Ontological Questions about Operational Set Theory

Masterarbeit

der Philosophisch-naturwissenschaftlichen Fakultät

der Universität Bern

vorgelegt von

Rico Zumbrunnen

2009

Leiter der Arbeit:

Prof. Dr. Gerhard Jäger, Institut für Informatik und angewandte Mathematik

Contents

1 Introduction **2**

2 Feferman's theory OST **3**

2.1 The language $\mathcal{L}^\circ$ of OST . . . 3

2.2 The logic of OST . . . 5

2.3 The axioms of OST . . . 5

2.3.1 Applicative axioms . . . 5

2.3.2 Basic set-theoretic axioms . . . 6

2.3.3 Logical operations axioms . . . 8

2.3.4 Operational set-theoretic axioms . . . 8

2.4 The strengthening OST($\mathbb{P}$) of OST . . . 9

2.5 The strengthening OST($\mathbf{E}, \mathbb{P}$) of OST . . . 9

2.6 First consequences of OST . . . 9

3 Totality **14**

4 Sets of operations and operational extensionality **16**

5 Functions as operations **21**

5.1 The theory KPω . . . 21

5.2 The embedding of OST plus **FO**1/2 into KPω plus $(V = L)$ . . . 23

6 Beeson's theory ZFR **34**

6.1 The theory ZFR . . . 34

6.1.1 Language and logic . . . 34

6.1.2 Non-logical axioms . . . 35

6.2 Sets of operations . . . 35

6.3 Logical operations . . . 37

References **39**

Acknowledgement

I am very grateful to Prof. Gerhard Jäger for introducing me to the interesting subject of this thesis and for guiding me during my work. Furthermore, I thank Sebastian Eberhard for his careful proofreading. Many thanks go to my parents who taught me to think for myself and thinking was surely the main acquirement for completing my degree. And I like to thank Kathrin Weyermann for motivating me and for everyday support. Finally, I would like to thank all the people who helped me in any way to complete this thesis.

1 Introduction

Feferman introduces the theory OST of operational set theory in [6]. On the one hand OST is a set theory, that is, OST is a theory about sets. On the other hand OST allows us to treat every object as operation and to apply it to each other. So every object in a model of OST is a set and an operation at the same time. Thereby operations are not necessarily total.
In section 2 we will present a minor syntactic variant of Fefermans's original formulation of OST like in [8] and we give some first useful properties.

The main aim of this thesis is to respond to some ontological questions about OST.
We would like to know whether we can consistently assume that all operations are total. In section 3, which is about a few questions concerning the totality of operations, we show that we can not.
Further we ask about the consistency of the existence of the set of all operations from a set a to another set b. We will see in section 4 that such a set can not exist (if a and b contain at least one element, two elements respectively). Then we will get that objects as operations are not extensional (i.e. all values of two different operations can agree) as a corollary.
Another question is about set-theoretic functions and operations. We are interested whether we can consistently assume that the values of every set-theoretic function f agree with the values of f as operation. In section 5 we will show that we can. For this we will use some results from various sources without proving them.

Beeson presents in [3] a computation system based on set theory which has some similarities to OST. Section 6 is about this system. We will see that we can prove also within this system with the same approach as in section 4, that there is no set of all operations from a set a to another set b (if we make some constraints regarding a and b). In addition we will show that we can prove some axioms of OST, which are not axioms of Beesons system.

2 Feferman's theory OST

The content of this section is adopted from Feferman [6, 7] and Jäger [8, 9] and elaborated. There is some additional information from other sources which are mentioned directly in the text.

2.1 The language $\mathcal{L}^\circ$ of OST

By $\mathcal{L}$ we denote the language of classical set theory. The only two relation symbols of $\mathcal{L}$ are $\in$ (element) and $=$ (identity). Further we have set variables $a, b, c, f, g, u, v, w, x, y, z, ...$ (possibly with subscripts) and the constant ω for the smallest infinite ordinal. There are no function symbols of $\mathcal{L}$, so the only terms are the variables and the constant ω. The formulas of $\mathcal{L}$ are defined as usual.

The theory OST is formulated in the language $\mathcal{L}^\circ$. $\mathcal{L}^\circ$ is an extension of $\mathcal{L}$ which gives us the possibility to treat all objects as operations and to apply them to each other. In addition to the relation symbols of $\mathcal{L}$ we have the unary symbol $\downarrow$ (defined). Moreover $\mathcal{L}^\circ$ possess the binary function symbol $\circ$ for partial term application. $\mathcal{L}^\circ$ possess, in addition to the constant of $\mathcal{L}$, these constant symbols:

(i) k and s (combinators),

(ii) $\top$, $\bot$, el, non, dis and e (logical operations),

(iii) $\mathbb{S}$, $\mathbb{R}$ and $\mathbb{C}$ (set-theoretic operations).

The meaning of these constants is described by the axioms of OST (see below).

Definition 2.1 (Terms of $\mathcal{L}^\circ$)**.** The terms of $\mathcal{L}^\circ$ (the $\mathcal{L}^\circ$ terms) are inductively defined as follows:

(i) If t is a variable or a constant of $\mathcal{L}^\circ$, then t is a term.

(ii) If s and t are terms, then so is $\circ(s, t)$.

We use the letters r, s and t (possibly with subscripts) to denote terms. We will often abbreviate $\circ(s, t)$ as $(s \circ t)$, (st), or st and we will often write $s(t_1, ..., t_n)$ instead of $st_1...t_n$.

Convention 2.2. The abbreviation $t_1t_2t_3...t_n$ stands for $((...((t_1t_2)t_3)...)t_n)$.

Sometimes we write $\vec{t}$ for a finite string $t_1, ..., t_n$ of terms, but only if the length is not important or evident from the context.
Because the partial term application $\circ$ is not necessarily total, there may be terms which do not denote an object. Therefore we need the definedness predicate $\downarrow$: The formula $(t\downarrow)$ means „the term t is defined" or „the term t denotes an object".

