm{Th}(\|\neg\alpha\|) \cap \mathrm{Th}(\|\neg\beta\|) \\ &= \mathrm{Th}(\|\neg\alpha\| \cup \|\neg\beta\|) \\ &= \mathrm{Th}(\|\neg\alpha \vee \neg\beta\|) \\ &= \mathrm{Th}(\|\neg(\alpha \wedge \beta)\|) \\ &= \mathrm{Th}(S \ominus^{ic} (\alpha \wedge \beta)). \end{aligned}$$

The proofs of the other postulates are similar. □

Similar to revision, Lemma 3.1.12 also holds if we only take the basic contraction postulates. The proof of Lemma 3.1.12 shows that this is indeed the case.

Lemma 3.1.13. *We have the following special cases of Lemma 3.1.12.*

1. *If a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C6}_{\mathcal{M}}$), then so also does the function $\ominus^{cc}$.*
2. *If a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{C2}_{\mathcal{M}}$), ($\mathsf{C3}_{\mathcal{M}}$), ($\mathsf{C5}_{\mathcal{M}}$) and ($\mathsf{C6}_{\mathcal{M}}$), then so also does the function $\ominus^{ic}$. Furthermore, if $\ominus$ additionally satisfies ($\mathsf{C1}_{\mathcal{M}}$) and ($\mathsf{C4}_{\mathcal{M}}$), then so also does the function $\ominus^{ic}$.*

In order to get some further results, we will now introduce a new postulate for contraction functions.

Definition 3.1.14. A function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies the following additional contraction postulate if for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$(\mathsf{C9}_{\mathcal{M}})$ $S = \emptyset \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) = \mathrm{Th}(\|\neg\alpha\|)$.

Observe that $(\mathsf{C9}_{\mathcal{M}})$ is satisfied by the full meet contraction function $\ominus_{\mathsf{fm}}$ from Example 2.2.5. We believe that $(\mathsf{C9}_{\mathcal{M}})$ has been intended by the authors of the AGM postulates [2]. Although one could argue that requiring

$$S = \emptyset \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) = \mathrm{Th}(S) \tag{3.2}$$

would be closer to the idea of minimal change, we think that it is better to quit an inconsistent belief state as soon as possible. We want to mention here that with property (3.2) instead of $(\mathsf{C9}_{\mathcal{M}})$, we would have defined the function $\ominus^{ic}$ differently, and all the results with $(\mathsf{C9}_{\mathcal{M}})$ replaced by (3.2) would still have been provable. The following lemma is a first application of the definition of $(\mathsf{C9}_{\mathcal{M}})$.

Lemma 3.1.15. *We have the following interchangeability results.*

1. *If a function* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{C9}_{\mathcal{M}})$*, then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\ (\ominus^{cc})^{ic}\ \alpha) = \mathrm{Th}(S \ominus \alpha)$.
2. *Let* $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *be given. Then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\ (\ominus^{ic})^{cc}\ \alpha) = \mathrm{Th}(S \ominus \alpha)$.

Proof. We show how to prove the first assertion, so let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. First, if $S \neq \emptyset$, then we have $S\ (\ominus^{cc})^{ic}\ \alpha = S \ominus^{cc} \alpha = S \ominus \alpha$ by definition, and the claim easily follows. Second, if $S = \emptyset$, then we have $\mathrm{Th}(S\ (\ominus^{cc})^{ic}\ \alpha) = \mathrm{Th}(\|\neg\alpha\|)$ by definition, which is the same as $\mathrm{Th}(S \ominus \alpha)$ by $(\mathsf{C9}_{\mathcal{M}})$. The proof of the second assertion is trivial. □

The requirements for update functions on consistent model sets are similar to the ones for revision functions. Of course, they are slightly different because they are associated with the KM postulates.

Definition 3.1.16. A function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent KM update*, if it satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U2}_{\mathcal{M}})$, and $(\mathsf{U4}_{\mathcal{M}})$–$(\mathsf{U8}_{\mathcal{M}})$, as well as the following modified update postulates,

$(\mathsf{U1}_{\mathcal{M}^c})$ $\not\models \neg\alpha \;\Rightarrow\; S \odot \alpha \models \alpha$,

$(\mathsf{U3}_{\mathcal{M}^c})$ $S \odot \alpha \neq \emptyset$.

We will now define the translation of an update function operating on model sets to an update function defined on consistent model sets, and the other way round. Observe that if $S = \emptyset$, then for all $\alpha \in \mathcal{L}_0$ we need $S \odot^{iu} \alpha := \emptyset$, because this is directly implied by ($\mathsf{U2}_{\mathcal{M}}$).

Definition 3.1.17. Given a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the function

$$\odot^{cu}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S \odot \alpha & \text{if } \not\models \neg\alpha, \\ S & \text{otherwise} \end{cases}$$

is the appropriate consistency preserving update function. On the other hand, if the function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is given, then the function

$$\odot^{iu}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S \odot \alpha & \text{if } \not\models \neg\alpha \text{ and } S \neq \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

is the adequate update function on possibly inconsistent model sets.

Clearly, we have a close relationship between KM updates and consistent KM updates, as we are going to prove.

Lemma 3.1.18. *We have the following correspondences between the two sets of update postulates.*

1. *If a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a KM update, then the function $\odot^{cu}$ is a consistent KM update.*
2. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent KM update, then the function $\odot^{iu}$ is a KM update.*

Proof. We first want to mention that $\odot^{cu}$ satisfies property (3.1), whenever $\odot$ satisfies ($\mathsf{U3}_{\mathcal{M}}$). Now, we will prove the validity of one postulate for each assertion.

1. We show that ($\mathsf{U7}_{\mathcal{M}}$) holds for $\odot^{cu}$. Let $S \in \mathcal{M}^c$ and α, β be given and assume $\mathrm{Card}(S) \leq 1$, that is $\mathrm{Card}(S) = 1$ because $S \neq \emptyset$. If α and β are both satisfiable, then so also is $\alpha \vee \beta$ and we have $S \odot^{cu} \alpha = S \odot \alpha$, $S \odot^{cu} \beta = S \odot \beta$, and $S \odot^{cu} (\alpha \vee \beta) = S \odot (\alpha \vee \beta)$, and the claim directly follows from ($\mathsf{U7}_{\mathcal{M}}$) for $\odot$. If α and β are both unsatisfiable, then so also is $\alpha \vee \beta$ and we have $S \odot^{cu} \alpha = S \odot^{cu} \beta = S \odot^{cu} (\alpha \vee \beta) = S$, hence the postulate trivially holds. If exactly one of the formulas α and β is satisfiable, then we proceed as follows. Without loss of generality, we assume that α is satisfiable and β is not. Thus, $\alpha \vee \beta$ is satisfiable and

we get $S \odot^{cu} \alpha = S \odot \alpha$, $S \odot^{cu} \beta = S$, and $S \odot^{cu} (\alpha \vee \beta) = S \odot (\alpha \vee \beta)$. If $(S \odot \alpha) \cap S = \emptyset$, then the claim trivially holds. If $(S \odot \alpha) \cap S \neq \emptyset$, then we know that $(S \odot \alpha) \cap S = S$ because $\mathrm{Card}(S) = 1$, and we get $S \models \alpha$ because $S \odot \alpha \models \alpha$ by $(\mathsf{U1}_{\mathcal{M}})$. So we also have $S \models \alpha \vee \beta$ and therefore, we have $\mathrm{Th}(S \odot (\alpha \vee \beta)) = \mathrm{Th}(S)$ by $(\mathsf{U2}_{\mathcal{M}})$.

2. We show that $(\mathsf{U8}_{\mathcal{M}})$ holds for $\odot^{iu}$. Let $S, S' \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. If α is not satisfiable, then $S \odot^{iu} \alpha = S' \odot^{iu} \alpha = (S \cup S') \odot^{iu} \alpha = \emptyset$, and the postulate is obviously satisfied. If α is satisfiable, then we distinguish three cases. If $S = S' = \emptyset$, then we have the same updated model sets as in the previous case. If S and S' are both non empty, then we immediately have $S \odot^{iu} \alpha = S \odot \alpha$, $S' \odot^{iu} \alpha = S' \odot \alpha$, and $(S \cup S') \odot^{iu} \alpha = (S \cup S') \odot \alpha$, and the claim follows from $(\mathsf{U8}_{\mathcal{M}})$ for $\odot$. In the last case, if exactly one model set of S and S' is non empty, we proceed as follows. Without loss of generality, we assume $S \neq \emptyset$ and $S' = \emptyset$, and we get $(S \cup S') \odot^{iu} \alpha = S \odot^{iu} \alpha = S \odot \alpha$ and $S' \odot^{iu} \alpha = \emptyset$, hence the proof is finished.

The other postulates can similarly be proved. □

The following lemma will be needed for some further results. Its proof directly follows from the proof of Lemma 3.1.18.

Lemma 3.1.19. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}^c})$, $(\mathsf{U3}_{\mathcal{M}^c})$, and $(\mathsf{U4}_{\mathcal{M}})$, then the function $\odot^{iu}$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, and $(\mathsf{U4}_{\mathcal{M}})$. Moreover, if $\odot$ additionally satisfies $(\mathsf{U2}_{\mathcal{M}})$, then so also does the function $\odot^{iu}$.*

There is another postulate for update functions operating on consistent model sets, that we have to define for the purpose of some further results.

Definition 3.1.20. A function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies the following additional update postulate if for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have

$(\mathsf{U9}_{\mathcal{M}^c})$ $\models \neg\alpha \;\Rightarrow\; \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S)$.

$(\mathsf{U9}_{\mathcal{M}^c})$ is exactly the same as $(\mathsf{R9}_{\mathcal{M}^c})$ for consistency preserving revision functions. Following the idea of minimal change, we require to keep the current belief state, if it is impossible to consistently learn some information.

Similar to revision, we think that updating the inconsistent model set $\emptyset$ with a formula α should result in the the model set $\|\alpha\|$. We cannot require this by a ninth update postulate, because we have $\mathrm{Th}(\emptyset \odot \alpha) = \mathcal{L}_0$ by $(\mathsf{U2}_{\mathcal{M}})$. Due to this fact and $(\mathsf{U9}_{\mathcal{M}^c})$, we are able to prove the following result.

Lemma 3.1.21. *We have the following interchangeability results.*

1. *If a function* $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{U1}_{\mathcal{M}})$ *and* $(\mathsf{U2}_{\mathcal{M}})$*, then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\,(\odot^{cu})^{iu}\,\alpha) = \mathrm{Th}(S \odot \alpha)$*.*
2. *If a function* $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{U9}_{\mathcal{M}^c})$*, then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S\,(\odot^{iu})^{cu}\,\alpha) = \mathrm{Th}(S \odot \alpha)$*.*

Proof. We give a proof of the first assertion. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. First, if $\not\models \neg\alpha$ and $S \neq \emptyset$, then we have $S\,(\odot^{cu})^{iu}\,\alpha = S \odot^{cu} \alpha = S \odot \alpha$ by definition, and the claim easily follows. Second, if $\models \neg\alpha$ or $S = \emptyset$, then we have $\mathrm{Th}(S\,(\odot^{cu})^{iu}\,\alpha) = \mathrm{Th}(\emptyset) = \mathcal{L}_0$ by definition, which is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U1}_{\mathcal{M}})$ and $(\mathsf{U2}_{\mathcal{M}})$. The second assertion is even easier to prove. □

The results of this section will be useful in the following sections of this chapter, where we are going to explore the relationship between revision and contraction functions, as well as between revision and update functions.

3.2 Revision and contraction

The revision and contraction processes are related to each other because they both formalise belief change in a *static world*. The former process consistently adds some information, whereas the latter removes some data. This relatedness has been the reason why people have defined translations from contraction to revision function functions and vice versa. Levi [51] has claimed that revising with a formula α should result in the same belief state as contracting with $\neg\alpha$ and then adding α with the (unique) expansion function. This intuitive definition of revision has been called the *Levi identity*. Accordingly, Harper [35] has claimed that contracting with a formula α should have the same effect as taking those of the original beliefs that remain if we revise with $\neg\alpha$. This description of the contraction process has been called the *Harper identity*. We will now give the formal definitions of the translations that follow from the above mentioned identities.

Definition 3.2.1. Given a function $\dot{-}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$, its Levi translation is defined as follows,

$$\dot{+}_{\dot{-}}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0), \qquad (T, \alpha) \mapsto (T \dot{-} \neg\alpha) + \alpha.$$

The Harper translation of a function $\dot{+}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ is defined by

$$\dot{-}_{\dot{+}}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0), \qquad (T, \alpha) \mapsto T \cap (T \dot{+} \neg\alpha).$$

For instance, the full meet revision function $\dotplus_{\mathsf{fm}}$ from Example 2.1.4 and the full meet contraction function $\dot{-}_{\mathsf{fm}}$ from Example 2.2.3 correspond to each other through the above defined identities.

Example 3.2.2. For all $T \in \mathcal{B}$ and all $\alpha \in \mathcal{L}_0$ we have

$$T \dotplus_{(\dot{-}_{\mathsf{fm}})} \alpha = T \dotplus_{\mathsf{fm}} \alpha, \qquad T \dot{-}_{(\dotplus_{\mathsf{fm}})} \alpha = T \dot{-}_{\mathsf{fm}} \alpha.$$

Originally, the full meet contraction function $\dot{-}_{\mathsf{fm}}$ has been defined as an example of a function satisfying all AGM postulates for contraction, see [2]. The full meet revision function $\dotplus_{\mathsf{fm}}$ has then been defined from $\dot{-}_{\mathsf{fm}}$ by use of the Levi identity in order to illustrate that the AGM postulates are preserved by this translation.

The following theorem shows that these identities conform to the AGM postulates and that they are strongly interchangeable. A proof has been presented by Gärdenfors in [27].

Theorem 3.2.3. *The identities by Levi and Harper preserve the AGM postulates and, given the first six postulates, they are inverse to each other.*

1. *If a function* $\dot{-}\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ *satisfies* $(\mathsf{C1}_{\mathcal{B}})$–$(\mathsf{C6}_{\mathcal{B}})$*, then the function* $\dotplus_{\dot{-}}$ *satisfies* $(\mathsf{R1}_{\mathcal{B}})$–$(\mathsf{R6}_{\mathcal{B}})$ *and for all* $T \in \mathcal{B}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $T \dot{-}_{(\dotplus_{\dot{-}})} \alpha = T \dot{-} \alpha$*. If* $\dot{-}$ *also satisfies* $(\mathsf{C7}_{\mathcal{B}})$ *and* $(\mathsf{C8}_{\mathcal{B}})$*, then* $\dotplus_{\dot{-}}$ *satisfies* $(\mathsf{R7}_{\mathcal{B}})$ *and* $(\mathsf{R8}_{\mathcal{B}})$*, as well.*
2. *If a function* $\dotplus\colon \mathcal{B} \times \mathcal{L}_0 \to \mathrm{Pow}(\mathcal{L}_0)$ *satisfies* $(\mathsf{R1}_{\mathcal{B}})$–$(\mathsf{R6}_{\mathcal{B}})$*, then the function* $\dot{-}_{\dotplus}$ *satisfies* $(\mathsf{C1}_{\mathcal{B}})$–$(\mathsf{C6}_{\mathcal{B}})$ *and for all* $T \in \mathcal{B}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $T \dotplus_{(\dot{-}_{\dotplus})} \alpha = T \dotplus \alpha$*. If* $\dotplus$ *additionally satisfies* $(\mathsf{R7}_{\mathcal{B}})$ *and* $(\mathsf{R8}_{\mathcal{B}})$*, then* $\dot{-}_{\dotplus}$ *also satisfies* $(\mathsf{C7}_{\mathcal{B}})$ *and* $(\mathsf{C8}_{\mathcal{B}})$*.*

We have almost the same results in the context of model sets. In a first step, we are going to translate the Levi and Harper identities to the notion of model sets. The only difference occurs in the definition of the Harper identity, because the intersection of two belief sets corresponds to the union of the corresponding model sets.

Definition 3.2.4. Given a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, its Levi translation is defined as follows,

$$\otimes^c_{\ominus}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto (S \ominus \neg\alpha) \oplus \alpha.$$

The Harper translation of a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$\ominus^r_{\otimes}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto S \cup (S \otimes \neg\alpha).$$

We are now working towards the equivalent of Theorem 3.2.3 in the context of model sets. For technical reasons, we will prove it step by step in three separate parts.

Lemma 3.2.5. *The basic revision and contraction postulates are preserved by the Levi and Harper identities.*

1. *If a function* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$*, then the function* $\otimes^c_\ominus$ *satisfies* $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$.
2. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$*, then the function* $\ominus^r_\otimes$ *satisfies* $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$.

Proof. For both assertions, we will show how to prove one postulate.

1. We show that the function $\otimes^c_\ominus$ satisfies $(\mathsf{R3}_{\mathcal{M}})$. For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\begin{aligned}
\mathrm{Th}(S \otimes^c_\ominus \alpha) &= \mathrm{Th}((S \ominus \neg\alpha) \oplus \alpha) \\
&= \mathrm{Th}(\overline{S \ominus \neg\alpha} \oplus \alpha) && \text{by Lemma 1.2.6} \\
&= \mathrm{Th}(\overline{S \ominus \neg\alpha} \cap \|\alpha\|) \\
&\subseteq \mathrm{Th}(\overline{S} \cap \|\alpha\|) && \text{by } (\mathsf{C2}_{\mathcal{M}}) \\
& && \text{and Lemma 1.1.14} \\
&= \mathrm{Th}(\overline{S} \oplus \alpha) \\
&= \mathrm{Th}(S \oplus \alpha) && \text{by Lemma 1.2.6.}
\end{aligned}$$

2. Now, we show that the function $\ominus^r_\otimes$ satisfies $(\mathsf{C5}_{\mathcal{M}})$. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given and assume $S \vDash \alpha$. Then we have

$$\begin{aligned}
\mathrm{Th}(S) &= \mathrm{Th}(S \cap \|\alpha\|) && \text{by assumption} \\
&= \mathrm{Th}((S \cap \|\alpha\|) \cup \emptyset) \\
&= \mathrm{Th}((S \cap \|\alpha\|) \cup ((S \otimes \neg\alpha) \cap \|\alpha\|)) && \text{by } (\mathsf{R2}_{\mathcal{M}}) \\
&= \mathrm{Th}((S \cup (S \otimes \neg\alpha)) \cap \|\alpha\|) \\
&= \mathrm{Th}((S \cup (S \otimes \neg\alpha)) \oplus \alpha) \\
&= \mathrm{Th}((S \ominus^r_\otimes \alpha) \oplus \alpha).
\end{aligned}$$

The proof of the other postulates is similar. ☐

Lemma 3.2.5 also holds for the whole sets of revision and contraction postulates, as we state in the following theorem. Although we could directly prove it, we will give a more elegant proof by using Theorem 3.2.3 as well as the translation lemmas from Section 2.1 and Section 2.2.

Theorem 3.2.6. *We have the following relationships between revision and contraction functions due to the Levi and Harper identities.*

1. *If a function* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *is an AGM contraction, then the function* $\otimes^c_\ominus$ *is an AGM revision.*
2. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *is an AGM revision, then the function* $\ominus^r_\otimes$ *is an AGM contraction.*

Proof. We show how to prove the first assertion. If $\ominus$ is an AGM contraction on model sets, then the function $\dot{-}_\ominus$ is an AGM contraction on belief sets by Lemma 2.2.6. Now, by Theorem 3.2.3, we get that $\dot{+}_{(\dot{-}_\ominus)}$ is an AGM revision on belief sets. Therefore, we get that the function

$$\otimes_{(\dot{+}_{(\dot{-}_\ominus)})}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \overline{S \ominus \neg\alpha} \oplus \alpha$$

is an AGM revision on model sets by Lemma 2.1.8. Observe that we have $\mathrm{Th}(\overline{S \ominus \neg\alpha} \oplus \alpha) = \mathrm{Th}((S \ominus \neg\alpha) \oplus \alpha)$ by Lemma 1.2.6, hence we get that the function $\otimes^c_\ominus$ is an AGM revision on model sets, as well. The proof of the second assertion is analogous. □

The following result slightly differs from the corresponding result in the context of belief sets. This is because the model sets need not be closed sets of models.

Lemma 3.2.7. *The Levi identity is the inverse of the Harper identity modulo* Th *and vice versa.*

1. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the revision postulates* $(\mathsf{R2}_\mathcal{M})$, $(\mathsf{R3}_\mathcal{M})$, *and* $(\mathsf{R6}_\mathcal{M})$, *then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes^c_{(\ominus^r_\otimes)} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the contraction postulates* $(\mathsf{C2}_\mathcal{M})$, $(\mathsf{C3}_\mathcal{M})$, $(\mathsf{C5}_\mathcal{M})$, *and* $(\mathsf{C6}_\mathcal{M})$, *then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \ominus^r_{(\otimes^c_\ominus)} \alpha) = \mathrm{Th}(S \ominus \alpha)$.

Proof. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. Then the first assertion can be proved as follows,

$$\begin{aligned}
\mathrm{Th}(S \otimes^c_{(\ominus^r_\otimes)} \alpha) &= \mathrm{Th}((S \ominus^r_\otimes \neg\alpha) \oplus \alpha) \\
&= \mathrm{Th}((S \cup (S \otimes \neg\neg\alpha)) \oplus \alpha) \\
&= \mathrm{Th}((S \cup (S \otimes \neg\neg\alpha)) \cap \|\alpha\|) \\
&= \mathrm{Th}((S \cap \|\alpha\|) \cup ((S \otimes \neg\neg\alpha) \cap \|\alpha\|)) \\
&= \mathrm{Th}((S \oplus \alpha) \cup ((S \otimes \neg\neg\alpha) \cap \|\alpha\|))
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Th}(S \oplus \alpha) \cap \mathrm{Th}((S \otimes \neg\neg\alpha) \cap \|\alpha\|) && \text{by Lemma 1.1.11}\\
&= \mathrm{Th}(S \oplus \alpha) \cap \mathrm{Th}(S \otimes \neg\neg\alpha) && \text{by } (\mathsf{R2}_{\mathcal{M}})\\
&= \mathrm{Th}(S \oplus \alpha) \cap \mathrm{Th}(S \otimes \alpha) && \text{by } (\mathsf{R6}_{\mathcal{M}})\\
&= \mathrm{Th}(S \otimes \alpha) && \text{by } (\mathsf{R3}_{\mathcal{M}}).
\end{aligned}
$$

For the second assertion, we first observe that

$$
\begin{aligned}
\mathrm{Th}(S \ominus^{r}_{(\otimes^{c}_{\ominus})} \alpha) &= \mathrm{Th}(S \cup (S \otimes^{c}_{\ominus} \neg\alpha))\\
&= \mathrm{Th}(S \cup ((S \ominus \neg\neg\alpha) \oplus \neg\alpha))\\
&= \mathrm{Th}(S) \cap \mathrm{Th}((S \ominus \neg\neg\alpha) \oplus \neg\alpha) && \text{by Lemma 1.1.11}\\
&= \mathrm{Th}(S) \cap \mathrm{Th}(\overline{S \ominus \neg\neg\alpha} \oplus \neg\alpha) && \text{by Lemma 1.2.6}\\
&= \mathrm{Th}(S) \cap \mathrm{Th}(\overline{S \ominus \neg\neg\alpha} \cap \|\neg\alpha\|)\\
&= \mathrm{Th}(S) \cap \mathrm{Th}(\overline{S \ominus \alpha} \cap \|\neg\alpha\|) && \text{by } (\mathsf{C6}_{\mathcal{M}}) \text{ and}\\
& && \text{Lemma 1.1.14}\\
&= \mathrm{Th}(S) \cap \mathrm{Th}(\overline{S \ominus \alpha} \oplus \neg\alpha)\\
&= \mathrm{Th}(S) \cap \mathrm{Th}((S \ominus \alpha) \oplus \neg\alpha) && \text{by Lemma 1.2.6}\\
&= \mathrm{Th}(S) \cap \mathrm{Th}((S \ominus \alpha) \cap \|\neg\alpha\|).
\end{aligned}
$$

Now, we distinguish two cases. In the first case, if $S \vDash \alpha$, then we have $\mathrm{Th}(S) = \mathrm{Th}(S \cap \|\alpha\|) = \mathrm{Th}((S \ominus \alpha) \cap \|\alpha\|)$ by $(\mathsf{C2}_{\mathcal{M}})$ and $(\mathsf{C5}_{\mathcal{M}})$, and we can continue the proof with

$$
\begin{aligned}
\ldots &= \mathrm{Th}(S) \cap \mathrm{Th}((S \ominus \alpha) \cap \|\neg\alpha\|)\\
&= \mathrm{Th}((S \ominus \alpha) \cap \|\alpha\|) \cap \mathrm{Th}((S \ominus \alpha) \cap \|\neg\alpha\|)\\
&= \mathrm{Th}(((S \ominus \alpha) \cap \|\alpha\|) \cup ((S \ominus \alpha) \cap \|\neg\alpha\|)) && \text{by Lemma 1.1.11}\\
&= \mathrm{Th}(S \ominus \alpha).
\end{aligned}
$$

Second, if $S \nvDash \alpha$, then we can finish the proof with

$$
\begin{aligned}
\ldots &= \mathrm{Th}(S) \cap \mathrm{Th}((S \ominus \alpha) \cap \|\neg\alpha\|)\\
&= \mathrm{Th}(S \ominus \alpha) \cap \mathrm{Th}((S \ominus \alpha) \cap \|\neg\alpha\|) && \text{by } (\mathsf{C3}_{\mathcal{M}})\\
&= \mathrm{Th}((S \ominus \alpha) \cup ((S \ominus \alpha) \cap \|\neg\alpha\|)) && \text{by Lemma 1.1.11}\\
&= \mathrm{Th}(S \ominus \alpha).
\end{aligned}
$$

We have now proved both assertions. □

As we have seen in Section 3.1, we have slightly different postulates for revision functions defined on consistent model sets. The definitions of the Levi and Harper identities, however, are very similar in this setting.

Definition 3.2.8. Given a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, its Levi translation is defined as follows,

$$\otimes_{\ominus}^{cc}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S,\alpha) \mapsto \begin{cases} (S \ominus \neg\alpha) \oplus \alpha & \text{if } \nvDash \neg\alpha, \\ S & \text{otherwise.} \end{cases}$$

The Harper translation of a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$\ominus_{\otimes}^{cr}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S,\alpha) \mapsto \begin{cases} S \cup (S \otimes \neg\alpha) & \text{if } \nvDash \alpha, \\ S & \text{otherwise.} \end{cases}$$

We want to mention that if a function $\ominus$ satisfies ($\mathsf{C4}_{\mathcal{M}}$), then for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have $S \otimes_{\ominus}^{cc} \alpha \neq \emptyset$. On the other hand, it is immediate that the function $\ominus_{\otimes}^{cr}$ satisfies property (3.1) for all functions $\otimes$. The translations from Definition 3.2.8 are compositions of some previous translations, as we state in the following lemma.

Lemma 3.2.9. *Let the function $\circledast\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \otimes_{\circledast}^{cc} \alpha = S\, (\otimes_{(\circledast^{ic})}^{c})^{cr}\, \alpha, \qquad S \ominus_{\circledast}^{cr} \alpha = S\, (\ominus_{(\circledast^{ir})}^{r})^{cc}\, \alpha.$$

Proof. Both equalities directly follow from Definition 3.1.5, Definition 3.1.11, and Definition 3.2.4. □

The following result is the same as Lemma 3.2.5, but in the context of consistent model sets. The proof is rather short because it is based on certain previous results.

Lemma 3.2.10. *The translations from Definition 3.2.8 preserve the basic postulates in the context of consistent model sets.*

1. *If a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C6}_{\mathcal{M}}$), then the function $\otimes_{\ominus}^{cc}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), and ($\mathsf{R6}_{\mathcal{M}}$).*
2. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), and ($\mathsf{R6}_{\mathcal{M}}$), then the function $\ominus_{\otimes}^{cr}$ satisfies the postulates ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C6}_{\mathcal{M}}$).*

Proof. For the first assertion, we can apply Lemma 3.1.13, Lemma 3.2.5, and Lemma 3.1.7 in order to show that the function $(\otimes_{(\ominus^{ic})}^{c})^{cr}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), and ($\mathsf{R6}_{\mathcal{M}}$). By Lemma 3.2.9, we know that this function is equal to $\otimes_{\ominus}^{cc}$. For the second assertion, we can use the same lemmas to get that the function $(\ominus_{(\otimes^{ir})}^{r})^{cc}$ satisfies ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C6}_{\mathcal{M}}$). By Lemma 3.2.9, we get that this is the same function as $\ominus_{\otimes}^{cr}$. □

We are now going to state the analogue of Theorem 3.2.6, but in the context of consistent model sets. The proof of the following theorem is also based on some previous results.

Theorem 3.2.11. *We have the following relationships between consistent AGM revisions and consistent AGM contractions.*

1. *If a function* $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *is a consistent AGM contraction, then the function* $\otimes_{\ominus}^{cc}$ *is a consistent AGM revision.*
2. *If a function* $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *is a consistent AGM revision, the function* $\ominus_{\otimes}^{cr}$ *is a consistent AGM contraction.*

Proof. For the first assertion, we can apply Lemma 3.1.12, Theorem 3.2.6, and Lemma 3.1.6 in order to get that the function $(\otimes_{(\ominus^{ic})}^{c})^{cr}$ is a consistent AGM revision. But this function is the same as $\otimes_{\ominus}^{cc}$ by Lemma 3.2.9. For the second assertion, we can use the same lemmas and theorem to prove that the function $(\ominus_{(\otimes^{ir})}^{r})^{cc}$ is a consistent AGM contraction. Applying Lemma 3.2.9 again, we get that this is the same function as $\ominus_{\otimes}^{cr}$. □

We conclude this section by proving that the Levi and Harper identities are inverse to each other in the context of consistent model sets, provided that the underlying belief change functions satisfy the right postulates.

Lemma 3.2.12. *In the context of consistent model sets, the Levi identity is the inverse of the Harper identity modulo* Th *and vice versa.*

1. *If a function* $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the postulates* $(\mathsf{R2}_{\mathcal{M}^c})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R6}_{\mathcal{M}})$, *and* $(\mathsf{R9}_{\mathcal{M}^c})$, *then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes_{(\ominus_{\otimes}^{cr})}^{cc} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the postulates* $(\mathsf{C2}_{\mathcal{M}})$, $(\mathsf{C3}_{\mathcal{M}})$, $(\mathsf{C5}_{\mathcal{M}})$, *and* $(\mathsf{C6}_{\mathcal{M}})$, *then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \ominus_{(\otimes_{\ominus}^{cc})}^{cr} \alpha) = \mathrm{Th}(S \ominus \alpha)$.

Proof. Let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given. For the first assertion, we distinguish two cases as follows. First, if $\not\models \neg\alpha$, then we get that the function $\otimes^{ir}$ satisfies $(\mathsf{R2}_{\mathcal{M}})$, $(\mathsf{R3}_{\mathcal{M}})$, and $(\mathsf{R6}_{\mathcal{M}})$ by Lemma 3.1.7, and we have

$$\begin{aligned}
\mathrm{Th}(S \otimes_{(\ominus_{\otimes}^{cr})}^{cc} \alpha) &= \mathrm{Th}((S \ominus_{\otimes}^{cr} \neg\alpha) \oplus \alpha) \\
&= \mathrm{Th}((S \cup (S \otimes \neg\neg\alpha)) \oplus \alpha) \\
&= \mathrm{Th}((S \cup (S \otimes^{ir} \neg\neg\alpha)) \oplus \alpha) \qquad \text{by assumption} \\
&= \mathrm{Th}((S \ominus_{(\otimes^{ir})}^{r} \neg\alpha) \oplus \alpha)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Th}(S \otimes^{c}_{(\ominus^{r}_{(\otimes^{ir})})} \alpha) && \\
&= \mathrm{Th}(S \otimes^{ir} \alpha) && \text{by Lemma 3.2.7} \\
&= \mathrm{Th}(S \otimes \alpha) && \text{by assumption.}
\end{aligned}
$$

Second, if $\vDash \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{cc}_{(\ominus^{cr}_{\otimes})} \alpha) = \mathrm{Th}(S)$, which is the same as $\mathrm{Th}(S \otimes \alpha)$ by ($\mathsf{R9}_{\mathcal{M}^c}$). For the second assertion, we distinguish the following two cases. First, if $\nvDash \alpha$, then we get that the function $\ominus^{ic}$ satisfies ($\mathsf{C2}_{\mathcal{M}}$), ($\mathsf{C3}_{\mathcal{M}}$), ($\mathsf{C5}_{\mathcal{M}}$), and ($\mathsf{C6}_{\mathcal{M}}$) by Lemma 3.1.13, and we have

$$
\begin{aligned}
\mathrm{Th}(S \ominus^{cr}_{(\otimes^{cc}_{\ominus})} \alpha) &= \mathrm{Th}(S \cup (S \otimes^{cc}_{\ominus} \neg\alpha)) && \\
&= \mathrm{Th}(S \cup ((S \ominus \neg\neg\alpha) \oplus \neg\alpha)) && \\
&= \mathrm{Th}(S \cup ((S \ominus^{ic} \neg\neg\alpha) \oplus \neg\alpha)) && \text{by assumption} \\
&= \mathrm{Th}(S \cup (S \otimes^{c}_{(\ominus^{ic})} \neg\alpha)) && \\
&= \mathrm{Th}(S \ominus^{r}_{(\otimes^{c}_{(\ominus^{ic})})} \alpha) && \\
&= \mathrm{Th}(S \ominus^{ic} \alpha) && \text{by Lemma 3.2.7} \\
&= \mathrm{Th}(S \ominus \alpha) && \text{by assumption.}
\end{aligned}
$$

Second, if $\vDash \alpha$, then we have $\mathrm{Th}(S \ominus^{cr}_{(\otimes^{cc}_{\ominus})} \alpha) = \mathrm{Th}(S)$. Since $S \vDash \alpha$, this is equal to $\mathrm{Th}(S \ominus \alpha)$ by ($\mathsf{C2}_{\mathcal{M}}$) and ($\mathsf{C5}_{\mathcal{M}}$). □

3.3 Revision and update

The revision and update processes are related to each other, because they both describe how to consistently integrate new information into a given belief state representation. The only difference in the definitions of these processes is that the former is applied in a *static world*, whereas the latter is used in a *dynamic world*. For this reason, we have tried to find two identities that allow for the same results as the Levi and Harper identities do in Section 3.2. So our goal is to define a revision function from a given update function and vice versa. In the following definition, we will present such identities between revision and update functions. As we will see later in this section, they nearly fulfil our requirements.

Definition 3.3.1. Given a function $\odot \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, its translation from update to revision is defined as follows,

$$
\otimes^{u}_{\odot} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S \oplus \alpha & \text{if } S \nvDash \neg\alpha, \\ S \odot \alpha & \text{otherwise.} \end{cases}
$$

The translation of a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ from revision to update is defined by

$$\odot^r_\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \bigcup_{w \in \overline{S}} (\{w\} \otimes \alpha).$$

The case $S = \emptyset$ in the definition of $\otimes^u_\odot$ corresponds to $(\mathsf{R9}_\mathcal{M})$, which we have already motivated in Section 3.2. The case $S \not\models \neg\alpha$ has to be defined this way due to $(\mathsf{R3}_\mathcal{M})$ and $(\mathsf{R4}_\mathcal{M})$. So the only challenging part in the definition of $\otimes^u_\odot$ is the case $S \models \neg\alpha$ and $S \neq \emptyset$. We claim that in this case a revision must have the same impact on a belief state as an update, because the new information is believed to be false. We think that it makes no difference whether the original beliefs have been false or the original beliefs have become false due to some change.

The definition of the translation $\odot^r_\otimes$ follows the general approach of defining update functions model by model. For further explanations, we refer to the work of Winslett [64, 65], Katsuno and Mendelzon [46, 47, 48], as well as Herzig and Rifi [38, 39]. It has turned out that translating revision to update like that is the right way with respect to the KM postulates. However, there is also a disadvantage of this definition. If for some $S \in \mathcal{M}$ and some $\alpha \in \mathcal{L}_0$ we have $S \models \neg\alpha$ and $S \neq \emptyset$, it can happen that $\mathrm{Th}(S \odot^r_\otimes \alpha) \neq \mathrm{Th}(S \otimes \alpha)$. We think that this should not be possible, because of our claim that in this case update and revision should be the same. Therefore, we will redefine this translation in Section 4.2 for our purposes.

The translations from Definition 3.3.1 preserve almost every postulate. Given an AGM revision $\otimes$, its translation $\odot^r_\otimes$ is always a KM update. On the other hand, the translation $\otimes^u_\odot$ of a KM update $\odot$ always satisfies the revision postulates $(\mathsf{R1}_\mathcal{M})$–$(\mathsf{R7}_\mathcal{M})$, but will not necessarily satisfy $(\mathsf{R8}_\mathcal{M})$. The reason for this lacking is that $(\mathsf{R8}_\mathcal{M})$ has no corresponding update postulate. We think that this is not a big disadvantage, because $(\mathsf{R7}_\mathcal{M})$ and $(\mathsf{R8}_\mathcal{M})$ imply that the change is too minimal in some cases. We will discuss this point later in Section 4.2.

Theorem 3.3.2. *We have the following relationships between AGM revisions and KM updates due to the translations of Definition 3.3.1.*

1. *If a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a KM update, then the function $\otimes^u_\odot$ satisfies $(\mathsf{R1}_\mathcal{M})$–$(\mathsf{R7}_\mathcal{M})$.*
2. *If a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an AGM revision, then the function $\odot^r_\otimes$ is a KM update.*

Proof. For the first assertion, we show how to prove that $\otimes^u_\odot$ satisfies $(\mathsf{R7}_\mathcal{M})$. Let $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given. If $S = \emptyset$, then the proof is easy because we have $S \otimes^u_\odot (\alpha \wedge \beta) = \|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\| = (S \otimes^u_\odot \alpha) \oplus \beta$. If $S \neq \emptyset$, then we distinguish three cases. First, if $S \not\models \neg(\alpha \wedge \beta)$, then we also have $S \not\models \neg\alpha$. So the proof is straightforward because we have $S \otimes^u_\odot (\alpha \wedge \beta) = S \oplus (\alpha \wedge \beta) = (S \oplus \alpha) \oplus \beta = (S \otimes^u_\odot \alpha) \oplus \beta$. Second, if $S \models \neg(\alpha \wedge \beta)$ and $S \not\models \neg\alpha$, then we have $S \models \alpha \to \neg\beta$ and therefore we get $S \otimes^u_\odot \alpha = S \oplus \alpha \models \alpha \wedge \neg\beta$. Hence, we have $\mathrm{Th}((S \otimes^u_\odot \alpha) \oplus \beta) = \mathcal{L}_0$, and we are done. In the last case, if $S \models \neg(\alpha \wedge \beta)$ and $S \models \neg\alpha$, then we have

$$\begin{aligned}
\mathrm{Th}(S \otimes^u_\odot (\alpha \wedge \beta)) &= \mathrm{Th}(S \odot (\alpha \wedge \beta)) \\
&\subseteq \mathrm{Th}((S \odot \alpha) \oplus \beta) && \text{by } (\mathsf{U5}_\mathcal{M}) \\
&= \mathrm{Th}((S \otimes^u_\odot \alpha) \oplus \beta).
\end{aligned}$$

For the second assertion, we first show that $\odot^r_\otimes$ satisfies $(\mathsf{U6}_\mathcal{M})$. Let $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$ be given and suppose that $S \odot^r_\otimes \alpha \models \beta$ and $S \odot^r_\otimes \beta \models \alpha$. Then for all $w \in \overline{S}$ we have $\{w\} \otimes \alpha \subseteq \|\beta\|$ and $\{w\} \otimes \beta \subseteq \|\alpha\|$, and we get

$$\begin{aligned}
\mathrm{Th}(S \odot^r_\otimes \alpha) &= \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \otimes \alpha)\Big) \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}(\{w\} \otimes \alpha) && \text{by Lemma 1.1.11} \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}((\{w\} \otimes \alpha) \cap \|\beta\|) && \text{by assumption} \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}(\{w\} \otimes (\alpha \wedge \beta)) && \text{by } (\mathsf{R7}_\mathcal{M}) \text{ and } (\mathsf{R8}_\mathcal{M}) \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}(\{w\} \otimes (\beta \wedge \alpha)) && \text{by } (\mathsf{R6}_\mathcal{M}) \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}((\{w\} \otimes \beta) \cap \|\alpha\|) && \text{by } (\mathsf{R7}_\mathcal{M}) \text{ and } (\mathsf{R8}_\mathcal{M}) \\
&= \bigcap_{w \in \overline{S}} \mathrm{Th}(\{w\} \otimes \beta) && \text{by assumption} \\
&= \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \otimes \beta)\Big) && \text{by Lemma 1.1.11} \\
&= \mathrm{Th}(S \odot^r_\otimes \beta).
\end{aligned}$$

Now, we show that the function $\odot^r_\otimes$ satisfies $(\mathsf{U7}_\mathcal{M})$. Assume $\mathrm{Card}(S) \leq 1$, that is we either have $S = \emptyset$ or $S = \{w\}$ for some $w \in \mathrm{Pow}(\mathcal{P})$. If $S = \emptyset$, then we have $\overline{S} = \emptyset$ by Lemma 1.1.17. Hence, we get $S \odot^r_\otimes (\alpha \vee \beta) = \emptyset$ and $(S \odot^r_\otimes \alpha) \cap (S \odot^r_\otimes \beta) = \emptyset$, and the claim easily follows. If $S = \{w\}$ for some model w, then we have $\overline{S} = \{w\}$ by Lemma 1.1.19. Therefore, by the definition of the translation $\odot^r_\otimes$, we get $S \odot^r_\otimes (\alpha \vee \beta) = \{w\} \otimes (\alpha \vee \beta)$ and $(S \odot^r_\otimes \alpha) \cap (S \odot^r_\otimes \beta) = (\{w\} \otimes \alpha) \cap (\{w\} \otimes \beta)$. We need to show that $\mathrm{Th}(\{w\} \otimes (\alpha \vee \beta)) \subseteq \mathrm{Th}((\{w\} \otimes \alpha) \cap (\{w\} \otimes \beta))$. By $(\mathsf{R2}_\mathcal{M})$, we have $\{w\} \otimes (\alpha \vee \beta) \models \alpha \vee \beta$, and we either get $\{w\} \otimes (\alpha \vee \beta) \not\models \neg\alpha$ or $\{w\} \otimes (\alpha \vee \beta) \not\models \neg\beta$. Without loss of generality, let $\{w\} \otimes (\alpha \vee \beta) \not\models \neg\alpha$. Then we have

$$\begin{aligned}
\mathrm{Th}(\{w\} \otimes (\alpha \vee \beta)) &\subseteq \mathrm{Th}((\{w\} \otimes (\alpha \vee \beta)) \cap \|\alpha\|) && \text{by Lemma 1.1.14} \\
&= \mathrm{Th}((\{w\} \otimes (\alpha \vee \beta)) \oplus \alpha) && \\
&\subseteq \mathrm{Th}(\{w\} \otimes ((\alpha \vee \beta) \wedge \alpha)) && \text{by } (\mathsf{R8}_\mathcal{M}) \\
&= \mathrm{Th}(\{w\} \otimes \alpha) && \text{by } (\mathsf{R6}_\mathcal{M}) \\
&\subseteq \mathrm{Th}((\{w\} \otimes \alpha) \cap (\{w\} \otimes \beta)) && \text{by Lemma 1.1.14.}
\end{aligned}$$

We want to mention that the proof of the first assertion only makes use of $(\mathsf{U0}_\mathcal{M})$, $(\mathsf{U1}_\mathcal{M})$, and $(\mathsf{U3}_\mathcal{M})$–$(\mathsf{U5}_\mathcal{M})$, where $(\mathsf{U5}_\mathcal{M})$ is exclusively used for proving $(\mathsf{R7}_\mathcal{M})$. Furthermore, $(\mathsf{R1}_\mathcal{M})$ is not used in order to prove the second assertion due to the definition of $\odot^r_\otimes$ and Lemma 1.1.19. □

Even if a function $\odot$ satisfies $(\mathsf{U0}_\mathcal{M})$–$(\mathsf{U8}_\mathcal{M})$, it can happen that its translation $\otimes^u_\odot$ from update to revision does not satisfy $(\mathsf{R8}_\mathcal{M})$. The following example illustrates this fact by making use of the *possible models approach* by Winslett [64].

Example 3.3.3. The update function $\odot_{\mathsf{pma}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S \odot_{\mathsf{pma}} \alpha := \bigcup_{w \in \overline{S}} \{v \in \|\alpha\| : \text{for all } u \in \|\alpha\|,\ w \,\Delta\, u \not\subset w \,\Delta\, v\}$$

satisfies $(\mathsf{U0}_\mathcal{M})$–$(\mathsf{U8}_\mathcal{M})$, see Example 2.3.5. Let $\mathcal{P} = \{p, q, r\}$, $S = \{\{q\}, \{r\}\}$, $\alpha = p$, and $\beta = q$. Then we have $S \neq \emptyset$, $S \models \neg\alpha$, and $S \models \neg(\alpha \wedge \beta)$, as well as $\overline{S} = S$ by Corollary 1.1.24. Therefore, we get

$$S \otimes^u_{(\odot_{\mathsf{pma}})} \alpha = S \odot_{\mathsf{pma}} \alpha = \{\{p, q\}, \{p, r\}\} \not\models \neg\beta,$$
$$S \otimes^u_{(\odot_{\mathsf{pma}})} (\alpha \wedge \beta) = S \odot_{\mathsf{pma}} (\alpha \wedge \beta) = \{\{p, q\}, \{p, q, r\}\}.$$

Thus, we have $(S \otimes^u_{(\odot_{\mathsf{pma}})} \alpha) \oplus \beta = \{\{p, q\}\} \not\supseteq S \otimes^u_{(\odot_{\mathsf{pma}})} (\alpha \wedge \beta)$, and we get $\mathrm{Th}((S \otimes^u_{(\odot_{\mathsf{pma}})} \alpha) \oplus \beta) \not\subseteq \mathrm{Th}(S \otimes^u_{(\odot_{\mathsf{pma}})} (\alpha \wedge \beta))$ by Corollary 1.1.24. For instance, we have $\neg r \in \mathrm{Th}((S \otimes^u_{(\odot_{\mathsf{pma}})} \alpha) \oplus \beta)$ but $\neg r \notin \mathrm{Th}(S \otimes^u_{(\odot_{\mathsf{pma}})} (\alpha \wedge \beta))$.

For some later result, the following lemma will be useful. Its proof is directly implied by the proof of Theorem 3.3.2.

Lemma 3.3.4. *If a function* $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, *and* $(\mathsf{U4}_{\mathcal{M}})$, *then the function* $\otimes^u_\odot$ *satisfies* $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$.

In order to get that the functions $\otimes$ and $\otimes^u_{(\odot^r_\otimes)}$ as well as the functions $\odot$ and $\odot^r_{(\otimes^u_\odot)}$ are the same (modulo Th), we need some additional properties, which are quite strong. One reason for this need is the fact that some of the original KM postulates for updates have no corresponding revision postulate and vice versa. Another reason is the following. If the set $\mathcal{P}$ is finite, then $(\mathsf{U8}_{\mathcal{M}})$ implies that a model set S can be updated by separately updating every model in S, and then taking the union. On the other hand, if $\mathcal{P}$ is infinite, we do not have this identity. Therefore, in order to obtain the desired result, we define the missing properties as follows,

$$\mathrm{Th}(S \odot \alpha) = \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \odot \alpha)\Big), \tag{3.3}$$

$$\emptyset \neq S \subseteq \|\neg\alpha\| \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \otimes \alpha)\Big). \tag{3.4}$$

From our point of view, the properties (3.3) and (3.4) are not directly related to the definition of the update and revision process respectively. We have just defined them in order to get the desired interchangeability results. However, in Section 4.2 we will replace the translation $\odot^r_\otimes$ by a new one, and we will get similar results without making use of (3.3) or (3.4).

Lemma 3.3.5. *We have the following interchangeability results.*

1. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the revision postulates* $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, *and* $(\mathsf{R9}_{\mathcal{M}})$ *as well as property* (3.4), *then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes^u_{(\odot^r_\otimes)} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the update postulate* $(\mathsf{U2}_{\mathcal{M}})$ *as well as property* (3.3), *then for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \odot^r_{(\otimes^u_\odot)} \alpha) = \mathrm{Th}(S \odot \alpha)$.

Proof. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. The first assertion can be proved as follows. If $S = \emptyset$, then we have $\mathrm{Th}(S \otimes^u_{(\odot^r_\otimes)} \alpha) = \mathrm{Th}(\|\alpha\|)$ by definition, which is equal to $\mathrm{Th}(S \otimes \alpha)$ by postulate $(\mathsf{R9}_{\mathcal{M}})$. If $S \not\models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^u_{(\odot^r_\otimes)} \alpha) = \mathrm{Th}(S \oplus \alpha)$ by definition, which is the same as $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R3}_{\mathcal{M}})$ and $(\mathsf{R4}_{\mathcal{M}})$. In the last case, if $S \neq \emptyset$ and $S \models \neg\alpha$, then we have

$$\mathrm{Th}(S \otimes^u_{(\odot^r_\otimes)} \alpha) = \mathrm{Th}(S \odot^r_\otimes \alpha) = \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \otimes \alpha)\Big)$$

by definition, which is equal to $\mathrm{Th}(S \otimes \alpha)$ by property (3.4). For the second assertion, we have

$$\begin{aligned}
\mathrm{Th}(S \odot^r_{(\otimes^u_\odot)} \alpha) &= \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \otimes^u_\odot \alpha)\Big) \\
&= \mathrm{Th}\Big(\Big(\bigcup_{w \in \overline{S} \cap \|\alpha\|} (\{w\} \otimes^u_\odot \alpha)\Big) \cup \Big(\bigcup_{w \in \overline{S} \setminus \|\alpha\|} (\{w\} \otimes^u_\odot \alpha)\Big)\Big) \\
&= \mathrm{Th}\Big(\Big(\bigcup_{w \in \overline{S} \cap \|\alpha\|} (\{w\} \oplus \alpha)\Big) \cup \Big(\bigcup_{w \in \overline{S} \setminus \|\alpha\|} (\{w\} \odot \alpha)\Big)\Big) \\
&= \mathrm{Th}\Big(\Big(\bigcup_{w \in \overline{S} \cap \|\alpha\|} \{w\}\Big) \cup \Big(\bigcup_{w \in \overline{S} \setminus \|\alpha\|} (\{w\} \odot \alpha)\Big)\Big)
\end{aligned}$$

by definition. Applying Lemma 1.1.11 and ($\mathsf{U2}_{\mathcal{M}}$), we get

$$\begin{aligned}
&= \Big(\bigcap_{w \in \overline{S} \cap \|\alpha\|} \Big(\mathrm{Th}(\{w\})\Big)\Big) \cap \Big(\bigcap_{w \in \overline{S} \setminus \|\alpha\|} \Big(\mathrm{Th}(\{w\} \odot \alpha)\Big)\Big) \\
&= \Big(\bigcap_{w \in \overline{S} \cap \|\alpha\|} \Big(\mathrm{Th}(\{w\} \odot \alpha)\Big)\Big) \cap \Big(\bigcap_{w \in \overline{S} \setminus \|\alpha\|} \Big(\mathrm{Th}(\{w\} \odot \alpha)\Big)\Big) \\
&= \mathrm{Th}\Big(\Big(\bigcup_{w \in \overline{S} \cap \|\alpha\|} (\{w\} \odot \alpha)\Big) \cup \Big(\bigcup_{w \in \overline{S} \setminus \|\alpha\|} (\{w\} \odot \alpha)\Big)\Big) \\
&= \mathrm{Th}\Big(\bigcup_{w \in \overline{S}} (\{w\} \odot \alpha)\Big),
\end{aligned}$$

which is the same as $\mathrm{Th}(S \odot \alpha)$ by property (3.3). □

If the set $\mathcal{P}$ is finite, then we have that property (3.3) is equivalent to ($\mathsf{U8}_{\mathcal{M}}$), as we state in the following lemma.

Lemma 3.3.6. *Let $\mathcal{P}$ be finite and the function $\odot \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then we have that $\odot$ satisfies* (3.3) *if and only if $\odot$ satisfies* ($\mathsf{U8}_{\mathcal{M}}$).

Proof. The fact that property (3.3) implies ($\mathsf{U8}_{\mathcal{M}}$) is an immediate consequence of Lemma 1.1.17 and even holds for infinite $\mathcal{P}$. We show how to prove the converse direction. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. Since $\mathcal{P}$ is finite, we also get that $S \subseteq \mathrm{Pow}(\mathcal{P})$ is finite. In addition, we have that $\overline{S} = S$ by Corollary 1.1.24. Therefore, it will be enough to prove

$$\mathrm{Th}(S \odot \alpha) = \mathrm{Th}\Big(\bigcup_{w \in S} (\{w\} \odot \alpha)\Big).$$

This can be done by induction on $\mathrm{Card}(S)$, where ($\mathsf{U8}_{\mathcal{M}}$) is only used in the induction step. □

The following result is directly implied by Lemma 3.3.5 and Lemma 3.3.6. There is no similar result for revision functions.

Corollary 3.3.7. *Let $\mathcal{P}$ be finite. If a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies* ($\mathsf{U2}_{\mathcal{M}}$) *and* ($\mathsf{U8}_{\mathcal{M}}$)*, then for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$\mathrm{Th}(S \odot^{r}_{(\otimes^{u}_{\odot})} \alpha) = \mathrm{Th}(S \odot \alpha).$$

The functions $\otimes_{\mathsf{fm}}$ and $\odot_{\mathsf{pma}}$ from Example 2.1.6 and Example 2.3.5 respectively satisfy all of the conditions we need to apply Lemma 3.3.5.

Example 3.3.8. The revision function $\otimes_{\mathsf{fm}}$ satisfies property (3.4) as well as ($\mathsf{R9}_{\mathcal{M}}$) by definition. In addition, we have seen in Example 2.1.6 that $\otimes_{\mathsf{fm}}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R8}_{\mathcal{M}}$). Therefore, we can apply Lemma 3.3.5 and for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(S \otimes^{u}_{(\odot^{r}_{(\otimes_{\mathsf{fm}})})} \alpha) = \mathrm{Th}(S \otimes_{\mathsf{fm}} \alpha).$$

The update function $\odot_{\mathsf{pma}}$ satisfies property (3.3) by definition. Moreover, from Example 2.3.5 we know that $\odot_{\mathsf{pma}}$ satisfies ($\mathsf{U0}_{\mathcal{M}}$)–($\mathsf{U8}_{\mathcal{M}}$). Now, by Lemma 3.3.5, we get that for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(S \odot^{r}_{(\otimes^{u}_{(\odot_{\mathsf{pma}})})} \alpha) = \mathrm{Th}(S \odot_{\mathsf{pma}} \alpha).$$

We are now going to explore the relationship between revision and update in the context of consistent model sets.

Definition 3.3.9. Given a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, its translation from update to revision is defined as follows,

$$\otimes^{cu}_{\odot}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S & \text{if } \models \neg\alpha, \\ S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ S \odot \alpha & \text{otherwise.} \end{cases}$$

The translation of a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ from revision to update is defined by

$$\odot^{cr}_{\otimes}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} S & \text{if } \models \neg\alpha, \\ \bigcup_{w \in \overline{S}} (\{w\} \otimes \alpha) & \text{otherwise.} \end{cases}$$

Observe that if a function $\odot$ satisfies ($\mathsf{U3}_{\mathcal{M}^c}$), then the function $\otimes^{cu}_{\odot}$ obviously satisfies property (3.1). This is also the case for the function $\odot^{cr}_{\otimes}$, whenever a given function $\otimes$ satisfies ($\mathsf{R5}_{\mathcal{M}^c}$). Like in Lemma 3.2.9, the translations from Definition 3.3.9 are just compositions of some earlier defined translations. In order to see this, we state the following lemma.

Lemma 3.3.10. *Let the function $\circledast\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ be given. Then for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \otimes^{cu}_{\circledast} \alpha = S\, (\otimes^{u}_{(\circledast^{iu})})^{cr}\, \alpha, \qquad\qquad S \odot^{cr}_{\circledast} \alpha = S\, (\odot^{r}_{(\circledast^{ir})})^{cu}\, \alpha.$$

Proof. Both equalities directly follow from Definition 3.1.5, Definition 3.1.17, and Definition 3.3.1. □

The following result is the same as Theorem 3.3.2, but in the context of consistent model sets. Again, the function $\otimes^{cu}_{\odot}$ is not necessarily a consistent AGM revision, if a given function $\odot$ is a consistent KM update. But ($\mathsf{R8}_{\mathcal{M}^c}$) is the only postulate that does not always hold.

Theorem 3.3.11. *We have the following relationships between consistent AGM revisions and consistent KM updates due to the translations of Definition 3.3.9.*

1. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent KM update, then the function $\otimes^{cu}_{\odot}$ satisfies* ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), ($\mathsf{R6}_{\mathcal{M}}$), *and* ($\mathsf{R7}_{\mathcal{M}}$).
2. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent AGM revision, then the function $\odot^{cr}_{\otimes}$ is a consistent KM update.*

Proof. For the first assertion, we can apply Lemma 3.1.18 and Theorem 3.3.2 in order to show that the function $\otimes^{u}_{\odot^{iu}}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R7}_{\mathcal{M}}$). Now, by Lemma 3.1.7, we get that the function $(\otimes^{u}_{(\odot^{iu})})^{cr}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), ($\mathsf{R6}_{\mathcal{M}}$), and ($\mathsf{R7}_{\mathcal{M}}$). By Lemma 3.3.10, we get that this is the same function as $\otimes^{cu}_{\odot}$. For the second assertion, we can use Lemma 3.1.6, Theorem 3.3.2, and Lemma 3.1.18 to prove that the function $(\odot^{r}_{(\otimes^{ir})})^{cu}$ is a consistent KM update. Due to Lemma 3.3.10, we know that this function is the same as $\odot^{cr}_{\otimes}$. □

We will now give an example for a consistent KM update $\odot$, such that its translation $\otimes^{cu}_{\odot}$ does not satisfy ($\mathsf{R8}_{\mathcal{M}^c}$). The following example is based on Example 3.3.3.

Example 3.3.12. The update function $\odot_{\mathsf{pma}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies all of the update postulates, see Example 2.3.5. Therefore, the translated function $(\odot_{\mathsf{pma}})^{cu} \colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, which is given by

$$S\ (\odot_{\mathsf{pma}})^{cu}\ \alpha := \begin{cases} S \odot_{\mathsf{pma}} \alpha & \text{if } \not\models \neg\alpha, \\ S & \text{otherwise}, \end{cases}$$

is a consistent KM update by Lemma 3.1.18. Again, we take $\mathcal{P} = \{p, q, r\}$, $S = \{\{q\}, \{r\}\}$, $\alpha = p$, and $\beta = q$, like in Example 3.3.3. Then we have $\not\models \neg\alpha$, $\not\models \neg(\alpha \wedge \beta)$, $S \models \neg\alpha$, as well as $S \models \neg(\alpha \wedge \beta)$, and we get

$$S \otimes^{cu}_{(\odot_{\mathsf{pma}})^{cu}} \alpha = S\ (\odot_{\mathsf{pma}})^{cu}\ \alpha = S \odot_{\mathsf{pma}} \alpha,$$
$$S \otimes^{cu}_{(\odot_{\mathsf{pma}})^{cu}} (\alpha \wedge \beta) = S\ (\odot_{\mathsf{pma}})^{cu}\ (\alpha \wedge \beta) = S \odot_{\mathsf{pma}} (\alpha \wedge \beta).$$

Therefore, we now have $\mathrm{Th}((S \otimes^{cu}_{(\odot_{\mathsf{pma}}{}^{cu})} \alpha) \oplus \beta) \nsubseteq \mathrm{Th}(S \otimes^{cu}_{(\odot_{\mathsf{pma}})^{cu}} (\alpha \wedge \beta))$ by Example 3.3.3, hence $(\mathsf{R8}_{\mathcal{M}^c})$ is not satisfied.

The following lemma will be useful for some further results. It is the restriction of the first assertion in Theorem 3.3.11 to a subset of postulates.

Lemma 3.3.13. *If a function* $\odot \colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}^c})$, $(\mathsf{U3}_{\mathcal{M}^c})$, *and* $(\mathsf{U4}_{\mathcal{M}})$, *then the function* $\otimes^{cu}_{\odot}$ *satisfies* $(\mathsf{R1}_{\mathcal{M}})$, $(\mathsf{R2}_{\mathcal{M}^c})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R5}_{\mathcal{M}^c})$, *and* $(\mathsf{R6}_{\mathcal{M}})$.

Proof. By applying Lemma 3.1.19, Lemma 3.3.4, and Lemma 3.1.7, we get that the function $(\otimes^{u}_{(\odot^{iu})})^{cr}$ satisfies $(\mathsf{R1}_{\mathcal{M}})$, $(\mathsf{R2}_{\mathcal{M}^c})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R5}_{\mathcal{M}^c})$, and $(\mathsf{R6}_{\mathcal{M}})$. But this is the same function as $\otimes^{cu}_{\odot}$ by Lemma 3.3.10. □

Like Lemma 3.3.5, the following result is a consequence of the properties (3.3) and (3.4).

Lemma 3.3.14. *We have the following interchangeability results.*

1. *If a function* $\otimes \colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the postulates* $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, *and* $(\mathsf{R9}_{\mathcal{M}^c})$ *as well as property* (3.4), *then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes^{cu}_{(\odot^{cr}_{\otimes})} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\odot \colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies the postulates* $(\mathsf{U2}_{\mathcal{M}})$ *and* $(\mathsf{U9}_{\mathcal{M}^c})$ *as well as property* (3.3), *then for all* $S \in \mathcal{M}^c$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \odot^{cr}_{(\otimes^{cu}_{\odot})} \alpha) = \mathrm{Th}(S \odot \alpha)$.

Proof. Let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given. For both assertions we have exactly the same proof as of Lemma 3.3.5, except for the case $\models \neg\alpha$. In this case, we first have $\mathrm{Th}(S \otimes^{cu}_{(\odot^{cr}_{\otimes})} \alpha) = \mathrm{Th}(S)$ by definition, which is equal to $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R9}_{\mathcal{M}^c})$. Second, we have $\mathrm{Th}(S \odot^{cr}_{(\otimes^{cu}_{\odot})} \alpha) = \mathrm{Th}(S)$ by definition, and this is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U9}_{\mathcal{M}^c})$. □

The following result follows from Lemma 3.3.14 and Lemma 3.3.6. We have no similar result for revision functions on consistent model sets.

