Proof-Systems for PLTL

Cycling Sequents and their Use in a Finitization for PLTL

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Roger Traber

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Leiter der Arbeit:

Prof. Dr. Gerhard Jäger
Institut für Informatik und angewandte Mathematik

und

Dr. Luca Alberucci
Institut für Informatik und angewandte Mathematik

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Introduction

This work presents various proof-systems for Propositional Linear Time Logic, henceforth PLTL. Special attention is paid to the possibilities of giving a "classical" sequent calculus for PLTL. There are many sound and complete systems based on tableau methods, as for example in [5], [9], [14] or [12]. Still there is not much satisfying work on sequent-calculi for PLTL. The problem about most of the known systems is, that they make use of a cut rule or they contain an infinitary rule, that is, a rule containing infinitely many premisses. Both properties are critical if we are interested in systematic proof-search. A system containing a cut for example is found in Peach [10]. The calculus given by Dax, Hofmann and Lange in [4] is cut-free but has the problem that it is not wellfounded. A system containing an infinitary rule can be found in Kawai [8], another one based on the infinitary calculus for the modal μ-calculus will be presented in this work.
So let us call a calculus to be "classical" when its proof-trees are wellfounded and when its proofs do not make use of any infinitary rule or a cut rule. It seem that the system LT1 given by Brünnler and Lange in [3] is so far the only calculus for PLTL that can be called "classical" in this sense.
In this work an alternative approach that may lead as well to a finitary, cut-free and wellfounded sequent system is presented. We give a calculus that contains like LT1 additional to the standard rules, rules that allow to test for cycles on branches. But in contrary to LT1 the system given here is not so much oriented on [4], but is rather based on work on the infinitary calculus by Alberucci and Jäger [1], as well on work by Jäger, Studer and Kretz [7] on the infinitary μ-calculus and a paper by Studer on the proof-theory of the modal μ-calculus [11].
The basic idea is to exploit the full power of the premisses of the infinitary rule of the calculus K^∞

$$\frac{\Gamma, (\bigcirc\top \wedge \alpha)^n \ \ (\forall n \in \mathbb{N})}{\Gamma, \Box\alpha}\ (\omega - \Box).$$

We want to show that there is no need to have premisses for all approximants of the formula $\Box\alpha$, but that it is enough to have only one approximant which is of a sufficiently high degree. This property seems to be strong enough to ensure the existence of repeating or cycling sequents on a branch of a proof-tree. This fact shall then be used to drop the infinitary rule and replace it by a rule of the form:

$$\frac{[\Gamma, \alpha]^{\emptyset} \quad [\Gamma, \bigcirc x^{\Box\alpha}]^{\{\Gamma, x^{\Box\alpha}\}}}{[\Gamma, \Box\alpha]} \ (Cyc)$$

The premise on the right of this rule has to be read as a conditional statement: That is, given the label $\{\Gamma, x^{\Box\alpha}\}$ holds then also the basic sequent $[\Gamma, \bigcirc x^{\Box\alpha}]$ must be true. This amounts to establish in the premise of the rule (Cyc) a kind of induction step, that can be used to derive a formula $\Box\alpha$.

In the first chapter some preliminary concepts and notations on PLTL will be given. In the second chapter some existing calculi for PLTL will be discussed, such of them including a cut rule or an infinitary rule. A tableau algorithm will be presented and difficulties about inverting a tableau to a corresponding sequent calculus are discussed. Then the mentioned cut-free, but non-wellfounded system DHL is presented and discussed with respect to the problem of finding a sequent calculus for PLTL. Then the finitary calculus LT1 is presented that is based on DHL. It makes use of labels that are annotated to formulas and that contain information about cycling sequents in the branch of a proof-tree.
In the third section the system K^{∞} is presented. Soundness and completeness of this system will be proven in full detail.
In the fourth and fifth section the question is addressed about how a finitization of K^{∞} could be given. First the basic ideas for a finitization will informally be discussed and then put in the strict formal framework of the calculus K1. Some difficulties about this calculus will be stated and a stronger calculus K2 will be introduced. Soundness of both calculi is proven, whereas as a conjecture in the last section a proof for the completeness is given.

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Chapter 1

Preliminaries

1.1 Propositional Linear Time Logic

In this chapter we introduce syntax and semantics of PLTL as well as some notions that will be frequently used throughout this work. The infinitary calculus K^∞ presented in chapter three, as well as the calculi given in chapter four and five will be defined relative to the language $\mathcal{L}_{\mathsf{PLTL}}$, resp. to an extension of this language by propositional variables. Therefore we are going to define $\mathcal{L}_{\mathsf{PLTL}}$ as the basic language. Apart from the language $\mathcal{L}_{\mathsf{PLTL}}$, also the language $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ will be introduced in this chapter. Languages of the calculi we come across in chapter two will be defined there as extensions or reductions of $\mathcal{L}_{\mathsf{PLTL}}$, resp. $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$.

1.1.1 Syntax

Definition 1.1.1. *The alphabet of the language $\mathcal{L}_{\mathsf{PLTL}}$ for PLTL, contains the following **basic syntactical symbols**:*

1. *An enumerable number of positive atomic propositions $p_1, p_2, p_3, \dots$*
2. *A symbol $\sim$ to form negative atomic propositions.*
3. *The logical symbols $\wedge$ (and), $\vee$ (or).*
4. *The temporal connectives $\Box$ (always), $\Diamond$ (eventual), $\mathcal{U}$ (until) and $\bigcirc$ (next).*
5. *Parentheses, brackets and commas.*

The alphabet of the language $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ is identical to $\mathcal{L}_{\mathsf{PLTL}}$, but contains the additional symbol $\mathcal{R}$ (release).

The **negative atomic propositions of $\mathcal{L}_{\mathsf{PLTL}}$ (resp. $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$)** are all expressions of the form $\sim p$ such that p is a positive atomic proposition.
We use the usual conventions $\bot := p \wedge \neg p$ and $\top := p \vee \neg p$ for a fixed atomic proposition p.

Definition 1.1.2. *The set of $\mathcal{L}_{\mathsf{PLTL}}$**-formulas** is defined inductively in the following way:*

1. *Every positive and negative atomic proposition is $\mathcal{L}_{\mathsf{PLTL}}$-formula.*
2. *If φ and ψ are $\mathcal{L}_{\mathsf{PLTL}}$-formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \mathcal{U} \psi)$ are $\mathcal{L}_{\mathsf{PLTL}}$-formulas.*
3. *If φ is a $\mathcal{L}_{\mathsf{PLTL}}$-formula, then $\Box\varphi$, $\Diamond\varphi$ and $\bigcirc\varphi$ are $\mathcal{L}_{\mathsf{PLTL}}$-formulas.*

The set of $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formulas is defined in the same way as the set of $\mathcal{L}_{\mathsf{PLTL}}$-formulas, but contains the additional clause:

4. *If φ and ψ are $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formulas, then also $(\varphi \mathcal{R} \psi)$ is $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formula.*

If clear from the context we omit parentheses in formulas, that is, we rather write $\varphi \wedge \psi$ instead of $(\varphi \wedge \psi)$.
The languages $\mathcal{L}_{\mathsf{PLTL}}$ and $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ generalize classical propositional logic with temporal operators, and thus contain the standard propositional connectives $\wedge$ (and) and $\vee$ (or). The negation $\neg$ (not) will be introduced in the next definition, whereas the remaining connectives $\rightarrow$ (implies) and $\leftrightarrow$ (if and only if) can then be assumed to be defined as the usual abbreviations.
Negation is defined in a way that allows to interpret negated formulas as macros for formulas just containing positive and negative atomic propositions.

Definition 1.1.3. *The **negation** $\neg\varphi$ of a $\mathcal{L}_{\mathsf{PLTL}}$-formula φ is inductively defined as follows:*

1. *If φ is a positive atomic proposition p, then $\neg\varphi :=\sim p$; if φ is a negative atomic proposition $\sim p$, then $\neg\varphi := p$.*
2. *If φ is of the form $(\alpha \wedge \beta)$, then $\neg\varphi := (\neg\alpha \vee \neg\beta)$; if φ is of the form $(\alpha \vee \beta)$, then $\neg\varphi := (\neg\alpha \wedge \neg\beta)$.*
3. *If φ is of the form $\bigcirc\alpha$, then $\neg\varphi := \bigcirc\neg\alpha$; if φ is of the form $\Box\alpha$, then $\neg\varphi := \Diamond\neg\alpha$; if φ is of the form $\Diamond\alpha$, then $\neg\varphi := \Box\neg\alpha$.*
4. *If φ is of the form $(\alpha \mathcal{U} \beta)$, then $\neg\varphi := (\neg\beta \mathcal{U} (\neg\alpha \wedge \neg\beta)) \vee \Box\neg\beta$.*

In the case of $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ negation for until-formulas $(\alpha\mathcal{U}\beta)$ is defined differently from above by help of the release-operator; further negation for release-formulas $(\alpha\mathcal{R}\beta)$ has to be defined. That is, to define negation for $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ we have clauses *1.-3.* form above and the additional clauses:

4. *If φ is of the form $(\alpha\,\mathcal{R}\beta)$, then $\neg\varphi := (\neg\alpha\,\mathcal{U}\neg\beta)$.*

5. *If φ is of the form $(\alpha\,\mathcal{U}\beta)$, then $\neg\varphi := (\neg\alpha\,\mathcal{R}\neg\beta)$.*

Remark: The language $\mathcal{L}_{\mathsf{PLTL}}$ does not contain the operator $\mathcal{R}$, therefore negation of until-formulas $(\alpha\mathcal{U}\beta)$ cannot be defined by its dual as in the case of $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$. In the next section on the semantics of PLTL we show that this definition of $\neg(\alpha\mathcal{U}\beta)$ for $\mathcal{L}_{\mathsf{PLTL}}$ has the intended meaning.
The reason for that we do not uniformly use a language containing the release-operator $\mathcal{R}$ is that the finitary calculi K1 and K2 presented in the last two chapters are based on a method to check cycles for greatest fixpoints. If there was a release operator in the language then there would be additional greatest fixpoints corresponding to formulas $(\alpha\mathcal{R}\beta)$. This would require to give also a method to check for cycles involving release-formulas.

Definition 1.1.4. *A finite set of $\mathcal{L}_{\mathsf{PLTL}}$-formulas (resp. $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formulas) will be called a $\mathcal{L}_{\mathsf{PLTL}}$-**sequent** (resp. $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-**sequent**).*

If clear from the context we will often talk just about formulas and sequents and drop the prefix indicating the language.

Convention 1.1.5. *As syntactic metavariables we are going to use (often with sub- and superscripts):*

1. *Small Greek letters for individual formulas*

$$\alpha, \beta, \gamma, \varphi, \psi, \ldots$$

2. *Capital Greek letters for sequents*

$$\Gamma, \Delta, \Pi, \ldots$$

Definition 1.1.6. *Given a $\mathcal{L}_{\mathsf{PLTL}}$-formula α, then $\Box\alpha$-**approximants** are defined inductively:*

1. $(\bigcirc\top \wedge \alpha)^0 := \top$.

2. $(\bigcirc\top \wedge \alpha)^{n+1} := (\bigcirc(\bigcirc\top \wedge \alpha)^n \wedge \alpha)$.

Definition 1.1.7. *Given a $\mathcal{L}_{\mathsf{PLTL}}$-formula β, then $\Diamond\beta$-**approximants** are defined inductively:*

1. $(\bigcirc\bot \vee \beta)^0 := \bot$.
2. $(\bigcirc\bot \vee \beta)^{n+1} := (\bigcirc(\bigcirc\bot \vee \beta)^n \vee \beta)$.

Definition 1.1.8. *Given $\mathcal{L}_{\mathsf{PLTL}}$-formulas α and β, then $(\alpha\mathcal{U}\beta)$-**approximants** are defined inductively:*

1. $((\alpha \wedge \bigcirc\bot) \vee \beta)^0 := \bot$.
2. $((\alpha \wedge \bigcirc\bot) \vee \beta)^{k+1} := ((\alpha \wedge \bigcirc((\alpha \wedge \bigcirc\bot) \vee \beta)^k) \vee \beta)$.

Remark: In our notation for approximants we keep the expression $\top$ also in approximants of higher degrees. The reason is that in the finitary calculi $\mathsf{K1}$ and $\mathsf{K2}$ we will work with transformations that replace the approximants by expressions containing propositional variables. The notation above will facilitate an understanding of how this transformation procedure works.

Definition 1.1.9. *The **rank** $\mathsf{rank}(\varphi)$ of a $\mathcal{L}_{\mathsf{PLTL}}$-formula φ is the ordinal number defined by the following recursion:*

1. *If φ is a positive atomic proposition p, then $\mathsf{rank}(\varphi) := 0$; if φ is a negative atomic proposition $\sim p$, then $\mathsf{rank}(\varphi) := 0$.*
2. *If φ is of the form $(\alpha\wedge\beta)$ or $(\alpha\vee\beta)$, then $\mathsf{rank}(\varphi) := max(\mathsf{rank}(\alpha), \mathsf{rank}(\beta))+1$.*
3. *If φ is of the form $\bigcirc\alpha$, then $\mathsf{rank}(\varphi) := \mathsf{rank}(\alpha) + 1$.*
4. *If φ is of the form $\Box\alpha$ or $\Diamond\alpha$, then $\mathsf{rank}(\varphi) := \mathsf{rank}(\alpha) + \omega$.*
5. *If φ is of the form $(\alpha\mathcal{U}\beta)$, then $\mathsf{rank}(\varphi) := max(\mathsf{rank}(\alpha), \mathsf{rank}(\beta)) + \omega$.*

The **rank** $\mathsf{rank}(\varphi)$ of a $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formula φ is defined as in the case of $\mathcal{L}_{\mathsf{PLTL}}$-formulas, but requires the additional clause

6. If φ is of the form $(\alpha\mathcal{R}\beta)$, then $\mathsf{rank}(\varphi) := \mathsf{rank}(\alpha \wedge \beta) + \omega$.

Lemma 1.1.10. *For all $\mathcal{L}_{\mathsf{PLTL}}$-formulas α, β and $k \in \mathbb{N}$ we have that*

1. $\mathsf{rank}(\Box\alpha) > \mathsf{rank}((\bigcirc\top \wedge \alpha)^k)$,
2. $\mathsf{rank}(\Diamond\alpha) > \mathsf{rank}((\bigcirc\bot \vee \alpha)^k)$, *and*
3. $\mathsf{rank}(\alpha\mathcal{U}\beta) > \mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^k)$.

Proof. **Part 1:** The proof goes by induction on $k \in \mathbb{N}$. First consider the case where $k = 0$, then

$$\mathsf{rank}((\bigcirc\top \wedge \alpha)^0) = \mathsf{rank}(\top) = \mathsf{rank}(p \vee \sim p) = \max(\mathsf{rank}(p), \mathsf{rank}(\sim p)) + 1 = 1.$$

But then

$$\mathsf{rank}((\bigcirc\top \wedge \alpha)^0) = 1 < \mathsf{rank}(\alpha) + \omega = \mathsf{rank}(\Box\alpha).$$

For the induction step we assume

$$\mathsf{rank}((\bigcirc\top \wedge \alpha)^k) < \mathsf{rank}(\alpha) + \omega.$$

We know that $\mathsf{rank}(\alpha) + \omega$ must be a limit ordinal number and therefore there must an ordinal number $n \in \omega$ such that

$$\mathsf{rank}((\bigcirc\top \wedge \alpha)^k) < \mathsf{rank}(\alpha) + n.$$

From this we get

$$\mathsf{rank}((\bigcirc\top \wedge \alpha)^{k+1}) = \mathsf{rank}((\bigcirc\top \wedge \alpha)^k) + 2 < \mathsf{rank}(\alpha) + \omega = \mathsf{rank}(\Box\alpha)$$

and we are done.

Proof of 2: This case is done analogously to part 1.

Proof of 3: If $k = 0$ then

$$\mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^0) = 1$$

and therefore clearly

$$\mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^0) < \mathsf{rank}(\alpha \wedge \beta) + \omega = \mathsf{rank}(\alpha \mathcal{U} \beta).$$

Now assume that $\mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^k) < \mathsf{rank}(\alpha \wedge \beta) + \omega = \mathsf{rank}(\alpha \mathcal{U} \beta)$. Again we know that $\mathsf{rank}(\alpha \mathcal{U} \beta)$ must be a limit ordinal and therefore cannot be reached by a successor of $\mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^k)$. Therefore we have

$$\mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^{k+1}) = \mathsf{rank}(((\alpha \wedge \bigcirc((\alpha \wedge \bigcirc\bot) \vee \beta)^k) \vee \beta))$$

$$= \mathsf{rank}((\alpha \wedge \bigcirc((\alpha \wedge \bigcirc\bot) \vee \beta)^k)) = \mathsf{rank}(\bigcirc((\alpha \wedge \bigcirc\bot) \vee \beta)^k)$$

$$= \mathsf{rank}(((\alpha \wedge \bigcirc\bot) \vee \beta)^k) + 1 < \mathsf{rank}(\alpha \mathcal{U} \beta).$$

Proof of 4: This case is done analogously to part 3. $\Box$

1.1.2 Semantics

The temporal connectives are interpreted over a flow of time that is linear, discrete, bounded in the past and infinite in the future. An obvious choice for such a timeline is any structure that is order isomorphic[1] to the natural numbers $\mathbb{N}$ ordered by the standard $<$ relation. So PLTL-models are defined in the following way:

Definition 1.1.11. *A* PLTL-***model*** *is a pair* $\mathcal{M} = (N, \pi)$*, where:*

- *N is a set of a copy of the natural numbers (that is, N and $\mathbb{N}$ are order isomorphic).*
- *π is a valuation function such that for every propositional variable p we have $\pi(p) \subseteq N$.*

The set $\pi(p)$ is understood as the set of all states where the propositional variable p is true.
Since the set of states, N, is a copy of $\mathbb{N}$ we will often denote a given state as n^N or simply write n, where $n \in \mathbb{N}$.