Definition 2.3 (Formulas of $\mathcal{L}^\circ$)**.** The formulas of $\mathcal{L}^\circ$ (the $\mathcal{L}^\circ$ formulas) are inductively defined as follows:

(i) If s and t are terms of $\mathcal{L}^\circ$, then $(s \in t)$, $(s = t)$ and $(t\downarrow)$ are formulas of $\mathcal{L}^\circ$.

(ii) If A and B are formulas of $\mathcal{L}^\circ$, then so are $\neg A$ and $(A \wedge B)$.

(iii) If A is a formula and x is a variable of $\mathcal{L}^\circ$, then $\forall x A$ is a formula of $\mathcal{L}^\circ$.

We use the capitals $A, B, C, D, ...$ and the Greek letter ϕ (possibly with subscripts) to denote formulas. Parentheses and brackets will be often omitted (if there is no danger of confusion).
The free variables of terms and formulas are defined as usual. The closed terms and formulas of $\mathcal{L}^\circ$ are those which do not contain any free variable. Closed $\mathcal{L}^\circ$ formulas are called $\mathcal{L}^\circ$ sentences.

Definition 2.4 (Abbreviations for some formulas)**.** Let A and B be formulas, x a variable and t a term of $\mathcal{L}^\circ$ which does not contain x. We define the remaining logical connectives, quantifier and the symbol $\uparrow$ as follows:

(i) $(A \vee B) := \neg(\neg A \wedge \neg B)$,

(ii) $(A \rightarrow B) := (\neg A \vee B)$,

(iii) $(A \leftrightarrow B) := ((A \rightarrow B) \wedge (B \rightarrow A))$,

(iv) $\exists x A := \neg \forall x (\neg A)$,

(v) $(\forall x \in t) A := \forall x (x \in t \rightarrow A)$,

(vi) $(\exists x \in t) A := \exists x (x \in t \wedge A)$,

(vii) $(t\uparrow) := \neg(t\downarrow)$.

We will often write $\forall x_1, ..., x_n A$ instead of $\forall x_1 ... \forall x_n A$ and $(\forall x_1, ..., x_n \in t) A$ instead of $(\forall x_1 \in t)...(\forall x_n \in t) A$ (and the same for the existential quantifier).

Convention 2.5. Let A be a formula of $\mathcal{L}^\circ$, u a variable which does not occur in A. Then we write A^u for the result of replacing each unbounded set quantifier $\forall x(...)$ by $(\forall x \in u)(...)$ in A.

Convention 2.6. Let $\vec{u} = u_1, ..., u_n$ and $\vec{t} = t_1, ..., t_n$ be a string of variables and a string of $\mathcal{L}^\circ$ terms respectively. Then $s[\vec{t}/\vec{u}]$ is the $\mathcal{L}^\circ$ term which is obtained from the $\mathcal{L}^\circ$ term s by simultaneously replacing all occurrences of the variables $\vec{u}$ by the terms $\vec{t}$. And $A[\vec{t}/\vec{u}]$ is the $\mathcal{L}^\circ$ formula which is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{u}$ by the terms $\vec{t}$ (in order to avoid collision of variables, a renaming of bound variables may be necessary). We often write $B[\vec{t}]$ instead of $B[\vec{t}/\vec{u}]$, if A is written as $B[u]$.

2.2 The logic of OST

The underlying logic of OST is the (classical) *logic of partial terms* due to Beeson [2]. In this logic each of the formulas $(st)\downarrow$, $(s = t)$ and $(s \in t)$ implies both $s\downarrow$ and $t\downarrow$.

Convention 2.7 ($\neq$ and $\notin$)**.** Let s and t be $\mathcal{L}^\circ$ terms. We will use the following abbreviations:

(i) $(s \neq t)$ for $(\neg(s = t) \wedge s\downarrow \wedge t\downarrow)$,

(ii) $(s \notin t)$ for $(\neg(s \in t) \wedge s\downarrow \wedge t\downarrow)$.

Definition 2.8 (Partial equality of terms)**.** Let s and t be $\mathcal{L}^\circ$ terms. We introduce partial equality ($\simeq$) of terms by

$$(s \simeq t) := ((s\downarrow \vee t\downarrow) \to s = t).$$

In classical logic assertions like $\forall x A[x]$ implies assertions like $A[t]$ for every term t. In the logic of partial terms this is not the case, since it is not assured that the term t is defined. To deduce $A[t]$ from $\forall x A[x]$ we need $t\downarrow$ as additional premise.

2.3 The axioms of OST

The non-logical axioms of OST are divided into four groups: applicative axioms, basic set-theoretic axioms, logical operations axioms and operational set-theoretic axioms.

2.3.1 Applicative axioms

The first axioms are about the applicative structure of the universe:

(1) $\mathsf{k} \neq \mathsf{s}$,

(2) $\mathsf{k}xy = x$,

(3) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq (xz)(yz)$.

These axioms express that we have a partial combinatory algebra. In the following we define like in [10] for each $\mathcal{L}^\circ$ term t a lambda expression $(\lambda x.t)$ which is an $\mathcal{L}^\circ$ term too and a so-called fixed point operator.

Definition 2.9 (Lambda expression)**.** Given an $\mathcal{L}^\circ$ term t and a variable x, we define inductively the $\mathcal{L}^\circ$ term $(\lambda x.t)$ as follows:

(i) If t is x, then $(\lambda x.t) := \mathsf{skk}$.

(ii) If t is a variable different from x or a constant of $\mathcal{L}^\circ$, then $(\lambda x.t) := \mathsf{k}t$.

(iii) If t is (rs) for $\mathcal{L}^\circ$ terms r and s, then $(\lambda x.t) := \mathsf{s}(\lambda x.r)(\lambda x.s)$.

We will often use the abbreviation $(\lambda x_1 ... x_n.t)$ for $(\lambda x_1.(\lambda x_2.(...(\lambda x_n.t)...)))$.