Corollary 3.3.15. *Let $\mathcal{P}$ be finite. If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies* ($\mathsf{U2}_{\mathcal{M}}$)*,* ($\mathsf{U8}_{\mathcal{M}}$)*, and* ($\mathsf{U9}_{\mathcal{M}^c}$)*, then for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$\mathrm{Th}(S \odot^{cr}_{(\otimes^{cu}_{\odot})} \alpha) = \mathrm{Th}(S \odot \alpha).$$

We conclude this section by discussing some common properties for revision and update functions. From our point of view, there are differences in the revision and update postulates that should not occur. For instance, given the inconsistent model set $S = \emptyset$ and a satisfiable formula α, we have

$$\begin{aligned} S \otimes \alpha &\neq \emptyset && \text{by } (\mathsf{R5}_{\mathcal{M}}), \\ S \odot \alpha &= \emptyset && \text{by } (\mathsf{U2}_{\mathcal{M}}). \end{aligned}$$

Why should the inconsistent belief state be differently revised than updated? We have not found any plausible explanation for this difference in the literature. Like ($\mathsf{R9}_{\mathcal{M}}$) for revision, we think that an update of the inconsistent belief state with a formula α should result in the model set $\|\alpha\|$,

$$S = \emptyset \ \Rightarrow\ \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(\|\alpha\|). \tag{3.5}$$

As a consequence of requirement (3.5), we come to the conclusion that the condition to end up in an inconsistent state has to be the same in both revision and update theory,

$$\models \neg\alpha \ \Leftrightarrow\ S \otimes \alpha = \emptyset \ \Leftrightarrow\ S \odot \alpha = \emptyset. \tag{3.6}$$

The following common behaviour of revision and update functions is a consequence of ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), and ($\mathsf{U2}_{\mathcal{M}}$),

$$S \models \alpha \text{ and } S \neq \emptyset \ \Rightarrow\ \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S), \tag{3.7}$$

and we believe that this is a suitable interpretation of minimal change. Last but not least, if the new information contradicts our beliefs, there is no connection between revision and update functions according to the original sets of postulates. In this case, we propose that the resulting belief state should again be the same,

$$S \models \neg\alpha \ \Rightarrow\ \mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S \odot \alpha). \tag{3.8}$$

So the only difference between revision and update functions occurs if $S \not\models \alpha$ and $S \not\models \neg\alpha$. Then we have $\mathrm{Th}(S \otimes \alpha) = \mathrm{Th}(S \oplus \alpha)$ by ($\mathsf{R3}_{\mathcal{M}}$) and ($\mathsf{R4}_{\mathcal{M}}$), and $\mathrm{Th}(S \odot \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha))$ by ($\mathsf{U2}_{\mathcal{M}}$) and ($\mathsf{U8}_{\mathcal{M}}$). In Section 4.2 we will change some of the update postulates so that the properties (3.5), (3.6), (3.7), and (3.8) are all satisfied.

Chapter 4

New functions

This chapter provides new belief change functions that correspond to our understanding of minimal change. In Section 4.1 we will define three variants of the standard update function and we will investigate which update postulates they satisfy. In addition, we will define the notion of strength of a belief change function and show that the new functions are all stronger than the standard update function. We will compare their strength to the strength of other update functions like the possible models approach, as well. Finally, we will translate the new update functions into revision functions and verify which revision postulates they satisfy. In Section 4.2 we will commit ourselves to new sets of postulates for revision, contraction, and update functions. The new correspondences between these sets of postulates will then be presented. They do not differ very much from the relationships between the original sets of postulates. Moreover, we will give examples for revision, contraction, and update functions that satisfy the new sets of postulates. Instead of "minimal change" we will introduce the notion of "minimax change", and we will argue that it leads to more reasonable belief change functions.

4.1 New update functions

From our point of view, the syntax independent standard update function $\odot_{\mathsf{su}}$ from Definition 2.3.9 does the right thing if the new information contradicts our beliefs. The main reason for this preference is the fact that there is no problem with disjunctive input, see [39] for a detailed discussion. But if the new information is consistent with our beliefs, we think that the amount of change performed by the function $\odot_{\mathsf{su}}$ is far too big. Especially, if we already believe that the new information is true, why should we change our beliefs? We believe that the only problem is given by the fact that $\odot_{\mathsf{su}}$ does not satisfy

($\mathsf{U2}_{\mathcal{M}}$), which we have proved in Lemma 2.3.10. In order to fix this lacking, we propose three variants of the standard update function, which all satisfy ($\mathsf{U2}_{\mathcal{M}}$). The *selective standard update* function $\odot_{\mathsf{ssu}}$ solves the problem by performing a simple case distinction. The *partial standard update* function $\odot_{\mathsf{psu}}$ only operates on one part of a given model set. The *cautious standard update* function $\odot_{\mathsf{csu}}$ only considers models that are "compatible" with the original beliefs. We want to mention that we use our personal understanding of compatibility in this definition.

Definition 4.1.1. The new functions $\odot_{\mathsf{ssu}}, \odot_{\mathsf{psu}}, \odot_{\mathsf{csu}} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ are defined as follows,

$$S \odot_{\mathsf{ssu}} \alpha := \begin{cases} S & \text{if } S \vDash \alpha, \\ S \odot_{\mathsf{su}} \alpha & \text{otherwise,} \end{cases}$$

$$S \odot_{\mathsf{psu}} \alpha := (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha),$$

$$S \odot_{\mathsf{csu}} \alpha := \begin{cases} (S \oplus \alpha) \cup \{w \in (\overline{S} \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha : \text{for some} \\ \quad v \in \overline{S} \oplus \alpha,\ w \cap \operatorname{atm}^\sharp(\alpha) = v \cap \operatorname{atm}^\sharp(\alpha)\} & \text{if } S \nvDash \neg\alpha, \\ S \odot_{\mathsf{su}} \alpha & \text{otherwise.} \end{cases}$$

Observe that the case $S \nvDash \neg\alpha$ in the definition of the function $\odot_{\mathsf{csu}}$ encodes our intuition about a model of the formula α being compatible with the model set S.

Lemma 4.1.2. *The functions $\odot_{\mathsf{ssu}}$ and $\odot_{\mathsf{csu}}$ only satisfy* ($\mathsf{U0}_{\mathcal{M}}$)–($\mathsf{U4}_{\mathcal{M}}$)*, whereas the function $\odot_{\mathsf{psu}}$ additionally satisfies* ($\mathsf{U8}_{\mathcal{M}}$).

Proof. We prove that $\odot_{\mathsf{psu}}$ satisfies ($\mathsf{U0}_{\mathcal{M}}$). For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\begin{aligned}
\operatorname{Th}(\overline{S} \odot_{\mathsf{psu}} \alpha) &= \operatorname{Th}((\overline{S} \oplus \alpha) \cup ((\overline{S} \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha)) \\
&= \operatorname{Th}(\overline{S} \oplus \alpha) \cap \operatorname{Th}((\overline{S} \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) && \text{by Lemma 1.1.11} \\
&= \operatorname{Th}(\overline{S} \oplus \alpha) \cap \operatorname{Th}(\overline{\overline{S} \setminus \|\alpha\|} \odot_{\mathsf{su}} \alpha) && \text{by Lemma 2.3.10} \\
&= \operatorname{Th}(\overline{S} \oplus \alpha) \cap \operatorname{Th}(\overline{\overline{S} \oplus \neg\alpha} \odot_{\mathsf{su}} \alpha) \\
&= \operatorname{Th}(S \oplus \alpha) \cap \operatorname{Th}(\overline{S \oplus \neg\alpha} \odot_{\mathsf{su}} \alpha) && \text{by Lemma 1.2.6} \\
&= \operatorname{Th}(S \oplus \alpha) \cap \operatorname{Th}(\overline{S \setminus \|\alpha\|} \odot_{\mathsf{su}} \alpha) \\
&= \operatorname{Th}(S \oplus \alpha) \cap \operatorname{Th}((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) && \text{by Lemma 2.3.10} \\
&= \operatorname{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha)) && \text{by Lemma 1.1.11} \\
&= \operatorname{Th}(S \odot_{\mathsf{psu}} \alpha).
\end{aligned}$$

The proofs of the other postulates and for the other functions are similar or a direct consequence of Lemma 2.3.10. In order to see that the functions $\odot_{\mathsf{ssu}}$, $\odot_{\mathsf{psu}}$, and $\odot_{\mathsf{csu}}$ all do not satisfy ($\mathsf{U5}_{\mathcal{M}}$), ($\mathsf{U6}_{\mathcal{M}}$), and ($\mathsf{U7}_{\mathcal{M}}$), we can use exactly the same counterexamples as for the function $\odot_{\mathsf{su}}$ in the proof of Lemma 2.3.10. We conclude this proof by showing that the functions $\odot_{\mathsf{ssu}}$ and $\odot_{\mathsf{csu}}$ both do not satisfy ($\mathsf{U8}_{\mathcal{M}}$). Let $\mathcal{P} = \{p, q, r\}$, $S = \{\{p\}\}$, $S' = \{\{r\}\}$, and $\alpha = p \vee q$. Then we obviously have $S \vDash \alpha$, $S' \nvDash \alpha$, and $S \cup S' \nvDash \alpha$, and we immediately get $(S \odot_{\mathsf{ssu}} \alpha) \cup (S' \odot_{\mathsf{ssu}} \alpha) = \{\{p\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$ and $(S \cup S') \odot_{\mathsf{ssu}} \alpha = \{\{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$. We get that ($\mathsf{U8}_{\mathcal{M}}$) is violated because we have $q \rightarrow r \in \mathrm{Th}((S \odot_{\mathsf{ssu}} \alpha) \cup (S' \odot_{\mathsf{ssu}} \alpha))$ but $q \rightarrow r \notin \mathrm{Th}((S \cup S') \odot_{\mathsf{ssu}} \alpha)$. On the other hand, we have $S \nvDash \neg\alpha$, $S' \vDash \neg\alpha$, and $(S \cup S') \nvDash \neg\alpha$, hence $(S \odot_{\mathsf{csu}} \alpha) \cup (S' \odot_{\mathsf{csu}} \alpha) = \{\{p\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$ and $(S \cup S') \odot_{\mathsf{csu}} \alpha = \{\{p\}, \{p, r\}\}$. We can see that ($\mathsf{U8}_{\mathcal{M}}$) is not satisfied because we have $\neg q \in \mathrm{Th}((S \cup S') \odot_{\mathsf{csu}} \alpha)$ but $\neg q \notin \mathrm{Th}((S \odot_{\mathsf{csu}} \alpha) \cup (S' \odot_{\mathsf{csu}} \alpha))$. □

There is a property that all update and revision functions have in common: for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ the models in $S \cap \|\neg\alpha\|$ are dropped. This is a direct consequence of ($\mathsf{U1}_{\mathcal{M}}$) and ($\mathsf{R2}_{\mathcal{M}}$) respectively. The only action that gives rise to different update and revision functions is the adding of some new models of α. Therefore, the update and revision functions can easily be ordered by set inclusion, see Herzig and Rifi [39] for the comparative strength of update functions on belief bases. Our way of ordering the functions is slightly different because of the notion of model sets.

Definition 4.1.3. Let the functions $\circledast, \circledast' \colon \mathcal{M} \times \mathcal{L}_0 \rightarrow \mathcal{M}$ be given. Then we say that the function $\circledast$ is *at least as strong as* $\circledast'$, denoted by $\circledast \lesssim \circledast'$, if for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(S \circledast \alpha) \supseteq \mathrm{Th}(S \circledast' \alpha).$$

We write $\circledast \approx \circledast'$ to express that the two functions have the *same strength*, and $\circledast < \circledast'$ to denote that the function $\circledast$ is *stronger* than $\circledast'$. These notions are defined by

$$\begin{aligned} \circledast \approx \circledast' \; &:\Leftrightarrow \; \circledast \lesssim \circledast' \text{ and } \circledast' \lesssim \circledast, \\ \circledast < \circledast' \; &:\Leftrightarrow \; \circledast \lesssim \circledast' \text{ and } \circledast \not\approx \circledast'. \end{aligned}$$

We have chosen the symbol $<$ ("less than") in order to indicate that a stronger function performs less change. That is, the stronger update or revision is adding less models to a model set. The next lemma states that the relation $\lesssim$ is a *preorder* (*reflexive* and *transitive*), $\approx$ is an *equivalence relation* (*reflexive* and *Euclidean*), and $<$ is a *strict partial order* (*irreflexive* and *transitive*).

Lemma 4.1.4. *For all functions* $\circledast, \circledast', \circledast'' \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *we have*

$$\begin{array}{ll} \circledast \lesssim \circledast, & \circledast \lesssim \circledast' \text{ and } \circledast' \lesssim \circledast'' \;\Rightarrow\; \circledast \lesssim \circledast'', \\ \circledast \approx \circledast, & \circledast \approx \circledast' \text{ and } \circledast \approx \circledast'' \;\Rightarrow\; \circledast' \approx \circledast'', \\ \circledast \not< \circledast, & \circledast < \circledast' \text{ and } \circledast' < \circledast'' \;\Rightarrow\; \circledast < \circledast''. \end{array}$$

Proof. Since the relation $\supseteq$ is reflexive and transitive, we easily get the same properties for the relation $\lesssim$. The properties of the relations $\approx$ and $<$ now directly follow from the reflexivity and transitivity of $\lesssim$. □

Note that if for some $S \in \mathcal{M}$ and some $\alpha \in \mathcal{L}_0$ we have $S \vDash \alpha$, then we get

$$S \odot_{\mathsf{csu}} \alpha = S \odot_{\mathsf{psu}} \alpha = S \odot_{\mathsf{ssu}} \alpha = S$$

by Definition 4.1.1. Moreover, it is easy to see that if $S \vDash \neg\alpha$, then we have

$$S \odot_{\mathsf{csu}} \alpha = S \odot_{\mathsf{psu}} \alpha = S \odot_{\mathsf{ssu}} \alpha = S \odot_{\mathsf{su}} \alpha.$$

So the new functions can only perform a different update if $S \nvDash \alpha$ and $S \nvDash \neg\alpha$. Indeed, we can show that the update functions we have met so far all have different strength, including $\odot_{\mathsf{pma}}$. In addition, all but two are comparable with respect to $\lesssim$, as we show in the following theorem.

Theorem 4.1.5. *We have the following comparability results,*

1. $\odot_{\mathsf{csu}} < \odot_{\mathsf{psu}} < \odot_{\mathsf{ssu}} < \odot_{\mathsf{su}}$,
2. $\odot_{\mathsf{pma}} < \odot_{\mathsf{psu}}$,
3. $\odot_{\mathsf{pma}} \not\lesssim \odot_{\mathsf{csu}}$ *and* $\odot_{\mathsf{csu}} \not\lesssim \odot_{\mathsf{pma}}$.

Proof. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. For the first assertion, we will show how to prove $\odot_{\mathsf{psu}} \lesssim \odot_{\mathsf{ssu}}$. Due to Lemma 1.1.14, it will do to prove $S \odot_{\mathsf{psu}} \alpha \subseteq \overline{S \odot_{\mathsf{ssu}} \alpha}$. We distinguish two cases. First, if $S \vDash \alpha$, then we have

$$S \odot_{\mathsf{psu}} \alpha = S \oplus \alpha = S = S \odot_{\mathsf{ssu}} \alpha \subseteq \overline{S \odot_{\mathsf{ssu}} \alpha}$$

by Lemma 1.1.14. Second, if $S \nvDash \alpha$, then we get

$$\begin{aligned} S \odot_{\mathsf{psu}} \alpha &= (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) \\ &\subseteq ((S \oplus \alpha) \odot_{\mathsf{su}} \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) && \text{by Corollary 2.3.12} \\ &\subseteq \overline{((S \oplus \alpha) \odot_{\mathsf{su}} \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha)} && \text{by Lemma 1.1.14} \\ &= \overline{((S \oplus \alpha) \cup (S \setminus \|\alpha\|)) \odot_{\mathsf{su}} \alpha} && \text{by Lemma 2.3.10} \\ &= \overline{S \odot_{\mathsf{su}} \alpha}, \end{aligned}$$

and this is equal to $\overline{S \odot_{\mathsf{ssu}} \alpha}$ by definition. The proofs of $\odot_{\mathsf{csu}} \lesssim \odot_{\mathsf{psu}}$ and $\odot_{\mathsf{ssu}} \lesssim \odot_{\mathsf{su}}$ are similar, so we have $\odot_{\mathsf{csu}} \lesssim \odot_{\mathsf{psu}} \lesssim \odot_{\mathsf{ssu}} \lesssim \odot_{\mathsf{su}}$. The following examples imply that these four functions all have different strength.

- Let $\mathcal{P} = \{p, q\}$, $S = \{\emptyset, \{p\}\}$, and $\alpha = p \vee q$. Then we can easily get $S \odot_{\mathsf{csu}} \alpha = \{\{p\}\}$ and $S \odot_{\mathsf{psu}} \alpha = \{\{p\}, \{q\}, \{p, q\}\}$. Therefore, we have $\neg q \in \mathrm{Th}(S \odot_{\mathsf{csu}} \alpha)$ but $\neg q \notin \mathrm{Th}(S \odot_{\mathsf{psu}} \alpha)$.
- Let $\mathcal{P} = \{p, q, r\}$, $S = \{\{p\}, \{r\}\}$, and $\alpha = p \vee q$. Then we immediately get $S \odot_{\mathsf{psu}} \alpha = \{\{p\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$. On the other hand, we get $S \odot_{\mathsf{ssu}} \alpha = \{\{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\}$ and we can easily see that $q \rightarrow r \in \mathrm{Th}(S \odot_{\mathsf{psu}} \alpha)$ but $q \rightarrow r \notin \mathrm{Th}(S \odot_{\mathsf{ssu}} \alpha)$.
- Let $\mathcal{P} = \{p, q\}$, $S = \{\{p\}\}$, and $\alpha = p \vee q$. Then we directly get $S \odot_{\mathsf{ssu}} \alpha = \{\{p\}\}$ and $S \odot_{\mathsf{su}} \alpha = \{\{p\}, \{q\}, \{p, q\}\}$. We can now observe that $\neg q \in \mathrm{Th}(S \odot_{\mathsf{ssu}} \alpha)$ but $\neg q \notin \mathrm{Th}(S \odot_{\mathsf{su}} \alpha)$.

We have now also shown $\odot_{\mathsf{csu}} \not\approx \odot_{\mathsf{psu}}$, $\odot_{\mathsf{psu}} \not\approx \odot_{\mathsf{ssu}}$, and $\odot_{\mathsf{ssu}} \not\approx \odot_{\mathsf{su}}$. For the second assertion, we will now prove $\odot_{\mathsf{pma}} \lesssim \odot_{\mathsf{psu}}$. It will be sufficient to show $S \odot_{\mathsf{pma}} \alpha \subseteq \overline{S \odot_{\mathsf{psu}} \alpha}$. We take any $v \in S \odot_{\mathsf{pma}} \alpha$ and distinguish two cases. First, if $v \in \overline{S}$, then we have $v \in \overline{S \oplus \alpha}$ by Lemma 1.2.6 and because $\odot_{\mathsf{pma}}$ satisfies $(\mathsf{U1}_{\mathcal{M}})$. Applying the fact that $S \oplus \alpha \subseteq S \odot_{\mathsf{psu}} \alpha$, we also get that $v \in \overline{S \odot_{\mathsf{psu}} \alpha}$ by Lemma 1.1.16. Second, if $v \notin \overline{S}$, then for some $w \in \overline{S}$ and for all $u \in \|\alpha\|$ we have $w \,\Delta\, u \not\subset w \,\Delta\, v$ by the definition of $\odot_{\mathsf{pma}}$ in Example 2.3.5. Let us suppose that $w \,\Delta\, v \not\subseteq \mathrm{atm}^\sharp(\alpha)$. We construct a model v' by changing in v the value of the propositions in $w \,\Delta\, v \setminus \mathrm{atm}^\sharp(\alpha)$. Obviously, we get $v' \in \|\alpha\|$ and $w \,\Delta\, v' \subset w \,\Delta\, v$, which is a contradiction. Therefore, we have $w \,\Delta\, v \subseteq \mathrm{atm}^\sharp(\alpha)$, that is $v \in S \odot_{\mathsf{psu}} \alpha$, hence $v \in \overline{S \odot_{\mathsf{psu}} \alpha}$ by Lemma 1.1.14. In order to prove $\odot_{\mathsf{pma}} \not\approx \odot_{\mathsf{psu}}$, let $\mathcal{P} = \{p, q\}$, $S = \{\emptyset\}$, and $\alpha = p \vee q$. Then we have $S \odot_{\mathsf{pma}} \alpha = \{\{p\}, \{q\}\}$ and $S \odot_{\mathsf{psu}} \alpha = \{\{p\}, \{q\}, \{p, q\}\}$. It is now easy to see that $\neg(p \wedge q) \in \mathrm{Th}(S \odot_{\mathsf{pma}} \alpha)$ but $\neg(p \wedge q) \notin \mathrm{Th}(S \odot_{\mathsf{psu}} \alpha)$. For the examples that prove the third assertion, we refer to the table of update results in Appendix A. □

Now, we have two ways of measuring the amount of change that belief change functions perform: the update postulates and the relation $\lesssim$. Lemma 4.1.2 and Theorem 4.1.5 show that these two measures are not compatible. According to the KM postulates for updates, the function $\odot_{\mathsf{psu}}$ is closer to the notion of minimal change than the functions $\odot_{\mathsf{csu}}$ and $\odot_{\mathsf{ssu}}$, because it satisfies one more postulate. On the other hand, with respect to $\lesssim$, the function $\odot_{\mathsf{csu}}$ performs less change than $\odot_{\mathsf{psu}}$ and $\odot_{\mathsf{ssu}}$.

Remark 4.1.6. According to the KM postulates, the possible models approach $\odot_{\mathsf{pma}}$ satisfies the conditions for minimal change, whereas the cautious standard update $\odot_{\mathsf{csu}}$ only satisfies four of the eight original postulates (remember that $(\mathsf{U0}_{\mathcal{M}})$ and $(\mathsf{U4}_{\mathcal{M}})$ have originally been stated in one postulate). Due to Theorem 4.1.5, we have that the functions $\odot_{\mathsf{csu}}$ and $\odot_{\mathsf{pma}}$

are not comparable with respect to the relation $\lesssim$. However, we believe that "in average", the amount of change performed by $\odot_{\mathsf{csu}}$ is less than the one performed by $\odot_{\mathsf{pma}}$. Since it is not clear how the "average amount of change" can be defined, we will discuss some empirical results in Appendix A.

We can show that with respect to $\lesssim$, the function $\odot_{\mathsf{psu}}$ is maximal in the following sense. All functions that are stronger than $\odot_{\mathsf{su}}$ and satisfy $(\mathsf{U2}_{\mathcal{M}})$ and $(\mathsf{U8}_{\mathcal{M}})$ are at least as strong as $\odot_{\mathsf{psu}}$.

Lemma 4.1.7. *If a function* $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *satisfies* $(\mathsf{U2}_{\mathcal{M}})$ *and* $(\mathsf{U8}_{\mathcal{M}})$*, then we have*

$$\odot \lesssim \odot_{\mathsf{su}} \;\Rightarrow\; \odot \lesssim \odot_{\mathsf{psu}}.$$

Proof. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. Then we have

$$\begin{aligned}
\mathrm{Th}(S \odot \alpha) &= \mathrm{Th}((S \cap \|\alpha\|) \cup (S \setminus \|\alpha\|) \odot \alpha) \\
&= \mathrm{Th}(((S \cap \|\alpha\|) \odot \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha)) && \text{by } (\mathsf{U8}_{\mathcal{M}}) \\
&= \mathrm{Th}((S \cap \|\alpha\|) \odot \alpha) \cap \mathrm{Th}((S \setminus \|\alpha\|) \odot \alpha) && \text{by Lemma 1.1.11} \\
&= \mathrm{Th}(S \cap \|\alpha\|) \cap \mathrm{Th}((S \setminus \|\alpha\|) \odot \alpha) && \text{by } (\mathsf{U2}_{\mathcal{M}}) \\
&\supseteq \mathrm{Th}(S \cap \|\alpha\|) \cap \mathrm{Th}((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) && \text{by assumption} \\
&= \mathrm{Th}((S \cap \|\alpha\|) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha)) && \text{by Lemma 1.1.11} \\
&= \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha)),
\end{aligned}$$

which is the same as $\mathrm{Th}(S \odot_{\mathsf{psu}} \alpha)$. □

Lemma 4.1.7 is helpful for the comparison of $\odot_{\mathsf{psu}}$ and $\odot_{\mathsf{ssu}}$ with other update functions from literature, as the following example illustrates.

Example 4.1.8. The function $\diamond_{\mathsf{mce}}\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ by Zhang and Foo [66] and the function $\diamond_{\mathsf{mcd}^*}\colon \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_0$ by Herzig and Rifi [38] are defined by

$$\|\varphi \diamond_{\mathsf{mce}} \alpha\| := \bigcup_{w \,\in\, \|\varphi\|} w \cdot_{\mathsf{mce}} \alpha, \qquad \|\varphi \diamond_{\mathsf{mcd}^*} \alpha\| := \bigcup_{w \,\in\, \|\varphi\|} w \cdot_{\mathsf{mcd}^*} \alpha,$$

for some functions $\cdot_{\mathsf{mce}}, \cdot_{\mathsf{mcd}^*}\colon \mathrm{Pow}(\mathcal{P}) \times \mathcal{L}_0 \to \mathcal{M}$. We can now define the corresponding functions $\odot_{\mathsf{mce}}, \odot_{\mathsf{mcd}^*}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ on model sets by

$$S \odot_{\mathsf{mce}} \alpha := \bigcup_{w \,\in\, \overline{S}} w \cdot_{\mathsf{mce}} \alpha, \qquad S \odot_{\mathsf{mcd}^*} \alpha := \bigcup_{w \,\in\, \overline{S}} w \cdot_{\mathsf{mcd}^*} \alpha.$$

Due to the results and examples by Herzig and Rifi [39], we immediately get $\odot_{\mathsf{mce}} < \odot_{\mathsf{su}}$ and $\odot_{\mathsf{mcd}^*} < \odot_{\mathsf{su}}$. Lemma 4.1.7 directly implies $\odot_{\mathsf{mce}} \lesssim \odot_{\mathsf{psu}}$ and $\odot_{\mathsf{mcd}^*} \lesssim \odot_{\mathsf{psu}}$, and the examples from [39] can also be used to show $\odot_{\mathsf{mce}} \not\approx \odot_{\mathsf{psu}}$ and $\odot_{\mathsf{mcd}^*} \not\approx \odot_{\mathsf{psu}}$.

We end this section by analysing the functions that we get by translating the update functions we have met so far to revision. We already know from Theorem 3.3.2 and Example 3.3.3 that the function

$$\otimes^u_{(\odot_{\mathsf{pma}})}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}, \qquad (S, \alpha) \mapsto \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ S \odot_{\mathsf{pma}} \alpha & \text{otherwise,} \end{cases}$$

satisfies the postulates ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R7}_{\mathcal{M}}$) but not ($\mathsf{R8}_{\mathcal{M}}$). It also satisfies the new postulate ($\mathsf{R9}_{\mathcal{M}}$) by definition. However, in the case $S \neq \emptyset$ and $S \models \neg\alpha$, we have that $\otimes^u_{(\odot_{\mathsf{pma}})}$ is too restrictive because of the problem of disjunctive input, cf. [39]. Therefore, the function $\otimes^u_{(\odot_{\mathsf{pma}})}$ is not a possible candidate for a revision function that satisfies our requirements. So we will concentrate on the translations of the variants of the standard update function to revision. It turns out that they all translate to the same function.

Lemma 4.1.9. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \otimes^u_{(\odot_{\mathsf{csu}})} \alpha = S \otimes^u_{(\odot_{\mathsf{psu}})} \alpha = S \otimes^u_{(\odot_{\mathsf{ssu}})} \alpha = S \otimes^u_{(\odot_{\mathsf{su}})} \alpha.$$

Proof. Every equality is an immediate consequence of Definition 3.3.1 and Definition 4.1.1. □

Due to Lemma 2.3.10 and Lemma 3.3.4 we get that the function $\otimes^u_{(\odot_{\mathsf{su}})}$ satisfies the postulates ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R6}_{\mathcal{M}}$). Clearly, $\otimes^u_{(\odot_{\mathsf{su}})}$ also satisfies ($\mathsf{R9}_{\mathcal{M}}$) by definition. In the following example we show that ($\mathsf{R7}_{\mathcal{M}}$) and ($\mathsf{R8}_{\mathcal{M}}$) are not satisfied.

Example 4.1.10. First, let $\mathcal{P} = \{p, q\}$, $S = \{\emptyset\}$, $\alpha = p \vee q$, and $\beta = p$. Then we have $S \models \neg\alpha$ and $S \models \neg(\alpha \wedge \beta)$, hence we get

$$S \otimes^u_{(\odot_{\mathsf{su}})} (\alpha \wedge \beta) = S \odot_{\mathsf{su}} (\alpha \wedge \beta) = \{\{p\}\},$$
$$(S \otimes^u_{(\odot_{\mathsf{su}})} \alpha) \oplus \beta = (S \odot_{\mathsf{su}} \alpha) \oplus \beta = \{\{p\}, \{p, q\}\}.$$

It is now easy to see that ($\mathsf{R7}_{\mathcal{M}}$) is violated because $\neg p \in \mathrm{Th}(S \otimes^u_{(\odot_{\mathsf{su}})} (\alpha \wedge \beta))$ but $\neg p \notin \mathrm{Th}((S \otimes^u_{(\odot_{\mathsf{su}})} \alpha) \oplus \beta)$. Second, let $\mathcal{P} = \{p, q, r\}$, $S = \{\{q\}\}$, $\alpha = p$, and $\beta = q \vee r$. Then we have $S \models \neg\alpha$ and $S \models \neg(\alpha \wedge \beta)$, thus we get

$$(S \otimes^u_{(\odot_{\mathsf{su}})} \alpha) \oplus \beta = (S \odot_{\mathsf{su}} \alpha) \oplus \beta = \{\{p, q\}\} \cap \|\beta\| = \{\{p, q\}\},$$
$$S \otimes^u_{(\odot_{\mathsf{su}})} (\alpha \wedge \beta) = S \odot_{\mathsf{su}} (\alpha \wedge \beta) = \{\{p, q\}, \{p, r\}, \{p, q, r\}\}.$$

We can now easily see that $\otimes^u_{(\odot_{\mathsf{su}})}$ does not satisfy ($\mathsf{R8}_{\mathcal{M}}$), because we have $(S \otimes^u_{(\odot_{\mathsf{su}})} \alpha) \not\models \neg\beta$, $\neg r \in \mathrm{Th}((S \otimes^u_{(\odot_{\mathsf{su}})} \alpha) \oplus \beta)$, and $\neg r \notin \mathrm{Th}(S \otimes^u_{(\odot_{\mathsf{su}})} (\alpha \wedge \beta))$.

In the next section we will formulate new requirements for revision, contraction, and update functions. Moreover, we will investigate the relationships between functions that satisfy the new conditions.

4.2 Minimal change reconsidered

We have already mentioned that the AGM postulates for revision and contraction as well as the KM postulates for update are too restrictive, cf. [26, 39]. That is, the functions that satisfy all of the respective postulates cannot be used in applications. So we will only accept the postulates that correspond to our intuition of the revision, contraction, and update process respectively. In the following, we will present our minimal sets of postulates that we think every revision, contraction, and update function should satisfy.

We have seen in Chapter 2 that the revision postulate ($\mathsf{R7}_{\mathcal{M}}$) and the update postulate ($\mathsf{U5}_{\mathcal{M}}$) are identical. Moreover, it has been shown in [39] that ($\mathsf{U5}_{\mathcal{M}}$) causes the problem of disjunctive input, which should be avoided. This problem also appears if the new information contradicts our beliefs. In this case we have claimed with (3.8) that update and revision should do the same. Therefore, in addition to ($\mathsf{U5}_{\mathcal{M}}$), we also reject ($\mathsf{R7}_{\mathcal{M}}$).

The situation is similar with the revision postulate ($\mathsf{R8}_{\mathcal{M}}$), which we reject for the following reason. We have already argued that if the new information contradicts our beliefs, then revision and update should be the same. Furthermore, we have argued that in this case the standard update function seems to be the most reasonable function. Now, as we have seen in Example 4.1.10, such a revision function does not satisfy ($\mathsf{R8}_{\mathcal{M}}$).

Definition 4.2.1. We commit ourselves to the following sets of revision postulates.

1. A function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a *revision candidate*, if it satisfies the revision postulates ($\mathsf{R1}_{\mathcal{M}}$)–($\mathsf{R6}_{\mathcal{M}}$) and ($\mathsf{R9}_{\mathcal{M}}$).
2. A function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent revision candidate*, if it satisfies the revision postulates ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), ($\mathsf{R6}_{\mathcal{M}}$), and ($\mathsf{R9}_{\mathcal{M}^c}$).

The translations from Definition 3.1.5 transform a revision candidate into a consistent revision candidate and vice versa.

Theorem 4.2.2. *We have the following relationships between revision candidates and consistent revision candidates.*

1. *If a function $\otimes\colon \mathcal{M}\times\mathcal{L}_0 \to \mathcal{M}$ is a revision candidate, then the function $\otimes^{cr}$ is a consistent revision candidate and for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have* $\mathrm{Th}(S\ (\otimes^{cr})^{ir}\ \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent revision candidate, then the function $\otimes^{ir}$ is a revision candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have* $\mathrm{Th}(S\ (\otimes^{ir})^{cr}\ \alpha) = \mathrm{Th}(S \otimes \alpha)$.

Proof. We show how to prove the first assertion. If $\otimes$ is a revision candidate, then it satisfies $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$ and the function $\otimes^{cr}$ satisfies $(\mathsf{R1}_{\mathcal{M}})$, $(\mathsf{R2}_{\mathcal{M}^c})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R5}_{\mathcal{M}^c})$, and $(\mathsf{R6}_{\mathcal{M}})$ by Lemma 3.1.7. In addition, the function $\otimes^{cr}$ satisfies $(\mathsf{R9}_{\mathcal{M}^c})$ by Definition 3.1.5, hence it is a consistent revision candidate. Moreover, for all $S \in \mathcal{M}$ all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\otimes^{cr})^{ir}\ \alpha) = \mathrm{Th}(S \otimes \alpha)$ by Lemma 3.1.9 because $\otimes$ satisfies $(\mathsf{R2}_{\mathcal{M}})$ and $(\mathsf{R9}_{\mathcal{M}})$. The second assertion can similarly be proved using the same two lemmas. $\square$

In addition to the postulates for revision candidates, we think that a revision function $\otimes\colon \mathcal{M} \to \mathcal{M}$ should satisfy the following property in order to avoid the problem of disjunctive input. For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we require

$$S \models \neg\alpha \text{ and } S \neq \emptyset \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) \subseteq \mathrm{Th}(S \odot_{\mathsf{su}} \alpha). \tag{4.1}$$

A revision candidate that satisfies property (4.1) corresponds to our idea of *minimax change*: if the new information contradicts our beliefs, the amount of change should be at least as big as the one performed by the standard update function, in all other cases the model set should be minimally changed. In the context of consistent model sets, the corresponding requirement for minimax change is given as follows ($S \in \mathcal{M}^c$, $\alpha \in \mathcal{L}_0$),

$$S \models \neg\alpha \text{ and } \not\models \neg\alpha \;\Rightarrow\; \mathrm{Th}(S \otimes \alpha) \subseteq \mathrm{Th}(S \odot_{\mathsf{su}} \alpha). \tag{4.2}$$

In the following example we are going to present a revision candidate and a consistent revision candidate that perform minimax change according to property (4.1) and property (4.2) respectively.

Example 4.2.3. The *minimax revision* function $\otimes_{\mathsf{mm}}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$S \otimes_{\mathsf{mm}} \alpha := S \otimes^{u}_{(\odot_{\mathsf{su}})} \alpha = \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ S \odot_{\mathsf{su}} \alpha & \text{otherwise.} \end{cases}$$

Due to Lemma 2.3.10 and Lemma 3.3.4 we get that the function $\otimes_{\mathsf{mm}}$ satisfies the postulates $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$. Postulate $(\mathsf{R9}_{\mathcal{M}})$ and property (4.1) are obviously satisfied, as well. The function $(\otimes_{\mathsf{mm}})^{cr}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S\ (\otimes_{\mathsf{mm}})^{cr}\ \alpha = \begin{cases} S & \text{if } \models \neg\alpha, \\ S \oplus \alpha & \text{if } S \not\models \neg\alpha, \\ S \odot_{\mathsf{su}} \alpha & \text{otherwise} \end{cases}$$

is a consistent revision candidate by Theorem 4.2.2. Moreover, it is immediate that $(\otimes_{\mathsf{mm}})^{cr}$ also satisfies property (4.2).

We have mentioned in Section 2.2 that the contraction postulates $(\mathsf{C7}_{\mathcal{M}})$ and $(\mathsf{C8}_{\mathcal{M}})$ are just the technical counterpart to $(\mathsf{R7}_{\mathcal{M}})$ and $(\mathsf{R8}_{\mathcal{M}})$ respectively. We reject $(\mathsf{C7}_{\mathcal{M}})$ and $(\mathsf{C8}_{\mathcal{M}})$ for the following reason. We accept the basic contraction postulates $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$. If a contraction function additionally satisfies $(\mathsf{C7}_{\mathcal{M}})$ and $(\mathsf{C8}_{\mathcal{M}})$, then its Levi translation satisfies $(\mathsf{R7}_{\mathcal{M}})$ and $(\mathsf{R8}_{\mathcal{M}})$ by Theorem 3.2.6. But we have already argued against these two revision postulates.

Definition 4.2.4. We commit ourselves to the following sets of contraction postulates.

1. A function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a *contraction candidate*, if it satisfies $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$ and $(\mathsf{C9}_{\mathcal{M}})$.
2. A function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent contraction candidate*, if it satisfies $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$.

The functions from Definition 3.1.11 translate a contraction candidate into a consistent contraction candidate and vice versa.

Theorem 4.2.5. *We have the following relationships between contraction candidates and consistent contraction candidates.*

1. *If a function $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a contraction candidate, then the function $\ominus^{cc}$ is a consistent contraction candidate and for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\ominus^{cc})^{ic}\ \alpha) = \mathrm{Th}(S \ominus \alpha)$.*
2. *If a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent contraction candidate, then the function $\ominus^{ic}$ is a contraction candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\ominus^{ic})^{cc}\ \alpha) = \mathrm{Th}(S \ominus \alpha)$.*

Proof. We show how to prove the second assertion. If $\ominus$ is a consistent contraction candidate, then it satisfies $(\mathsf{C1}_{\mathcal{M}})$–$(\mathsf{C6}_{\mathcal{M}})$ and the function $\ominus^{ic}$ so also does by Lemma 3.1.13. By Definition 3.1.11, $\ominus^{ic}$ also satisfies $(\mathsf{C9}_{\mathcal{M}})$, so it is a contraction candidate. Furthermore, for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\ominus^{ic})^{cc}\ \alpha) = \mathrm{Th}(S \ominus \alpha)$ by Lemma 3.1.15. The proof of the first assertion is similar and uses the same two lemmas. □

The Levi and Harper identities from Definition 3.2.4 are suitable to translate a revision candidate into a contraction candidate and vice versa.

Theorem 4.2.6. *We have the following relationships between revision and contraction candidates.*

1. *If a function* $\otimes\colon \mathcal{M}\times\mathcal{L}_0 \to \mathcal{M}$ *is a revision candidate, then the function* $\ominus^r_\otimes$ *is a contraction candidate and for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes^c_{(\ominus^r_\otimes)} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\ominus\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *is a contraction candidate, then the function* $\otimes^c_\ominus$ *is a revision candidate and for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \ominus^r_{(\otimes^c_\ominus)} \alpha) = \mathrm{Th}(S \ominus \alpha)$.

Proof. We show how to prove the first assertion. If $\otimes$ is a revision candidate, then it satisfies ($\mathsf{R1}_\mathcal{M}$)–($\mathsf{R6}_\mathcal{M}$) and the function $\ominus^r_\otimes$ satisfies ($\mathsf{C1}_\mathcal{M}$)–($\mathsf{C6}_\mathcal{M}$) by Lemma 3.2.5. Because $\otimes$ also satisfies ($\mathsf{R9}_\mathcal{M}$) by assumption, the function $\ominus^r_\otimes$ satisfies ($\mathsf{C9}_\mathcal{M}$), so it is a contraction candidate. In addition, for all $S \in \mathcal{M}$ all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(S \otimes^c_{(\ominus^r_\otimes)} \alpha) = \mathrm{Th}(S \otimes \alpha)$$

by Lemma 3.2.7, because $\otimes$ satisfies ($\mathsf{R2}_\mathcal{M}$), ($\mathsf{R3}_\mathcal{M}$), and ($\mathsf{R6}_\mathcal{M}$). The second assertion similarly follows from the same two lemmas. □

Due to the Levi and Harper identities we get that our notion of minimax change can also be stated for contraction functions. Property (4.1) translates to the following requirement ($S \in \mathcal{M}$, $\alpha \in \mathcal{L}_0$),

$$S \models \alpha \text{ and } S \neq \emptyset \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) \subseteq \mathrm{Th}(S \cup (S \odot_{\mathsf{su}} \neg\alpha)). \tag{4.3}$$

Accordingly, the notion of minimax change for consistent contraction functions that corresponds to property (4.2) can be formulated as follows. For all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we require

$$S \models \alpha \;\Rightarrow\; \mathrm{Th}(S \ominus \alpha) \subseteq \mathrm{Th}(S \cup (S \odot_{\mathsf{su}} \neg\alpha)). \tag{4.4}$$

If we apply the Harper translation from Definition 3.2.4 to the minimax revision function $\otimes_{\mathsf{mm}}$ from Example 4.2.3, we get the corresponding minimax contraction function.

Example 4.2.7. The *minimax contraction* function $\ominus_{\mathsf{mm}}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined as follows,

$$S \ominus_{\mathsf{mm}} \alpha := S \ominus^r_{(\otimes_{\mathsf{mm}})} \alpha = \begin{cases} \|\neg\alpha\| & \text{if } S = \emptyset, \\ S & \text{if } S \not\models \alpha, \\ S \cup (S \odot_{\mathsf{su}} \neg\alpha) & \text{otherwise.} \end{cases}$$

By Theorem 4.2.6, we get that the function $\ominus_{\mathsf{mm}}$ is a contraction candidate. Clearly, we also get that the function $(\ominus_{\mathsf{mm}})^{cc}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ defined by

$$S\ (\ominus_{\mathsf{mm}})^{cc}\ \alpha = \begin{cases} S & \text{if } S \not\models \alpha, \\ S \cup (S \odot_{\mathsf{su}} \neg\alpha) & \text{otherwise.} \end{cases}$$

is a consistent contraction candidate by Theorem 4.2.5.

The translations from Definition 3.2.8 transform a consistent revision candidate into a consistent contraction candidate and vice versa.

Theorem 4.2.8. *We have the following relationships between consistent revision and consistent contraction candidates.*

1. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent revision candidate, then the function $\ominus^{cr}_{\otimes}$ is a consistent contraction candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have* $\mathrm{Th}(S \otimes^{cc}_{(\ominus^{cr}_{\otimes})} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function $\ominus\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent contraction candidate, then the function $\otimes^{cc}_{\ominus}$ is a consistent revision candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have* $\mathrm{Th}(S \ominus^{cr}_{(\otimes^{cc}_{\ominus})} \alpha) = \mathrm{Th}(S \ominus \alpha)$.

Proof. We show how to prove the second assertion. If $\ominus$ is a consistent contraction candidate, then it satisfies ($\mathsf{C1}_{\mathcal{M}}$)–($\mathsf{C6}_{\mathcal{M}}$) and the function $\otimes^{cc}_{\ominus}$ satisfies ($\mathsf{R1}_{\mathcal{M}}$), ($\mathsf{R2}_{\mathcal{M}^c}$), ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), ($\mathsf{R5}_{\mathcal{M}^c}$), and ($\mathsf{R6}_{\mathcal{M}}$) by Lemma 3.2.10. By Definition 3.2.8, $\otimes^{cc}_{\ominus}$ also satisfies ($\mathsf{R9}_{\mathcal{M}^c}$), thus it is a consistent revision candidate. Moreover, for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have

$$\mathrm{Th}(S \ominus^{cr}_{(\otimes^{cc}_{\ominus})} \alpha) = \mathrm{Th}(S \ominus \alpha)$$

by Lemma 3.2.12, because $\ominus$ satisfies ($\mathsf{C2}_{\mathcal{M}}$), ($\mathsf{C3}_{\mathcal{M}}$), ($\mathsf{C5}_{\mathcal{M}}$), and ($\mathsf{C6}_{\mathcal{M}}$). The first assertion can similarly be proved using the same two lemmas. □

We are now going to define new update postulates. According to requirement (3.5), we want the revision postulate ($\mathsf{R9}_{\mathcal{M}}$) to hold for update functions, as well. Therefore, we define the following additional update postulate,

($\mathsf{U9}_{\mathcal{M}}$) $S = \emptyset \;\Rightarrow\; \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(\|\alpha\|)$.

As a consequence of ($\mathsf{U9}_{\mathcal{M}}$), we have to slightly weaken the update postulate ($\mathsf{U2}_{\mathcal{M}}$). The following modified postulate corresponds to requirement (3.7),

($\mathsf{U2}^{\star}_{\mathcal{M}}$) $S \models \alpha$ and $S \neq \emptyset \;\Rightarrow\; \mathrm{Th}(S \odot \alpha) = \mathrm{Th}(S)$.

Furthermore, we believe that $(\mathsf{U8}_{\mathcal{M}})$ has not much to do with minimal change. Remember that according to this postulate, an update has to be performed model by model. By Theorem 4.1.5, we now that the variants of the standard update function can be ordered as follows,

$$\odot_{\mathsf{csu}} < \odot_{\mathsf{psu}} < \odot_{\mathsf{ssu}} < \odot_{\mathsf{su}}.$$

By Lemma 2.3.10 and Lemma 4.1.2, we get that only $\odot_{\mathsf{su}}$ and $\odot_{\mathsf{psu}}$ satisfy $(\mathsf{U8}_{\mathcal{M}})$. We therefore suggest a weaker version of this postulate,

$$(\mathsf{U8}^{\star}_{\mathcal{M}}) \quad S \not\models \alpha \;\Rightarrow\; \mathrm{Th}(S \odot \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha)).$$

Observe that the premise $S \not\models \alpha$ of the modified postulate $(\mathsf{U8}^{\star}_{\mathcal{M}})$ is necessary because of property (3.5). We think that $(\mathsf{U8}^{\star}_{\mathcal{M}})$ corresponds to the idea of minimal change because it requires that only one partition of the original model set is updated.

We have now added one additional update postulate and modified two of the original KM postulates for update. Moreover, we reject $(\mathsf{U5}_{\mathcal{M}})$, $(\mathsf{U6}_{\mathcal{M}})$, and $(\mathsf{U7}_{\mathcal{M}})$ for the following reason. If the new information contradicts our beliefs, we think that the standard update function $\odot_{\mathsf{su}}$ from Definition 2.3.9 performs an adequate change because we have no problem with disjunctive input, cf. [39]. The counterexamples in the proof of Lemma 2.3.10 show that in this case the function $\odot_{\mathsf{su}}$ violates the postulates $(\mathsf{U5}_{\mathcal{M}})$, $(\mathsf{U6}_{\mathcal{M}})$, and $(\mathsf{U7}_{\mathcal{M}})$.

Definition 4.2.9. We commit ourselves to the following sets of update postulates.

1. A function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an *update candidate*, if it satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U2}^{\star}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, $(\mathsf{U4}_{\mathcal{M}})$, $(\mathsf{U8}^{\star}_{\mathcal{M}})$, and $(\mathsf{U9}_{\mathcal{M}})$.
2. A function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a *consistent update candidate*, if it satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}^c})$, $(\mathsf{U2}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}^c})$, $(\mathsf{U4}_{\mathcal{M}})$, $(\mathsf{U8}^{\star}_{\mathcal{M}})$, and $(\mathsf{U9}_{\mathcal{M}^c})$.

Note that we do not need to replace $(\mathsf{U2}_{\mathcal{M}})$ by $(\mathsf{U2}^{\star}_{\mathcal{M}})$ in the context of consistent model sets because a consistent model set is always non empty. We can now show that revision and update candidates satisfy most of our requirements from the end of Section 3.3.

Lemma 4.2.10. *If a function* $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *is a revision candidate and a function* $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *is an update candidate, then the two functions satisfy the properties* (3.5), (3.6), *and* (3.7).

Proof. Property (3.5) is directly implied by ($\mathsf{R9}_{\mathcal{M}}$) and ($\mathsf{U9}_{\mathcal{M}}$), property (3.7) is satisfied due to ($\mathsf{R3}_{\mathcal{M}}$), ($\mathsf{R4}_{\mathcal{M}}$), and ($\mathsf{U2}^{\star}_{\mathcal{M}}$). We show that property (3.6) is also satisfied. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. By ($\mathsf{R5}_{\mathcal{M}}$), we immediately get $\models \neg\alpha \;\Leftrightarrow\; S \otimes \alpha$. For the update candidate, we proceed as follows. If $\models \neg\alpha$, then we get $S \odot \alpha \models \alpha$ by ($\mathsf{U1}_{\mathcal{M}}$), hence $S \odot \alpha = \emptyset$. For the converse direction, if $S \odot \alpha = \emptyset$, then we have either $S = \emptyset$ or $\models \neg\alpha$ by ($\mathsf{U3}_{\mathcal{M}}$). But $S = \emptyset$ implies $\mathrm{Th}(\|\alpha\|) = \mathrm{Th}(S \odot \alpha)$ by ($\mathsf{U9}_{\mathcal{M}}$). Hence, we have $\models \neg\alpha$ in both cases, and the proof is finished. ☐

Due to the significant changes in the sets of update postulates, we have to introduce a new translation from update functions operating on consistent model sets to functions on possibly inconsistent model sets.

Definition 4.2.11. Given a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, its translation $\odot^{iu\star}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by

$$S \odot^{iu\star} \alpha := \begin{cases} S \odot \alpha & \text{if } \not\models \neg\alpha \text{ and } S \neq \emptyset, \\ \|\alpha\| & \text{otherwise.} \end{cases}$$

We can now show that the translations $\odot^{cu}$ from Definition 3.1.17 and $\odot^{iu\star}$ are in the same relationship as $\otimes^{cr}$ and $\otimes^{ir}$.

Theorem 4.2.12. *We have the following relationships between update candidates and consistent update candidates.*

1. *If a function $\odot\colon \mathcal{M}\times\mathcal{L}_0 \to \mathcal{M}$ is an update candidate, then the function $\odot^{cu}$ is a consistent update candidate and for all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\odot^{cu})^{iu\star}\ \alpha) = \mathrm{Th}(S \odot \alpha)$.*
2. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent update candidate, then the function $\odot^{iu\star}$ is an update candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S\ (\odot^{iu\star})^{cu}\ \alpha) = \mathrm{Th}(S \odot \alpha)$.*

Proof. For the first assertion, let $\odot$ be an update candidate. We show how to prove that $\odot^{cu}$ satisfies ($\mathsf{U8}^{\star}_{\mathcal{M}}$). Let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given and suppose that $S \not\models \alpha$. Now, we distinguish two cases. In the first case, if $\models \neg\alpha$, then we have $\mathrm{Th}(S \odot^{cu} \alpha) = \mathrm{Th}(S)$ by Definition 3.1.17. This is the same as $\mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot^{cu} \alpha))$, because $S \oplus \alpha = \emptyset$ and $S \setminus \|\alpha\| = S$. In the other case, if $\not\models \neg\alpha$, then we have

$$\begin{aligned} \mathrm{Th}(S \odot^{cu} \alpha) &= \mathrm{Th}(S \odot \alpha) && \text{by Definition 3.1.17} \\ &= \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha)) && \text{by } (\mathsf{U8}^{\star}_{\mathcal{M}}) \text{ for } \odot \\ &= \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot^{cu} \alpha)) && \text{by Definition 3.1.17,} \end{aligned}$$

because $S \setminus \|\alpha\| \neq \emptyset$. The proof of the other postulates is similar and we get that the function $\odot^{cu}$ is a consistent update candidate. Now, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. We distinguish three cases. First, if $\not\models \neg\alpha$ and $S \neq \emptyset$, then we have $\mathrm{Th}(S\ (\odot^{cu})^{iu\star}\ \alpha) = \mathrm{Th}(S \odot^{cu} \alpha)$. This is equal to $\mathrm{Th}(S \odot \alpha)$ by Definition 3.1.17, because $\not\models \neg\alpha$. Second, if $\models \neg\alpha$, then we have $\mathrm{Th}(S\ (\odot^{cu})^{iu\star}\ \alpha = \mathrm{Th}(\|\alpha\|) = \mathrm{Th}(\emptyset)$, which is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U1}_{\mathcal{M}})$. In the last case, if $S = \emptyset$, then we have $\mathrm{Th}(S\ (\odot^{cu})^{iu\star}\ \alpha) = \mathrm{Th}(\|\alpha\|)$. By $(\mathsf{U9}_{\mathcal{M}})$, we get that this equals $\mathrm{Th}(S \odot \alpha)$ and the first assertion is proved.

For the second assertion, let $\odot$ be a consistent update candidate. We show how to prove that $\odot^{iu\star}$ satisfies $(\mathsf{U2}^{\star}_{\mathcal{M}})$. Let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given and suppose that $S \models \alpha$ and $S \neq \emptyset$. Then we immediately get $\not\models \neg\alpha$, hence we have $\mathrm{Th}(S \odot^{iu\star} \alpha) = \mathrm{Th}(S \odot \alpha)$. But this is the same as $\mathrm{Th}(S)$ by $(\mathsf{U2}_{\mathcal{M}})$ for $\odot$. The other postulates can similarly be proved and we get that the function $\odot^{iu\star}$ is an update candidate. Now, let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given. We distinguish two cases. First, if $\not\models \neg\alpha$, then we have $\mathrm{Th}(S\ (\odot^{iu\star})^{cu}\ \alpha) = \mathrm{Th}(S \odot^{iu\star} \alpha)$ by Definition 3.1.17. Since $S \neq \emptyset$, this is equal to $\mathrm{Th}(S \odot \alpha)$. Second, if $\models \neg\alpha$, then we have $\mathrm{Th}(S\ (\odot^{iu\star})^{cu}\ \alpha) = \mathrm{Th}(S)$, which equals $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U9}_{\mathcal{M}^c})$, and we are done. □

We will now redefine the translation from revision to update in order to get the intended correspondences between revision and update candidates.

Definition 4.2.13. Given a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, its translation $\odot^{r\star}_{\otimes}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ from revision to update is defined as follows,

$$S \odot^{r\star}_{\otimes} \alpha := \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S & \text{if } S \neq \emptyset \text{ and } S \models \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha) & \text{otherwise.} \end{cases}$$

If a revision and contraction candidate are related by one of the translations from Definition 4.2.13 or Definition 3.3.1, then the two functions satisfy property (3.8).

Lemma 4.2.14. *We have the following relationships between revision and update candidates.*

1. *If a function $\otimes\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is a revision candidate, then the functions $\otimes$ and $\odot^{r\star}_{\otimes}$ satisfy property* (3.8).
2. *If a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an update candidate, then the functions $\odot$ and $\otimes^{u}_{\odot}$ satisfy property* (3.8).

Proof. Both assertions directly follow from the definition of the translations as well as the postulates $(\mathsf{R9}_{\mathcal{M}})$ and $(\mathsf{U9}_{\mathcal{M}})$. □

Due to Definition 4.2.13 and Definition 3.3.1, we can translate a revision candidate into an update candidate and vice versa.

Theorem 4.2.15. *We have the following relationships between revision and update candidates.*

1. *If a function* $\otimes\colon \mathcal{M}\times\mathcal{L}_0 \to \mathcal{M}$ *is a revision candidate, then the function* $\odot^{r\star}_{\otimes}$ *is an update candidate and for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \otimes^{u}_{(\odot^{r\star}_{\otimes})} \alpha) = \mathrm{Th}(S \otimes \alpha)$.
2. *If a function* $\odot\colon \mathcal{M}\times\mathcal{L}_0 \to \mathcal{M}$ *is an update candidate, then the function* $\otimes^{u}_{\odot}$ *is a revision candidate and for all* $S \in \mathcal{M}$ *and all* $\alpha \in \mathcal{L}_0$ *we have* $\mathrm{Th}(S \odot^{r\star}_{(\otimes^{u}_{\odot})} \alpha) = \mathrm{Th}(S \odot \alpha)$.

Proof. For the first assertion, let $\otimes$ be a revision candidate and let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. We show how to prove that $\odot^{r\star}_{\otimes}$ satisfies $(\mathsf{U3}_{\mathcal{M}})$. Assume $S \neq \emptyset$ and $\not\models \neg\alpha$. We distinguish two cases. First, if $S \models \alpha$, then we have $S \odot^{r\star}_{\otimes} \alpha = S$, which is non empty by assumption. Second, if $S \not\models \alpha$, then we have $S \odot^{r\star}_{\otimes} \alpha = (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha)$. Since $S \setminus \|\alpha\| \neq \emptyset$ by assumption, we get that $(S \setminus \|\alpha\|) \otimes \alpha \neq \emptyset$ by $(\mathsf{R5}_{\mathcal{M}})$, hence we immediately get $S \odot^{r\star}_{\otimes} \alpha \neq \emptyset$. The proofs of the other postulates are similar and we get that the function $\odot^{r\star}_{\otimes}$ is an update candidate. For the second claim of the first assertion, we distinguish three cases. First, if $S = \emptyset$, then we have $\mathrm{Th}(S \otimes^{u}_{(\odot^{r\star}_{\otimes})} \alpha) = \mathrm{Th}(\|\alpha\|)$, which is the same as $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R9}_{\mathcal{M}})$. Second, if $S \not\models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{u}_{(\odot^{r\star}_{\otimes})} \alpha) = \mathrm{Th}(S)$, and this equals $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R3}_{\mathcal{M}})$ and $(\mathsf{R4}_{\mathcal{M}})$. In the last case, if $S \neq \emptyset$ and $S \models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{u}_{(\odot^{r\star}_{\otimes})} \alpha) = \mathrm{Th}(S \odot^{r\star}_{\otimes} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha))$, because we have $S \not\models \alpha$ by assumption. This is equal to $\mathrm{Th}(S \otimes \alpha)$, because $S \oplus \alpha = \emptyset$ and $S \setminus \|\alpha\| = S$ by assumption.

For the second assertion, let $\odot$ be an update candidate. Then we have that $\odot$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, and $(\mathsf{U4}_{\mathcal{M}})$. Therefore, by Lemma 3.3.4, we get that the function $\otimes^{u}_{\odot}$ satisfies $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$. Since $\otimes^{u}_{\odot}$ also satisfies $(\mathsf{R9}_{\mathcal{M}})$ by Definition 3.3.1, we have that the function $\otimes^{u}_{\odot}$ is a revision candidate. For the second claim of the second assertion, let $S \in \mathcal{M}$ and $\alpha \in \mathcal{L}_0$ be given. We distinguish three cases. First, if $S = \emptyset$, then we have $\mathrm{Th}(S \odot^{r\star}_{(\otimes^{u}_{\odot})} \alpha) = \mathrm{Th}(\|\alpha\|)$, which is equal to $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U9}_{\mathcal{M}})$. Second, if $\emptyset \neq S \subseteq \|\alpha\|$, then we have $\mathrm{Th}(S \odot^{r\star}_{(\otimes^{u}_{\odot})} \alpha) = \mathrm{Th}(S)$, which is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U2}^{\star}_{\mathcal{M}})$. In the last case, if $S \not\models \alpha$, then we have $\mathrm{Th}(S \odot^{r\star}_{(\otimes^{u}_{\odot})} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes^{u}_{\odot} \alpha))$, which is the same as $\mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha))$ by Definition 3.3.1, because we have $\emptyset \neq S \setminus \|\alpha\| \subseteq \|\neg\alpha\|$ by assumption. And since we have $S \not\models \alpha$ by assumption, this equals $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U8}^{\star}_{\mathcal{M}})$, hence the proof is now complete. ☐

Applying the translation $\odot^{r\star}_{\otimes}$ to the minimax revision function $\otimes_{\mathsf{mm}}$ from Example 4.2.3, we get the corresponding minimax update function. Note that the requirements of minimax change for update candidates and consistent update candidates are given by (4.1) and (4.2).

Example 4.2.16. The *minimax update* function $\odot_{\mathsf{mm}}\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined by $S \odot_{\mathsf{mm}} \alpha := S \odot^{r\star}_{(\otimes_{\mathsf{mm}})} \alpha$, that is by

$$S \odot_{\mathsf{mm}} \alpha := \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S & \text{if } S \neq \emptyset \text{ and } S \vDash \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) & \text{otherwise.} \end{cases}$$

By Theorem 4.2.15, we get that the function $\odot_{\mathsf{mm}}$ is an update candidate. Moreover, $\odot_{\mathsf{mm}}$ obviously satisfies property (4.1). Consequently, we get that the function $(\odot_{\mathsf{mm}})^{cu}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, defined by

$$S\, (\odot_{\mathsf{mm}})^{cu}\, \alpha := \begin{cases} S & \text{if } \vDash \neg\alpha \text{ or } S \vDash \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot_{\mathsf{su}} \alpha) & \text{otherwise} \end{cases}$$

is a consistent update candidate by Theorem 4.2.12. Clearly, the function $(\odot_{\mathsf{mm}})^{cu}$ satisfies property (4.2).

We will now also redefine the translation from revision to update in the context of consistent model sets.

Definition 4.2.17. Given a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, its translation $\odot^{cr\star}_{\otimes}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ from revision to update is defined by

$$S \odot^{cr\star}_{\otimes} \alpha := \begin{cases} S & \text{if } S \vDash \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha) & \text{otherwise.} \end{cases}$$

Due to Definition 4.2.17 and Definition 3.3.9, we can translate a consistent revision candidate into a consistent update candidate and vice versa.

Theorem 4.2.18. *We have the following relationships between consistent revision and consistent update candidates.*

1. *If a function $\otimes\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent revision candidate, then the function $\odot^{cr\star}_{\otimes}$ is a consistent update candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S \otimes^{cu}_{(\odot^{cr\star}_{\otimes})} \alpha) = \mathrm{Th}(S \otimes \alpha)$.*
2. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is a consistent update candidate, then the function $\otimes^{cu}_{\odot}$ is a consistent revision candidate and for all $S \in \mathcal{M}^c$ and all $\alpha \in \mathcal{L}_0$ we have $\mathrm{Th}(S \odot^{cr\star}_{(\otimes^{cu}_{\odot})} \alpha) = \mathrm{Th}(S \odot \alpha)$.*

Proof. For the first assertion, let $\otimes$ be a consistent revision candidate and let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given. We show how to prove that $\odot_{\otimes}^{cr\star}$ satisfies $(\mathsf{U8}^\star_{\mathcal{M}})$ and assume $S \not\models \alpha$. Then we have $\mathrm{Th}(S \odot_{\otimes}^{cr\star} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha))$. As an immediate consequence of Definition 4.2.17 we get that $(S \setminus \|\alpha\|) \otimes \alpha = (S \setminus \|\alpha\|) \odot_{\otimes}^{cr\star} \alpha$ and $(\mathsf{U8}^\star_{\mathcal{M}})$ holds. The proofs of the other postulates are similar and we get that the function $\odot_{\otimes}^{cr\star}$ is a consistent update candidate. For the second claim of the first assertion, we distinguish three cases. First, if $\models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{cu}_{(\odot_{\otimes}^{cr\star})} \alpha) = \mathrm{Th}(S)$, which equals $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R9}_{\mathcal{M}^c})$. Second, if $S \not\models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{cu}_{(\odot_{\otimes}^{cr\star})} \alpha) = \mathrm{Th}(S \oplus \alpha)$, and this is the same as $\mathrm{Th}(S \otimes \alpha)$ by $(\mathsf{R3}_{\mathcal{M}})$ and $(\mathsf{R4}_{\mathcal{M}})$. In the last case, if $\not\models \neg\alpha$ and $S \models \neg\alpha$, then we have $\mathrm{Th}(S \otimes^{cu}_{(\odot_{\otimes}^{cr\star})} \alpha) = \mathrm{Th}(S \odot_{\otimes}^{cr\star} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes \alpha))$, because we have $S \not\models \alpha$ by assumption. This is equal to $\mathrm{Th}(S \otimes \alpha)$, because $S \oplus \alpha = \emptyset$ and $S \setminus \|\alpha\| = S$ by assumption.