Definition 1.1.12. *Given a* PLTL*-model* $\mathcal{M} = (N, \pi)$*, the set* $\|\varphi\|^{\mathcal{M}}$ *of states satisfying the* $\mathcal{L}_{\mathsf{PLTL}}$*-formula* φ *is defined inductively as follows:*

1. *If $\varphi = p$, then $\| p \|^{\mathcal{M}} = \pi(p)$.*
2. *If $\varphi = \sim p$, then $\| \sim p \|^{\mathcal{M}} = N \setminus \pi(p)$.*
3. *If $\varphi = (\alpha \wedge \beta)$, then $\| (\alpha \wedge \beta) \|^{\mathcal{M}} = \| \alpha \|^{\mathcal{M}} \cap \| \beta \|^{\mathcal{M}}$.*
4. *If $\varphi = (\alpha \vee \beta)$, then $\| (\alpha \vee \beta) \|^{\mathcal{M}} = \| \alpha \|^{\mathcal{M}} \cup \| \beta \|^{\mathcal{M}}$.*
5. *If $\varphi = (\alpha \mathcal{U} \beta)$, then*

   $$\|(\alpha\mathcal{U}\beta)\|^{\mathcal{M}} = \{n \in N \mid \exists m \geq n\ (m \in \|\beta\|^{\mathcal{M}} \textit{ and } \forall\, n \leq n' < m\ n' \in \|\alpha\|^{\mathcal{M}})\}$$

6. *If $\varphi = \Box\alpha$, then $\| \Box\alpha \|^{\mathcal{M}} = \{n \in N \mid \forall m \geq n\ m \in \| \alpha \|^{\mathcal{M}}\}$.*
7. *If $\varphi = \Diamond\alpha$, then $\| \Diamond\alpha \|^{\mathcal{M}} = \{n \in N \mid \exists m \geq n\ m \in \| \alpha \|^{\mathcal{M}}\}$.*

[1] Two posets $(A, <_A)$ and $(B, <_B)$ are said to be **order isomorphic** if there exists a bijective mapping $f : A \to B$ such that for all u and v in A we have

$$u <_A v \quad \Leftrightarrow \quad f(u) <_B f(v).$$

8. *If* $\varphi = \bigcirc\alpha$, *then* $\| \bigcirc\alpha \|^{\mathcal{M}} = \{n \in N \mid n+1 \in \| \alpha \|^{\mathcal{M}}\}$.

The set $\|\varphi\|^{\mathcal{M}}$ of states satisfying the $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formula φ if defined by adding to the definition above a further clause to interpret release formulas $(\alpha\mathcal{R}\beta)$:

9. $\|(\alpha\mathcal{R}\beta)\|^{\mathcal{M}} = \{n \in N \mid \forall m \geq n\, (m \in \|\beta\|^{\mathcal{M}}$ *or* $\exists\, n \leq n' < m\; n' \in \|\alpha\|^{\mathcal{M}})\}$.

Let us informally characterize the meaning of the temporal connectives. First consider the unary connectives $\bigcirc$, $\Diamond$ and $\Box$. The connective $\bigcirc$ can be read as "at the next timepoint". Thus the formula $\bigcirc\alpha$ will be satisfied at some timepoint if and only if α is satisfied at the next timepoint. $\Diamond$ means "either now, or at some time in the future". Thus the formula $\Diamond\alpha$ is satisfied at some time if and only if α is satisfied at that timepoint or at some timepoint in the future. The connective $\Box$ is interpreted as "now, and at all times in the future. So $\Box\alpha$ is satisfied at some timepoint, if α is satisfied at that timepoint and at every timepoint in the future. The binary connective $\mathcal{U}$ is interpreted as "until". The formula $(\alpha\mathcal{U}\beta)$ is said to be satisfied at some timepoint if and only if the formula β is satisfied at that timepoint or at some state in the future, and α is satisfied at every timepoint until the timepoint that β is satisfied. The binary connective $\mathcal{R}$ is interpreted as "release". Thus the formula $(\alpha\mathcal{R}\beta)$ is satisfied at some timepoint if and only if the formula β is satisfied until the formula α is satisfied at some timepoint, or forever if such a state does not exist.

Definition 1.1.13. *Let* φ *be a* $\mathcal{L}_{\mathsf{PLTL}}$*-formula, then:*

1. *The formula* φ *is said to be* ***satisfiable in a*** PLTL***-model*** $\mathcal{M} = (N, \pi)$, *if there exists a state* $n \in N$ *such that* $n \in \|\varphi\|^{\mathcal{M}}$, *we then write* $\mathcal{M}, n \models \varphi$, *or simply* $n \models \varphi$.

2. *The formula* φ *is said to be* ***valid in a*** PLTL***-model*** $\mathcal{M} = (N, \pi)$, *if we have for all* $n \in N$ *that* $\mathcal{M}, n \models \varphi$, *we then write* $\mathcal{M} \models \varphi$.

3. *The formula* φ *is said to be* ***valid****, if we have* $\mathcal{M} \models \varphi$ *for all* PLTL*-models* $\mathcal{M} = (N, \pi)$, *we then write* $\models \varphi$.

Definition 1.1.14. *Let* Γ *be a* $\mathcal{L}_{\mathsf{PLTL}}$*-sequent, then:*

1. *The sequent* Γ *is said to be* ***satisfiable in a*** PLTL***-model*** $\mathcal{M} = (N, \pi)$, *if there exists a state* $n \in N$ *such that* $n \models \bigvee\Gamma$, *in this case we write* $\mathcal{M}, n \models \Gamma$, *or simply* $n \models \Gamma$.

2. *The sequent* Γ *is said to be* ***valid in a*** PLTL***-model*** $\mathcal{M} = (N, \pi)$, *if we have for all* $n \in N$ *that* $\mathcal{M}, n \models \bigvee\Gamma$, *we then write* $\mathcal{M} \models \Gamma$.

3. *The sequent Γ is said to be **valid**, if we have $\mathcal{M} \models \Gamma$ for all* PLTL*-models $\mathcal{M} = (N, \pi)$, we then write $\models \Gamma$.*

The definitions of satisfaction and validity for $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ are analogue.
If we have for a formula not $n \models \varphi$, then we write $n \not\models \varphi$. Analogously we define $M \not\models \varphi$, $\not\models \varphi$, $n \not\models \Gamma$, $\mathcal{M} \not\models \Gamma$ and $\not\models \Gamma$.

Lemma 1.1.15. *For all states n in a* PLTL*-model $\mathcal{M}$ and for all $\mathcal{L}_{\mathsf{PLTL}}$-formulas φ we have that*

$$n \models \varphi \quad \Leftrightarrow \quad n \not\models \neg\varphi.$$

Proof. The proof goes by induction on $\mathsf{rank}(\varphi)$. In the case where $\mathsf{rank}(\varphi) = 0$, then φ must be equal to a positive atomic proposition p or to a negative atomic proposition $\sim p$. We have $n \models p$ if and only if $n \in \pi(p)$. But $n \in \pi(p)$ if and only if $n \notin N \setminus \pi(p) = \| \sim p\|^{\mathcal{M}}$. Thus we have $n \models p$ if and only if $n \not\models \sim p$. The case for $\sim p$ is analogue.
In the case where $\mathsf{rank}(\varphi)$ is a successor ordinal, then φ must be of the form $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\bigcirc\alpha$. We have $n \models \alpha \wedge \beta$ if and only if $n \models \alpha$ and $n \models \beta$. By induction hypothesis this is the case if and only if $n \not\models \neg\alpha$ and $n \not\models \neg\beta$, and so $n \not\models \neg\alpha \vee \neg\beta$, but this is the same as $n \not\models \neg(\alpha \wedge \beta)$. The case for $\alpha \vee \beta$ is analogue. If we have $n \models \bigcirc\alpha$, then by the semantics of the nexttime operator this is the case if and only if $n + 1 \models \alpha$ and therefore by induction hypothesis if and only if $n + 1 \not\models \neg\alpha$. Thus $n \models \bigcirc\alpha$ if and only if $n \not\models \bigcirc\neg\alpha$ what is the same as $n \not\models \neg \bigcirc \alpha$.
Then in the case where $\mathsf{rank}(\varphi)$ is a limit ordinal, φ must be of the form $\Box\alpha$, $\Diamond\alpha$ or $\alpha\mathcal{U}\beta$. We have $n \models \Box\alpha$ if and only if $n + l \models \alpha$ for all $l \in \mathbb{N}$. By induction hypothesis that is $n \models \Box\alpha$ if and only if $n + l \not\models \neg\alpha$ for all $l \in \mathbb{N}$. But this is equivalent to $n \not\models \Diamond\alpha$ what is the same as $n \not\models \neg\Box\alpha$. The case for $\Diamond\alpha$ is analogue. Suppose that $n \models (\alpha\mathcal{U}\beta)$ then there is a natural number $m \geq n$ such that $m \models \beta$ and for all n' with $n \leq n' < m$ we have $n' \models \alpha$. From this we get by induction hypothesis that there is a natural number $m \geq n$ such that $m \not\models \neg\beta$ and for all n' with $n \leq n' < m$ we have $n' \not\models \neg\alpha$. Now choose the smallest such $m \geq n$. As we have $m \not\models \neg\beta$ we get $n \not\models \Box\neg\beta$. If $m = n$ then clearly we have $n \not\models \neg\beta\mathcal{U}(\neg\alpha \wedge \neg\beta)$. If $m > n$, then we have for all n' with $n \leq n' < m$ that $n' \models \neg\beta$ and $m \not\models \neg\alpha \wedge \neg\beta$, but then we have as well $n \not\models \neg\beta\mathcal{U}(\neg\alpha \wedge \neg\beta)$. Therefore we have shown that

$$n \models (\alpha\mathcal{U}\beta) \quad \Rightarrow \quad n \not\models (\neg\beta\mathcal{U}(\neg\alpha \wedge \neg\beta)) \vee \Box\neg\beta.$$

But as the right side of the implication is exactly the definition of the negation for until-formulas, we get what we want

$$n \models (\alpha\mathcal{U}\beta) \quad \Rightarrow \quad n \not\models \neg(\alpha\mathcal{U}\beta).$$

For the other direction, by contraposition we assume $n \not\models (\alpha\mathcal{U}\beta)$. If there is no $n' > n$ such that $n' \models \beta$, then $n \models \Box\neg\beta$ and we are done. Suppose there is a least natural number l such that $n + l \models \beta$, then for all n' with $n \leq n' < n + l$ we have $n' \models \neg\beta$ and there must be a natural number n'' with $n \leq n'' < n + l$ such that $n'' \models \neg\alpha$. But from this we get $n \models \neg\beta\mathcal{U}(\neg\alpha \wedge \neg\beta)$ and so we are done.

□

The corresponding lemma for $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$-formulas can be proven analogue.

1.2 Tait-Style Sequent Calculi

In this work we will often use Tait-style sequent calculi. Let K stand for such an arbitrary calculus, then it is used to derive finite sets (or multisets) of formulas of a given language $\mathcal{L}$. These finite sets (or multisets) are called sequents. The rules of inference of K are configurations of the form

$$\frac{\Gamma_i, \gamma_i \quad i \in I}{\Gamma, \gamma}\ (\theta)$$

The sequents Γ_i, γ_i above the line are called the premisses of the rule θ, the K-sequent Γ, γ below the line is called the conclusion of the rule θ. If $I = \emptyset$ then the rule (θ) does not have any premisses, such rules are denoted as axioms. The particular formula γ displayed explicitly in the conclusion of the rule is called the **principle** or **distinguished formula**. The **active formulas** of the rule are those formulas that are explicitly shown in the rule, that is, the formulas γ_i and γ. The formulas contained in the sequents Γ and Γ_i are called side formulas.

Definition 1.2.1. *For all sequents Γ and all ordinal numbers α, we define $\vdash^{\alpha} \Gamma$ by induction on α:*

1. *If Γ is an axiom of K, then we have $\mathsf{K} \vdash^{\alpha} \Gamma$ for all ordinals α.*
2. *If there is a Tait rule (θ) with conclusion Γ such that for all premisses Γ_i $i \in I$ we have $\mathsf{K} \vdash^{\alpha_i} \Gamma_i$ with $\alpha_i < \alpha$, then we have $\mathsf{K} \vdash^{\alpha} \Gamma$.*

Hence the meaning of the expression $\mathsf{K} \vdash^{\alpha} \Gamma$ is that there exists a proof for the K-sequent Γ in the system K whose depth is bounded by the ordinal number α. We write $\mathsf{K} \vdash \Gamma$ if there is an ordinal number α such that $\mathsf{K} \vdash^{\alpha} \Gamma$. The **prooflength** is the minimal α such that we have $\mathsf{K} \vdash^{\alpha} \Gamma$. We write $\mathsf{K} \vdash^{<\alpha} \Gamma$ if there is an ordinal number $\beta < \alpha$ such that $\mathsf{K} \vdash^{\beta} \Gamma$.

Definition 1.2.2. *Given a language $\mathcal{L}$, a class of models $\mathcal{C}$ for that language and a calculus K, then a Tait rule of the form*

$$\frac{\Gamma_i, \gamma_i \ \ i \in I}{\Gamma, \gamma} \ (\theta)$$

is said to be ***sound with respect to*** $\mathcal{C}$ *if for all models $\mathcal{M} \in \mathcal{C}$ we have*

$$\mathcal{M} \models \Gamma_i, \gamma_i \quad \textit{for all } i \in I \quad \Rightarrow \quad \mathcal{M} \models \Gamma, \gamma.$$

If $\mathcal{C}$ is clear from the context we will just say that a rule (θ) is sound.

Chapter 2

Proof-Systems for PLTL

2.1 Hilbert-style Calculi and Calculi with Cut

Lichtenstein and Pnueli present in [9] a sound and complete Hilbert-style axiom system for PLTL. In a Hilbert-style system a finite list of axiom-schemas determines which formulas are taken as axioms and so by definition as true. These axioms together with a finite number of inference rules constitute the logic. Speaking more exactly in a Hilbert-style system a proof of a formula φ is a finite sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$ such that every φ_i is either an axiom or it is the result of an application of an inference rule to some of the previous formulas $\varphi_j \quad j < i$. If there is a proof for a formula φ, then φ is said to be a theorem. The logic of a given Hilbert-style axiom system then consits in the set of theorems.

The Hilbert-style system for PLTL given by Pnueli and Lichtenstein contains the following axiom-schemas, that is, for arbitrary $\mathcal{L}_{\mathsf{PLTL}}$-formulas φ and ψ the following formulas are valid:

- A0. φ, if φ is the substitution instance of a propositional tautology.
- A1. $\vdash \bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$
- A2. $\vdash \bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$
- A3. $\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- A4. $\vdash \Box(\varphi \rightarrow \bigcirc\varphi) \rightarrow (\varphi \rightarrow \Box\varphi)$
- A5. $\vdash (\varphi\mathcal{U}\psi) \leftrightarrow (\psi \vee (\varphi \wedge (\varphi\mathcal{U}\psi)))$
- A6. $\vdash (\varphi\mathcal{U}\psi) \rightarrow \Diamond\psi$

Note that A0. also includes temporal instances of tautologies; for example also the formula $\neg\Box\alpha \vee \Box\alpha$ would be the instance of a propositional tautology. A detailed comment on the axioms can be found in [9]. The system contains the following rules of inference:

- $(NEC_{\bigcirc})$ is the necessitation rule for the next-operator $\bigcirc$

$$\frac{\alpha}{\bigcirc\alpha}\ (NEC_{\bigcirc}).$$

- $(NEC_{\Box})$ is the necessitation rule for the always-operator $\Box$

$$\frac{\alpha}{\Box\alpha}\ (NEC_{\Box}).$$

- (MP) is the usual rule for the modus ponens

$$\frac{\varphi \to \psi \quad \varphi}{\psi}\ (MP).$$

If φ is a theorem of the given Hilbert-style axiom system, then we write $\mathsf{HS} \vdash \varphi$. It is an easy exercise to check the soundness of the axioms and of the inference rules. More difficult is a proof of completeness: The proof in [9] is based on a decision procedure using a semantic tableau that allows to check for the satisfiability of a given formula. The crux is that for checking the satisfiability it is enough to check for the existence of so called "fulfilling paths" on maximal strongly connected graphs that are constructed by the tableau procedure. Thus the following theorem holds for the Hilbert-style axiom system:

Theorem 2.1.1. *For every $\mathcal{L}_{\mathsf{PLTL}}$-formula φ we have*

$$\mathsf{HS} \vdash \varphi \quad \Leftrightarrow \quad \models \varphi.$$

From a proof theoretic standpoint the Hilbert-style system above has the disadvantage that Modus Ponens is like a cut rule. Any system containing an (unrestricted) cut rule makes it impossible to go for a systematic proof-search: Given an instance of an application of a cut rule, then there is no way to step back from the conclusion to its premisses. In the premisses there is always a formula that does not appear in the conclusion, therefore to check for all possible premisses in an application of (MP) would require to check for infinitely many premisses. Thus, the system above does not guide us from a formula φ to a proof of this formula. It is in some way forward and not backward oriented: That is, we always start with axioms and say because $\varphi_1, \ldots, \varphi_n$ are theorems

we can infer that also φ must be a theorem, but there is no way back from the conclusion φ to its premisses. The structure of the formula that should be proven does not come into play or at least not systematically. Proofs in Hilbert-style systems require experience about the best way to handle the axioms and inference rules and they force us to make proofs sometimes by trial and error.
In the literature also sequent-systems for PLTL can be found that contain a cut rule. Such as in Brünnler and Steiner [2] where the cut rule is inbuilt in the induction rule for formulas $\Box\alpha$:

$$\frac{\Gamma, \psi \quad \neg\psi, \bigcirc\psi \quad \neg\psi, \alpha}{\Gamma, \Box\alpha} \ (\Box - Ind).$$

Similarly we find in Peach [10] a cut rule hidden in the induction rule for the weak-until-operator $\mathcal{W}$[1]:

$$\frac{\Gamma, \psi \qquad \neg\psi, \bigcirc(\beta \vee (\alpha \wedge \psi))}{\Gamma, \alpha\mathcal{W}\beta} \ (\mathcal{R} - Ind).$$

2.2 Tableau Calculi

According to Schwendimann [12] the standard approach to work with tableaux for PLTL proceeds in two steps: There is a first step to create a graph followed by a second step to check for the fulfillment of the eventuality formulas. This second step usually involves an analysis of the strongly connected components contained in the graph. Work of this kind can be found in Pnueli and Lichtenstein [9], but as well in Wolper [14]. A different approach is found in Schwendimann [12], where a tableau method is given that performs the second step locally.
Here we present a tableau calculus that follows the standard method and is worked out in detail by Dixon, Fisher and Wooldridge in [5]. There a tableau-algorithm is presented that in a first step expands a structure in a way that later maybe a model can be extracted. Then in a second step this structure is contracted such that states are deleted that can not be part of a model. Finally this algorithm amounts to a decision procedure for the satisfiability of a given formula φ.