Lemma 2.10 (λ-abstraction)**.** *Given an arbitrary variable x and terms s and t of $\mathcal{L}^\circ$,* OST *proves the formulas*

$$(\lambda x.t)\downarrow,$$

$$(\lambda x.t)x \simeq t \text{ and}$$

$$s\downarrow \to (\lambda x.t)s \simeq t[s/x].$$

The variables of $(\lambda x.t)$ are those of t other than x.

Proof. The proof is a simple induction on the complexity of the term t. $\square$

Lemma 2.11. *Given two different variables x and y and two terms s and t of $\mathcal{L}^\circ$,* OST *proves the formula $(\lambda x.t)[s/y]x \simeq t[s/y]$.*

Proof. Again the proof is a simple induction on the complexity of the term t (see [10] p. 16-17). $\square$

The first axioms also guarantee the existence of a fixed point operator.

Definition 2.12 (Fixed point operator)**.** Let t be the $\mathcal{L}^\circ$ term $(\lambda yx.f(yy)x)$. Then we define the fixed point operator as follows:

$$\mathsf{fix} := (\lambda f.tt).$$

Lemma 2.13 (Recursion theorem)**.** *For three variables x, f and g,* OST *proves this formula:*

$$\mathsf{fix}(f)\downarrow \wedge (\mathsf{fix}(f) = g \to gx \simeq f(g, x)).$$

Proof. Let t be the $\mathcal{L}^\circ$ term $(\lambda yx.f(yy)x)$. By Lemma 2.10 we have $t\downarrow$ and

$$\mathsf{fix}(f) \simeq tt \simeq (\lambda yx.f(yy)x)t \simeq (\lambda x.f(yy)x)[t/y].$$

Because both t and $(\lambda x.f(yy)x)$ are defined, we have $(\lambda x.f(yy)x)[t/y]\downarrow$ (this follows from the logical axioms of the logic of partial terms) and so $\mathsf{fix}(f)\downarrow$. If we set $g := \mathsf{fix}(f)$, then we have by Lemma 2.11

$$gx \simeq (\lambda x.f(yy)x)[t/y]x \simeq f(tt)x \simeq f(g, x).$$ $\square$

2.3.2 Basic set-theoretic axioms

From now on we will use standard set-theoretic terminology. Some examples of abbreviations, wich are used to formulate the next axioms, are given in the table below.

Table 1: Some abbreviations

Abb.	*Formula*	*Meaning*
$a \neq \emptyset$	$(\exists x \in a)(x = x)$	*a is not empty.*
$a \subseteq b$	$(\forall x \in a)(x \in b)$	*a is a subset of b.*
$y = x \cup \{x\}$	$(\forall z \in x)(z \in y) \wedge x \in y \wedge (\forall z \in y)(z \in x \vee z = x)$	
$\mathrm{Tran}(a)$	$(\forall x \in a)(x \subseteq a)$	*a is transitiv.*
$\mathrm{Ord}(a)$	$\mathrm{Tran}(a) \wedge (\forall x \in a)(\mathrm{Tran}(x))$	*a is an ordinal.*
$\mathrm{Lim}(a)$	$\mathrm{Ord}(a) \wedge a \neq \emptyset \wedge (\forall x \in a)((\exists y \in a)(y = x \cup \{x\}))$	*a is a limit ordinal.*

The axioms of the second group are classical set-theoretic axioms:

(1) $\exists x \forall y (y \notin x)$ (there is the empty set),

(2) $\forall u \forall v \exists x \forall z (z \in x \leftrightarrow z = u \vee z = v)$ (there are unordered pairs),

(3) $\forall x \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge z \in u))$ (there are unions),

(4) $\mathrm{Lim}(\omega) \wedge (\forall x \in \omega)(\neg \mathrm{Lim}(x))$ (ω is the first infinite ordinal),

(5) $\forall x \forall y (\forall u (u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$ (all objects are extensional),

(6) $\in$-induction ($\mathcal{L}^\circ$-$\mathsf{I}_\in$) is available for arbitrary formulas $A[x]$ of $\mathcal{L}^\circ$.

We say that $\in$-induction is available for an $\mathcal{L}^\circ$ formula $A[x]$ if

($\mathcal{L}^\circ$-$\mathsf{I}_\in$) $$\forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x]$$

holds.

Definition 2.14 (Class)**.** Let $A[x]$ be an $\mathcal{L}^\circ$ formula. With $\{x : A[x]\}$ we denote the class of all sets satisfying A.

An expression of the form $\{x \in t : A[x]\}$ is used as a shorthand for $\{x : x \in t \wedge A[x]\}$. Some classes $\{x : A[x]\}$ are extensionally equal to a set, but this is not the case for all classes.

Definition 2.15. Given a variable x, a term t and a formula $A[x]$ of $\mathcal{L}^\circ$ we define

(i) $(t \in \{x : A[x]\}) := t\downarrow \wedge A[t]$,

(ii) $(t = \{x : A[x]\}) := t\downarrow \wedge \forall x (x \in t \leftrightarrow A[x])$,

(iii) $\mathbb{B} := \{x : x = \top \vee x = \bot\}$,

(iv) $\mathbb{V} := \{x : x\downarrow\}$.

So $\mathbb{B}$ stands for the unordered pair consisting of the two truth values $\top$ and $\bot$. By the previous axioms $\mathbb{B}$ is a set. $\mathbb{V}$ is the class of all sets, it is not a set itself (see lemma 4.1).

Definition 2.16. For an arbitrary natural number n and variables a, b, f, x, $x_1, ..., x_{n+1}$ we define

(i) $(f : a \to b) := (\forall x \in a)(fx \in b)$,

(ii) $(f : a^{n+1} \to b) := (\forall x_1, ..., x_{n+1} \in a)(f(x_1, ..., x_{n+1}) \in b)$.