For the second assertion, let $\odot$ be a consistent update candidate. Then we have that $\odot$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}^c})$, $(\mathsf{U3}_{\mathcal{M}^c})$, and $(\mathsf{U4}_{\mathcal{M}})$. Therefore, by Lemma 3.3.13, we get that the function $\otimes_{\odot}^{cu}$ satisfies $(\mathsf{R1}_{\mathcal{M}})$, $(\mathsf{R2}_{\mathcal{M}^c})$, $(\mathsf{R3}_{\mathcal{M}})$, $(\mathsf{R4}_{\mathcal{M}})$, $(\mathsf{R5}_{\mathcal{M}^c})$, and $(\mathsf{R6}_{\mathcal{M}})$. Since $\otimes_{\odot}^{cu}$ also satisfies $(\mathsf{R9}_{\mathcal{M}^c})$ by Definition 3.3.9, we have that the function $\otimes_{\odot}^{cu}$ is a consistent revision candidate. For the second claim of the second assertion, let $S \in \mathcal{M}^c$ and $\alpha \in \mathcal{L}_0$ be given. We distinguish three cases. First, if $S \models \alpha$, then we have $\mathrm{Th}(S \odot^{cr\star}_{(\otimes_{\odot}^{cu})} \alpha) = \mathrm{Th}(S)$, which is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U2}_{\mathcal{M}})$. Second, if $S \not\models \alpha$ and $\models \neg\alpha$, then we have $\mathrm{Th}(S \odot^{cr\star}_{(\otimes_{\odot}^{cu})} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes_{\odot}^{cu} \alpha)) = \mathrm{Th}(S \otimes_{\odot}^{cu} \alpha)$, because we have $S \oplus \alpha = \emptyset$ and $S \setminus \|\alpha\| = S$ by assumption. By Definition 3.3.9, we get $\mathrm{Th}(S)$ because we have $\models \neg\alpha$ by assumption, and this equals $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U9}_{\mathcal{M}^c})$. In the last case, if $S \not\models \alpha$ and $\not\models \neg\alpha$, then we have $\mathrm{Th}(S \odot^{cr\star}_{(\otimes_{\odot}^{cu})} \alpha) = \mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \otimes_{\odot}^{cu} \alpha))$, which is equal to $\mathrm{Th}((S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha))$ by Definition 3.3.9, because we have $\emptyset \neq S \setminus \|\alpha\| \subseteq \|\neg\alpha\|$ by assumption. And since we have $S \not\models \alpha$ by assumption, this is the same as $\mathrm{Th}(S \odot \alpha)$ by $(\mathsf{U8}^\star_{\mathcal{M}})$ and we are done. □

We end this chapter by translating the update functions from Section 4.1 into update candidates.

Definition 4.2.19. Given a function $\odot \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$, the possible update candidate $\odot^{cand} \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is defined to be the function $\odot^{r\star}_{(\otimes_{\odot}^{u})}$, that is

$$S \odot^{cand} \alpha := \begin{cases} \|\alpha\| & \text{if } S = \emptyset, \\ S & \text{if } S \neq \emptyset \text{ and } S \models \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha) & \text{otherwise.} \end{cases}$$

Given a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$, the possible consistent update candidate $\odot^{ccand}\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ is defined to be the function $\odot^{cr\star}_{(\otimes^{cu}_{\odot})}$, that is

$$S \odot^{ccand} \alpha := \begin{cases} S & \text{if } \models \neg\alpha \text{ or } S \models \alpha, \\ (S \oplus \alpha) \cup ((S \setminus \|\alpha\|) \odot \alpha) & \text{otherwise.} \end{cases}$$

In order to get an update candidate and a consistent update candidate by using the translations from Definition 4.2.19, there are minimal conditions that the original update functions have to satisfy.

Theorem 4.2.20. *We have the following conditions for the translation to update candidates.*

1. *If a function $\odot\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, and $(\mathsf{U4}_{\mathcal{M}})$, then the function $\odot^{cand}$ is an update candidate.*
2. *If a function $\odot\colon \mathcal{M}^c \times \mathcal{L}_0 \to \mathcal{M}$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}^c})$, $(\mathsf{U3}_{\mathcal{M}^c})$, and $(\mathsf{U4}_{\mathcal{M}})$, then the function $\odot^{ccand}$ is a consistent update candidate.*

Proof. We show how to prove the first assertion. If $\odot$ satisfies $(\mathsf{U0}_{\mathcal{M}})$, $(\mathsf{U1}_{\mathcal{M}})$, $(\mathsf{U3}_{\mathcal{M}})$, and $(\mathsf{U4}_{\mathcal{M}})$, then the function $\otimes^{u}_{\odot}$ satisfies $(\mathsf{R1}_{\mathcal{M}})$–$(\mathsf{R6}_{\mathcal{M}})$ by Lemma 3.3.4 and $(\mathsf{R9}_{\mathcal{M}})$ by Definition 3.3.1. Hence, $\otimes^{u}_{\odot}$ is a revision candidate. By Theorem 4.2.15, we get that the function $\odot^{r\star}_{(\otimes^{u}_{\odot})}$ is an update candidate, which is defined to be the function $\odot^{cand}$. The proof of the second assertion works exactly the same way by first using Lemma 3.3.13 and Definition 3.3.9, and then applying Theorem 4.2.18. □

As an immediate consequence of Theorem 4.2.20 we get that the possible models approach $\odot_{\mathsf{pma}}$ and the variants of the standard update function $\odot_{\mathsf{su}}$ can be translated into update candidates.

Corollary 4.2.21. *We have that all the functions $(\odot_{\mathsf{pma}})^{cand}$, $(\odot_{\mathsf{csu}})^{cand}$, $(\odot_{\mathsf{psu}})^{cand}$, $(\odot_{\mathsf{ssu}})^{cand}$, and $(\odot_{\mathsf{su}})^{cand}$ are update candidates.*

The translations $\odot^{cand}$ and $\odot^{ccand}$ do not have an inverse function, because two belief change functions of different strength can result in the same candidate.

Lemma 4.2.22. *For all $S \in \mathcal{M}$ and all $\alpha \in \mathcal{L}_0$ we have*

$$S \odot_{\mathsf{mm}} \alpha = S\,(\odot_{\mathsf{mm}})^{cand}\,\alpha = S\,(\odot_{\mathsf{su}})^{cand}\,\alpha = S\,(\odot_{\mathsf{ssu}})^{cand}\,\alpha = S\,(\odot_{\mathsf{psu}})^{cand}\,\alpha = S\,(\odot_{\mathsf{csu}})^{cand}\,\alpha.$$

Proof. All equalities are directly implied by the definition of the respective functions. $\square$

It has turned out that the relations $\lesssim$ and $\approx$ from Definition 4.1.3 are preserved by the translation $\odot^{cand}$.

Lemma 4.2.23. *For all functions* $\odot, \odot' \colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ *we have*

$$\odot \lesssim \odot' \;\Rightarrow\; \odot^{cand} \lesssim (\odot')^{cand},$$
$$\odot \approx \odot' \;\Rightarrow\; \odot^{cand} \approx (\odot')^{cand}.$$

Proof. Both assertions immediately follow from Definition 4.1.3 and Definition 4.2.19. $\square$

We have seen in Lemma 4.2.22 that most of the update functions we have defined so far translate to the same update candidate $\odot_{\mathsf{mm}}$. This is not surprising, because they all perform the standard update if the new formula contradicts the actual beliefs. However, we can show that the update candidate $(\odot_{\mathsf{pma}})^{cand}$ that we get from the possible models approach is stronger than the function $\odot_{\mathsf{mm}}$.

Theorem 4.2.24. *We have the following comparability result,*

$$(\odot_{\mathsf{pma}})^{cand} < \odot_{\mathsf{mm}}.$$

Proof. We know from Theorem 4.1.5 that $\odot_{\mathsf{pma}} \lesssim \odot_{\mathsf{su}}$, hence we directly get $(\odot_{\mathsf{pma}})^{cand} \lesssim (\odot_{\mathsf{su}})^{cand}$ by Lemma 4.2.23. Lemma 4.2.22 now immediately implies $(\odot_{\mathsf{pma}})^{cand} \lesssim \odot_{\mathsf{mm}}$. The following example shows that the functions $(\odot_{\mathsf{pma}})^{cand}$ and $\odot_{\mathsf{mm}}$ have different strength. Let $\mathcal{P} = \{p, q\}$, $S = \{\emptyset\}$, and $\alpha = p \vee q$. Then we have $S \ (\odot_{\mathsf{pma}})^{cand} \ \alpha = S \odot_{\mathsf{pma}} \alpha = \{\{p\}, \{q\}\}$ and $S \odot_{\mathsf{mm}} \alpha = S \odot_{\mathsf{su}} \alpha = \{\{p\}, \{q\}, \{p, q\}\}$. We can now easily see that $\neg(p \wedge q) \in \mathrm{Th}(S \ (\odot_{\mathsf{pma}})^{cand} \ \alpha)$ but $\neg(p \wedge q) \notin \mathrm{Th}(S \odot_{\mathsf{mm}} \alpha)$, and we get $(\odot_{\mathsf{pma}})^{cand} \not\approx \odot_{\mathsf{mm}}$ by Definition 4.1.3. $\square$

Part II

Belief change in modal logic

Introduction to Part II

In the second part of this thesis we are going to present different approaches to expansion for multi-agent modal logic. We set a number n of rational agents, whose beliefs we want to reason about. For a short survey of the reasoning power of such agents, we refer to the book of Fagin et al. [25]. In addition to the language of classical propositional logic from Chapter 1, the formal language of epistemic (doxastic) logic contains modal operators $K_1, \ldots, K_n$ for each agent. A formula of the form $K_i\alpha$ is read as "agent i knows (believes) α". For the semantics, the agents' beliefs are encoded in Kripke structures that contain a set of possible worlds. These worlds are models of classical propositional logic augmented with some additional information. Therefore, belief change functions on model sets can be adapted to functions on Kripke structures. The resulting functions are called model transformations. In the following chapters we will define different model transformations in order to formalise different belief change functions in modal logic.

In multi-agent modal logic we can reason about the formulas that an agent believes to be true. This reasoning takes place in a static world. By use of a model transformation, we can simulate a dynamic world and observe how the agents' beliefs change. An example of this process has been given in [25] with the well-known Muddy Children Puzzle. But belief change in modal logic encompasses more: we also want to be able to reason about the process of learning. That is, we want to enrich the logical language with new operators in order to express sentences like "after the agents learnt a formula α, the formula β holds". So, unlike in Part I, the act of learning should be contained in the formal language. For this purpose we will introduce languages that extend the language of modal logic with dynamic style operators.

A modal logic that contains operators for formalising the act of learning has been originally presented by Plaza [57]. The publication of this work can be seen as the foundation of dynamic epistemic logic. Plaza has extended the language of modal logic by public announcement operators. The new formulas of the form $[\alpha!]\beta$ are read as "β holds after the public announcement of α". Of course, other readings are possible, for instance, "β holds after the agents have observed α". The notion of an announcement formula $[\alpha!]\beta$ being satisfied is defined via a model transformation: the formula $[\alpha!]\beta$ holds at some possible world, if the formula β holds at some world in the transformed Kripke structure. The transformed Kripke structure is constructed from the original one and, in general, depends on α. Since communication is not always public, Gerbrandy and Groeneveld have introduced group announcement operators in [30]. For all non empty groups G of agents we have formulas of the form $[\alpha!_G]\beta$. They are read as "β holds after the announcement of α to the group G". We will provide model transformations for the semantical definition of both public and group announcement operators. In addition, we will give axiomatisations for each announcement logic and prove soundness and completeness.

Outline

It is the aim of Chapter 5 to introduce the syntax and semantics of multi-agent modal logics that we are going to use in Part II. First, we will define the formal language that extends the language of classical propositional logic with knowledge operators K_i for each agent ($1 \leq i \leq n$). We will then give the definition of Kripke semantics. We are interested in four properties of knowledge: consistency of knowledge, truth, positive introspection, and negative introspection. Each of these properties can be formalised by an axiom. Nine different subsets of such axioms lead to the Hilbert systems of our interest, and we will present the well-known soundness and completeness results for these nine deductive systems.

Common knowledge is an important concept in epistemic logic. For all non empty groups G of agents we will add a modal operator C_G to the language of modal logic. The formulas of the form $C_G\alpha$ are read as "α is common knowledge among the agents in G". This sentence can be seen as an abbreviation for the following infinite statement, "everybody in G knows α and everybody in G knows that everybody in G knows α and ...". The extended possible worlds semantics corresponds to this interpretation of the common knowledge formulas. The nine deductive systems can be extended with an axiom and an inference rule for common knowledge. Again, we will state the soundness and completeness results for the Hilbert systems.

As a generalisation of common knowledge, we will also introduce relativised common knowledge operators. That is, for all non empty groups G of agents we will add the binary modal operator RC_G to the language of modal logic. The interpretation of the formulas of the form $RC_G(\alpha, \beta)$ is related to the interpretation of the until operator from temporal logic. Furthermore, common knowledge can be defined by use of relativised common knowledge operators. Our main contribution in Chapter 5 is the definition of a new axiomatisation for the logic of relativised common knowledge. We will give a soundness and completeness proof for the new Hilbert systems.

Another extension of epistemic logic is the logic of knowledge and belief. That is, we will add the modal operators $B_1, \ldots, B_n$ to the basic language of modal logic. A formula of the form $B_i\alpha$ is read as "agent i believes α". There are several properties that describe the interrelation between knowledge and belief. We will choose three sets of such properties and will define the corresponding Hilbert systems. Each of these systems will also be extended with axioms and rules for common knowledge and common belief. We will then present the soundness and completeness results for our six bimodal deductive systems.

In the end of Chapter 5 we will define the languages that extend the language of epistemic logic by public announcement and group announcement operators respectively. These announcement operators are related to the operators from dynamic logic. We will introduce the semantics of the logic of public communications by Plaza [57] as an introductory example. The semantics of an announcement operator is given by a belief change function on Kripke structures, also called model transformation. We will define five important properties of announcements, and test each type of announcement for them. For instance, one such property is called fact preservation and requires that propositional facts be unaffected by any announcements. Finally, we will introduce the concept of announcement resistant formulas. As the name suggests, a true announcement resistant formula always remains true after any announcement.

In Chapter 6 we will define two different semantics for group announcement operators. First, we will define a model transformation that is related to the expansion function on model sets from Chapter 1. That is, the agents will always accept the announced formula, which can lead to inconsistent belief. We will show that the beliefs of an agent remain unchanged if this agent does not belong to the group that the new information is announced to. We will provide three different Hilbert systems that extend three of our nine deductive systems of modal logic. For these three systems we will prove soundness and completeness. We have the same deductive systems as Gerbrandy and Groeneveld have in [30]. Therefore, it follows that our interpretation of group announcements is equivalent to their non well-founded semantics, cf. [30].

Our second model transformation for the semantics of group announcement operators is related to the consistent expansion function from Chapter 3. Therefore, the agents reject an announcement if the new information contradicts their beliefs. Again, we will show that an agent does not learn anything, if this agent is not in the group that receives the announcement. In addition to our previous work [61] where we have defined one deductive system for the new semantics, we will now define four Hilbert systems. We will prove soundness and completeness for these four systems and investigate the properties of the new interpretation of group announcements.

We will end Chapter 6 with a thorough analysis of the logic of common knowledge and group announcements. To this end, we will use our first semantics of group announcements, where the agents always accept the new information. The system of Baltag, Moss, and Solecki [7] is an axiomatisation of a similar semantics, but within a much stronger language. We will provide a similar axiomatisation and prove soundness. Completeness remains an open problem.

Chapter 7 is about public announcement logics. First, we will introduce the well-known semantics of truthful public announcements, cf. [22]. These public announcements are partial, because the syntax is defined by a partial model transformation. Since only true formulas can be truthfully announced, an announcement with a false formula leads to an inconsistent epistemic state. We will then present six axiomatisations extending six of our nine Hilbert systems for modal logic. In addition, we will give the corresponding axiomatisations for the logic of common knowledge and relativised common knowledge respectively augmented with public announcement operators. The results for all these logics are not new, but we will provide new proofs similar to the ones in Chapter 6. That is, most of our proofs are syntactic.

We will then define a new public announcement semantics by using the total model transformation. This total public announcement semantics works as follows. If the announced formula is true, then the model transformation is the same as for truthful public announcements. If the announced formula is false, then the model transformation has no effect on the Kripke structure, and the knowledge of the agent remains unchanged. Compared to our previous contribution in [62], where we have given one axiomatisation, we will now provide six Hilbert systems for total public announcements. Again, we will prove soundness and completeness for these six systems. Moreover, we will thoroughly investigate the properties of total public announcements. We will be able to formally prove that a total public announcement with a true announcement-free formula has the same effect on announcement-free formulas as the truthful public announcement with the same formula does.

In a next step we will study total public announcements in the presence of common knowledge and relativised common knowledge respectively. Again, we will define six deductive systems for both logics. The completeness proofs are different. With relativised common knowledge we can define a translation to the announcement-free fragment of the language so that the translation of a formula is provably equivalent to the formula itself. Then completeness follows from the completeness of the announcement-free fragment. This implies that total public announcements do not add expressive strength to the logic of relativised common knowledge. The situation is different with common knowledge, because there is no such translation. Therefore, the completeness proof has to be worked out like for the logic of common knowledge, cf. [25]. However, the proof is similar to the one for the logic of common knowledge and truthful public announcements, cf. [22].

Finally, we will present an application of total public announcements in Chapter 8. We will define a model transformation for the logic of knowledge, belief, and public announcements. The agents' beliefs are affected by every public announcement, whereas only true facts will be learnt on the knowledge level. We will prove soundness and completeness for a Hilbert system that extends one of our three systems for the logic of knowledge and belief.

Chapter 5

Multi-agent modal logics

This is an introductory chapter where we will introduce the standard notions and results on normal modal logics as well as the basic concepts of announcement logics. In Section 5.1, we will define the epistemic logics of our interest. We will give the standard semantics and state the well-known soundness and completeness results. We will define the notion of common knowledge and relativised common knowledge in Section 5.2. While the first has been thoroughly investigated, the latter is quite new, and we will give a new axiomatisation. Section 5.3 deals with deductive systems for the bi-modal logics of knowledge and belief. There are several axioms that describe the interaction between knowledge and belief. We will have to make a choice of such axioms, because some combinations of these axioms lead to triviality results. The chapter ends with a short introduction to announcement logics in Section 5.4. We will define both private and public announcement operators and discuss some important properties of announcement logics. We will conclude the section with defining distinguished sets of formulas.

5.1 Epistemic logics

In this part of the thesis, we are dealing with a number $n \geq 1$ of *rational agents*. We do not fix the exact application area and capabilities of our fictive agents; they can be seen as subsystems in a general system, e. g. computers in a network, processors in a computer, processes in an operating system, or simply players in a game. For a more precise description of rational agents, we refer to the introduction of the book of Fagin, Halpern, Moses, and Vardi [25].

In conjunction with rational agents, it is useful to work with the concept of knowledge that such agents can have, see [25]. This is where the name *epistemic logic* comes from. For this purpose, we are going to define the language of multi-agent modal logic with n agents. From now on, we will use the set $\mathcal{A} := \{1, \ldots, n\}$ to denote the group of all agents. The language contains a modal operator K_i for every agent $i \in \mathcal{A}$. The countable set $\mathcal{P} \neq \emptyset$ is still our collection of propositions.

Definition 5.1.1. The language $\mathcal{L}_n$ of epistemic logic for n agents is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha.$$

The propositional constants $\top$ and $\bot$, as well as the binary connectives $\vee$, $\rightarrow$, and $\leftrightarrow$, are defined like in Definition 1.1.1. The formula $K_i\alpha$ stands for "agent i knows α". We will now define the notion of *length*, *subformula*, and *substitution* for the language $\mathcal{L}_n$.

Definition 5.1.2. The following defining clauses extend Definition 1.1.2, Definition 1.1.3, and Definition 1.1.4 respectively,

$$|K_i\beta| := |\beta| + 1,$$

$$\mathrm{sub}(K_i\beta) := \{K_i\beta\} \cup \mathrm{sub}(\beta),$$

$$(K_i\beta)[\varphi/\psi] := \begin{cases} \varphi & \text{if } \psi = K_i\beta, \\ K_i(\beta[\varphi/\psi]) & \text{otherwise.} \end{cases}$$

Remember that we will write "induction on α" if we do an induction on the length of a formula α.

The commonly used semantics for multi-agent modal logics is called *possible worlds semantics*. A brief overview on the history of this semantics can be found in the book of Fagin et al. [25] or in the book of Blackburn, de Rijke, and Venema [15]. The semantical objects are directed labelled graphs where the vertices (*possible worlds* or *information states*) are models of classical propositional logic and for every agent we have a different type of edges (*accessibility relations*). At any possible world, the accessibility relation of an agent points to a (possibly empty) set of worlds that this agent considers possible.

Definition 5.1.3. A *Kripke structure* (for n agents) is defined to be a tuple $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$, where $S \neq \emptyset$ is a set of possible worlds, $R_i \subseteq S^2$ is a binary relation for each agent $i \in \mathcal{A}$, and $V : \mathcal{P} \to \mathrm{Pow}(S)$ is a *valuation function*. A Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ can be seen as a relational structure[1] over S, so we call S the *universe* of $\mathfrak{K}$, denoted by $|\mathfrak{K}|$. A *pointed structure* is a pair $\mathfrak{K}, s$ where $\mathfrak{K}$ is a Kripke structure and $s \in |\mathfrak{K}|$.

[1] Every Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ can be described by the relational structure $(S, R_1, \ldots, R_n, V(p_0), V(p_1), \ldots)$, see [15].

We will use the infix notation sR_it for the expression $(s,t) \in R_i$, which means that at world s, agent i considers the world t possible. Sometimes, we will use the set

$$R_i(s) := \{t \in S : sR_it\}$$

to denote the worlds that agent i considers possible at world s.

As a usual question in mathematical theories, we want to know when two Kripke structures are the same, that is, when do two pointed structures represent the same information states of the agents. It has turned out that the suitable answer to this question is the notion of bisimulation. An overview on different kinds of bisimulation relations can be found in Gerbrandy's thesis [29].

Definition 5.1.4. Let $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ and $\mathfrak{K}' = (S', R'_1, \ldots, R'_n, V')$ be two given Kripke structures. Then a binary relation $B \subseteq S \times S'$ is called a *bisimulation* between $\mathfrak{K}$ and $\mathfrak{K}'$, if for all $s \in S$ and all $s' \in S'$ we have that sBs' implies

1. for all $p \in \mathcal{P}$ we have $s \in V(p)$ if and only if $s' \in V'(p)$,
2. for all $i \in \mathcal{A}$ and all $t \in R_i(s)$ there is a $t' \in R'_i(s')$ such that tBt',
3. for all $i \in \mathcal{A}$ and all $t' \in R'_i(s')$ there is a $t \in R_i(s)$ such that tBt'.

Two pointed structures $\mathfrak{K}, s$ and $\mathfrak{K}', s'$ are *bisimilar*, denoted by $\mathfrak{K}, s \simeq \mathfrak{K}', s'$, if there is a bisimulation B between $\mathfrak{K}$ and $\mathfrak{K}'$ such that sBs'. Sometimes we will write $\mathfrak{K}, s \simeq_B \mathfrak{K}', s'$ to express that B is a bisimulation between $\mathfrak{K}$ and $\mathfrak{K}'$ that *connects* s and s', that is sBs'.

Satisfaction of formulas is locally evaluated within Kripke structures: at each world in a given Kripke Structure, a formula either holds or does not hold. The corresponding satisfaction definition is the following.

Definition 5.1.5. Let the Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ be given and let $s \in S$. Then the notion of an $\mathcal{L}_n$ formula being *satisfied* in the pointed structure $\mathfrak{K}, s$ is inductively defined as follows,

$$\begin{aligned}
\mathfrak{K}, s \models p \;&:\Leftrightarrow\; s \in V(p),\\
\mathfrak{K}, s \models \neg\alpha \;&:\Leftrightarrow\; \mathfrak{K}, s \not\models \alpha,\\
\mathfrak{K}, s \models \alpha \wedge \beta \;&:\Leftrightarrow\; \mathfrak{K}, s \models \alpha \text{ and } \mathfrak{K}, s \models \beta,\\
\mathfrak{K}, s \models K_i\alpha \;&:\Leftrightarrow\; \text{for all } t \in R_i(s),\ \mathfrak{K}, t \models \alpha.
\end{aligned}$$

If a formula α is satisfied at a world u, we sometimes use the terms α *holds* at u, or α is *true* at world u. We say that a formula α is *valid* in a Kripke structure $\mathfrak{K}$ ($\mathfrak{K} \models \alpha$), if for all $s \in |\mathfrak{K}|$ we have $\mathfrak{K}, s \models \alpha$. Accordingly, a formula α is *valid with respect to* a class $\mathcal{X}$ of Kripke structures ($\mathcal{X} \models \alpha$), if for all $\mathfrak{K} \in \mathcal{X}$ we have $\mathfrak{K} \models \alpha$. Sometimes, it is useful to have an expression for the subset of $|\mathfrak{K}|$ where a formula α is satisfied. Hence, we define the *extension* $\|\alpha\|_{\mathfrak{K}}$ of a formula α in a Kripke structure $\mathfrak{K}$ by

$$\|\alpha\|_{\mathfrak{K}} := \{s \in |\mathfrak{K}| : \mathfrak{K}, s \models \alpha\}.$$

Throughout this thesis, the deductive systems for modal logics are given by a so-called *Hilbert calculus*. The usual way of stating a Hilbert system for modal logics is to focus on the non propositional part of the language, that is the propositional tautologies do not need to be derived.

Definition 5.1.6. The system K_n consists of the *tautology axiom* and the *distribution axiom*,

(PT) Every instance of a propositional tautology,

(K) $K_i(\alpha \to \beta) \to (K_i\alpha \to K_i\beta)$,

as well as the *modus ponens rule* and the *necessitation rule*,

$$(\mathsf{MP})\ \frac{\alpha \qquad \alpha \to \beta}{\beta}, \qquad\qquad (\mathsf{NEC})\ \frac{\alpha}{K_i\alpha}.$$

A *proof* of a formula α in a Hilbert system X is a finite sequence $\alpha_1, \dots, \alpha_m$ of formulas, such that $\alpha_m = \alpha$, and every α_i is either an instance of an axiom of X or the conclusion of an inference rule of X with the premisses being elements of the sequence $\alpha_1, \dots, \alpha_{i-1}$. We say that a formula α is *provable* in a Hilbert system X ($\mathsf{X} \vdash \alpha$), if there is a proof of α in X. Furthermore, a system X is called *sound* with respect to a class $\mathcal{X}$ of Kripke structures, if for all formulas α we have

$$\mathsf{X} \vdash \alpha \ \Rightarrow\ \mathcal{X} \models \alpha.$$

On the other hand, X is called *complete* with respect to $\mathcal{X}$, if we have

$$\mathcal{X} \models \alpha \ \Rightarrow\ \mathsf{X} \vdash \alpha$$

for all formulas α. A Hilbert system X is called *consistent*, if there is no proof of $\bot$ in X, that is if $\mathsf{X} \nvdash \bot$. Clearly, if X is sound with respect to some class $\mathcal{X}$ of Kripke structures, then it is also consistent. On the other hand, if a Hilbert system X is consistent, it is not immediate how to find its corresponding class of Kripke structures. In Remark 6.2.8 we will point to such a consistent system. For every Hilbert system X, there is the notion of maximal X-consistent sets, which we will use in some completeness proofs.

Definition 5.1.7. Let X be a Hilbert system for a language $\mathcal{L}$ extending $\mathcal{L}_n$. Then we call a set $Z \subseteq \mathcal{L}$ of formulas X-*consistent*, if for every finite subset $\{\alpha_1, \ldots, \alpha_m\} \subseteq Z$ of Z we have

$$\mathsf{X} \nvdash \neg(\alpha_1 \wedge \ldots \wedge \alpha_m).$$

An $\mathcal{L}$ formula α is called X-consistent, if the set $\{\alpha\}$ is X-consistent. $Z \subseteq \mathcal{L}$ is called a *maximal* X-*consistent* set, if Z is X-consistent and for every $\alpha \in \mathcal{L} \setminus Z$ we have that $Z \cup \{\alpha\}$ is not X-consistent.

Throughout this thesis, every language is closed under the propositional connectives $\neg$ and $\wedge$. In addition, every Hilbert calculus in this thesis is consistent and contains the axiom (PT) and the inference rule (MP). For all these reasons the following lemma holds, a proof can be found in [25].

Lemma 5.1.8. *Let $\mathcal{L}$ be a language extending $\mathcal{L}_n$ and X be a Hilbert system containing K_n. Then every X-consistent set of $\mathcal{L}$ formulas can be extended to a maximal X-consistent set. In addition, every maximal X-consistent set $Z \subseteq \mathcal{L}$ satisfies the following conditions for all $\alpha, \beta \in \mathcal{L}$,*

1. $\alpha \in Z \;\Leftrightarrow\; \neg\alpha \notin Z$,
2. $\alpha \wedge \beta \in Z \;\Leftrightarrow\; \alpha \in Z$ *and* $\beta \in Z$,
3. $\alpha \in Z$ *and* $\alpha \to \beta \in Z \;\Rightarrow\; \beta \in Z$,
4. $\mathsf{X} \vdash \alpha \;\Rightarrow\; \alpha \in Z$.

Observe that the first assertion of Lemma 5.1.8 is equivalent to saying that exactly one of the formulas α and $\neg\alpha$ is an element of Z.

There have been many discussions about what properties rational agents should have in what situations, see Blackburn et al. [15] as well as Fagin et al. [25] for an overview. It is not surprising that the only consensus is that the properties and capabilities of the agents always depend on the application. For our purposes, we will choose from the following options,

(D)	$K_i\neg\alpha \to \neg K_i\alpha$	(*consistency*),
(T)	$K_i\alpha \to \alpha$	(*truth/knowledge*),
($\mathsf{4}$)	$K_i\alpha \to K_iK_i\alpha$	(*positive introspection*),
($\mathsf{5}$)	$\neg K_i\alpha \to K_i\neg K_i\alpha$	(*negative introspection*),

where we choose ($\mathsf{5}$) only if we choose ($\mathsf{4}$). Clearly, we will always choose the same properties (axioms) for all agents. The names of the different axioms and systems are given by traditional habits and are commonly known in the community, cf. [15, 25].

Definition 5.1.9. The system K_n as well as the following extensions of K_n are the deductive systems in our focus,

$$\begin{array}{ccc} & \mathsf{K4}_n := \mathsf{K}_n + (\mathsf{4}), & \mathsf{K45}_n := \mathsf{K4}_n + (\mathsf{5}), \\ \mathsf{KD}_n := \mathsf{K}_n + (\mathsf{D}), & \mathsf{KD4}_n := \mathsf{KD}_n + (\mathsf{4}), & \mathsf{KD45}_n := \mathsf{KD4}_n + (\mathsf{5}), \\ \mathsf{T}_n := \mathsf{K}_n + (\mathsf{T}), & \mathsf{S4}_n := \mathsf{T}_n + (\mathsf{4}), & \mathsf{S5}_n := \mathsf{S4}_n + (\mathsf{5}). \end{array}$$

Observe that $(\mathsf{4})$ is included in $\mathsf{S5}_n$ for traditional reasons, it can easily be proved in the system $\mathsf{T}_n + (\mathsf{5})$.

Although we use the terms knowledge and epistemic logic, we want to mention here that the systems without the truth axiom (T) are often called systems of belief. In this terminology, the expression *doxastic logic* would be more appropriate. For simplicity reasons, we do not make this distinction.

It has turned out that each of the above mentioned axioms can be satisfied by restricting the accessibility relations R_i correspondingly. Therefore, each of our deductive systems has its characterising class of Kripke structures. The restrictions of the accessibility relations are called *frame conditions*, because the semantics of modal logics is often defined via *frames*, that is a Kripke structure without the valuation, see e. g. Blackburn et al. [15]. For our deductive systems it is sufficient to take the right combinations of the following frame conditions,

s: for all $u \in |\mathfrak{K}|$, $R_i(u) \neq \emptyset$ (*seriality*),

r: for all $u \in |\mathfrak{K}|$, $u \in R_i(u)$ (*reflexivity*),

t: for all $u, v, w \in |\mathfrak{K}|$, uR_iv and $vR_iw \;\Rightarrow\; uR_iw$ (*transitivity*),

u: for all $u, v, w \in |\mathfrak{K}|$, uR_iv and $uR_iw \;\Rightarrow\; vR_iw$ (*Euclideanity*).

Clearly, we always take the same frame conditions for all agents. We will write $\mathcal{K}_n$ for the class of all Kripke structures. Furthermore, for a subclass $\mathcal{X}$ of Kripke structures we will write $\mathcal{K}_n^{\vec{x}}$, where $\vec{x}$ consists of the one-letter names of the frame conditions that define $\mathcal{X}$. For instance, $\mathcal{K}_n^{st}$ denotes the class of Kripke structures that have accessibility relations that are both serial and transitive.

We are now ready to state the soundness and completeness theorems for all of our deductive systems. Soundness is proved by induction on the length of the proof, see [34]. A Completeness proof for K_n, T_n, $\mathsf{S4}_n$, $\mathsf{S5}_n$, and $\mathsf{KD45}_n$ can be found in [34], the proof for the other systems requires only slight modifications.

Theorem 5.1.10. *For all $\alpha \in \mathcal{L}_n$ we have*

$$\mathsf{K}_n \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n \vDash \alpha, \qquad\qquad \mathsf{K4}_n \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^t \vDash \alpha,$$

$$\begin{aligned} \mathsf{K45}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{tu} \models \alpha, & \mathsf{KD}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{s} \models \alpha, \\ \mathsf{KD4}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{st} \models \alpha, & \mathsf{KD45}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{stu} \models \alpha, \\ \mathsf{T}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{r} \models \alpha, & \mathsf{S4}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{rt} \models \alpha, \\ \mathsf{S5}_n \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{rtu} \models \alpha. \end{aligned}$$

Observe that the letter t for transitivity in the superscript of $\mathcal{K}_n^{rtu}$ would not be required, because every reflexive and Euclidean relation is also transitive. But we want to write all the properties that correspond to an axiom.

5.2 Logics of common knowledge

It is the aim of this section to introduce the concepts of common knowledge and relativised common knowledge. Relativised common knowledge is a generalisation of common knowledge, and we will show completeness of our axiomatisation of relativised common knowledge from [62]. We start with extending the language $\mathcal{L}_n$ with common knowledge operators.

Definition 5.2.1. The language $\mathcal{L}_n^C$ of epistemic logic for n agents with *common knowledge* is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ C_G\alpha.$$

The formula $C_G\alpha$ is read as "α is common knowledge among the members in G". The following defining clauses extend Definition 5.1.2,

$$\begin{aligned} |C_G\beta| &:= |\beta| + 1, \\ \mathrm{sub}(C_G\beta) &:= \{C_G\beta\} \cup \mathrm{sub}(\beta), \\ (C_G\beta)[\varphi/\psi] &:= \begin{cases} \varphi & \text{if } \psi = C_G\beta, \\ C_G(\beta[\varphi/\psi]) & \text{otherwise.} \end{cases} \end{aligned}$$

It will be useful to define the notion of *mutual knowledge* for an arbitrary group $G \subseteq \mathcal{A}$ of agents, which is the following abbreviation,

$$E_G\alpha := \bigwedge_{i \in G} K_i\alpha.$$

The formula $E_G\alpha$ is read as "everybody in G knows α". Observe, that the operator E_G is also defined for $G = \emptyset$, in that case it is defined to be $\top$. Iterated mutual knowledge is inductively defined by

$$E_G^0\alpha := \alpha, \qquad\qquad E_G^{k+1} := E_G E_G^k\alpha.$$

The intended semantics of common knowledge is the following. A formula α is common knowledge, if and only if everybody knows α ($E_G\alpha$), everybody knows that everybody knows α ($E_G^2\alpha$), and so on. This means that a formula $C_G\alpha$ is satisfied in a pointed structure $\mathfrak{K}, s$ if and only if, informally speaking, the infinite expression

$$E_G(\alpha \wedge E_G(\alpha \wedge E_G(\alpha \wedge \dots)))$$

is supposed to hold in $\mathfrak{K}, s$. It has turned out that a formula α is common knowledge at world u, if and only if α holds at every world v reachable from u by a *G-path*, see Fagin et al. [25] for a proof. These G-paths can be described by first defining the union R_G of the accessibility relations from the agents belonging to G,

$$R_G := \bigcup_{i \in G} R_i,$$

and then taking the *transitive closure* R_G^+ of R_G. In order to define the transitive closure of a binary relation, we introduce the product

$$RQ := \{(x, z) : \text{there is a } y,\ xRy \text{ and } yQz\}$$

of any two binary relations R and Q on the same set. The transitive closure R^+ of a binary relation R can now be obtained by taking the union of all the powers of R,

$$R^1 := R, \qquad R^{k+1} := RR^k, \qquad R^+ := \bigcup_{k \geq 1} R^k.$$

The expressions sR_Gt and $R_G(s)$ as well as sR_G^+t and $R_G^+(s)$ are defined like sR_it and $R_i(s)$ respectively. We are now ready to add the defining clause for common knowledge formulas to Definition 5.1.5.

Definition 5.2.2. The definition of an $\mathcal{L}_n^C$ formula $C_G\alpha$ being *satisfied* in a pointed structure $\mathfrak{K}, s$ is the following,

$$\mathfrak{K}, s \vDash C_G\alpha \ :\Leftrightarrow\ \text{for all } t \in R_G^+(s),\ \mathfrak{K}, t \vDash \alpha.$$

The deductive systems for the logic of common knowledge can be obtained by adding the *co-closure axiom* (C) and the *induction rule* (CI) to the systems of epistemic logic from Section 5.1.

Definition 5.2.3. If X is one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, KD_n, $\mathsf{KD4}_n$, $\mathsf{KD45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$, then the system X^C is defined to be X augmented with the co-closure axiom for common knowledge,

(C) $\quad C_G\alpha \to E_G(\alpha \wedge C_G\alpha)$,

as well as the induction rule for common knowledge,

$$(\mathsf{CI})\ \frac{\alpha \to E_G(\alpha \wedge \beta)}{\alpha \to C_G\beta}\ .$$

We have soundness and completeness results for these systems like in Section 5.1. For the systems K_n^C, T_n^C, $\mathsf{S4}_n^C$, $\mathsf{S5}_n^C$, and $\mathsf{KD45}_n^C$ we refer to Fagin et al. [25]. For the other systems the proof requires only slight modifications.

Theorem 5.2.4. *For all $\mathcal{L}_n^C$ formulas α we have*

$$\begin{aligned}
\mathsf{K}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n \models \alpha, & \mathsf{K4}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^t \models \alpha,\\
\mathsf{K45}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{tu} \models \alpha, & \mathsf{KD}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^s \models \alpha,\\
\mathsf{KD4}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{st} \models \alpha, & \mathsf{KD45}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{stu} \models \alpha,\\
\mathsf{T}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^r \models \alpha, & \mathsf{S4}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{rt} \models \alpha,\\
\mathsf{S5}_n^C \vdash \alpha &\Leftrightarrow \mathcal{K}_n^{rtu} \models \alpha.
\end{aligned}$$

In the context of dynamic epistemic logics (see Section 5.4), the notion of relativised common knowledge has been introduced in order to compare the expressive strength of various logics, cf. van Benthem et al. [12] and Kooi [49]. It has turned out that relativised common knowledge is more expressive than common knowledge, see [12] for a proof.

Definition 5.2.5. The language $\mathcal{L}_n^{RC}$ of epistemic logic for n agents and *relativised common knowledge* is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha) \ \mid \ K_i\alpha \ \mid \ RC_G(\alpha, \alpha).$$

The following defining clauses extend Definition 5.1.2,

$$\begin{aligned}
|RC_G(\beta,\gamma)| &:= |\beta| + |\gamma| + 1,\\
\mathrm{sub}(RC_G(\beta,\gamma)) &:= \{RC_G(\beta,\gamma)\} \cup \mathrm{sub}(\beta) \cup \mathrm{sub}(\gamma),\\
RC_G(\beta,\gamma)[\varphi/\psi] &:= \begin{cases} \varphi & \text{if } \psi = RC_G(\beta,\gamma),\\ RC_G(\beta[\varphi/\psi], \gamma[\varphi/\psi]) & \text{otherwise.} \end{cases}
\end{aligned}$$

The semantics for relativised common knowledge results from *relativising* the semantics of common knowledge. That is, a formula $RC_G(\alpha, \beta)$ holds at a world u, if β holds at all worlds v accessible from u by a G-path such that α

holds at every world of this path (but not necessarily at u). We call such a path a *G-path relativised to α*. From this informal definition, it is immediate how to define common knowledge from relativised common knowledge. The following definition has been given by van Benthem et al. [11, 12],

$$C_G\alpha := RC_G(\top, \alpha).$$

We will now add the formal defining clause for relativised common knowledge formulas to Definition 5.1.5.

Definition 5.2.6. The definition of an $\mathcal{L}_n^{RC}$ formula $RC_G(\alpha, \beta)$ being *satisfied* in a pointed structure $\mathfrak{K}, s$ is the following,

$$\mathfrak{K}, s \vDash RC_G(\alpha, \beta) \;:\Leftrightarrow\; \text{for all } t \in (R_G \cap (|\mathfrak{K}| \times \|\alpha\|_{\mathfrak{K}}))^+(s),\ \mathfrak{K}, t \vDash \beta.$$

It has often been mentioned that relativised common knowledge is related to the *until operator* from temporal logic, cf. [12]. We will now state some concrete properties of this relationship.

Remark 5.2.7. First, we will show how the until operator can be defined in the logic of relativised common knowledge. The expression "α *until* β" from linear time temporal logics corresponds to the formula

$$\beta \vee (\alpha \wedge RC(\neg\beta, \alpha) \wedge \neg RC(\top, \neg\beta)),$$

where we have omitted the group subscript ($\mathcal{A} = \{1\}$). On the other hand, the formula $RC(\alpha, \beta)$ can be expressed in temporal logics by

$$\text{“}\mathit{next}(\beta \ \mathit{until}\ \neg\alpha) \vee \mathit{next}(\mathit{always}(\alpha \wedge \beta))\text{”}.$$

For a precise definition of the syntax and semantics of propositional temporal logics, we refer to the article of Lichtenstein and Pnueli [52].

The deductive systems for the logic of relativised common knowledge can be obtained by adding the co-closure axiom (RC) and the induction rule (RCI) to the systems of epistemic logic from Section 5.1.

Definition 5.2.8. If X is one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, KD_n, $\mathsf{KD4}_n$, $\mathsf{KD45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$, then the system X^{RC} is defined to be X augmented with the *co-closure axiom* for relativised common knowledge,

(RC) $\quad RC_G(\alpha, \beta) \to E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta))$,

as well as the *induction rule* for relativised common knowledge,

$$(\mathsf{RCI})\ \frac{\alpha \to E_G(\beta \to \alpha \wedge \gamma)}{\alpha \to RC_G(\beta, \gamma)}\,.$$

We will now prove soundness and completeness for these systems. For this purpose, we will show that our system K_n^{RC} is equivalent to the system of van Benthem et al. from [12]. The direct proof is not hard but tedious, we did it for $\mathsf{S5}_n^{RC}$ in [62].

Theorem 5.2.9. *For all $\mathcal{L}_n^{RC}$ formulas α we have*

$$\begin{array}{rclrcl}
\mathsf{K}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n \models \alpha, & \mathsf{K4}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^t \models \alpha,\\
\mathsf{K45}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{tu} \models \alpha, & \mathsf{KD}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^s \models \alpha,\\
\mathsf{KD4}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{st} \models \alpha, & \mathsf{KD45}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{stu} \models \alpha,\\
\mathsf{T}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^r \models \alpha, & \mathsf{S4}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rt} \models \alpha,\\
\mathsf{S5}_n^{RC} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rtu} \models \alpha. & & &
\end{array}$$

Proof. The system EL-RC from van Benthem et al. [12] is defined to be K_n augmented with the following *relativised common knowledge axioms*,

(RC-Dist) $RC_G(\alpha, \beta \to \gamma) \to (RC_G(\alpha, \beta) \to RC_G(\alpha, \gamma))$,

(RC-Mix) $RC_G(\alpha, \beta) \leftrightarrow E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta))$,

(RC-Ind) $(E_G(\alpha \to \beta) \wedge RC_G(\alpha, \beta \to E_G(\alpha \to \beta))) \to RC_G(\alpha, \beta)$,

and the *necessitation rule* for relativised common knowledge,

$$\text{(RC-Nec)}\ \frac{\alpha}{RC_G(\beta, \alpha)}\,.$$

We know from [12] that EL-RC is sound and complete with respect to $\mathcal{K}_n$.

For the soundness proof of K_n^{RC}, we show that the axiom (RC) is derivable and the rule (RCI) is admissible in EL-RC. Since (RC) is contained in this system, we only have to show the admissibility of (RCI). Suppose, that we have a proof of the formula

$$\alpha \to E_G(\beta \to \alpha \wedge \gamma)$$

for some formulas α, β, and γ. Then we can easily derive

$$\alpha \wedge \gamma \to E_G(\beta \to \alpha \wedge \gamma),$$

and with (RC-Nec) we get

$$RC_G(\beta, \alpha \wedge \gamma \to E_G(\beta \to \alpha \wedge \gamma)).$$

This immediately yields

$$\alpha \to RC_G(\beta, \alpha \wedge \gamma \to E_G(\beta \to \alpha \wedge \gamma)),$$

and together with $\alpha \to E_G(\beta \to \alpha \wedge \gamma)$ from the assumption, (RC-Ind) and some tautological derivations, we get

$$\alpha \to RC_G(\beta, \alpha \wedge \gamma).$$

By use of (RC-Nec) and (RC-Dist) we finally get $\alpha \to RC_G(\beta, \gamma)$, and soundness of K_n^{RC} is proved.

In order to prove completeness of K_n^{RC}, we need to show that the axioms (RC-Dist), (RC-Mix), and (RC-Ind) are derivable and the rule (RC-Nec) is admissible in K_n^{RC}. For the derivation of (RC-Dist), we take some arbitrary $\mathcal{L}_n^{RC}$ formulas α, β, and γ, and we start with the following instances of (RC),

$$RC_G(\alpha, \beta \to \gamma) \to E_G(\alpha \to (\beta \to \gamma) \wedge RC_G(\alpha, \beta \to \gamma)),$$
$$RC_G(\alpha, \beta) \to E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta)).$$

From those we can derive the formula

$$RC_G(\alpha, \beta \to \gamma) \wedge RC_G(\alpha, \beta) \to E_G(\alpha \to RC_G(\alpha, \beta \to \gamma) \wedge RC_G(\alpha, \beta) \wedge \gamma).$$

Now, we can apply the rule (RCI) to get the formula

$$RC_G(\alpha, \beta \to \gamma) \wedge RC_G(\alpha, \beta) \to RC_G(\alpha, \gamma),$$

which is equivalent to (RC-Dist) by tautological reasoning.

In order to derive (RC-Mix), we can see that one direction of the equivalence is exactly the axiom (RC). For the other direction, we start with (RC),

$$RC_G(\alpha, \beta) \to E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta)),$$

from which we derive the formula

$$E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta)) \to E_G(\alpha \to E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta)) \wedge \beta).$$

Applying the rule (RCI) now does the job.

For the derivation of (RC-Ind), we start again with an instance of (RC),

$$RC_G(\alpha, \beta \to E_G(\alpha \to \beta)) \to E_G(\alpha \to (\beta \to E_G(\alpha \to \beta)) \wedge RC_G(\alpha, \beta \to E_G(\alpha \to \beta))).$$

Doing some derivations using (PT), (K), and (NEC), we get the formula

$$E_G(\alpha \to \beta) \land RC_G(\alpha, \beta \to E_G(\alpha \to \beta)) \to$$
$$E_G(\alpha \to E_G(\alpha \to \beta) \land RC_G(\alpha, \beta \to E_G(\alpha \to \beta)) \land \beta),$$

and we finish again with an application of (RCI).

For the admissibility of the rule (RC-Nec), we start with a provable formula α, derive the formula

$$\top \to E_G(\beta \to \top \land \alpha),$$

and after applying (RCI) we end with $\top \to RC_G(\beta, \alpha)$, which is equivalent to the formula $RC_G(\beta, \alpha)$. Hence, we have shown completeness of K_n^{RC}.

For the soundness and completeness proofs of the other systems, one can add the corresponding axioms to EL-RC and make slight modifications in the soundness and completeness proof for EL-RC. Then, the above derivations are still valid and we are done. □

We will need soundness and completeness of the systems K_n^{RC}, $\mathsf{K4}_n^{RC}$, $\mathsf{K45}_n^{RC}$, T_n^{RC}, $\mathsf{S4}_n^{RC}$, and $\mathsf{S5}_n^{RC}$ in conjunction with public announcement operators in Section 7.3.

5.3 Combining knowledge and belief

We have already mentioned in Section 5.1 that it is common to talk about knowledge in the presence of the truth axiom (T), the other systems are rather describing belief. This habit already indicates that belief can be seen as sort of *weak knowledge*. Like in natural language, there are applications where we want to talk about knowledge and belief in the same context, see Hintikka [40] for a detailed discussion. The sentence "I believe that she is on holiday, but I do not know it for sure" is a typical example. Moreover, there are sentences where knowledge and belief are combined, e. g. "If I know some fact, then I also believe it" or "I believe that she knows that dolphins are mammals". The first system for the modal logic of knowledge and belief was presented and proved complete by Kraus and Lehmann [50]. Van der Hoek has then shown completeness for all possible systems one can build by combining certain properties of knowledge and belief, see [41]. For our purposes, we will only choose three systems.

First, we will define the language for the logic of knowledge and belief, as well as the logic of common knowledge and common belief.

Definition 5.3.1. The language $\mathcal{L}_n^B$ of the logic of knowledge and belief is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ B_i\alpha,$$

and the language $\mathcal{L}_n^{BCD}$ of the logic of common knowledge and common belief is defined as follows ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ B_i\alpha \ | \ C_G\alpha \ | \ D_G\alpha.$$

We define the notion of length, subformula, and substitution for the new operators as follows,

$$\begin{aligned}
|B_i\beta| &:= |\beta| + 1, \\
|D_G\beta| &:= |\beta| + 1, \\
\mathrm{sub}(B_i\beta) &:= \{B_i\beta\} \cup \mathrm{sub}(\beta), \\
\mathrm{sub}(D_G\beta) &:= \{D_G\beta\} \cup \mathrm{sub}(\beta), \\
(B_i\beta)[\varphi/\psi] &:= \begin{cases} \varphi & \text{if } \psi = B_i\beta, \\ B_i(\beta[\varphi/\psi]) & \text{otherwise,} \end{cases} \\
(D_G\beta)[\varphi/\psi] &:= \begin{cases} \varphi & \text{if } \psi = D_G\beta, \\ D_G(\beta[\varphi/\psi]) & \text{otherwise.} \end{cases}
\end{aligned}$$

The operators D_G are used for *common belief*, which is exactly the same concept as common knowledge. Like in Section 5.2, we have the abbreviation $E_G\alpha$ to express that everybody in G knows α. Accordingly, we define the notion of *mutual belief* for every group $G \subseteq \mathcal{A}$ of agents by

$$F_G\alpha := \bigwedge_{i \in G} B_i\alpha.$$

The semantics is given by Kripke structures with two accessibility relations for each agent. Agent i's accessibility relations for knowledge and belief are usually named R_i and Q_i respectively.

Definition 5.3.2. Let $\mathfrak{K} = (S, R_1, \ldots, R_n, Q_1, \ldots, Q_n, V)$ and $s \in S$ be given. As expected, the notion of an $\mathcal{L}_n^B$ formula being *satisfied* in the pointed structure $\mathfrak{K}, s$ is inductively defined by

$$\begin{aligned}
\mathfrak{K}, s \models p \ &:\Leftrightarrow \ s \in V(p), \\
\mathfrak{K}, s \models \neg\alpha \ &:\Leftrightarrow \ \mathfrak{K}, s \not\models \alpha,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{K}, s \vDash \alpha \wedge \beta &:\Leftrightarrow \mathfrak{K}, s \vDash \alpha \text{ and } \mathfrak{K}, s \vDash \beta, \\
\mathfrak{K}, s \vDash K_i\alpha &:\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha, \\
\mathfrak{K}, s \vDash B_i\alpha &:\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha.
\end{aligned}$$

In addition, we have the following defining clauses for the notion of an $\mathcal{L}_n^{BCD}$ formula of the form $C_G\alpha$ and $D_G\alpha$ being satisfied in $\mathfrak{K}, s$,

$$\begin{aligned}
\mathfrak{K}, s \vDash C_G\alpha &:\Leftrightarrow \text{for all } t \in R_G^+(s),\ \mathfrak{K}, t \vDash \alpha, \\
\mathfrak{K}, s \vDash D_G\alpha &:\Leftrightarrow \text{for all } t \in Q_G^+(s),\ \mathfrak{K}, t \vDash \alpha.
\end{aligned}$$

The basic systems KK_n and KK_n^{CD} for the languages $\mathcal{L}_n^B$ and $\mathcal{L}_n^{BCD}$ are just the union of K_n and K_n^C with the basic systems for belief and common belief respectively.

Definition 5.3.3. The system KK_n consists of the *tautology axiom* and the two *distribution axioms*,

(PT) Every instance of a propositional tautology,

(K) $K_i(\alpha \to \beta) \to (K_i\alpha \to K_i\beta)$,

(K′) $B_i(\alpha \to \beta) \to (B_i\alpha \to B_i\beta)$,

as well as the *moduls ponens rule* and the two *necessitation rules*,

$$(\mathsf{MP})\ \frac{\alpha \qquad \alpha \to \beta}{\beta}, \qquad (\mathsf{NEC})\ \frac{\alpha}{K_i\alpha}, \qquad (\mathsf{NEC}')\ \frac{\alpha}{B_i\alpha}.$$

We obtain the system KK_n^{CD} by adding to KK_n the following *co-closure axioms*,

(C) $C_G\alpha \to E_G(\alpha \wedge C_G\alpha)$,

(C′) $D_G\alpha \to F_G(\alpha \wedge D_G\alpha)$,

and the following *induction rules*,

$$(\mathsf{CI})\ \frac{\alpha \to E_G(\alpha \wedge \beta)}{\alpha \to C_G\beta}, \qquad (\mathsf{CI}')\ \frac{\alpha \to F_G(\alpha \wedge \beta)}{\alpha \to D_G\beta}.$$

Like in the previous sections of this chapter, we will sometimes add a combination of the axioms (T), (4), and (5) to the basic systems. Furthermore, we will add some of the following axioms,

(D′) $B_i\neg\alpha \to \neg B_i\alpha$ (*consistency*),

(4′) $B_i\alpha \to B_iB_i\alpha$ (*positive introspection*),

(5′) $\neg B_i\alpha \to B_i\neg B_i\alpha$ (*negative introspection*), that we obtain by replacing K_i by B_i in (D), $(\mathsf{4})$, and $(\mathsf{5})$ respectively. For the names of the resulting systems we just write the name of the system for knowledge succeeded by the name of the system for belief. The following example definitions illustrate how we name the different combinations of axioms,

$$\mathsf{S4K4}_n := \mathsf{KK}_n + (\mathsf{T}) + (\mathsf{4}) + (\mathsf{4}'),$$
$$\mathsf{S5KD4}_n^{CD} := \mathsf{KK}_n^{CD} + (\mathsf{T}) + (\mathsf{4}) + (\mathsf{5}) + (\mathsf{D}') + (\mathsf{4}').$$

In addition, the following axioms describing the interrelation between knowledge and belief are under consideration,

(I)	$K_i\alpha \to B_i\alpha$	(*entailment*),
(A)	$B_i\alpha \to K_iB_i\alpha$	(*positive consciousness*),
(A^-)	$\neg B_i\alpha \to K_i\neg B_i\alpha$	(*negative consciousness*),
(G)	$B_i\alpha \to B_iK_i\alpha$	(*positive certainty*),
(G^-)	$\neg B_i\alpha \to B_i\neg K_i\alpha$	(*negative certainty*).

The labelling of these axioms is different in almost every contribution to the logic of knowledge and belief. Our choice of labels ensures that no conflicts arise. The letter I stands for *Inclusion*, A for *Awareness*, and G for the German word *Gewissheit*, which is the translation of certainty.

We want to mention here that some of the axioms for knowledge, belief and some of the interrelation axioms are dependent.

Lemma 5.3.4. *We have the following dependencies,*

$$\mathsf{KK}_n + (\mathsf{I}) + (\mathsf{A}) \vdash (\mathsf{4}'), \qquad \mathsf{KK}_n + (\mathsf{I}) + (\mathsf{G}) \vdash (\mathsf{4}'),$$
$$\mathsf{KK}_n + (\mathsf{I}) + (\mathsf{A}^-) \vdash (\mathsf{5}'), \qquad \mathsf{KKD45}_n + (\mathsf{A}) \vdash (\mathsf{A}^-),$$
$$\mathsf{KK}_n + (\mathsf{5}) + (\mathsf{I}) \vdash (\mathsf{G}^-), \qquad \mathsf{KK}_n + (\mathsf{5}') + (\mathsf{I}) \vdash (\mathsf{G}^-).$$

Proof. The only nontrivial part is the proof of (A^-) in $\mathsf{KKD45}_n + (\mathsf{A})$. First, we take the instance $B_i\neg B_i\alpha \to \neg B_iB_i\alpha$ of (D') and the contraposition $\neg B_iB_i\alpha \to \neg B_i\alpha$ of $(\mathsf{4})$ to derive

$$K_iB_i\neg B_i\alpha \to K_i\neg B_i\alpha. \tag{5.1}$$

The following chain of instances of $(\mathsf{5}')$, (A), and (5.1),

$$\neg B_i\alpha \to B_i\neg B_i\alpha \to K_iB_i\neg B_i\alpha \to K_i\neg B_i\alpha,$$

finishes the proof. ∎

There are systems of knowledge and belief that are too strong in the sense that they prove undesired interrelation properties. One of these properties is $B_i K_i \alpha \to \alpha$, which has been mentioned by Nguyen [55]. Another undesired property is $B_i \alpha \leftrightarrow K_i \alpha$, which has already been discussed by Kraus and Lehmann [50].

Lemma 5.3.5. *For all $\mathcal{L}_n^B$ formulas α we have*

$$\begin{aligned} \mathsf{KK}_n + (\mathsf{T}) + (\mathsf{5}) + (\mathsf{D}') + (\mathsf{I}) &\vdash B_i K_i \alpha \to \alpha, \\ \mathsf{KK}_n + (\mathsf{5}) + (\mathsf{D}') + (\mathsf{I}) + (\mathsf{G}) &\vdash B_i \alpha \leftrightarrow K_i \alpha. \end{aligned}$$

Proof. The following chain of instances of the contraposition of (D'), (I), and $(\mathsf{5})$, followed by an instance of (T) proves the first assertion,

$$B_i K_i \alpha \to \neg B_i \neg K_i \alpha \to \neg K_i \neg K_i \alpha \to K_i \alpha \to \alpha.$$

An instance of (G), followed by a sequence of instances of the contraposition of (D'), (I), and $(\mathsf{5})$,

$$B_i \alpha \to B_i K_i \alpha \to \neg B_i \neg K_i \alpha \to \neg K_i \neg K_i \alpha \to K_i \alpha,$$

shows how we can prove the direction from left to right in the second assertion. The direction from right to left is just an instance of (I). ☐

In order to avoid these undesired properties, Halpern has suggested the *axiom of objective entailment* in [32], which allows $K_i \alpha \to B_i \alpha$ only if α is an *objective* formula, that is if $\alpha \in \mathcal{L}_0$. He proves completeness with respect to a non standard semantics, and we will therefore not take this axiom into account.

We will now define the three systems of our interest. They are maximal in the sense that they do not prove the two undesirable properties from Lemma 5.3.5, but they have as many axioms as possible. The only exception is that we omit the axiom (G), which would cause problems in conjunction with public announcements.Observe, that we add even the provable axioms, because we want to have transparency over the valid properties.

Definition 5.3.6. We define the following systems of knowledge and belief,

$$\begin{aligned} \mathbb{KB}\mathsf{DI}_n &:= \mathsf{S4KD4}_n + (\mathsf{I}) + (\mathsf{A}) + (\mathsf{G}^-), \\ \mathbb{KB}\mathsf{5I}_n &:= \mathsf{S5K45}_n + (\mathsf{I}) + (\mathsf{A}) + (\mathsf{A}^-) + (\mathsf{G}^-), \\ \mathbb{KB}\mathsf{5D}_n &:= \mathsf{S5KD45}_n + (\mathsf{A}) + (\mathsf{A}^-) + (\mathsf{G}^-). \end{aligned}$$

The systems $\mathbb{KB}\mathsf{DI}_n^{CD}$, $\mathbb{KB}\mathsf{5I}_n^{CD}$, and $\mathbb{KB}\mathsf{5D}_n^{CD}$ are accordingly defined.

Van der Hoek has shown in [41], that not only the axioms of knowledge and belief, but also the interrelation axioms can be characterised via restrictions on the accessibility relations. In order to define the new classes of Kripke structures, we introduce the following properties,

e: $Q_i \subseteq R_i$ (*inclusion*),
c: for all $u, v, w \in |\mathfrak{K}|$, uR_iv and $vQ_iw \Rightarrow uQ_iw$ (*transitivity*[2]),
d: for all $u, v, w \in |\mathfrak{K}|$, uR_iv and $uQ_iw \Rightarrow vQ_iw$ (*Euclideanity*[3]),
g: for all $u, v, w \in |\mathfrak{K}|$, uQ_iv and $vR_iw \Rightarrow uQ_iw$ (*transitivity*[4]),
h: for all $u, v, w \in |\mathfrak{K}|$, uQ_iv and $uQ_iw \Rightarrow vR_iw$ (*Euclideanity*[5]).

We will write $\mathcal{K}_{2n}$ for the class of all Kripke structures with two accessibility relations for each of the n agents. Furthermore, for a subclass $\mathcal{X}$ of this kind of Kripke structures, we will write $\mathcal{K}_{2n}^{\vec{x},\vec{y},\vec{z}}$, where $\vec{x}$ consists of the one-letter names of the frame conditions for knowledge, $\vec{y}$ characterises the properties of belief, and $\vec{z}$ denotes the interrelation properties in $\mathcal{X}$. For instance, $\mathcal{K}_{2n}^{rt,st,e}$ denotes the class of Kripke structures that have accessibility relations with the following properties: the accessibility relations for knowledge are both reflexive and transitive, the accessibility relations for belief are both serial and transitive, and the accessibility relations for belief are subsets of the corresponding accessibility relations for knowledge.

The following theorem states soundness and completeness of the systems in our focus. It is an immediate consequence of van der Hoek's results in [41].

Theorem 5.3.7. *For all $\mathcal{L}_n^B$ formulas α and all $\mathcal{L}_n^{BCD}$ formulas β we have*

$$\mathbb{KBD}\mathsf{I}_n \vdash \alpha \Leftrightarrow \mathcal{K}_{2n}^{rt,st,ech} \models \alpha, \qquad \mathbb{KBD}\mathsf{I}_n^{CD} \vdash \beta \Leftrightarrow \mathcal{K}_{2n}^{rt,st,ech} \models \beta,$$
$$\mathbb{KB}5\mathsf{I}_n \vdash \alpha \Leftrightarrow \mathcal{K}_{2n}^{rtu,tu,ecdh} \models \alpha, \qquad \mathbb{KB}5\mathsf{I}_n^{CD} \vdash \beta \Leftrightarrow \mathcal{K}_{2n}^{rtu,tu,ecdh} \models \beta,$$
$$\mathbb{KB}5\mathsf{D}_n \vdash \alpha \Leftrightarrow \mathcal{K}_{2n}^{rtu,stu,cdh} \models \alpha, \qquad \mathbb{KB}5\mathsf{D}_n^{CD} \vdash \beta \Leftrightarrow \mathcal{K}_{2n}^{rtu,stu,cdh} \models \beta.$$

Observe that we again write all the properties that correspond to an axiom, even if they are redundant.