[1] The weak-until $\mathcal{W}$ is similar to the regular until, but differs from that insofar a formula of the form $(\alpha\mathcal{W}\beta)$ does not require that in the future the formula β must be satisfied. Formulas containing a weak-until can be introduced as the abbreviation

$$(\alpha\mathcal{W}\beta) := (\alpha\mathcal{U}\beta) \vee \Box\alpha.$$

The tableau system presented in [5] is in fact designed for a combined logic of knowledge and time (resp. belief and time), that is, it extends the standard decision procedure for $\mathcal{L}_{\mathsf{PLTL}}$-formulas to a language that contains an additional operator for knowledge (resp. for belief). We just have a closer look at the temporal part of this tableau system.

2.2.1 A Tableau-Based Decision Procedure

The tableau procedure is defined with respect to the language $\mathcal{L}_{\mathsf{PLTL}}$. It makes use of the following tableau rules for the classical and temporal operators:

Tableau rules:

$$\frac{\alpha \wedge \beta}{\alpha, \beta}\ (\wedge) \qquad \frac{\alpha \vee \beta}{\alpha \quad | \quad \beta}\ (\vee)$$

$$\frac{\Box\varphi}{\varphi, \bigcirc\Box\varphi}\ (\Box) \qquad \frac{\Diamond\varphi}{\varphi \quad | \quad \neg\varphi \wedge \bigcirc\Diamond\varphi}\ (\Diamond)$$

$$\frac{\varphi\mathcal{U}\psi}{\psi \quad | \quad \neg\psi \wedge \varphi \wedge \bigcirc(\varphi\mathcal{U}\psi)}\ (\mathcal{U})$$

These rules will be used to construct the set of propositional tableaux.
There is need to ensure that the states constructed in the tableau procedure are consistent, this is done by checking if a state is proper or not.

Definition 2.2.1. *A set of $\mathcal{L}_{\mathsf{PLTL}}$-formula Δ is said to be* ***proper*** *if*

$$\varphi \in \Delta \quad \Rightarrow \quad \neg\varphi \notin \Delta.$$

Let us now give the algorithm for the construction of the set of propositional tableaux for a given set of formulas Δ. There are two steps in the algorithm: First the tableau rules above are used to expand the set $\mathcal{F} = \{\Delta\}$ to a set $\mathcal{F} = \{\Delta_1, \dots, \Delta_n\}$. In a second step improper sets $\Delta_i \in \mathcal{F}$ are deleted.

The Set of Propositional Tableaux of Δ: Start with the set $\mathcal{F} = \{\Delta\}$ and apply the next step until no further application is possible, then move to step two.

1. For any proper $\Delta' \in \mathcal{F}$, we take a formula $\varphi \in \Delta'$ on which one of the tableau rules has not yet been applied, then

   - If $\varphi = \alpha \wedge \beta$, then

   $$\mathcal{F} = \mathcal{F} \setminus \{\Delta'\} \cup \{\Delta' \cup \{\alpha, \beta\}\}$$

- If $\varphi = \Box\alpha$, then
$$\mathcal{F} = \mathcal{F} \setminus \{\Delta'\} \cup \{\Delta' \cup \{\alpha, \bigcirc\Box\alpha\}\}$$
- If $\varphi = \Diamond\alpha$, then
$$\mathcal{F} = \mathcal{F} \setminus \{\Delta'\} \cup \{\Delta' \cup \{\varphi\}\} \cup \{\Delta' \cup \ \{\neg\varphi \wedge \bigcirc\Diamond\varphi\}\}$$
- If $\varphi = \alpha\mathcal{U}\beta$, then
$$\mathcal{F} = \mathcal{F} \setminus \{\Delta'\} \cup \{\Delta' \cup \{\psi\}\} \cup \{\Delta' \cup \ \{\neg\psi \wedge \varphi \wedge \bigcirc(\varphi\mathcal{U}\psi)\}\}$$

2. Every improper $\Delta_i \in \mathcal{F}$ is deleted.

The set $\mathcal{F} = \{\Delta_1, \ldots, \Delta_n\}$ resulting from the application of this algorithm to Δ is called the **set of propositional tableaux of** Δ. Note that a tableau rule is only applied if it changes the set of propositional tableaux. For example there is no iterated application of the $(\Box)$-rule when just the same sets are obtained.

Given we can construct the set of propositional tableaux for a given set Δ, a further algorithm can be defined that allows to check for the satisfiability of a given formula φ_0. This algorithm will construct a tableau structure from which a $\mathcal{L}_{\mathsf{PLTL}}$-model for φ_0 can be extracted.

Definition 2.2.2. *We say a **tableau structure** is a triple $\mathcal{H} = (S, \eta, L)$, where:*

- *S is a set of states.*
- *$\eta \subset S \times S$ is a binary relation, that is interpreted as the nexttime-relation on S.*
- *$L : S \to \mathcal{P}(Fml(\mathcal{L}_{\mathsf{PLTL}}))$ is a function that labels each state with a set of formulas.*

We give in detail the algorithm for the construction of the tableau structure $\mathcal{H} = (S, \eta, L)$ for a given formula φ_0.

Tableau Algorithm: Given the $\mathcal{L}_{\mathsf{PLTL}}$-formula φ_0 , then proceed by the following steps.

1. **Initialization:**

   First, set
$$S = \eta = L = \emptyset.$$
   Construct $\mathcal{F}$, the set of propositional tableaux for $\{\varphi_0\}$. For each $\Delta_i \in \mathcal{F}$ create a new state s_i and let $L(s_i) = \Delta_i$ and $S = S \cup \{s_i\}$. For each $\Delta_i \in \mathcal{F}$ proceed with the next step until none apply.

2. **Creating $\bigcirc$-successors**:

   For any state s labelled by formulas $L(s)$, where $L(s)$ is proper and a propositional tableau, if $\bigcirc\psi \in L(s)$ create the set of formulas $\Delta = \bigcirc(L(s))^2$. For each Δ construct $\mathcal{F}$ the set of propositional tableaux for Δ, and for each member $\Delta' \in \mathcal{F}$ if there is a state $s'' \in S$ such that $\Delta' = L(s'')$ then add (s, s'') to η, otherwise add a state s' to the set of states , labelled by $L(s') = \Delta'$ and add (s, s') to η.

3. **Contraction:**

   Continue deleting any state s where:

   - There is a $\Diamond\varphi \in L(s)$ such that there is no s' with $(s, s') \in \eta^*$ and $\varphi \in L(s')$.[3]
   - There is $\bigcirc\varphi \in L(s)$ such that there is no s' with $(s, s') \in \eta$.

   Continue with this step until no further deletions are possible.

It is important that expansion steps and deletion steps are not interleaved, otherwise states may be wrongly deleted. For a formula φ_0 the tableau algorithm is said to be **successful** if the constructed structure contains a state s such that $\varphi_0 \in L(s)$. The following theorem relates the notion of satisfaction to the tableau algorithm.

Theorem 2.2.3. *If φ_0 is a $\mathcal{L}_{\mathsf{PLTL}}$-formula then φ_0 is satisfiable if and only if the tableau algorithm on φ_0 returns a structure $\mathcal{H} = (S, \eta, L)$ such that $\varphi_0 \in L(s)$.*

In the proof of this theorem it is shown that from the constructed tableau structure $\mathcal{H} = (S, \eta, L)$ for φ_0 an ordinary $\mathcal{L}_{\mathsf{PLTL}}$-model $\mathcal{M} = (N, \pi)$ for φ_0 can be extracted. The set of states S together with the successor relation η defines a timeline N, whereas the labelling function L can be used to determine a valuation function π. This is worked out in full detail in [5].
That the tableau algorithm establishes effectively a decision procedure there is need to make sure that it terminates, this is stated by the following theorem.

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[2]We follow the convention: $\bigcirc(L(s)) := \{\varphi \mid \bigcirc\varphi \in L(s)\}$.

[3]Given the relation $\eta \subset S \times S$ we define:

(a) The transitive hull of η is given by:

$$(x, y) \in \ \eta^+ \ :\Leftrightarrow \exists n \geq 0 \ \exists x = x_1, \ldots, x_n = y \in S \ \text{ such that } \ (x_i, x_{i+1}) \in \eta.$$

(b) Whereas the transitive closure of η is:

$$(x, y) \in \ \eta^* \ :\Leftrightarrow x = y \ \text{ or } \ (x, y) \in \ \eta^+.$$

Theorem 2.2.4. *If φ_0 is a $\mathcal{L}_{\mathsf{PLTL}}$-formula, then the tableau algorithm applied to φ_0 terminates.*

2.2.2 From Tableau to Sequent Calculi

Often a tableau can easily be converted into a sequent calculus. Unfortunately this is not true for PLTL and it is hard to see how a corresponding sequent calculus to a tableau like the one presented above should look like. To give the corresponding sequent-system for this tableau would require us not only to invert the tableau rules, but also to simulate the deletion process that is involved in the tableau algorithm. It is far from clear how this deletion process can be incorporated in a sequent calculus that can still be understood as “classical”.
Although, it is elucidating to have a closer look to what happens when we are just standardly inverting the rules of the tableau presented in [5]. For the eventual operator $\Diamond$ and the $\Box$-operator we would get as corresponding sequent-rules the unfolding rules for the fixpoints:

$$\frac{\Gamma, \alpha, \bigcirc\Diamond\alpha}{\Gamma, \Diamond\alpha}\ (\Diamond) \qquad \frac{\Gamma, \alpha \qquad \Gamma \bigcirc \Box\alpha}{\Gamma, \Box\alpha}\ (\Box).$$

Brünnler and Lange show in [3] that a system containing a $\Box$-rule of this form can lead into a branch that never ends on an axiomatic sequent of the form $\Gamma, p, \sim p$. This case occurs when in the countermodel construction in the case of an unfolding of a formula $\Box\alpha$ always the right side of the premisses is chosen. For example suppose we want to give a proof of the induction axiom

$$\varphi \wedge \Box(\varphi \to \bigcirc\varphi) \to \Box\varphi.$$

A proof-tree could be of the form:

$$\dfrac{\Diamond(\varphi \wedge \bigcirc\neg\varphi), \neg\varphi, \varphi \qquad \dfrac{\dfrac{\bigcirc\Diamond(\varphi \wedge \bigcirc\neg\varphi), \varphi, \neg\varphi, \bigcirc\Box\varphi \qquad \dfrac{\Diamond(\varphi \wedge \bigcirc\neg\varphi), \neg\varphi, \Box\varphi}{\bigcirc\Diamond(\varphi \wedge \bigcirc\neg\varphi), \bigcirc\neg\varphi, \neg\varphi, \bigcirc\Box\varphi}\ (\bigcirc)}{\bigcirc\Diamond(\varphi \wedge \bigcirc\neg\varphi), \varphi \wedge \bigcirc\neg\varphi, \neg\varphi, \bigcirc\Box\varphi}\ (\wedge)}{\Diamond(\varphi \wedge \bigcirc\neg\varphi), \neg\varphi, \bigcirc\Box\varphi}\ (\Diamond)}{\Diamond(\varphi \wedge \bigcirc\neg\varphi), \neg\varphi, \Box\varphi}\ (\Box)$$

In this case the sequent appearing in the root is repeated in the branch on the right. This proof can infinitely often be repeated and therefore there would be a branch in the proof-tree that does not close. So there is need for an additional criteria to deal with cycling branches of this form. An obvious answer would be to say a branch is closed if we have established a cycle, that is, we would call a branch closed after the first repetition of a sequent. As Brünnler and Lange show this would lead to an unsound calculus, they present a counterexample of a derivation of a formula that is not valid:

$$\dfrac{\dfrac{\dfrac{\Box\varphi, \bigcirc\Diamond\Box\varphi}{\Diamond\Box\varphi}\,(\Diamond)}{\varphi, \bigcirc\Diamond\Box\varphi}\,(\bigcirc) \qquad \dfrac{\dfrac{\Box\varphi, \bigcirc\Diamond\Box\varphi}{\Box\varphi, \Diamond\Box\varphi}\,(\Diamond)}{\bigcirc\Box\varphi, \bigcirc\Diamond\Box\varphi}\,(\bigcirc)}{\Box\varphi, \bigcirc\Diamond\Box\varphi}\,(\Box)$$

This indicates that cycles involving lowest fixpoints as $\Diamond\varphi$ and cycles involving greatest fixpoints $\Box\varphi$ behave in a different manner. This point will be made clear in the next section where the system DHL will be presented that allows to check systematically for cycles in a proof-tree. We will see that in fact it is sufficient to check for cycles that occur by the unfolding of greatest fixpoints.

2.3 The non-wellfounded System DHL

Dax, Hofmann and Lange exploit in [4] properties of cycling sequents in infinite branches to develop a sound and complete proof-system for the linear time μ-calculus. We will refer to this system henceforth as the system DHL. Instead of particular temporal operators the language of the linear time μ-calculus $\mathcal{L}_\mu^{lin}$ contains greatest and least fixpoint operators to construct arbitrary fixpoints. The language $\mathcal{L}_\mu^{lin}$ can then be understood as the μ-calculus interpreted over the PLTL-models introduced in the first chapter. As $\mathcal{L}_{\mathsf{PLTL}}$-formulas containing temporal connectives can be viewed as fixpoints, PLTL can be seen as a fragment of the linear time μ-calculus. Hence the system DHL for the linear time μ-calculus is relevant for our work insofar it induces also a proof-system for PLTL. In this section we are going to have a closer look at the system DHL, especially at the use of so called threads in order to proof completeness.

2.3.1 Syntax and Semantics

Definition 2.3.1. *The alphabet of the language $\mathcal{L}_\mu^{lin}$ contains the following **basic syntactical symbols**:*

1. *An enumerable number of positive atomic propositions $p_1, p_2, p_3, \ldots$*
2. *An enumerable number of propositional variables $x_1, x_2, x_3, \ldots$*
3. *A symbol $\sim$.*
4. *The connectives $\wedge$ (and), $\vee$ (or) and $\bigcirc$ (next).*
5. *The operators μ and ν to construct least, resp. greatest fixpoints.*

6. *Parentheses, brackets, dots and commas.*

Definition 2.3.2. *The* $\mathcal{L}_\mu^{lin}$***-formulas*** *are defined inductively in the following way:*

1. *Every positive atomic proposition* p *and every negative atomic proposition* $\sim p$ *is* $\mathcal{L}_\mu^{lin}$*-formula.*
2. *Every positive propositional variable* x *and every negative propositional variable* $\sim x$ *is* $\mathcal{L}_\mu^{lin}$*-formula.*
3. *If* φ *and* ψ *are* $\mathcal{L}_\mu^{lin}$*-formulas then* $(\varphi \wedge \psi)$ *and* $(\varphi \vee \psi)$ *are* $\mathcal{L}_\mu^{lin}$*-formulas.*
4. *If* φ *is a* $\mathcal{L}_\mu^{lin}$*-formula, then* $\bigcirc\varphi$ *is a* $\mathcal{L}_\mu^{lin}$*-formula.*
5. *If* φ *is a* $\mathcal{L}_\mu^{lin}$*-formula, then* $\nu x.\varphi(x)$ *and* $\mu x.\varphi(x)$ *are* $\mathcal{L}_\mu^{lin}$*-formulas.*

Definition 2.3.3. *A finite set of* $\mathcal{L}_\mu^{lin}$*-formulas will be called a* $\mathcal{L}_\mu^{lin}$***-sequent****.*

If clear from the context we will often talk just about formulas and sequents, rather than $\mathcal{L}_\mu^{lin}$-formulas and $\mathcal{L}_\mu^{lin}$-sequents. We will often use the symbol σ meaning that we have either ν or μ. The substitution of all occurrences of a propositional variable x by a formula ψ is written as $\varphi[\psi/x]$.
Negation $\neg\varphi$ for any $\mathcal{L}_\mu^{lin}$-formula φ is introduced by definition, where we make use of de Morgan's laws and of the duality of μ and ν.

Definition 2.3.4. *The* ***negation*** $\neg\varphi$ *of a* $\mathcal{L}_\mu^{lin}$*-formula* φ *is inductively defined as follows:*

1. *If* φ *is a positive atomic proposition* p*, then* $\neg\varphi :=\sim p$*; if* φ *is a negative atomic proposition* $\sim p$*, then* $\neg\varphi := p$*.*
2. *If* φ *is a positive propositional variable* x*, then* $\neg\varphi :=\sim x$*; if* φ *is a negative propositional variable* $\sim x$*, then* $\neg\varphi := x$*.*
3. *If* φ *is of the form* $(\alpha \wedge \beta)$*, then* $\neg\varphi := (\neg\alpha \vee \neg\beta)$*; if* φ *is of the form* $(\alpha \vee \beta)$*, then* $\neg\varphi := (\neg\alpha \wedge \neg\beta)$*.*
4. *If* φ *is of the form* $\bigcirc\alpha$*, then* $\neg\varphi := \bigcirc\neg\alpha$*.*
5. *If* α *is of the form* $\mu x.\varphi(x)$*, then* $\neg\varphi := \nu x.\neg\varphi(\neg x)$
6. *If* α *is of the form* $\nu x.\varphi(x)$*, then* $\neg\varphi := \mu x.\neg\varphi(\neg x)$

The connectives $\rightarrow$ and $\leftrightarrow$ can now be defined in the usual way by the use of $\neg$ and $\wedge$.