The variables a and/or b may be replaced by $\mathbb{V}$ and/or $\mathbb{B}$.

This formulas express that f is a unary and $(n + 1)$-ary mapping from a to b in the operational sense, respectively. But they do not say that f is a function in the set-theoretic sense (see below)! The formula $(f : a \to \mathbb{V})$ means that f is total on a (id est $(\forall x \in a)fx\downarrow$), $(f : \mathbb{V} \to b)$ means that f maps all sets into b. The formula $(f : a \to \mathbb{B})$ expresses that $fx \in \mathbb{B}$ if $x \in a$. The n-ary *Boolean operations* are those f for which $(f : \mathbb{B}^n \to \mathbb{B})$ and so on.

2.3.3 Logical operations axioms

The third group of axioms describes the representation of the element relation, elementary logical connectives and bounded existential quantification as operations:

(1) $\top \neq \bot$,

(2) $(\mathsf{el} : \mathbb{V}^2 \to \mathbb{B}) \wedge \forall x \forall y(\mathsf{el}(x, y) = \top \leftrightarrow x \in y)$,

(3) $(\mathsf{non} : \mathbb{B} \to \mathbb{B}) \wedge (\forall x \in \mathbb{B})(\mathsf{non}(x) = \top \leftrightarrow x = \bot)$,

(4) $(\mathsf{dis} : \mathbb{B}^2 \to \mathbb{B}) \wedge (\forall x, y \in \mathbb{B})(\mathsf{dis}(x, y) = \top \leftrightarrow (x = \top \vee y = \top))$,

(5) $(f : a \to \mathbb{B}) \to (\mathsf{e}(f, a) \in \mathbb{B} \wedge (\mathsf{e}(f, a) = \top \leftrightarrow (\exists x \in a)(fx = \top)))$.

2.3.4 Operational set-theoretic axioms

The last axiom group consists of three special operational set-theoretic axioms:

(1) Separation for definite operations:

$$(f : a \to \mathbb{B}) \to (\mathbb{S}(f, a)\downarrow \wedge \forall x(x \in \mathbb{S}(f, a) \leftrightarrow (x \in a \wedge fx = \top))).$$

(2) Replacement:

$$(f : a \to \mathbb{V}) \to (\mathbb{R}(f, a)\downarrow \wedge \forall x(x \in \mathbb{R}(f, a) \leftrightarrow (\exists y \in a)(x = fy))).$$

(3) Choice:

$$\exists x(fx = \top) \to (\mathbb{C}f\downarrow \wedge f(\mathbb{C}f) = \top).$$

2.4 The strengthening $\mathsf{OST}(\mathbb{P})$ of OST

We get a significant strengthening of OST if we add the operational form of the power set axiom to the other ones.

Definition 2.17 ($\mathcal{L}^\circ(\mathbb{P})$)**.** The language $\mathcal{L}^\circ(\mathbb{P})$ is the language $\mathcal{L}^\circ$ extended by the new constant $\mathbb{P}$.

The operational power set axiom is stated as follows:

$$(\mathbb{P}) \qquad (\mathbb{P} : \mathbb{V} \to \mathbb{V}) \land \forall x \forall y (x \in \mathbb{P}y \leftrightarrow x \subseteq y).$$

Definition 2.18 ($\mathsf{OST}(\mathbb{P})$)**.** The axioms of the operational set theory $\mathsf{OST}(\mathbb{P})$ are the axioms of OST, now formulated for all $\mathcal{L}^\circ(\mathbb{P})$ formulas, plus the operational power set axiom $(\mathbb{P})$.

2.5 The strengthening $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ of OST

We achieve a further strengthening of OST, if we let a new constant $\mathbf{E}$ act as the unbounded analogue of the constant $\mathbf{e}$.

Definition 2.19 ($\mathcal{L}^\circ(\mathbf{E})$ and $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$)**.** The languages $\mathcal{L}^\circ(\mathbf{E})$ and $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ are the languages $\mathcal{L}^\circ$ and $\mathcal{L}^\circ(\mathbb{P})$ respectively extended by the new constant $\mathbf{E}$.

The role of $\mathbf{E}$ is specified by the new axiom

$$(\mathbf{E}) \qquad (f : \mathbb{V} \to \mathbb{B}) \to (\mathbf{E}(f) \in \mathbb{B} \land (\mathbf{E}(f) = \top \leftrightarrow \exists x (fx = \top))).$$

Definition 2.20 ($\mathsf{OST}(\mathbf{E})$ and $\mathsf{OST}(\mathbf{E}, \mathbb{P})$)**.** The axioms of the theory $\mathsf{OST}(\mathbf{E})$ and $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ are the axioms of OST and $\mathsf{OST}(\mathbb{P})$ respectively, now formulated for all $\mathcal{L}^\circ(\mathbf{E})$ and $\mathcal{L}^\circ(\mathbf{E}, \mathbb{P})$ formulas respectively, plus the axiom $(\mathbf{E})$ about unbounded quantification.

2.6 First consequences of OST

There are two sorts of formulas of $\mathcal{L}^\circ$ (and of $\mathcal{L}^\circ(\mathbb{P})$) which can be represented by constant $\mathcal{L}^\circ$ terms (and $\mathcal{L}^\circ(\mathbb{P})$ terms respectively).

Definition 2.21 (Δ_0 formulas)**.** A formula of $\mathcal{L}^\circ$ (or of $\mathcal{L}^\circ(\mathbb{P})$) is called Δ_0 formula if it does not contain the function symbol $\circ$, the relation symbol $\downarrow$ or unbounded quantifiers.

That is, they are the usual Δ_0 formulas of set theory, possibly containing additional constants.

Example 1. All formulas listed in table 1 are Δ_0 formulas of $\mathcal{L}^\circ$.

Definition 2.22 ($e\Sigma$ formulas)**.** The $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$ are inductively defined as follows:

(i) If s and t are $\mathcal{L}^\circ(\mathbb{P})$ terms, then $(s \in t)$, $(s = t)$ and $(t\downarrow)$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.