5.4 Announcement logics

At the end of the eighties, Plaza published his famous article about logics of public communications [57]. In this work, Plaza adds new operators to

[2]transitivity of Q_i over (R_i, Q_i)
[3]Euclideanity of Q_i over (R_i, Q_i)
[4]transitivity of Q_i over (Q_i, R_i)
[5]Euclideanity of R_i over (Q_i, Q_i)

modal logic in order to formalise the communication of true formulas to the agents. Inspired by this idea, many authors further developed the theory of belief and knowledge change caused by incoming information in a modal logic setting. At the beginning of Chapter 6 and Chapter 7 we will give a short survey of the particular literature. Usually, announcement operators similar to action operators from dynamic logic are added to a language of multi-agent modal logics. Therefore, the term dynamic epistemic logic is commonly used for such constructs, see the book from van Ditmarsch et al. [22] of the same title. It has turned out that information can be privately told to arbitrary groups of agents within the typical systems of belief (Gerbrandy and Groeneveld [30]), whereas in deductive systems containing the truth axiom (T) announcements can exclusively be broadcasted to all of the agents (van Ditmarsch [17]). The former kind of announcements is called *private announcements* or *group announcements*, the latter kind is called *public announcements*. For both types of announcements, we will provide languages for epistemic logic extended with announcement operators of this type.

Definition 5.4.1. The language $\mathcal{L}_n^{PA}$ for epistemic logic with public announcement operators is the set of formulas that is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$),

$$\alpha \ ::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha) \ \mid \ K_i\alpha \ \mid \ [\alpha!]\alpha.$$

The languages $\mathcal{L}_n^{BPA}$, $\mathcal{L}_n^{CPA}$, $\mathcal{L}_n^{RCPA}$, and $\mathcal{L}_n^{BCDPA}$ are the corresponding extensions of $\mathcal{L}_n^{B}$, $\mathcal{L}_n^{C}$, $\mathcal{L}_n^{RC}$, and $\mathcal{L}_n^{BCD}$ respectively with public announcement operators.

The language $\mathcal{L}_n^{GA}$ for epistemic logic with group announcements is defined as follows ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha) \ \mid \ K_i\alpha \ \mid \ [\alpha!_G]\alpha.$$

The definition of the language $\mathcal{L}_n^{CGA}$ is more elaborate: it additionally contains operators for finite sequences of group announcements. For this purpose, we simultaneously define the formulas and announcements of $\mathcal{L}_n^{CGA}$ by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\begin{aligned} \alpha \ &::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha) \ \mid \ K_i\alpha \ \mid \ C_G\alpha \ \mid \ [\pi]\alpha, \\ \pi \ &::= \ \alpha!_G \ \mid \ (\pi \,;\, \alpha!_G). \end{aligned}$$

The formula $[\alpha!_G]\beta$ stands for "β holds after the private announcement of α to the group G", the formula $[\alpha!]\beta$ means "β holds after the public announcement of α". We define the notion of length, subformula, and substitution for the announcement operators as follows,

$$
\begin{aligned}
|[\beta!]\gamma| &:= |\beta| + |\gamma| + 2,\\
|[\beta!_G]\gamma| &:= |\beta| + |\gamma| + 2,\\
\mathrm{sub}([\beta!]\gamma) &:= \{[\beta!]\gamma\} \cup \mathrm{sub}(\beta) \cup \mathrm{sub}(\gamma),\\
\mathrm{sub}([\beta!_G]\gamma) &:= \{[\beta!_G]\gamma\} \cup \mathrm{sub}(\beta) \cup \mathrm{sub}(\gamma),\\
([\beta!]\gamma)[\varphi/\psi] &:= \begin{cases}\varphi & \text{if } \psi = [\beta!]\gamma,\\ [(\beta[\varphi/\psi])!](\gamma[\varphi/\psi]) & \text{otherwise,}\end{cases}\\
([\beta!_G]\gamma)[\varphi/\psi] &:= \begin{cases}\varphi & \text{if } \psi = [\beta!_G]\gamma,\\ [(\beta[\varphi/\psi])!_G](\gamma[\varphi/\psi]) & \text{otherwise.}\end{cases}
\end{aligned}
$$

The definition of length, subformula, and substitution for the language $\mathcal{L}_n^{CGA}$ is more complex, and will be introduced in Section 6.3.

We are now able to give an overview of the languages defined in this thesis. Figure 5.1 shows the hierarchy of our languages, where the arrows stand for set inclusion. In the following, we will simply write π or ρ for an arbitrary

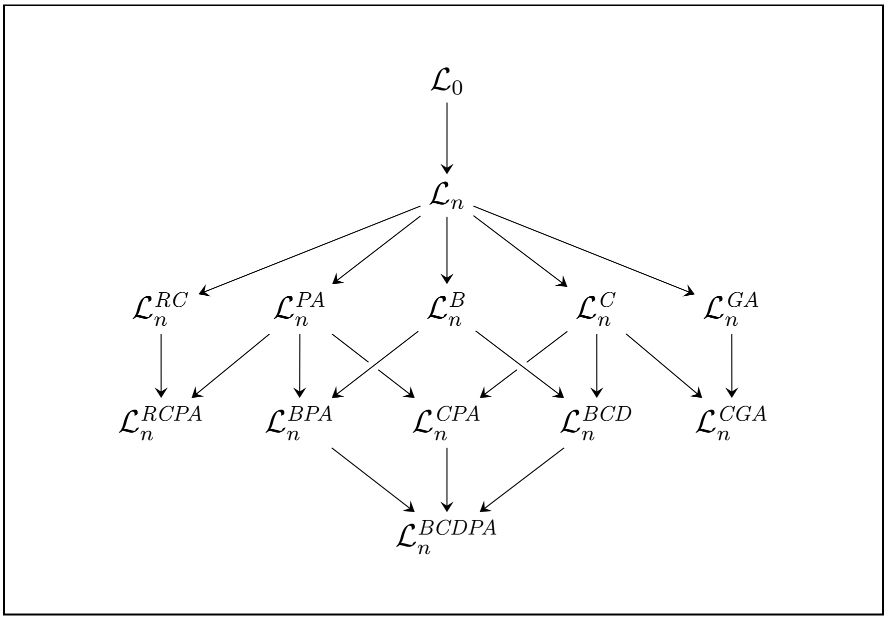

Figure 5.1: The languages in this thesis

announcement, if we do not care about the announcement being private or public. Iterated announcements are inductively defined by

$$[\pi]^0\beta := \beta, \qquad\qquad [\pi]^{k+1}\beta := [\pi][\pi]^k\beta,$$

for all announcements π. The semantics of an announcement formula $[\pi]\alpha$ is usually defined via *model transformations*. In order to illustrate this procedure, we will give the semantics of the logics of public communications from Plaza [57]. For this purpose, we add the defining clause for public announcement formulas to Definition 5.1.5.

Definition 5.4.2. Let $\mathfrak{K} = (S, R_1, \dots, R_n, V)$ and $s \in S$ be given. Then the notion of an $\mathcal{L}_n^{PA}$ formula $[\alpha!]\beta$ being *satisfied* in the pointed structure $\mathfrak{K}, s$ is defined by

$$\mathfrak{K}, s \vDash [\alpha!]\beta \;:\Leftrightarrow\; \mathfrak{K}, s \vDash \alpha \text{ and } \mathfrak{K}^\alpha, s \vDash \beta,$$

where $\mathfrak{K}^\alpha := (S^\alpha, R_1^\alpha, \dots, R_n^\alpha, V^\alpha)$ is the structure $\mathfrak{K}$ restricted to the worlds where α holds. This relativised structure is given by defining

$$\begin{aligned} S^\alpha &:= \|\alpha\|_{\mathfrak{K}}, \\ R_i^\alpha &:= R_i \cap \|\alpha\|_{\mathfrak{K}}^2, \\ V^\alpha(p) &:= V(p) \cap \|\alpha\|_{\mathfrak{K}}, \end{aligned}$$

for every agent $i \in \mathcal{A}$ and all propositions $p \in \mathcal{P}$.

We want to mention here that Definition 5.4.2 is a bit problematic: the structure $\mathfrak{K}^\alpha$ is only defined if there exists an $s \in |\mathfrak{K}|$ such that $\mathfrak{K}, s \vDash \alpha$, that is if $\|\alpha\|_{\mathfrak{K}} \neq \emptyset$. Therefore, we suggest to give this definition via the extension of a public announcement formula $[\alpha!]\beta$ by

$$\|[\alpha!]\beta\|_{\mathfrak{K}} := \begin{cases} \emptyset & \text{if } \|\alpha\|_{\mathfrak{K}} = \emptyset, \\ \|\beta\|_{\mathfrak{K}^\alpha} & \text{otherwise.} \end{cases}$$

During our studies of announcement logics we found many interesting properties that characterise the different approaches.

Definition 5.4.3. An announcement logic satisfies one of the following properties with respect to a class $\mathcal{X}$ of Kripke structures, if the property holds for all announcements π of this logic.

Fact preservation: For all $\beta \in \mathcal{L}_0$, $\mathcal{X} \vDash \beta \to [\pi]\beta$.
This property is very important for information change in a *static world*. Only knowledge and belief can be changed by incoming information, but not propositional facts.

Adequacy: $\mathcal{X} \models [\pi]\top$.
In every modal logic like alethic[6], epistemic, temporal, or dynamic logic there is an adequate modal operator.

Totality: $\mathcal{X} \models \neg[\pi]\bot$.
Totality is a term from dynamic logic. It means that an announcement can always be executed at any state. If we interpret announcements as communication acts between agents, we think that totality is a nice property. But there are also interpretations of announcements where we agree that they must be partial, for instance truthful announcements, see Section 7.1.

Self-duality: For all formulas α, $\mathcal{X} \models \neg[\pi]\alpha \leftrightarrow [\pi]\neg\alpha$.
This property is an additional requirement for total announcements. It characterises the fact that there is always exactly one way of executing an announcement.

Normality: For all formulas α, β, $\mathcal{X} \models [\pi](\alpha \to \beta) \to ([\pi]\alpha \to [\pi]\beta)$ and $\mathcal{X} \models \alpha \;\Rightarrow\; \mathcal{X} \models [\pi]\alpha$.
Normal modal logics are the modal logics having a Kripke semantics and they all have the normality property. Although the announcement semantics is not an original Kripke semantics, we call an announcement logic normal whenever it satisfies these two properties.

We are now going to discuss the above defined properties for Plaza's public announcement logic with respect to $\mathcal{K}_n^{rtu}$, because in Plaza's original work the accessibility relations were defined to be equivalence relations.

Lemma 5.4.4. *Plaza's public announcements are total with respect to $\mathcal{K}_n^{rtu}$. The other properties from Definition 5.4.3 are all violated.*

Proof. We have totality because the formula $[\alpha!]\bot$ is not satisfiable for all α, which immediately follows from Definition 5.4.2. Fact preservation is violated, because the formula $[\neg p!]p$ is not satisfied whenever p is true. Adequacy fails because the formula $[\bot!]\top$ is not even satisfiable and thus not valid with respect to any $\mathcal{X}$. Self-duality does not hold, because the formula $\neg[p!]\bot \wedge \neg[p!]\neg\bot$ is satisfied whenever p is false. We do not have normality, because we have $\mathcal{K}_n^{rtu} \models \top$ but $\mathcal{K}_n^{rtu} \not\models [\bot!]\top$. □

In Section 7.1 we will introduce the dual of Plaza's public announcements as truthful public announcements. As we will see, more properties hold for this widely discussed logic.

[6]For example, the *necessity* operator is called an alethic modality.

We end this section with a short note about the crucial benefit of announcement operators. In real life, we communicate in order to learn new information, therefore we expect that the knowledge or the belief of an agent will change after having performed an announcement. For instance, in Plaza's logic we have

$$\mathcal{K}_n^r \models \alpha \rightarrow [\alpha!]K_i\alpha \tag{5.2}$$

for all $\alpha \in \mathcal{L}_0$ and all $i \in \mathcal{A}$. Thus, we have that a true propositional fact will always be learnt by the agents after one public announcement of this fact. However, the above mentioned implication does not hold for all formulas, but it holds for a bigger set than $\mathcal{L}_0$. For instance, for all formulas of the form K_ip and for all formulas valid in $\mathcal{K}_n^r$. It is a nontrivial task to find out which set of formulas is defined by a property like (5.2)

It has turned out that there is one property that gives rise to an interesting set of formulas in every announcement logic we are going to use in this thesis. It is the notion of announcement resistant[7] formulas, which we have introduced in [61, 62]. For instance, the announcement resistant formulas all satisfy property 5.2 in all announcement logics we will define in the following chapters.

Definition 5.4.5. Let $\mathcal{X}$ be an arbitrary class of Kripke structures and $\mathcal{L}$ be a language containing public announcement operators. A formula $\alpha \in \mathcal{L}$ is called *announcement resistant* in $\mathcal{X}$, if for all $\beta \in \mathcal{L}$ we have

$$\mathcal{X} \models \alpha \rightarrow [\beta!]\alpha.$$

Now, let $\mathcal{L}$ be a language containing group announcement operators and $G \subseteq \mathcal{A}$ be a non empty group of agents. A formula $\alpha \in \mathcal{L}$ is called *announcement resistant for G* in $\mathcal{X}$, if for all $\beta \in \mathcal{L}$ we have

$$\mathcal{X} \models \alpha \rightarrow [\beta!_G]\alpha.$$

In many contributions to public announcement logics, there is the notion of *successful formulas*, which has been thoroughly studied by van Ditmarsch [17, 18, 19] and van Ditmarsch et al. [21, 22, 23]. We will define this property in Section 7.1 because this property defines an interesting set of formulas only in the logic of truthful public announcements.

[7]Sometimes the term *preserved* formulas is used, see van Benthem [10] as well as van Ditmarsch and Kooi [23]. But this means "preserved under submodels" and is not only used in the context of announcement logics.

Chapter 6

Belief expansion

In this chapter, we will explore two different belief expansion functions in group announcement logics. We will show that these functions are not expansion functions for arbitrary beliefs, but there is a big set of formulas that will always be learnt by the group they are announced to. In Section 6.1 we will give a straightforward semantics for the expansion of trustful agents' beliefs. This means that the agents believe every incoming information without any constraints. We will prove soundness and completeness for the system of Gerbrandy and Groeneveld from [30]. This allows us to conclude that our semantics is equivalent to their non well-founded approach. We will then present our work from [61] about a new way of expanding the belief of rational agents in Section 6.2. The agents' beliefs will always remain consistent, even after performing an announcement that contradicts their beliefs. In this case, the agents do not accept the announced formula, but they learn that other agents have received the same information. Due to this behaviour, we call them sceptical agents. They act the same way as we have already proposed with the consistent expansion function, see Definition 3.1.2. Section 6.3 deals with the challenge of axiomatising group announcement logics augmented with common belief operators. Although there exist systems for extensions of these logics (cf. [7]), we are working towards an axiomatisation within a simpler language.

6.1 Trustful agents

First of all, we repeat the definition of the language $\mathcal{L}_n^{GA}$ of group announcement logic from Definition 5.4.1. The formulas of this language are defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ [\alpha!_G]\alpha.$$

Since we only consider systems without the truth axiom (T) in this chapter, we will read a formula $K_i\alpha$ as "agent i believes α". Now, we want to extend the standard Kripke semantics to the language $\mathcal{L}_n^{GA}$. Similar constructions can be found in Hommersom et al. [44, 45], Roorda et al. [43], and van Linder et al. [42]. They have all defined some model transformations which makes the agents belief the announced formula. However, they either work within a richer language [43, 42], or have some constraints on the incoming information [44, 45].

Definition 6.1.1. We extend Definition 5.1.5 by the following clause,

$$\mathfrak{K}, s \vDash [\alpha!_G]\beta \;:\Leftrightarrow\; \mathfrak{K}^{\alpha,G}, s_1 \vDash \beta,$$

where the Kripke structure $\mathfrak{K}^{\alpha,G} = (S', R_1^{\alpha,G}, \dots, R_n^{\alpha,G}, V')$ is defined by

$$\begin{aligned} S' &:= S \times \{0,1\}, \\ R_i^{\alpha,G} &:= \begin{cases} \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_1) : sR_it \text{ and } \mathfrak{K}, t \vDash \alpha\} & \text{if } i \in G, \\ \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_0) : sR_it\} & \text{if } i \notin G, \end{cases} \\ V'(p) &:= V(p) \times \{0,1\} \end{aligned}$$

for all $i \in \mathcal{A}$ and all $p \in \mathcal{P}$. The expressions s_0 and s_1 are abbreviations for the worlds $(s,0)$ and $(s,1)$ respectively.

The essential feature of the above definition is the following. The agents not in G keep their original beliefs, while the beliefs of the agents belonging to G are affected by the announced formula. We call such agents *trustful agents*, because they always believe the announced formula. We will now give an example in order to illustrate how this model transformation works.

Example 6.1.2. Alice, Bob, and Charlie meet in a pub, we will also call them agents 1, 2, and 3. Bob and Charlie wonder whether Alice has got a sister or not. She actually does and, of course, she believes this fact. The situation can be illustrated with a simple Kripke structure $\mathfrak{K} = (\{u,v\}, R_1, R_2, R_3, V)$, see Figure 6.1 (the actual world u is underlined). At world u Alice has got a sister, while she does not at world v. For this purpose, we choose a proposition p with the meaning "Alice has got a sister". Therefore, we have $V(p) = \{u\}$.

While Charlie is going to get some drinks, Alice tells Bob that she has got a sister. We have $\alpha = p$, $G = \{1,2\}$, and Bob has learnt p at world u_1 from the transformed Kripke structure $\mathfrak{K}^{p,\{1,2\}}$, see Figure 6.2. As we can see, Alice has learnt that Bob has learnt p, since she belongs to the group G. Moreover, Charlie's beliefs remain unchanged, hence Alice's and Bob's beliefs about Charlie's beliefs remain unchanged, as well.

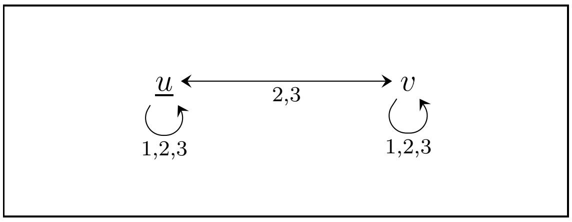

Figure 6.1: Alice has got a sister

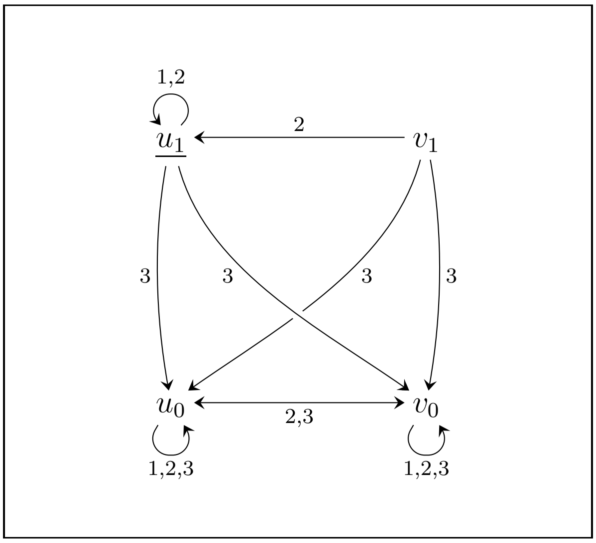

Figure 6.2: Bob has learnt that Alice has got a sister

It seems to be obvious that the beliefs of the agents that do not hear an announcement remain unchanged. In order to prove this fact, the following lemma is useful.

Lemma 6.1.3. *Let the two Kripke structures* $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ *and* $\mathfrak{K}' = (S', R'_1, \ldots, R'_n, V')$ *as well as the worlds* $s \in S$ *and* $s' \in S'$ *be given. If* $\mathfrak{K}, s \simeq \mathfrak{K}', s'$*, then for all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathfrak{K}, s \vDash \alpha \;\;\Leftrightarrow\;\; \mathfrak{K}', s' \vDash \alpha.$$

Proof. By induction on α. We will show how to prove the last case of the induction step, where α is of the form $[\beta!_G]\gamma$. Let B be the bisimulation relation that connects s and s'. Then we define a new binary relation by

$$B^+ := \{(u_0, u'_0) : uBu'\} \cup \{(u_1, u'_1) : uBu'\}.$$

Note that we have $B^+ \subseteq (S \times \{0,1\}) \times (S' \times \{0,1\})$. It is not hard to show that B^+ is a bisimulation between $\mathfrak{K}^{\beta,G}$ and $(\mathfrak{K}')^{\beta,G}$ that connects s_1 and s_1' using the induction hypothesis for β. By induction hypothesis for γ, we get $\mathfrak{K}^{\beta,G}, s_1 \vDash \gamma$ if and only if $(\mathfrak{K}')^{\beta,G}, s_1' \vDash \gamma$, and we are done. □

We will now prove that the model transformation retains transitivity and Euclideanity of the accessibility relations. It follows from Example 6.1.2 that this is not the case for seriality or reflexivity.

Lemma 6.1.4. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^t$ or $\mathcal{K}_n^{tu}$. Then for all Kripke structures $\mathfrak{K}$, all $\alpha \in \mathcal{L}_n^{GA}$, and all non empty $G \subseteq \mathcal{A}$ we have*

$$\mathfrak{K} \in \mathcal{X} \;\Rightarrow\; \mathfrak{K}^{\alpha,G} \in \mathcal{X}.$$

Proof. We will show how to prove that transitivity is preserved. So let R_i be transitive. Further, let $i \in G$, $u_1 R_i^{\alpha,G} v_1$, and $v_1 R_i^{\alpha,G} w_1$. This implies $u R_i w$ and $\mathfrak{K}, w \vDash \alpha$, hence we get $u_1 R_i^{\alpha,G} w_1$. The other cases are similar and transitivity preservation is proved, no matter if R_i is Euclidean or not. The proof for Euclideanity preservation is similar. □

We are now able to state three deductive systems for our group announcements extending K_n, $\mathsf{K4}_n$, and $\mathsf{K45}_n$. Lemma 6.1.4 will be crucial in order to prove soundness of these systems.

Definition 6.1.5. The deductive systems K_n^{GA}, $\mathsf{K4}_n^{GA}$, and $\mathsf{K45}_n^{GA}$ are the systems K_n, $\mathsf{K4}_n$, and $\mathsf{K45}_n$ respectively augmented with the following *group announcement axioms*,

(GA1) $[\alpha!_G]p \leftrightarrow p$,
(GA2) $[\alpha!_G](\beta \to \gamma) \to ([\alpha!_G]\beta \to [\alpha!_G]\gamma)$,
(GA3) $[\alpha!_G]\neg\beta \leftrightarrow \neg[\alpha!_G]\beta$,
(GA4) $[\alpha!_G]K_i\beta \leftrightarrow K_i\beta \quad (i \notin G)$,
(GA5) $[\alpha!_G]K_i\beta \leftrightarrow K_i(\alpha \to [\alpha!_G]\beta) \quad (i \in G)$.

as well as the *group announcement necessitation rule*,

$$(\mathsf{GAN})\ \frac{\alpha}{[\beta!_G]\alpha}\,.$$

We will first prove soundness of the three Hilbert systems, which allows us to formally prove properties of our model transformation.

Lemma 6.1.6. *For all $\alpha \in \mathcal{L}_n^{GA}$ we have*

$$\begin{aligned} \mathsf{K}_n^{GA} \vdash \alpha &\Rightarrow \mathcal{K}_n \vDash \alpha, \\ \mathsf{K4}_n^{GA} \vdash \alpha &\Rightarrow \mathcal{K}_n^{t} \vDash \alpha, \\ \mathsf{K45}_n^{GA} \vdash \alpha &\Rightarrow \mathcal{K}_n^{tu} \vDash \alpha. \end{aligned}$$

Proof. By induction on the length of the proof. In the base case, we first show that axiom (GA4) is valid in $\mathcal{K}_n$. Let $\mathfrak{K} \in \mathcal{K}_n$, $s \in |\mathfrak{K}|$, $\alpha, \beta \in \mathcal{L}_n^{GA}$, $\emptyset \neq G \subseteq \mathcal{A}$, and $i \in \mathcal{A} \setminus G$, be given. Then we define a binary relation by

$$B := \{(u, u_0) : u \in |\mathfrak{K}|\}.$$

It is now easy to show that B is a bisimulation between $\mathfrak{K}$ and $\mathfrak{K}^{\alpha,G}$. By Lemma 6.1.3, we get

$$\begin{aligned} \mathfrak{K}, s \vDash [\alpha!_G]K_i\beta &\Leftrightarrow \mathfrak{K}^{\alpha,G}, s_1 \vDash K_i\beta \\ &\Leftrightarrow \text{for all } t_0 \in R_i^{\alpha,G}(s_1),\ \mathfrak{K}^{\alpha,G}, t_0 \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}^{\alpha,G}, t_0 \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \beta \\ &\Leftrightarrow \mathfrak{K}, s \vDash K_i\beta. \end{aligned}$$

Second, we show that axiom (GA5) is valid in $\mathcal{K}_n$. For all $\mathfrak{K} \in \mathcal{K}_n$, all $s \in \mathfrak{K}$, all $\alpha, \beta \in \mathcal{L}_n^{GA}$, all non empty $G \subseteq \mathcal{A}$, and all $i \in G$ we have

$$\begin{aligned} \mathfrak{K}, s \vDash [\alpha!_G]K_i\beta &\Leftrightarrow \mathfrak{K}^{\alpha,G}, s_1 \vDash K_i\beta \\ &\Leftrightarrow \text{for all } t_1 \in R_i^{\alpha,G}(s_1),\ \mathfrak{K}^{\alpha,G}, t_1 \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}, t \vDash [\alpha!_G]\beta \\ &\Leftrightarrow \mathfrak{K}, s \vDash K_i(\alpha \to [\alpha!_G]\beta). \end{aligned}$$

In the induction step, soundness of the rule (GAN) is proved as follows. Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, or $\mathcal{K}_n^{tu}$. For all $\alpha, \beta \in \mathcal{L}_n^{GA}$ and all non empty $G \subseteq \mathcal{A}$, we have to show that $\mathcal{X} \vDash \alpha$ implies $\mathcal{X} \vDash [\beta!_G]\alpha$. For given $\mathfrak{K} \in \mathcal{X}$ and $s \in |\mathfrak{K}|$, we have $\mathfrak{K}^{\beta,G} \in \mathcal{X}$ by Lemma 6.1.4, thus we get $\mathfrak{K}^{\beta,G}, s_1 \vDash \alpha$ by assumption. This yields $\mathfrak{K}, s \vDash [\beta!_G]\alpha$, and we are done. □

The following lemma states a so-called *reduction axiom*, which is provable in our three systems.

Lemma 6.1.7. *Let* X *be one of the deductive systems* K_n^{GA}, $\mathsf{K4}_n^{GA}$, *or* $\mathsf{K45}_n^{GA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$ *and all non empty* $G \subseteq \mathcal{A}$ *we have that* X *proves*

$$[\alpha!_G](\beta \wedge \gamma) \leftrightarrow [\alpha!_G]\beta \wedge [\alpha!_G]\gamma.$$

Proof. The proof is like in normal modal logic for the K_i modality, that is we only need (GA2) and (GAN) in addition to the corresponding systems of belief. □

As an immediate consequence of axiom (GA3), we could replace axiom (GA2) by the reduction axiom from Lemma 6.1.7 to get equivalent deductive systems. In addition, the converse direction of (GA2) is also provable by use of axiom (GA3) and Lemma 6.1.7.

Since we have soundness, we can formally prove that every property from Definition 5.4.3 is satisfied by our group announcement semantics.

Lemma 6.1.8. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, or $\mathcal{K}_n^{tu}$. Then the group announcements for trustful agents are fact preserving, adequate, total, self-dual, and normal with respect to $\mathcal{X}$.*

Proof. Let X^{GA} be the deductive system that corresponds to $\mathcal{X}$. In order to prove fact preservation, we can show for all $\alpha \in \mathcal{L}_0$ and all $\beta \in \mathcal{L}_n^{GA}$ that X^{GA} proves $\alpha \leftrightarrow [\beta!_G]\alpha$. This can be done by induction on α using the axioms (GA1) and (GA3) as well as Lemma 6.1.7. Due to the prove of fact preservation, we easily get adequacy and totality, the latter can be proved using axiom (GA3). Self-duality and normality are part of our systems. All of the properties now follow from soundness. □

For the completeness proof, we are going to define a translation from $\mathcal{L}_n^{GA}$ to $\mathcal{L}_n$. This translation will be established in two stages. In the first stage, we define an auxiliary function h, which exclusively operates on announcement formulas.

Definition 6.1.9. We inductively define the auxiliary function h that maps from $\{[\alpha!_G]\beta : \alpha, \beta \in \mathcal{L}_n^{GA}, \emptyset \neq G \subseteq \mathcal{A}\}$ to $\mathcal{L}_n^{GA}$ as follows,

$$\begin{aligned}
h([\alpha!_G]p) &:= p, \\
h([\alpha!_G]\neg\beta) &:= \neg h([\alpha!_G]\beta), \\
h([\alpha!_G](\beta \wedge \gamma) &:= h([\alpha!_G]\beta) \wedge h([\alpha!_G]\gamma), \\
h([\alpha!_G]K_i\beta) &:= \begin{cases} K_i(\alpha \to h([\alpha!_G]\beta)) & \text{if } i \in G, \\ K_i\beta & \text{if } i \notin G, \end{cases} \\
h([\alpha!_G][\beta!_H]\gamma) &:= [\alpha!_G][\beta!_H]\gamma.
\end{aligned}$$

It is an easy induction on β to show that for all $\alpha, \beta \in \mathcal{L}_n$ and all non empty $G \subseteq \mathcal{A}$ we have $h([\alpha!_G]\beta) \in \mathcal{L}_n$. Moreover, the function h is equivalence preserving in the following sense.

Lemma 6.1.10. *Let* X *be one of the deductive systems* K_n^{GA}*,* $\mathsf{K4}_n^{GA}$*, or* $\mathsf{K45}_n^{GA}$*. Then for all* $\alpha, \beta \in \mathcal{L}_n^{GA}$ *and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathsf{X} \vdash h([\alpha!_G]\beta) \leftrightarrow [\alpha!_G]\beta.$$

Proof. By induction on β. In the base case, if $\beta = p$ for some $p \in \mathcal{P}$, we have that X proves $p \leftrightarrow [\alpha!_G]p$ by axiom ($\mathsf{GA1}$). In the induction step, we show how to prove the case $\beta = K_i\gamma$ for $i \in G$. We start with a proof of $h([\alpha!_G]\gamma) \leftrightarrow [\alpha!_G]\gamma$ by induction hypothesis. By standard modal logic reasoning, we can now prove $K_i(\alpha \to h([\alpha!_G]\gamma)) \leftrightarrow K_i(\alpha \to [\alpha!_G]\gamma)$. Since the left expression is the formula $h([\alpha!_G]K_i\gamma)$ by definition, and the right expression is provably equivalent to $[\alpha!_G]K_i\gamma$ by ($\mathsf{GA5}$), we are done. □

In order to prove completeness, the following lemma will be very useful.

Lemma 6.1.11. *Let* X *be one of the deductive systems* K_n^{GA}*,* $\mathsf{K4}_n^{GA}$*, or* $\mathsf{K45}_n^{GA}$*. Then for all* $\alpha, \beta \in \mathcal{L}_n^{GA}$*, all* $\varphi \in \mathcal{L}_n$*, and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!_G]\varphi \leftrightarrow [\beta!_G]\varphi.$$

Proof. By induction on φ. Again, the only interesting case is in the induction step, if $\varphi = K_i\psi$ and $i \in G$. We have $\mathsf{X} \vdash \alpha \leftrightarrow \beta$ by assumption, and $\mathsf{X} \vdash [\alpha!_G]\psi \leftrightarrow [\beta!_G]\psi$ by induction hypothesis. This immediately yields a proof of $K_i(\alpha \to [\alpha!_G]\psi) \leftrightarrow K_i(\beta \to [\beta!_G]\psi)$ by standard modal logic reasoning. Two applications of axiom ($\mathsf{GA5}$) end the proof. □

Using the auxiliary function h, we are now able to define the translation f from $\mathcal{L}_n^{GA}$ to $\mathcal{L}_n$.

Definition 6.1.12. The function $f\colon \mathcal{L}_n^{GA} \to \mathcal{L}_n^{GA}$ is inductively defined by

$$\begin{aligned} f(p) &:= p, \\ f(\neg\alpha) &:= \neg f(\alpha), \\ f(\alpha \land \beta) &:= f(\alpha) \land f(\beta), \\ f(K_i\alpha) &:= K_i f(\alpha), \\ f([\alpha!_G]\beta) &:= h([f(\alpha)!_G]f(\beta)). \end{aligned}$$

Again, it is not hard to show that $f(\alpha) \in \mathcal{L}_n$ for all $\alpha \in \mathcal{L}_n^{GA}$ by induction on α. Furthermore, we can show that f is equivalence preserving like the function h.

Lemma 6.1.13. *Let* X *be one of the deductive systems* K_n^{GA}*,* $\mathsf{K4}_n^{GA}$*, or* $\mathsf{K45}_n^{GA}$*. Then for all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{X} \vdash f(\alpha) \leftrightarrow \alpha.$$

Proof. By induction on α. We show how to prove the case $\alpha = [\beta!_G]\gamma$ in the induction step. By induction hypothesis, we have that X proves both formulas $f(\beta) \leftrightarrow \beta$ and $f(\gamma) \leftrightarrow \gamma$. Therefore, the formula α is provably equivalent to $[\beta!_G]f(\gamma)$ by (GAN) and ($\mathsf{GA2}$), which is provably equivalent to $[f(\beta)!_G]f(\gamma)$ by Lemma 6.1.11. But now, by Lemma 6.1.10, this formula is provably equivalent to $h([f(\beta)!_G]f(\gamma))$, which is defined to be $f(\alpha)$. □

Due to Lemma 6.1.13 and Lemma 6.1.11, we can prove that group announcements for trustful agents are syntax independent.

Lemma 6.1.14. *Let* X *be one of the deductive systems* K_n^{GA}*,* $\mathsf{K4}_n^{GA}$*, or* $\mathsf{K45}_n^{GA}$*. Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$ *and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!_G]\gamma \leftrightarrow [\beta!_G]\gamma.$$

Proof. Let $\mathsf{X} \vdash \alpha \leftrightarrow \beta$. We have that X proves the equivalence of $[\alpha!_G]\gamma$ and $[\alpha!_G]f(\gamma)$ by Lemma 6.1.13, axiom ($\mathsf{GA2}$), and the rule (GAN). By assumption and Lemma 6.1.11, the latter formula is provably equivalent to $[\beta!_G]f(\gamma)$, which is provably equivalent to $[\beta!_G]\gamma$ by again Lemma 6.1.13, axiom ($\mathsf{GA2}$), and the rule (GAN). □

Due to Lemma 6.1.14, we can now prove the *Replacement Theorem* for our three Hilbert systems.

Theorem 6.1.15 (Replacement)**.** *Let* X *be one of the deductive systems* K_n^{GA}*,* $\mathsf{K4}_n^{GA}$*, or* $\mathsf{K45}_n^{GA}$*. Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash \gamma \leftrightarrow \gamma[\alpha/\beta].$$

Proof. By induction on γ. In the base case, if $\gamma = p$ for some $p \in \mathcal{P}$, we distinguish two cases. First, if $\beta \neq p$, then we immediately get $\gamma[\alpha/\beta] = \gamma$ and the claim easily follows. Second, if $\beta = p = \gamma$, then we immediately get $\gamma[\alpha/\beta] = \alpha$ and the claim follows by assumption. In the induction step, we show how to prove the case $\gamma = [\delta!_G]\varphi$. By axiom ($\mathsf{GA2}$), the rule ($\mathsf{GAN}$), and the induction hypothesis for φ, we get $\mathsf{X} \vdash [\delta!_G]\varphi \leftrightarrow [\delta!_G](\varphi[\alpha/\beta])$. In addition, we have that the formulas $[\delta!_G](\varphi[\alpha/\beta])$ and $[(\delta[\alpha/\beta])!_G](\varphi[\alpha/\beta])$ are provably equivalent by Lemma 6.1.14 and the induction hypothesis for δ. The latter formula is defined to be $([\delta!_G]\varphi)[\alpha/\beta]$, and we are done. □

Another consequence of Lemma 6.1.13 is that the logic of group announcements has the same expressive strength as epistemic logic. This fact gives rise to a short and elegant completeness proof, as we show in the proof of the following theorem. We will see in Section 6.3 that this is not the case in the presence of common knowledge operators.

Theorem 6.1.16. *For all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{K}_n^{GA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n \vDash \alpha,$$
$$\mathsf{K4}_n^{GA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^t \vDash \alpha,$$
$$\mathsf{K45}_n^{GA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{tu} \vDash \alpha.$$

Proof. Let X be one of the systems K_n, $\mathsf{K4}_n$, or $\mathsf{K45}_n$ and $\mathcal{X}$ be the class of Kripke structures that corresponds to X. The direction from left to right has already been proved. For completeness of X^{GA}, we assume $\mathcal{X} \vDash \alpha$. By soundness and Lemma 6.1.13, we have $\mathcal{X} \vDash f(\alpha)$. By completeness of X and the fact that $f(\alpha) \in \mathcal{L}_n$, we get $\mathsf{X} \vdash f(\alpha)$. Since the system X^{GA} extends X, we immediately get $\mathsf{X}^{GA} \vdash f(\alpha)$. Finally, again by Lemma 6.1.13, we have that X^{GA} proves α, which concludes the proof. □

We want to mention that we have soundness and completeness for exactly the same deductive system presented by Gerbrandy and Groeneveld [30] and Gerbrandy [29]. Their semantics is completely different because the models are non well-founded sets. In fact, their semantics is similar to the non well-founded semantics introduced by Barwise and Moss [9]. They need the Anti Foundation Axiom and the Solution Lemma for their results, see Aczel [1] for further details. So we have given a semantics with exactly the same valid formulas without claiming the existence of non well-founded sets. Since the fully introspective models of Gerbrandy and Groeneveld have the same valid formulas as Veltman's update semantics [63], we get the same result for $\mathcal{K}_n^{tu}$ as Proposition 3.9 in [30].

The announcement resistant formulas are a suitable tool for the purpose of investigating how announcements affect the agent's beliefs. There is a big set of announcement resistant formulas for every non empty $G \subseteq \mathcal{A}$ in all three classes of Kripke structures.

Lemma 6.1.17. *Let* $G \subseteq \mathcal{A}$ *be a non empty set of agents and* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^t$, *or* $\mathcal{K}_n^{tu}$. *Then we have the following sufficient conditions for a formula* $\alpha \in \mathcal{L}_n^{GA}$ *to be announcement resistant for* G *in* $\mathcal{X}$,

1. $\alpha \in \mathcal{L}_0$,
2. $\mathcal{X} \vDash \alpha$ *or* $\mathcal{X} \vDash \neg\alpha$,

3. *$\alpha = K_i\beta$ or $\alpha = \neg K_i\beta$ for some $i \in \mathcal{A} \setminus G$ and some $\beta \in \mathcal{L}_n^{GA}$,*
4. *$\alpha = \beta \wedge \gamma$ or $\alpha = \beta \vee \gamma$ for some β, γ announcement resistant for G in $\mathcal{X}$,*
5. *$\alpha = K_i\beta$ for some $i \in G$ and some β announcement resistant for G in $\mathcal{X}$.*

Proof. The first assertion is a direct consequence of Lemma 6.1.8 (fact preservation). The two claims in the second assertion follow from soundness of the rule (GAN) and tautological reasoning respectively. The third assertion is directly implied by soundness of the axioms (GA3) and (GA4). The fourth assertion is a consequence of Lemma 6.1.7, axiom (GA3), and soundness. Now, we show how to prove the fifth assertion. Let X^{GA} be the deductive system that corresponds to $\mathcal{X}$, $\beta \in \mathcal{L}_n^{GA}$ be announcement resistant for G in $\mathcal{X}$, and $\gamma \in \mathcal{L}_n^{GA}$ be arbitrarily given. By completeness, we have a proof of $\beta \to [\gamma!_G]\beta$ in X^{GA}. By tautological reasoning, we get that X^{GA} proves $\beta \to (\gamma \to [\gamma!_G]\beta)$, which leads to a proof of $K_i\beta \to K_i(\gamma \to [\gamma!_G]\beta)$ in X^{GA} by normal modal logic reasoning. Applying axiom (GA5), we finally get $\mathsf{X}^{GA} \vdash K_i\beta \to [\gamma!_G]K_i\beta$. Due to soundness, we are done. □

We conclude this section by stating that the announcement resistant formulas get common belief after one single announcement.

Theorem 6.1.18. *Let $G \subseteq \mathcal{A}$ be a non empty set of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, or $\mathcal{K}_n^{tu}$. Further, let $\alpha \in \mathcal{L}_n^{GA}$ be announcement resistant for G in $\mathcal{X}$. Then for all $l \geq 1$ and all $i_1, \ldots, i_l \in G$ we have*

$$\mathcal{X} \vDash [\alpha!_G]K_{i_l} \ldots K_{i_1}\alpha.$$

Proof. Let X^{GA} be the deductive system that corresponds to $\mathcal{X}$. We will show by induction on l that the formula $[\alpha!_G]K_{i_l} \ldots K_{i_1}\alpha$ is provable in X^{GA}. In the base case, we have that X^{GA} proves $\alpha \to [\alpha!_G]\alpha$ by assumption and completeness. Applying the rule (NEC) and axiom (GA5), we get a proof of $[\alpha!_G]K_{i_1}\alpha$ in X^{GA}. For the induction step, we have $\mathsf{X}^{GA} \vdash [\alpha!_G]K_{i_l} \ldots K_{i_1}\alpha$ by induction hypothesis. We can now get $\mathsf{X}^{GA} \vdash K_{i_{l+1}}(\alpha \to [\alpha!_G]K_{i_l} \ldots K_{i_1}\alpha)$ by standard modal logic reasoning. The final step is an application of axiom (GA5), which results in a proof of $[\alpha!_G]K_{i_{l+1}}K_{i_l} \ldots K_{i_1}\alpha$ in X^{GA}. By soundness, we get the desired result. □

Observe that if $i \in G$, we have $\mathcal{X} \vDash [\alpha!_G]K_i\alpha$ for all unsatisfiable formulas $\alpha \in \mathcal{L}_n^{GA}$. This problem cannot be fixed by adding the axiom (D), because the systems would become inconsistent. In order to see this, we start with the fact that the formula $[\bot!_G]K_i\bot$ is provable for all groups $G \subseteq \mathcal{A}$ satisfying

$i \in G$. On the other hand, (D) allows us to prove $\neg K_i\bot$, applying (GAN) and axiom (GA3) results in $\neg[\bot!_G]K_i\bot$. In the next section we will provide a more sophisticated announcement semantics, which allows us to avoid this problem.

6.2 Sceptical agents

It is the aim of this section to present our semantics for the logic of group announcements from [61]. The language $\mathcal{L}_n^{GA}$ from the Section 6.1 is still our set of formulas. We will implement belief expansion in modal logic as proposed in Section 3.1. That is, the agents will only accept new information that is consistent with their beliefs, cf. Definition 3.1.2. Due to our definition of the new model transformation, we will be able to provide deductive systems containing the *consistency axiom*

(D) $\quad K_i\neg\alpha \rightarrow \neg K_i\alpha$,

without giving up the other nice properties from Section 6.1. In addition, we have that for all $\alpha \in \mathcal{L}_n^{GA}$ and all non empty $G \subseteq \mathcal{A}$ the formula $\neg[\alpha!_G]K_i\bot$, which is equivalent to $[\alpha!_G]\neg K_i\bot$, will be provable in the new deductive systems that contain axiom (D). The consistency of beliefs is a widely accepted requirement for rational agents, who are always aware of all the consequences of their beliefs. Therefore, we regard it as very natural to require that property to hold after any announcement.

For the new semantics, we are going to slightly modify the definition of the transformed Kripke structure $\mathfrak{K}^{\alpha,G}$ from Section 6.1.

Definition 6.2.1. We extend Definition 5.1.5 by the following clause,

$$\mathfrak{K}, s \vDash [\alpha!_G]\beta \;:\Leftrightarrow\; \mathfrak{K}^{\alpha,G}, s_1 \vDash \beta,$$

where the Kripke structure $\mathfrak{K}^{\alpha,G} = (S', R_1^{\alpha,G}, \dots, R_n^{\alpha,G}, V')$ is defined by

$$S' := S \times \{0,1\},$$

$$R_i^{\alpha,G} := \begin{cases} \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_1) : sR_it \text{ and } \mathfrak{K}, t \vDash \alpha\} \cup \\ \quad \{(s_1,t_1) : sR_it \text{ and } \mathfrak{K}, s \vDash K_i\neg\alpha\} & \text{if } i \in G, \\ \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_0) : sR_it\} & \text{if } i \notin G. \end{cases}$$

$$V'(p) := V(p) \times \{0,1\}$$

for all $i \in \mathcal{A}$ and all $p \in \mathcal{P}$. The expressions s_0 and s_1 are abbreviations for the worlds $(s,0)$ and $(s,1)$ respectively.

In the model transformation of Section 6.1, all the worlds that do not satisfy α become inaccessible by performing an announcement. With the new model transformation, such worlds stay accessible for the agents if there is no accessible world where α holds. This is the reason why there is always at least one accessible world, hence the beliefs of all agents remain consistent. From the agents' perspective, this means that an agent rejects the incoming information if he believes in its negation. We call such agents *sceptical agents*, because they stick to their beliefs if they consider the new information to be false. But it is not the case that these agents do not learn anything by rejecting an announcement. They learn that the other agents belonging to G have learnt or rejected the announced formula. We think that this strategy is quite good if the source of the information is not known to be reliable.

In order to get a system retaining the consistency of beliefs, there have also been different proposals in literature, cf. Hommersom et al. [44, 45] and Roorda et al. [43]. Our approach is completely different and works for announcements with arbitrary formulas, not only for propositional information. The following example is from [61] and illustrates how the new semantics works.

Example 6.2.2. Imagine a game with the players 1, 2, and 3, as well as three cards, one Ace and two indistinguishable Queens. The players know which cards are in play and one card is dealt to each player in such a way that he can only see his own card. Then the only player who knows the deal is the player with the Ace and the situation is represented by the Kripke structure $\mathfrak{K}$ as is illustrated in Figure 6.3. For each player i we take a proposition p_i,

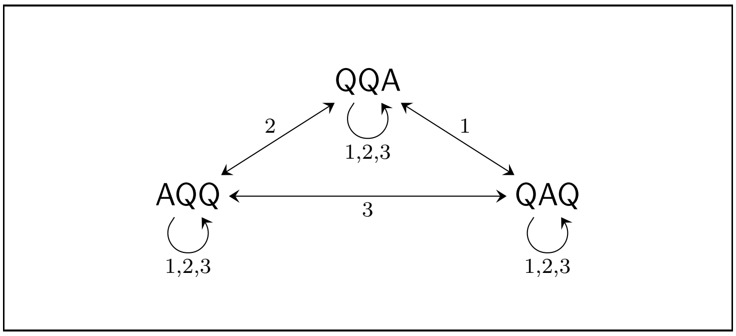

Figure 6.3: Every possible hand

which is defined to be true if and only if player i has got the Ace. Thus, the valuation function V is given by

$$V(p_1) = \{\mathsf{AQQ}\}, \qquad V(p_2) = \{\mathsf{QAQ}\}, \qquad V(p_3) = \{\mathsf{QQA}\} .$$

Now, player 1 secretly tells player 2 that he does not have the Ace. No matter in which state we are, we can perform this announcement as described in Definition 6.2.1. This means that $\alpha = \neg p_1$, $G = \{1, 2\}$, hence $\mathfrak{K}^{\neg p_1, \{1,2\}}$ is the resulting structure from Figure 6.4. Observe that, no matter in which

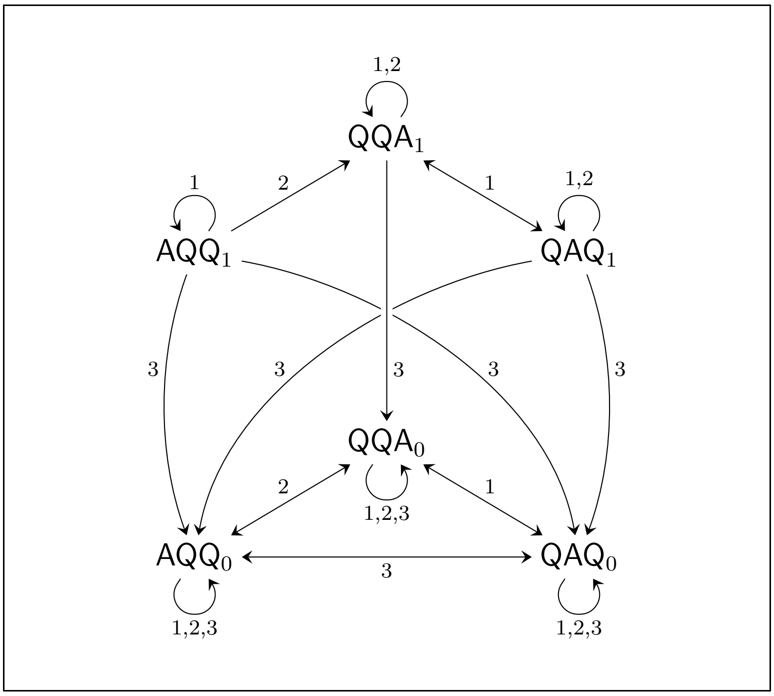

Figure 6.4: Does player 1 have the Ace?

state we are before the announcement, player 2 accepts the announcement and believes $\neg p_1$ afterwards. In the state QQA, player 2 learns a true fact and believes that player 3 has got the Ace, which is also true. Player 2 learns nothing new in the state QAQ, because she has already believed that player 1 does not have the Ace. On the other hand, in the state AQQ, player 2 believes a lie, and player 1 rejects his own announcement, because he believes its negation p_1. In addition, player 1 learns that player 2 believes his lie. This fact illustrates the essential difference of the new procedure to the approach from Definition 6.1.1.

Like in Section 6.1, it is not immediate that the beliefs of the agents that do not hear an announcement remain unchanged. For this purpose, we will again prove that two bisimilar pointed structures satisfy the same formulas.

Lemma 6.2.3. *Let the two Kripke structures* $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ *and* $\mathfrak{K}' = (S', R'_1, \ldots, R'_n, V')$ *as well as the worlds* $s \in S$ *and* $s' \in S'$ *be given. If* $\mathfrak{K}, s \simeq \mathfrak{K}', s'$*, then for all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathfrak{K}, s \vDash \alpha \;\Leftrightarrow\; \mathfrak{K}', s' \vDash \alpha.$$

Proof. By induction on α. We will again show how to prove the last case of the induction step, where α is of the form $[\beta!_G]\gamma$. Let B be the bisimulation relation that connects s and s'. Then we define a new binary relation by

$$B^+ := \{(u_0, u'_0) : uBu'\} \cup \{(u_1, u'_1) : uBu'\}.$$

Again, we have $B^+ \subseteq (S \times \{0,1\}) \times (S' \times \{0,1\})$. We can show that B^+ is a bisimulation between $\mathfrak{K}^{\beta,G}$ and $(\mathfrak{K}')^{\beta,G}$ that connects s_1 and s'_1 using the induction hypothesis for both β and $K_i\neg\beta$. By induction hypothesis for γ, we get $\mathfrak{K}^{\beta,G}, s_1 \vDash \gamma$ if and only if $(\mathfrak{K}')^{\beta,G}, s'_1 \vDash \gamma$, and we are done. □

In Example 6.2.2, the initial Kripke structure belongs to $\mathcal{K}_n^{rtu}$, that is every accessibility relation is an equivalence relation. But the resulting structure is an element of $\mathcal{K}_n^{stu}$, so reflexivity of the accessibility relation is in general not preserved. We have already argued why seriality must be retained. Unfortunately, transitivity is preserved only if the accessibility relations are also Euclidean. This means that we will not be able to define a deductive system for the classes of Kripke structures with transitive but not Euclidean accessibility relations.

Lemma 6.2.4. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, *or* $\mathcal{K}_n^{stu}$. *Then for all Kripke structures* $\mathfrak{K}$, *all* $\alpha \in \mathcal{L}_n^{GA}$, *and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathfrak{K} \in \mathcal{X} \;\Rightarrow\; \mathfrak{K}^{\alpha,G} \in \mathcal{X}.$$

Proof. First, we will show that seriality is preserved. So let R_i be serial and assume $i \in G$. Then for all $u \in |\mathfrak{K}|$ we have $R_i(u) \neq \emptyset$. Of course, we have $R_i^{\alpha,G}(u_0) = \{v_0 : v \in R_i(u_0)\} \neq \emptyset$. We have to show that $R_i^{\alpha,G}(u_1)$ is also non empty. If $\mathfrak{K}, u \nvDash K_i\neg\alpha$, then we have $\mathfrak{K}, v \vDash \alpha$ for some $v \in R_i(u)$, and we get $u_1 R_i^{\alpha,G} v_1$. Otherwise, if $\mathfrak{K}, u \vDash K_i\neg\alpha$, then we immediately get $R_i^{\alpha,G}(u_1) = \{v_1 : v \in R_i(u)\} \neq \emptyset$. The case $i \notin G$ is trivial and seriality preservation is proved. Second, we will show that transitivity is preserved in the presence of Euclideanity. So let R_i be transitive and Euclidean and assume $i \in G$. Clearly, $u_0 R_i^{\alpha,G} v_0$, and $v_0 R_i^{\alpha,G} w_0$ directly imply $u_0 R_i^{\alpha,G} w_0$. So let $u_1 R_i^{\alpha,G} v_1$ and $v_1 R_i^{\alpha,G} w_1$. This implies uR_iw and either $\mathfrak{K}, w \vDash \alpha$ or $\mathfrak{K}, v \vDash K_i\neg\alpha$. If $\mathfrak{K}, w \vDash \alpha$, then we immediately have $u_1 R_i^{\alpha,G} w_1$, and we are done. If $\mathfrak{K}, v \vDash K_i\neg\alpha$, then we have $\mathfrak{K}, u \vDash \neg K_i\neg K_i\neg\alpha$ because uR_iv. By

Euclideanity of R_i, we now get $\mathfrak{K}, u \vDash K_i\neg\alpha$, hence we have $u_1 R_i^{\alpha,G} w_1$. The case $i \notin G$ is trivial, and transitivity preservation is proved. The proof for Euclideanity preservation in the presence of transitivity is similar. □

The following example shows that transitivity is not necessarily preserved, if the accessibility relations are not Euclidean.

Example 6.2.5. Let $p \in \mathcal{P}$ and $\mathfrak{K} = (\{s, t, u\}, R_1, \ldots, R_n, V)$ be defined by

$$R_1 = \{(s,t), (t,u), (s,u), (u,u)\} \qquad V : q \mapsto \{t\},$$

such that the accessibility relations $R_2, \ldots R_n$ are serial and transitive. Then we have $\mathfrak{K} \in \mathcal{K}_n^{st}$, hence $\mathfrak{K} \in \mathcal{K}_n^{t}$. If p is announced to agent 1, we get

$$R_i^{p,\{1\}} = \{(s_0,t_0), (t_0,u_0), (s_0,u_0), (u_0,u_0), (s_1,t_1), (t_1,u_1), (u_1,u_1)\}.$$

That is, we have $\mathfrak{K}^{p,\{1\}} \notin \mathcal{K}_n^t$, thus $\mathfrak{K}^{p,\{1\}} \notin \mathcal{K}_n^{st}$.

We are now able to define four deductive systems for the logic of group announcements for sceptical agents.

Definition 6.2.6. The Hilbert systems $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, and $\mathsf{KD45}_n^{GA_c}$ are the systems K_n, $\mathsf{K45}_n$, KD_n, and $\mathsf{KD45}_n$ respectively augmented with the following *group announcement axioms*,

(GA1) $[\alpha!_G]p \leftrightarrow p$,
(GA2) $[\alpha!_G](\beta \to \gamma) \to ([\alpha!_G]\beta \to [\alpha!_G]\gamma)$,
(GA3) $[\alpha!_G]\neg\beta \leftrightarrow \neg[\alpha!_G]\beta$,
(GA4) $[\alpha!_G]K_i\beta \leftrightarrow K_i\beta$ $\quad (i \notin G)$,
(GA5$_c$) $\neg K_i\neg\alpha \to ([\alpha!_G]K_i\beta \leftrightarrow K_i(\alpha \to [\alpha!_G]\beta))$ $\quad (i \in G)$,
(GA6$_c$) $K_i\neg\alpha \to ([\alpha!_G]K_i\beta \leftrightarrow K_i[\alpha!_G]\beta)$ $\quad (i \in G)$,

as well as the *group announcement necessitation rule*,

$$(\mathsf{GAN})\ \frac{\alpha}{[\beta!_G]\alpha}\,.$$

Again, the above defined systems are sound with respect to the corresponding classes of Kripke structures. The proof makes use of Lemma 6.2.4 and therefore, it would not work for the respective extensions of $\mathsf{K4}_n$ and $\mathsf{KD4}_n$.

Lemma 6.2.7. *For all $\alpha \in \mathcal{L}_n^{GA}$ we have*

$$\mathsf{K}_n^{GA_c} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n \vDash \alpha, \qquad \mathsf{K45}_n^{GA_c} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{tu} \vDash \alpha,$$
$$\mathsf{KD}_n^{GA_c} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{s} \vDash \alpha, \qquad \mathsf{KD45}_n^{GA_c} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{stu} \vDash \alpha.$$

Proof. Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, or $\mathcal{K}_n^{stu}$ and X^{GA_c} be the deductive system that corresponds to $\mathcal{X}$. The proof is by induction on the length of the derivation. In the base case, we first want to mention that axiom ($\mathsf{GA4}$) is valid in $\mathcal{X}$ due to Lemma 6.2.3. The proof works exactly the same way as for trustful agents, where we replace Lemma 6.1.3 by Lemma 6.2.3 in the proof of Lemma 6.1.6. The proof of the validity of axiom ($\mathsf{GA5}_c$) is similar to the proof for axiom ($\mathsf{GA5}$) in the context of trustful agents, see again the proof of Lemma 6.1.6. Now, we show how to prove the correctness of ($\mathsf{GA6}_c$). Let $i \in G$, $\mathfrak{K} \in \mathcal{X}$, and $s \in |\mathfrak{K}|$, and assume that $\mathfrak{K}, s \vDash K_i \neg\alpha$. Then we have

$$\begin{aligned} \mathfrak{K}, s \vDash [\alpha!_G]K_i\beta &\Leftrightarrow \mathfrak{K}^{\alpha,G}, s_1 \vDash K_i\beta \\ &\Leftrightarrow \text{for all } t_1 \in R_i^{\alpha,G}(s_1),\ \mathfrak{K}^{\alpha,G}, t_1 \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash [\alpha!_G]\beta \\ &\Leftrightarrow \mathfrak{K}, s \vDash K_i[\alpha!_G]\beta. \end{aligned}$$

In the induction step, soundness of the rule (GAN) is proved the same way as in Lemma 6.1.6, but it uses Lemma 6.2.4 instead of Lemma 6.1.4. ☐

We have mentioned in Section 5.1 that if a Hilbert system X is consistent, it is not immediate how to find its corresponding class of Kripke structures. In this section, we have all the tools we need to illustrate this fact.

Remark 6.2.8. Let $\mathsf{K4}_n^{GA_c}$ and $\mathsf{KD4}_n^{GA_c}$ be the systems $\mathsf{K4}_n$ and $\mathsf{KD4}_n$ respectively extended by the group announcement axioms and the group announcement necessitation rule from Definition 6.2.6. Then we know that $\mathsf{K4}_n^{GA_c}$ and $\mathsf{KD4}_n^{GA_c}$ are consistent because they are both contained in the consistent system $\mathsf{KD45}_n^{GA_c}$. But, since transitivity is not retained without Euclideanity, cf. Example 6.2.5, we have that $\mathsf{K4}_n^{GA_c}$ and $\mathsf{KD4}_n^{GA_c}$ are not sound with respect to $\mathcal{K}_n^{t}$ and $\mathcal{K}_n^{st}$ respectively. In order to see this, let $p \in \mathcal{P}$ be arbitrarily given. Then we have that the formula $[p!_{\{1\}}](K_1p \rightarrow K_1K_1p)$ is obviously provable in $\mathsf{K4}_n^{GA_c}$ and $\mathsf{KD4}_n^{GA_c}$, but not valid in $\mathcal{K}_n^{t}$ and $\mathcal{K}_n^{st}$. For instance, it is not satisfied at world s from the Kripke structure $\mathfrak{K} = (\{s,t,u\}, R_1, \dots, R_n, V)$, where

$$R_1 = \{(s,t),(t,u),(s,u),(u,u)\}, \qquad V\colon q \mapsto \{t\},$$

and the accessibility relations $R_2, \dots, R_n$ are arbitrary serial and transitive relations. Clearly, we have $\mathfrak{K} \in \mathcal{K}_n^{st}$, hence $\mathfrak{K} \in \mathcal{K}_n^{t}$.

In order to define the new translation from $\mathcal{L}_n^{GA}$ to $\mathcal{L}_n$, it will be useful to have the following *reduction axioms*, which are both provable in our four systems.

Lemma 6.2.9. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$, *all non empty* $G \subseteq \mathcal{A}$, *and all* $i \in G$ *we have that* X *proves*

$$[\alpha!_G](\beta \wedge \gamma) \leftrightarrow [\alpha!_G]\beta \wedge [\alpha!_G]\gamma,$$
$$[\alpha!_G]K_i\beta \leftrightarrow K_i[\alpha!_G]\beta \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to [\alpha!_G]\beta)).$$

Proof. The proof of the first assertion works exactly the same way as the proof of Lemma 6.1.7. We show how to prove the second assertion. By the axioms ($\mathsf{GA5}_c$) and ($\mathsf{GA6}_c$) as well as tautological reasoning, we get that the formula $[\alpha!_G]K_i\beta$ is provably equivalent to

$$(K_i\neg\alpha \wedge K_i[\alpha!_G]\beta) \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to [\alpha!_G]\beta)) \tag{6.1}$$

in X. Using the fact that X proves $K_i[\alpha!_G]\beta \to K_i(\alpha \to [\alpha!_G]\beta)$ and again propositional reasoning, we get that formula (6.1) is now provably equivalent to $K_i[\alpha!_G]\beta \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to [\alpha!_G]\beta))$ in X, hence we are done. □

The group announcements for sceptical agents also satisfy all the properties from Definition 5.4.3.

Lemma 6.2.10. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, *or* $\mathcal{K}_n^{stu}$. *Then the group announcements for sceptical agents are fact preserving, adequate, total, self-dual, and normal with respect to* $\mathcal{X}$.

Proof. These assertions can all be formally proved in the deductive system that corresponds to $\mathcal{X}$. The proof is identical to the proof of Lemma 6.1.8. □

In addition to the properties of Lemma 6.2.10, we have that the group announcements for sceptical agents are *consistency preserving*. That is, the consistency of beliefs is preserved after any announcement, as we show in the following lemma.