Definition 2.3.5. *The **set of subformula** $sub(\varphi)$ for a $\mathcal{L}_{\mu}^{lin}$-formula φ is defined inductively:*

1. *If φ is a positive atomic proposition p, then $sub(\varphi) := \{p\}$; if φ is a negative atomic proposition $\sim p$, then $sub(\sim p) := \{\sim p\}$.*

2. *If φ is a positive propositional variable x, then $sub(\varphi) := \{x\}$, if φ is a negative propositional variable $\sim x$, then $sub(\varphi) := \{\sim x\}$*

3. *If φ is of the form $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ or $(\alpha \mathcal{U} \beta)$, then $sub(\varphi) := \{\varphi\} \cup sub(\alpha) \cup sub(\beta)$.*

4. *If φ is of the form $\mu x.\alpha(x)$ or $\nu x.\alpha(x)$, then $sub(\varphi) := \{\varphi\} \cup sub(\alpha)$*

Definition 2.3.6. *The semantics for the system* DHL *is defined relative to a **linear time model** $\mathcal{T} = (\mathcal{M}, \mathcal{V})$, where:*

- *$\mathcal{M}$ is an ordinary* PLTL*-model $\mathcal{M} = (N, \pi)$.*
- *$\mathcal{V}$ is a valuation function $\mathcal{V} : X \to \mathcal{P}(N)$ for the propositional variables.*

Definition 2.3.7. *Given a linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V})$, the set $\| \varphi \|_{\mathcal{V}}^{\mathcal{M}}$ of states satisfying the formula φ is defined inductively as follows:*

1. *If $\varphi = p$, then $\|p\|_{\mathcal{V}}^{\mathcal{M}} = \pi(p)$.*

2. *If $\varphi = \sim p$, then $\|\sim p\|_{\mathcal{V}}^{\mathcal{M}} = N \setminus \pi(p)$.*

3. *If $\varphi = x$, then $\|x\|_{\mathcal{V}}^{\mathcal{M}} = \mathcal{V}(x)$.*

4. *If $\varphi = \sim x$, then $\|\sim x\|_{\mathcal{V}}^{\mathcal{M}} = N \setminus \mathcal{V}(x)$.*

5. *If $\varphi = (\alpha \wedge \beta)$, then $\|(\alpha \wedge \beta)\|_{\mathcal{V}}^{\mathcal{M}} = \|\alpha\|_{\mathcal{V}}^{\mathcal{M}} \cap \|\beta\|_{\mathcal{V}}^{\mathcal{M}}$.*

6. *If $\varphi = (\alpha \vee \beta)$, then $\|(\alpha \vee \beta)\|_{\mathcal{V}}^{\mathcal{M}} = \|\alpha\|_{\mathcal{V}}^{\mathcal{M}} \cup \|\beta\|_{\mathcal{V}}^{\mathcal{M}}$.*

7. *If $\varphi = \bigcirc\alpha$, then $\|\bigcirc\alpha\|_{\mathcal{V}}^{\mathcal{M}} = \{n \mid n + 1 \in \|\alpha\|_{\mathcal{V}}^{\mathcal{M}}\}$.*

8. *If $\varphi = \mu x.\alpha(x)$, then $\|\mu x.\alpha(x)\|_{\mathcal{V}}^{\mathcal{M}} = \bigcap\{S \subset N \mid \ \|\alpha\|_{\mathcal{V}[x:=S]}^{\mathcal{M}} \subseteq S\}$.*

9. *If $\varphi = \nu x.\alpha(x)$, then $\|\nu x.\alpha(x)\|_{\mathcal{V}}^{\mathcal{M}} = \bigcup\{S \subset N \mid S \subseteq \|\alpha\|_{\mathcal{V}[x:=S]}^{\mathcal{M}}\}$.*

Where $\mathcal{V}[x := S]$ is the valuation function that maps x on S and is identical to $\mathcal{V}$ in all the other cases.

Definition 2.3.8. *Given a* PLTL*-model* $\mathcal{M} = (N, \pi)$ *and a valuation function* $\mathcal{V}$ *for the propositional variables, then a* $\mathcal{L}_\mu^{lin}$*-formula* φ *is said*

1. *to be* ***satisfiable in the linear time model*** $\mathcal{T} = (\mathcal{M}, \mathcal{V})$ *if there is a state* $n \in N$ *such that* $n \in \| \varphi \|_{\mathcal{V}}^{\mathcal{M}}$*, we then write* $\mathcal{T}, n \models \varphi$.

2. *to be* ***valid in the linear time model*** $\mathcal{T} = (\mathcal{M}, \mathcal{V})$*, if for all states* $n \in N$ *we have* $\mathcal{T}, n \models \varphi$*, we then write* $\mathcal{T} \models \varphi$.

3. *to be* ***valid*** *if for all linear time models* $\mathcal{T} = (\mathcal{M}, \mathcal{V})$ *we have* $\mathcal{T} \models \varphi$*, we then write* $\models \varphi$.

Definition 2.3.9. *Given a* PLTL*-model* $\mathcal{M} = (N, \pi)$ *and a valuation function* $\mathcal{V}$ *for the propositional variables, then a* $\mathcal{L}_\mu^{lin}$*-sequent* Γ *is said*

1. *to be* ***satisfiable in*** $\mathcal{T} = (\mathcal{M}, \mathcal{V})$*, if there is a state* n *such that* $n \in \| \bigvee \Gamma \|_{\mathcal{V}}^{\mathcal{M}}$*, we then write* $\mathcal{T}, n \models \Gamma$.

2. *to be* ***valid in*** $\mathcal{T} = (\mathcal{M}, \mathcal{V})$*, if for all states* $n \in N$*, we have* $\mathcal{T}, n \models \Gamma$*, we then write* $\mathcal{T} \models \Gamma$.

3. *to be* ***valid****, if for all linear time structures* $\mathcal{T} = (\mathcal{M}, \mathcal{V})$ *we have* $\mathcal{T} \models \Gamma$*, we then write* $\models \Gamma$.

2.3.2 Proof Definition

A Tait-style proof system for DHL can be established by the following rules of inference:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge) \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee)$$

$$\frac{\Gamma}{\bigcirc\Gamma, \Delta}\ (\bigcirc) \qquad \frac{\Gamma, \varphi[\sigma x.\varphi/x]}{\Gamma, \sigma x.\varphi}\ (\sigma)$$

Definition 2.3.10. *A* ***pre-proof for*** φ *is a possibly infinite tree whose nodes are labeled with sequents, whose root is labeled with* φ *and which is built according the proof rules* $(\wedge)$, $(\vee)$, $(\bigcirc)$, (σ).

Definition 2.3.11. *Let* Γ *be a set of* $\mathcal{L}_\mu^{lin}$*-formulas. The* ***Fischer-Ladner Closure*** $\mathbb{FL}(\Gamma)$ *of* Γ *is defined to be the smallest set such that:*

1. $\Gamma \subseteq \mathbb{FL}(\Gamma)$.

2. *If* $(\alpha \wedge \beta)$ *or* $(\alpha \vee \beta)$ *is in* $\mathbb{FL}(\Gamma)$*, then* α *and* β *are in* $\mathbb{FL}(\Gamma)$.

3. *If $\bigcirc\alpha$ is in $\mathbb{FL}(\Gamma)$, then α is in $\mathbb{FL}(\Gamma)$.*
4. *If $\mu x.\psi(x)$ is in $\mathbb{FL}(\Gamma)$, then $\psi(x)[\mu x.\psi(x)/x]$ is in $\mathbb{FL}(\Gamma)$.*
5. *If $\nu x.\psi(x)$ is in $\mathbb{FL}(\Gamma)$, then $\psi(x)[\nu x.\psi(x)/x]$ is in $\mathbb{FL}(\Gamma)$.*

By Fischer and Ladner [6] the cardinality $|\mathbb{FL}(\Gamma)|$ is bound by the syntactical length of the formulas $\varphi \in \Gamma$.
In the sequel we are going to introduce the notion of a thread in a pre-proof tree.

Definition 2.3.12. *Given a rule application (θ_i) occurring in a pre-proof for φ_0. We define a* ***connection relation*** *$Con(\theta_i) \subset \mathbb{FL}(\{\varphi_0\}) \times \mathbb{FL}(\{\varphi_0\})$ as follows, we have $(\varphi, \psi) \in Con(\theta_i)$ if:*

1. *$\varphi = \psi$ is a side formula of (θ_i).*
2. *φ is an active formula in the conclusion and ψ is an active formula in the premise of (θ_i).*

Definition 2.3.13. *Given a branch $\Gamma_0, \Gamma_1, \ldots$ in a pre-proof tree [4], let (θ_i) be the rule application that infers Γ_i from Γ_{i+1}. A* ***thread*** *in this branch is a sequence of formula $\varphi_0, \varphi_1, \ldots$ such that $(\varphi_i, \varphi_{i+1}) \in Con(\theta_i)$ and $\varphi_i \in \Gamma_i$ for all i.*

Definition 2.3.14. *We say that a thread $\varphi_0, \varphi_i, \ldots$ is* ***ν-thread*** *if there is a $\nu x.\psi(x) \in \mathbb{FL}(\varphi_0)$ such that $\varphi_i = \nu x.\psi(x)$ for infinitely many $i \in \mathbb{N}$ and for all $\mu y.\psi'(y)$ s.t. $\nu x.\psi \in sub(\mu y.\psi'(y))$ there are only finitely many $\varphi_i = \mu y.\psi'(y)$. A μ-thread is defined analogously.*

To ensure that the proof-definition is well-defined the following lemma is given in [4].

Lemma 2.3.15. *Every thread is either a ν-thread or a μ-thread.*

Proofs will now be defined as possibly infinite trees in which the infinite branches satisfy some global condition.

Definition 2.3.16. *A* ***proof for*** *φ_0 is a pre-proof such that every finite branch ends in a sequent $\Gamma, p, \sim p$, and every infinite branch contains a ν-thread. If there is a proof for φ_0 we write $\mathsf{DHL} \vdash \varphi_0$.*

Theorem 2.3.17. *For every $\mathcal{L}_\mu^{lin}$-formula φ, we have*

$$\mathsf{DHL} \vdash \varphi \quad \Leftrightarrow \quad \models \varphi$$

Proof. The proof can be found in [4]. □

[4] When talking about branches we always have a bottom up perspective on a proof tree, that is, in the branch $\Gamma_0, \Gamma_1 \ldots$ the sequent Γ_0 is the root of the proof-tree.

2.3.3 Relating PLTL to DHL

Formulas of $\mathcal{L}_{\mathsf{PLTL}}$ containing temporal operators correspond to fixpoints-formulas in the language $\mathcal{L}_\mu^{lin}$. For example the $\mathcal{L}_{\mathsf{PLTL}}$-formula $\Box\varphi$ can be interpreted as the greatest fixpoint of the $\mathcal{L}_\mu^{lin}$-formula $\varphi \wedge \bigcirc x$. Therefore the $\mathcal{L}_\mu^{lin}$-formula corresponding to the always-operator would be

$$\nu.\varphi \wedge \bigcirc x.$$

Analogue the $\mathcal{L}_{\mathsf{PLTL}}$-formula $\Diamond\varphi$ corresponds to the formula

$$\mu.\varphi \vee \bigcirc x$$

and the $\mathcal{L}_{\mathsf{PLTL}}$-formula $\varphi\mathcal{U}\psi$ would be interpreted as

$$\mu.(\psi \vee (\varphi \wedge \bigcirc x)).$$

Therefore $\mathcal{L}_{\mathsf{PLTL}}$ can be viewed as a fragment of the linear-time μ-calculus. In this spirit the system DHL restricted to the unfolding rules for the fixpoints corresponding to $\Box\varphi$, $\Diamond\varphi$ and $\varphi\mathcal{U}\psi$ induces a proof system for PLTL that is sound and complete. The problem about this restriction of DHL to PLTL is that the resulting proof system is not wellfounded and therefore not "classical" in our sense.

Still there are interesting observations to make that might be helpful to give a finitary, cut-free and wellfounded calculus for PLTL. Suppose we are given a restriction of DHL as indicated above, that is, we have a calculus for the language $\mathcal{L}_{\mathsf{PLTL}}$ that contains apart from the rules for the classical connectives and a rule for the next time operator the unfolding rules for the temporal operators $\mathcal{U}$, $\Diamond$ and $\Box$:

$$\frac{\Gamma, \alpha \wedge \bigcirc\Box\alpha}{\Gamma, \Box\alpha}\ (\Box)$$

$$\frac{\Gamma, (\alpha \wedge \bigcirc(\alpha\mathcal{U}\beta)) \vee \beta}{\Gamma, \alpha\mathcal{U}\beta}\ (\mathcal{U}) \qquad \frac{\Gamma, \alpha \vee \bigcirc\Diamond\alpha}{\Gamma, \Diamond\alpha}\ (\Diamond)$$

Adapting the proof-definition of DHL to this fragment would amount to say that in a proof-tree either a branch will end on a regular axiom of the form $\Gamma, p, \sim p$ or either it will be ¡ and contain a thread belonging to a greatest fixpoint. As both $\mathcal{U}$ and $\Diamond$ generate only lowest fixpoints, we are just required to check for the existence of threads corresponding to $\Box$-formulas, let us call them $\Box$-threads. This says, given an infinite branch $\Gamma_0, \Gamma_1, \ldots$ then there must be a formula $\Box\alpha$ occurring infinitely often in a thread of this branch. Therefore there must be infinitely many Γ_k with $k \in \mathbb{N}$ on this branch such that

$$\Gamma_k = \Delta_k, \Box\alpha.$$

We know that all $\Delta_k \subseteq \mathbb{FL}(\Gamma_0)$ and we know as well that $|\mathbb{FL}(\Gamma_0)| < \infty$. But from this we infer by an easy combinatorial exercise that on the branch there must be sequents Γ_i and Γ_j with $i < j$ such that

$$\Gamma_i = \Gamma_j = \Delta_i, \Box\alpha.$$

That is, on infinite branches containing a $\Box$-thread there must be a repetition of at least one sequent. This fact is similarly also stated in Studer [11].
In the next section a system will be presented that uses theses observations about the cycling behavior of sequents to give a finitary calculus for PLTL.

2.4 The Labelled System $\mathsf{LT1}$

Finally let us have a look at the sequent calculus $\mathsf{LT1}$ that is based on the system DHL and is truly finitary and cut free. It was developed by Brünnler and Lange and is found in [3]. As we have seen in the last section the completeness of DHL depends on the existence of cycling sequents on infinite branches. Having this in mind Brünnler and Lange present a proof-system for PLTL that closes infinite branches after the first repetition of a cycling sequent.
The system $\mathsf{LT1}$ makes use of the language $\mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}$: That is, the language $\mathcal{L}^{\mathcal{R}}_{\mathsf{PLTL}}$ first reduced by the temporal operators for eventual $\Diamond$ and always $\Box$ and then extended by labelled or how Brünnler and Lange call it annotated release-formulas. These are release-formulas that possess a subscript H which stands for a finite set of sequents. Formally they are defined in the following way: If ψ and φ are $\mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}$-formulas, then also

$$(\varphi \mathcal{R}_H \psi) \quad \text{and} \quad \bigcirc(\psi \mathcal{R}_H \varphi) \quad \text{are } \mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}\text{-formulas}$$

where the label or annotation H is a finite set of finite sets of formulas. In other words, a labelled formula in $\mathsf{LT1}$ is a pair of a label H and a formula $\varphi\mathcal{R}\psi$, resp. $\bigcirc(\varphi\mathcal{R}\psi)$.
The fundamental idea is that the label H is used to store all the contexts on the branch where an unfolding rule to a release formula has been applied. In this way it becomes possible to detect if there is a cycling sequent on the branch or not, if yes then the branch will be closed. As shown in DHL there is only need to establish the existence of cycles generated by the unfolding of greatest fixpoints. Therefore there is no need to annotated formulas corresponding to lowest fixpoints.

Definition 2.4.1. *A $\mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}$-**sequent** is said to be a finite set of $\mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}$-formulas that contains at most one annotated formula.*

The interpretation of $\mathcal{L}^{\mathcal{R}+}_{\mathsf{PLTL}}$-formulas is the standard semantics presented in the first chapter. Whereas the semantics of an annotated formula $(\varphi \mathcal{R}_H \psi)$ is given by the interpretation of its corresponding formula

$$(\varphi \vee \overline{H}) \mathcal{R} (\psi \vee \overline{H})$$

where

$$\overline{H} := \bigwedge_{\gamma_i \in \Gamma_1} \neg\gamma_i \vee \cdots \vee \bigwedge_{\gamma_i \in \Gamma_n} \neg\gamma_i \quad \Gamma_i \in H.$$

The system $\mathsf{LT1}$ is determined by the following rules:

I. Axioms of $\mathsf{LT1}$:

$$\frac{}{\Gamma, p, \sim p}\ (Ax.1) \qquad \frac{}{\Gamma, \varphi \mathcal{R}_{H,\Gamma} \psi}\ (Ax.2)$$

II. Classical rules of $\mathsf{LT1}$:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge) \qquad \frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee)$$

$$\frac{\Gamma, (\varphi \wedge \bigcirc(\varphi \mathcal{U} \psi)) \vee \psi}{\Gamma, \varphi \mathcal{U} \psi}\ (\mathcal{U}) \qquad \frac{\Gamma, \psi \wedge (\bigcirc(\varphi \mathcal{R} \psi) \vee \varphi)}{\Gamma, \varphi \mathcal{R} \psi}\ (\mathcal{R})$$

$$\frac{\Gamma}{\bigcirc\Gamma, \Delta}\ (\bigcirc) \qquad \bigcirc\,\Gamma := \{\bigcirc\gamma \mid \gamma \in \Gamma\}$$

III. Cycle testing rules of $\mathsf{LT1}$:

$$\frac{\Gamma, \varphi \mathcal{R}_{\emptyset} \psi}{\Gamma, \varphi \mathcal{R} \psi}\ (foc)$$

$$\frac{\Gamma, \psi \wedge (\bigcirc(\varphi \mathcal{R}_{H,\Gamma} \psi) \vee \varphi)}{\Gamma, \varphi \mathcal{R}_H \psi}\ (RN_H)$$

Let us have a closer look at the non standard axioms and rules that are introduced to deal with cycling sequents:

- (RN_H): In a top down reading of the proof-tree this rule eliminates sequents that are contained in the label H. In the more intuitive bottom up reading of the proof tree this rule stores the side formulas that are involved in every application of an unfolding of a release formula. Note that the rule looks like the ordinary unfolding rule ($\mathcal{R}$) except that the release formula is labelled by all the sets that occurred as side formulas on this branch while applying the unfolding rule to the release formula. Remember that the crucial point is to close a branch $\Gamma_0, \Gamma_1 \ldots$ after we have found a repetition of a sequent on it, that is, if we have found Γ_i, Γ_j with $i < j$ such that

$$\Gamma_i = \Gamma_j = \Delta, \varphi \mathcal{R} \psi.$$

As the release formula in a sequent contains all the information about former applications of the unfolding rule, it enables us to check for such repetitions of sequents on the branch.