(ii) If s and t are variables or constants, then $(s \notin t)$ and $(s \neq t)$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.

(iii) If A and B are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$, then so are $(A \vee B)$ and $(A \wedge B)$.

(iv) If A is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$ and t a term of $\mathcal{L}^\circ(\mathbb{P})$ which does not contain the variable x, then $(\exists x \in t)A$ and $\exists x A$ are $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$.

(v) If A is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$ and t a constant or a variable other than the variable x, then $(\forall x \in t)A$ is an $e\Sigma$ formula of $\mathcal{L}^\circ(\mathbb{P})$.

The $e\Sigma$ formulas of $\mathcal{L}^\circ$ are exactly the $e\Sigma$ formulas of $\mathcal{L}^\circ(\mathbb{P})$ in which the constant $\mathbb{P}$ does not occur.

So the $e\Sigma$ formulas (the extended Σ formulas) of $\mathcal{L}^\circ$ (and $\mathcal{L}^\circ(\mathbb{P})$) are as the Σ formulas of set theory (see definition 5.3) with positive occurrences of arbitrary $\mathcal{L}^\circ$ terms (and $\mathcal{L}^\circ(\mathbb{P})$ terms) permitted as well.

Lemma 2.23. *Let $\vec{u}$ be the sequence of variables $u_1, ..., u_n$ and $A[\vec{u}]$ an $\mathcal{L}^\circ$ formula with at most the variables $\vec{u}$ free.*

(i) If $A[\vec{u}]$ is a Δ_0 formula of $\mathcal{L}^\circ$, then there exists a closed $\mathcal{L}^\circ$ term t_A such that OST *proves the formula*

$$t_A\downarrow \wedge (t_A : \mathbb{V}^n \to \mathbb{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

(ii) If $A[\vec{u}]$ is an $e\Sigma$ formula of $\mathcal{L}^\circ$, then there exists a closed $\mathcal{L}^\circ$ term t_A such that OST *proves the formula*

$$t_A\downarrow \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

We also have the analogous result for $A[\vec{u}]$ is an $\mathcal{L}^\circ(\mathbb{P})$ formula and the theory $\mathsf{OST}(\mathbb{P})$.

The proof of this Lemma is an elaboration of the proof-idea in [6].

Proof. First we define an operation **eq** such that

$$(\mathbf{eq} : \mathbb{V}^2 \to \mathbb{B}) \wedge (\mathbf{eq}(x, y) = \top \leftrightarrow x = y).$$

By the basic set theoretic axiom of extensionality we have $x = y \leftrightarrow (x \subseteq y \wedge y \subseteq x)$ and thus

$$\begin{aligned} x \simeq y \quad &\text{iff} \quad (\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x) \\ &\text{iff} \quad \neg(\exists z \in x)\neg(z \in y) \wedge \neg(\exists z \in y)\neg(z \in x) \\ &\text{iff} \quad \neg((\exists z \in x)\neg(z \in y) \vee (\exists z \in y)\neg(z \in x)). \end{aligned}$$

So by the logical operations axioms

$$\mathbf{eq} := (\lambda xy.\mathsf{non}(\mathsf{dis}(\mathsf{e}((\lambda z.\mathsf{non}(\mathsf{el}(z,y))),x),\mathsf{e}((\lambda z.\mathsf{non}(\mathsf{el}(z,x))),y))))$$

has the desired properties. Using λ-abstraction and equipped with the terms el, non, dis, e and $\mathbf{eq}$ we can define the term t_A with the desired properties formulated in the first part of the lemma. This can be showed by induction on the complexity of the Δ_0 formula $A[\vec{u}]$.

If $A[\vec{u}]$ is a general $e\Sigma$ formula, it can contain three new things: positive occurrences of formulas which contain terms with the function symbol $\circ$, positive occurrences of the formula $(t\downarrow)$ where t is an arbitrary $\mathcal{L}^\circ$ term and unrestricted existential quantification. For manage formulas like $xy = z$ define $\mathbf{ap} := (\lambda xy.xy)$ and observe that $\mathbf{eq}(\mathbf{ap}(x,y),z) = \top \leftrightarrow xy = z$. For manage formulas like $(t\downarrow)$ observe that $t\downarrow \leftrightarrow t = t$ (notice that in the second part of the lemma it is only necessary that $t_A(\vec{u})$ has a value if $A[\vec{u}]$ is the case). If $A[\vec{u}]$ is the $e\Sigma$ formula $\exists x B[\vec{u},x]$ and t_B is the term for the formula $B[\vec{u},x]$, we set $s := (\lambda x.t_B(\vec{u},x))$. Then the term $t_A := (\lambda\vec{u}.s(\mathbb{C}(s)))$ has the desired properties. If we have noticed that, the rest of the proof can be done by induction on the complexity of the $e\Sigma$ formula $A[\vec{u}]$. $\square$

Ordered pairs and products are defined as usual, i.e. ordered pairs are Kuratowski pairs and products sets of ordered pairs.

Lemma 2.24. *There exist the following closed $\mathcal{L}^\circ$ terms:*

(i) $\emptyset$ *for the empty set,*

(ii) $\mathbf{uopa}$ *for forming unordered pairs,*

(iii) $\mathbf{un}$ *for forming unions,*

(iv) $\mathbf{p}$ *for forming ordered pairs,*

(v) $\mathbf{prod}$ *for forming products,*

(vi) $\mathbf{p}_L$ *and* $\mathbf{p}_R$ *as projection operations with respect to* $\mathbf{p}$*, (i.e.* $\mathbf{p}_L(\mathbf{p}(a,b)) = a$ *and* $\mathbf{p}_R(\mathbf{p}(a,b)) = b$*).*

Proof. The basic set-theoretic axioms guarantee the existence of the empty set, of the unordered pair of two given sets and of the union of a given set. If a is one of this sets, then there is a Δ_0 formula $A[x]$ of $\mathcal{L}^\circ$ such that we have by extensionality $A[x]$ iff. $x = a$ (see [1]). Thus, with the term t_A and the operation $\mathbb{C}$ we can pick the set a (like we do it below for the sixth assertion). This finishes the proof of the first three assertions.