Lemma 6.2.11. *Let* X *be one of the Hilbert systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha \in \mathcal{L}_n^{GA}$, *all non empty* $G \subseteq \mathcal{A}$, *and all* $i \in \mathcal{A}$ *we have that* X *proves*

$$\neg K_i\bot \to [\alpha!_G]\neg K_i\bot.$$

Proof. For $i \in \mathcal{A} \setminus G$, the proof is trivial. We show how to prove the claim for $i \in G$. By axiom ($\mathsf{GA3}$) and the second assertion of Lemma 6.2.9, we get

$$\mathsf{X} \vdash [\alpha!_G]\neg K_i\bot \leftrightarrow \neg(K_i[\alpha!_G]\bot \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to [\alpha!_G]\bot))).$$

From the proof of Lemma 6.2.10 (fact preservation), we know that $[\alpha!_G]\bot$ is provably equivalent to $\bot$ in X, hence we get

$$\mathsf{X} \vdash [\alpha!_G]\neg K_i\bot \leftrightarrow \neg(K_i\bot \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to \bot)))$$

by normal modal logic reasoning. Again by standard modal logic reasoning, it is easy to see that $\mathsf{X} \vdash \neg K_i\bot \to \neg(K_i\bot \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \to \bot)))$, hence we are done. □

We have seen in Example 6.1.2 that Lemma 6.2.11 does not hold in the context of trustful agents. In the presence of axiom (D), we can even stat a stronger statement as Lemma 6.2.11. As an immediate consequence of axiom (D) and the rule (GAN), we get that for all $\alpha \in \mathcal{L}_n^{GA}$, all non empty $G \subseteq \mathcal{A}$, and all $i \in \mathcal{A}$ we have

$$\mathsf{KD}_n^{GA_c} \vdash [\alpha!_G]\neg K_i\bot, \qquad \mathsf{KD45}_n^{GA_c} \vdash [\alpha!_G]\neg K_i\bot.$$

Although the consistency preserving announcements have a more sophisticated semantics, we can still establish an equivalence preserving translation to epistemic logic. The only difference is in the definition of the auxiliary function h, which makes use of both assertions from Lemma 6.2.9.

Definition 6.2.12. We inductively define the auxiliary function h that maps from $\{[\alpha!_G]\beta : \alpha, \beta \in \mathcal{L}_n^{GA}, \emptyset \neq G \subseteq \mathcal{A}\}$ to $\mathcal{L}_n^{GA}$ as follows,

$$\begin{aligned}
h([\alpha!_G]p) &:= p,\\
h([\alpha!_G]\neg\beta) &:= \neg h([\alpha!_G]\beta),\\
h([\alpha!_G](\beta \wedge \gamma) &:= h([\alpha!_G]\beta) \wedge h([\alpha!_G]\gamma),\\
h([\alpha!_G]K_i\beta) &:= \begin{cases} K_i h([\alpha!_G]\beta) \vee & \\ \quad (\neg K_i\neg\alpha \wedge K_i(\alpha \to h([\alpha!_G]\beta))) & \text{if } i \in G,\\ K_i\beta & \text{if } i \notin G,\end{cases}\\
h([\alpha!_G][\beta!_H]\gamma) &:= [\alpha!_G][\beta!_H]\gamma.
\end{aligned}$$

It is again easy to see that for all $\alpha, \beta \in \mathcal{L}_n$ and all non empty $G \subseteq \mathcal{A}$ we have $h([\alpha!_G]\beta) \in \mathcal{L}_n$. Moreover, the new function h is equivalence preserving like in Section 6.1.

Lemma 6.2.13. *Let X be one of the Hilbert systems $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, or $\mathsf{KD45}_n^{GA_c}$. Then for all $\alpha, \beta \in \mathcal{L}_n^{GA}$ and all non empty $G \subseteq \mathcal{A}$ we have*

$$\mathsf{X} \vdash h([\alpha!_G]\beta) \leftrightarrow [\alpha!_G]\beta.$$

Proof. By induction on β. The only difference to the proof of Lemma 6.1.10 is the case $\beta = K_i\gamma$ for $i \in G$ in the induction step. We start with a proof of $h([\alpha!_G]\gamma) \leftrightarrow [\alpha!_G]\gamma$ by induction hypothesis. By normal modal logic reasoning, we get that $K_i h([\alpha!_G]\gamma) \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \rightarrow h([\alpha!_G]\gamma)))$ and $K_i[\alpha!_G]\gamma \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \rightarrow [\alpha!_G]\gamma))$ are provably equivalent in X. Since the former formula is defined to be $h([\alpha!_G]K_i\gamma)$, and the latter formula is provably equivalent to $[\alpha!_G]K_i\gamma$ by the second assertion of Lemma 6.2.9, we are done. ☐

Like in Section 6.1, we will now prove a restricted version of syntax independence for the consistency preserving group announcements.

Lemma 6.2.14. *Let* X *be one of the Hilbert systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{GA}$, *all* $\varphi \in \mathcal{L}_n$, *and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!_G]\varphi \leftrightarrow [\beta!_G]\varphi.$$

Proof. By induction on φ. We show how to prove the case $\varphi = K_i\psi$ for $i \in G$ in the induction step. By assumption and the induction hypothesis, we have $\mathsf{X} \vdash \alpha \leftrightarrow \beta$ and $\mathsf{X} \vdash [\alpha!_G]\psi \leftrightarrow [\beta!_G]\psi$. By normal modal logic reasoning, we get that $K_i[\alpha!_G]\psi \vee (\neg K_i\neg\alpha \wedge K_i(\alpha \rightarrow [\alpha!_G]\psi))$ is provably equivalent to $K_i[\beta!_G]\psi \vee (\neg K_i\neg\beta \wedge K_i(\beta \rightarrow [\beta!_G]\psi))$ in X. By two applications of the second assertion of Lemma 6.2.9, we are done. ☐

The translation f from $\mathcal{L}_n^{GA}$ to $\mathcal{L}_n$ is defined as in Section 6.1, but now it uses the redefined auxiliary function h.

Definition 6.2.15. The function $f\colon \mathcal{L}_n^{GA} \rightarrow \mathcal{L}_n^{GA}$ is inductively defined by

$$\begin{aligned} f(p) &:= p, \\ f(\neg\alpha) &:= \neg f(\alpha), \\ f(\alpha \wedge \beta) &:= f(\alpha) \wedge f(\beta), \\ f(K_i\alpha) &:= K_i f(\alpha), \\ f([\alpha!_G]\beta) &:= h([f(\alpha)!_G]f(\beta)). \end{aligned}$$

Clearly, we again have that for all $\alpha \in \mathcal{L}_n^{GA}$ we have $f(\alpha) \in \mathcal{L}_n$. Like in Section 6.1, the function f is equivalence preserving.

Lemma 6.2.16. *Let* X *be one of the Hilbert systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{X} \vdash f(\alpha) \leftrightarrow \alpha.$$

Proof. The proof is the complete analogue to the proof of Lemma 6.1.13, but it uses Lemma 6.2.13 and Lemma 6.2.14 instead of Lemma 6.1.10 and Lemma 6.1.11 respectively. □

As an immediate consequence of Lemma 6.2.16, we get that the logic of group announcements for sceptical agents also has the same expressive strength as normal modal logic. And we can now prove syntax independence for the consistency preserving group announcements.

Lemma 6.2.17. *Let* X *be one of the Hilbert systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$ *and all non empty* $G \subseteq \mathcal{A}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \ \Rightarrow\ \mathsf{X} \vdash [\alpha!_G]\gamma \leftrightarrow [\beta!_G]\gamma.$$

Proof. The proof is identical to the proof of Lemma 6.1.14, but it uses Lemma 6.2.14 and Lemma 6.2.16 instead of Lemma 6.1.11 and Lemma 6.1.13 respectively. □

We are now able to state the *Replacement Theorem* for the logic of group announcements for sceptical agents.

Theorem 6.2.18 (Replacement)**.** *Let* X *be one of the deductive systems* $\mathsf{K}_n^{GA_c}$, $\mathsf{K45}_n^{GA_c}$, $\mathsf{KD}_n^{GA_c}$, *or* $\mathsf{KD45}_n^{GA_c}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \ \Rightarrow\ \mathsf{X} \vdash \gamma \leftrightarrow \gamma[\alpha/\beta].$$

Proof. By induction on γ. The proof works exactly the same way as the proof of Theorem 6.1.15. In the last case of the induction step, we use Lemma 6.2.17 instead of Lemma 6.1.14. □

Due to Lemma 6.2.16, we have a short completeness proof for the logic of consistency preserving group announcements.

Theorem 6.2.19. *For all* $\alpha \in \mathcal{L}_n^{GA}$ *we have*

$$\mathsf{K}_n^{GA_c} \vdash \alpha \ \Leftrightarrow\ \mathcal{K}_n \vDash \alpha, \qquad \mathsf{K45}_n^{GA_c} \vdash \alpha \ \Leftrightarrow\ \mathcal{K}_n^{tu} \vDash \alpha,$$
$$\mathsf{KD}_n^{GA_c} \vdash \alpha \ \Leftrightarrow\ \mathcal{K}_n^{s} \vDash \alpha, \qquad \mathsf{KD45}_n^{GA_c} \vdash \alpha \ \Leftrightarrow\ \mathcal{K}_n^{stu} \vDash \alpha.$$

Proof. Let X be one of the systems K_n, $\mathsf{K45}_n$, KD_n, or $\mathsf{KD45}_n$ and $\mathcal{X}$ be the class of Kripke structures that corresponds to X. Soundness has already been proved. For the direction from right to left, we assume $\mathcal{X} \vDash \alpha$. By soundness and Lemma 6.2.16, we have $\mathcal{X} \vDash f(\alpha)$. By completeness of X and the fact that $f(\alpha) \in \mathcal{L}_n$, we get $\mathsf{X} \vdash f(\alpha)$. Since the system X^{GA_c} extends X, we immediately get $\mathsf{X}^{GA_c} \vdash f(\alpha)$. Finally, again by Lemma 6.2.16, we have that X^{GA_c} proves α, which concludes the proof. □

Due to Lemma 6.2.11, we have that the formulas of the form $\neg K_i \bot$ are announcement resistant in the logic of group announcements for sceptical agents. We have seen in Example 6.1.2, that this is not the case with trustful agents, so we can say that we have more announcement resistant formulas for any G in $\mathcal{K}_n$ and $\mathcal{K}_n^{tu}$ with the consistency preserving announcements. The following lemma states that we have more conditions for a formula to be announcement resistant than in Section 6.1.

Lemma 6.2.20. *Let $G \subseteq \mathcal{A}$ be a non empty group of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, or $\mathcal{K}_n^{stu}$. Then we have the following sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{GA}$ to be announcement resistant for G in $\mathcal{X}$,*

1. *$\alpha \in \mathcal{L}_0$,*
2. *$\mathcal{X} \models \alpha$ or $\mathcal{X} \models \neg\alpha$,*
3. *$\alpha = K_i\beta$ or $\alpha = \neg K_i\beta$ for some $i \in \mathcal{A} \setminus G$ and some $\beta \in \mathcal{L}_n^{GA}$,*
4. *$\alpha = \beta \wedge \gamma$ or $\alpha = \beta \vee \gamma$ for some β, γ announcement resistant for G in $\mathcal{X}$,*
5. *$\alpha = K_i\beta$ for some $i \in G$ and some β announcement resistant for G in $\mathcal{X}$.*
6. *$\alpha = \neg K_i\beta$ for some $i \in G$ and some $\beta \in \mathcal{L}_n^{GA}$ satisfying $\mathcal{X} \models \neg\beta$.*

Proof. The first four assertions can be proved exactly the same way as in Lemma 6.1.17. The last assertion is a consequence of Lemma 6.2.11 and the fact that $\neg K_i\bot$ is equivalent to $\neg K_i\beta$ whenever β is not satisfiable in $\mathcal{X}$. We show how to prove the fifth assertion. Let X^{GA_c} be the deductive system that corresponds to $\mathcal{X}$. Furthermore, let β be announcement resistant for G in $\mathcal{X}$, $i \in G$ and $\gamma \in \mathcal{L}_n^{GA}$ be arbitrarily given. By completeness, we have $\mathsf{X}^{GA_c} \vdash \beta \to [\gamma!_G]\beta$. Hence, we easily get that X^{GA_c} proves $K_i\beta \to K_i[\gamma!_G]\beta$ by normal modal logic reasoning. By the second assertion of Lemma 6.2.9, we can derive $\mathsf{X}^{GA_c} \vdash K_i[\gamma!_G]\beta \to [\gamma!_G]K_i\beta$. We finally get that X^{GA_c} proves $K_i\beta \to [\gamma!_G]K_i\beta$ by tautological reasoning. By soundness, we get the desired result. □

Lemma 6.2.20 implies that an announcement resistant formula can never be forgotten by the agents. This is because if α is announcement resistant for G in $\mathcal{X}$, then for all $\beta \in \mathcal{L}_n^{GA}$ and all $i \in \mathcal{A}$ we have $\mathcal{X} \models K_i\alpha \to [\beta!_G]K_i\alpha$. That is, the agents will never revise the belief in announcement resistant formulas by accepting an announcement. Since the objective formulas are announcement resistant, we have belief expansion for factual belief.

With sceptical agents, it is not always the case that an announcement resistant formula gets common belief after it has been announced. It can happen that an agent does not learn a true announcement resistant formula that has been announced, because he believes that the new information is false. The following example illustrates this fact.

Example 6.2.21. Let $p \in \mathcal{P}$ and $\mathfrak{K} = (\{s,t\}, R_1, \dots, R_n, V)$ be defined by

$$R_1 = \{(s,t),(t,t)\}, \qquad V : q \mapsto \{s\},$$

such that the accessibility relations $R_2, \dots, R_n$ are serial, transitive, and Euclidean. Then we have $\mathfrak{K}, s \vDash p \wedge \neg[p!_{\{1\}}]K_1 p$. Since $\mathfrak{K} \in \mathcal{K}_n^{stu}$, we have that the formulas of the form $q \to [q!_G]K_i q$ are not provable in any of our four deductive systems.

For a big set of formulas, namely all negations of an announcement resistant formula, we have the following phenomenon. If an agent i believes that another agent j will reject an announcement, then agent i will believe after the announcement that agent j still believes the announced formula to be false.

Lemma 6.2.22. *Let $G \subseteq \mathcal{A}$ be a non empty group of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, or $\mathcal{K}_n^{stu}$. If $\neg\alpha$ is announcement resistant for G in $\mathcal{X}$, then for all $i, j \in \mathcal{A}$ we have*

$$\mathcal{X} \vDash K_i K_j \neg\alpha \to [\alpha!_G]K_i K_j \neg\alpha.$$

Proof. If $\neg\alpha$ is announcement resistant for G in $\mathcal{X}$, then so also is the formula $K_i K_j \neg\alpha$ by Lemma 6.2.20. □

Although an immediate consequence of Lemma 6.2.20, Lemma 6.2.22 implies that many announcement resistant formulas will not necessarily be learnt by the agents, because many of them are closed under negation. The following theorem states the best we can get with announcement resistant formulas for sceptical agents.

Theorem 6.2.23. *Let $G \subseteq \mathcal{A}$ be a non empty set of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, or $\mathcal{K}_n^{stu}$. Further, let $\alpha \in \mathcal{L}_n^{GA}$ be announcement resistant for G in $\mathcal{X}$. Then for all $l \geq 1$ and all $i_1, \dots, i_l \in G$ we have*

$$\mathcal{X} \vDash K_{i_l} \dots K_{i_2} \neg K_{i_1} \neg\alpha \to [\alpha!_G]K_{i_l} \dots K_{i_2} K_{i_1} \alpha.$$

Proof. Let X^{GA_c} be the deductive system that corresponds to $\mathcal{X}$. We will show that X^{GA_c} proves the $K_{i_l} \dots K_{i_2} \neg K_{i_1} \neg\alpha \to [\alpha!_G]K_{i_l} \dots K_{i_2} K_{i_1} \alpha$ by induction on l. In the base case, we have a proof of $\alpha \to [\alpha!_G]\alpha$ by assumption and completeness, and we get $\mathsf{X}^{GA_c} \vdash K_{i_1}(\alpha \to [\alpha!_G]\alpha)$ by an application of the rule (NEC). As a consequence of ($\mathsf{GA5}_c$), we have that X^{GA_c} proves $K_{i_1}(\alpha \to [\alpha!_G]\alpha) \to (\neg K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_1}\alpha)$, and we directly get $\mathsf{X}^{GA_c} \vdash \neg K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_1}\alpha$. In the induction step, we have a proof of $K_{i_l}\ldots K_{i_2}\neg K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ by induction hypothesis, thus we get $\mathsf{X}^{GA_c} \vdash K_{i_{l+1}}K_{i_l}\ldots K_{i_2}\neg K_{i_1}\neg\alpha \to K_{i_{l+1}}[\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ by normal modal logic reasoning. Since the second assertion of Lemma 6.2.9 implies $\mathsf{X}^{GA_c} \vdash K_{i_{l+1}}[\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha \to [\alpha!_G]K_{i_{l+1}}K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$, the claim now easily follows. Due to soundness, we get the desired result. □

In presence of axiom (4), we get that an agent will know that he has learnt an announcement resistant formula. Moreover, the other agents know this fact if they have known that this agent will accept the announced formula.

Corollary 6.2.24. *Let $G \subseteq \mathcal{A}$ be a non empty set of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n^{tu}$ or $\mathcal{K}_n^{stu}$. Further, let $\alpha \in \mathcal{L}_n^{GA}$ be announcement resistant for G in $\mathcal{X}$. Then for all $k, l \geq 1$ and all $i_1, \ldots, i_l \in G$ we have*

$$\mathcal{X} \vDash K_{i_l}\ldots K_{i_2}\neg K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}^k\alpha.$$

Proof. Let X^{GA_c} be one of the deductive systems $\mathsf{K45}_n^{GA_c}$ or $\mathsf{KD45}_n^{GA_c}$. Then we have $\mathsf{X}^{GA_c} \vdash K_{i_l}\ldots K_{i_2}\neg K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ by Theorem 6.2.23 and completeness. It is easy to see that for all $m \geq 1$ we have that X^{GA_c} proves $[\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}^m\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}^{m+1}\alpha$ by the axioms (K), (4), and (GA2) as well as the rules (NEC) and (PAN). Hence, we can prove $[\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}^l\alpha$ in X^{GA_c} by induction on l. Finally, we get that X^{GA_c} proves $\neg K_{i_l}\ldots K_{i_2}K_{i_1}\neg\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}^l\alpha$ by tautological reasoning. Due to soundness, we are done. □

As another consequence of Theorem 6.2.23, we get that an announcement resistant formula will always remain in the beliefs of the agents, if this formula is repeatedly told to the same group.

Corollary 6.2.25. *Let $G \subseteq \mathcal{A}$ be a non empty set of agents and $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{s}$, or $\mathcal{K}_n^{stu}$. Further, let $\alpha \in \mathcal{L}_n^{GA}$ be announcement resistant for G in $\mathcal{X}$. Then for all $k, l \geq 1$ and all $i_1, \ldots, i_l \in G$ we have*

$$\mathcal{X} \vDash K_{i_l}\ldots K_{i_2}\neg K_{i_1}\neg\alpha \to [\alpha!_G]^k K_{i_l}\ldots K_{i_2}K_{i_1}\alpha.$$

Proof. Let X^{GA_c} be the deductive system that corresponds to $\mathcal{X}$. Then we have that the formula $K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ is announcement resistant for G in $\mathcal{X}$ by Lemma 6.2.20. Therefore, by completeness, we get that the formula $K_{i_l}\ldots K_{i_2}K_{i_1}\alpha \to [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ is provable in X^{GA_c}. Thus, we can prove $\mathsf{X}^{GA_c} \vdash [\alpha!_G]K_{i_l}\ldots K_{i_2}K_{i_1}\alpha \to [\alpha!_G]^k K_{i_l}\ldots K_{i_2}K_{i_1}\alpha$ by induction on

k, using axiom (PA2) and the rule (PAN). By Theorem 6.2.23, we get that X^{GA_c} proves $K_{i_l} \dots K_{i_2} \neg K_{i_1} \neg \alpha \rightarrow [\alpha!_G]^k K_{i_l} \dots K_{i_2} K_{i_1}$, and by soundness, we get the desired result. □

6.3 Adding common belief

In this section, we are working towards an axiomatisation for the logic of common knowledge and group announcements. For this purpose, we also add action composition to the language. The reason for this technical detail is that we were not able to find an axiom like

$$[\alpha!_G][\beta!_H]\gamma \leftrightarrow [\delta!_I]\gamma,$$

even without the presence of common knowledge operators. There is an axiomatisation of a much more powerful logic where group announcements and common knowledge are included by Baltag, Moss, and Solecki [7]. However, an announcement is represented by a Kripke structure and can hardly be written in a one line formula. We consider it to be an interesting and important task to come up with an axiomatisation within a simpler language. First, we repeat the language $\mathcal{L}_n^{CGA}$ of the logic of common belief and group announcements from Definition 5.4.2. The formulas and actions are simultaneously defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\begin{aligned} \alpha &::= p \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid K_i\alpha \mid C_G\alpha \mid [\pi]\alpha, \\ \pi &::= \alpha!_G \mid (\pi \,;\, \alpha!_G). \end{aligned}$$

Observe that an action is just a sequence of group announcements. We are now going to introduce the notion of length, subformula and substitution for the language $\mathcal{L}_n^{CGA}$.

Definition 6.3.1. The length $|\pi|$ of an action π is just the number of announcements it contains and is defined by induction on π as follows,

$$|\alpha!_G| := 1, \qquad |\rho \,;\, \alpha!_G| := |\rho| + 1.$$

On the other hand, the length of formulas of the form $[\pi]\beta$ is defined to be the sum of the lengths of the formulas occurring in π plus $|\beta| + 2$,

$$|[\alpha!_G]\beta| := |\alpha| + |\beta| + 2, \qquad |[\rho \,;\, \alpha!_G]\beta| := |[\rho]\beta| + |\alpha|.$$

The length of a formula $[\pi]\beta$ is defined by induction on the length of the action π, which we will also call *induction on* π. The set $\mathrm{af}(\pi)$ of announced formulas of π is defined by induction on π by

$$\mathrm{af}(\alpha!_G) := \{\alpha\}, \qquad \mathrm{af}(\rho \,;\, \alpha!_G) := \mathrm{af}(\rho) \cup \{\alpha\}.$$

Due to the definition of $\mathrm{af}(\pi)$, we are now able to define the set of subformulas occurring in a formula of the form $[\pi]\beta$,

$$\mathrm{sub}([\pi]\beta) := \{[\pi]\beta\} \cup \mathrm{sub}(\beta) \cup \{\mathrm{sub}(\gamma) : \gamma \in \mathrm{af}(\pi)\}.$$

Clearly, the number of subformulas of a formula α is bound by the length of α, like in the simpler languages. Although we have a limited action composition, the composition of two arbitrary actions can easily be defined.

Definition 6.3.2. For all actions π, ρ we define the composed action $\pi\,;\rho$ by induction on ρ,

$$\begin{aligned} \pi\,;\alpha!_G & \quad \text{is already defined,} \\ \pi\,;(\sigma\,;\alpha!_G) &:= (\pi\,;\sigma)\,;\alpha!_G. \end{aligned}$$

Note that we overload the action composition operator ; because the resulting action will always be a sequence of announcements and no confusion can arise. In this section, we will work with the semantics for trustful agents like in Section 6.1. Yet, the definition of the notion of satisfaction for announcement formulas is a bit improved. The following defining clauses are added to Definition 5.1.5.

Definition 6.3.3. Given a Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ and a world $s \in S$, the notion of common knowledge formulas and announcement formulas of $\mathcal{L}_n^{CGA}$ being *satisfied* in the pointed structure $\mathfrak{K}, s$ is defined by

$$\begin{aligned} \mathfrak{K}, s \vDash C_G\alpha &\;:\Leftrightarrow\; \text{for all } t \in S,\ sR_G^+t \;\Rightarrow\; \mathfrak{K}, t \vDash \alpha, \\ \mathfrak{K}, s \vDash [\pi]\alpha &\;:\Leftrightarrow\; (\mathfrak{K}, s)^\pi \vDash \alpha, \end{aligned}$$

where the transformed pointed structure $(\mathfrak{K}, s)^\pi$ is simultaneously defined by induction on π,

$$(\mathfrak{K}, s)^{\alpha!_G} := \mathfrak{K}^{\alpha,G}, s_1, \qquad\qquad (\mathfrak{K}, s)^{\rho;\alpha!_G} := ((\mathfrak{K}, s)^\rho)^{\alpha!_G},$$

and $\mathfrak{K}^{\alpha,G} = (S', R_1^{\alpha,G}, \ldots, R_n^{\alpha,G}, V')$ is exactly the same structure as in Definition 6.1.1. We repeat this definition here,

$$\begin{aligned} S' &:= S \times \{0,1\}, \\ R_i^{\alpha,G} &:= \begin{cases} \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_1) : sR_it \text{ and } \mathfrak{K}, t \vDash \alpha\} & \text{if } i \in G, \\ \{(s_0,t_0) : sR_it\} \cup \{(s_1,t_0) : sR_it\} & \text{if } i \notin G, \end{cases} \\ V'(p) &:= V(p) \times \{0,1\}. \end{aligned}$$

for all $i \in \mathcal{A}$ and all $p \in \mathcal{P}$. Again, we define $s_0 := (s,0)$ and $s_1 := (s,1)$ for all $s \in S$.

If $\mathfrak{K}$ is a Kripke structure and π is an action, then we will also define the notion of the transformed structure $\mathfrak{K}^\pi$.

Definition 6.3.4. Let the Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ and the action π be given. Then we define the transformed Kripke structure $\mathfrak{K}^\pi$ by induction on π as follows,

$$\mathfrak{K}^{\alpha!_G} := \mathfrak{K}^{\alpha,G},$$
$$\mathfrak{K}^{\pi;\alpha!_G} := (\mathfrak{K}^\pi)^{\alpha,G}.$$

The universe, the accessibility relations, and the valuation function in the structure $\mathfrak{K}^\pi$ are denoted by S^π, R_i^π, and V^π respectively. In order to denote the worlds in $\mathfrak{K}^\pi$, we are going to use binary words of length $|\pi|$. If π is of the form $\alpha!_G$ and $s \in |\mathfrak{K}|$, then the worlds $s_w = (s, w)$ denote the elements of $|\mathfrak{K}^{\alpha!_G}|$ for the binary word $w \in \{0, 1\}$. If π is a composed action $\rho\,;\alpha!_G$ and w is a binary word of length $|\rho|$, then we will use the following denotations,

$$s_{w\star 0} := (s_w, 0), \qquad\qquad s_{w\star 1} := (s_w, 1),$$

where $\star$ is the concatenation of binary words. Hence, we can now write the pointed structure $(\mathfrak{K}, s)^\pi$ as $\mathfrak{K}^\pi, s_{\vec{1}}$.

Since the length of a binary word is equal to the length of the action, it will always be clear from the context what the length of a binary word w in an expression s_w is. Due to the above defined notions, we can now express that $\mathfrak{K}, s \vDash [\pi] C_G \alpha$ holds if and only if

$$\text{for all } t \in S \text{ and all } w \text{ with length } |\pi|,\ s_{\vec{1}} (R_G^\pi)^+ t_w \ \Rightarrow\ \mathfrak{K}^\pi, t_w \vDash \alpha.$$

The relation R_G^π is defined to be the union of $\{R_i^\pi : i \in G\}$, as we would expect. Like in Section 6.1, we have that transitivity and Euclideanity are preserved by the above defined model transformations.

Lemma 6.3.5. *For all Kripke structures $\mathfrak{K}$ and all actions π we have*

$$\mathfrak{K} \in \mathcal{K}_n^t \ \Rightarrow\ \mathfrak{K}^\pi \in \mathcal{K}_n^t, \qquad\qquad \mathfrak{K} \in \mathcal{K}_n^{tu} \ \Rightarrow\ \mathfrak{K}^\pi \in \mathcal{K}_n^{tu}.$$

Proof. By induction on π. The base case ($\pi = \alpha!_G$) and the induction step ($\pi = \rho\,;\alpha!_G$) both have exactly the same proof as Lemma 6.1.4. □

For every action π, all possible worlds $s_v, t_w \in S^\pi$, and all $i \in \mathcal{A}$, it is not immediate whether or not $s_v R_i^\pi t_w$ can possibly hold. In the following definitions, we are going to prepare a characterisation lemma.

Definition 6.3.6. Let the action π and $i \in \mathcal{A}$ be given. Then for all binary words v, w of length $|\pi|$ we define the relation $v \xrightarrow{\pi}_i w$ by induction on π by

$$v \xrightarrow{\alpha!_G}_i w \;:\Leftrightarrow \begin{cases} v = w & \text{if } i \in G, \\ w = 0 & \text{if } i \notin G, \end{cases}$$

$$v \star k \xrightarrow{\rho;\alpha!_G}_i w \star l \;:\Leftrightarrow \begin{cases} v \xrightarrow{\rho}_i w \text{ and } k = l & \text{if } i \in G \\ v \xrightarrow{\rho}_i w \text{ and } l = 0 & \text{if } i \notin G. \end{cases}$$

Observe that we have defined this relation without semantical objects like possible worlds, accessibility relations, or valuation functions. Clearly, the relation $\xrightarrow{\pi}_i$ is not enough to characterise the relation R_i^π. The following definition of the characteristic formula will be very useful for the whole section.

Definition 6.3.7. Given an action π and a binary word w of length $|\pi|$, the *characteristic formula* χ_w^π and the *restriction* π_w *of* π *to* w are simultaneously defined by induction on π,

$$\chi_w^{\alpha!_G} := \begin{cases} \alpha & \text{if } w = 1, \\ \top & \text{if } w = 0, \end{cases} \qquad \chi_{w\star k}^{\rho;\alpha!_G} := \begin{cases} \chi_w^\rho \wedge [\pi_w]\alpha & \text{if } k = 1, \\ \chi_w^\rho & \text{if } k = 0, \end{cases}$$

$$(\alpha!_G)_w := \begin{cases} \alpha!_G & \text{if } w = 1, \\ \top!_{\mathcal{A}} & \text{if } w = 0, \end{cases} \qquad (\rho\,;\alpha!_G)_{w\star k} := \begin{cases} \rho_w\,;\alpha!_G & \text{if } k = 1, \\ \rho_w & \text{if } k = 0. \end{cases}$$

Again, we have defined formulas and actions without using semantical terms. The following lemma is important for the understanding of the complex semantics.

Lemma 6.3.8. *Let $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ and $s, t \in S$ be given. Then for all actions π, all binary words v, w of length $|\pi|$, and all $i \in \mathcal{A}$ we have*

$$s_v R_i^\pi t_w \;\Leftrightarrow\; sR_it \text{ and } v \xrightarrow{\pi}_i w \text{ and } \mathfrak{K}, t \vDash \chi_w^\pi.$$

Moreover, for all $\alpha \in \mathcal{L}_n^{CGA}$ we have

$$\mathfrak{K}^\pi, s_w \vDash \alpha \;\Leftrightarrow\; \mathfrak{K}, s \vDash [\pi_w]\alpha.$$

Proof. Both assertions can be proved by induction on π. □

He have seen that the formulas χ_w^π and actions π_w are very useful for semantical considerations. However, for syntactical purposes, it is easier to deal with similar formulas and actions that are defined without the notion of binary words.

Definition 6.3.9. Given an action π and an agent $i \in \mathcal{A}$, the *characteristic formula* χ_i^π and the *restriction* π_i *of* π *to* i are simultaneously defined by induction on π,

$$\chi_i^{\alpha!_G} := \begin{cases} \alpha & \text{if } i \in G, \\ \top & \text{if } i \notin G, \end{cases} \qquad \chi_i^{\rho;\alpha!_G} := \begin{cases} \chi_i^\rho \wedge [\pi_i]\alpha & \text{if } i \in G, \\ \chi_i^\rho & \text{if } i \notin G, \end{cases}$$

$$(\alpha!_G)_i := \begin{cases} \alpha!_G & \text{if } i \in G, \\ \top!_{\mathcal{A}} & \text{if } i \notin G, \end{cases} \qquad (\rho\,;\alpha!_G)_i := \begin{cases} \rho_i\,;\alpha!_G & \text{if } i \in G, \\ \rho_i & \text{if } i \notin G. \end{cases}$$

The following lemma shows how we can use the formulas χ_i^π and actions π_i.

Lemma 6.3.10. *For all actions π, all $i \in \mathcal{A}$, and all formulas α we have*

$$\mathcal{K}_n \vDash [\pi]K_i\alpha \leftrightarrow K_i(\chi_i^\pi \rightarrow [\pi_i]\alpha).$$

Proof. By induction on π. In the base case, we use the trivial fact that $\mathcal{K}_n \vDash [\top!_{\mathcal{A}}]\beta \leftrightarrow \beta$ for all formulas β. ☐

Observe that π_i is the sequence of announcements from π that affect agent i. Therefore, we obviously have $(\pi_i)_i = \pi_i$ and $\chi_i^{\pi_i} = \chi_i^\pi$. The following lemma shows how χ_i^π and π_i are related to χ_w^π and π_w respectively.

Lemma 6.3.11. *Let the action π and the agent $i \in \mathcal{A}$ be given. Then for all binary words v, w of length $|\pi|$ we have*

$$v \xrightarrow{\pi}_i w \;\Rightarrow\; \chi_i^{\pi_v} = \chi_w^\pi \text{ and } (\pi_v)_i = \pi_w.$$

Proof. By induction on π. ☐

With the fact that $\pi_{\vec{1}} = \pi$ for all actions π we immediately get the following corollary.

Corollary 6.3.12. *For all actions π, all agents $i \in \mathcal{A}$, and all binary words w of length $|\pi|$ we have*

$$\vec{1} \xrightarrow{\pi}_i w \;\Rightarrow\; \chi_i^\pi = \chi_w^\pi \text{ and } \pi_i = \pi_w.$$

Before we can give the Hilbert system, we have to define a normal form for all formulas of the form $\chi_w^\pi \rightarrow [\pi_w]\alpha$ and $\chi_i^\pi \rightarrow [\pi_i]\alpha$, where the artificial announcements $\top!_{\mathcal{A}}$ do not occur.

Definition 6.3.13. Given an action π and a binary word w of length $|\pi|$, the formula η^π_w and the function $f^\pi_w \colon \mathcal{L}^{CGA}_n \to \mathcal{L}^{CGA}_n$ are simultaneously defined by induction on π,

$$\eta^{\alpha!_G}_w := \begin{cases} \alpha & \text{if } w = 1, \\ \top & \text{if } w = 0, \end{cases} \qquad \eta^{\rho;\alpha!_G}_{w\star k} := \begin{cases} \eta^\rho_w \wedge f^\rho_w(\alpha) & \text{if } k = 1, \\ \eta^\rho_w & \text{if } k = 0, \end{cases}$$

$$f^{\alpha!_G}_w(\beta) := \begin{cases} [\alpha!_G]\beta & \text{if } w = 1, \\ \beta & \text{if } w = 0, \end{cases} \qquad f^{\rho;\alpha!_G}_{w\star k}(\beta) := \begin{cases} f^\rho_w([\alpha!_G]\beta) & \text{if } k = 1, \\ f^\rho_w(\beta) & \text{if } k = 0. \end{cases}$$

Accordingly, for a given agent $i \in \mathcal{A}$ the formula η^π_i and the function $f^\pi_i \colon \mathcal{L}^{CGA}_n \to \mathcal{L}^{CGA}_n$ are simultaneously defined by induction on π,

$$\eta^{\alpha!_G}_i := \begin{cases} \alpha & \text{if } i \in G, \\ \top & \text{if } i \notin G, \end{cases} \qquad \eta^{\rho;\alpha!_G}_{i} := \begin{cases} \eta^\rho_i \wedge f^\rho_i(\alpha) & \text{if } i \in G, \\ \eta^\rho_i & \text{if } i \notin G, \end{cases}$$

$$f^{\alpha!_G}_i(\beta) := \begin{cases} [\alpha!_G]\beta & \text{if } i \in G, \\ \beta & \text{if } i \notin G, \end{cases} \qquad f^{\rho;\alpha!_G}_{i}(\beta) := \begin{cases} f^\rho_i([\alpha!_G]\beta) & \text{if } i \in G, \\ f^\rho_i(\beta) & \text{if } i \notin G. \end{cases}$$

Clearly the above defined notions are equivalent to these normal forms in the following sense.

Lemma 6.3.14. *For all actions π, all agents $i \in \mathcal{A}$, all binary words w of length $|\pi|$, and all formulas α we have*

$$\mathcal{K}_n \vDash \chi^\pi_w \leftrightarrow \eta^\pi_w, \qquad \mathcal{K}_n \vDash [\pi_w]\alpha \leftrightarrow f^\pi_w(\alpha),$$
$$\mathcal{K}_n \vDash \chi^\pi_i \leftrightarrow \eta^\pi_i, \qquad \mathcal{K}_n \vDash [\pi_i]\alpha \leftrightarrow f^\pi_i(\alpha).$$

Proof. All of the assertions are an immediate consequence of the fact that we have $\mathcal{K}_n \vDash [\top!_\mathcal{A}]\beta \leftrightarrow \beta$ and $\mathcal{K}_n \vDash [\rho\,;\gamma!_G]\beta \leftrightarrow [\rho][\gamma!_G]\beta$ for all actions $\rho, \gamma!_G$ and all formulas β. ☐

We are now ready to define the Hilbert systems for the logic of common knowledge and group announcements. They contain an inference rule with a variable number of premises, which is not very nice.

Definition 6.3.15. The deductive systems K^{CGA}_n, $\mathsf{K4}^{CGA}_n$, and $\mathsf{K45}^{CGA}_n$ are the systems K^C_n, $\mathsf{K4}^C_n$, and $\mathsf{K45}^C_n$ respectively augmented with the following *group announcement axioms*,

(GA1) $[\alpha!_G]p \leftrightarrow p$,

(GA2) $[\alpha!_G](\beta \to \gamma) \to ([\alpha!_G]\beta \to [\alpha!_G]\gamma)$,
(GA3) $[\alpha!_G]\neg\beta \leftrightarrow \neg[\alpha!_G]\beta$,
(GA4) $[\alpha!_G]K_i\beta \leftrightarrow K_i\beta \quad (i \notin G)$,
(GA5) $[\alpha!_G]K_i\beta \leftrightarrow K_i(\alpha \to [\alpha!_G]\beta) \quad (i \in G)$,
(GA7) $[\pi][\alpha!_G]\beta \leftrightarrow [\pi\,;\,\alpha!_G]\beta$,

as well as the *group announcement necessitation rule* and the *group announcement induction rule*,

$$(\mathsf{GAN})\ \frac{\alpha}{[\beta!_G]\alpha}\,,$$

$$(\mathsf{GAI})\ \frac{\alpha \to E_G(\eta^\pi_w \to \alpha \wedge f^\pi_w(\beta)) \qquad \text{for all } w \text{ satisfying } \vec{1}(\xrightarrow{\pi}_G)^+ w}{\alpha \to [\pi]C_G\beta}\,.$$

Clearly, the relation $\xrightarrow{\pi}_G$ is the union of $\{\xrightarrow{\pi}_i : i \in G\}$, as we would expect. We have now made all the preparations we need in order to prove soundness of our three Hilbert systems.

Lemma 6.3.16. *For all $\alpha \in \mathcal{L}^{CGA}_n$ we have*

$$\mathsf{K}^{CGA}_n \vdash \alpha \;\Rightarrow\; \mathcal{K}_n \models \alpha,$$
$$\mathsf{K4}^{CGA}_n \vdash \alpha \;\Rightarrow\; \mathcal{K}^t_n \models \alpha,$$
$$\mathsf{K45}^{CGA}_n \vdash \alpha \;\Rightarrow\; \mathcal{K}^{tu}_n \models \alpha.$$

Proof. By induction on the length of the proof. In the base case, the only new case is axiom (GA7). For all pointed structures $\mathfrak{K}, s$ we have

$$\begin{aligned} \mathfrak{K}, s \models [\pi][\alpha!_G]\beta &\Leftrightarrow (\mathfrak{K}, s)^\pi \models [\alpha!_G]\beta \\ &\Leftrightarrow ((\mathfrak{K}, s)^\pi)^{\alpha!_G} \models \beta \\ &\Leftrightarrow (\mathfrak{K}, s)^{\pi;\alpha!_G} \models \beta \\ &\Leftrightarrow \mathfrak{K}, s \models [\pi\,;\,\alpha!_G]\beta. \end{aligned}$$

In the induction step, the rule (GAN) is sound with respect to $\mathcal{K}^t_n$ and $\mathcal{K}^{tu}_n$ due to Lemma 6.3.5. We will now prove soundness of the rule (GAI). Suppose the premise of the rule is valid in one of the three classes of Kripke structures, that is

for all w with length $|\pi|$ and all $i \in G$,
$$\vec{1}(\xrightarrow{\pi}_G)^+ w \;\Rightarrow\; \mathsf{K}_n \models \alpha \to K_i(\eta^\pi_w \to \alpha \wedge f^\pi_w(\beta)),$$

and suppose that $\mathfrak{K}, s \vDash \alpha$. In order to show that $\mathfrak{K}, s \vDash [\pi]C_G\beta$, we will show by induction on k that

$$\text{for all } k \geq 1, \text{ all } t \in |\mathfrak{K}|, \text{ and all } w \text{ of length } |\pi|, \quad s_{\vec{1}}(R^\pi_G)^k t_w \;\Rightarrow\; \mathfrak{K}^\pi, t_w \vDash \beta \text{ and } \mathfrak{K}, t \vDash \alpha.$$

In the base case ($k = 1$), we have $s_{\vec{1}} R^\pi_G t_w$ implies $sR_i t$, $\vec{1} \xrightarrow{\pi}_i w$, and $\mathfrak{K}, t \vDash \chi^\pi_w$ for some $i \in G$ by Lemma 6.3.8. By Lemma 6.3.14, we get $\mathfrak{K}, t \vDash \eta^\pi_w$, and the premise yields $\mathfrak{K}, t \vDash \alpha \wedge f^\pi_w(\beta)$. Again by Lemma 6.3.14, we now get $\mathfrak{K}, t \vDash [\pi_w]\beta$, which is equivalent to $\mathfrak{K}^\pi, t_w \vDash \beta$ by Lemma 6.3.8. In the induction step ($k \mapsto k+1$), we have $s_{\vec{1}}(R^\pi_G)^{k+1} t_w$ implies $s_{\vec{1}}(R^\pi_G)^k u_v R^\pi_i t_w$ for some $u_v \in |\mathfrak{K}^\pi|$ and some $i \in G$. We get $uR_i t$, $\vec{1}(\xrightarrow{\pi}_G)^+ v \xrightarrow{\pi}_i w$, and $\mathfrak{K}, t \vDash \chi^\pi_w$ by Lemma 6.3.8. Thus, we have $\mathfrak{K}, t \vDash \eta^\pi_w$ by Lemma 6.3.14. By induction hypothesis, we have $\mathfrak{K}, u \vDash \alpha$, thus the valid premise yields $\mathfrak{K}, t \vDash \alpha \wedge f^\pi_w(\beta)$. Like in the base case, we now get $\mathfrak{K}^\pi, t_w \vDash \beta$, and we are done. □

We will now derive some axioms and rules that contain arbitrary announcements.

Lemma 6.3.17. *In* K_n^{CGA}, $\mathsf{K4}_n^{CGA}$, *and* $\mathsf{K45}_n^{CGA}$ *we can prove the following action axioms,*

(A1) $[\pi]p \leftrightarrow p$,

(A2) $[\pi](\alpha \wedge \beta) \leftrightarrow [\pi]\alpha \wedge [\pi]\beta$,

(A3) $[\pi]\neg\alpha \leftrightarrow \neg[\pi]\alpha$,

(A4) $[\pi]K_i\alpha \leftrightarrow K_i(\eta^\pi_i \to f^\pi_i(\alpha))$

(A6) $[\pi][\rho]\alpha \leftrightarrow [\pi\,;\rho]\alpha$,

(A7) $[\pi]E_G\alpha \leftrightarrow \bigwedge_{i\in G} K_i(\eta^\pi_i \to f^\pi_i(\alpha))$,

(A8) $[\pi]C_G\alpha \leftrightarrow \bigwedge_{i\in G} K_i(\eta^\pi_i \to f^\pi_i(\alpha \wedge C_G\alpha))$,

as well as the following action necessitation rule,

$$(\mathsf{AN})\ \frac{\alpha}{[\pi]\alpha}\,.$$

Proof. All of these properties can be proved by induction on π. (A7) and (A8) are an immediate consequence of (A4). □

Although our deductive systems are similar to the more complex system in [7], we do not know whether or not our systems are complete. For instance, we have not managed to syntactically prove $[\top!_{\mathcal{A}}]\alpha \leftrightarrow \alpha$.

Chapter 7

Knowledge expansion

The first logic with operators for public communication has been presented by Plaza [57]. We have introduced this logic in Section 5.4, where we have shown some of its properties. Inspired by this idea, many authors further developed the theory of belief and knowledge change caused by incoming information in a modal logic setting. We confine ourselves to mentioning just a few typical articles by Baltag, van Benthem, van Ditmarsch, van Eijck, van der Hoek, Kooi, Moss, Renne, and Solecki [5, 6, 7, 8, 10, 11, 12, 13, 17, 18, 19, 20, 21, 22, 23, 59, 60]. In this chapter we will present two different public announcement semantics where announcements can expand the knowledge of the agents. Some announcements can also retract knowledge, but not knowledge about propositional formulas. In Section 7.1 we will present the well-known logic of truthful public announcements. These announcements are partial, which is very natural because a truthful announcement can only be made if the new information is true. However, for some applications like expansion of knowledge and belief, we need a logic with total announcements. This is what we have in Section 7.2, where we will present our results from [62] in all details. In Section 7.3, we will enrich the language of public announcements with operators for relativised common knowledge and common knowledge respectively, like we already do at the end of Section 7.1. We will give axiomatisations for the public announcement logics augmented with these operators and prove soundness and completeness.

7.1 Truthful public announcements

In this section we will give an introduction to the logic of truthful public announcements, which has the dual announcement semantics of Plaza's logic from [57]. Most of the results in this section are not new, but we sometimes give a proof in order to point to the differences with other announcement logics. We will quickly repeat the language $\mathcal{L}_n^{PA}$ from Section 5.4, which will be relevant for the whole chapter. The language $\mathcal{L}_n^{PA}$ of epistemic logic and public announcements is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ [\alpha!]\alpha.$$

For the semantics, we can just add the defining clause for public announcement formulas $[\alpha!]\beta$ to Definition 5.1.5.

Definition 7.1.1. Let the Kripke structure $K = (S, R_1, \ldots, R_n, V)$ be given and $s \in S$. Then the notion of an $\mathcal{L}_n^{PA}$ formula of the form $[\alpha!]\beta$ being *satisfied* in the pointed structure $\mathfrak{K}, s$ is defined as follows,

$$\mathfrak{K}, s \vDash [\alpha!]\beta \ :\Leftrightarrow \ \mathfrak{K}, s \vDash \alpha \text{ implies } \mathfrak{K}^\alpha, s \vDash \beta.$$

The relativised Kripke structure $\mathfrak{K}^\alpha = (S^\alpha, R_1^\alpha, \ldots, R_n^\alpha, V^\alpha)$ is exactly defined the same way as in Definition 5.4.2,

$$\begin{aligned} S^\alpha &:= \|\alpha\|_{\mathfrak{K}}, \\ R_i^\alpha &:= R_i \cap \|\alpha\|_{\mathfrak{K}}^2, \\ V^\alpha(p) &:= V(p) \cap \|\alpha\|_{\mathfrak{K}}, \end{aligned}$$

for all agents $i \in \mathcal{A}$ and all propositions $p \in \mathcal{P}$. Definition 7.1.1 is a bit problematic for the following reasons. First, the structure $\mathfrak{K}^\alpha$ is not defined if $\|\alpha\|_{\mathfrak{K}} = \emptyset$. Second, if $\mathfrak{K}, s \nvDash \alpha$ then we have $s \notin |\mathfrak{K}^\alpha|$. Therefore, we suggest to give this definition by specifying the worlds in $|\mathfrak{K}|$ that satisfy $[\alpha!]\beta$,

$$\|[\alpha!]\beta\|_{\mathfrak{K}} := \begin{cases} S & \text{if } \|\alpha\|_{\mathfrak{K}} = \emptyset, \\ (S \setminus \|\alpha\|_{\mathfrak{K}}) \cup \|\beta\|_{\mathfrak{K}^\alpha} & \text{otherwise,} \end{cases}$$

That is, we have $\mathfrak{K}, s \vDash [\alpha!]\beta$ if and only if $s \in \|[\alpha!]\beta\|_{\mathfrak{K}}$. As an immediate consequence of this definition, we get that the dual operator $\neg[\,\cdot\,!]\neg$ is equivalent to Plaza's announcement operator, cf. Definition 5.4.2.

Seriality of the accessibility relations is in general not preserved by the above defined model transformation. We will now give a counterexample in order to illustrate this fact.

Example 7.1.2. Let $\mathfrak{K} = (\{s, t\}, R_1, \ldots, R_n, V)$ be defined by

$$R_1 = \{(s,t), (t,t)\}, \qquad\qquad V : p \mapsto \{s\},$$

such that the accessibility relations $R_2, \ldots, R_n$ are serial, transitive, and Euclidean. Then we have $\mathfrak{K} \in \mathcal{K}_n^{stu}$, hence $\mathfrak{K} \in \mathcal{K}_n^{st}$ and $\mathfrak{K} \in \mathcal{K}_n^{s}$. Since for all $p \in \mathcal{P}$ we have $S^p = \{s\}$ and $R_1^p = \emptyset$, we get that $\mathfrak{K}^p \notin \mathcal{K}_n^s$, therefore $\mathfrak{K}^p \notin \mathcal{K}_n^{st}$ and $\mathfrak{K}^p \notin \mathcal{K}_n^{stu}$.

Due to Example 7.1.2, we cannot consider the classes $\mathcal{K}_n^s$, $\mathcal{K}_n^{st}$, and $\mathcal{K}_n^{stu}$, because the axiom (D) would not be satisfied after every public announcement. The following lemma shows that we do not have this problem with the other classes of Kripke structures.

Lemma 7.1.3. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then for all Kripke structures $\mathfrak{K}$ and all $\alpha \in \mathcal{L}_n^{PA}$ we have*

$$\|\alpha\|_{\mathfrak{K}} \neq \emptyset \text{ and } \mathfrak{K} \in \mathcal{X} \;\Rightarrow\; \mathfrak{K}^\alpha \in \mathcal{X}.$$

Proof. We show that Euclideanity is preserved. Let $sR_i^\alpha t$ and $sR_i^\alpha u$. Then we have $sR_i t$ and $sR_i u$, hence we get $tR_i u$ by Euclideanity of R_i. By assumption, we have that $t, u \in |\mathfrak{K}^\alpha|$, thus we get $tR_i^\alpha u$. The proofs of reflexivity and transitivity preservation are similar. □

We have chosen the following axioms and rules of the Hilbert systems for truthful public announcement logics.

Definition 7.1.4. The deductive systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, and $\mathsf{S5}_n^{PA}$ are the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, T_n, $\mathsf{S4}_n$, and $\mathsf{S5}_n$ respectively augmented with the following *public announcement axioms*,

(PA1) $[\alpha!]p \leftrightarrow (\alpha \to p)$,

(PA2) $[\alpha!](\beta \to \gamma) \to ([\alpha!]\beta \to [\alpha!]\gamma)$,

(PA3) $[\alpha!]\neg\beta \leftrightarrow (\alpha \to \neg[\alpha!]\beta)$,

(PA4) $[\alpha!]K_i\beta \leftrightarrow (\alpha \to K_i[\alpha!]\beta))$,

as well as the *public announcement necessitation rule*,

$$(\mathsf{PAN})\ \frac{\alpha}{[\beta!]\alpha}\,.$$

Axiom (PA2) is usually not part of the Hilbert systems for truthful public announcements, see e. g. van Benthem et al. [12]. There is a so-called *reduction axiom* instead, which is provable in all of our systems.

Lemma 7.1.5. *Let X be one of the deductive systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, or $\mathsf{S5}_n^{PA}$. Then for all $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ we have*

$$\mathsf{X} \vdash [\alpha!](\beta \wedge \gamma) \leftrightarrow [\alpha!]\beta \wedge [\alpha!]\gamma.$$

Proof. Since we have distribution (PA2) and necessitation (PAN) for public announcement operators, we get exactly the same proof as for the formula $K_i(\beta \wedge \gamma) \leftrightarrow K_i\beta \wedge K_i\gamma$ in normal modal logic. □

Each of our six systems is sound with respect to its corresponding class of Kripke structures.

Lemma 7.1.6. *For all* $\alpha \in \mathcal{L}_n^{PA}$ *we have*

$$\begin{aligned} \mathsf{K}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n \vDash \alpha, & \mathsf{K4}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n^t \vDash \alpha, \\ \mathsf{K45}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n^{tu} \vDash \alpha, & \mathsf{T}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n^r \vDash \alpha, \\ \mathsf{S4}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n^{rt} \vDash \alpha, & \mathsf{S5}_n^{PA} \vdash \alpha &\Rightarrow \mathcal{K}_n^{rtu} \vDash \alpha. \end{aligned}$$

Proof. We show some cases of the induction on the length on the proof. In the base case, we first prove that axiom (PA2) is valid. Let the pointed structure $\mathfrak{K}, s$ be given and assume $\mathfrak{K}, s \vDash [\alpha!](\beta \to \gamma) \wedge [\alpha!]\beta$. If $\mathfrak{K}, s \nvDash \alpha$, then we immediately have $\mathfrak{K}, s \vDash [\alpha!]\gamma$. If $\mathfrak{K}, s \vDash \alpha$, then we have $\mathfrak{K}^\alpha, s \vDash (\beta \to \gamma) \wedge \beta$, thus we have $\mathfrak{K}^\alpha, s \vDash \gamma$. Therefore, we get $\mathfrak{K}, s \vDash [\alpha!]\gamma$. Now, we prove the correctness of axiom (PA4). Let the pointed structure $\mathfrak{K}, s$ be given and assume $\mathfrak{K}, s \vDash \alpha$. Then we have

$$\begin{aligned} \mathfrak{K}, s \vDash [\alpha!]K_i\beta &\Leftrightarrow \mathfrak{K}^\alpha, s \vDash K_i\beta \\ &\Leftrightarrow \text{for all } t \in R_i^\alpha(s),\ \mathfrak{K}^\alpha, t \vDash \beta \\ &\Leftrightarrow \text{for all } t \in (R_i \cap \|\alpha\|_{\mathfrak{K}}^2)(s),\ \mathfrak{K}^\alpha, t \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s) \cap \|\alpha\|_{\mathfrak{K}},\ \mathfrak{K}^\alpha, t \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}^\alpha, t \vDash \beta \\ &\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash [\alpha!]\beta \\ &\Leftrightarrow \mathfrak{K}, s \vDash K_i[\alpha!]\beta. \end{aligned}$$

We have now proved that $\mathcal{K}_n \vDash \alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i[\alpha!]\beta)$. Together with the fact that $\mathcal{K}_n \vDash \neg\alpha \to [\alpha!]K_i\beta$, we get the desired axiom by propositional reasoning. In the induction step, the only interesting case is the rule (PAN). This is the only case where we have to distinguish the six classes of Kripke structures, and the correctness of this rule follows from Lemma 7.1.3. □

The following result is about the impact of truthful public announcements on propositional formulas.

Lemma 7.1.7. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha \in \mathcal{L}_n^{PA}$ *and all* $\beta \in \mathcal{L}_0$ *we have*

$$\mathsf{X} \vdash [\alpha!]\beta \leftrightarrow (\alpha \to \beta).$$

Proof. By induction on β. The base case is directly implied by axiom (PA1). The two cases in the induction step easily follow from axiom (PA3) and Lemma 7.1.5 respectively. □

We are now going to state which properties from Definition 5.4.3 are satisfied by the truthful public announcement semantics.

Lemma 7.1.8. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then the truthful public announcements are*

1. *fact preserving with respect to $\mathcal{X}$,*
2. *adequate with respect to $\mathcal{X}$,*
3. *not total with respect to $\mathcal{X}$,*
4. *not self-dual with respect to $\mathcal{X}$,*
5. *normal with respect to $\mathcal{X}$.*

Proof. Let X^{PA} be the deductive system that corresponds to $\mathcal{X}$. In order to show fact preservation, we can prove that for all $\alpha \in \mathcal{L}_0$ and all $\beta \in \mathcal{L}_n^{PA}$ we have $\mathsf{X}^{PA} \vdash \alpha \rightarrow [\beta!]\alpha$ by induction on α. The most interesting case is in the induction step, where α is of the form $\neg\gamma$. We have that the following formulas are all provable in X^{PA},

$$\begin{aligned} &\gamma \rightarrow [\beta!]\gamma && \text{by induction hypothesis,} \\ &[\beta!]\neg\gamma \leftrightarrow (\beta \rightarrow \neg[\beta!]\gamma) && \text{by (PA3),} \\ &[\beta!]\gamma \leftrightarrow (\beta \rightarrow \gamma) && \text{by Lemma 7.1.7,} \\ &[\beta!]\neg\gamma \leftrightarrow (\beta \rightarrow \neg\gamma) && \text{by Lemma 7.1.7.} \end{aligned}$$

By propositional reasoning, we get that X^{PA} proves $\neg\gamma \rightarrow [\beta!]\neg\gamma$ and fact preservation follows by soundness of X^{PA}. Adequacy is an immediate consequence of fact preservation. In order to show that totality does not hold, we can prove $\mathsf{X}^{PA} \vdash [\bot!]\bot$ as a consequence Lemma 7.1.7. By soundness, we get $\mathcal{X} \vDash [\bot!]\bot$, hence we have $\mathcal{X} \nvDash \neg[\bot!]\bot$. For the fourth assertion, we can prove $\mathsf{X} \vdash [\bot!]\neg\bot$ again by Lemma 7.1.7 and we get $\mathcal{X} \vDash [\bot!]\neg\bot$ by soundness. Since we know $\mathcal{X} \vDash [\bot!]\bot$ from the proof of the third assertion, we immediately get $\mathcal{X} \vDash \neg(\neg[\bot!]\bot \leftrightarrow [\bot!]\neg\bot)$. Therefore, we have shown $\mathcal{X} \nvDash \neg[\bot!]\bot \leftrightarrow [\bot!]\neg\bot$. Normality directly follows from the soundness of axiom (PA2) and the rule (PAN), which are both part of X^{PA}. □

Since we do not have totality, the truthful public announcements are called *partial*, see also van Ditmarsch et al. [22], Proposition 4.11.

As a preparatory step for the completeness proof, we are going to state the following lemma.

Lemma 7.1.9. *Let X be one of the deductive systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, or $\mathsf{S5}_n^{PA}$. Then for all $\alpha, \beta \in \mathcal{L}_n^{PA}$ and all $\varphi \in \mathcal{L}_n$ we have*

$$\mathsf{X} \vdash \neg\alpha \rightarrow [\alpha!]\varphi, \qquad \mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\varphi \leftrightarrow [\beta!]\varphi.$$

Proof. Both assertions can be proved by induction on φ, using axiom ($\mathsf{PA1}$), axiom ($\mathsf{PA3}$), Lemma 7.1.5, and axiom ($\mathsf{PA4}$). $\square$

By Lemma 7.1.5, we can already get completeness of $\mathsf{S5}_n^{PA}$ by using the completeness result of van Benthem et al. [12]. In order to prove completeness for all of our six systems, we will now define a translation from $\mathcal{L}_n^{PA}$ to $\mathcal{L}_n$, which will also be useful for additional results. Again, this translation is defined in two steps.

Definition 7.1.10. The two functions $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ and $f\colon \mathcal{L}_n^{PA} \to \mathcal{L}_n^{PA}$ are inductively defined by

$$\begin{aligned} h([\alpha!]p) &:= \alpha \to p, & f(p) &:= p, \\ h([\alpha!]\neg\beta) &:= \alpha \to \neg h([\alpha!]\beta), & f(\neg\alpha) &:= \neg f(\alpha), \\ h([\alpha!](\beta \wedge \gamma) &:= h([\alpha!]\beta) \wedge h([\alpha!]\gamma), & f(\alpha \wedge \beta) &:= f(\alpha) \wedge f(\beta), \\ h([\alpha!]K_i\beta) &:= \alpha \to K_i h([\alpha!]\beta), & f(K_i\alpha) &:= K_i f(\alpha), \\ h([\alpha!][\beta!]\gamma) &:= [\alpha!][\beta!]\gamma, & f([\alpha!]\beta) &:= h([f(\alpha)!]f(\beta)). \end{aligned}$$

It is easy to see that for all $\alpha, \beta \in \mathcal{L}_n$ we have $h([\alpha!]\beta) \in \mathcal{L}_n$ and therefore, for all $\gamma \in \mathcal{L}_n^{PA}$ we have $f(\gamma) \in \mathcal{L}_n$. In addition, we can now show that the translations h and f are strongly equivalence preserving in the following sense.

Lemma 7.1.11. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash h([\alpha!]\beta) \leftrightarrow [\alpha!]\beta, \qquad \mathsf{X} \vdash f(\beta) \leftrightarrow \beta.$$

Proof. Both assertions can be proved by induction on β. The only interesting part of the proof is the last case of the induction step in the second assertion: if $\beta = [\gamma!]\delta$ for some $\gamma, \delta \in \mathcal{L}_n^{PA}$, then we have $f([\gamma!]\delta) = h([f(\gamma)!]f(\delta))$ by definition, which is provably equivalent to $[f(\gamma)!]f(\delta)$ by an application of the first assertion. Since $f(\delta) \in \mathcal{L}_n$, we can apply Lemma 7.1.9 and the induction hypothesis, and we get that X proves the equivalence of $[f(\gamma)!]f(\delta)$ and $[\gamma!]f(\delta)$. But this can be proved equivalent to $[\gamma!]\delta$ by again making use of the induction hypothesis as well as ($\mathsf{PA2}$) and (PAN). $\square$

Due to Lemma 7.1.11, we can now generalise Lemma 7.1.9 to arbitrary $\mathcal{L}_n^{PA}$ formulas.

Lemma 7.1.12. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash \neg\alpha \to [\alpha!]\beta, \qquad \mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. We show how to prove the second assertion. Let $\alpha, \beta \in \mathcal{L}_n^{PA}$ be given and $\mathsf{X} \vdash \alpha \leftrightarrow \beta$. Since $f(\gamma) \in \mathcal{L}_n$, we have $\mathsf{X} \vdash [\alpha!]f(\gamma) \leftrightarrow [\beta!]f(\gamma)$ by Lemma 7.1.9. In addition, we get that X proves both $[\alpha!]\gamma \leftrightarrow [\alpha!]f(\gamma)$ and $[\beta!]\gamma \leftrightarrow [\beta!]f(\gamma)$ by Lemma 7.1.11, axiom ($\mathsf{PA2}$), and the rule (PAN). By propositional reasoning, we get the desired result. □

Due to Lemma 7.1.12, we are now able to proof the so-called *Replacement Theorem*.

Theorem 7.1.13 (Replacement)**.** *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash \gamma \leftrightarrow \gamma[\alpha/\beta].$$

Proof. By induction on γ. For the base case, let $\gamma = p$ for some $p \in \mathcal{P}$. If $\beta \neq p$, then we immediately get $\gamma[\alpha/\beta] = \gamma$ and the claim easily follows. On the other hand, if $\beta = p = \gamma$, then we have $\gamma[\alpha/\beta] = \alpha$ and the claim follows by assumption. In the induction step, the only nontrivial case is if γ is of the form $[\delta!]\varphi$. The claim can then be proved using axiom ($\mathsf{PA2}$), the rule (PAN), and Lemma 7.1.12 as well as the induction hypothesis for both formulas δ and φ. □

Lemma 7.1.12 also allows us to formulate equivalent formulas for formulas of the form $[\alpha!]\beta$ and $\neg[\alpha!]\beta$.