- (foc): This rule says that if we have a proof of a release formula annotated with an empty label $(\varphi \mathcal{R}_{\emptyset} \psi)$, then we have a proof of the label-free formula $(\varphi \mathcal{R} \psi)$. In a top-down reading of the proof-tree this means that if we were able to eliminate all sets contained in the annotation by sufficiently many applications of the unfolding rule, then we are allowed to switch from an annotated release formula to an ordinary release formula.

- $(Ax.2)$: This additional axiom is needed to close a branch if we were able to establish the existence of a cycle.

The labelled system $\mathsf{LT1}$ is sound and complete for sequents that do not contain annotated formulas:

Theorem 2.4.2. *Let Γ be a $\mathsf{LT1}$-sequent such that it does not contain an annotated formula, then we have*

$$\mathsf{LT1} \vdash \Gamma \quad \Leftrightarrow \quad \models \Gamma.$$

Chapter 3

The Infinitary Calculus K^∞

The core element about an infinitary system for PLTL is that it makes use of an ω-rule, that is, a rule containing infinitely many premisses allowing to derive a sequent containing a formula $\Box\alpha$. There are several ways such an infinitary rule can be formulated, a prominent version is

$$\frac{\Gamma, \bigcirc^k\alpha \ \ (\forall k \in \mathbb{N})}{\Gamma, \Box\alpha} \ (\omega).$$

Where the expression $\bigcirc^k\alpha$ is defined inductively as

$$\bigcirc^k\alpha := \begin{cases} \alpha \quad \text{if} \quad k = 0 \\ \bigcirc(\bigcirc^{k-1}\alpha) \quad \text{if} \quad k > 0. \end{cases}$$

A sound and complete proof-system containing an ω-rule of this form is presented by Brünnler and Steiner in [2]. The infinitary system K^∞ presented here is based on work by Jäger and Alberucci in [1] and by Jäger, Studer and Kretz in [7]. The ω-rule it contains differs from the one above as it is not formulated by the $\bigcirc$-operator, but with respect to the approximants of a formula $\Box\alpha$.

In the following we present the relevant Tait-style inference rules for the system K^∞. The calculus is defined with respect to the language $\mathcal{L}_{\mathsf{PLTL}}$, where we are working with $\mathcal{L}_{\mathsf{PLTL}}$-sequents.

I. Axiom of K^∞:

$$\frac{}{\Gamma, p, {\sim}p} \ (Ax)$$

II. Classical rules of K^∞:

$$\frac{\Gamma, \alpha \quad \Gamma, \beta}{\Gamma, \alpha \wedge \beta}\ (\wedge) \qquad \frac{\Gamma, \alpha, \beta}{\Gamma, \alpha \vee \beta}\ (\vee)$$

$$\frac{\Gamma}{\bigcirc\Gamma, \Delta}\ (\bigcirc) \qquad \bigcirc\,\Gamma := \{\bigcirc\gamma \mid \gamma \in \Gamma\}$$

$$\frac{\Gamma, ((\alpha \wedge \bigcirc(\alpha\mathcal{U}\beta)) \vee \beta)}{\Gamma, \alpha\mathcal{U}\beta}\ (\mathcal{U}) \qquad \frac{\Gamma, (\alpha \vee \bigcirc\Diamond\alpha)}{\Gamma, \Diamond\alpha}\ (\Diamond)$$

III. ω-rule of K^∞:

$$\frac{\Gamma, (\bigcirc\top \wedge \alpha)^n \;\; (\forall n \in \mathbb{N})}{\Gamma, \Box\alpha}\ (\omega - \Box)$$

Theorem 3.0.3. *Every K^∞-rule is sound.*

Proof of (Ax): Choose any PLTL-model $\mathcal{M} = (N, \pi)$, clearly in every state n either p or $\sim p$ must hold, therefore the (Ax) must be valid.

Proof of $(\wedge)$: Choose any PLTL-model $\mathcal{M} = (N, \pi)$. By assumption we have for any state n we have $\mathcal{M}, n \models \Gamma, \alpha$ and $\mathcal{M}, n \models \Gamma, \beta$, therefore by definition of the semantics for $(\wedge)$ we have $\mathcal{M}, n \models \Gamma, \alpha \wedge \beta$ and so we are done.

Proof of $(\vee)$: Choose any PLTL-model $\mathcal{M} = (N, \pi)$. By assumption for any state n we have $\mathcal{M}, n \models \Gamma, \alpha, \beta$, therefore by definition of the semantics for $(\vee)$ we have $\mathcal{M}, n \models \Gamma, \alpha \vee \beta$ and so we are done.

Proof for $(\Diamond)$: Choose any PLTL-model $\mathcal{M} = (N, \pi)$. By assumption we have for any state n we have $\mathcal{M}, n \models \Gamma, \alpha, \bigcirc\Diamond\alpha$. If there is a $\gamma \in \Gamma$ that is satisfied in n then clearly $\mathcal{M}, n \models \Gamma, \Diamond\alpha$. If $\mathcal{M}, n \models \alpha$, then by definition of the $\Diamond$-operator we must have as well $\mathcal{M}, n \models \Diamond\alpha$. If $\mathcal{M}, n \models \bigcirc\Diamond\alpha$ then we have $\mathcal{M}, n+1 \models \Diamond\alpha$ and so $\mathcal{M}, n \models \Diamond\alpha$.

Proof for $(\mathcal{U})$: Choose any PLTL-model $\mathcal{M} = (N, \pi)$. In the case where no formula $\gamma \in \Gamma$ is satisfied, $\mathcal{M}, n \models ((\alpha \wedge \bigcirc(\alpha\,\mathcal{U}\beta)) \vee \beta)$ must hold and therefore $\mathcal{M}, n \models (\alpha \wedge \bigcirc(\alpha\,\mathcal{U}\beta)), \beta$ must be true as well. If $\mathcal{M}, n \models \beta$ holds, we have clearly as well $\mathcal{M}, n \models (\alpha\mathcal{U}\beta)$. On the other hand if $\mathcal{M}, n \not\models \beta$, then $\mathcal{M}, n \models (\alpha \wedge \bigcirc(\alpha\,\mathcal{U}\beta))$ must hold. But from this we get that $\mathcal{M}, n \models \alpha$ and $\mathcal{M}, n+1 \models \bigcirc(\alpha\,\mathcal{U}\beta)$ and therefore we have that $\mathcal{M}, n \models (\alpha\mathcal{U}\beta)$.

Proof for ($\bigcirc$): For any PLTL-model $\mathcal{M} = (N, \pi)$ and for any state n we know that $\mathcal{M}, n \models \Gamma$ holds. Therefore we know that Γ holds as well in any $\bigcirc$-successor of n, i.e. $\mathcal{M}, n+1 \models \Gamma$, but then we have $\mathcal{M}, n \models \Delta, \bigcirc\Gamma$.

Proof for ($\omega - \Box$): Choose any PLTL-model $\mathcal{M} = (N, \pi)$. In the case where no formula $\gamma \in \Gamma$ is satisfied, $\mathcal{M}, n \models (\bigcirc\top \wedge \alpha)^k$ $(\forall k \in \mathbb{N})$ must hold. Therefore $\mathcal{M}, n \models \bigcirc^k \alpha$ must hold for any $k \in \mathbb{N}$ and so $\mathcal{M}, m \models \alpha$ for all $m \geq n$. But then $\mathcal{M}, n \models \Box\alpha$.

Theorem 3.0.4. *For all $\mathcal{L}_{\mathsf{PLTL}}$-sequents Γ we have that*

$$\mathsf{K}^\infty \vdash \Gamma \quad \Rightarrow \quad \models \Gamma.$$

Proof. Suppose that $\mathsf{K}^\infty \vdash \Gamma$, then we know that there is an ordinal α such that $\mathsf{K}^\infty \vdash^\alpha \Gamma$. We proceed by induction on the prooflength α. If $\alpha = 0$ then Γ must be of the form of (Ax) and therefore by the preceding theorem Γ must be valid. In the case where $\alpha > 0$, then Γ must be the conclusion of an application of a K^∞ rule (θ) with premisses Γ_i for $i \in I$. As Γ is provable, also the premisses must be provable and we have $\mathsf{K}^\infty \vdash^{<\alpha} \Gamma_i$ for all $i \in I$. By induction hypothesis this yields $\models \Gamma_i$ for all $i \in I$. Therefore by the preceding theorem we get $\models \Gamma$. $\Box$

Definition 3.0.5. *A (possibly infinite) set Δ of $\mathcal{L}_{\mathsf{PLTL}}$-formulas is called **not provable** if for all finite subsets $\Gamma \subseteq \Delta$ we have that*

$$\mathsf{K}^\infty \nvdash \Gamma.$$

Definition 3.0.6. *A not provable set of $\mathcal{L}_{\mathsf{PLTL}}$-formulas Γ is called **saturated** if the following holds:*

1. *If $\alpha \wedge \beta \in \Gamma$ then $\alpha \in \Gamma$ or $\beta \in \Gamma$.*
2. *If $\alpha \vee \beta \in \Gamma$ then $\alpha, \beta \in \Gamma$.*
3. *If $\Box\alpha \in \Gamma$ then there is a natural number $k \in \mathbb{N}$ such that $(\bigcirc\top \wedge \alpha)^k \in \Gamma$.*
4. *If $\Diamond\alpha \in \Gamma$ then $(\alpha \vee \bigcirc\Diamond\alpha) \in \Gamma$.*
5. *If $\alpha\mathcal{U}\beta \in \Gamma$ then $(\beta \vee (\alpha \wedge \bigcirc(\alpha\mathcal{U}\beta))) \in \Gamma$.*

Lemma 3.0.7. *For any not provable $\mathcal{L}_{\mathsf{PLTL}}$-sequent Γ there is a saturated set Γ^s such that $\Gamma \subseteq \Gamma^s$.*

Proof. The saturated superset Γ^s can be constructed by systematically adding formulas to Γ such that all the saturation conditions are satisfied. We fix an enumeration $\delta_0, \delta_1, \ldots$ of all $\mathcal{L}_{\mathsf{PLTL}}$-formulas. If the formula α is the formula δ_i, then i will be called the index of α. Then we define for each non-provable sequent Δ the sequent $\Delta' \supseteq \Delta$ in the following way:

- If Δ is saturated then $\Delta = \Delta'$.
- If Δ is not saturated then we take the formula $\alpha \in \Delta$ with the smallest index for which one of the conditions of the definition of saturated sets is violated. Then Δ' is constructed as following:

  1. If $\alpha = (\beta \vee \gamma) \in \Delta$ then $\Delta' := \{\beta, \gamma\} \cup \Delta$.
  2. If $\alpha = (\beta \wedge \gamma) \in \Delta$, then as Δ is not provable, we know

     $$\mathsf{K}^\infty \nvdash \Delta, \beta \quad \text{or} \quad \mathsf{K}^\infty \nvdash \Delta, \gamma.$$

     We set

     $$\Delta' := \begin{cases} \{\beta\} \cup \Delta & \text{if } \mathsf{K}^\infty \nvdash \Delta, \beta. \\ \{\gamma\} \cup \Delta & \text{if } \mathsf{K}^\infty \nvdash \Delta, \gamma. \end{cases}$$

  3. If $\alpha = \Box\beta \in \Delta$ and as Δ is not provable, we know

     $$\mathsf{K}^\infty \nvdash \Delta, (\bigcirc\top \wedge \beta)^k \quad \text{for a natural number} \quad k \in \mathbb{N}.$$

     We choose the smallest such k and set

     $$\Delta' := \{(\bigcirc\top \wedge \beta)^k\} \cup \Delta.$$

  4. If $\alpha = \Diamond\beta \in \Delta$ then $\Delta' := \{\beta \vee \bigcirc\Diamond\beta\} \cup \Delta$.
  5. If $\alpha = (\beta\mathcal{U}\gamma) \in \Delta$ then $\Delta' = \{(\gamma \vee (\beta \wedge (\beta\mathcal{U}\gamma)))\} \cup \Delta$.

Obviously this construction ensures that every Δ' remains unprovable. Now we set:

$$\Gamma_n = \begin{cases} \Gamma & \text{if} \quad n = 0 \\ \Gamma'_{n-1} & \text{if} \quad n > 0. \end{cases}$$

We define

$$\Gamma^s := \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

Clearly $\Gamma \subset \Gamma^s$. It remains to show that Γ^s is not provable and saturated. Given a finite $\Delta \subset \Gamma^s$, then there must be a Γ_n such that $\Delta \subset \Gamma_n$. As Γ_n is not provable, we get that $\mathsf{K}^\infty \nvdash \Delta$. Therefore Γ^s is not provable. The saturation of Γ^s follows by the construction: Suppose for example that $\alpha \wedge \beta \in \Gamma^s$. Then there is a natural number n such that already $\alpha \wedge \beta \in \Gamma_n$, but then $\alpha \in \Gamma_{n+1}$ or $\beta \in \Gamma_{n+1}$. As $\Gamma_{n+1} \subset \Gamma^s$ we get therefore that $\alpha \in \Gamma^s$ or $\beta \in \Gamma^s$ and we are done. The other cases for saturation follow in the same way. $\Box$

$\cdot^s$ can be seen like a function mapping a not provable sequent Γ to a minimal saturated set $\Gamma^s \supseteq \Gamma$. Given a not provable sequent Γ we define its **canonical countermodel**, $\mathsf{cm}(\Gamma) = (N^{\mathsf{cm}}, \pi^{\mathsf{cm}})$, as follows:

- The states $N^{\mathsf{cm}} = \{0^{\mathsf{cm}}, 1^{\mathsf{cm}}, \ldots\}$ of the canonical countermodel are defined inductively such that $0^{\mathsf{cm}} = \Gamma^s$ and such that if $n^{\mathsf{cm}} = \Delta$ then

$$(n+1)^{\mathsf{cm}} = \{\alpha \mid \bigcirc \alpha \in \Delta\}^s.$$

- Further, for all atomic propositions p we have that

$$\pi^{\mathsf{cm}}(p) = \{n^{\mathsf{cm}} \mid p \notin n^{\mathsf{cm}}\}.$$

Lemma 3.0.8. *For any not provable sequent* Γ *the canonical countermodel* $\mathsf{cm}(\Gamma)$ *is a well-defined* PLTL*-model.*

Proof. By a straightforward induction on n it can be shown that all states n^{cm} are saturated sets of formulas and, therefore, that the construction using $\cdot^s$ works, that is, we have $N^{cm} \cong \mathbb{N}$. Further, since all states are not provable the valuation π^{cm} is well-defined. $\square$

Lemma 3.0.9. *Let* Γ *be a not provable set of* $\mathcal{L}_{\mathsf{PLTL}}$*-formulas. For all states* n^{cm} *of the canonical countermodel* $\mathsf{cm}(\Gamma)$ *we have that*

$$\alpha \in n^{\mathsf{cm}} \quad \Rightarrow \quad n^{\mathsf{cm}} \models \neg\alpha.$$

Proof. By induction on $\mathsf{rank}(\alpha)$. The case where $\mathsf{rank}(\alpha) = 0$ is clear. If $\mathsf{rank}(\alpha)$ is a successor ordinal then α is of the form $\beta \wedge \gamma, \beta \vee \gamma$ of $\bigcirc\beta$. For α being of the form $\beta \wedge \gamma$ or $\beta \vee \gamma$ the induction step follows by the properties of saturated sets. If α is of the form $\bigcirc\beta$ then by construction we have that $\beta \in (n+1)^{\mathsf{cm}}$ and by induction hypothesis we have that $(n+1)^{\mathsf{cm}} \models \neg\beta$ and, therefore, $n \models \neg\alpha$. If α is a limit ordinal then it is of the form $\Box\beta$, $\Diamond\beta$ or $\beta\mathcal{U}\gamma$. If it is of the form $\Box\beta$ then by construction of saturated sets there is $k \in \mathbb{N}$ such that $(\bigcirc\top\wedge\beta)^k \in n^{\mathsf{cm}}$. Since by Lemma 1.1.10 we have that $\mathsf{rank}(\bigcirc\top \wedge \beta)^k < \mathsf{rank}(\Box\beta)$ by induction hypothesis we have that $n^{\mathsf{cm}} \models \neg(\bigcirc\top \wedge \beta)^k$ and, therefore, that $n^{\mathsf{cm}} \models \neg\Box\beta$. If α is of the form $\Diamond\beta$ then by construction of saturated sets we have $(\beta \vee \bigcirc(\Diamond\beta)) \in n^{\mathsf{cm}}$, and by the properties of saturated sets we have that $\beta, \bigcirc(\Diamond\beta)) \in n^{\mathsf{cm}}$. With the induction hypothesis this yields that $n^{\mathsf{cm}} \models \neg\beta$, and by construction of the canonical model we have that $\beta \in (n+1)^{\mathsf{cm}}$, and, therefore, that $(n+1)^{\mathsf{cm}} \models \neg\beta$. By iterating this argument we easily infer that for all $i \geq 0$ we have that

$$(n+i)^{\mathsf{cm}} \models \neg\beta$$

and, therefore, that $n^{\mathsf{cm}} \models \neg\Diamond\beta$.
If α is of the form $\beta\,\mathcal{U}\gamma$ then by construction of saturated sets we have $(\gamma \vee (\beta \wedge \bigcirc(\beta\,\mathcal{U}\gamma)) \in n^{\mathsf{cm}}$, and by the properties of saturated sets we have that

$$\gamma, \beta \in n^{\mathsf{cm}} \text{ or } \gamma, \bigcirc(\beta\mathcal{U}\gamma) \in n^{\mathsf{cm}}$$

If $\gamma, \beta \in n^{\mathsf{cm}}$ then by induction hypothesis we have that

$$n^{\mathsf{cm}} \models \neg\gamma \wedge \neg\beta$$

and, therefore, that $n^{\mathsf{cm}} \models \neg(\beta\,\mathcal{U}\gamma)$. In the second case, we must have that $n^{\mathsf{cm}} \models \neg\gamma$ and, by construction of the canonical model, that

$$\beta\mathcal{U}\gamma \in (n+1)^{\mathsf{cm}}.$$

By iterating this argument we either get an $l \in \mathbb{N}$ such that

$$(n+l)^{\mathsf{cm}} \models \neg\gamma \wedge \neg\beta \text{ and for all } k < l \;\; (n+k)^{\mathsf{cm}} \models \neg\gamma$$

or, we have that for all $n' \geq n$ it holds that

$$n'^{\mathsf{cm}} \models \neg\gamma.$$

In both cases we have that

$$n^{\mathsf{cm}} \models \neg(\beta\mathcal{U}\gamma).$$

□

Theorem 3.0.10. *For all $\mathcal{L}_{\mathsf{PLTL}}$-sequents Γ we have that*

$$\mathsf{K}^{\infty} \vdash \Gamma \quad \Leftrightarrow \quad \models \Gamma.$$