For the fourth assertion take $\mathbf{p} := \lambda xy.\mathbf{uopa}(\mathbf{uopa}(x,x), \mathbf{uopa}(x,y))$.

Let t be the term $\lambda b.(\lambda x.\mathbb{R}(\lambda y.\mathbf{p}(x,y), b))$. Then we take $\mathbf{prod} := \lambda ab.\mathbf{un}(\mathbb{R}(t(b), a))$.

For the sixth assertion let $A[a,x]$ be a Δ_0 formula of $\mathcal{L}^\circ$ which expresses that a is an ordered pair and its first component is x (for the existence of $A[a,x]$ see [1]). Then set $\mathbf{p}_L := \lambda a.\mathbb{C}(\lambda x.t_A(a,x))$. The term $\mathbf{p}_R$ is defined analogous. □

Sometimes we will write {a, b} for $\mathbf{uopa}(a,b)$, $\cup a$ for $\mathbf{un}(a)$, $\langle a, b\rangle$ for $\mathbf{p}(a,b)$ and $a \times b$ for $\mathbf{prod}(a,b)$. Now we have introduced some abbreviations which stand for two different formulas. E.g. $a \neq \emptyset$ stands for the formula specified in table 1, but it also stands for the formula $(\neg(a = t) \wedge a{\downarrow} \wedge t{\downarrow})$ where t is the term for the empty set of lemma 2.24. If we need an abbreviation which stands for two different formulas, then mostly it doesn't make any difference which of them we mean by it, because the two formulas always have the same meaning. If it matters although which formula we mean, we will declare which one it is.

We denote by $Rel(a)$ and $Fun(a)$ Δ_0 formulas of the basic language $\mathcal{L}$ which denote that the set a is a binary relation and function, respectively, in the set-theoretic sense (it is well-known that such formulas exist, for example see [1]). Further we need $Dom(a) = b$ and $Ran(a) = b$ as abbreviations for Δ_0 formulas which express that a is a relation with domain b and a is a relation with range b respectively (surly such formulas exist). If $Fun(a)$ holds and u belongs to the domain of a we write $a'u$ for the unique v such that $\langle u, v\rangle \in a$.

Lemma 2.25. *There exist closed $\mathcal{L}^\circ$ terms* **dom**, **ran**, **op** *and* **fun** *such that* OST *proves the following assertions:*

(i) $\mathbf{dom}(f){\downarrow} \wedge \mathbf{ran}(f){\downarrow} \wedge \mathbf{op}(f){\downarrow}$.

(ii) $Rel(a) \to (Dom(a) = \mathbf{dom}(a) \wedge Ran(a) = \mathbf{ran}(a))$.

(iii) $(Fun(f) \wedge a \in \mathbf{dom}(f)) \to f'a = \mathbf{op}(f,a)$.

(iv) $(f : a \to \mathbb{V}) \to (Fun(\mathbf{fun}(f,a)) \wedge \mathbf{dom}(\mathbf{fun}(f,a)) = a)$.

(v) $(f : a \to \mathbb{V}) \to (\forall x \in a)(\mathbf{fun}(f,a)'x = fx)$.

PROOF. Let $A[a,x]$ and $B[a,x]$ be two Δ_0 formulas of $\mathcal{L}^\circ$, with exactly the variables a and x free, which expresses that x is in the domain of a and x is in the range of a respectively. Then we set $\mathbf{dom} := \lambda a.\mathbb{S}(\lambda x.t_A(a,x), \mathbf{un}(\mathbf{un}(a)))$ and $\mathbf{ran} := \lambda a.\mathbb{S}(\lambda x.t_B(a,x), \mathbf{un}(\mathbf{un}(a)))$.
Let $C[f,x,y]$ be a Δ_0 formula of $\mathcal{L}^\circ$, with exactly the variables f, x and y free, which expresses that the ordered pair with the first component x and the second component y is an element of f. Then take $\mathbf{op} := \lambda f.(\lambda x.\mathbb{C}(\lambda y.t_C(f,x,y)))$. This finishes the proof of the first three assertions.

Assume $(f : a \to \mathbb{V})$. If we set $s = \mathbb{R}(f,a)$, then of course $(f : a \to s)$. Let $\mathbf{ap}$ and $\mathbf{eq}$ be the operations defined in the proof of lemma 2.23 and $t := \lambda f x_1 x_2.\mathbf{eq}(\mathbf{ap}(f,x_1),x_2)$. Thus $t(f,x_1,x_2) = \top \leftrightarrow fx_1 = x_2$. Then set $\mathbf{fun} := \lambda fa.\mathbb{S}(\lambda x.t(f,\mathbf{p}_L(x),\mathbf{p}_R(x)), \mathbf{prod}(a,s))$ and check that the assertions four and five hold. □

So we have that each set-theoretic function can be translated into an operation which yields the same values on the domain of the function. On the other hand there corresponds to each operation, which is total on a set a, a set-theoretic function with domain a such that the values of this operation and of this function agree on a.

Definition 2.26 (Axiom of choice)**.** The axiom of choice is given by

$$(\mathbf{AC}) \qquad (\forall x \in a)(x \neq \emptyset) \to \exists f(Fun(f) \wedge Dom(f) = a \wedge (\forall x \in a)(f'x \in x)).$$

Lemma 2.27 (The axiom of choice)**.** *The axiom of choice holds in* OST.

PROOF. Set $t := \lambda x.\mathbb{C}(\lambda y.\mathbf{el}(y,x))$. Then for every non-empty set x we have $t(x) \in x$. Thus if a is a set of non-empty sets, then $\mathbf{fun}(t,a)$ is a choice function on a with the desired properties. □

3 Totality

In this section we are interested in questions about totality for operations.