Lemma 7.1.14. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash [\alpha!]\beta \leftrightarrow (\alpha \rightarrow [\alpha!]\beta), \qquad \mathsf{X} \vdash \neg[\alpha!]\beta \leftrightarrow \alpha \wedge [\alpha!]\neg\beta.$$

Proof. The first assertion is tautologically implied by the first assertion of Lemma 7.1.12. For the second assertion, we have that $\neg[\alpha!]\beta$ can be proved equivalent to $\neg[\alpha!]\neg\neg\beta$ by axiom ($\mathsf{PA2}$) and the rule (PAN). By axiom ($\mathsf{PA3}$), we now get that this is provably equivalent to $\alpha \wedge [\alpha!]\neg\beta$. □

Like in Section 6.1 and Section 6.2, we have an elegant completeness proof for our six systems due to Lemma 7.1.12.

Theorem 7.1.15. *For all* $\alpha \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{K}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n \vDash \alpha, \qquad \mathsf{K4}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{t} \vDash \alpha,$$
$$\mathsf{K45}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{tu} \vDash \alpha, \qquad \mathsf{T}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{r} \vDash \alpha,$$
$$\mathsf{S4}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{rt} \vDash \alpha, \qquad \mathsf{S5}_n^{PA} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{rtu} \vDash \alpha.$$

Proof. Soundness has already been proved. We show the direction from right to left. Let X be one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$, and $\mathcal{X}$ be its corresponding class of Kripke structures. For a given formula $\alpha \in \mathcal{L}_n^{PA}$, we assume that $\mathcal{X} \vDash \alpha$. Then we have that $\mathcal{X} \vDash f(\alpha)$ by Lemma 7.1.11 and soundness. By completeness of X, we get that $\mathsf{X} \vdash f(\alpha)$ and, obviously, $\mathsf{X}^{PA} \vdash f(\alpha)$. Again by Lemma 7.1.11, we get $\mathsf{X}^{PA} \vdash \alpha$, and we are done. □

Unlike group announcement logic, it is possible in public announcement logic to encode every sequence of public announcements by one single announcement. For this purpose, the following lemma is very useful.

Lemma 7.1.16. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash ([\alpha!]\beta \to [\alpha!]\gamma) \to [\alpha!](\beta \to \gamma), \qquad \mathsf{X} \vdash [\alpha!](\beta \vee \gamma) \leftrightarrow [\alpha!]\beta \vee [\alpha!]\gamma.$$

Proof. The first property can be proved by making use of axiom (PA3), Lemma 7.1.5 and propositional reasoning, since the formula $\alpha \to \beta$ is defined to be $\neg(\neg\neg\alpha \wedge \neg\beta)$. We show how to prove the second one. First, the formula $[\alpha!](\beta \vee \gamma)$ is provably equivalent to $(\alpha \to [\alpha!]\beta) \vee (\alpha \to [\alpha!]\gamma)$ due to axiom (PA3), Lemma 7.1.5, and propositional reasoning. But now, we can apply the first assertion of Lemma 7.1.14, hence we are done. □

We are now ready to prove that compositional public announcements can be replaced by a single one.

Lemma 7.1.17. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash [\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma.$$

Proof. This result can be established in two steps. First, we can prove a restricted version where γ has to be an element of $\mathcal{L}_n$. This can be done by induction on γ. In the induction step, if γ is of the form $\neg\delta$, we proceed as follows. By (PA3), (PA2), and (PAN), we have that $[\alpha!][\beta!]\neg\delta$ is provably equivalent to $[\alpha!](\beta \to \neg[\beta!]\delta)$, which is provably equivalent to the formula $[\alpha!]\beta \to [\alpha!]\neg[\beta!]\delta$ by (PA2) and Lemma 7.1.16. This formula can now be proved equivalent to $[\alpha!]\beta \to (\alpha \to \neg[\alpha!][\beta!]\delta)$ by (PA3), and this is provably equivalent to $\alpha \wedge [\alpha!]\beta \to \neg[(\alpha \wedge [\alpha!]\beta)!]\delta$ by induction hypothesis and propositional reasoning. Finally, we get that this is provably equivalent to $[(\alpha \wedge [\alpha!]\beta)!]\neg\delta$ by (PA3). The restricted result can then be used to prove the general result for arbitrary formulas $\gamma \in \mathcal{L}_n^{PA}$ using Lemma 7.1.11, axiom (PA2), and the rule (PAN). □

As a preparation for the next step, we will give a reduction axiom for formulas of the form $[\alpha!]E_G\beta$. Observe that mutual knowledge can be defined within the language $\mathcal{L}_n$, cf. Section 5.2.

Lemma 7.1.18. *Let* X *be one of the deductive systems* K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, *or* $\mathsf{S5}_n^{PA}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash [\alpha!]E_G\beta \leftrightarrow (\alpha \rightarrow E_G[\alpha!]\beta).$$

Proof. The claim follows from Lemma 7.1.5, axiom (PA4), and tautological reasoning. □

In a next step, we are going to provide some results concerning announcement resistant formulas. In the context of truthful public announcements, there is also the set of successful formulas, cf. [22, 23], that we will now define.

Definition 7.1.19. Let $\mathcal{X}$ be an arbitrary class of Kripke structures and $\mathcal{L}$ be a language containing public announcement operators. A formula $\alpha \in \mathcal{L}$ is called *successful* in $\mathcal{X}$, if we have

$$\mathcal{X} \vDash [\alpha!]\alpha.$$

First, we want to mention that every announcement resistant formula is also successful.

Lemma 7.1.20. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *or* $\mathcal{K}_n^{rtu}$ *and* $\alpha \in \mathcal{L}_n^{PA}$ *be given. If* α *is announcement resistant in* $\mathcal{X}$, *then* α *is also successful in* $\mathcal{X}$.

Proof. Let X^{PA} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have $\mathsf{X}^{PA} \vdash \alpha \rightarrow [\beta!]\alpha$ for all $\beta \in \mathcal{L}_n^{PA}$. Thus, we obviously have $\mathsf{X}^{PA} \vdash \alpha \rightarrow [\alpha!]\alpha$, and we get $\mathsf{X}^{PA} \vdash [\alpha!]\alpha$ by Lemma 7.1.14. Due to soundness, we get the desired result. □

In general, the converse direction of Lemma 7.1.20 does not hold. The following result has partly been proved by van Ditmarsch and Kooi in [23].

Lemma 7.1.21. *Let* $p \in \mathcal{P}$ *and* $i \in \mathcal{A}$ *be given. Then we have that the formula* $\neg K_i p$ *is successful in* $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *and* $\mathcal{K}_n^{rtu}$, *but not announcement resistant in* $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *and* $\mathcal{K}_n^{rtu}$.

Proof. For the successfulness of $\neg K_i p$, we can get $\mathsf{K45}_n^{PA} \vdash [\neg K_i p!]\neg K_i p$ and $\mathsf{T}_n^{PA} \vdash [\neg K_i p!]\neg K_i p$ by showing that the formula $f([\neg K_i p!]\neg K_i p)$ is provable in both $\mathsf{K45}_n$ and T_n. It is not hard to see that $\mathsf{K45}_n$ and T_n both prove the formula $\neg K_i p \rightarrow \neg(\neg K_i p \rightarrow K_i(\neg K_i p \rightarrow p))$, which is the translation of

$[\neg K_i p!]\neg K_i p$. Observe that axiom (4) is not used in the proof in $\mathsf{K45}_n$. In order to see that the formula $\neg K_i p$ is not announcement resistant in any of the six classes of Kripke structures, it is enough to get a proof of the formula $p \wedge \neg K_i p \rightarrow \neg[p!]\neg K_i p$ in K_n^{PA} by showing that K_n proves its translation $f(p \wedge \neg K_i p \rightarrow \neg[p!]\neg K_i p)$. It is not hard to see that this translation given by $p \wedge K_i p \rightarrow \neg(p \rightarrow \neg(p \rightarrow K_i(p \rightarrow p)))$ is provable in K_n. □

So there are more successful formulas than announcement resistant ones in $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{r}$, $\mathcal{K}_n^{rt}$, and $\mathcal{K}_n^{rtu}$. We do not know whether this is also the case for the other two classes of Kripke structures, because the formulas of the form $\neg K_i p$ are not successful in $\mathcal{K}_n$ and $\mathcal{K}_n^{t}$, as we show in the following example.

Example 7.1.22. Let $p \in \mathcal{P}$ and $i \in \mathcal{A}$ be given and let the Kripke structure $\mathfrak{K} = (\{s,t,u\}, R_1, \ldots, R_n, V)$ be defined by

$$R_1 = \{(s,t),(t,u),(s,u)\}, \qquad\qquad V : q \mapsto \{w\},$$

such that the accessibility relations $R_2, \ldots, R_n$ are transitive. Then we have $\mathfrak{K} \in \mathcal{K}_n^t$ and $\mathfrak{K}, s \vDash \neg[\neg K_i p!]\neg K_i p$, because $\mathfrak{K}, s \vDash \neg K_i p \wedge [\neg K_i p!] K_i p$. Therefore, we get $\mathcal{K}_n^t \nvDash [\neg K_i p!]\neg K_i p$ and, obviously, $\mathcal{K}_n \nvDash [\neg K_i p!]\neg K_i p$.

For every class $\mathcal{X}$, there is a huge set of formulas that are announcement resistant in $\mathcal{X}$. The following lemma illustrates this fact.

Lemma 7.1.23. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have the following sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{PA}$ to be announcement resistant in $\mathcal{X}$,*

1. *$\alpha \in \mathcal{L}_0$,*
2. *$\mathcal{X} \vDash \alpha$ or $\mathcal{X} \vDash \neg\alpha$,*
3. *$\alpha = \beta \wedge \gamma$ or $\alpha = \beta \vee \gamma$ for some β, γ announcement resistant in $\mathcal{X}$,*
4. *$\alpha = K_i\beta$ for some $i \in \mathcal{A}$ and some β announcement resistant in $\mathcal{X}$.*

Proof. The first assertion directly follows from fact preservation, which we have proved in Lemma 7.1.8. We show how to prove the fourth assertion. Let X^{PA} be the deductive system that corresponds to $\mathcal{X}$. Further, let β be announcement resistant in $\mathcal{X}$ and $\gamma \in \mathcal{L}_n^{PA}$ be arbitrarily given. Then we have $\mathsf{X}^{PA} \vdash \beta \rightarrow [\gamma!]\beta$ by completeness, and we get $\mathsf{X}^{PA} \vdash K_i\beta \rightarrow K_i[\gamma!]\beta$ by normal modal logic reasoning. Therefore, since we have that X^{PA} proves $K_i[\gamma!]\beta \rightarrow [\gamma!]K_i\beta$ as a consequence of (PA4), we get $\mathsf{X}^{PA} \vdash K_i\beta \rightarrow [\gamma!]K_i\beta$ by tautological reasoning. Due to soundness, we get the desired result. □

We want to mention here that van Ditmarsch and Kooi have proved in [23] that if α and β are both preserved under submodels, then so also is the formula $[\neg\alpha!]\beta$. We believe that this closure condition also holds for announcement resistance, but we have not yet found a syntactical proof.

The successful formulas do not in general satisfy the closure conditions from Lemma 7.1.23. The following result has partly been proved in [23].

Lemma 7.1.24. *Let $p \in \mathcal{P}$, $i \in \mathcal{A}$, and $\mathcal{X}$ be one of the classes $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{r}$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have that the formulas p and $\neg K_i p$ are both successful in $\mathcal{X}$, but the formula $p \wedge \neg K_i p$ is not successful in $\mathcal{K}_n$, $\mathcal{K}_n^{t}$, and $\mathcal{X}$.*

Proof. The formula $\neg K_i p$ is successful in $\mathcal{X}$ by Lemma 7.1.21 and so also is p by the first assertion of Lemma 7.1.23. In order to see that $p \wedge \neg K_i p$ is not successful in any of the six classes of Kripke structures, we can show that K_n^{PA} proves $p \wedge \neg K_i p \to \neg[(p \wedge \neg K_i p)!](p \wedge \neg K_i p)$. This can be done by showing that K_n proves the translation $f(p \wedge \neg K_i p \to \neg[(p \wedge \neg K_i p)!](p \wedge \neg K_i p))$, which is given by the formula

$$\begin{aligned} & p \wedge \neg K_i p \to \\ & \quad \neg((p \wedge \neg K_i p \to p) \wedge (p \wedge \neg K_i p \to \neg(p \wedge K_i p \to K_i(p \wedge \neg K_i p \to p)))). \end{aligned}$$

By soundness, we get $\mathcal{K}_n \vDash p \wedge \neg K_i p \to \neg[(p \wedge \neg K_i p)!](p \wedge \neg K_i p)$. Since we have that the formula $p \wedge \neg K_i p$ is satisfiable in every of our six classes of Kripke structures, we get that the formula $[(p \wedge \neg K_i p)!](p \wedge \neg K_i p)$ is not valid in any of the six classes of Kripke structures. ☐

Due to Lemma 7.1.24, we can now easily show that the logic of truthful public announcements does not have the *substitution property*.

Corollary 7.1.25. *Let X be one of the deductive systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, or $\mathsf{S5}_n^{PA}$. Then for all $p \in \mathcal{P}$ we have*

$$\mathsf{X} \vdash [p!]p, \qquad \mathsf{X} \nvdash [(p \wedge \neg K_i p)!](p \wedge \neg K_i p).$$

Similar to group announcement logics with trustful agents, we have that the announcement resistant formulas get common knowledge after one single public announcement. We can even state this result for the set of successful formulas, which contains the announcement resistant formulas.

Theorem 7.1.26. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^{t}$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^{r}$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{PA}$ be given. If α is successful in $\mathcal{X}$, then for all $l \geq 1$ and all $i_1, \ldots, i_l \in \mathcal{A}$ we have*

$$\mathcal{X} \vDash [\alpha!]K_{i_l} \ldots K_{i_1}\alpha.$$

Proof. Let X^{PA} be the deductive system that corresponds to $\mathcal{X}$. Due to soundness, it is enough to show $\mathsf{X}^{PA} \vdash [\alpha!]K_{i_l} \dots K_{i_1}\alpha$. The proof is by induction on l. In the base case, we have a proof of $[\alpha!]\alpha$ by assumption and completeness, and we get a proof of $K_{i_1}[\alpha!]\alpha$ by (NEC). Applying axiom (PA4) now results in a proof of $[\alpha!]K_{i_1}\alpha$. In the induction step, we start with a proof of $[\alpha!]K_{i_l} \dots K_{i_1}\alpha$ by induction hypothesis. By (NEC) we get a proof of $K_{i_{l+1}}[\alpha!]K_{i_l} \dots K_{i_1}\alpha$, which implies $[\alpha!]K_{i_{l+1}}K_{i_l} \dots K_{i_1}\alpha$ by (PA4). □

It has been mentioned in [22] that the converse direction of Theorem 7.1.26 also holds. That is, the successful formulas are exactly those that get common knowledge after being announced once. We cannot prove this because we define common knowledge via the transitive closure of the accessibility relations, not via the reflexive transitive closure. However, we have the same result for the three classes of Kripke structures where the accessibility relations are all reflexive.

Lemma 7.1.27. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{PA}$ be given. If for some $i \in \mathcal{A}$ we have $\mathcal{X} \vDash [\alpha!]K_i\alpha$, then α is successful in $\mathcal{X}$.*

Proof. Let X^{PA} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have $\mathsf{X}^{PA} \vdash [\alpha!]K_i\alpha$. We can now show that X^{PA} proves $[\alpha!]\alpha$ by an easy application of the axioms (T) and (PA2) as well as the rule (PAN). By soundness, we are done. □

We have not yet found formulas that are not successful in $\mathcal{K}_n$, $\mathcal{K}_n^t$, and $\mathcal{K}_n^{tu}$, but get common knowledge after being announced once.

We are now going to show that truthful public announcements and group announcements for trustful agents are closely related, if the group announcements are told to all agents. We can prove that a public announcement with a true announcement free formula has the same impact on announcement free formulas in both approaches.

Theorem 7.1.28. *Let $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ be the translation from Definition 7.1.10, and $h'\colon \{[\alpha!_G]\beta : \alpha, \beta \in \mathcal{L}_n^{GA}, \emptyset \neq G \subseteq \mathcal{A}\} \to \mathcal{L}_n^{GA}$ denote the corresponding function defined in Definition 6.1.9. Further, let X be one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$. Then for all $\alpha, \beta \in \mathcal{L}_n$ we have*

$$\mathsf{X} \vdash \alpha \to (h([\alpha!]\beta) \leftrightarrow h'([\alpha!_{\mathcal{A}}]\beta)).$$

Proof. By induction on β. We show how to prove the last case of the induction step, where β is of the form $K_i\gamma$. By induction hypothesis, we have that

X proves $\alpha \to (h([\alpha!]\gamma) \leftrightarrow h'([\alpha!_{\mathcal{A}}]\gamma))$. By normal modal logic reasoning, we immediately get that X proves

$$\alpha \to ((\alpha \to K_i(\alpha \to h([\alpha!]\gamma))) \leftrightarrow K_i(\alpha \to h'([\alpha!_{\mathcal{A}}]\gamma))).$$

Now, we only have to show that the formula $\alpha \to K_i(\alpha \to h([\alpha!]\gamma))$ is provably equivalent to $\alpha \to K_i(h([\alpha!]\gamma))$. For this purpose, one can easily prove that for all $\varphi, \psi \in \mathcal{L}_n$ we have $\mathsf{X} \vdash h([\varphi!]\psi) \leftrightarrow (\varphi \to h([\varphi!]\psi))$ by induction on ψ. □

Observe that Theorem 7.1.28 only holds if the announced formula is true, we have no similar relationship between the two approaches otherwise. If the group announcements are not told to every agent, there is no such result either.

We will now state some results about truthful public announcement logic augmented with common knowledge operators. First, we are going to extend the language $\mathcal{L}_n^{PA}$ with operators for relativised common knowledge. We recall the grammar of the language $\mathcal{L}_n^{RCPA}$ from Definition 5.4.1, which is given as follows ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ RC_G(\alpha, \alpha) \ | \ [\alpha!]\alpha.$$

For the semantics of relativised common knowledge formulas we extend Definition 7.1.1 with the following clause,

$$\mathfrak{K}, s \vDash RC_G(\alpha, \beta) \ :\Leftrightarrow \ \text{for all } t \in (R_G \cap (|\mathfrak{K}| \times \|\alpha\|_{\mathfrak{K}}))^+(s),\ \mathfrak{K}, t \vDash \beta,$$

which we have already given in Definition 5.2.6. We get the Hilbert systems by combining the systems for truthful public announcements with the systems for relativised common knowledge and adding one axiom concerning relativised common knowledge after a public announcement. The following version of the *reduction axiom* (PA5) has been presented in [22].

Definition 7.1.29. The systems K_n^{RCPA}, $\mathsf{K4}_n^{RCPA}$, $\mathsf{K45}_n^{RCPA}$, T_n^{RCPA}, $\mathsf{S4}_n^{RCPA}$, and $\mathsf{S5}_n^{RCPA}$ are the Hilbert systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, and $\mathsf{S5}_n^{PA}$ respectively augmented with the *co-closure axiom* and the *public announcement axiom* for relativised common knowledge,

(RC) $RC_G(\alpha, \beta) \to E_G(\alpha \to \beta \wedge RC_G(\alpha, \beta))$,
(PA5) $[\alpha!]RC_G(\beta, \gamma) \leftrightarrow (\alpha \to RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\gamma))$,

as well as the *induction rule* for relativised common knowledge,

$$(\mathsf{RCI})\ \frac{\alpha \to E_G(\beta \to \alpha \wedge \gamma)}{\alpha \to RC_G(\beta, \gamma)}\ .$$

Observe that Lemma 7.1.3 and Lemma 7.1.5 obviously hold in the extended setting. The soundness proof of the extended Hilbert systems is a straightforward extension of the proof of Lemma 7.1.6, cf. [22] for a detailed proof of the validity of (PA5). Of course, the proofs of Lemma 7.1.7, Lemma 7.1.8, and Lemma 7.1.18 are still proofs in the extended systems. Moreover, it is not hard to extend Lemma 7.1.9 to relativised common knowledge operators.

In order to prove completeness, we can extend the functions h and f from Definition 7.1.10 to formulas of the form $RC_G(\beta, \gamma)$ as follows,

$$h([\alpha!]RC_G(\beta,\gamma)) := \alpha \to RC_G(\alpha \wedge h([\alpha!]\beta), h([\alpha!]\gamma)),$$
$$f(RC_G(\beta,\gamma)) := RC_G(f(\beta), f(\gamma)).$$

This is because axiom (PA5) is a reduction axiom and shows that adding public announcements to the logic of relativised common knowledge does not increase its expressive strength. This fact has already been mentioned by van Benthem, van Eijck, and Kooi [11, 12]. Clearly, the extended functions h and f are still equivalence preserving in the sense of Lemma 7.1.11. Therefore, the results from Lemma 7.1.12, Theorem 7.1.13, Lemma 7.1.14, and Lemma 7.1.16 also hold in the extended framework. So we have an elegant completeness proof almost identical to the proof of Theorem 7.1.15.

Theorem 7.1.30. *For all $\alpha \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathsf{K}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n \vDash \alpha, \qquad \mathsf{K4}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n^t \vDash \alpha,$$
$$\mathsf{K45}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n^{tu} \vDash \alpha, \qquad \mathsf{T}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n^r \vDash \alpha,$$
$$\mathsf{S4}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n^{rt} \vDash \alpha, \qquad \mathsf{S5}_n^{RCPA} \vdash \alpha \Leftrightarrow \mathcal{K}_n^{rtu} \vDash \alpha.$$

Due to the new translation f from $\mathcal{L}_n^{RCPA}$ to $\mathcal{L}_n^{RC}$, we have that Lemma 7.1.17 still holds. Furthermore, it is still true that every announcement resistant formula is successful, as we have shown in Lemma 7.1.20. For announcement resistant formulas, we have an additional closure condition in addition to the ones in Lemma 7.1.23.

Lemma 7.1.31. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have the same sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{RCPA}$ to be announcement resistant in $\mathcal{X}$ as in Lemma 7.1.23 plus the following one,*

5. *$\alpha = RC_G(\neg\beta, \gamma)$ for some non empty $G \subseteq \mathcal{A}$ and some β, γ announcement resistant in $\mathcal{X}$.*

Proof. It is easy to see that the conditions from Lemma 7.1.23 also hold for $\mathcal{L}_n^{RCPA}$ formulas. For the fifth assertion, let X^{RCPA} be the deductive system that corresponds to $\mathcal{X}$, β, γ be announcement resistant in $\mathcal{X}$, and $\delta \in \mathcal{L}_n^{RCPA}$ be arbitrarily given. By completeness, we get that the formulas $\beta \to [\delta!]\beta$ and $\gamma \to [\delta!]\gamma$ are both provable in X^{RCPA}. Therefore, we get that X^{RCPA} proves

$$(\neg\beta \to \gamma \wedge RC_G(\neg\beta, \gamma)) \to (\delta \wedge [\delta!]\neg\beta \to [\delta!]\gamma \wedge RC_G(\neg\beta, \gamma))$$

by Lemma 7.1.14 and propositional reasoning. By normal modal reasoning and the following instance of (RC),

$$RC_G(\neg\beta, \gamma) \to E_G(\neg\beta \to \gamma \wedge RC_G(\neg\beta, \gamma)),$$

we immediately get that the formula

$$(\neg\beta \to \gamma \wedge RC_G(\neg\beta, \gamma)) \to (\delta \wedge [\delta!]\neg\beta \to [\delta!]\gamma \wedge RC_G(\neg\beta, \gamma))$$

is provable in X^{RCPA}. Therefore, again by normal modal logic reasoning, we get that X^{RCPA} proves

$$RC_G(\neg\beta, \gamma) \to E_G(\delta \wedge [\delta!]\neg\beta \to [\delta!]\gamma \wedge RC_G(\neg\beta, \gamma)).$$

Applying the rule (RCI) now results in a proof of the formula

$$RC_G(\neg\beta, \gamma) \to RC_G(\delta \wedge [\delta!]\neg\beta, [\delta!]\gamma),$$

and we get $\mathsf{X}^{RCPA} \vdash RC_G(\neg\beta, \gamma) \to [\delta!]RC_G(\neg\beta, \gamma)$ by an application of the axiom (PA5). By soundness, we get the desired result. □

As we have seen in Theorem 7.1.26, an announced successful formula gets common knowledge among all agents after one public announcement. With relativised common knowledge, we are able to express this fact within the logical language. Remember that the formula $RC_G(\top, \alpha)$ means "α is common knowledge among G", as we have already mentioned in Section 5.2. The following theorem even states a stronger result.

Theorem 7.1.32. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{RCPA}$ be given. If α is successful in $\mathcal{X}$, then for all non empty $G \subseteq \mathcal{A}$ and all $\beta \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathcal{X} \vDash [\alpha!]RC_G(\beta, \alpha).$$

Proof. Let X^{RCPA} be the Hilbert system that corresponds to $\mathcal{X}$ and $\beta \in \mathcal{L}_n^{RCPA}$ be arbitrarily given. We will show that $\mathsf{X}^{RCPA} \vdash [\alpha!]RC_G(\beta, \alpha)$. By assumption and completeness, we have $\mathsf{X}^{RCPA} \vdash [\alpha!]\alpha$. Since we know from the proof of Theorem 5.2.9 that the inference rule (RC-Nec) is admissible in X^{RCPA}, we easily get a proof of $RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\alpha)$. But this formula now implies $[\alpha!]RC_G(\beta, \alpha)$ by (PA5). Due to soundness, we are done. □

Similar to Lemma 7.1.27, if the accessibility relations are all reflexive, then we also have the converse direction of Theorem 7.1.32.

Lemma 7.1.33. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{RCPA}$ be given. If for some non empty $G \subseteq \mathcal{A}$ we have $\mathcal{X} \vDash [\alpha!]RC_G(\top, \alpha)$, then α is successful in $\mathcal{X}$.*

Proof. Let X^{RCPA} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have $\mathsf{X}^{RCPA} \vdash [\alpha!]RC_G(\top, \alpha)$. We can now show that X^{RCPA} proves $[\alpha!]\alpha$ by an easy application of the axioms (T), (RC), and (PA2) as well as the rule (PAN). By soundness, we get the desired result. □

Now, we will state similar results with common knowledge instead of relativised common knowledge. For this purpose, we repeat the grammar of the language $\mathcal{L}_n^{CPA}$ from Definition 5.4.1 ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ C_G\alpha \ | \ [\alpha!]\alpha.$$

Moreover, we recall the semantics for common knowledge formulas from Definition 5.2.2. From now on, the following clause extends Definition 7.1.1,

$$\mathfrak{K}, s \vDash C_G\alpha \ :\Leftrightarrow \ \text{for all } t \in R_G^+(s),\ \mathfrak{K}, t \vDash \alpha.$$

Baltag, Moss, and Solecki have shown in [8] that the logic of common knowledge and truthful public announcements has more expressive strength than the logic of common knowledge. Therefore, the Hilbert systems cannot only have reduction axioms. Baltag and Moss have introduced a deductive system in [6], which contains the *public announcement composition axiom* from Lemma 7.1.17,

$$[\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma,$$

as well as the following *announcement rule*,

$$\frac{\alpha \to [\beta!]\gamma \qquad \alpha \wedge \beta \to E_G\alpha}{\alpha \to [\beta!]C_G\gamma}.$$

A detailed completeness proof can be found in the book of van Ditmarsch, van der Hoek, and Kooi [22]. In our setting, we have to slightly change the announcement rule, because we have defined common knowledge via the transitive closure of the accessibility relations. Due to the announcement composition axiom and the new inference rule, we are able to prove all of the other results about truthful public announcements where we have used the translation f from $\mathcal{L}_n^{PA}$ to $\mathcal{L}_n$, as we will show later. We will now define the new Hilbert systems.

Definition 7.1.34. The deductive systems K_n^{CPA}, $\mathsf{K4}_n^{CPA}$, $\mathsf{K45}_n^{CPA}$, T_n^{CPA}, $\mathsf{S4}_n^{CPA}$, and $\mathsf{S5}_n^{CPA}$ are the systems K_n^{PA}, $\mathsf{K4}_n^{PA}$, $\mathsf{K45}_n^{PA}$, T_n^{PA}, $\mathsf{S4}_n^{PA}$, and $\mathsf{S5}_n^{PA}$ respectively augmented with the *co-closure axiom* for common knowledge and the *public announcement composition axiom*,

(C) $C_G\alpha \to E_G(\alpha \wedge C_G\alpha)$,

(PA6) $[\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma$,

as well as the *induction rules* for common knowledge and public announcements,

$$(\mathsf{CI})\ \frac{\alpha \to E_G(\alpha \wedge \beta)}{\alpha \to C_G\beta}, \qquad (\mathsf{PAI})\ \frac{\alpha \to [\beta!]E_G\gamma \qquad \alpha \wedge \beta \to E_G\alpha}{\alpha \to [\beta!]C_G\gamma}.$$

Clearly, Lemma 7.1.3 and Lemma 7.1.5 still hold in the extended framework and soundness can be proved as usual. For the correctness of the rule (PAI), one can make slight modifications in the proof from [22]. Again, we can prove Lemma 7.1.7, Lemma 7.1.8, and Lemma 7.1.18 in the extended systems.

Completeness of the above defined systems follows from the results in [6, 22]. We do not give the completeness proof here, because in Section 7.2 we will give a detailed proof for *total public announcements* and common knowledge, which is similar.

Theorem 7.1.35. *For all $\alpha \in \mathcal{L}_n^{CPA}$ we have*

$$\begin{array}{rclrcl} \mathsf{K}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n \vDash \alpha, & \mathsf{K4}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^t \vDash \alpha, \\ \mathsf{K45}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{tu} \vDash \alpha, & \mathsf{T}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^r \vDash \alpha, \\ \mathsf{S4}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rt} \vDash \alpha, & \mathsf{S5}_n^{CPA} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rtu} \vDash \alpha. \end{array}$$

We are now going to state a few properties of the logic of truthful public announcements and common knowledge. The following results concerning common knowledge after a public announcement are useful for our syntactical proofs.

Lemma 7.1.36. *Let* X *be one of the Hilbert systems K_n^{CPA}, $\mathsf{K4}_n^{CPA}$, $\mathsf{K45}_n^{CPA}$, T_n^{CPA}, $\mathsf{S4}_n^{CPA}$, or $\mathsf{S5}_n^{CPA}$. Then for all non empty $G \subseteq \mathcal{A}$ and all $\alpha, \beta \in \mathcal{L}_n^{CPA}$ we have that the following formulas are provable in* X*,*

$$[\alpha!]C_G\beta \leftrightarrow [\alpha!]E_G(\beta \wedge C_G\beta), \qquad C_G[\alpha!]\beta \to [\alpha!]C_G\beta.$$

Proof. The first assertion is a consequence of the axioms (C) and (PA2), as well as the rules (PAN) and (CI). For the second assertion, we have that X proves $C_G[\alpha!]\beta \to [\alpha!]E_G\beta$ by axiom (C) and Lemma 7.1.18, and X proves $C_G[\alpha!]\beta \wedge \alpha \to E_GC_G[\alpha!]\beta$ by axiom (C) and tautological reasoning. An application of the rule (PAI) now finishes the proof. □

Although we do not have a translation from $\mathcal{L}_n^{CPA}$ to $\mathcal{L}_n$, we can still prove Lemma 7.1.12 in the extended systems.

Lemma 7.1.37. *Let* X *be one of the Hilbert systems* K_n^{CPA}, $\mathsf{K4}_n^{CPA}$, $\mathsf{K45}_n^{CPA}$, T_n^{CPA}, $\mathsf{S4}_n^{CPA}$, *or* $\mathsf{S5}_n^{CPA}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ *we have*

$$\mathsf{X} \vdash \neg\alpha \to [\alpha!]\beta, \qquad \mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. We show how to prove the second assertion by induction on γ, so assume $\mathsf{X} \vdash \alpha \leftrightarrow \beta$. The base case as well as the cases $\gamma = \neg\delta$, $\gamma = \delta \wedge \varphi$, and $\gamma = K_i\delta$ in the induction step are identical to the proof of Lemma 7.1.9. First, let γ be of the form $C_G\delta$. We will only show $\mathsf{X} \vdash [\alpha!]C_G\delta \to [\beta!]C_G\delta$, the other direction is analogous. We have the following chain of provable implications by the first assertion of Lemma 7.1.36, Lemma 7.1.18, the induction hypothesis, and the assumption,

$$[\alpha!]C_G\delta \to [\alpha!]E_G\delta \to (\alpha \to E_G[\alpha!]\delta) \to (\beta \to E_G[\beta!]\delta) \to [\beta!]E_G\delta.$$

Therefore, we have that X proves $[\alpha!]C_G\delta \to [\beta!]E_G\delta$ by tautological reasoning. On the other hand, we have the following chain of provable implications by the first assertion of Lemma 7.1.36, Lemma 7.1.18, and the assumption,

$$[\alpha!]C_G\delta \wedge \beta \to [\alpha!]E_GC_G\delta \wedge \beta \to (\alpha \to E_G[\alpha!]C_G\delta) \wedge \alpha.$$

Hence, we have $\mathsf{X} \vdash [\alpha!]C_G\delta \wedge \beta \to E_G[\alpha!]C_G\delta$ by tautological reasoning. An application of the rule (PAI) now finishes this part of the proof. In the last case of the induction step, if $\gamma = [\delta!]\varphi$, we proceed as follows. We have that X proves $\alpha \wedge [\alpha!]\delta \leftrightarrow \beta \wedge [\beta!]\delta$ by induction hypothesis and assumption. Hence, we get $\mathsf{X} \vdash [(\alpha \wedge [\alpha!]\delta)!]\varphi \leftrightarrow [(\beta \wedge [\beta!]\delta)!]\varphi$ by again applying the induction hypothesis. By (PA6), we get the desired equivalence. □

The proof of Lemma 7.1.37 illustrates how axiom (PA6) and the rule (PAI) can deal with the non existence of a translation from $\mathcal{L}_n^{CPA}$ to $\mathcal{L}_n^{C}$. As an immediate consequence of Lemma 7.1.37, we also get that Theorem 7.1.13, Lemma 7.1.14, and Lemma 7.1.16 still hold in the extended setting. Thus, we again get that every announcement resistant formula is successful, which can be proved like Lemma 7.1.20. Similar to Lemma 7.1.31, there is an additional condition for an $\mathcal{L}_n^{CPA}$ formula to be announcement resistant.

Lemma 7.1.38. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *or* $\mathcal{K}_n^{rtu}$. *Then we have the same sufficient conditions for a formula* $\alpha \in \mathcal{L}_n^{CPA}$ *to be announcement resistant in* $\mathcal{X}$ *as in Lemma 7.1.23 plus the following one,*

5. $\alpha = C_G\beta$ *for some non empty* $G \subseteq \mathcal{A}$ *and some* β *announcement resistant in* $\mathcal{X}$.

Proof. It is not hard to show that the conditions from Lemma 7.1.23 also hold for $\mathcal{L}_n^{CPA}$ formulas. For the fifth assertion, let X^{CPA} be the deductive system that corresponds to $\mathcal{X}$, $\beta \in \mathcal{L}_n^{CPA}$ be announcement resistant in $\mathcal{X}$, and γ be arbitrarily given. By assumption and completeness, we have that X^{CPA} proves $\beta \to [\gamma!]\beta$, and we get $\mathsf{X}^{CPA} \vdash E_G\beta \to E_G[\gamma!]\beta$ by normal modal logic reasoning. Together with $C_G\beta \to E_G\beta$, which is easily derivable from (C), we get that X^{CPA} proves $C_G\beta \to E_G[\gamma!]\beta$ by tautological reasoning. Since we have that the formula $E_G[\gamma!]\beta \to [\gamma!]E_G\beta$ is provable as an immediate consequence of Lemma 7.1.18, we now get a proof of $C_G\beta \to [\gamma!]E_G\beta$, again by tautological reasoning. On the other hand, it is easy to get a proof of $C_G\beta \land \gamma \to E_G C_G\beta$ from (C). Now, we can apply the rule (PAI) in order to get a proof of $C_G\beta \to [\gamma!]C_G\beta$. By soundness, we get the desired result. □

In the logic of common knowledge and truthful public announcements it is also the case that every successful formula is commonly known by the agents after being announced once. The following theorem is the natural extension of Theorem 7.1.26 to common knowledge operators.

Theorem 7.1.39. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{CPA}$ be given. If α is successful in $\mathcal{X}$, then for all non empty $G \subseteq \mathcal{A}$ we have*

$$\mathcal{X} \vDash [\alpha!]C_G\alpha.$$

Proof. Let X^{CPA} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have that the formula $[\alpha!]\alpha$ is provable in X^{CPA}, and we can easily derive the formula $E_G[\alpha!]\alpha$ by normal modal logic reasoning. Since we have that the formula $E_G[\alpha!]\alpha \to [\alpha!]E_G\alpha$ is provable by Lemma 7.1.18, we get a proof of $[\alpha!]E_G\alpha$, hence we have that X^{CPA} proves $\top \to [\alpha!]E_G\alpha$. Together with the formula $\top \land \alpha \to E_G\top$, which is obviously provable in X^{CPA}, we can apply the rule (PAI) and get a proof of $\top \to [\alpha!]C_G\alpha$ in X^{CPA}. But this formula is provably equivalent to $[\alpha!]C_G\alpha$, and due to soundness, we are done. □

Similar to Lemma 7.1.27, if the accessibility relations are all reflexive, then we also have the converse direction of Theorem 7.1.39.

Lemma 7.1.40. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{CPA}$ be given. If for some non empty $G \subseteq \mathcal{A}$ we have $\mathcal{X} \vDash [\alpha!]C_G\alpha$, then α is successful in $\mathcal{X}$.*

Proof. Let X^{CPA} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have that $\mathsf{X}^{CPA} \vdash [\alpha!]C_G\alpha$. We can now show that X^{CPA} proves $[\alpha!]\alpha$ by an easy application of the axioms (T), (C), and ($\mathsf{PA2}$) as well as the rule (PAN). By soundness, we get the desired result. □

It has been proved in [22] that all $\mathcal{L}_n^{CPA}$ formulas of the form $C_{\mathcal{A}}\alpha$ are successful formulas. But for some non empty $G \subset \mathcal{A}$ and some formulas $\alpha \in \mathcal{L}_n^{CPA}$ we have that the formula $C_G\alpha$ is not successful in $\mathcal{K}_n$. The following example illustrates this fact.

Example 7.1.41. Let $n \geq 2$ and $p \in \mathcal{P}$ be given. Furthermore, let the Kripke structure $\mathfrak{K} = (\{s,t\}, R_1, \ldots, R_n, V)$ be defined by

$$R_1 = \{(s,s),(t,t)\}, \qquad R_2 = \{(s,t),(t,s)\}, \qquad V\colon q \mapsto \{s\},$$

the accessibility relations $R_3, \ldots, R_n$ can be arbitrarily defined. Then we have $\mathfrak{K}, s \vDash C_{\{1\}}\neg K_2 p$ and $\mathfrak{K}, t \nvDash C_{\{1\}}\neg K_2 p$, hence we get $|\mathfrak{K}^{C_{\{1\}}\neg K_2 p}| = \{s\}$. Now, it is not hard to see that $\mathfrak{K}, s \vDash [C_{\{1\}}\neg K_2 p!]\neg C_{\{1\}}\neg K_2 p$, which implies $\mathfrak{K}, s \vDash \neg[C_{\{1\}}\neg K_2 p!]C_{\{1\}}\neg K_2 p$ by the second assertion of Lemma 7.1.14 and soundness. Therefore, we have $\mathcal{K}_n \nvDash [C_{\{1\}}\neg K_2 p!]C_{\{1\}}\neg K_2 p$.

We do not know the conditions for a formula of the form $C_G\alpha$ to be successful in a class $\mathcal{X}$ of Kripke structures. We believe that there are two possible solutions, either $\mathcal{X} \subseteq \mathcal{K}_n^r$ or $G = \mathcal{A}$.

7.2 Total public announcements

In this section, we present a system in which public announcements are *total*, that is new information can always be announced. Therefore, public announcements need not be truthful—they can be true or false. As usual, a true announcement will lead to the update of an agent's epistemic state. However, a false announcement will not lead to an inconsistent epistemic state like in Section 7.1, it will automatically be ignored by the agents. That is, after a false announcement, every agent will have the same epistemic state as before the announcement.

Throughout this section, we are working within the language $\mathcal{L}_n^{PA}$ from Definition 5.4.1. First, we introduce the semantics of the logic of *total public announcements* by adding the defining clause for the formulas of the form $[\alpha!]\beta$ to Definition 5.1.5. As usual, the *model transformation* is simultaneously defined.

Definition 7.2.1. Let $\mathfrak{K} = (S, R_1, \ldots, R_n, V)$ be an arbitrary Kripke structure and $s \in S$ be given. The notion of a public announcement formula $[\alpha!]\beta$ being *satisfied* in the pointed structure $\mathfrak{K}, s$ is defined as follows,

$$\mathfrak{K}, s \vDash [\alpha!]\beta \;:\Leftrightarrow\; \mathfrak{K}^{\alpha,s}, s \vDash \beta,$$

where the Kripke structure $\mathfrak{K}^{\alpha,s} := (S^{\alpha,s}, R_1^{\alpha,s}, \ldots, R_n^{\alpha,s}, V^{\alpha,s})$ is simultaneously defined by

$$\mathfrak{K}^{\alpha,s} := \begin{cases} \mathfrak{K}^{\alpha} & \text{if } \mathfrak{K}, s \vDash \alpha, \\ \mathfrak{K} & \text{otherwise.} \end{cases}$$

The Kripke structure $\mathfrak{K}^{\alpha} = (S^{\alpha}, R_1^{\alpha}, \ldots, R_n^{\alpha}, V^{\alpha})$ has already been defined in Definition 5.4.2.

We want to mention that the semantics from Definition 7.2.1 is slightly different from the one we have given in [62]. However, as we will see, the two notions give rise to the same axiomatisations. The following example illustrates how the new semantics of total public announcements works. It is related to the *muddy children puzzle*, cf. [25].

Example 7.2.2. Alice, Bob, and Charlie (agents 1–3) each wear a hat and cannot see its colour. But they can see, of course, the colour of the others' hats. There are three blue hats but only two red hats, and it is common knowledge that this is the case. We have $\mathcal{A} = \{1, 2, 3\}$ and we take some propositions $b_i \in \mathcal{P}$ with the meaning "agent i wears a blue hat". If agent i wears a red hat, proposition b_i is false. A state is named $c_1c_2c_3$ where c_i is the colour (b or r) of agent i's hat. Suppose Alice wears a red hat, while Bob and Charlie both wear blue hats. Figure 7.1 shows the Kripke structure $\mathfrak{K}$ that represents this situation, where the actual world is underlined (all of the accessibility relations are reflexive, but the loops are omitted in the figure). Observe that we have $rbb \notin V(b_1)$, whereas $rbb \in V(b_2) \cap V(b_3)$. Now, Alice

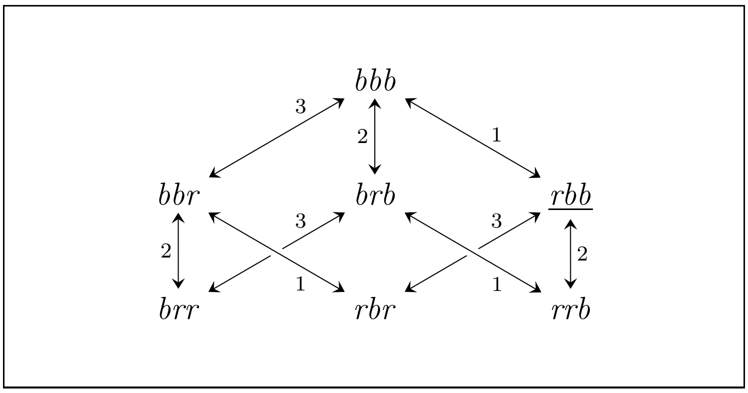

Figure 7.1: The initial Kripke structure

publicly announces that she does not know the colour of her hat, which is true. This information can be expressed by the formula $\neg K_1 b_1 \wedge \neg K_1 \neg b_1$. After that, Bob announces that he still does not know the colour of his hat, again a true announcement. We use the formula $\neg K_2 b_2 \wedge \neg K_2 \neg b_2$ to encode this fact. After these two announcements, we get the Kripke structure

$$(\mathfrak{K}^{\neg K_1 b_1 \wedge \neg K_1 \neg b_1, rbb})^{\neg K_2 b_2 \wedge \neg K_2 \neg b_2, rbb},$$

which is illustrated in Figure 7.2 (all of the accessibility relations are still reflexive). Now, Charlie knows that he is wearing a blue hat. Observe that a

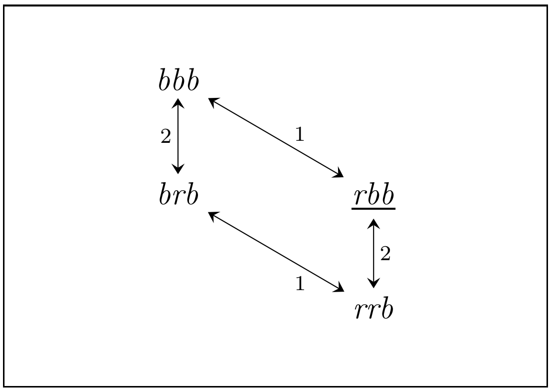

Figure 7.2: The situation after two announcements

false announcement at any time would have no effect. For instance, imagine that after the announcement of Alice, Charlie announced that he knows that he wears a red hat. This is represented by the formula $K_3 \neg b_3$, which is false at state rbb, and the situation would be the same as before,

$$((\mathfrak{K}^{\neg K_1 b_1 \wedge \neg K_1 \neg b_1, rbb})^{K_3 \neg b_3, rbb})^{\neg K_2 b_2 \wedge \neg K_2 \neg b_2, rbb} = (\mathfrak{K}^{\neg K_1 b_1 \wedge \neg K_1 \neg b_1, rbb})^{\neg K_2 b_2 \wedge \neg K_2 \neg b_2, rbb}.$$

We have proved in Lemma 7.1.3 that the model transformation from Definition 5.4.2 preserves reflexivity, transitivity, and Euclideanity of the accessibility relations. Therefore, we have the same result also for the new model transformation.

Lemma 7.2.3. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then for all Kripke structures $\mathfrak{K}$, all $s \in |\mathfrak{K}|$, and all $\alpha \in \mathcal{L}_n^{PA}$ we have*

$$\mathfrak{K} \in \mathcal{X} \;\Rightarrow\; \mathfrak{K}^{\alpha,s} \in \mathcal{X}.$$

Proof. If $\mathfrak{K}, s \vDash \neg\alpha$, then we have $\mathfrak{K}^{\alpha,s} = \mathfrak{K}$ and the lemma trivially holds. On the other hand, if $\mathfrak{K}, s \vDash \alpha$, then we have $\mathfrak{K}^{\alpha,s} = \mathfrak{K}^{\alpha}$ and the claim directly follows from Lemma 7.1.3. □

We want to mention that seriality of the accessibility relations is in general not preserved by the new model transformation. In order to see this, one can take the same Kripke structure and the same formula as in Example 7.1.2. Therefore, we cannot get axiomatisations for the classes $\mathcal{K}_n^s$, $\mathcal{K}_n^{st}$, and $\mathcal{K}_n^{stu}$.

The following deductive systems for total public announcements are closer related to the systems for group announcements than to the systems for truthful public announcements, which was one of the original motivations to develop the new semantics.

Definition 7.2.4. The Hilbert systems $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, and $\mathsf{S5}_n^{PA_t}$ are the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$ T_n, $\mathsf{S4}_n$, and $\mathsf{S5}_n$ respectively augmented with the following *total public announcement axioms*,

(TPA) $\neg\alpha \to ([\alpha!]\beta \leftrightarrow \beta)$,
(PA1$_t$) $[\alpha!]p \leftrightarrow p$,
(PA2) $[\alpha!](\beta \to \gamma) \to ([\alpha!]\beta \to [\alpha!]\gamma)$,
(PA3$_t$) $[\alpha!]\neg\beta \leftrightarrow \neg[\alpha!]\beta$,
(PA4$_t$) $\alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i(\alpha \to [\alpha!]\beta))$,

and the *public announcement necessitation rule*,

$$(\mathsf{PAN})\ \frac{\alpha}{[\beta!]\alpha}\,.$$

Observe that the instances of axiom (TPA) of the form

$$\neg\alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i\beta) \tag{7.1}$$

show that the announcement with a false formula can never affect the knowledge of the agents. This makes sure that the agents never learn false formulas, and they will never be in an inconsistent state. Note that we could formulate all of the four systems with (7.1) instead of (TPA). Then (TPA) would be provable in the resulting systems. However, later we will consider extensions of these systems by common knowledge operators. There, things get much simpler if (TPA) is already included as an axiom.

Due to Lemma 7.2.3, our systems are sound with respect to the corresponding classes of Kripke structures.

Lemma 7.2.5. *For all* $\alpha \in \mathcal{L}_n^{PA}$ *we have*

$$\begin{array}{rclrcl} \mathsf{K}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n \vDash \alpha, & \mathsf{K4}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n^t \vDash \alpha, \\ \mathsf{K45}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n^{tu} \vDash \alpha, & \mathsf{T}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n^r \vDash \alpha, \\ \mathsf{S4}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n^{rt} \vDash \alpha, & \mathsf{S5}_n^{PA_t} \vdash \alpha & \Rightarrow & \mathcal{K}_n^{rtu} \vDash \alpha. \end{array}$$

Proof. By induction on the length of the proof. In the base case, we show how to prove that axiom ($\mathsf{PA4}_t$) is valid. Let $\mathfrak{K}$ be an arbitrary Kripke structure, $s \in |\mathfrak{K}|$, and $i \in \mathcal{A}$ be given and assume that $\mathfrak{K}, s \vDash \alpha$. Then we have

$$\begin{array}{rcl} \mathfrak{K}, s \vDash [\alpha!]K_i\beta & \Leftrightarrow & \mathfrak{K}^{\alpha,s}, s \vDash K_i\beta \\ & \Leftrightarrow & \mathfrak{K}^{\alpha}, s \vDash K_i\beta \\ & \Leftrightarrow & \text{for all } t \in R_i^{\alpha}(s),\ \mathfrak{K}^{\alpha}, t \vDash \beta \\ & \Leftrightarrow & \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}^{\alpha}, t \vDash \beta \\ & \Leftrightarrow & \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}^{\alpha,t}, t \vDash \beta \\ & \Leftrightarrow & \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}, t \vDash [\alpha!]\beta \\ & \Leftrightarrow & \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash \alpha \rightarrow [\alpha!]\beta \\ & \Leftrightarrow & \mathfrak{K}, s \vDash K_i(\alpha \rightarrow [\alpha!]\beta). \end{array}$$

In the induction step, soundness of the rule (PAN) immediately follows from Lemma 7.2.3. ☐

We will now present the *reduction axioms* that will be helpful for the definition of a new translation from $\mathcal{L}_n^{PA}$ to $\mathcal{L}_n$.

Lemma 7.2.6. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash [\alpha!](\beta \wedge \gamma) \leftrightarrow ([\alpha!]\beta \wedge [\alpha!]\gamma),$$
$$\mathsf{X} \vdash [\alpha!]K_i\beta \leftrightarrow (\neg\alpha \wedge K_i\beta) \vee (\alpha \wedge K_i(\alpha \rightarrow [\alpha!]\beta)).$$

Proof. The first assertion can be proved the same way like Lemma 7.1.5 using axiom (PA2) and the rule (PAN). The second one is tautologically implied by axiom ($\mathsf{PA4}_t$) and the instance (7.1) of axiom (TPA). ☐

Due to Lemma 7.2.6, we can directly prove the analogue of Lemma 7.1.16.

Lemma 7.2.7. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash ([\alpha!]\beta \rightarrow [\alpha!]\gamma) \rightarrow [\alpha!](\beta \rightarrow \gamma), \quad \mathsf{X} \vdash [\alpha!](\beta \vee \gamma) \leftrightarrow ([\alpha!]\beta \vee [\alpha!]\gamma).$$

Proof. Both assertions are an immediate consequence of axiom ($\mathsf{PA3}_t$) and the first assertion of Lemma 7.2.6. □

Lemma 7.2.6 is also useful for the proof that total public announcements do not affect propositional facts, as we state in the following lemma.

Lemma 7.2.8. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha \in \mathcal{L}_n^{PA}$ *and all* $\beta \in \mathcal{L}_0$ *we have*

$$\mathsf{X} \vdash [\alpha!]\beta \leftrightarrow \beta.$$

Proof. By induction on β using the axioms ($\mathsf{PA1}_t$) and ($\mathsf{PA3}_t$), as well as the first assertion of Lemma 7.2.6. □

Due to Lemma 7.2.8, we have that total public announcements perform knowledge change in a *static world*. That is, propositional facts will never change after having announced new information. Observe that Lemma 7.2.8 obviously implies fact preservation of total public announcements. Moreover, we can show that all of the properties from Definition 5.4.3 are satisfied.

Lemma 7.2.9. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *or* $\mathcal{K}_n^{rtu}$. *Then the total public announcements are fact preserving, adequate, total, self-dual, and normal with respect to* $\mathcal{X}$.

Proof. Let X^{PA_t} be the deductive system that corresponds to $\mathcal{X}$. Fact preservation is an immediate consequence of Lemma 7.2.8 and soundness. Adequacy trivially follows from fact preservation, because $\top \in \mathcal{L}_0$. For totality, we can show that for all $\alpha \in \mathcal{L}_n^{PA}$ we have $\mathsf{X}^{PA_t} \vdash [\alpha!]\neg\bot$ by Lemma 7.2.8 and propositional reasoning. By an application of axiom ($\mathsf{PA3}_t$) and soundness, we get $\mathcal{X} \vDash \neg[\alpha!]\bot$, and totality is proved. Self-duality is given by axiom ($\mathsf{PA3}_t$) and soundness. Due to axiom ($\mathsf{PA2}$), the rule (PAN), and soundness we directly get normality. □

The fact that total public announcements are self-dual also shows a difference to truthful public announcements, namely in the way an announcement formula is read. That is $[\alpha!]\beta$ means "β holds after *the* public announcement of α". In the context of truthful public announcements, it is read as "β holds after *every* truthful public announcement of α", and its dual has the meaning "β holds after *some* truthful public announcement of α", see [22].

Like truthful public announcements, we have that total public announcements are syntax independent. Again, we will first state a restricted version of this fact.

Lemma 7.2.10. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{PA}$ *and all* $\gamma \in \mathcal{L}_n$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. We can prove this lemma by induction on γ, using the axioms ($\mathsf{PA1}_t$) and ($\mathsf{PA3}_t$), as well as both assertions of Lemma 7.2.6. □

In order to show completeness of the six systems for total public announcements, we will again define a two step translation from $\mathcal{L}_n^{PA}$ to $\mathcal{L}_n$.

Definition 7.2.11. The function $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ is inductively defined by

$$\begin{aligned} h([\alpha!]p) &:= p, \\ h([\alpha!]\neg\beta) &:= \neg h([\alpha!]\beta), \\ h([\alpha!](\beta \wedge \gamma)) &:= h([\alpha!]\beta) \wedge h([\alpha!]\gamma), \\ h([\alpha!]K_i\beta) &:= (\neg\alpha \wedge K_i\beta) \vee (\alpha \wedge K_i(\alpha \to h([\alpha!]\beta))), \\ h([\alpha!][\beta!]\gamma) &:= [\alpha!][\beta!]\gamma. \end{aligned}$$

Again, it is easy to see that for all $\alpha, \beta \in \mathcal{L}_n$ we have $h([\alpha!]\beta) \in \mathcal{L}_n$, and the function h is equivalence preserving in the following sense.

Lemma 7.2.12. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash h([\alpha!]\beta) \leftrightarrow [\alpha!]\beta.$$

Proof. This lemma can be proved by induction on β using the axioms ($\mathsf{PA1}_t$) and ($\mathsf{PA3}_t$), as well as both assertions of Lemma 7.2.6. □

The translation that eliminates the announcement operator in every $\mathcal{L}_n^{PA}$ formula is defined the same way as in the previous sections about announcement logics.

Definition 7.2.13. The function $f\colon \mathcal{L}_n^{PA} \to \mathcal{L}_n^{PA}$ is inductively defined by

$$\begin{aligned} f(p) &:= p, \\ f(\neg\alpha) &:= \neg f(\alpha), \\ f(\alpha \wedge \beta) &:= f(\alpha) \wedge f(\beta), \\ f(K_i\alpha) &:= K_i f(\alpha), \\ f([\alpha!]\beta) &:= h([f(\alpha)!]f(\beta)). \end{aligned}$$

It is obvious that for all $\alpha \in \mathcal{L}_n^{PA}$ we have that $f(\alpha) \in \mathcal{L}_n$. In addition, we can prove the equivalence of α and $f(\alpha)$ in all of our six systems.

Lemma 7.2.14. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash f(\alpha) \leftrightarrow \alpha.$$

Proof. By induction on α. The only nontrivial case is in the induction step, where α is of the form $[\beta!]\gamma$. In this case, the claim can be proved using Lemma 7.2.12 and Lemma 7.2.10. It works exactly the same way as the proof of Lemma 7.1.11. □

Lemma 7.2.14 is very helpful not only for the completeness proof, but also for generalising results on $\mathcal{L}_n$ formulas to $\mathcal{L}_n^{PA}$ formulas. Making use of it and of Lemma 7.2.10, we can now easily prove the general version of syntax independence for total public announcements.

Lemma 7.2.15. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. First, we have that X proves the equivalence of $[\alpha!]\gamma$ and $[\alpha!]f(\gamma)$ by Lemma 7.2.14 using axiom ($\mathsf{PA2}$) and the rule (PAN). Now, using the fact that $f(\gamma) \in \mathcal{L}_n$ and $\mathsf{X} \vdash \alpha \leftrightarrow \beta$, we get that $[\alpha!]f(\gamma)$ is provably equivalent to $[\beta!]f(\gamma)$ by Lemma 7.2.10. Again, we can apply Lemma 7.2.14 and get that $\mathsf{X} \vdash [\beta!]f(\gamma) \leftrightarrow [\beta!]\gamma$, hence we are done. □

Due to Lemma 7.2.15, we can now prove the *Replacement Theorem* for the logic of total public announcements.

Theorem 7.2.16 (Replacement)**.** *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash \gamma \leftrightarrow \gamma[\alpha/\beta].$$

Proof. The proof is by induction on γ and is identical to the proof of Theorem 7.1.13. In the induction step, we can apply Lemma 7.2.15 instead of Lemma 7.1.12. □

As another consequence of Lemma 7.2.14 we get the following equivalence concerning consecutive announcement operators.

Lemma 7.2.17. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{PA}$ *we have*

$$\mathsf{X} \vdash \alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma).$$

Proof. This result can be established in two steps. First, we can prove a restricted version where γ has to be an $\mathcal{L}_n$ formula. This can be done by induction on γ. We show how to prove the last case of the induction step, where γ is of the form $K_i\delta$. By Lemma 7.2.6 and Lemma 7.2.7, the axioms ($\mathsf{PA2}$) and ($\mathsf{PA3}_t$), as well as the rule (PAN), we get that $[\alpha!][\beta!]K_i\delta$ is provably equivalent to the following formula,

$$\begin{aligned}&(\neg[\alpha!]\beta \wedge [\alpha!]K_i\delta) \vee \\ &\qquad([\alpha!]\beta \wedge ((\neg\alpha \wedge K_i(\beta \to [\beta!]\delta)) \vee (\alpha \wedge K_i(\alpha \wedge [\alpha!]\beta \to [\alpha!][\beta!]\delta)))).\end{aligned}$$

Therefore, we immediately get that X proves

$$\alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]K_i\delta \leftrightarrow K_i(\alpha \wedge [\alpha!]\beta \to [\alpha!][\beta!]\delta)) \tag{7.2}$$

by tautological reasoning. On the other hand, we have the following instance of axiom ($\mathsf{PA4}_t$),

$$\alpha \wedge [\alpha!]\beta \to ([(\alpha \wedge [\alpha!]\beta)!]K_i\delta \leftrightarrow K_i(\alpha \wedge [\alpha!]\beta \to [(\alpha \wedge [\alpha!]\beta)!]\delta)). \tag{7.3}$$

Using the induction hypothesis, which states that the formula

$$\alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]\delta \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\delta)$$

is provable in X, we can derive

$$K_i(\alpha \wedge [\alpha!]\beta \to [\alpha!][\beta!]\delta) \leftrightarrow K_i(\alpha \wedge [\alpha!]\beta \to [(\alpha \wedge [\alpha!]\beta)!]\delta) \tag{7.4}$$

by normal modal logic reasoning. Now, by the provability of the formulas (7.2), (7.3), and (7.4), as well as tautological reasoning, we get

$$\mathsf{X} \vdash \alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]K_i\delta \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]K_i\delta).$$

We have now proved that the assertion holds for all $\gamma \in \mathcal{L}_n$. In order to prove it for arbitrary $\gamma \in \mathcal{L}_n^{PA}$, we can make use of the restricted result and Lemma 7.2.14. ∎

Like in Section 7.1, we get an elegant completeness proof for our six deductive systems due to Lemma 7.2.14.

Theorem 7.2.18. *For all $\alpha \in \mathcal{L}_n^{PA}$ we have*

$$\begin{array}{rclrcl} \mathsf{K}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n \vDash \alpha, & \mathsf{K4}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^t \vDash \alpha, \\ \mathsf{K45}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{tu} \vDash \alpha, & \mathsf{T}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^r \vDash \alpha, \\ \mathsf{S4}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rt} \vDash \alpha, & \mathsf{S5}_n^{PA_t} \vdash \alpha & \Leftrightarrow & \mathcal{K}_n^{rtu} \vDash \alpha. \end{array}$$

Proof. Soundness has already been proved. We show the direction from right to left. Let X be one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$, and $\mathcal{X}$ be its corresponding class of Kripke structures. For a given formula $\alpha \in \mathcal{L}_n^{PA}$, we assume that $\mathcal{X} \vDash \alpha$. Then we have that $\mathcal{X} \vDash f(\alpha)$ by Lemma 7.2.14 and soundness. By completeness of X, we get that $\mathsf{X} \vdash f(\alpha)$ and, obviously, $\mathsf{X}^{PA_t} \vdash f(\alpha)$. Again by Lemma 7.2.14, we get $\mathsf{X}^{PA_t} \vdash \alpha$, and we are done. ☐

As a preparation for Section 7.3, we will now prove a *reduction axiom* for the notion of mutual knowledge.

Lemma 7.2.19. *Let X be one of the deductive systems $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, or $\mathsf{S5}_n^{PA_t}$. Then for all $\alpha, \beta \in \mathcal{L}_n^{PA}$ we have*

$$\mathsf{X} \vdash \alpha \to ([\alpha!]E_G\beta \leftrightarrow E_G(\alpha \to [\alpha!]\beta)).$$

Proof. By a simple application of axiom ($\mathsf{PA4}_t$) and Lemma 7.2.6. ☐

In a next step, we are going to present some results about the announcement resistant formulas in total public announcement logic. Compared to truthful public announcements, it does not make sense to consider the successful formulas, because in this setting not even propositions would be successful. On the other hand, Lemma 7.2.8 directly implies that all $\mathcal{L}_0$ formulas are announcement resistant. We will now show that we have the same sufficient conditions for a formula to be announcement resistant as with truthful public announcements.