Proof. The direction from the left to the right is Theorem 3.0.4. For the other direction assume that $\mathsf{K}^{\infty} \not\vdash \Gamma$. This means that Γ is not provable and there is a canonical countermodel $\mathsf{cm}(\Gamma)$. Since by construction we have that $\Gamma \subseteq 0^{\mathsf{cm}}$ by Lemma 3.0.9 we have that

$$0^{\mathsf{cm}} \models \bigwedge_{\alpha \in \Gamma} \neg\alpha$$

and, therefore, we get that $\not\models \Gamma$ and finish the proof. □

Chapter 4

Finitising the infinitary Calculus

It is tempting to go for a finitization of the ω-rule of K^∞, that is, to find a way to replace the rule by some weaker rules that rely only on a finite number of premisses. The resulting system would then be cut-free, wellfounded and finite, what is all we want. In this section we are going to sketch how such a finitization could look like. The basic idea is on the one hand to introduce some new rules that allow to check for cycles in a proof tree and on the other hand to exploit the full power of the premisses of the ω-rule. That is to say that there is no need for having a proof of all approximants $(\bigcirc\top \wedge \alpha)^k$ to derive $\Box\alpha$, but that it is enough to have a derivation of an approximant of a sufficiently high degree. Roughly speaking the overall strategy amounts to find a new calculus $\mathsf{K}_{\mathsf{new}}$ in which we can establish a lemma of the following form :

Finitization Lemma 4.0.11. *For a sequent Γ and a natural number k big enough*

$$\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k \qquad \Rightarrow \quad \mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha.$$

How does this lemma lead to a finitization of the ω-rule? Suppose that our new calculus $\mathsf{K}_{\mathsf{new}}$ is identical to K^∞ except that the ω-rule is replaced by one or more finitary rules such that the lemma above holds. Then we can establish a proof of a lemma that says:

$$\mathsf{K}^\infty \vdash \Gamma \quad \Rightarrow \quad \mathsf{K}_{\mathsf{new}} \vdash \Gamma. \tag{4.1}$$

The proof can be done by an induction on the prooflength: As K^∞ and $\mathsf{K}_{\mathsf{new}}$ are identical up to the ω-rule the only critical induction step is an application of the ω-rule. But in this case there are premisses of the form $\mathsf{K}^\infty \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ for all $k \in \mathbb{N}$ and therefore by the induction hypothesis $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ for all $k \in \mathbb{N}$. But form this we get by the finitization lemma $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha$ and so finally 4.1 must hold.

This fact together with the completeness of K^∞ then yields in the end the completeness of the finitary calculus $\mathsf{K}_{\mathsf{new}}$:

$$\models \Gamma \quad \Rightarrow \quad \mathsf{K}^\infty \vdash \Gamma \quad \Rightarrow \quad \mathsf{K}_{\mathsf{new}} \vdash \Gamma.$$

4.1 The general Idea for a Finitization of the ω-Rule

How are we going to find the calculus $\mathsf{K}_{\mathsf{new}}$ such that the finitization lemma above holds? The idea is to introduce cycle-testing rules that allow to transform fully syntactically a proof $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ into a proof $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha$. More concretely we want to take a proof tree for $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ and give instructions about how to transform it to get a new proof-tree for $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha$. What follows is an informal sketch about how such a transformation could look like and how it motivates the introduction of the finitary rules that will in the end replace the ω-rule.
It is clear from what we said above that we can choose the degree of the approximant $(\bigcirc\top \wedge \alpha)^k$ arbitrarily big and therefore we can choose it big enough. But how big is big enough? Big enough for k means that it must be bigger than the cardinality of the powerset $\mathcal{P}(\mathbb{FL}(\Gamma))$ where the set Γ are the sideformulas contained in the root of the proof-tree $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$. So we must have

$$k > 2^{|\mathbb{FL}(\Gamma)|}.$$

The reason is, that in this case on branches with sufficiently many unfoldings of the approximant $(\bigcirc\top \wedge \alpha)^k$ there will be some kind of cycle: Given a branch $\Gamma_0, \dots \Gamma_n$ in the proof tree of $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ with sufficiently many unfoldings of $(\bigcirc\top \wedge \alpha)^k$, then there would be sequents Γ_i, Γ_j with $0 \leq i < j \leq n$ such that

$$\Gamma_i = \Delta, \bigcirc(\bigcirc\top \wedge \alpha)^{k'} \quad \text{and} \quad \Gamma_j = \Delta, (\bigcirc\top \wedge \alpha)^{k''} \quad \text{where} \quad k' > k''.$$

Suppose now we are given the proof tree of the sequent $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ where $k > 2^{|\mathbb{FL}(\Gamma)|}$ then we are going to check on every branch if we can find a cycle of the form above and if yes, the branch will be cut after the second occurrence of the repeating sequent[1]. Schematically after cutting a branch we find the following situation:

[1] We have a bottom up reading of the proof tree.

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Delta_i, \alpha\end{array} \qquad \begin{array}{c}\Delta_i, (\bigcirc\top \wedge \alpha)^{k_i-l}\\ \vdots \\ \Delta_i, \bigcirc(\bigcirc\top\wedge\alpha)^{k_i-1}\end{array}}{\Delta_i, (\bigcirc\top\wedge\alpha)^{k_i}}\ (\wedge)\\
\hline
\vdots\\
\Gamma, (\bigcirc\top\wedge\alpha)^{k}
\end{array}
$$

Obviously the cut branch is not ending on an axiomatic sequent. Therefore we have to construct the new calculus $\mathsf{K}_{\mathsf{new}}$ in a way that this branch will be closed. The idea is to make use of propositional variables simulating the behavior of the $\Box\alpha$ approximants. This is to say: An occurrence of $(\bigcirc\top\wedge\alpha)^k$ is replaced by a formula x and an occurrence of $\bigcirc(\bigcirc\top\wedge\alpha)^k$ by a formula $\bigcirc x$. Additionally labels are introduced that store the cycling sequent and allow to close the branch. Let us illustrate this practice, transforming what we had before would yield:

$$
\begin{array}{c}
\dfrac{[\Delta_i, \alpha]^{\emptyset} \qquad \begin{array}{c}[\Delta_i, x]^{\{\Delta_i, x\}}\\ \vdots \\ [\Delta_i, \bigcirc x]^{\{\Delta_i, x\}}\end{array}}{[\Delta_i, x]^{\{\Delta_i, x\}}}\\
\hline
\vdots\\
[\Gamma, x]^{\{\Delta_i, x\}}
\end{array}
$$

The intended interpretation of the expression $[\Delta, x]^{\{\Delta,x\}}$ is the following: If the sequent contained in the label $\{\Delta_i, x\}$ is valid in a model then also the basic sequent $[\Delta_i, x]$ must be valid in that model. Clearly for an expression of the form $[\Delta, x]^{\{\Delta,x\}}$ this holds for every Δ. Therefore they will be called pseudo axioms and we are going to say that a branch is closed if it ends on such a pseudo axiom.
The remaining challenge is to introduce the formula $\Box\alpha$. This will be done by help of a so called cycle rule:

$$\dfrac{[\Gamma, \alpha]^{\emptyset} \quad [\Gamma, \bigcirc x^{\Box\alpha}]^{\{\Gamma, x^{\Box\alpha}\}}}{[\Gamma, \Box\alpha]}\ (Cyc).$$

The intuitive reading of this rule is the following: Having established the premisses of the cycle rule is in fact the same as having established some kind of induction step, that is, it allows to proceed from any approximant of a certain degree to the next higher approximant of the formula $\Box\alpha$.
In the case illustrated above the cycle rule can be applied to eliminate the label that we introduced with the axiom. So we get:

$$\dfrac{\begin{array}{cc} \vdots & \begin{array}{c}[\Delta_i, x]^{\{\Delta_i,x\}}\\ \vdots\end{array} \\ [\Delta_i,\alpha]^{\emptyset} & [\Delta_i, \bigcirc x]^{\{\Delta_i,x\}} \end{array}}{\dfrac{[\Delta_i,\Box\alpha]^{\emptyset}}{\begin{array}{c}\vdots\\ [\Gamma,\Box\alpha]^{\emptyset}\end{array}}}\ (Cyc)$$

Finally this amounts to introduce a formula $\Box\alpha$ after having established the existence of a cycle on a branch of the proof-tree.

Is the introduction of a cycle rule enough to give a full transformation of a proof of $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ into a proof of $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha$? Above we just replaced naively all formulas of the form $(\bigcirc\top \wedge \alpha)^k$, resp. $\bigcirc(\bigcirc\top \wedge \alpha)^k$ by the propositional variable x, resp. $\bigcirc x$, we were making use of a pseudo-axiom to close the cycling branch and applied the cycle-rule to introduce the formula $\Box\alpha$, but essentially we did undergo the same proof as for $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$. Now imagine that we have somewhere in the original proof tree for $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$ a rule application of the form

$$\frac{\Gamma, \alpha \qquad \Gamma, \bigcirc(\bigcirc\top \wedge \alpha)^{k_i}}{\Gamma, (\bigcirc\top \wedge \alpha)^{k_i+1}}\ (\wedge).$$

So we consider a rule application that unfolds an approximant $(\bigcirc\top\wedge\alpha)^{k_i+1}$ that might be of a lower degree than the approximant $(\bigcirc\top \wedge \alpha)^k$ which is contained in the root, i.e. $k_{i+1} \leq k$. If we would simply transform this application of $(\wedge)$ in the way as indicated before we would get

$$\frac{[\Gamma, \alpha]^{\emptyset} \qquad [\Gamma, \bigcirc x]^{\emptyset}}{[\Gamma, x]^{\emptyset}}.$$

Obviously this is not any longer an application of $(\wedge)$ and therefore the naive transformation would not lead to a correct copy of the original proof, that is, the tree resulting from the naive transformation would in fact not be a proof. The problem with the naive approach was, that it did not take in account that the approximants in the original proof-tree do change their degrees. Therefore an unfolding rule for the propositional variables has to be introduced:

$$\frac{[\Gamma, \bigcirc x \wedge \alpha]^L}{[\Gamma, x]^L}\ (Unf).$$

Then the rule application

$$\frac{\Gamma, \alpha \qquad \Gamma, \bigcirc(\bigcirc\top \wedge \alpha)^{k_i}}{\Gamma, (\bigcirc\top \wedge \alpha)^{k_i+1}}\ (\wedge)$$

could be transformed as following

$$\dfrac{\dfrac{[\Gamma, \alpha]^{\emptyset} \qquad [\Gamma, \bigcirc x]^{\emptyset}}{[\Gamma, \bigcirc x \wedge \alpha]^{\emptyset}}\,(\wedge)}{[\Gamma, x]^{\emptyset}}\,(Unf) \quad .$$

Thus to transform the original proof-tree there is an additional rule application necessary to deal with the newly introduced propositional variables. The proof for $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, \Box\alpha$ will therefore in the end become longer than the proof of $\mathsf{K}_{\mathsf{new}} \vdash \Gamma, (\bigcirc\top \wedge \alpha)^k$.

4.2 First Attempt towards a Finitization - The System K1

Let us pin down those rather rough ideas in the strict formal framework of a sequent-calculus called $\mathsf{K1}$.

4.2.1 Syntax and Semantics of K1

In order to introduce the calculus $\mathsf{K1}$ we have slightly to extend the language $\mathcal{L}_{\mathsf{PLTL}}$ to the new language $\mathcal{L}^{+}_{\mathsf{PLTL}}$. This is done by introducing a new set of propositional variables X such that

$$X = \{x^{\Box\varphi} \mid \Box\varphi \in \mathcal{L}_{\mathsf{PLTL}}\}.$$

This means, that for each $\mathcal{L}_{\mathsf{PLTL}}$-formula of the form $\Box\varphi$ the extended language $\mathcal{L}^{+}_{\mathsf{PLTL}}$ contains a variable of the form $x^{\Box\varphi}$. A variable of this form will be called a $\Box\varphi$-variable. If we do not want to refer to a variable belonging to a specific formula, then we simply speak of a $\Box$-variable. The index $\Box\varphi$ indicates that the variable $x^{\Box\varphi}$ will be interpreted as an approximant of the fixpoint $\Box\varphi$.

Definition 4.2.1. *The $\mathcal{L}^{+}_{\mathsf{PLTL}}$-**formulas** are defined inductively in the following way:*

1. *Every positive and negative atomic proposition p, resp. $\sim p$ is $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formula.*
2. *Every propositional variable $x^{\Box\varphi}$ that is contained in X is $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formula.*
3. *If φ and ψ are $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi\,\mathcal{U}\psi)$ are $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formulas.*

4. *If φ is a $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formula, then $\Box\varphi$, $\Diamond\varphi$ and $\bigcirc\varphi$ are $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formulas.*

Note that in $\mathcal{L}^{+}_{\mathsf{PLTL}}$ we do not introduce negation. This is because negated occurrences of $\Box$-variables would be in conflict with the soundness of the unfolding rule (Unf). We say that a $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formula α is **pure** if it does not contain any $\Box$-variables. Clearly the set of pure $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formulas is equal to the set of all $\mathcal{L}_{\mathsf{PLTL}}$-formulas. Therefore we can say that for the fragment $\mathcal{L}_{\mathsf{PLTL}}$ of $\mathcal{L}^{+}_{\mathsf{PLTL}}$ negation is defined as given in the first chapter.

Definition 4.2.2. *Sequents and pseudo-sequents for the system* $\mathsf{K1}$ *are introduced in the following way:*

1. *Γ is said to be a* $\mathsf{K1}$***-sequent*** *if it is a multiset of $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formulas.*

2. *L is said to be a* $\mathsf{K1}$***-label*** *if it is a multiset of* $\mathsf{K1}$*-sequents $L = \{\Gamma_1, \ldots, \Gamma_n\}$.*

3. *A* $\mathsf{K1}$***-pseudo sequent*** *will be called an expression of the form*

$$[\Gamma]^L$$

where Γ is a $\mathsf{K1}$*-sequent and L is a* $\mathsf{K1}$*-label. We refer to the multiset Γ which is part of the expression $[\Gamma]^L$ as the* $\mathsf{K1}$***-basic sequent*** *of the given* $\mathsf{K1}$*-pseudo sequent.*

An ordinary $\mathsf{K1}$-sequent Γ can be understood as a pseudo sequent with empty label $[\Gamma]^{\emptyset}$, in these cases we will also write just Γ.
If clear from the context, we will often talk just about sequents, labels and pseudo-sequents rather than about $\mathsf{K1}$-sequents, $\mathsf{K1}$-labels and $\mathsf{K1}$-pseudo sequents.

Definition 4.2.3. *The* ***set of subformula*** *$sub(\varphi)$ for a $\mathcal{L}^{+}_{\mathsf{PLTL}}$-formula φ is defined inductively:*

1. *If φ is a positive atomic proposition p, then $sub(\varphi) := \{p\}$; if φ is a negative atomic proposition $\sim p$, then $sub(\sim p) := \{\sim p\}$.*

2. *If φ is a $\Box$-variable $x^{\Box\alpha}$, then $sub(\varphi) := \{x^{\Box\alpha}\}$.*

3. *If φ is of the form $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ or $(\alpha\ \mathcal{U}\beta)$, then $sub(\varphi) := \{\varphi\} \cup sub(\alpha) \cup sub(\beta)$.*

4. *If φ is of the form $\Diamond\alpha$, $\Box\alpha$ or $\bigcirc\alpha$, then $sub(\varphi) := \{\varphi\} \cup sub(\alpha)$.*

Convention 4.2.4. *Let Γ be a* K1*-sequent and L be a* K1*-label:*

1. *If there is a $\varphi \in \Gamma$ such that $\alpha \in sub(\varphi)$, then we write*
$$\alpha \leq \Gamma.$$
2. *If there is a $\Gamma_i \in L$ such that $\alpha \leq \Gamma_i$, then we write*
$$\alpha \leq L.$$

If not $\alpha \leq \Gamma$, resp. not $\alpha \leq L$, we write $\alpha \not\leq \Gamma$, resp. not $\alpha \not\leq L$.
We are just interested in a semantics for K1 where the variable $x^{\Box\alpha}$ is interpreted as approximant of the fixpoint $\Box\alpha$. This aspect will be captured by the introduction of so called consistent valuation functions.

Definition 4.2.5. *We say that a given valuation function $\mathcal{V} : X \to \mathcal{P}(N)$ is* ***consistent*** *if for all variables $x^{\Box\alpha}$ we have that:*

$$\mathcal{V}(x^{\Box\alpha}) = \|(\bigcirc\top \wedge \alpha)^k\|^{\mathcal{M}}_{\mathcal{V}_c} \quad \text{for a } k \in \mathbb{N}.$$

We will write $\mathcal{V}_c$ if a valuation function $\mathcal{V}$ is consistent.