Definition 3.1 (Totality)**.** Totality is the assertion

$$(\mathbf{TOT}) \qquad \forall x \forall y (xy\downarrow).$$

First we are asking whether OST plus totality is consistent. Cantini and Crosilla showed in [5] and [4] via a fixed point argument that the theories COST (constructive operational set theory) and EST (elementary constructive operational set theory), which have similarities with OST, refutes $\mathbf{TOT}$. We choose another approach (another fixed point argument) to show that the same holds also for OST.

Theorem 3.2. *There is an operation which is nowhere defined. More precisely, there exists a closed $\mathcal{L}^\circ$ term t such that OST proves $t\downarrow$ and $\forall x(tx\uparrow)$.*

Proof. We define s as the term given by $(\lambda xy.\mathbf{uopa}(xy, xy))$ and we set $t := \mathsf{fix}(s)$. Then we have $t\downarrow$ and for an arbitrary set x

$$tx \simeq s(t, x) \simeq \mathbf{uopa}(tx, tx) \simeq \{tx\}.$$

If $tx\downarrow$ held for any x, we would have $tx = \{tx\}$. It would follow that $tx \in tx$ which is not possible in OST (this can be showed by $\in$-induction). Hence there is no x such that $tx\downarrow$. $\square$

So we have immediately:

Corollary 3.3. OST *refutes* $\mathbf{TOT}$.

The second question in this section is: Is there an operation which checks if an arbitrary operation is total, i.e. does an operation f with the properties

$$(f : \mathbb{V} \to \mathbb{B}) \ \wedge \ \forall x(fx = \top \leftrightarrow \forall y(xy\downarrow)) \tag{1}$$

exist? For the proof that such an operation does not exist we need this lemma:

Lemma 3.4. *Let $\vec{x}$ be a sequence of variables $x_1, ..., x_n$. For every Δ_0 formula $A[\vec{x}]$ with at most the variables $\vec{x}$ free and every two terms t and s of $\mathcal{L}^\circ$ (or of $\mathcal{L}^\circ(\mathbb{P})$), there exists an $\mathcal{L}^\circ$ term (or $\mathcal{L}^\circ(\mathbb{P})$ term) $\mathrm{mod}_A^{t,s}$ such that OST (or $\mathsf{OST}(\mathbb{P})$) proves*

$$(t\downarrow \wedge s\downarrow) \ \to \ \mathrm{mod}_A^{t,s}\downarrow \ \wedge \ (\mathrm{mod}_A^{t,s} : \mathbb{V}^n \to \{t, s\}) \ \ \textit{and}$$

$$(t\downarrow \wedge s\downarrow) \ \to \ ((A[\vec{x}] \to \mathrm{mod}_A^{t,s}(\vec{x}) = t) \ \wedge \ (\neg A[\vec{x}] \to \mathrm{mod}_A^{t,s}(\vec{x}) = s)).$$

If t and s are closed terms, then so is $\mathrm{mod}_A^{t,s}$.

PROOF. By Lemma 2.23 there is a closed $\mathcal{L}^\circ$ term (or $\mathcal{L}^\circ(\mathbb{P})$ term) t_A such that

$$t_A\downarrow \wedge (t_A : \mathbb{V}^n \to \mathbb{B}) \wedge \forall \vec{x}(A[\vec{x}] \leftrightarrow t_A(\vec{x}) = \top).$$

Let r be the term $\{\langle \top, t\rangle, \langle \bot, s\rangle\}$. We define

$$\mathrm{mod}_A^{t,s} := (\lambda x_1 ... x_n.\mathbf{op}(r, t_A(\vec{x}))).$$

It is easy to check that $\mathrm{mod}_A^{t,s}$ has the desired properties. $\square$

Theorem 3.5. OST *refutes the existence of an operation f with the properties given by (1).*

PROOF. Let us assume that there is an operation f with the properties given by (1). We define the operation mod as

$$\mathrm{mod} := (\lambda xy.(\mathrm{mod}_A^{(\lambda xy.\mathbf{uopa}(xy,xy)),(\lambda xy.\emptyset)}(fx))xy),$$

where $A[x]$ is the Δ_0 formula $x = \top$. Thus we have

$$\mathrm{mod}(x, y) = \begin{cases} \{xy\} & \text{if } fx = \top, \\ \emptyset & \text{if } fx = \bot. \end{cases}$$

Now we set $g := \textsf{\textbf{fix}}(\mathrm{mod})$, and so we have for an arbitrary x

$$gx \simeq \mathrm{mod}(g, x).$$

In the case that $xy\uparrow$ we have $\mathrm{mod}(x, y) = \emptyset$ and thus $\forall x \forall y(\mathrm{mod}(x, y)\downarrow)$ and so $\forall x(gx\downarrow)$. It follows that

$$gx = \mathrm{mod}(g, x) = \{gx\}$$

like in the proof of theorem 3.2 which is not possible. Thus our assumption is false. $\square$

4 Sets of operations and operational extensionality

In classical set theory there exists for arbitrary sets a and b the set of all functions from a to b. This is also the case in operational set theory with the operational powerset axiom if we mean by it the set of all functions in the typical set-theoretic sense. How about the set of all operations from a to b ($\{f : (f : a \to b)\}$), is the assumption of its existence consistent? This chapter is mainly about this question. Cantini and Crosilla discussed in [5] and [4] a similar question. They showed via a fixed point argument (another one than we will use below) that the theories COST and EST together with the assertion

$$\forall a \forall b \exists c (c = \{f : (\forall x \in a)(\exists y \in b)(fx \simeq y)\})$$

are inconsistent.
Before we can answer the question of this section, we need the following lemmas.