Lemma 7.2.20. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have the following sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{PA}$ to be announcement resistant in $\mathcal{X}$,*

1. *$\alpha \in \mathcal{L}_0$,*
2. *$\mathcal{X} \vDash \alpha$ or $\mathcal{X} \vDash \neg\alpha$,*
3. *$\alpha = \beta \wedge \gamma$ or $\alpha = \beta \vee \gamma$ for some β, γ announcement resistant in $\mathcal{X}$,*
4. *$\alpha = K_i\beta$ for some $i \in \mathcal{A}$ and some β announcement resistant in $\mathcal{X}$.*

Proof. We show how to prove the fourth assertion. Let $\mathcal{X}$ be given and X^{PA_t} be its corresponding system. Further, let β be announcement resistant in $\mathcal{X}$ and $\gamma \in \mathcal{L}_n^{PA}$ be arbitrarily given. By completeness, we have that X^{PA_t} proves $\beta \to [\gamma!]\beta$, and we get $\mathsf{X}^{PA_t} \vdash K_i\beta \to K_i(\gamma \to [\gamma!]\beta)$ by normal modal logic reasoning. Now, by tautological reasoning, we easily get

$$\mathsf{X}^{PA_t} \vdash K_i\beta \to (\neg\gamma \wedge K_i\beta) \vee (\gamma \wedge K_i(\gamma \to [\gamma!]\beta)).$$

By the second assertion of Lemma 7.2.6, we immediately get that X proves $K_i\beta \to [\gamma!]K_i\beta$. By soundness, we are done. □

As an immediate consequence of Lemma 7.2.20, we get that for all $\alpha \in \mathcal{L}_0$, the formula $K_i\alpha$ is announcement resistant. That is, knowledge of propositional facts can never be retracted by public announcements. We can therefore say that the logic of total public announcements formalises *expansion* for propositional knowledge. As we have seen in Example 7.2.2, agents can really expand their knowledge due to some announcements.

Now, we will prove that the logic of total public announcements does not have the *substitution property*. The proof uses the fact that the formulas of the form $p \wedge \neg K_i p$ are not announcement resistant and is similar to the one for truthful public announcements.

Lemma 7.2.21. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, *or* $\mathsf{S5}_n^{PA_t}$. *Then for all* $p \in \mathcal{P}$ *we have*

$$\mathsf{X} \vdash p \to [p!]p, \qquad \mathsf{X} \nvdash p \wedge \neg K_i p \to [(p \wedge \neg K_i p)!](p \wedge \neg K_i p).$$

Proof. The first assertion trivially follows from axiom ($\mathsf{PA1}_t$). We show how to prove the second assertion. By Lemma 7.2.14, we have that the formula $p \wedge \neg K_i p \to [(p \wedge \neg K_i p)!](p \wedge \neg K_i p)$ is provably equivalent to its translation $f(p \wedge \neg K_i p \to [(p \wedge \neg K_i p)!](p \wedge \neg K_i p))$, that is to

$$p \wedge \neg K_i p \to p \wedge \neg K_i(p \wedge \neg K_i p \to p).$$

But this formula is provably equivalent to $p \to K_i p$ by normal modal logic reasoning, which is obviously not provable in X. □

The following theorem states that true announcement resistant formulas get common knowledge after being announced once.

Theorem 7.2.22. *Let* $\mathcal{X}$ *be one of the classes* $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, *or* $\mathcal{K}_n^{rtu}$ *and* $\alpha \in \mathcal{L}_n^{PA}$ *be given. If* α *is announcement resistant in* $\mathcal{X}$, *then for all* $l \geq 1$ *and all* $i_1, \ldots, i_l \in \mathcal{A}$ *we have*

$$\mathcal{X} \vDash \alpha \to [\alpha!]K_{i_l} \ldots K_{i_1}\alpha.$$

Proof. Let X^{PA_t} be the system that corresponds to $\mathcal{X}$. We will prove by induction on l that X^{PA_t} proves $\alpha \to [\alpha!]K_{i_l} \dots K_{i_1}\alpha$. In the base case, we have a proof of $\alpha \to [\alpha!]\alpha$ by assumption, and an application of the rule (NEC) results in a proof of $K_{i_1}(\alpha \to [\alpha!]\alpha)$. Applying axiom ($\mathsf{PA4}_t$) and some tautology now results in a proof of $\alpha \to [\alpha!]K_{i_1}\alpha$. In the induction step, we start with a proof of $\alpha \to [\alpha!]K_{i_l} \dots K_{i_1}\alpha$ by induction hypothesis, and we get that X^{PA_t} proves $K_{i_{l+1}}(\alpha \to [\alpha!]K_{i_l} \dots K_{i_1}\alpha)$ by applying the rule (NEC). Finally, we get $\mathsf{X}^{PA_t} \vdash \alpha \to [\alpha!]K_{i_{l+1}}K_{i_l} \dots K_{i_1}\alpha$ by axiom ($\mathsf{PA4}_t$) and tautological reasoning. Due to soundness, we get the desired result. □

We have now seen that we have a big set of announcement resistant formulas in the logic of total public announcements, like in the logic of truthful public announcements. Moreover, a true announcement resistant formula gets common knowledge after being announced once, similar to the results in Section 7.1. From a semantical point of view, it is immediate that the transformed Kripke structure is the same in both approaches, if the announced formula is true. But there are formulas, of course, that are true in one semantics, and false in the other, and vice versa. We end this section by proving that an announcement with a true announcement free formula has the same impact on announcement free formulas in both approaches.

Theorem 7.2.23. *Let $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ be the translation from Definition 7.2.11, and $h'\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ denote the corresponding function defined in Definition 7.1.10. Further, let X be one of the systems K_n, $\mathsf{K4}_n$, $\mathsf{K45}_n$, T_n, $\mathsf{S4}_n$, or $\mathsf{S5}_n$. Then for all $\alpha, \beta \in \mathcal{L}_n$ we have*

$$\mathsf{X} \vdash \alpha \to (h([\alpha!]\beta) \leftrightarrow h'([\alpha!]\beta)).$$

Proof. By induction on β. The only nontrivial case is where β is of the form $K_i\gamma$ in the induction step. By induction hypothesis, we can assume that we have a proof of $\alpha \to (h([\alpha!]\gamma) \leftrightarrow h'([\alpha!]\gamma))$ in X. By normal modal logic reasoning, we immediately get that X proves

$$\alpha \to ((\neg\alpha \wedge K_i\gamma) \vee (\alpha \wedge K_i(\alpha \to h([\alpha!]\gamma))) \leftrightarrow (\alpha \to K_i(\alpha \to h'([\alpha!]\gamma)))).$$

We only need to show now that the formula $\alpha \to K_i(\alpha \to h'([\alpha!]\gamma))$ is provably equivalent to $\alpha \to K_i(h'([\alpha!]\gamma))$. For this purpose, one can show that for all $\varphi, \psi \in \mathcal{L}_n$ we have $\mathsf{X} \vdash h'([\varphi!]\psi) \leftrightarrow (\varphi \to h'([\varphi!]\psi))$ by a simple induction on ψ, like in the proof of Theorem 7.1.28. □

So the crucial difference between total public announcement and truthful public announcement semantics is the following. If an announced formula is false, we have no impact on the agent's knowledge in the former, and an inconsistent epistemic state in the latter case. This fact could be useful for defining public announcements in systems of knowledge and belief.

Due to Theorem 7.1.28 and Theorem 7.2.23, we easily get that the total public announcements are in the same relationship to group announcements for trustful agents like the truthful public announcements.

Theorem 7.2.24. *Let* $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{PA}\} \to \mathcal{L}_n^{PA}$ *be the translation from Definition 7.2.11, and* $h'\colon \{[\alpha!_G]\beta : \alpha, \beta \in \mathcal{L}_n^{GA}, \emptyset \neq G \subseteq \mathcal{A}\} \to \mathcal{L}_n^{GA}$ *denote the corresponding function defined in Definition 6.1.9. Further, let* X *be one of the systems* K_n*,* $\mathsf{K4}_n$*,* $\mathsf{K45}_n$*,* T_n*,* $\mathsf{S4}_n$*, or* $\mathsf{S5}_n$*. Then for all* $\alpha, \beta \in \mathcal{L}_n$ *we have*

$$\mathsf{X} \vdash \alpha \to (h([\alpha!]\beta) \leftrightarrow h'([\alpha!_{\mathcal{A}}]\beta)).$$

7.3 Adding common knowledge operators

In this section, we will provide deductive systems for the logic of total public announcements augmented with common knowledge operators. First, we will add relativised common knowledge and show completeness via reduction axioms. That is, we will extend the equivalence preserving translation from Section 7.2 to a function mapping from $\mathcal{L}_n^{RCPA}$ to $\mathcal{L}_n^{RC}$. Moreover, we will extend the results about announcement resistant formulas. In a second step, we will add common knowledge and give a completeness proof in full detail. Since truthful public announcement logic augmented with common knowledge operators is more expressive than the logic of common knowledge (cf. Section 7.1), and total public announcements have the same effect in case the announced formula is true (cf. Theorem 7.2.23), we can conclude that there is no translation from $\mathcal{L}_n^{CPA}$ to $\mathcal{L}_n^{C}$ that is equivalence preserving. Therefore, the deductive systems contain more than just reduction axioms, cf. [62], where we have presented such a system based on $\mathsf{S5}_n^C$. Again, we will do some discussion on announcement resistant formulas, and we will generalise some of the previous results such as Theorem 7.2.22.

In the first part of this section, we are dealing with the language $\mathcal{L}_n^{RCPA}$, which we have defined in Definition 5.4.1 ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ \mid \ \neg\alpha \ \mid \ (\alpha \wedge \alpha) \ \mid \ K_i\alpha \ \mid \ RC_G(\alpha, \alpha) \ \mid \ [\alpha!]\alpha.$$

For public announcement operators, we will use the semantics of total public announcements from Section 7.2. We recall the semantics for the operators extending the language $\mathcal{L}_n$ of modal logic,

$$\mathfrak{K}, s \vDash RC_G(\alpha, \beta) \ :\Leftrightarrow \ \text{for all } t \in (R_G \cap (|\mathfrak{K}| \times \|\alpha\|_{\mathfrak{K}}))^+(s),\ \mathfrak{K}, t \vDash \beta,$$

$$\mathfrak{K}, s \vDash [\alpha!]\beta \;:\Leftrightarrow\; \mathfrak{K}^{\alpha,s}, s \vDash \beta,$$

where the transformed Kripke structure $\mathfrak{K}^{\alpha,s} = (S^{\alpha,s}, R_1^{\alpha,s}, \ldots, R_n^{\alpha,s}, V^{\alpha,s})$ is defined as in Definition 7.2.1.

Since we have the same model transformation as in Section 7.2, we have that Lemma 7.2.3 still holds in the extended framework, that is the model transformation preserves reflexivity, transitivity, and Euclideanity of the accessibility relations.

Lemma 7.3.1. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then for all Kripke structures $\mathfrak{K}$, all $s \in |\mathfrak{K}|$, and all $\alpha \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathfrak{K} \in \mathcal{X} \;\Rightarrow\; \mathfrak{K}^{\alpha,s} \in \mathcal{X}.$$

The new deductive systems can be obtained by combining the systems for total public announcements with the systems for relativised common knowledge. In addition, there is a new reduction axiom for relativised common knowledge after a public announcement.

Definition 7.3.2. The Hilbert systems $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, and $\mathsf{S5}_n^{RCPA_t}$ are the systems $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, and $\mathsf{S5}_n^{PA_t}$ respectively augmented with the *co-closure axiom* and the *public announcement axiom* for relativised common knowledge,

(RC) $\quad RC_G(\alpha,\beta) \to E_G(\alpha \to \beta \wedge RC_G(\alpha,\beta))$,

($\mathsf{PA5}_t$) $\quad \alpha \to ([\alpha!]RC_G(\beta,\gamma) \leftrightarrow RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\gamma))$,

as well as the *induction rule* for relativised common knowledge,

$$(\mathsf{RCI})\ \frac{\alpha \to E_G(\beta \to \alpha \wedge \gamma)}{\alpha \to RC_G(\beta,\gamma)}\ .$$

Like in Section 7.2, Lemma 7.3.1 is essential for proving soundness of our six deductive systems.

Lemma 7.3.3. *For all $\alpha \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathsf{K}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n \vDash \alpha, \qquad \mathsf{K4}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^t \vDash \alpha,$$
$$\mathsf{K45}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{tu} \vDash \alpha, \qquad \mathsf{T}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^r \vDash \alpha,$$
$$\mathsf{S4}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{rt} \vDash \alpha, \qquad \mathsf{S5}_n^{RCPA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_n^{rtu} \vDash \alpha.$$

Proof. By induction on the length of the proof. The only new case is in the base case, and we show that axiom ($\mathsf{PA5}_t$) is valid in all Kripke structures.

Let $\mathfrak{K} \in \mathcal{K}_n$, $s \in |\mathfrak{K}|$, $\emptyset \neq G \subseteq \mathcal{A}$, and $\alpha, \beta, \gamma \in \mathcal{L}_n^{RCPA}$ be given and assume $\mathfrak{K}, s \vDash \alpha$. Then we have $\mathfrak{K}^{\alpha,s} = \mathfrak{K}^{\alpha}$, and we get

$$\begin{aligned}
&\mathfrak{K}, s \vDash [\alpha!]RC_G(\beta,\gamma)\\
&\Leftrightarrow\ \mathfrak{K}^{\alpha,s}, s \vDash RC_G(\beta,\gamma)\\
&\Leftrightarrow\ \mathfrak{K}^{\alpha}, s \vDash RC_G(\beta,\gamma)\\
&\Leftrightarrow\ \text{for all } t \in (R_G^{\alpha} \cap (|\mathfrak{K}^{\alpha}| \times \|\beta\|_{\mathfrak{K}^{\alpha}}))^{+}(s),\ \mathfrak{K}^{\alpha}, t \vDash \gamma\\
&\Leftrightarrow\ \text{for all } t \in ((R_G \cap \|\alpha\|_{\mathfrak{K}}^2)) \cap (\|\alpha\|_{\mathfrak{K}} \times \|\alpha \wedge [\alpha!]\beta\|_{\mathfrak{K}}))^{+}(s),\ \mathfrak{K}^{\alpha}, t \vDash \gamma\\
&\Leftrightarrow\ \text{for all } t \in (R_G \cap (\|\alpha\|_{\mathfrak{K}} \times \|\alpha \wedge [\alpha!]\beta\|_{\mathfrak{K}}))^{+}(s),\ \mathfrak{K}^{\alpha}, t \vDash \gamma\\
&\Leftrightarrow\ \text{for all } t \in (R_G \cap (|\mathfrak{K}| \times \|\alpha \wedge [\alpha!]\beta\|_{\mathfrak{K}}))^{+}(s),\ \mathfrak{K}^{\alpha,t}, t \vDash \gamma\\
&\Leftrightarrow\ \text{for all } t \in (R_G \cap (|\mathfrak{K}| \times \|\alpha \wedge [\alpha!]\beta\|_{\mathfrak{K}}))^{+}(s),\ \mathfrak{K}, t \vDash [\alpha!]\gamma,
\end{aligned}$$

which is equivalent to $\mathfrak{K}, s \vDash RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\gamma)$. ☐

Since the deductive systems X^{RCPA_t} are an extension of the systems X^{PA_t}, we immediately get that Lemma 7.2.6, Lemma 7.2.7, Lemma 7.2.8, Lemma 7.2.9, and Lemma 7.2.19 all still hold for the language $\mathcal{L}_n^{RCPA}$ instead of $\mathcal{L}_n^{PA}$. Moreover, we can prove an additional *reduction axiom* in the extended systems.

Lemma 7.3.4. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{RCPA}$ *we have that* X *proves*

$$[\alpha!]RC_G(\beta,\gamma) \leftrightarrow (\neg\alpha \wedge RC_G(\beta,\gamma)) \vee (\alpha \wedge RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\gamma)).$$

Proof. The claim directly follows from the axioms (TPA) and ($\mathsf{PA5}_t$) by tautological reasoning. ☐

Due to Lemma 7.3.4, we have that Lemma 7.2.10 still holds in the extended setting:

Lemma 7.3.5. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{RCPA}$ *and all* $\varphi \in \mathcal{L}_n^{RC}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \ \Rightarrow\ \mathsf{X} \vdash [\alpha!]\varphi \leftrightarrow [\beta!]\varphi.$$

Proof. We can prove this lemma by induction on φ like Lemma 7.2.10. In the new case of the induction step, we can apply the new reduction axiom from Lemma 7.3.4. ☐

As another consequence of Lemma 7.3.4, we know how to extend the translation h from Definition 7.2.11 to a function mapping from $\{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{RCPA}\}$ to $\mathcal{L}_n^{RCPA}$. For this purpose, we add the following clause to Definition 7.2.11,

$$h([\alpha!]RC_G(\beta,\gamma)) := \\ (\neg\alpha \wedge RC_G(\beta,\gamma)) \vee (\alpha \wedge RC_G(\alpha \wedge h([\alpha!]\beta), h([\alpha!]\gamma))).$$

Again, it is obvious that for all $\alpha, \beta \in \mathcal{L}_n^{RC}$ we have that the formula $h([\alpha!]\beta)$ is an element of $\mathcal{L}_n^{RC}$. Moreover, we have that the function h is still equivalence preserving in the following sense.

Lemma 7.3.6. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha, \beta \in \mathcal{L}_n^{RCPA}$ *we have*

$$\mathsf{X} \vdash h([\alpha!]\beta) \leftrightarrow [\alpha!]\beta.$$

Proof. The claim can be proved by induction on β and is similar to the proof of Lemma 7.2.12. The only new case in the induction step can be proved using Lemma 7.3.4. □

Now, we will also extend the function f from Definition 7.2.13 to a function mapping from $\mathcal{L}_n^{RCPA}$ to $\mathcal{L}_n^{RCPA}$. This can be done by adding the following clause to Definition 7.2.13,

$$f(RC_G(\alpha,\beta)) := RC_G(f(\alpha), f(\beta)).$$

It is easy to see that for all $\alpha \in \mathcal{L}_n^{RCPA}$ we have that the formula $f(\alpha)$ is an element of $\mathcal{L}_n^{RC}$. Furthermore, the translation f is equivalence preserving in the sense of Lemma 7.2.14:

Lemma 7.3.7. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha \in \mathcal{L}_n^{RCPA}$ *we have*

$$\mathsf{X} \vdash f(\alpha) \leftrightarrow \alpha.$$

Proof. By induction on α. In the last case of the induction step, where α is of the form $[\beta!]\gamma$, the proof is identical to the proof of Lemma 7.2.14. That is, we can apply Lemma 7.3.5 and Lemma 7.3.6 instead of Lemma 7.2.10 and Lemma 7.2.12 respectively. □

As a direct consequence of the semantics we have that total public announcements are syntax independent. Due to the previous lemmas, we have a syntactical proof of this fact.

Lemma 7.3.8. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{RCPA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. The proof of this assertion works exactly the same way as the proof of Lemma 7.2.15. The only difference is that we apply Lemma 7.3.5 and Lemma 7.3.7 instead of Lemma 7.2.10 and Lemma 7.2.14 respectively. □

Due to Lemma 7.3.8, we get that the Replacement Theorem still holds, as we have proved in Theorem 7.2.16. We are now going to show that Lemma 7.2.17 also holds in the extended framework.

Lemma 7.3.9. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{RCPA_t}$, $\mathsf{K4}_n^{RCPA_t}$, $\mathsf{K45}_n^{RCPA_t}$, $\mathsf{T}_n^{RCPA_t}$, $\mathsf{S4}_n^{RCPA_t}$, *or* $\mathsf{S5}_n^{RCPA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{RCPA}$ *we have*

$$\mathsf{X} \vdash \alpha \land [\alpha!]\beta \rightarrow ([\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \land [\alpha!]\beta)!]\gamma).$$

Proof. Like Lemma 7.2.17, this result can be established in two steps. First, we can prove a restricted version where γ has to be an $\mathcal{L}_n^{RC}$ formula. This can be done by induction on γ. We show how to prove the last case of the induction step, where γ is of the form $RC_G(\delta, \varphi)$. By Lemma 7.2.6, Lemma 7.2.7, and Lemma 7.3.4, the axioms ($\mathsf{PA2}$) and ($\mathsf{PA3}_t$), as well as the rule (PAN), we get that $[\alpha!][\beta!]RC_G(\delta, \varphi)$ is provably equivalent to

$$(\neg[\alpha!]\beta \land [\alpha!]RC_G(\delta, \varphi)) \lor ([\alpha!]\beta \land ((\neg\alpha \land RC_G(\beta \land [\beta!]\delta, [\beta!]\varphi)) \lor \\ (\alpha \land RC_G(\alpha \land [\alpha!]\beta \land [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi)))).$$

Therefore, we immediately get that X proves

$$\alpha \land [\alpha!]\beta \rightarrow ([\alpha!][\beta!]RC_G(\delta, \varphi) \leftrightarrow \\ RC_G(\alpha \land [\alpha!]\beta \land [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi)) \quad (7.5)$$

by tautological reasoning. On the other hand, as an instance of ($\mathsf{PA5}_t$), we have that X proves

$$\alpha \land [\alpha!]\beta \rightarrow ([(\alpha \land [\alpha!]\beta)!]RC_G(\delta, \varphi) \leftrightarrow \\ RC_G(\alpha \land [\alpha!]\beta \land [(\alpha \land [\alpha!]\beta)!]\delta, [(\alpha \land [\alpha!]\beta)!]\varphi)). \quad (7.6)$$

Using the induction hypothesis, which states that the formula

$$\alpha \land [\alpha!]\beta \rightarrow ([\alpha!][\beta!]\delta \leftrightarrow [(\alpha \land [\alpha!]\beta)!]\delta) \land ([\alpha!][\beta!]\varphi \leftrightarrow [(\alpha \land [\alpha!]\beta)!]\varphi)$$

is provable in X, we can derive the formula

$$E_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta \to \psi \wedge [\alpha!][\beta!]\varphi) \to \\ E_G(\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\delta \to \psi \wedge [(\alpha \wedge [\alpha!]\beta)!]\varphi)$$

for every formula $\psi \in \mathcal{L}_n^{RCPA}$ by normal modal logic reasoning. If we let ψ be the formula $RC_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi)$, we can now apply axiom (RC) in order to get a proof of

$$RC_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi) \to E_G(\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\delta \to \\ RC_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi) \wedge [(\alpha \wedge [\alpha!]\beta)!]\varphi).$$

Finally, by an application of the rule (RCI), we get that X proves

$$RC_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi) \to \\ RC_G(\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\delta, [(\alpha \wedge [\alpha!]\beta)!]\varphi).$$

The converse direction can similarly be proved, hence we get a proof of

$$RC_G(\alpha \wedge [\alpha!]\beta \wedge [\alpha!][\beta!]\delta, [\alpha!][\beta!]\varphi) \leftrightarrow \\ RC_G(\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\delta, [(\alpha \wedge [\alpha!]\beta)!]\varphi) \quad (7.7)$$

in X. Now, by the provability of the formulas (7.5), (7.6), and (7.7), as well as tautological reasoning, we get

$$\mathsf{X} \vdash \alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]RC_G(\delta, \varphi) \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]RC_G(\delta, \varphi)).$$

We have now proved that the assertion holds for all $\gamma \in \mathcal{L}_n^{RC}$. In order to prove it for arbitrary $\gamma \in \mathcal{L}_n^{RCPA}$, we can make use of the restricted result and Lemma 7.3.7. □

Due to Lemma 7.3.7, we also get an elegant completeness proof for our six Hilbert systems like in Section 7.1 and Section 7.2.

Theorem 7.3.10. *For all $\alpha \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathsf{K}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n \models \alpha, \qquad \mathsf{K4}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n^t \models \alpha,$$
$$\mathsf{K45}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n^{tu} \models \alpha, \qquad \mathsf{T}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n^r \models \alpha,$$
$$\mathsf{S4}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n^{rt} \models \alpha, \qquad \mathsf{S5}_n^{RCPA_t} \vdash \alpha \iff \mathcal{K}_n^{rtu} \models \alpha.$$

Proof. Soundness has already been proved. For the direction from right to left, we can proceed exactly the same way like in the proof of Theorem 7.2.18 using Lemma 7.3.7 instead of Lemma 7.2.14. □

It is not surprising that we have the same conditions for a formula to be announcement resistant like in the logic of truthful public announcements and relativised common knowledge. That is, Lemma 7.1.31 also holds for total public announcements and relativised common knowledge.

Lemma 7.3.11. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have the same sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{RCPA}$ to be announcement resistant in $\mathcal{X}$ as in Lemma 7.1.31.*

Proof. The first four conditions have the same proof as Lemma 7.2.20. We show how to prove the fifth one, that is

5. $\alpha = RC_G(\neg\beta, \gamma)$ for some non empty $G \subseteq \mathcal{A}$ and some β, γ announcement resistant in $\mathcal{X}$.

Let X^{RCPA_t} be the Hilbert system that corresponds to $\mathcal{X}$ and $\delta \in \mathcal{L}_n^{RCPA}$ be arbitrarily given. By assumption and completeness, we have that the formulas $\beta \to [\delta!]\beta$ and $\gamma \to [\delta!]\gamma$ are both provable in X^{RCPA_t}. By normal modal logic reasoning, we easily get a proof of

$$E_G(\neg\beta \to RC_G(\neg\beta, \gamma) \wedge \gamma) \to E_G(\delta \wedge \neg[\delta!]\beta \to RC_G(\neg\beta, \gamma) \wedge [\delta!]\gamma)$$

in X^{RCPA_t}. Applying the axioms $(\mathsf{PA3}_t)$ and (RC), we get that X^{RCPA_t} proves

$$RC_G(\neg\beta, \gamma) \to E_G(\delta \wedge [\delta!]\neg\beta \to RC_G(\neg\beta, \gamma) \wedge [\delta!]\gamma).$$

An application of the rule (RCI) now results in a proof of

$$RC_G(\neg\beta, \gamma) \to RC_G(\delta \wedge [\delta!]\neg\beta, [\delta!]\gamma),$$

and we get that X^{RCPA_t} proves

$$RC_G(\neg\beta, \gamma) \to (\neg\delta \wedge RC_G(\neg\beta, \gamma)) \vee (\delta \wedge RC_G(\delta \wedge [\delta!]\neg\beta, [\delta!]\gamma))$$

by tautological reasoning. An application of Lemma 7.3.4 implies

$$\mathsf{X}^{RCPA_t} \vdash RC_G(\neg\beta, \gamma) \to [\delta!]RC_G(\neg\beta, \gamma),$$

and by soundness, we get the desired result. □

As we have proved in Theorem 7.2.22, a true announcement resistant formula gets common knowledge among any group of agents after being announced once. Like in Theorem 7.1.32, we can express this fact within the extended language, and we are able to prove a slightly stronger result.

Theorem 7.3.12. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{RCPA}$ be given. If α is announcement resistant in $\mathcal{X}$, then for all non empty $G \subseteq \mathcal{A}$ and all $\beta \in \mathcal{L}_n^{RCPA}$ we have*

$$\mathcal{X} \vDash \alpha \to [\alpha!]RC_G(\beta, \alpha).$$

Proof. Let X^{RCPA_t} be the Hilbert system that corresponds to $\mathcal{X}$ and $\beta \in \mathcal{L}_n^{RCPA}$ be arbitrarily given. By assumption and completeness, we get that X^{RCPA_t} proves $\alpha \to [\alpha!]\alpha$. By normal modal logic reasoning, we easily get a proof of $\alpha \to E_G(\alpha \wedge [\alpha!]\beta \to \alpha \wedge [\alpha!]\alpha)$ in X^{RCPA_t}. An application of the rule (RCI) now results in a proof of $\alpha \to RC_G(\alpha \wedge [\alpha!]\beta, [\alpha!]\alpha)$. Finally, we get that X^{RCPA_t} proves $\alpha \to [\alpha!]RC_G(\beta, \alpha)$ by axiom ($\mathsf{PA5}_t$) and tautological reasoning. Due to soundness, we are done. □

Now, we are going to add common knowledge instead of relativised common knowledge operators to the logic of total public announcements. As we will see, we do not have reduction axioms that eliminate the announcement operators, like in the logic of truthful public announcements and common knowledge. Therefore, the completeness proof is not that easy, and we will prove it in full detail.

First, we will recall the language $\mathcal{L}_n^{CPA}$ from Definition 5.4.1, which is given by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ C_G\alpha \ | \ [\alpha!]\alpha.$$

The semantics is given by adding the following clause to Definition 7.2.1,

$$\mathfrak{K}, s \vDash C_G\alpha \ :\Leftrightarrow \ \text{for all } t \in R_G^+(s),\ \mathfrak{K}, t \vDash \alpha,$$

where the relation R_G^+ is the transitive closure of $\bigcup_{i \in G} R_i$. Since we have the same model transformation as in Definition 7.2.1, it is not surprising that seriality, transitivity, and Euclideanity are still preserved within the extended setting, cf. Lemma 7.2.3 and Lemma 7.3.1.

Lemma 7.3.13. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then for all Kripke structures $\mathfrak{K}$, all $s \in |\mathfrak{K}|$, and all $\alpha \in \mathcal{L}_n^{CPA}$ we have*

$$\mathfrak{K} \in \mathcal{X} \ \Rightarrow \ \mathfrak{K}^{\alpha,s} \in \mathcal{X}.$$

We get the Hilbert systems for the extended logic by combining the systems for total public announcements and for common knowledge. In addition, we have an axiom for announcement composition as well as an inference rule for common knowledge after a public announcement.

Definition 7.3.14. The deductive systems $\mathsf{K}_n^{CPA_t}$, $\mathsf{K4}_n^{CPA_t}$, $\mathsf{K45}_n^{CPA_t}$, $\mathsf{T}_n^{CPA_t}$, $\mathsf{S4}_n^{CPA_t}$, and $\mathsf{S5}_n^{CPA_t}$ are the systems $\mathsf{K}_n^{PA_t}$, $\mathsf{K4}_n^{PA_t}$, $\mathsf{K45}_n^{PA_t}$, $\mathsf{T}_n^{PA_t}$, $\mathsf{S4}_n^{PA_t}$, and $\mathsf{S5}_n^{PA_t}$ respectively augmented with the co-closure axiom for common knowledge and the *public announcement composition axiom*,

(C) $\quad C_G\alpha \to E_G(\alpha \land C_G\alpha)$,
($\mathsf{PA6}_t$) $\quad \alpha \land [\alpha!]\beta \to ([\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \land [\alpha!]\beta)!]\gamma)$,

as well as the following *induction rules*,

$$(\mathsf{CI})\ \frac{\alpha \to E_G(\alpha \land \beta)}{\alpha \to C_G\beta}, \quad (\mathsf{PAI}_t)\ \frac{\alpha \to [\beta!]E_G\gamma \qquad \alpha \land \beta \to E_G(\beta \to \alpha)}{\alpha \land \beta \to [\beta!]C_G\gamma}.$$

Observe that the rule (PAI_t) is slightly different from our rule in [62], because we also have systems without the knowledge axiom (T). Axiom ($\mathsf{PA6}_t$) is provable in the systems without common knowledge and the systems with relativised common knowledge, see Lemma 7.2.17 and Lemma 7.3.9. However, this is not the case for the systems with common knowledge, because there is no translation available that eliminates the announcement operators.

As in the previous sections, soundness can be proved due to Lemma 7.3.13.

Lemma 7.3.15. *For all* $\alpha \in \mathcal{L}_n^{CPA}$ *we have*

$$\begin{aligned} \mathsf{K}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n \models \alpha, & \mathsf{K4}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n^t \models \alpha,\\ \mathsf{K45}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n^{tu} \models \alpha, & \mathsf{T}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n^r \models \alpha,\\ \mathsf{S4}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n^{rt} \models \alpha, & \mathsf{S5}_n^{CPA_t} \vdash \alpha \ &\Rightarrow\ \mathcal{K}_n^{rtu} \models \alpha. \end{aligned}$$

Proof. By induction on the length of the proof. Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and X^{CPA_t} be the deductive system that corresponds to $\mathcal{X}$. Further, let $\mathfrak{K} \in \mathcal{X}$, $s \in \mathfrak{K}$, and $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ be given. First, for the base case, we first show how to prove that axiom ($\mathsf{PA6}_t$) is valid in $\mathcal{X}$. Assume $\mathfrak{K}, s \models \alpha \land [\alpha!]\beta$. Then we have

$$\begin{aligned} \mathfrak{K}, s \models [\alpha!][\beta!]\gamma \ &\Leftrightarrow\ \mathfrak{K}^{\alpha,s}, s \models [\beta!]\gamma\\ &\Leftrightarrow\ (\mathfrak{K}^{\alpha,s})^{\beta,s}, s \models \gamma\\ &\Leftrightarrow\ (\mathfrak{K}^{\alpha})^{\beta}, s \models \gamma\\ &\Leftrightarrow\ \mathfrak{K}^{\alpha\land[\alpha!]\beta}, s \models \gamma\\ &\Leftrightarrow\ \mathfrak{K}^{\alpha\land[\alpha!]\beta,s}, s \models \gamma\\ &\Leftrightarrow\ \mathfrak{K}, s \models [(\alpha \land [\alpha!]\beta)!]\gamma. \end{aligned}$$

Now, we show how to prove soundness of the rule (PAI_t) in the induction step. Suppose the formula $\alpha \wedge \beta \rightarrow [\beta!]C_G\gamma$ has been derived in X^{CPA_t} by an application of the rule (PAI_t). Then, by induction hypothesis, we have

$$\mathcal{X} \vDash \alpha \rightarrow [\beta!]E_G\gamma, \qquad \mathcal{X} \vDash \alpha \wedge \beta \rightarrow E_G(\beta \rightarrow \alpha).$$

Given $\mathfrak{K}, s \vDash \alpha \wedge \beta$, we now have to show that $\mathfrak{K}, s \vDash [\beta!]C_G\gamma$. By side induction on k, it is not hard to prove

$$\text{for all } k \geq 1, \text{ for all } t \in (R_G^{\beta,s})^k(s),\ \mathfrak{K}^{\beta,s}, t \vDash \gamma \text{ and } \mathfrak{K}, t \vDash \alpha \wedge \beta.$$

The base case and the induction step can be proved exactly the same way using both implications from the induction hypothesis. Observe that we have $R_G^{\beta,s} = R_G^{\beta}$ and $\mathfrak{K}^{\beta,s} = \mathfrak{K}^{\beta}$ because $\mathfrak{K}, s \vDash \beta$. Soundness of the rule (PAN) follows from Lemma 7.3.13. □

Again, we have that the Hilbert systems X^{CPA_t} are an extension of the systems X^{PA_t}, and therefore, we immediately get that Lemma 7.2.6, Lemma 7.2.7, Lemma 7.2.8, Lemma 7.2.9, and Lemma 7.2.19 all still hold for the language $\mathcal{L}_n^{CPA}$ instead of $\mathcal{L}_n^{PA}$. Analogous to Lemma 7.1.36, the following result is useful for further syntactical proofs.

Lemma 7.3.16. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{CPA_t}$, $\mathsf{K4}_n^{CPA_t}$, $\mathsf{K45}_n^{CPA_t}$, $\mathsf{T}_n^{CPA_t}$, $\mathsf{S4}_n^{CPA_t}$, *or* $\mathsf{S5}_n^{CPA_t}$. *Then for all non empty* $G \subseteq \mathcal{A}$ *and all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ *we have that* X *proves*

$$[\alpha!]C_G\beta \leftrightarrow [\alpha!]E_G(\beta \wedge C_G\beta), \qquad \alpha \wedge C_G(\alpha \rightarrow [\alpha!]\beta) \rightarrow [\alpha!]C_G\beta.$$

Proof. The proof of the first assertion is identical to the proof of the first assertion of Lemma 7.1.36. For the second assertion, we have that X proves $\alpha \wedge C_G(\alpha \rightarrow [\alpha!]\beta) \rightarrow [\alpha!]E_G\beta$ by axiom (C) and Lemma 7.2.19, and X proves $\alpha \wedge C_G(\alpha \rightarrow [\alpha!]\beta) \wedge \alpha \rightarrow E_G(\alpha \rightarrow \alpha \wedge C_G(\alpha \rightarrow [\alpha!]\beta))$ by axiom (C) and normal modal logic reasoning. By an application of the rule (PAI_t) we get $\mathsf{X} \vdash \alpha \wedge C_G(\alpha \rightarrow [\alpha!]\beta) \wedge \alpha \rightarrow [\alpha!]C_G\beta$, and we get the desired result by tautological reasoning. □

The second assertion of Lemma 7.3.16 can be seen as one half of a reduction axiom. The missing part of the full reduction axiom, which is given by the formula $\alpha \wedge [\alpha!]C_G\beta \rightarrow C_G(\alpha \rightarrow [\alpha!]\beta)$, is in general not valid and therefore not provable. The following example illustrates this fact.

Example 7.3.17. Let $p, q \in \mathcal{P}$ be given. Furthermore, let the Kripke structure $\mathfrak{K} = (\{s, t, u\}, R_1, \ldots, R_n, V)$ be defined by

$$R_1 = \{s, t, u\}^2, \qquad V(p) = \{s, u\} \qquad V(q) = \{s\},$$

the accessibility relations $R_2, \dots, R_n$ can be arbitrary equivalence relations, and the propositions in $\mathcal{P} \setminus \{p, q\}$ can have an arbitrary valuation. Then we have $\mathfrak{K}, s \vDash p \wedge [p!]C_{\{1\}}q$ but $\mathfrak{K}, s \vDash \neg C_{\{1\}}(p \rightarrow [p!]q)$, because u is accessible from s for $\{1\}$ and $\mathfrak{K}, u \vDash p \wedge \neg[p!]q$. Since $\mathfrak{K} \in \mathcal{K}_n^{rtu}$, we have that the formula $p \wedge [p!]C_{\{1\}}q \rightarrow C_{\{1\}}(p \rightarrow [p!]q)$ is not valid in any of the six classes of Kripke structures.

Due to axiom ($\mathsf{PA6}_t$), we can show that two consecutive announcements can always be reduced to a boolean combination of single announcements.

Lemma 7.3.18. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{CPA_t}$, $\mathsf{K4}_n^{CPA_t}$, $\mathsf{K45}_n^{CPA_t}$, $\mathsf{T}_n^{CPA_t}$, $\mathsf{S4}_n^{CPA_t}$, *or* $\mathsf{S5}_n^{CPA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ *we have that* X *proves*

$$[\alpha!][\beta!]\gamma \leftrightarrow (\neg\alpha \wedge [\beta!]\gamma) \vee (\neg[\alpha!]\beta \wedge [\alpha!]\gamma) \vee (\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\gamma).$$

Proof. We start our proof with the formula $\neg\beta \rightarrow ([\beta!]\gamma \leftrightarrow \gamma)$, which is an instance of (TPA). By the axioms ($\mathsf{PA2}$) and ($\mathsf{PA3}_t$), the rule (PAN), as well as tautological reasoning, we get that X proves

$$\neg[\alpha!]\beta \rightarrow ([\alpha!][\beta!]\gamma \leftrightarrow [\alpha!]\gamma).$$

Together with the following instances of (TPA) and ($\mathsf{PA6}_t$) respectively,

$$\neg\alpha \rightarrow ([\alpha!][\beta!]\gamma \leftrightarrow [\beta!]\gamma),$$
$$\alpha \wedge [\alpha!]\beta \rightarrow ([\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma),$$

we get the desired result by propositional reasoning. □

Although we do not have an equivalence preserving translation from $\mathcal{L}_n^{CPA}$ to $\mathcal{L}_n^{C}$, we can prove Lemma 7.2.15 in the presence of common knowledge operators:

Lemma 7.3.19. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{CPA_t}$, $\mathsf{K4}_n^{CPA_t}$, $\mathsf{K45}_n^{CPA_t}$, $\mathsf{T}_n^{CPA_t}$, $\mathsf{S4}_n^{CPA_t}$, *or* $\mathsf{S5}_n^{CPA_t}$. *Then for all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ *we have*

$$\mathsf{X} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathsf{X} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. By induction on γ. The base case and the cases $\gamma = \neg\delta$, $\gamma = \delta \wedge \varphi$, and $\gamma = K_i\delta$ in the induction step have exactly the same proof as Lemma 7.2.10. We show how to prove the two new cases in the induction step. First, let γ be of the form $C_G\delta$. By Lemma 7.3.16, Lemma 7.2.19 and axiom (TPA), as well as the induction hypothesis and the assumption, we have the following sequence of provable implications,

$$[\alpha!]C_G\delta \to [\alpha!]E_G\delta \to (\neg\alpha \wedge E_G\delta) \vee (\alpha \wedge E_G(\alpha \to [\alpha!]\delta)) \to$$
$$(\neg\beta \wedge E_G\delta) \vee (\beta \wedge E_G(\beta \to [\beta!]\delta)) \to [\beta!]E_G\delta,$$

so we immediately get $\mathsf{X} \vdash [\alpha!]C_G\delta \to [\beta!]E_G\delta$ by tautological reasoning. On the other hand, we have the following chain of provable implications by Lemma 7.3.16, Lemma 7.2.19, and the assumption,

$$[\alpha!]C_G\delta \to [\alpha!]E_GC_G\delta \to \Big(\alpha \to E_G(\alpha \to [\alpha!]C_G\delta)\Big) \to$$
$$\Big(\beta \to E_G(\beta \to [\alpha!]C_G\delta)\Big),$$

and we get $\mathsf{X} \vdash [\alpha!]C_G\delta \wedge \beta \to E_G(\beta \to [\alpha!]C_G\delta)$ by propositional reasoning. We can now apply the rule (PAI_t) and get a proof of $[\alpha!]C_G\delta \wedge \beta \to [\beta!]C_G\delta$ in X. Due to the assumption and axiom (TPA), we can also prove the formula $[\alpha!]C_G\delta \wedge \neg\beta \to [\beta!]C_G\delta$ in X. Therefore, by tautological reasoning, we get $\mathsf{X} \vdash [\alpha!]C_G\delta \to [\beta!]C_G\delta$. The converse direction is similar and the equivalence is proved. Second, let γ be of the form $[\delta!]\varphi$. Then, by Lemma 7.3.18, we get that the formula $[\alpha!][\delta!]\varphi$ is provably equivalent to

$$(\neg\alpha \wedge [\delta!]\varphi) \vee (\neg[\alpha!]\delta \wedge [\alpha!]\varphi) \vee (\alpha \wedge [\alpha!]\delta \wedge [(\alpha \wedge [\alpha!]\delta)!]\varphi).$$

By assumption, the induction hypothesis for both δ and φ, and tautological reasoning, this formula is provably equivalent to

$$(\neg\beta \wedge [\delta!]\varphi) \vee (\neg[\beta!]\delta \wedge [\beta!]\varphi) \vee (\beta \wedge [\beta!]\delta \wedge [(\beta \wedge [\beta!]\delta)!]\varphi),$$

which is provably equivalent to $[\beta!][\delta!]\varphi$ by Lemma 7.3.18. □

As an immediate consequence of Lemma 7.3.19, we get that the Replacement Theorem is still true in the logic of total public announcements and common knowledge, cf. Theorem 7.2.16.

In a next step, we are going to prepare the completeness proof for our six Hilbert systems. The used method is the same as for the logic of common knowledge, where one has to define the canonical structure via maximal consistent sets, cf. [25]. Similar to the logic of common knowledge, we will first define the *closure* of a formula using an extended notion of subformulas.

Definition 7.3.20. For all $\alpha \in \mathcal{L}_n^{CPA}$, the set $\mathrm{sub}^+(\alpha)$ is defined to be the smallest set satisfying the following conditions,

1. $\beta \in \mathrm{sub}^+(\alpha)$ and $\gamma \in \mathrm{sub}(\beta) \;\Rightarrow\; \gamma \in \mathrm{sub}^+(\alpha)$,
2. $C_G\beta \in \mathrm{sub}^+(\alpha) \;\Rightarrow\; E_G\beta \in \mathrm{sub}^+(\alpha)$ and $E_GC_G\beta \in \mathrm{sub}^+(\alpha)$,

3. $[\beta!]\neg\gamma \in \text{sub}^+(\alpha) \;\Rightarrow\; \neg[\beta!]\gamma \in \text{sub}^+(\alpha)$,
4. $[\beta!](\gamma \wedge \delta) \in \text{sub}^+(\alpha) \;\Rightarrow\; [\beta!]\gamma \wedge [\beta!]\delta \in \text{sub}^+(\alpha)$,
5. $[\beta!]K_i\gamma \in \text{sub}^+(\alpha) \;\Rightarrow\; (\neg\beta \wedge K_i\gamma) \vee (\beta \wedge K_i(\beta \to [\beta!]\gamma)) \in \text{sub}^+(\alpha)$,
6. $[\beta!]C_G\gamma \in \text{sub}^+(\alpha) \;\Rightarrow\; [\beta!]E_G\gamma \in \text{sub}^+(\alpha)$, $E_G(\beta \to [\beta!]\gamma) \in \text{sub}^+(\alpha)$, and $E_G(\beta \to [\beta!]C_G\gamma) \in \text{sub}^+(\alpha)$,
7. $[\beta!][\gamma!]\delta \in \text{sub}^+(\alpha) \;\Rightarrow$
   $(\neg\beta \wedge [\gamma!]\delta) \vee (\neg[\beta!]\gamma \wedge [\beta!]\delta) \vee (\beta \wedge [\beta!]\gamma \wedge [(\beta \wedge [\beta!]\gamma)!]\delta) \in \text{sub}^+(\alpha)$,

where $p \in \mathcal{P}$, $i \in \mathcal{A}$, $\emptyset \neq G \subseteq \mathcal{A}$, and $\beta, \gamma, \delta \in \mathcal{L}_n^{CPA}$. For all $\alpha \in \mathcal{L}_n^{CPA}$, the closure of α is now defined by

$$\text{cl}(\alpha) := \text{sub}^+(\alpha) \cup \{\neg\beta : \beta \in \text{sub}^+(\alpha)\}.$$

Note that the closure of a formula α is not closed under complements. But for all $\beta \in \text{cl}(\alpha)$, there is a formula $\sim\beta \in \text{cl}(\alpha)$ that is equivalent to $\neg\beta$. This formula is defined by

$$\sim\beta := \begin{cases} \neg\beta & \text{if } \beta \text{ is not a negation,} \\ \gamma & \text{if } \beta = \neg\gamma \text{ for some } \gamma \in \mathcal{L}_n^{CPA}. \end{cases}$$

The definition of the *rank* of a formula is useful for the inductive proof of the so-called Truth Lemma. We have proved this lemma in [62] without the notion of rank, but we needed a side induction and the proof was more complicated. The Truth Lemma for truthful public announcements and common knowledge has been proved by use of such a rank function in [22].

Definition 7.3.21. The rank of a formula is inductively defined by

$$\begin{aligned} \text{rk}(p) &:= 1, \\ \text{rk}(\neg\alpha) &:= \text{rk}(\alpha) + 1, \\ \text{rk}(\alpha \wedge \beta) &:= \max\{\text{rk}(\alpha), \text{rk}(\beta)\} + 1, \\ \text{rk}(K_i\alpha) &:= \text{rk}(\alpha) + 1, \\ \text{rk}(C_G\alpha) &:= \text{rk}(\alpha) + n + 1, \\ \text{rk}([\alpha!]\beta) &:= (\text{rk}(\alpha) + 8) \cdot \text{rk}(\beta). \end{aligned}$$

We want to mention that the rank of disjunctions, implications, and mutual knowledge formulas is evaluated as follows,

$$\begin{aligned} \text{rk}(\alpha \vee \beta) &= \max\{\text{rk}(\alpha), \text{rk}(\beta)\} + 3, \\ \text{rk}(\alpha \to \beta) &= \max\{\text{rk}(\alpha + 1), \text{rk}(\beta)\} + 3, \end{aligned}$$

$$\mathrm{rk}(E_G\alpha) = \mathrm{rk}(\alpha) + \mathrm{Card}(G).$$

Due to Definition 7.3.21, we have the following properties of our rank function, which are useful for proving the Truth Lemma.

Lemma 7.3.22. *For all $\alpha, \beta, \gamma \in \mathcal{L}_n^{CPA}$ we have*

1. $\beta \in \mathrm{sub}(\alpha) \setminus \{\alpha\} \;\Rightarrow\; \mathrm{rk}(\alpha) > \mathrm{rk}(\beta)$,
2. $\mathrm{rk}([\alpha!]\neg\beta) > \mathrm{rk}(\neg[\alpha!]\beta)$,
3. $\mathrm{rk}([\alpha!](\beta \wedge \gamma)) > \mathrm{rk}([\alpha!]\beta \wedge [\alpha!]\gamma)$,
4. $\mathrm{rk}([\alpha!]K_i\beta) > \mathrm{rk}((\neg\alpha \wedge K_i\beta) \vee (\alpha \wedge K_i(\alpha \to [\alpha!]\beta)))$,
5. $\mathrm{rk}([\alpha!]C_G\beta) > \max\{\mathrm{rk}([\alpha!]E_G\beta), \mathrm{rk}(E_G(\alpha \to [\alpha!]\beta))\}$,
6. $\mathrm{rk}([\alpha!][\beta!]\gamma) > \mathrm{rk}((\neg\alpha \wedge [\beta!]\gamma) \vee (\neg[\alpha!]\beta \wedge [\alpha!]\gamma) \vee$ $(\alpha \wedge [\alpha!]\beta \wedge [(\alpha \wedge [\alpha!]\beta)!]\gamma))$.

Proof. The first assertion can be proved by induction on α. We will show how to prove the fourth and the fifth one. For the fourth assertion, we proceed as follows. By Definition 7.3.21 as well as the definition of the connectives $\vee$ and $\to$, we get

$$\begin{aligned}\mathrm{rk}([\alpha!]K_i\beta) &= \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + 8 + \mathrm{rk}(\alpha)\\ &> \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + 8\\ &= \mathrm{rk}((\neg\alpha \wedge K_i\beta) \vee (\alpha \wedge K_i(\alpha \to [\alpha!]\beta))).\end{aligned}$$

For the fifth assertion, we have to prove two inequalities. First, by Definition 7.3.21, we directly get

$$\begin{aligned}\mathrm{rk}([\alpha!]C_G\beta) &= \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + 8\cdot n + 8 + (n+1)\cdot\mathrm{rk}(\alpha)\\ &> \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + 8\cdot n\\ &\geq \mathrm{rk}([\alpha!]E_G\beta).\end{aligned}$$

Second, by Definition 7.3.21 as well as the definition of the connective $\to$, we get

$$\begin{aligned}\mathrm{rk}([\alpha!]C_G\beta) &= \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + 8\cdot n + 8 + (n+1)\cdot\mathrm{rk}(\alpha)\\ &> \mathrm{rk}(\alpha)\cdot\mathrm{rk}(\beta) + 8\cdot\mathrm{rk}(\beta) + n + 3\\ &\geq \mathrm{rk}(E_G(\alpha \to [\alpha!]\beta)),\end{aligned}$$

and the proof is finished. □

In the completeness proof we will use the fact that the closure of a formula is always finite, which we will now prove.

Lemma 7.3.23. *For all* $\alpha \in \mathcal{L}_n^{CPA}$*, the set* $\mathrm{cl}(\alpha)$ *is finite.*

Proof. We prove by induction on the rank of α that the set $\mathrm{sub}^+(\alpha)$ is finite. We show the case where α is of the form $[\beta!]C_G\gamma$ in the induction step. It is not hard to see that we have

$$\mathrm{sub}^+([\beta!]C_G\gamma) = \mathrm{sub}^+(C_G\gamma) \cup \mathrm{sub}^+([\beta!]E_G\gamma) \cup \mathrm{sub}^+(E_G(\beta \to [\beta!]\gamma)) \cup \mathrm{sub}(E_G(\beta \to [\beta!]C_G\gamma)).$$

Due to Lemma 7.3.22, we can apply the induction hypothesis. Moreover, the set of subformulas is always finite, and we are done. ☐

The canonical structure of a formula α will be very important for the completeness proof. The worlds of this structure are given by all maximal consistent subsets of $\mathrm{cl}(\alpha)$.

Definition 7.3.24. Let X be one of the deductive systems $\mathsf{K}_n^{CPA_t}$, $\mathsf{K4}_n^{CPA_t}$, $\mathsf{K45}_n^{CPA_t}$, $\mathsf{T}_n^{CPA_t}$, $\mathsf{S4}_n^{CPA_t}$, or $\mathsf{S5}_n^{CPA_t}$ and $\alpha \in \mathcal{L}_n^{CPA}$ be given. Then the *canonical structure* $\mathfrak{C} := (\mathcal{S}, \mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{V})$ of α with respect to X is defined by

$$\mathcal{S} := \{U \cap \mathrm{cl}(\alpha) : U \text{ is a maximal } \mathsf{X}\text{-consistent set}\},$$

$$\mathcal{R}_i := \begin{cases} \{(X,Y) \in \mathcal{S}^2 : X/K_i \subseteq Y\} & \text{if } \mathsf{X} = \mathsf{K}_n^{CPA_t}, \\ \{(X,Y) \in \mathcal{S}^2 : X/K_i \subseteq Y \cap Y/K_i\} & \text{if } \mathsf{X} = \mathsf{K4}_n^{CPA_t}, \\ \{(X,Y) \in \mathcal{S}^2 : Y/K_i = X/K_i \subseteq Y\} & \text{if } \mathsf{X} = \mathsf{K45}_n^{CPA_t}, \\ \{(X,Y) \in \mathcal{S}^2 : X/K_i \subseteq Y\} & \text{if } \mathsf{X} = \mathsf{T}_n^{CPA_t}, \\ \{(X,Y) \in \mathcal{S}^2 : X/K_i \subseteq Y/K_i\} & \text{if } \mathsf{X} = \mathsf{S4}_n^{CPA_t}, \\ \{(X,Y) \in \mathcal{S}^2 : X/K_i = Y/K_i\} & \text{if } \mathsf{X} = \mathsf{S5}_n^{CPA_t}, \end{cases}$$

$$\mathcal{V}(p) := \{X \in \mathcal{S} : p \in X\},$$

for all $i \in \mathcal{A}$ and all $p \in \mathcal{P}$, where the set X/K_i is defined by

$$X/K_i := \{\beta : K_i\beta \in X\}$$

for all $X \in \mathcal{S}$ and all $i \in \mathcal{A}$.

Definition 7.3.24 is the same as the definition of the canonical structures for the logic of common knowledge without public announcement operators, cf. [25]. The following lemma states that the accessibility relations of the canonical structures have the intended properties.

Lemma 7.3.25. *Let* X *be one of the deductive systems* $\mathsf{K}_n^{CPA_t}$*,* $\mathsf{K4}_n^{CPA_t}$*,* $\mathsf{K45}_n^{CPA_t}$*,* $\mathsf{T}_n^{CPA_t}$*,* $\mathsf{S4}_n^{CPA_t}$*, or* $\mathsf{S5}_n^{CPA_t}$ *and* $\mathcal{X}$ *be the class of Kripke structures that corresponds to* X*. Then for all* $\alpha \in \mathcal{L}_n^{CPA}$ *we have that the canonical structure* $\mathfrak{C}$ *of* α *with respect to* X *is an element of* $\mathcal{X}$*.*

Proof. We will show how to prove $\mathfrak{C} \in \mathcal{K}_n^r$ for $\mathsf{X} = \mathsf{T}_n^{CPA_t}$, the other assertions are directly implied by the definition of the accessibility relations of the canonical structure. Let $X \in |\mathfrak{C}|$ and $\beta \in X/K_i$ be given, that is we have $K_i\beta \in X$ by definition. Now, let U be a maximal $\mathsf{T}_n^{CPA_t}$-consistent superset of X. By the fourth assertion of Lemma 5.1.8, we have $K_i\beta \rightarrow \beta \in U$, hence we get $\beta \in U$ by the third assertion of Lemma 5.1.8. Since we have $\beta \in \mathrm{cl}(\alpha)$, we get $\beta \in X$ and we are done. ☐

We are now ready to prove the Truth Lemma. It states that every world from the canonical structure of a formula α satisfies exactly those formulas from $\mathrm{cl}(\alpha)$ that are an element of that world.

Lemma 7.3.26 (Truth)**.** *Let* X *be one of the deductive systems* $\mathsf{K}_n^{CPA_t}$*,* $\mathsf{K4}_n^{CPA_t}$*,* $\mathsf{K45}_n^{CPA_t}$*,* $\mathsf{T}_n^{CPA_t}$*,* $\mathsf{S4}_n^{CPA_t}$*, or* $\mathsf{S5}_n^{CPA_t}$ *and* $\alpha \in \mathcal{L}_n^{CPA}$ *be given. In addition, let* $\mathfrak{C} = (\mathcal{S}, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{V})$ *be the canonical structure of* α *with respect to* X*. Then for all* $\beta \in \mathrm{cl}(\alpha)$ *and all* $X \in \mathcal{S}$ *we have*

$$\beta \in X \;\Leftrightarrow\; \mathfrak{C}, X \models \beta.$$

Proof. We show how to prove the claim for $\mathsf{X} = \mathsf{K}_n^{CPA_t}$, and we make an induction on the rank of β. In the base case, if β is a proposition p, then we have

$$p \in X \;\Leftrightarrow\; X \in \mathcal{V}(p) \;\Leftrightarrow\; \mathfrak{C}, X \models \beta$$

by Definition 7.3.24. In the induction step, we proceed as follows. The cases where β does not begin with an announcement operator are worked out in full detail for the logic of common knowledge in [25]. We will show two cases where β begins with an announcement operator. First, let β be of the form $[\gamma!]K_i\delta$. Then, by Lemma 5.1.8, Lemma 7.2.6, and Definition 7.3.20, Lemma 7.3.22 and the induction hypothesis, as well as again Lemma 7.2.6 and soundness, we have

$$\begin{aligned} [\gamma!]K_i\delta \in X \;&\Leftrightarrow\; (\neg\gamma \wedge K_i\delta) \vee (\gamma \wedge K_i(\gamma \rightarrow [\gamma!]\delta)) \in X \\ &\Leftrightarrow\; \mathfrak{C}, X \models (\neg\gamma \wedge K_i\delta) \vee (\gamma \wedge K_i(\gamma \rightarrow [\gamma!]\delta)) \\ &\Leftrightarrow\; \mathfrak{C}, X \models [\gamma!]K_i\delta. \end{aligned}$$

Now, we let β be of the form $[\gamma!]C_G\delta$. We distinguish two cases. If $\gamma \notin X$, then we have $\mathfrak{C}, X \vDash \neg\gamma$ by induction hypothesis, and we get

$$[\gamma!]C_G\delta \in X \;\Leftrightarrow\; C_G\delta \in X \;\Leftrightarrow\; \mathfrak{C}, X \vDash C_G\delta \;\Leftrightarrow\; \mathfrak{C}, X \vDash [\gamma!]C_G\delta$$

by Lemma 5.1.8, axiom (TPA), the induction hypothesis, and soundness. On the other hand, if $\gamma \in X$, then we get $\mathfrak{C}, X \vDash \gamma$ by induction hypothesis, and we give a separate proof for each direction of the claim. For the direction from left to right, let $[\gamma!]C_G\delta \in X$. Since we have $\gamma \in X$ by assumption, we get $E_G(\gamma \to [\gamma!]\delta) \in X$ and $E_G(\gamma \to [\gamma!]C_G\delta) \in X$ by Lemma 5.1.8, Lemma 7.2.19, and Lemma 7.3.16. Therefore, we can prove

$$\text{for all } k \geq 1, \text{for all } Y \in (\mathcal{R}_G^\gamma)^k(X),\; [\gamma!]\delta \in Y \text{ and } [\gamma!]C_G\delta \in Y$$

by side induction on k, only using the induction hypothesis for γ. By induction hypothesis for $[\gamma!]\delta$, we now immediately get

$$\text{for all } Y \in (\mathcal{R}_G^\gamma)^+(X),\; \mathfrak{C}, Y \vDash [\gamma!]\delta.$$

But this is equivalent to $\mathfrak{C}, X \vDash [\gamma!]C_G\delta$ by the definition of the semantics. For the direction from right to left, let $\mathfrak{C}, X \vDash [\gamma!]C_G\delta$. Remember that we have $\gamma \in X$ by assumption and $\mathfrak{C}, X \vDash \gamma$ by induction hypothesis. As a preparatory step, we define the following abbreviations,

$$\psi_Y := \bigwedge_{\xi \in Y} \xi, \qquad \chi_B := \bigvee_{Y \in B} \psi_Y, \qquad \chi_{\overline{B}} := \bigvee_{Y \in \overline{B}} \psi_Y,$$

for all $Y \in \mathcal{S}$, the set $B := \{Z \in \mathcal{S} : \mathfrak{C}, Z \vDash [\gamma!]C_G\delta\}$, and the set $\overline{B} := \mathcal{S} \setminus B$. Our aim is to show that the following formulas are provable in X,

1. $\chi_B \to [\gamma!]E_G\delta$,
2. $\chi_B \wedge \gamma \to E_G(\gamma \to \chi_B)$.

Once we have proved these two formulas, we can apply the rule (PAI_t), and we get that X proves $\chi_B \wedge \gamma \to [\gamma!]C_G\delta$. Since $X \in B$ and $\gamma \in X$ by assumption, we get that X proves $\psi_X \to [\gamma!]C_G\delta$. Therefore, we get $[\gamma!]C_G\delta \in X$ because the set X is X-consistent. So let us prove the above mentioned formulas.

1. Let $Y \in B$. Then we have $\mathfrak{C}, Y \vDash [\gamma!]C_G\delta$, and we get $\mathfrak{C}, Y \vDash [\gamma!]E_G\delta$ by Lemma 7.3.16 and soundness. By Lemma 7.3.22, we can apply the induction hypothesis, and we get $[\gamma!]E_G\delta \in Y$, which immediately implies $\mathsf{X} \vdash \psi_Y \to [\gamma!]E_G\delta$. Because this holds for all $Y \in B$, we get $\mathsf{X} \vdash \chi_B \to [\gamma!]E_G\delta$.

2. Let $Y \in B$. If $\gamma \notin Y$, then we easily get $\mathsf{X} \vdash \psi_Y \wedge \gamma \rightarrow E_G(\gamma \rightarrow \chi_B)$. So assume $\gamma \in Y$, which implies $\mathfrak{C}, Y \vDash \gamma$ by induction hypothesis. Together with the fact that $\mathfrak{C}, Y \vDash [\gamma!]C_G\delta$, it is now easy to show that $\mathfrak{C}, Y \vDash E_G(\gamma \rightarrow [\gamma!]C_G\delta)$ by Lemma 7.2.19, Lemma 7.3.16, and soundness. Now, let $Z \in \overline{B}$. We distinguish two cases. First, if $Y\mathcal{R}_G Z$, then we have $\mathfrak{C}, Z \vDash \gamma \rightarrow [\gamma!]C_G\delta$ by the definition of the semantics, and we get $\mathfrak{C}, Z \nvDash \gamma$ because we have $\mathfrak{C}, Z \nvDash [\gamma!]C_G\delta$. By induction hypothesis, we get $\gamma \notin Z$, which implies $\mathsf{X} \vdash \gamma \rightarrow \neg\psi_Z$. Hence, we get $\mathsf{X} \vdash \psi_Y \rightarrow E_G(\gamma \rightarrow \neg\psi_Z)$ by normal modal logic reasoning. Second, if not $Y\mathcal{R}_G Z$, then for all $i \in G$ we have $Y/K_i \nsubseteq Z$. That is, for all $i \in G$ there is some $\xi \in \mathrm{cl}(\alpha)$ satisfying $K_i\xi \in Y$ and $\xi \notin Z$. Therefore, we get $\mathsf{X} \vdash \xi \rightarrow \neg\psi_Z$, which implies $\mathsf{X} \vdash K_i\xi \rightarrow K_i\neg\psi_Z$. Thus, we have $\mathsf{X} \vdash \psi_Y \rightarrow K_i\neg\psi_Z$ for all $i \in G$, and we get that X proves $\psi_Y \rightarrow E_G(\gamma \rightarrow \neg\psi_Z)$ by normal modal logic reasoning. Because this holds for all $Z \in \overline{B}$, we have $\mathsf{X} \vdash \psi_Y \rightarrow E_G(\gamma \rightarrow \neg\chi_{\overline{B}})$. Similar to the completeness proof for the logic of common knowledge in [25], we can show $\mathsf{X} \vdash \chi_B \vee \chi_{\overline{B}}$, hence we get $\mathsf{X} \vdash \psi_Y \wedge \gamma \rightarrow E_G(\gamma \rightarrow \chi_B)$ by normal modal logic reasoning. Because this holds for all $Y \in B$, we finally get $\mathsf{X} \vdash \chi_B \wedge \gamma \rightarrow E_G(\gamma \rightarrow \chi_B)$.