Definition 4.2.6. *The semantics for the system* K1 *is defined relative to a* ***linear time model****, this is a pair $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$, where:*

- *$\mathcal{M}$ is an ordinary* PLTL*-model $\mathcal{M} = (N, \pi)$.*
- *$\mathcal{V}_c$ is a consistent valuation function for the $\Box$-variables.*

Definition 4.2.7. *Given a linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$, the set $\|\varphi\|^{\mathcal{M}}_{\mathcal{V}_c}$ of states satisfying the formula φ is defined inductively:*

1. *If $\varphi = p$, then $\|p\|^{\mathcal{M}}_{\mathcal{V}_c} = \pi(p)$.*
2. *If $\varphi = \sim p$, then $\|\sim p\|^{\mathcal{M}}_{\mathcal{V}_c} = N \setminus \pi(p)$.*
3. *If $\varphi = x^{\Box\alpha}$, then $\| x^{\Box\alpha} \|^{\mathcal{M}}_{\mathcal{V}_c} = \mathcal{V}_c(x^{\Box\alpha})$.*
4. *If $\varphi = (\alpha \wedge \beta)$, then $\| (\alpha \wedge \beta) \|^{\mathcal{M}}_{\mathcal{V}_c} = \| \alpha \|^{\mathcal{M}}_{\mathcal{V}_c} \cap \| \beta \|^{\mathcal{M}}_{\mathcal{V}_c}$.*
5. *If $\varphi = (\alpha \vee \beta)$, then $\| (\alpha \vee \beta) \|^{\mathcal{M}}_{\mathcal{V}_c} = \| \alpha \|^{\mathcal{M}}_{\mathcal{V}_c} \cup \| \beta \|^{\mathcal{M}}_{\mathcal{V}_c}$.*
6. *If $\varphi = (\alpha\mathcal{U}\beta)$, then*
$$\| \alpha\mathcal{U}\beta \|^{\mathcal{M}}_{\mathcal{V}_c} = \{n \mid \exists m \geq n \; m \in \| \beta\|^{\mathcal{M}}_{\mathcal{V}_c} \text{ and } \forall n' \; n \leq n' < m \; n' \in \| \alpha \|^{\mathcal{M}}_{\mathcal{V}_c}\}.$$

7. *If* $\varphi = \Box\alpha$, *then* $\| \Box\alpha \|_{\mathcal{V}_c}^{\mathcal{M}} = \{n \mid \forall m \geq n \; m \in \| \alpha \|_{\mathcal{V}_c}^{\mathcal{M}}\}$.

8. *If* $\varphi = \Diamond\alpha$, *then* $\| \Diamond\alpha \|_{\mathcal{V}_c}^{\mathcal{M}} = \{n \mid \exists m \geq n \; m \in \| \alpha \|_{\mathcal{V}_c}^{\mathcal{M}}\}$.

9. *If* $\varphi = \bigcirc\alpha$, *then* $\| \bigcirc\alpha \|_{\mathcal{V}_c}^{\mathcal{M}} = \{n \mid n+1 \in \| \alpha \|_{\mathcal{V}_c}^{\mathcal{M}}\}$.

Notions of satisfaction and validity for $\mathcal{L}^+_{\mathsf{PLTL}}$-formulas φ and $\mathcal{L}^+_{\mathsf{PLTL}}$-multisets Γ with respect to a linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ can be defined analogously as it has been done for the system DHL. But there is need for a further concept: As the rules of the calculus K1 manipulate pseudo sequents a notion of validity with respect to pseudo sequents has to be introduced.

Definition 4.2.8. *Let* $[\Gamma]^L$ *be a pseudo sequent and* $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ *a linear time model. Then* $[\Gamma]^L$ *is said to be* ***valid in*** $\mathcal{T}$, *written as* $\mathcal{T} \models [\Gamma]^L$, *if the following holds*

$$\mathcal{T} \models \Gamma_i \quad \textit{for all} \quad \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models \Gamma.$$

Definition 4.2.9. *Let* $[\Gamma]^L$ *be a pseudo sequent, then* $[\Gamma]^L$ *is said to be* ***valid****, written as* $\models [\Gamma]^L$, *if the following holds*

$$\mathcal{T} \models [\Gamma]^L \quad \textit{for all linear time models} \quad \mathcal{T}.$$

4.2.2 Rules of K1

The classical Tait-style inference rules of K1 are labelled versions of the rules of K^∞. The ω-rule of K^∞ is replaced by the finitary cycle testing rules (Unf) and $(\mathsf{K1} - Cyc)$. Also a new kind of a axiom is introduced.

I. Axioms of K1:

$$\frac{}{[\Gamma, p, {\sim}p]^L}\;(Ax.1)$$

$$\frac{}{[\Gamma, \Delta, x^{\Box\alpha}]^{L,\{\Gamma, x^{\Box\alpha}\}}}\;(Ax.2) \qquad x^{\Box\alpha} \not\leq \Gamma$$

II. Logical rules of K1:

$$\frac{[\Gamma, \varphi]^L \quad [\Gamma, \psi]^L}{[\Gamma, (\varphi \wedge \psi)]^L}\;(\wedge) \qquad \frac{[\Gamma, \varphi, \psi]^L}{[\Gamma, (\varphi \vee \psi)]^L}\;(\vee)$$

$$\frac{[\Gamma, (\varphi \vee \bigcirc\Diamond\varphi)]^L}{[\Gamma, \Diamond\varphi]^L}\;(\Diamond) \qquad \frac{[\Gamma, (\psi \vee (\varphi \wedge \bigcirc(\varphi\mathcal{U}\psi)))]^L}{[\Gamma, (\varphi\mathcal{U}\psi)]^L}\;(\mathcal{U})$$

$$\frac{[\Gamma]^L}{[\Delta, \bigcirc\Gamma]^L} \, (\bigcirc) \quad \Gamma \neq \emptyset.$$

III. Cycle testing rules of $\mathsf{K1}$**:**

$$\frac{[\Gamma, (\bigcirc x^{\Box\alpha} \wedge \alpha)]^L}{[\Gamma, x^{\Box\alpha}]^L} \, (Unf)$$

$$\frac{[\Gamma, \alpha]^L \quad [\Gamma, \bigcirc x^{\Box\alpha}]^{L, \{\Gamma, x^{\Box\alpha}\}}}{[\Gamma, \Box\alpha]^L} \, (\mathsf{K1} - Cyc) \quad x^{\Box\alpha} \not\lesssim L \quad x^{\Box\alpha} \not\lesssim \Gamma.$$

Theorem 4.2.10. *The* $\mathsf{K1}$*-rules are sound.*

Proof of $(Ax.1)$**:** For any linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ holds $\mathcal{T}, n \models \Gamma, p, {\sim}p$ for every state n that is contained in the PLTL-model $\mathcal{M} = (N, \pi)$. Therefore if $\mathcal{T} \models \Gamma_i$ holds for all $\Gamma_i \in L$ then trivially $\mathcal{T} \models \Gamma, p, {\sim}p$ holds as well. And therefore the $\mathsf{K1}$-sequent $[\Gamma, p, {\sim}p]^L$ must be valid.

Proof of $(Ax.2)$**:** Given an arbitrary linear time model $\mathcal{T} = (M, \mathcal{V}_c)$, suppose that $\mathcal{T} \models \Gamma_i$ for all $\Gamma_i \in L$ and $\mathcal{T} \models \Gamma, x^{\Box\alpha}$. In this case trivially $\mathcal{T} \models \Gamma, \Delta, x^{\Box\alpha}$ must be true as well. As we have chosen no particular $\mathcal{T} = (M, \mathcal{V}_c)$ we must have $\models [\Gamma, \Delta, x^{\Box\alpha}]^{L, \{\Gamma, x^{\Box\alpha}\}}$.

Proof of $(\wedge)$**:** Suppose that $\models [\Gamma, \varphi]^L$ and $\models [\Gamma, \psi]^L$ hold. For an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ we assume $\mathcal{T} \models \Gamma_i$ for all $\Gamma_i \in L$. Then by the truth of the premisses of $(\wedge)$ it follows that $\mathcal{T} \models \Gamma, \varphi$ and $\mathcal{T} \models \Gamma, \psi$ must hold. By definition this is the same as to say $\mathcal{T}, n \models \Gamma, \varphi$ holds for all $n \in N$ and $\mathcal{T}, n \models \Gamma, \psi$ holds for all $n \in N$, where N is the universe belonging to the PLTL-model $\mathcal{M}$. But from this we get that $\mathcal{T}, n \models \varphi \wedge \psi$ for all $n \in N$ and therefore $\mathcal{T} \models \varphi \wedge \psi$ must be true. So we have shown that $\mathcal{T} \models \varphi \wedge \psi$ must hold by the assumption that $\mathcal{T} \models \Gamma_i$ is true for all $\Gamma_i \in L$, but this together with the fact that we have chosen an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ gives that $\models [\Gamma, \varphi \wedge \psi]^L$ and so $(\wedge)$ must be sound.

Proof of $(\vee)$**,** $(\Diamond)$**,** $(\Box)$**,** $(\bigcirc)$**,** $(\mathcal{U})$**:** As in the case of an application of $(\wedge)$ there is no interaction between the labels and the basic sequents, the labels in these rule applications remain untouched. Therefore as in the case of $(\wedge)$ the soundness of these rules is an immediate consequence of the soundness of the corresponding standard rules.

Proof of (Unf)**:** We know that $\models [\Delta, \bigcirc x^{\Box\alpha} \wedge \alpha]^L$ holds, which means that for any $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ we have:

$$\mathcal{T} \models \Gamma_i \quad \text{for all} \quad \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models (\bigcirc x^{\Box\alpha} \wedge \alpha).$$

Now suppose that the labels of the conclusion of (Unf) are valid, that is, $\mathcal{T} \models \Gamma_i$ for all $\Gamma_i \in L$. From this and the proof hypothesis we can infer $\mathcal{T} \models \Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha$. As we are dealing with a consistent valuation function $\mathcal{V}_c$ we know that $\mathcal{V}_c(x^{\Box\alpha}) = \| (\bigcirc \top \wedge \alpha)^k \|^{\mathcal{M}}_{\mathcal{V}_c}$ for a $k \in \mathbb{N}$. We know that approximants of a certain order are contained in the approximants of lower order and therefore we have:

$$\| x^{\Box\alpha} \|^{\mathcal{M}}_{\mathcal{V}_c} = \| (\bigcirc \top \wedge \alpha)^k \|^{\mathcal{M}}_{\mathcal{V}_c} \supseteq \| (\bigcirc \top \wedge \alpha)^{k+1} \|^{\mathcal{M}}_{\mathcal{V}_c} = \| (\bigcirc x^{\Box\alpha} \wedge \alpha)^k \|^{\mathcal{M}}_{\mathcal{V}_c}.$$

Therefore if $\mathcal{T} \models \Gamma, (\bigcirc x^{\Box\alpha} \wedge \alpha)$ holds, we have as well that $\mathcal{T} \models \Gamma, x^{\Box\alpha}$ must hold and we are done.

Convention 4.2.11. *In the proof of* $(\mathsf{K1} - Cyc)$ *we make use of the following notation for a so called k-modification of the valuation* $\mathcal{V}_c$ *with respect to the variable* $y^{\Box\beta}$*. We set*

$$\mathcal{V}_c^{y^{\Box\beta},k}(x^{\Box\alpha}) := \begin{cases} \mathcal{V}_c(x^{\Box\alpha}) & \text{if } x^{\Box\alpha} \neq y^{\Box\beta} \\ \|(\bigcirc \top \wedge \beta)^k\|^{\mathcal{M}}_{\mathcal{V}_c} & \text{if } x^{\Box\alpha} = y^{\Box\beta} \end{cases}$$

We often write $\mathcal{T}, \mathcal{V}_c^{y^{\Box\beta},k} \models \Gamma$ to indicate explicitly how the valuation function $\mathcal{V}_c$ is modified.

Proof $(\mathsf{K1} - Cyc)$**:** We show that

$$\models [\Gamma, (\bigcirc \top \wedge \alpha)^k]^L \quad \text{for} \quad k = 0, 1, 2, \ldots$$

We proceed by induction on k: The case for $k = 0$ is clear. Now let us consider the induction step: By the proof hypothesis we have $\models [\Gamma, \bigcirc x^{\Box\alpha}]^{L,\{\Gamma, x^{\Box\alpha}\}}$, what means that for every linear time model $\mathcal{T}$ the following implication holds

$$\left.\begin{array}{l} \mathcal{T} \models \Gamma_i \quad \text{for all} \quad \Gamma_i \in L, \quad x^{\Box\alpha} \not\leq L \\ \mathcal{T} \models \Gamma, x^{\Box\alpha} \end{array}\right\} \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \bigcirc x^{\Box\alpha}. \tag{4.2}$$

As induction hypothesis we assume $\models [\Gamma, (\bigcirc \top \wedge \alpha)^k]^L$, thus for every linear time model $\mathcal{T}$

$$\mathcal{T} \models \Gamma_i \quad \text{for all} \quad \Gamma_i \in L, \quad x^{\Box\alpha} \not\leq L \quad \Rightarrow \quad \mathcal{T} \models \Gamma, (\bigcirc \top \wedge \alpha)^k. \tag{4.3}$$

Suppose now that for an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V})$

$$\mathcal{T} \models \Gamma_i \quad \text{for all} \quad \Gamma_i \in L. \tag{4.4}$$

By this and the induction hypothesis we get

$$\mathcal{T} \models \Gamma, (\bigcirc\top \wedge \alpha)^k.$$

But then, as $x^{\Box\alpha} \not\leq L$ and $x^{\Box\alpha} \not\leq \Gamma$ the following must hold as well

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma_i \quad \text{for all} \quad \Gamma_i \in L \quad \text{and} \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma, x^{\Box\alpha}.$$

From this and 4.2 we get

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma, \bigcirc x^{\Box\alpha}.$$

Together with the second premise this yields

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha$$

what is the same as

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma, x^{\Box\alpha}.$$

But this is equivalent to

$$\mathcal{T} \models \Gamma, (\bigcirc\top \wedge \alpha)^{k+1}.$$

and we are done.

$\Box$

Theorem 4.2.12. *Let Γ be a* $\mathsf{K1}$*-sequent, then we have*

$$\mathsf{K1} \vdash [\Gamma]^L \quad \Rightarrow \quad \models [\Gamma]^L.$$

Proof. Suppose that $\mathsf{K1} \vdash [\Gamma]^L$, then we know that there is a natural number n such that $\mathsf{K1} \vdash^n [\Gamma]^L$. We proceed by induction on the prooflength n. If $n = 0$ then $[\Gamma]^L$ must be of the form of $(Ax.1)$ or $(Ax.2)$ and therefore by the preceding theorem $[\Gamma]^L$ must be valid. In the case where $n > 0$, then $[\Gamma]^L$ must be the conclusion of an application of a $\mathsf{K1}$ rule (θ) with premisses $[\Gamma_i]^{L_i}$ for $i \in I$. As $[\Gamma]^L$ is provable, also the premisses must be provable and we have $\mathsf{K1} \vdash^{n-1} [\Gamma_i]^{L_i}$ for all $i \in I$. By induction hypothesis this yields $\models [\Gamma_i]^{L_i}$ for all $i \in I$. Therefore by the preceding theorem we get $\models [\Gamma]^L$. $\Box$

Example of a derivation using $(\mathsf{K1} - Cyc)$**:**

$$\dfrac{\dfrac{[p, \sim p]^{\{p, x^{\Box\bigcirc\sim p}\}}}{[\bigcirc p, \bigcirc \sim p]^{\{p, x^{\Box\bigcirc\sim p}\}}}\,(\bigcirc) \qquad \dfrac{[p, x^{\Box\bigcirc\sim p}]^{\{\bigcirc p, x^{\Box\bigcirc\sim p}\},\{p, x^{\Box\bigcirc\sim p}\}}}{[\bigcirc p, \bigcirc x^{\Box\bigcirc\sim p}]^{\{\bigcirc p, x^{\Box\bigcirc\sim p}\},\{p, x^{\Box\bigcirc\sim p}\}}}\,(\bigcirc)}{[\bigcirc p, \Box\bigcirc \sim p]^{\{p, x^{\Box\bigcirc\sim p}\}}}\,(\mathsf{K1} - Cyc))$$

Let us verify that the pseudo sequent $[\bigcirc p, \Box\bigcirc \sim p]^{\{p, x^{\Box\bigcirc\sim p}\}}$ is valid in a linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V})$ with $\mathcal{V}(x^{\Box\bigcirc\sim p}) = \|(\bigcirc\top \wedge \bigcirc \sim p)^1\|^{\mathcal{M}}_{\mathcal{V}}$. Then we have to show the following implication

$$\mathcal{T} \models p \vee \bigcirc \sim p \quad \Rightarrow \quad \mathcal{T} \models \bigcirc p \vee \Box\bigcirc \sim p.$$

So suppose that for all states of the model

$$\mathcal{T}, s \models p \vee \bigcirc \sim p.$$

Then there is a state n such that

$$\mathcal{T}, m \models p \quad \text{for all} \quad m < n$$

and

$$\mathcal{T}, s \models \sim p \quad \text{for all} \quad m \geq n.$$

But then clearly we have

$$\mathcal{T} \models \bigcirc p, \Box\bigcirc \sim p.$$

This example illustrates that in a pseudo sequent $[\Gamma]^L$ the label L expresses a global and not a local condition for the basic sequent, locally the implication above would not hold.

Chapter 5

The System $\mathsf{K2}$

5.1 Refinement of $\mathsf{K1}$ - Motivating new Cycle Testing Rules

Using the notation of pseudo-sequents introduced for $\mathsf{K1}$ we can formulate more precisely what we are actually aiming at: Given we have a proof-tree for

$$\mathsf{K1} \vdash [\Gamma, (\bigcirc\top \wedge \alpha)^k]^L$$

then we want to transform it into a proof-tree of

$$\mathsf{K1} \vdash [\Gamma, \Box\alpha]^{L^*}.$$

Where L and L^* are labels with the same cardinality. The strategy for the transformation of the proof tree proceeds in two steps:

- First the original proof tree is cut on branches that contain cycling sequents.
- Then approximants of the form $(\bigcirc\top\wedge\alpha)^k$ or $\bigcirc(\bigcirc\top\wedge\alpha)^k$ occurring in the cut proof-tree are replaced in a suitable way by propositional variables. Additional applications of the cycle rules (Unf) and $(\mathsf{K1} - Cyc)$ will be necessary to deal with these newly introduced variables and to derive $\Box\alpha$.

The first step can be managed without great problem, but it is quiet a difficult task to find a transformation procedure such that the transformed proof-tree really becomes a proof in our new calculus.
Let us consider some further difficulties in the transformation-procedure that lead to the introduction of new cycle testing rules.

Consider the following scheme of a proof-tree which has branches that are already cut:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}[\Delta_1, (\bigcirc\top \wedge \alpha)^{k'}] \\ \vdots \\ [\Delta_1, \bigcirc(\bigcirc\top \wedge \alpha)^{k}] \\ \vdots \\ [\Gamma, \gamma_1, \bigcirc(\bigcirc\top \wedge \alpha)^{k}]\end{array} \qquad \begin{array}{c}[\Delta_2, (\bigcirc\top \wedge \alpha)^{k''}] \\ \vdots \\ [\Gamma, \gamma_2, \bigcirc(\bigcirc\top \wedge \alpha)^{k}]\end{array}}{[\Gamma, \gamma_1 \wedge \gamma_2, \bigcirc(\bigcirc\top \wedge \alpha)^{k}]}\ (\wedge) \\ \vdots \\ [\Delta_2, \bigcirc(\bigcirc\top \wedge \alpha)^{k}] \\ \vdots
\end{array}
$$

A naive transformation would just replace the approximants as we said, introduce labels for the cycling sequents and apply the rule $(\mathsf{K1} - Cyc)$ to introduce the formula $\Box\alpha$. Transforming the tree above in this way would then yield:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}[\Delta_1, x^{\Box\alpha}]^{\{\Delta_1, x^{\Box\alpha}\}} \\ \vdots \\ \dfrac{[\Delta_1, \bigcirc x^{\Box\alpha}]^{\{\Delta_1, x^{\Box\alpha}\}}}{[\Delta_1, \Box\alpha]^{\emptyset}}\ (\mathsf{K1} - Cyc) \\ \vdots \\ [\Gamma, \gamma_1, \Box\alpha]^{\emptyset}\end{array} \qquad \begin{array}{c}[\Delta_2, x^{\Box\alpha}]^{\{\Delta_2, x^{\Box\alpha}\}} \\ \vdots \\ [\Gamma, \gamma_2, \bigcirc x^{\Box\alpha}]^{\{\Delta_2, x^{\Box\alpha}\}}\end{array}}{[\Gamma, \gamma_1 \wedge \gamma_2, \bigcirc x^{\Box\alpha}]^{\{\Delta_2, x^{\Box\alpha}\}}}\ (\wedge) \\ \vdots \\ \dfrac{[\Delta_2, \bigcirc x^{\Box\alpha}]^{\{\Delta_2, x^{\Box\alpha}\}}}{[\Delta_2, \Box\alpha]^{\emptyset}} \\ \vdots
\end{array}
$$

In this case the cycle rule is always applied just right when we have arrived to establish a cycle. The problem with the transformation above is that the application of $(\wedge)$ is not any longer sound.
The transformation procedure should be designed in a way that the local soundness of the rule applications in the original proof tree is preserved. Therefore we split the cycle rule in a rule $(\mathsf{K2} - Cyc)$ that eliminates labels after the occurrence of a cycle and a rule (Dec) that introduces the formula $\Box\alpha$ after the elimination of all labels containing $x^{\Box\alpha}$:

$$
\dfrac{[\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^{L, \{\Gamma, x^{\Box\alpha}\}}}{[\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^{L}}\ (\mathsf{K2} - Cyc)
$$

$$\frac{[\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^L}{[\Gamma, \Box\alpha]^L}\ (Dec) \quad x^{\Box\alpha} \not\leq L.$$

Then we could transform the proof above in the following way:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[\Delta_1, x^{\Box\alpha}]^{\{\Delta_1, x^{\Box\alpha}\}}\\ \vdots \\ [\Delta_1, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\{\Delta_1, x^{\Box\alpha}\}}\end{array}}{[\Delta_1, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\emptyset}}\ (\mathsf{K2}-Cyc)}{\begin{array}{c}\vdots\\ [\Gamma, \gamma_1, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\emptyset}\end{array}} \qquad \begin{array}{c}[\Delta_2, x^{\Box\alpha}]^{\{\Delta_2, x^{\Box\alpha}\}}\\ \vdots \\ [\Gamma, \gamma_2, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\{\Delta_2, x^{\Box\alpha}\}}\end{array}}{\begin{array}{c}[\Gamma, \gamma_1 \wedge \gamma_2, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\{\Delta_2, x^{\Box\alpha}\}}\\ \vdots \\ \dfrac{\dfrac{[\Delta_2, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\{\Delta_2, x^{\Box\alpha}\}}}{[\Delta_2, \bigcirc x^{\Box\alpha} \wedge \alpha]^{\emptyset}}\ (\mathsf{K2}-Cyc)}{[\Delta_2, \Box\alpha]^{\emptyset}}\ (Dec)\\ \vdots\end{array}}\ (\wedge)$$

Further refinements of the cycle-testing rules are necessary. Rather technical problems in a proof of the finitization lemma require an even stronger version of the rule $(\mathsf{K2} - Cyc)$:

$$\frac{[\Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^{L, \{\Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n}\}}}{[\Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^L}\ (\mathsf{K2} - Cyc)$$

$$x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n} \not\leq L$$

The rules of K1 are too weak for a transformation of the original proof-tree of $[\Gamma, (\bigcirc x^{\Box\alpha} \wedge \alpha)^k]^L$. There is need for rules that enable to deal better with the propositional variables that replace the approximants in the original proof. The final calculus K2 which is presented in the next section might have these stronger rules.

5.2 The Calculus K2

Let us now present the final version of a finitary calculus for PLTL. In contrary to K1 in the system K2 basic sequents are defined to be sets and not multisets. The reason is that a proof of completeness seems to require the hidden contraction that is incorporated in a sequent-calculus with sets.

Definition 5.2.1. *Sequents and pseudo-sequents for the system* K2 *are introduced in the following way:*

1. Γ *is said to be a* K2-***sequent*** *if it is a set of* $\mathcal{L}^{+}_{\mathsf{PLTL}}$*-formulas.*

2. L *is said to be a* K2-***label*** *if it is a multiset of* K2*-sequents* $L = \{\Gamma_1, \ldots, \Gamma_n\}$.

3. *A* K2***-pseudo sequent*** *will be called an expression of the form*

$$[\Gamma]^L$$

where Γ *is a* K2*-sequent and* L *is a* K2*-label. We refer to the multiset* Γ *which is part of* $[\Gamma]^L$ *as the* K2***-basic sequent*** *of the given* K2*-pseudo sequent.*

The calculus K2 contains the same axioms as K1, as well as the same rules for the classical and temporal connectives. The rules form K2 differ to K1 insofar as they are defined with respect to basic-sequents that are sets and not multisets. As state above K2 contains stronger cycle-testing rules than K1.

Cycle testing rules of K2**:**

$$\frac{[\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^L}{[\Gamma, x^{\Box\alpha}]^L} \ (Unf)$$

$$\frac{[\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^L}{[\Gamma, \Delta, \Box\alpha]^L} \ (Dec) \quad x^{\Box\alpha} \not\leq \Gamma$$

$$\frac{[\Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^{L, \{\Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n}\}}}{[\Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^L} \ (\mathsf{K2} - Cyc)$$

$$x^{\Box\alpha_i} \not\leq \Gamma \quad i = 1, \ldots, n$$

Theorem 5.2.2. *The* K2*-rules are sound.*

Proof of $(Ax.1)$, $(Ax, 2)$, $(\wedge)$, $(\vee)$, $(\Diamond)$, $(\Box)$, $(\bigcirc)$, $(\mathcal{U})$, (Unf)**:** The proof goes like in the case of K1.

To make the proofs for (Dec) and $(\mathsf{K2} - Cyc)$ more readable we use again the k-modification of the valuation $\mathcal{V}_c$ with respect to the variable $y^{\Box\beta}$ that we introduced in the proof for the soundness of $(\mathsf{K1} - Cyc)$.

Proof of (Dec)**:** We are going to show that for any linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ and for all $k \in \mathbb{N}$ the following implication must hold:

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma_i \text{ for all } \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma, \Delta, x^{\Box\alpha} \tag{5.1}$$

From this statement we then get immediately that for an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ we he have:

$$\mathcal{T} \models \Gamma_i \text{ for all } \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \Delta, \Box\alpha$$

We are going to prove 5.1 by an induction on k. First we consider the case where $k = 0$. Then $\mathcal{V}_c^{x^{\Box\alpha},0}(x^{\Box\alpha}) = \| \top \|^{\mathcal{M}}_{\mathcal{V}_c}$ and we get the equivalence

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},0} \models \Gamma, \Delta, x^{\Box\alpha} \quad \Leftrightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},0} \models \Gamma, \Delta, \top.$$

Therefore in the case where $k = 0$ the statement 5.1 holds trivially.
Now we consider the step from k to $k + 1$. By proof hypothesis we know that the premise of (Dec) must hold, that is, we have $\models [\Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha]^L$. Therefore for an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ it holds that

$$\mathcal{T} \models \Gamma_i \text{ for all } \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha. \tag{5.2}$$

Suppose that

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma_i \text{ for all } \Gamma_i \in L.$$

As $x^{\Box\alpha}$ occurs only positively in L we can infer from this

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma_i \text{ for all } \Gamma_i \in L.$$

But from this together with 5.2 we get

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k} \models \Gamma, \bigcirc x^{\Box\alpha} \wedge \alpha.$$

As $x^{\Box\alpha} \not\leq \Gamma$ this is the same as

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma, x^{\Box\alpha}.$$

But then we have as well

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma, \Delta, x^{\Box\alpha}.$$

Therefore we have proven the implication for the case $k + 1$:

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma_i \text{ for all } \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha},k+1} \models \Gamma, \Delta, x^{\Box\alpha}.$$

Therefore we get

$$\mathcal{T} \models \Gamma_i \text{ for all } \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \Delta, \Box\alpha$$

and so we are done.

Proof of ($\mathsf{K2}-Cyc$)**:** We are going to show that if the premise for the cycle rule holds, that is, if we have that $\models [\Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^{L,\{\Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n}\}}$, then it must as well be true that $\models [\Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^L$ holds. The latter is equivalent to show that for an arbitrarily chosen linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ it must be true that

$$\mathcal{T} \models \Gamma_i \;\; \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

But that is the same as to say that for an arbitrary linear time model $\mathcal{T} = (\mathcal{M}, \mathcal{V}_c)$ and for every natural number k we have

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma_i \;\; \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha} \wedge \alpha \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n. \tag{5.3}$$

By the proof hypothesis we know that

$$\models [\Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n]^{L,\{\Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n}\}}$$

what can be written as the implication

$$\left.\begin{array}{l} \mathcal{T} \models \Gamma_i \;\; \Gamma_i \in L \\ \mathcal{T} \models \Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n} \end{array}\right\} \quad \Rightarrow \quad \mathcal{T} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n \tag{5.4}$$

where $\mathcal{T}$ is an arbitrary linear time model.
We are going to proof 5.3 by induction on k. First consider the case where $k = 0$, then $\mathcal{V}_c^{x^{\Box\alpha_n},0}(x^{\Box\alpha_n}) = \|\top\|_{\mathcal{V}_c}^{\mathcal{M}}$. Therefore we have the semantic equivalence

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n} \quad \Leftrightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_{n-1}}, \top.$$

As the expression on the right side holds trivially we know

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, x^{\Box\alpha}, \ldots, x^{\Box\alpha_n} \tag{5.5}$$

Now we suppose that

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma_i \;\; \Gamma_i \in L. \tag{5.6}$$

But 5.5 and 5.6 are the premisses of 5.4 and therefore we can infer

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n$$

But then clearly we have as well

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

So we have verified for the case where $k = 0$ that

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma_i \;\; \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},0} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

Now let us show the step from k to $k+1$. By induction hypothesis we have for a fixed k that

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma_i \;\; \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

As Δ can chosen to be empty we have

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma_i \;\; \Gamma_i \in L \quad \Rightarrow \quad \mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots \bigcirc x^{\Box\alpha_n} \wedge \alpha_n. \tag{5.7}$$

Now assume that for all $\Gamma_i \in L$ we have

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k+1} \models \Gamma_i \tag{5.8}$$

As $x^{\Box\alpha_i}$ for $i = 1, \ldots, n$ occurs only positively in L we get by semantic considerations

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma_i. \tag{5.9}$$

This together with 5.7 implies

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

But this is in fact the same as

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k+1} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_{n-1}} \wedge \alpha_{n-1}, x^{\Box\alpha_n}.$$

From this we can infer

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k+1} \models \Gamma, x^{\Box\alpha_1}, \ldots, x^{\Box\alpha_n}. \tag{5.10}$$

But 5.9 and 5.10 are in fact premisses for the proof hypothesis 5.4 and therefore we can infer

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k+1} \models \Gamma, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

From what we get clearly

$$\mathcal{T}, \mathcal{V}_c^{x^{\Box\alpha_n},k+1} \models \Gamma, \Delta, \bigcirc x^{\Box\alpha_1} \wedge \alpha_1, \ldots, \bigcirc x^{\Box\alpha_n} \wedge \alpha_n.$$

and we are done.

$\square$

An easy induction, like done for theorem 4.2.12 leads to the soundness of $\mathsf{K2}$.

Theorem 5.2.3. *Let* Γ *be a* $\mathsf{K2}$*-sequent, then we have*

$$\mathsf{K2} \vdash [\Gamma]^L \quad \Rightarrow \quad \models [\Gamma]^L.$$

5.3 Conjecture: Completeness

We conjecture the following version of the finitization lemma. Then completeness for the finitary calculus $\mathsf{K2}$ can be established.

Finitization Lemma 5.3.1. *For a* $\mathsf{K2}$*-pseudo sequent* Γ *of pure* $\mathcal{L}^+_{\mathsf{PLTL}}$*-formulas, labels* L *and* L^**, and a natural number* $k > 2^{|\mathbb{FL}(\Gamma)|}$ *we have:*

$$\mathsf{K2} \vdash [\Gamma, (\bigcirc\top \wedge \alpha)^k]^L \quad \Rightarrow \quad \mathsf{K2} \vdash [\Gamma, \Box\alpha]^{L^*}$$

where the labels have the same cardinality, i.e. $|L| = |L^*|$*.*

The claim about the cardinality of the labels is needed to ensure that a proof of the sequent $[\Gamma, (\bigcirc\top \wedge \alpha)^k]^L$ with $L = \emptyset$ implies a proof the sequent $[\Gamma, \Box\alpha]^{L^*}$ where the label remains the empty set, that is, $L^* = \emptyset$.

The next lemma relates the calculus $\mathsf{K2}$ to the infinitary calculus K^∞.

Lemma 5.3.2. *Given a* $\mathcal{L}_{\mathsf{PLTL}}$*-sequent* Γ*, then*

$$\mathsf{K}^\infty \vdash \Gamma \quad \Rightarrow \quad \mathsf{K2} \vdash \Gamma$$

Proof. We proceed by an induction on the prooflength n. First consider the case where $n = 0$. Then we are dealing with a K^∞-Axiom, that is, we have

$$\mathsf{K}^\infty \vdash^0 \Gamma, p, {\sim}p.$$

Clearly the pseudo-sequent $[\Gamma, p, {\sim}p]^\emptyset$ must be a $\mathsf{K2}$-axiom as well and therefore we have

$$\mathsf{K2} \vdash^0 [\Gamma, p, {\sim}p]^\emptyset$$

what by notational convention is the same as

$$\mathsf{K2} \vdash^0 \Gamma, p, {\sim}p.$$

For the induction step from $n-1$ to n we proceed by a case distinction on the last rule that has been applied in the proof.

Proof of $(\wedge)$**:** In this case we are given a conclusion of the form

$$\mathsf{K}^{\infty} \vdash^{n} \Gamma, \gamma_1 \wedge \gamma_2$$

where the premisses must be

$$\mathsf{K}^{\infty} \vdash^{n-1} \Gamma, \gamma_1 \quad \text{and} \quad \mathsf{K}^{\infty} \vdash^{n-1} \Gamma, \gamma_2.$$

By induction hypothesis we get

$$\mathsf{K2} \vdash^{n-1} \Gamma, \gamma_1 \quad \text{and} \quad \mathsf{K2} \vdash^{n-1} \Gamma, \gamma_2.$$

By notational convention this is the same as

$$\mathsf{K2} \vdash^{n-1} [\Gamma, \gamma_1]^{\emptyset} \quad \text{and} \quad \mathsf{K2} \vdash^{n-1} [\Gamma, \gamma_2]^{\emptyset}.$$

From this we get by use of the K2 rule $(\wedge)$ that the following must hold

$$\mathsf{K2} \vdash^{n} [\Gamma, \gamma_1 \wedge \gamma_2]^{\emptyset}.$$

And so we are done.

Proof of $(\vee)$**,** $(\Diamond)$**,** $(\bigcirc)$**,** $(\mathcal{U})$**:** The proof of these cases is analogue to the proof of $(\wedge)$. Note that if the labels of the pseudo-sequents are empty, then the rules of K^{∞}, except the omega-rule, are the same as the rules for K2.

Proof of $(\omega - \Box)$**:** The only critical case is when the last inference was an application of the ω-rule

$$\frac{\Gamma, (\bigcirc\top \wedge \alpha)^k \ \ (\forall k \in \mathbb{N})}{\Gamma, \Box\alpha}\ (\omega - \Box).$$

That is, we have a premise of the form

$$\mathsf{K}^{\infty} \vdash^{n-1} \Gamma, (\bigcirc\top \wedge \alpha)^k \ \ (\forall k \in \mathbb{N}).$$

So we get by the induction hypothesis

$$\mathsf{K2} \vdash^{n-1} \Gamma, (\bigcirc\top \wedge \alpha)^k \ \ (\forall k \in \mathbb{N})$$

what by notational convention is the same as

$$\mathsf{K2} \vdash^{n-1} [\Gamma, (\bigcirc\top \wedge \alpha)^k]^{\emptyset} \ \ (\forall k \in \mathbb{N}).$$

This means that we can choose an approximant for $\Box\alpha$ of such a large degree that we can apply the finitization lemma 5.3.1. That is, for sure we find a natural number k such that

$$\mathsf{K2} \vdash^{n-1} [\Gamma, (\bigcirc\top \wedge \alpha)^k]^{\emptyset} \ \text{for} \ \ k > 2^{|\mathbb{FL}(\Gamma)|}.$$

As that is the premise of the finitization lemma 5.3.1 we can infer (remember that if $L = \emptyset$ then also $L^* = \emptyset$)

$$\mathsf{K2} \vdash^n [\Gamma, \Box\alpha]^{\emptyset}$$

what is exactly what we want. $\Box$

We get the completeness of $\mathsf{K2}$ as an easy corollary:

Corollary 5.3.3. *For any pure* $\mathsf{K2}$*-sequent, we have:*

$$\models \Gamma \quad \Rightarrow \quad \mathsf{K2} \vdash \Gamma$$

Proof. By the completeness of K^∞ and the foregoing lemma 5.3.2:

$$\models \Gamma \quad \Rightarrow \quad \mathsf{K}^\infty \vdash \Gamma \quad \Rightarrow \quad \mathsf{K2} \vdash \Gamma.$$

$\Box$

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