We want to mention that for $\mathsf{X} \neq \mathsf{K}_n^{CPA_t}$, the only changes in the proof are in the cases $\beta = K_i\gamma$, $\beta = C_G\gamma$, and $\beta = [\gamma!]C_G\delta$. Some of the changes for the first two cases can be found in [25]. The changes for the last case are similar to the ones in the case $\beta = C_G\gamma$. ☐

Observe that Lemma 7.3.26 implies the finite model property and that the satisfiability problem is decidable. In addition, we get that every consistent formula is satisfiable, which implies completeness.

Theorem 7.3.27. *For all $\alpha \in \mathcal{L}_n^{CPA}$ we have*

$$\begin{array}{llll} \mathsf{K}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n \vDash \alpha, & & \mathsf{K4}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{t} \vDash \alpha, \\ \mathsf{K45}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{tu} \vDash \alpha, & & \mathsf{T}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{r} \vDash \alpha, \\ \mathsf{S4}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{rt} \vDash \alpha, & & \mathsf{S5}_n^{CPA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_n^{rtu} \vDash \alpha. \end{array}$$

Proof. Soundness has already been proved in Lemma 7.3.15. For the direction from right to left, let X^{CPA_t} be one of our six Hilbert systems and $\mathcal{X}$ be the class of Kripke structures that corresponds to X^{CPA_t}. We prove the contraposition, so assume $\mathsf{X}^{CPA_t} \nvdash \alpha$. Then we obviously have $\mathsf{X}^{CPA_t} \nvdash \neg\neg\alpha$, that is $\neg\alpha$ is X^{CPA_t}-consistent. By Lemma 5.1.8, we get that $\neg\alpha$ is an element of some maximal X^{CPA_t}-consistent set. Since we have $\neg\alpha \in \mathrm{cl}(\alpha)$, we have $\neg\alpha \in X$ for some world X from the canonical structure $\mathfrak{C}$ of α with respect to X. By Lemma 7.3.26, we get $\mathfrak{C}, X \vDash \neg\alpha$, thus $\neg\alpha$ is satisfiable in $\mathcal{X}$ by Lemma 7.3.25. This yields $\mathcal{X} \nvDash \alpha$, and we are done. □

We end this section by stating some results about announcement resistant formulas. First, we will prove that we have the same conditions for a formula to be announcement resistant as in the logic of truthful public announcements and common knowledge, cf. Lemma 7.1.38.

Lemma 7.3.28. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$. Then we have the same sufficient conditions for a formula $\alpha \in \mathcal{L}_n^{CPA}$ to be announcement resistant in $\mathcal{X}$ as in Lemma 7.2.20 plus the following one,*

5. *$\alpha = C_G\beta$ for some non empty $G \subseteq \mathcal{A}$ and some β announcement resistant in $\mathcal{X}$.*

Proof. Clearly, the four conditions from Lemma 7.2.20 also hold for $\mathcal{L}_n^{CPA}$ formulas. For the fifth assertion, let X^{CPA_t} be the deductive system that corresponds to $\mathcal{X}$, $\beta \in \mathcal{L}_n^{CPA}$ be announcement resistant in $\mathcal{X}$, and γ be arbitrarily given. By the fourth assertion, we get that for all $i \in G$ the formula $K_i\beta$ is announcement resistant in $\mathcal{X}$, and by completeness, we get $\mathsf{X}^{CPA_t} \vdash K_i\beta \to [\gamma!]K_i\beta$ for all $i \in G$. Therefore, we get that X^{CPA_t} proves $E_G\beta \to [\gamma!]E_G\beta$ by Lemma 7.2.6 and tautological reasoning. Together with the formula $C_G\beta \to E_G\beta$, which easily follows from axiom (C), we get that X^{CPA_t} proves $C_G\beta \to [\gamma!]E_G\beta$ by tautological reasoning. On the other hand, it is easy to get a proof of $C_G\beta \wedge \gamma \to E_G(\gamma \to C_G\beta)$ by axiom (C) and normal modal logic reasoning. Now, we can apply the rule (PAI_t) in order to get a proof of $C_G\beta \wedge \gamma \to [\gamma!]C_G\beta$ in X^{CPA_t}. Together with the formula $C_G\beta \wedge \neg\gamma \to [\gamma!]C_G\beta$, which is directly implied by axiom (TPA), we get that X^{CPA_t} proves $C_G\beta \to [\gamma!]C_G\beta$. By soundness, we get the desired result. □

Again, we will prove that every true announcement resistant formula is commonly known by the agents after one public announcement. The following theorem is an equivalent reformulation of Theorem 7.2.22 with common knowledge operators.

Theorem 7.3.29. *Let $\mathcal{X}$ be one of the classes $\mathcal{K}_n$, $\mathcal{K}_n^t$, $\mathcal{K}_n^{tu}$, $\mathcal{K}_n^r$, $\mathcal{K}_n^{rt}$, or $\mathcal{K}_n^{rtu}$ and $\alpha \in \mathcal{L}_n^{CPA}$ be given. If α is announcement resistant in $\mathcal{X}$, then for all non empty $G \subseteq \mathcal{A}$ we have*

$$\mathcal{X} \vDash \alpha \to [\alpha!]C_G\alpha.$$

Proof. Let X^{CPA_t} be the deductive system that corresponds to $\mathcal{X}$. By assumption and completeness, we have that the formula $\alpha \to [\alpha!]\alpha$ is provable in X^{CPA_t}, and we get that X^{CPA_t} proves $\alpha \to E_G(\alpha \to [\alpha!]\alpha)$ by normal modal logic reasoning. By Lemma 7.2.19 and tautological reasoning, we now get $\mathsf{X}^{CPA_t} \vdash \alpha \to [\alpha!]E_G\alpha$. Together with the formula $\alpha \wedge \alpha \to E_G(\alpha \to \alpha)$, which is easily provable in X^{CPA_t}, we can apply the rule (PAI_t) and get that X^{CPA_t} proves $\alpha \wedge \alpha \to [\alpha!]C_G\alpha$. But this formula is provably equivalent to $\alpha \to [\alpha!]C_G\alpha$, and due to soundness, we are done. □

Chapter 8

Expansion in bimodal systems

We end this thesis with a short chapter that illustrates how the model transformation for total public announcements from Section 7.2 can be used for public announcements in the logic of knowledge and belief. The idea is that true announced formulas will be learnt by the agents on the level of both knowledge and belief, whereas an announcement with a false formula will only affect the beliefs of the agents. In Section 8.1 we will introduce a new model transformation that operates on Kripke structures with accessibility relations for both knowledge and belief. We will provide a Hilbert system extending $\mathbb{KB}\mathsf{5I}_n$, which we have introduced in Section 5.3. That is, we will combine the approach for the logic of group announcements for trustful agents from Section 6.1 with the one for the logic of total public announcements from Section 7.2.

We will not provide deductive systems extending $\mathbb{KB}\mathsf{DI}_n$ and $\mathbb{KB}\mathsf{5D}_n$ for the following reasons. First, in the Kripke structures corresponding to the system $\mathbb{KB}\mathsf{DI}_n$, the accessibility relations for belief are serial and transitive, but not necessarily Euclidean. We do not have a model transformation which preserves this kind of structures. Second, we think that the total public announcements from Section 7.2 and the announcements for sceptical agents from Section 6.2 are incompatible. It could happen that the announcement with a true formula affects the agents' knowledge but not their belief. Although this would be technically possible in a system extending $\mathbb{KB}\mathsf{5D}_n$, we believe that such a behaviour is too unnatural.

8.1 Trustful behaviour

In this section, we will provide a system for public announcements in the logic of knowledge and belief. Like in Section 6.1 and Section 7.2, the public announcements we are going to define will be total. On the level of belief, the public announcements will have the same effect as the group announcements for trustful agents from Section 6.1. The only difference is that the formulas will always be announced to every agent. On the knowledge level, the public announcements will directly influence the agents' knowledge only if the new information is true. In this case, it has the same effect as the total public announcements from Section 7.2.

First, we are going to recall the language $\mathcal{L}_n^{BPA}$ for the logic of knowledge, belief, and public announcements from Definition 5.4.1, which is defined by the following grammar ($p \in \mathcal{P}$, $i \in \mathcal{A}$),

$$\alpha \ ::= \ p \ | \ \neg\alpha \ | \ (\alpha \wedge \alpha) \ | \ K_i\alpha \ | \ B_i\alpha \ | \ [\alpha!]\alpha.$$

For the semantics, we just add the defining clause for public announcement formulas to Definition 5.3.2.

Definition 8.1.1. Let $\mathfrak{K} = (S, R_1, \dots, R_n, Q_1, \dots, Q_n, V)$ and $s \in S$ be given. The notion of an $\mathcal{L}_n^{BPA}$ formula of the form $[\alpha!]\beta$ being *satisfied* in the pointed structure $\mathfrak{K}, s$ is defined by

$$\mathfrak{K}, s \models [\alpha!]\beta \ :\Leftrightarrow \ \mathfrak{K}^{\alpha,s} \models \beta,$$

where the Kripke structure $\mathfrak{K}^{\alpha,s} = (S^{\alpha,s}, R_1^{\alpha,s}, \dots, R_n^{\alpha,s}, Q_1^{\alpha,s}, \dots, Q_n^{\alpha,s}, V^{\alpha,s})$ is simultaneously defined as follows,

$$\mathfrak{K}^{\alpha,s} := \begin{cases} \mathfrak{K}^{\alpha} & \text{if } \mathfrak{K}, s \models \alpha, \\ \mathfrak{K}^{\alpha,\mathcal{A}} & \text{otherwise.} \end{cases}$$

The corresponding structures $\mathfrak{K}^{\alpha} = (S^{\alpha}, R_1^{\alpha}, \dots, R_n^{\alpha}, Q_1^{\alpha}, \dots, Q_n^{\alpha}, V^{\alpha})$ and $\mathfrak{K}^{\alpha,\mathcal{A}} = (S, R_1, \dots, R_n, Q_1^{\alpha,\mathcal{A}}, \dots, Q_n^{\alpha,\mathcal{A}}, V)$ are defined by

$$\begin{aligned} S^{\alpha} &:= S \cap \|\alpha\|_{\mathfrak{K}}, \\ R_i^{\alpha} &:= R_i \cap \|\alpha\|_{\mathfrak{K}}^2, \\ Q_i^{\alpha} &:= Q_i \cap \|\alpha\|_{\mathfrak{K}}^2, \qquad\qquad Q_i^{\alpha,\mathcal{A}} := Q_i \cap (|\mathfrak{K}| \times \|\alpha\|_{\mathfrak{K}}), \\ V^{\alpha}(p) &:= V(p) \cap \|\alpha\|_{\mathfrak{K}}. \end{aligned}$$

for all $i \in \mathcal{A}$ and all $p \in \mathcal{P}$.

Observe that the model transformations $\mathfrak{K}^{\alpha}$ and $\mathfrak{K}^{\alpha,\mathcal{A}}$ from Definition 8.1.1 are the canonical extensions of the transformations from Definition 5.4.2 and Definition 6.1.1 respectively.

We want to mention that a false announcement does not directly affect the agents' knowledge, but it can change the knowledge about knowledge or belief. The following example illustrates how this can happen.

Example 8.1.2. Let $\mathfrak{K} = (\{s,t\}, R_1, \dots, R_n, Q_1, \dots, Q_n, V)$ be defined by

$$R_1 = \{s,t\}^2, \qquad Q_1 = \{s,t\}^2, \qquad V : p \mapsto \{t\},$$

the other accessibility relations can be arbitrarily defined. For all $p \in \mathcal{P}$ we have $\mathfrak{K}, s \vDash \neg K_1 p \wedge \neg B_1 p$ and $\mathfrak{K}, s \vDash \neg K_1 B_1 p \wedge \neg K_1 K_1 B_1 p$. Since p is false at world s, agent 1 cannot learn p on the knowledge level. We get $\mathfrak{K}, s \vDash [p!](\neg K_1 p \wedge B_1 p)$ and $\mathfrak{K}, s \vDash [p!](K_1 B_1 p \wedge K_1 K_1 B_1 p)$.

It turns out that the model transformation from Definition 8.1.1 preserves most of the properties of the accessibility relations that we have introduced in Section 5.1 and Section 5.3.

Lemma 8.1.3. *Let $\mathfrak{K} = (S, R_1, \dots, R_n, Q_1, \dots, Q_n, V)$ be an arbitrarily defined Kripke structure. Then for all $s \in S$ and all $\alpha \in \mathcal{L}_n^{BPA}$ we have*

$$\mathfrak{K} \in \mathcal{K}_{2n}^{rtu,tu,ecdh} \;\Rightarrow\; \mathfrak{K}^{\alpha,s} \in \mathcal{K}_{2n}^{rtu,tu,ecdh}.$$

Proof. The nine properties are all independently preserved and the proof is straightforward. We show how to prove that property c, that is transitivity of Q_i over (R_i, Q_i) is preserved. Let $u, v, w \in S^{\alpha,s}$, $uR_i^{\alpha,s}v$, and $vQ_i^{\alpha,s}w$. Then we have uR_iv and vQ_iw by definition, and we get uQ_iw by assumption. Now, if $\mathfrak{K}, s \vDash \alpha$, then we have $u, v, w \in \|\alpha\|_{\mathfrak{K}}$ and we immediately get $uQ_i^{\alpha}w$. On the other hand, if $\mathfrak{K}, s \vDash \neg\alpha$, then we have $\mathfrak{K}, w \vDash \alpha$ and we directly get $uQ_i^{\alpha,\mathcal{A}}w$. So in both cases we have $uQ_i^{\alpha,s}w$, and we are done. ☐

The following example shows that seriality of Q_i and transitivity of Q_i over (Q_i, R_i) are in general not preserved by the new model transformation. This is the reason why we have defined the system $\mathbb{KB}5\mathsf{I}_n$ without the positive certainty axiom (G) in Definition 5.3.6.

Example 8.1.4. Let $\mathfrak{K} = (\{s,t\}, R_1, \dots R_n, Q_1, \dots, Q_n, V)$ be defined by

$$R_i = \{s,t\}^2, \qquad Q_i = \{s,t\}^2, \qquad V : p \mapsto \{t\}$$

for all $i \in \mathcal{A}$. Then we obviously have $\mathfrak{K} \in \mathcal{K}_{2n}^{rtu,stu,ecdgh}$. For all $p \in \mathcal{P}$ and all $i \in \mathcal{A}$ we have $R_i^{p,s} = R_i$ and $Q_i^{p,s} = \{(s,t),(t,t)\}$. Hence, $Q_i^{p,s}$ is not transitive over $(Q_i^{p,s}, R_i^{p,s})$, because we have $sQ_i^{p,s}tR_i^{p,s}s$ but not $sQ_i^{p,s}s$. On the other hand, if we change the valuation V in $\mathfrak{K}$ to $V : p \mapsto \emptyset$, then we have $Q_i^{p,s} = \emptyset$ and we lose seriality of Q_i.

In order to give an axiomatisation, it is useful to define an abbreviation for announcements that only affect the belief of the agents.

Definition 8.1.5. For all $\alpha, \beta \in \mathcal{L}_n^{BPA}$, the formula $[\![\alpha!]\!]\beta$ is defined by induction on β as follows,

$$
\begin{aligned}
[\![\alpha!]\!]p &:= p, \\
[\![\alpha!]\!]\neg\gamma &:= \neg[\![\alpha!]\!]\gamma, \\
[\![\alpha!]\!](\gamma \wedge \delta) &:= [\![\alpha!]\!]\gamma \wedge [\![\alpha!]\!]\delta, \\
[\![\alpha!]\!]K_i\gamma &:= K_i[\![\alpha!]\!]\gamma, \\
[\![\alpha!]\!]B_i\gamma &:= B_i(\alpha \rightarrow [\![\alpha!]\!]\gamma), \\
[\![\alpha!]\!][\gamma!]\delta &:= [\alpha!][\gamma!]\delta.
\end{aligned}
$$

It is not hard to show that for all $\alpha, \beta \in \mathcal{L}_n^B$ we have $[\![\alpha!]\!]\beta \in \mathcal{L}_n^B$, which can be proved by induction on β. Moreover, we can show that Definition 8.1.5 fulfills our requirements about the abbreviation $[\![\alpha!]\!]\beta$. The following lemma will be useful for proving soundness of the system we are going to define.

Lemma 8.1.6. *Let $\mathfrak{K} = (S, R_1, \ldots, R_n, Q_1, \ldots, Q_n, V)$ and $s \in S$ be given. Then for all $\alpha \in \mathcal{L}_n^{BPA}$ and all $\varphi \in \mathcal{L}_n^B$ we have*

$$\mathfrak{K}, s \vDash [\![\alpha!]\!]\varphi \;\Leftrightarrow\; \mathfrak{K}^{\alpha,\mathcal{A}}, s \vDash \varphi.$$

Proof. By induction on φ. We show how to prove two cases in the induction step. First, if φ is of the form $K_i\psi$, then we have

$$
\begin{aligned}
\mathfrak{K}, s \vDash [\![\alpha!]\!]K_i\psi &\Leftrightarrow \mathfrak{K}, s \vDash K_i[\![\alpha!]\!]\psi \\
&\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}, t \vDash [\![\alpha!]\!]\psi \\
&\Leftrightarrow \text{for all } t \in R_i(s),\ \mathfrak{K}^{\alpha,\mathcal{A}}, t \vDash \psi
\end{aligned}
$$

by induction hypothesis, which is equivalent to $\mathfrak{K}^{\alpha,\mathcal{A}}, s \vDash K_i\psi$. Second, if φ is of the form $B_i\psi$, then we have

$$
\begin{aligned}
\mathfrak{K}, s \vDash [\![\alpha!]\!]B_i\psi &\Leftrightarrow \mathfrak{K}, s \vDash B_i(\alpha \rightarrow [\![\alpha!]\!]\psi) \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \rightarrow [\![\alpha!]\!]\psi \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}, t \vDash [\![\alpha!]\!]\psi \\
&\Leftrightarrow \text{for all } t \in Q_i^{\alpha,\mathcal{A}}(s),\ \mathfrak{K}, t \vDash [\![\alpha!]\!]\psi \\
&\Leftrightarrow \text{for all } t \in Q_i^{\alpha,\mathcal{A}}(s),\ \mathfrak{K}^{\alpha,\mathcal{A}}, t \vDash \psi
\end{aligned}
$$

by induction hypothesis, which is equivalent to $\mathfrak{K}^{\alpha,\mathcal{A}}, s \vDash B_i\psi$. □

We will now define our Hilbert system for the logic of public announcements, knowledge, and belief, which extends $\mathbb{KB}\mathsf{5I}_n$.

Definition 8.1.7. The deductive system $\mathbb{KB}5\mathsf{I}_n^{PA_t}$ is the system $\mathbb{KB}5\mathsf{I}_n$ augmented with the following *public announcement axioms*,

(TPA′) $\neg\alpha \to ([\alpha!]\varphi \leftrightarrow [\![\alpha!]\!]\varphi) \quad (\varphi \in \mathcal{L}_n^B)$,
(PA1$_t$) $[\alpha!]p \leftrightarrow p$,
(PA2) $[\alpha!](\beta \to \gamma) \to ([\alpha!]\beta \to [\alpha!]\gamma)$,
(PA3$_t$) $[\alpha!]\neg\beta \leftrightarrow \neg[\alpha!]\beta$,
(PA4$_t$) $\alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i(\alpha \to [\alpha!]\beta))$,
(PA7$_t$) $\alpha \to ([\alpha!]B_i\beta \leftrightarrow B_i(\alpha \to [\alpha!]\beta))$,

as well as the *public announcement necessitation rule*,

$$\text{(PAN)}\ \frac{\alpha}{[\beta!]\alpha}\,.$$

Due to Lemma 8.1.3 and Lemma 8.1.6, it is now easy to prove soundness of the system $\mathbb{KB}5\mathsf{I}_n^{PA_t}$.

Lemma 8.1.8. *For all* $\alpha \in \mathcal{L}_n^{BPA}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \;\Rightarrow\; \mathcal{K}_{2n}^{rtu,tu,ecdh} \vDash \alpha.$$

Proof. By induction on the length of the proof. For the base case, let the Kripke structure $\mathfrak{K} = (S, R_1, \ldots, R_n, Q_1, \ldots, Q_n, V)$ and the world $s \in S$ be given. Furthermore, let $\alpha, \beta \in \mathcal{L}_n^{BPA}$ and $\varphi \in \mathcal{L}_n^B$. First, we show that axiom (TPA′) is valid, so assume $\mathfrak{K}, s \vDash \neg\alpha$. Then we have

$$\mathfrak{K}, s \vDash [\alpha!]\varphi \;\Leftrightarrow\; \mathfrak{K}^{\alpha,s}, s \vDash \varphi \;\Leftrightarrow\; \mathfrak{K}^{\alpha,\mathcal{A}}, s \vDash \varphi \;\Leftrightarrow\; \mathfrak{K}, s \vDash [\![\alpha!]\!]\varphi$$

by Lemma 8.1.6. Second, we prove that axiom (PA7$_t$) is valid, so assume $\mathfrak{K}, s \vDash \alpha$. Then we have

$$\begin{aligned}
\mathfrak{K}, s \vDash [\alpha!]B_i\beta &\Leftrightarrow \mathfrak{K}^{\alpha,s}, s \vDash B_i\beta \\
&\Leftrightarrow \mathfrak{K}^{\alpha}, s \vDash B_i\beta \\
&\Leftrightarrow \text{for all } t \in Q_i^{\alpha}(s),\ \mathfrak{K}^{\alpha}, t \vDash \beta \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}^{\alpha}, t \vDash \beta \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}^{\alpha,t}, t \vDash \beta \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \text{ implies } \mathfrak{K}, t \vDash [\alpha!]\beta \\
&\Leftrightarrow \text{for all } t \in Q_i(s),\ \mathfrak{K}, t \vDash \alpha \to [\alpha!]\beta \\
&\Leftrightarrow \mathfrak{K}, s \vDash B_i(\alpha \to [\alpha!]\beta).
\end{aligned}$$

In the induction step, soundness of the rule (PAN) can be proved using Lemma 8.1.3. □

Like in Section 6.1 and Section 7.2, the public announcements do not add expressive strength to the logic of knowledge and belief. But some of the *reduction axioms* are restricted to $\mathcal{L}_n^B$ formulas.

Lemma 8.1.9. *For all* $\alpha \in \mathcal{L}_n^{BPA}$*, all* $\varphi \in \mathcal{L}_n^B$*, and all* $i \in \mathcal{A}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]K_i\varphi \leftrightarrow (\neg\alpha \wedge [\![\alpha!]\!]K_i\varphi) \vee (\alpha \wedge K_i(\alpha \to [\alpha!]\varphi)),$$
$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]B_i\varphi \leftrightarrow (\neg\alpha \wedge [\![\alpha!]\!]B_i\varphi) \vee (\alpha \wedge B_i(\alpha \to [\alpha!]\varphi)).$$

Proof. Both assertions can be proved by axiom (TPA'), tautological reasoning, as well as axiom ($\mathsf{PA4}_t$) and ($\mathsf{PA7}_t$) respectively. □

Observe that the reduction axioms from Lemma 8.1.9 show how an announced formula affects the knowledge and belief of the agents. The only difference occurs if the announced formula is false, because of the difference between $[\![\alpha!]\!]K_i\beta$ and $[\![\alpha!]\!]B_i\beta$ in Definition 8.1.5. The following *reduction axiom* holds for all formulas.

Lemma 8.1.10. *For all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{BPA}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!](\beta \wedge \gamma) \leftrightarrow [\alpha!]\beta \wedge [\alpha!]\gamma.$$

Proof. This assertion can be proved by axiom ($\mathsf{PA2}$) and the rule (PAN) exactly the same way as in the other systems for announcement logics. □

The following result is an immediate consequence of Lemma 8.1.10 and axiom ($\mathsf{PA3}_t$).

Corollary 8.1.11. *For all* $\alpha, \beta, \gamma \in \mathcal{L}_n^{BPA}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!](\beta \vee \gamma) \leftrightarrow [\alpha!]\beta \vee [\alpha!]\gamma,$$
$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash ([\alpha!]\beta \to [\alpha!]\gamma) \to [\alpha!](\beta \to \gamma).$$

As we have already mentioned, a false announcement will never change the agents' knowledge about propositional facts.

Lemma 8.1.12. *For all* $\alpha \in \mathcal{L}_n^{BPA}$*, all* $\beta \in \mathcal{L}_0$*, and all* $i \in \mathcal{A}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \neg\alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i\beta).$$

Proof. It is not hard to prove by induction on β that $[\![\alpha!]\!]\beta = \beta$. Therefore, we directly get $[\![\alpha!]\!]K_i\beta = K_i[\![\alpha!]\!]\beta = K_i\beta$. This implies that the formula $\neg\alpha \to ([\alpha!]K_i\beta \leftrightarrow K_i\beta)$ is an instance of (TPA'), and we are done. □

Again, we have that the public announcements do not affect the truth value of propositional facts. That is, we have change of knowledge and belief in a *static world*.

Lemma 8.1.13. *For all* $\alpha \in \mathcal{L}_n^{BPA}$ *and all* $\beta \in \mathcal{L}_0$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]\beta \leftrightarrow \beta.$$

Proof. By induction on β using the axioms ($\mathsf{PA1}_t$) and ($\mathsf{PA3}_t$), as well as Lemma 8.1.10. □

Due to Lemma 8.1.8 and Lemma 8.1.13, it is now easy to prove that the public announcements in the logic of knowledge and belief satisfy all of the properties from Definition 5.4.3.

Lemma 8.1.14. *The total public announcements are fact preserving, adequate, total, self-dual, and normal with respect to* $\mathcal{K}_{2n}^{rtu,tu,ecdh}$.

Proof. Fact preservation follows from Lemma 8.1.13 and soundness. Adequacy and totality are an immediate consequence of fact preservation. Self-duality is given by axiom ($\mathsf{PA3}_t$) and soundness. Due to axiom ($\mathsf{PA2}$), the rule (PAN), and soundness, we immediately get normality. □

In order to define a translation from $\mathcal{L}_n^{BPA}$ to $\mathcal{L}_n^{B}$, we need a restricted version of syntax independence for public announcements.

Lemma 8.1.15. *For all* $\alpha, \beta \in \mathcal{L}_n^{BPA}$ *and all* $\varphi \in \mathcal{L}_n^{B}$ *we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\![\alpha!]\!]\varphi \leftrightarrow [\![\beta!]\!]\varphi,$$
$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]\varphi \leftrightarrow [\beta!]\varphi.$$

Proof. Both assertions can be proved by induction on φ. The proof of the first assertion is straightforward. For the second assertion, we show how to prove the case $\varphi = B_i\psi$ in the induction step. By Lemma 8.1.9, we have that the formula $[\alpha!]B_i\psi$ is provably equivalent to

$$(\neg\alpha \wedge [\![\alpha!]\!]B_i\psi) \vee (\alpha \wedge B_i(\alpha \rightarrow [\alpha!]\psi)). \tag{8.1}$$

By the first assertion, we have $\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\![\alpha!]\!]B_i\psi \leftrightarrow [\![\beta!]\!]B_i\psi$, and we have $\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]\psi \leftrightarrow [\beta!]\psi$ by induction hypothesis. Therefore, the formula (8.1) is provably equivalent to $(\neg\beta \wedge [\![\beta!]\!]B_i\psi) \vee (\beta \wedge B_i(\beta \rightarrow [\beta!]\psi))$ by normal modal logic reasoning. But this is provably equivalent to $[\beta!]B_i\psi$ by again applying Lemma 8.1.9. □

We are going to define the translation from $\mathcal{L}_n^{BPA}$ to $\mathcal{L}_n^B$ in two stages. Due to Lemma 8.1.9 and Lemma 8.1.10, it is immediate how to define the auxiliary function h, which is the first stage of the translation.

Definition 8.1.16. The function $h\colon \{[\alpha!]\beta : \alpha, \beta \in \mathcal{L}_n^{BPA}\} \to \mathcal{L}_n^{BPA}$ is inductively defined by

$$\begin{aligned} h([\alpha!]p) &:= p, \\ h([\alpha!]\neg\beta) &:= \neg h([\alpha!]\beta), \\ h([\alpha!](\beta \wedge \gamma)) &:= h([\alpha!]\beta) \wedge h([\alpha!]\gamma), \\ h([\alpha!]K_i\beta) &:= (\neg\alpha \wedge [\![\alpha!]\!]K_i\beta) \vee (\alpha \wedge K_i(\alpha \to h([\alpha!]\beta))), \\ h([\alpha!]B_i\beta) &:= (\neg\alpha \wedge [\![\alpha!]\!]B_i\beta) \vee (\alpha \wedge B_i(\alpha \to h([\alpha!]\beta))), \\ h([\alpha!][\beta!]\gamma) &:= [\alpha!][\beta!]\gamma. \end{aligned}$$

Remember that for all $\alpha, \beta \in \mathcal{L}_n^B$ we have $[\![\alpha!]\!]\beta \in \mathcal{L}_n^B$. Therefore, we get $h([\alpha!]\beta) \in \mathcal{L}_n^B$, whenever $\alpha, \beta \in \mathcal{L}_n^B$. This can be proved by induction on β. We can show that the function h is equivalence preserving on a subset of its domain.

Lemma 8.1.17. *For all $\alpha \in \mathcal{L}_n^{BPA}$ and all $\varphi \in \mathcal{L}_n^B$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash h([\alpha!]\varphi) \leftrightarrow [\alpha!]\varphi.$$

Proof. By induction on φ. We show how to prove the case $\varphi = B_i\psi$ in the induction step. By induction hypothesis, we have that $\mathbb{KB}5\mathsf{I}_n^{PA_t}$ proves $h([\alpha!]\psi) \leftrightarrow [\alpha!]\psi$. By normal modal logic reasoning, we immediately get that the formula $(\neg\alpha \wedge [\![\alpha!]\!]B_i\psi) \vee (\alpha \wedge B_i(\alpha \to h([\alpha!]\psi)))$ is provably equivalent to $(\neg\alpha \wedge [\![\alpha!]\!]B_i\psi) \vee (\alpha \wedge B_i(\alpha \to [\alpha!]\psi))$. The former formula is defined to be $h([\alpha!]B_i\psi)$, the latter is provably equivalent to $[\alpha!]B_i\psi$ by Lemma 8.1.9. ☐

For the translation that eliminates the announcement operator, we follow the ideas from Definition 6.1.12 and Definition 7.2.13.

Definition 8.1.18. The function $f\colon \mathcal{L}_n^{BPA} \to \mathcal{L}_n^{BPA}$ is inductively defined by

$$\begin{aligned} f(p) &:= p, \\ f(\neg\alpha) &:= \neg f(\alpha), \\ f(\alpha \wedge \beta) &:= f(\alpha) \wedge f(\beta), \\ f(K_i\alpha) &:= K_i f(\alpha), \\ f(B_i\alpha) &:= B_i f(\alpha), \\ f([\alpha!]\beta) &:= h([f(\alpha)!]f(\beta)). \end{aligned}$$

Again, it is not hard to see that for all $\alpha \in \mathcal{L}_n^{BPA}$ we have that the formula $f(\alpha)$ is an element of $\mathcal{L}_n^B$. Moreover, the translation f is equivalence preserving in the following sense.

Lemma 8.1.19. *For all $\alpha \in \mathcal{L}_n^{BPA}$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash f(\alpha) \leftrightarrow \alpha.$$

Proof. By induction on α. We show how to prove the case $\alpha = [\beta!]\gamma$ in the induction step. The formula $f([\beta!]\gamma)$ is defined to be the $h([f(\beta)!]f(\gamma))$, which is provably equivalent to $[f(\beta)!]f(\gamma)$ by Lemma 8.1.17 because $f(\gamma)$ is an element of $\mathcal{L}_n^B$. By Lemma 8.1.15 and the induction hypothesis for β, this formula is now provably equivalent to $[\beta!]f(\gamma)$, again because $f(\gamma) \in \mathcal{L}_n^B$. Finally, this formula is provably equivalent to $[\beta!]\gamma$ by axiom (PA2), the rule (PAN), as well as the induction hypothesis for γ. □

We can now prove syntax independence for the public announcement operator due to Lemma 8.1.19.

Lemma 8.1.20. *For all $\alpha, \beta, \gamma \in \mathcal{L}_n^{BPA}$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash [\alpha!]\gamma \leftrightarrow [\beta!]\gamma.$$

Proof. The proof is identical to the proof of Lemma 7.2.15, but we apply Lemma 8.1.15 and Lemma 8.1.19 instead of Lemma 7.2.10 and Lemma 7.2.14 respectively. □

Lemma 8.1.20 now implies the *Replacement Theorem* for the logic of knowledge, belief, and public announcements.

Theorem 8.1.21 (Replacement)**.** *For all $\alpha, \beta, \gamma \in \mathcal{L}_n^{BPA}$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \leftrightarrow \beta \;\Rightarrow\; \mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \gamma \leftrightarrow \gamma[\alpha/\beta].$$

Proof. The proof is by induction on γ and is similar to the proof of Theorem 7.1.13. In the induction step, we can apply Lemma 8.1.20 instead of Lemma 7.1.12. □

Due to Lemma 8.1.19, we can prove the same result concerning consecutive announcements as in Section 7.2.

Lemma 8.1.22. *For all $\alpha, \beta, \gamma \in \mathcal{L}_n^{BPA}$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \wedge [\alpha!]\beta \to ([\alpha!][\beta!]\gamma \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\gamma).$$

Proof. Like in the proof of Lemma 7.2.17, we will first prove a restricted version where γ is a formula in $\mathcal{L}_n^B$. This can be done by induction on γ, and we show the case in the induction step where γ is of the form $B_i\psi$. By induction hypothesis, we have that $\mathbb{KB}5\mathsf{I}_n^{PA_t}$ proves

$$\alpha \wedge [\alpha!]\beta \rightarrow ([\alpha!][\beta!]\psi \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]\psi),$$

and we get a proof of

$$B_i(\alpha \wedge [\alpha!]\beta \rightarrow [\alpha!][\beta!]\psi) \leftrightarrow B_i(\alpha \wedge [\alpha!]\beta \rightarrow [(\alpha \wedge [\alpha!]\beta)!]\psi) \tag{8.2}$$

by normal modal logic reasoning. Furthermore, by the axioms (PA2) and (PA7$_t$), as well as the rule (PAN), we get that $\mathbb{KB}5\mathsf{I}_n^{PA_t}$ proves

$$[\alpha!]\beta \rightarrow ([\alpha!][\beta!]B_i\psi \leftrightarrow [\alpha!]B_i(\beta \rightarrow [\beta!]\psi)),$$
$$\alpha \rightarrow ([\alpha!]B_i(\beta \rightarrow [\beta!]\psi) \leftrightarrow B_i(\alpha \rightarrow [\alpha!](\beta \rightarrow [\beta!]\psi))),$$

and we get a proof of

$$\alpha \wedge [\alpha!]\beta \rightarrow ([\alpha!][\beta!]B_i\psi \leftrightarrow B_i(\alpha \wedge [\alpha!]\beta \rightarrow [\alpha!][\beta!]\psi)) \tag{8.3}$$

by Corollary 8.1.11 and normal modal logic reasoning. On the other hand, we have the following instance of (PA7$_t$),

$$\alpha \wedge [\alpha!]\beta \rightarrow ([(\alpha \wedge [\alpha!]\beta)!]B_i\psi \leftrightarrow B_i(\alpha \wedge [\alpha!]\beta \rightarrow [(\alpha \wedge [\alpha!]\beta)!]\psi)). \tag{8.4}$$

Finally, by the provability of the formulas (8.2), (8.3), and (8.4), we get that $\mathbb{KB}5\mathsf{I}_n^{PA_t}$ proves

$$\alpha \wedge [\alpha!]\beta \rightarrow ([\alpha!][\beta!]B_i\psi \leftrightarrow [(\alpha \wedge [\alpha!]\beta)!]B_i\psi)$$

by tautological reasoning. We can now prove the assertion for arbitrary $\gamma \in \mathcal{L}_n^{BPA}$ by using the restricted result and Lemma 8.1.19. □

As another consequence of Lemma 8.1.19, we get a short completeness proof via the completeness of $\mathbb{KB}5\mathsf{I}_n$.

Theorem 8.1.23. *For all $\alpha \in \mathcal{L}_n^{BPA}$ we have*

$$\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha \;\Leftrightarrow\; \mathcal{K}_{2n}^{rtu,tu,ecdh} \models \alpha.$$

Proof. Soundness has already been proved in Lemma 8.1.8. For the direction from right to left, we assume $\mathcal{K}_{2n}^{rtu,tu,ecdh} \models \alpha$. By Lemma 8.1.19 and soundness, we get $\mathcal{K}_{2n}^{rtu,tu,ecdh} \models f(\alpha)$. Since $f(\alpha) \in \mathcal{L}_n^B$, we get $\mathbb{KB}5\mathsf{I}_n \vdash f(\alpha)$ by completeness of $\mathbb{KB}5\mathsf{I}_n$. Of course, $\mathbb{KB}5\mathsf{I}_n$ is contained in $\mathbb{KB}5\mathsf{I}_n^{PA_t}$, and we easily get $\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash f(\alpha)$. Finally, we get $\mathbb{KB}5\mathsf{I}_n^{PA_t} \vdash \alpha$ by again applying Lemma 8.1.19. □

Concluding remarks

It is the aim of this last chapter to briefly summarise the main issues of this thesis as well as to point to some open questions and to some possible future work.

Concerning Part I

We have presented the basic notions and concepts of classical propositional logic in Chapter 1. Furthermore, we have introduced three different ways of representing a belief state in propositional logic: model sets, belief sets, and belief bases. Our preferred kind of belief state representation is the concept of model sets for the following reasons. First, if the set of propositions is infinite, then we can represent more belief states with model sets than with belief bases. Second, if the set of propositions is finite, then a belief state is always represented by a finite model set, whereas every belief set is an infinite set. The third reason for our preference on model sets is the connection between Part I and Part II of this thesis: belief change functions operating on model sets can be translated into belief change functions in modal logic. For instance, the expansion functions $\oplus$ and $\oplus^c$ are implemented in a multi-agent setting in Definition 6.1.1 and Definition 6.2.1 respectively.

Compared to Chapter 1, where we have only given the definition of the expansion function for each type of belief state representation, we have shown how to translate an arbitrary belief change function from one notion to another in Chapter 2. We have proved that these translations are invertible, that is going forth and back results in an equivalent belief change function. Moreover, we have introduced three kinds of belief change functions: revision, contraction, and update. For each kind there are eight original postulates that specify the requirement of minimal change. For both revision and contraction, the postulates are stated in the context of belief sets, whereas the update postulates are formulated by means of belief bases. Due to our preference on the different kinds of belief state representations, we have given translations of the three sets of postulates in the context of model sets. We have then shown that the translated postulates are equivalent to the original ones.

Chapter 3 deals with the translation of belief change functions of some type into functions of another type. First, we have proposed the definition of an expansion function $\oplus^c$ operating on consistent model sets. This definition implements the idea of sceptical agents, who reject new information if they believe that it is false. We have also shown how to restrict revision, contraction, and update functions to consistent model sets. This cannot be done by the use of integrity constraints, so we have defined the corresponding translation operators. It was not hard to define the respective operators mapping functions on consistent model sets to functions on possibly inconsistent model sets. Again, we have shown that going forth and back results in a function equivalent to the original one. For this purpose, it has been useful to define a ninth postulate. Furthermore, we have modified our three sets of postulates to fit the context of consistent model sets.

Additional achievements of Chapter 3 are the translations of revision functions to contraction and update functions respectively. The translation from revision to contraction and vice versa has already been given by the Levi and Harper identities, cf. [51, 35]. We have adapted these translations to the notions of both model sets and consistent model sets. From revision to update, we have defined new translations with the following property. If a function satisfies all revision postulates, then its translation to update satisfies all update postulates. For the converse direction, we have only found a translation such that the resulting function satisfies all but one revision postulate. At the end of the chapter, we have postulated some common behaviour of revision and update functions. Due to these new requirements, we have modified some of the update postulates in Chapter 4.

At the beginning of Chapter 4, we have defined three variants of the standard update function that all satisfy postulate ($\mathsf{U2}_{\mathcal{M}}$). Like the standard update function, these three variants do not have the problem of disjunctive input. We have defined a preorder relation on belief change functions, hence we have got a strict partial order, which allows for talking about the comparative strength. We have shown that our three new update functions are all stronger than the standard update function. The cautious standard update function is the strongest one. Since we have proved that it is not comparable to the possible models approach, we have compared the two update functions by statistical means in Appendix A. We believe that the cautious standard update is a promising function that deserves a deeper analysis in the future, primarily on the examples from [39].

The second part of Chapter 4 is about the new concept of revision candidates, contraction candidates, and update candidates. We have modified the three sets of postulates in such a way that they are not too strong anymore. There are additional modifications on some update postulates because of our new requirements from the end of Chapter 3. It has turned out that most of the translations from Chapter 3 are still adequate. We have only had to modify three translations, for instance the translation from update to revision. Moreover, we have introduced the new concept of minimax change. Using the example of revision functions, we have required minimal change except for the case that the new information is inconsistent with our beliefs. In this case we have suggested to use the standard update function, which can be seen as performing maximal change. Finally, we have presented concrete functions that comply with the requirement of minimax change: minimax revision, minimax contraction, and minimax update. Of course, we have also given the restrictions of these minimax belief change functions to consistent model sets.

It would be interesting to know the computational complexity of the decision problem for our new belief change functions. If $\circledast\colon \mathcal{M} \times \mathcal{L}_0 \to \mathcal{M}$ is an arbitrary belief change function, the decision problem is to find out whether $S \circledast \alpha \vDash \beta$ for any given $S \in \mathcal{M}$ and $\alpha, \beta \in \mathcal{L}_0$. We know that this problem is coNP-complete for the standard update function, cf. [39]. Since our functions are all related to it, we think that the variants of the standard update function as well as the minimax belief change functions all have similar complexity of the decision problem.

Concerning Part II

We have started with introducing the syntax and semantics of normal modal logic in Chapter 5, and we have presented nine different Hilbert systems that correspond to nine different properties of knowledge/belief. In addition, we have extended this logic with common knowledge operators as well as operators for relativised common knowledge. Since the latter is a generalisation of the former logic, we have formulated the Hilbert systems for relativised common knowledge in such a way that the relationship to common knowledge is obvious. Furthermore, we have introduced the bimodal logic of knowledge and belief, where we have defined three maximal deductive systems that can possibly be extended with announcement operators. As we have argued in Chapter 8, we think that only one out of these three systems is suitable for this purpose. Finally, we have given the languages of announcement logics with either private or public announcement operators. We have introduced the idea of announcement resistant formulas, which is a slight modification of the successful formulas. Due to this modification, we can use the same concept for every announcement logic. Moreover, we have defined five properties that the new announcement logics defined in this thesis all have: fact preservation, adequacy, totality, self-duality, and normality.

In Chapter 6, we have introduced two different semantics for the logic of group announcements. As the name indicates, it is typically the case that some agents do not hear an announcement, which makes them believe that the beliefs of all agents remain unchanged. Mostly, this is a false belief, hence group announcements lead to Hilbert systems that are inconsistent with the knowledge axiom. We call this process belief expansion because the beliefs in propositional facts are never retracted after an announcement. The first approach is for trustful agents: they always accept new information, even if they end in an inconsistent epistemic state. The second semantics is for sceptical agents: they only accept new information if it is consistent with their beliefs, otherwise they refuse to learn the announced formula. We have presented three deductive systems for the expansion of trustful agents' beliefs, whereas we have been able to give four axiomatisations of the semantics for sceptical agents. These Hilbert systems are all systems of normal modal logic augmented with group announcement axioms and the group announcement necessitation rule. It has turned out that one of the Hilbert systems for trustful agents is identical to the one by Gerbrandy and Groeneveld [30]. We consider it as an advantage of our semantics that we do not need Aczel's anti-foundation axiom, cf. [1].

The last section of Chapter 6 is about the logic of group announcements and common belief. We have added announcement composition to the language such that not only formulas, but also sequences of formulas can be announced. These extra announcement operators are very convenient because the interpretation of consecutive group announcements is quite complicated, especially if the groups of agents are both differing and intersecting. We have also extended the semantics for trustful agents by the semantics of both announcement composition and common belief. Furthermore, we have developped purely syntactical notions that describe the semantical interpretation of announcement composition. These notions have given rise to a group announcement induction rule, which is part of our suggested Hilbert systems. We have proved soundness, but we have not succeeded in proving completeness for the presented deductive systems. Although there exists a complete axiomatisation that extends one of our systems, cf. [7], we do not know whether or not they are complete. This open problem certainly is a task for future work.

Chapter 7 deals with public announcement logic, where an announcement is always told to every agent. We call this process knowledge expansion because the deductive systems are consistent with the knowledge axiom and known propositional facts are still known after every announcement. Unlike group announcements, it is easy to express two consecutive announcements by a single one. Due to this difference, it is not that hard to prove the completeness of Hilbert systems for the logic of public announcements and common knowledge.

In the first section of Chapter 7, we have given a short survey about the well-known truthful public announcements. We have presented six Hilbert systems that we have proved sound and complete. Although these axiomatisations are not new, we have shown that our methods from group announcement logic can be applied. Furthermore, we have given new syntactical proofs for many known results. There remain some open problems concerning the successful formulas, mainly in systems without the knowledge axiom. These questions would not arise if the successful formulas were defined this way: a formula α is successful in $\mathcal{X}$ if and only if for all $i \in \mathcal{A}$ we have $\mathcal{X} \vDash [\alpha!]K_i\alpha$. We think that the concept of knowable formulas by Balbiani et al. [4] is another answer to the above-mentioned lacking of the successful formulas. We have not had similar problems with the announcement resistant formulas. Finally, we have studied the logic of truthful public announcements and relativised common knowledge, as well as the logic of truthful public announcements and common knowledge. We have managed to prove the majority of the results by extending the proofs for the language without any common knowledge operators.

In the remaining part of Chapter 7, we have thoroughly studied our total public announcements from [62]. In the definition of the semantics for formulas of the form $[\alpha!]\beta$ we have distinguished two cases: if the announced formula α is true at the actual world, we perform the same model transformation as with truthful public announcements. In case α is false, the public announcement has no effect on the interpretation of the succeeding formula β. This little difference to the truthful public announcement semantics indeed results in totality: all formulas of the form $[\alpha!]\bot$ are not satisfiable anymore. We have given six Hilbert systems for total public announcements, for which we have proved soundness and completeness. It has turned out that the public announcement axioms are closely related to the group announcement axioms from Chapter 6. Therefore, it is not surprising that most of the results as well as their proofs are similar to the ones for group announcements. Again, we have investigated the logic of total public announcements and relativised common knowledge, as well as the logic of truthful public announcements and common knowledge. For the latter, we have given the completeness proof in every detail using a rank function similar to the one from [22].

Finally, we have applied the idea from total public announcements to define a semantics for the logic of knowledge, belief, and public announcements in Chapter 8. On the knowledge level, an announcement is interpreted like in the logic of total public announcements: the announcement of a false formula does not directly affect the agents' knowledge. On the level of belief, a public announcement has the same effect as a public announcement to trustful agents. We have provided a Hilbert system for this logic, and we have again given a soundness and completeness proof. Some of the proofs are a bit more elaborate than in the previous chapters because of the differing impacts of an announcement on the agent's knowledge and their beliefs. Due to this phenomenon, we have not yet succeeded in proving that a true announcement resistant formula α is always known and believed by the agents after one public announcement of α.

Similar to belief expansion, where we have defined two announcement semantics based on the expansion functions $\oplus$ and $\oplus^c$, we think that we could define new group announcement semantics for belief revision. These definitions would be based on the minimax revision function $\otimes_{\mathsf{mm}}$ and its restriction $(\otimes_{\mathsf{mm}})^{cr}$ to consistent model sets, cf. Example 4.2.3. Furthermore, we could define a new public announcement semantics for knowledge update analogous to the total public announcement semantics. This new definition would be based on the restriction $(\odot_{\mathsf{mm}})^{cu}$ of the minimax update function to consistent model sets, cf. Example 4.2.16.

Appendix A

Comparing $\odot_{\mathsf{csu}}$ with $\odot_{\mathsf{pma}}$

In this appendix we will compare the cautious standard update function $\odot_{\mathsf{csu}}$ from Definition 4.1.1 with the possible models approach $\odot_{\mathsf{pma}}$ from Example 2.3.5. First, we will present some statistical results that indicate that the function $\odot_{\mathsf{pma}}$ does not perform less change than $\odot_{\mathsf{csu}}$. Second, we will provide a table that contains $S \odot_{\mathsf{csu}} (p \vee q)$ and $S \odot_{\mathsf{pma}} (p \vee q)$ for all model sets $S \subseteq \mathrm{Pow}(\{p, q, r\})$. This table illustrates how the function $\odot_{\mathsf{csu}}$ deals with disjunctive input and how it differs from $\odot_{\mathsf{pma}}$.

Some empirical data

The table on the next page presents a comparison of the amount of change performed by the two update functions $\odot_{\mathsf{csu}}$ and $\odot_{\mathsf{pma}}$. First, we are going to explain how the reader has to interpret the content of this table. We have defined $\mathcal{P} = \{p, q, r, s\}$, thus we have $\mathcal{M} = \mathrm{Pow}(\mathrm{Pow}(\{p, q, r, s\}))$. For all $S \subseteq \mathrm{Pow}(\mathcal{P})$ and for every formula α in the first column we have calculated $S \odot_{\mathsf{csu}} \alpha$ and $S \odot_{\mathsf{pma}} \alpha$. That is, we have performed $2 \cdot 23 \cdot 65536$ updates because we have 2 functions, 23 formulas, and 2^{16} model sets. Given a line in the table and the formula α from the first column of this line, we have that the next five columns contain the following information.

1. $\Sigma \odot_{\mathsf{csu}}$: the sum over all $S \in \mathcal{M}$ of $\mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{csu}} \alpha))$,
2. $\Sigma \odot_{\mathsf{pma}}$: the sum over all $S \in \mathcal{M}$ of $\mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{pma}} \alpha))$,
3. $\odot_{\mathsf{csu}} > \odot_{\mathsf{pma}}$: the number of model sets $S \in \mathcal{M}$ that satisfy $\mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{csu}} \alpha)) > \mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{pma}} \alpha))$,
4. $\odot_{\mathsf{csu}} < \odot_{\mathsf{pma}}$: the number of model sets $S \in \mathcal{M}$ that satisfy $\mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{csu}} \alpha)) < \mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{pma}} \alpha))$,

5. $\odot_{\mathsf{csu}} = \odot_{\mathsf{pma}}$: the number of model sets $S \in \mathcal{M}$ that satisfy $\mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{csu}} \alpha)) = \mathrm{Card}(S \,\Delta\, (S \odot_{\mathsf{pma}} \alpha))$.

It can be shown that for all $S \in \mathcal{M}$ and all conjunctions α of literals we have $S \odot_{\mathsf{csu}} \alpha = S \odot_{\mathsf{pma}} \alpha$. Therefore, we have chosen formulas that are not conjunctions of literals. We want to mention that the set $\mathcal{P}$ and the number of formulas are both too small for a thorough statistical analysis. However, we think that this table indicates that $\odot_{\mathsf{csu}}$ does not perform more change than the function $\odot_{\mathsf{pma}}$.

	$\Sigma \odot_{\mathsf{csu}}$	$\Sigma \odot_{\mathsf{pma}}$	$\odot_{\mathsf{csu}} > \odot_{\mathsf{pma}}$	$\odot_{\mathsf{csu}} < \odot_{\mathsf{pma}}$	$\odot_{\mathsf{csu}} = \odot_{\mathsf{pma}}$
$((((p \vee q) \vee r) \vee s) \leftrightarrow (((p \wedge q) \wedge r) \wedge s))$	491518	524284	0	32766	32770
$((p \wedge q) \wedge (r \leftrightarrow s))$	491518	524256	14	32752	32770
$(((p \wedge q) \wedge r) \vee ((p \wedge q) \wedge s))$	450557	519680	1128	47976	16432
$(((p \vee q) \vee r) \leftrightarrow ((q \wedge r) \wedge s))$	409596	521728	284	56982	8270
$(p \wedge ((q \wedge r) \leftrightarrow s))$	409596	518080	722	56500	8314
$((p \wedge q) \vee ((p \wedge r) \wedge s))$	370683	495360	1362	59034	5140
$((p \wedge q) \vee (p \wedge r))$	428864	475136	12198	32052	21286
$((p \vee q) \leftrightarrow (r \wedge s))$	333818	507904	358	62664	2514
$(((p \wedge q) \vee (p \wedge r)) \vee (p \wedge s))$	298489	446464	486	60894	4156
$((p \wedge q) \vee (r \wedge s))$	298489	466944	511	62639	2386
$(p \leftrightarrow q)$	435712	458752	0	7650	57886
$((p \leftrightarrow q) \leftrightarrow r)$	386944	491520	3460	43542	18534
$(((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow s)$	264184	507904	72	64538	926
$(p \vee ((q \wedge r) \wedge s))$	230519	376576	127	60779	4630
$((p \vee q) \wedge (r \vee s))$	230519	409600	127	62259	3150
$(p \vee (q \wedge r))$	340528	352256	22203	25173	18160
$((p \wedge q) \leftrightarrow (r \wedge s))$	197238	392704	63	62611	2862
$((p \vee (q \wedge r)) \vee (q \wedge s))$	164181	319488	31	59705	5800
$(((p \vee q) \vee r) \leftrightarrow ((p \vee q) \wedge (r \vee s)))$	164181	358400	31	61293	4212
$(p \vee q)$	303200	262144	36015	5925	23596
$((p \vee q) \vee (r \wedge s))$	98395	225280	7	53433	12096
$((p \vee q) \vee r)$	180252	163840	21947	12835	30754
$(((p \vee q) \vee r) \vee s)$	32783	98304	1	30719	34816
Average per formula	304859.3	409417.6	4397.7	45857.4	15280.9

Table with update results

We set $\mathcal{P} = \{p, q, r\}$, thus we have $\mathcal{M} = \text{Pow}(\text{Pow}(\{p, q, r\}))$, which contains 256 different model sets. For each $S \in \mathcal{M}$ we will list the updated model sets $S \odot_{\mathsf{csu}} (p \vee q)$ and $S \odot_{\mathsf{pma}} (p \vee q)$ in the following table. With this example we have that the total amount of change performed by $\odot_{\mathsf{csu}}$ is a bit less than the one performed by $\odot_{\mathsf{pma}}$,

$$\sum_{S \in \mathcal{M}} \text{Card}(S \,\Delta\, (S \odot_{\mathsf{csu}} (p \vee q))) = 460,$$

$$\sum_{S \in \mathcal{M}} \text{Card}(S \,\Delta\, (S \odot_{\mathsf{pma}} (p \vee q))) = 512.$$

In 39 (69) cases the change performed by $\odot_{\mathsf{csu}}$ is bigger (smaller) than the one performed by $\odot_{\mathsf{pma}}$. That is, in 148 cases out of 256 the two functions perform the same amount of change.

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{\emptyset\}$	$\{\{p\},\{q\},\{p,q\}\}$	$\{\{p\},\{q\}\}$
$\{\{p\}\}$	$\{\{p\}\}$	$\{\{p\}\}$
$\{\{q\}\}$	$\{\{q\}\}$	$\{\{q\}\}$
$\{\{r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\}\}$
$\{\{p,q\}\}$	$\{\{p,q\}\}$	$\{\{p,q\}\}$
$\{\{p,r\}\}$	$\{\{p,r\}\}$	$\{\{p,r\}\}$
$\{\{q,r\}\}$	$\{\{q,r\}\}$	$\{\{q,r\}\}$
$\{\{p,q,r\}\}$	$\{\{p,q,r\}\}$	$\{\{p,q,r\}\}$
$\{\emptyset,\{p\}\}$	$\{\{p\}\}$	$\{\{p\},\{q\}\}$
$\{\emptyset,\{q\}\}$	$\{\{q\}\}$	$\{\{p\},\{q\}\}$
$\{\emptyset,\{r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p,q\}\}$	$\{\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$
$\{\emptyset,\{p,r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$
$\{\emptyset,\{q,r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$
$\{\emptyset,\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q,r\}\}$
$\{\{p\},\{q\}\}$	$\{\{p\},\{q\}\}$	$\{\{p\},\{q\}\}$
$\{\{p\},\{r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{p,q\}\}$	$\{\{p\},\{p,q\}\}$	$\{\{p\},\{p,q\}\}$
$\{\{p\},\{p,r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{p,r\}\}$
$\{\{p\},\{q,r\}\}$	$\{\{p\},\{q,r\}\}$	$\{\{p\},\{q,r\}\}$
$\{\{p\},\{p,q,r\}\}$	$\{\{p\},\{p,q,r\}\}$	$\{\{p\},\{p,q,r\}\}$
$\{\{q\},\{r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{p,q\}\}$	$\{\{q\},\{p,q\}\}$	$\{\{q\},\{p,q\}\}$
$\{\{q\},\{p,r\}\}$	$\{\{q\},\{p,r\}\}$	$\{\{q\},\{p,r\}\}$
$\{\{q\},\{q,r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{q\},\{q,r\}\}$
$\{\{q\},\{p,q,r\}\}$	$\{\{q\},\{p,q,r\}\}$	$\{\{q\},\{p,q,r\}\}$
$\{\{r\},\{p,q\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,r\}\}$	$\{\{p,r\}\}$	$\{\{p,r\},\{q,r\}\}$
$\{\{r\},\{q,r\}\}$	$\{\{q,r\}\}$	$\{\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,q,r\}\}$	$\{\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p,q\},\{p,r\}\}$	$\{\{p,q\},\{p,r\}\}$	$\{\{p,q\},\{p,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\{p,q\},\{q,r\}\}$	$\{\{p,q\},\{q,r\}\}$	$\{\{p,q\},\{q,r\}\}$
$\{\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$
$\{\{p,r\},\{q,r\}\}$	$\{\{p,r\},\{q,r\}\}$	$\{\{p,r\},\{q,r\}\}$
$\{\{p,r\},\{p,q,r\}\}$	$\{\{p,r\},\{p,q,r\}\}$	$\{\{p,r\},\{p,q,r\}\}$
$\{\{q,r\},\{p,q,r\}\}$	$\{\{q,r\},\{p,q,r\}\}$	$\{\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\}\}$	$\{\{p\},\{q\}\}$	$\{\{p\},\{q\}\}$
$\{\emptyset,\{p\},\{r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{p,q\}\}$	$\{\{p\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$
$\{\emptyset,\{p\},\{p,r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$
$\{\emptyset,\{p\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$
$\{\emptyset,\{p\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q,r\}\}$
$\{\emptyset,\{q\},\{r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{q\},\{p,q\}\}$	$\{\{q\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$
$\{\emptyset,\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$
$\{\emptyset,\{q\},\{q,r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$
$\{\emptyset,\{q\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q,r\}\}$
$\{\emptyset,\{r\},\{p,q\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{r\},\{p,r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{r\},\{q,r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p,q\},\{p,r\}\}$	$\{\{p\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$
$\{\emptyset,\{p,q\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$
$\{\emptyset,\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$
$\{\emptyset,\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{p,q,r\}\}$
$\{\emptyset,\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{q\},\{r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{q\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$
$\{\{p\},\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$
$\{\{p\},\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$	$\{\{p\},\{q\},\{q,r\}\}$
$\{\{p\},\{q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q,r\}\}$
$\{\{p\},\{r\},\{p,q\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\{p\},\{r\},\{p,r\}\}$	$\{\{p\},\{p,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{p,q\},\{p,r\}\}$
$\{\{p\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{p,q\},\{q,r\}\}$
$\{\{p\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,q,r\}\}$
$\{\{p\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{p,q,r\}\}$
$\{\{p\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{r\},\{p,q\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{p,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{q,r\}\}$	$\{\{q\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{p,q,r\}\}$	$\{\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{p,q\},\{p,r\}\}$	$\{\{q\},\{p,q\},\{p,r\}\}$	$\{\{q\},\{p,q\},\{p,r\}\}$
$\{\{q\},\{p,q\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\}\}$
$\{\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,q,r\}\}$
$\{\{q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{p,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{p,q,r\}\}$
$\{\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{q,r\},\{p,q,r\}\}$
$\{\{r\},\{p,q\},\{p,r\}\}$	$\{\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,q\},\{q,r\}\}$	$\{\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{r\},\{p,r\},\{q,r\}\}$	$\{\{p,r\},\{q,r\}\}$	$\{\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,r\},\{p,q,r\}\}$	$\{\{p,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{r\},\{q,r\},\{p,q,r\}\}$	$\{\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{p,q,r\}\}$
$\{\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{q,r\},\{p,q,r\}\}$
$\{\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\}\}$
$\{\emptyset,\{p\},\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\emptyset, \{p\}, \{q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,r\}\}$	$\{\{p\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}\}$
$\{\emptyset, \{p\}, \{p,q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,q\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{q,r\}\}$	$\{\{q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}\}$
$\{\emptyset, \{q\}, \{p,q\}, \{q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{p,q\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{r\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{r\}, \{p,q\}, \{q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{r\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{r\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{r\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\{p\},\{q\},\{r\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{q\},\{r\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{q\},\{r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{q\},\{r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$
$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$
$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$
$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{p,q,r\}\}$
$\{\{p\},\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{r\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{r\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{r\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{r\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{r\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$
$\{\{p\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{q,r\},\{p,q,r\}\}$
$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{r\},\{p,q\},\{p,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{p,q\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{p,q\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{r\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{r\},\{p,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{r\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{q\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{p,r\},\{p,q,r\}\}$
$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$
$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{r\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\}\}$
$\{\{r\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\{r\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{r\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{r\},\{p,q\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{r\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{p,q\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
$\{\emptyset,\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{p,q\},\{p,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{p\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,r\},\{q,r\},\{p,q,r\}\}$
$\{\emptyset,\{q\},\{r\},\{p,q\},\{p,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\},\{p,q,r\}\}$	$\{\{p\},\{q\},\{p,q\},\{p,r\},\{q,r\}\}$
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$\{\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$

S	$S \odot_{\mathsf{csu}} (p \vee q)$	$S \odot_{\mathsf{pma}} (p \vee q)$
$\{\emptyset, \{p\}, \{q\}, \{r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{r\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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$\{\{p\}, \{q\}, \{r\}, \{p,q\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\{p\}, \{q\}, \{r\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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$\{\{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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$\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
$\{\emptyset, \{p\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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$\{\{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$	$\{\{p\}, \{q\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$
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Notation index

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Erklärung

gemäss Art. 28 Abs. 2 RSL 05

Name/Vorname:	Steiner David		
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Ich erkläre hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe o des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist.

Bern, 28.04.2009

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