Lemma 4.1. OST *proves that there is no set which contains all sets.*

Proof. Assume that there is the set of all sets and call it $\mathbb{V}$. Then we had $\mathbb{V} \in \mathbb{V}$ which is not possible (this can be showed by $\in$-induction). $\square$

In the proof of the next lemma we need the unbounded quantification operation E. It is an interesting question whether we can prove the assertion also for OST, i.e. without the operation E.

Lemma 4.2. *Let f be a total injective operation, i.e.*

$$(f : \mathbb{V} \to \mathbb{V}) \wedge \forall x \forall y (fx = fy \to x = y).$$

Then $\mathsf{OST}(\mathsf{E})$ *proves*

$$\neg \exists a \forall x (fx \in a).$$

Proof. Assume that there is a set a such that we have for all x that $fx \in a$. Now let t be the term $(\lambda yx.\mathbf{eq}(fx, y))$ where the term $\mathbf{eq}$ is defined as in the proof of lemma 2.23. Since f is total, we have $(t : \mathbb{V}^2 \to \mathbb{B})$ and

$$t(y, x) = \begin{cases} \top & \text{if } fx = y, \\ \bot & \text{if } fx \neq y. \end{cases}$$

So we have for every y that $(t(y) : \mathbb{V} \to \mathbb{B})$ and thus by the axiom (E) about unbounded quantification

$$\mathsf{E}(t(y)) \in \mathbb{B} \wedge (\mathsf{E}(t(y)) = \top \leftrightarrow \exists x (fx = y)).$$

Then by operational separation there is the set $a' = \mathbb{S}(\lambda y.\mathsf{E}(t(y)), a)$ which is equal to the class $\{y : \exists x(fx = y)\}$.

Now we define the operation f^{-1} by the term $(\lambda y.\mathbb{C}(t(y)))$. By the operational choice axiom it follows that $f^{-1} : a' \to \mathbb{V}$ and for every $y \in a'$

$$f^{-1}y = x \leftrightarrow fx = y,$$

because f is injective. So we have by the operational replacement axiom $\mathbb{R}(f^{-1}, a')\downarrow$ and from the totality of f it follows that $\mathbb{R}(f^{-1}, a')$ contains every set. This is not possible by reason of the previous lemma. Thus a set with the properties of a can not exist. $\square$

Theorem 4.3. *Given an arbitrary set a and a nonempty set b, $\mathsf{OST}(\mathsf{E})$ refutes the existence of a set which is extensionally equal to the class $\{f : (f : a \to b)\}$.*

Proof. We denote by w an element of b. Since there is no set which contains all sets, there is a set which is not in a. We pick such a set and call it x_0. Then we define f as the operation

$$\lambda z.\mathbf{un}(\mathbf{uopa}(\mathbf{prod}(a, \mathbf{uopa}(w, w)), \mathbf{uopa}(\mathbf{p}(x_0, z), \mathbf{p}(x_0, z))))$$
$$= \lambda z.(a \times \{w\}) \cup \{\langle x_0, z\rangle\}.$$

So we have that $f(z) = \{\langle x, w\rangle : x \in a\} \cup \{\langle x_0, z\rangle\}$ for every z. In particular f is total and injective (by the basic set-theoretic axioms) and we have $Fun(f(z))$ for every z. Further we define g as the operation given by $\lambda z.\mathbf{op}(f(z))$. Then also g is total and, because $g(z, x_0) = z$ for every z, it is also injective (i.e. $g(z_1) = g(z_2)$ iff $z_1 = z_2$). Further we have $g(z) \in \{f : (f : a \to b)\}$ for every set z. Thus, by the previous lemma there is no set equal to $\{f : (f : a \to b)\}$. $\square$

A set-theoretic function is only defined on its domain which is always a set. But the formula $(f : a \to b)$ doesn't state that f is only defined on a. If we translate a set-theoretic function with domain a into an operation yielding the same values on a, then there are many possibilities in order to which values the operation can yield outside of a. The proof of theorem 4.3 says that these are too many possibilities such that the class of operations $\{f : (f : a \to b)\}$ can not be a set. We can now restrict this class such that there is only one possibility in order to which values the operation can yield outside of a. So the next question is: is the class

$$\{f : (f : a \to b) \wedge \forall x(x \notin a \to fx = c)\},$$

where c is an arbitrary set, a set? The next theorem states that this is not the case if a and b contains at least one element, two elements respectively.

Theorem 4.4. *Given a set a containing at least one element, a set b containing at least two elements and an arbitrary set c. Then* OST *refutes that the class*

$$\{f : (f : a \to b) \wedge \forall x(x \notin a \to fx = c)\}$$

is a set.

Proof. Let a, b, c, v, w_1 and w_2 be sets where v is an element of a and w_1 and w_2 are two different elements of b.

Let's assume that

$$\{f : (f : a \to b) \wedge \forall x(x \notin a \to fx = c)\}$$

is a set and denote it by u.

We denote by $A[x, y]$ the formula $x = y$ and by $B[x, y]$ the formula $x \in y$. Both formulas are Δ_0. We define the term mod as

$$\mathrm{mod} := \left(\lambda f x.\mathrm{mod}_B^{\mathrm{mod}_A^{w_1,w_2}(fx,w_2),c}(x, a)\right),$$

(for the definition of $\mathrm{mod}_B^{\mathrm{mod}_A^{w_1,w_2}(fx,w_2),c}$ see Lemma 3.4) and we have

$$\mathrm{mod}(f, x) = \begin{cases} w_1 & \text{if } fx = w_2 \text{ and } x \in a, \\ w_2 & \text{if } fx \neq w_2 \text{ and } x \in a, \\ c & \text{if } x \notin a \text{ and } fx\downarrow. \end{cases}$$

It follows that if $f \in u$ then also $(\lambda x.\mathrm{mod} f x) \in u$.

Another term MOD is defined as

$$\mathrm{MOD} := \left(\lambda f x.(\mathrm{mod}_B^{\lambda x.\mathrm{mod}(f,x),\lambda x.\mathrm{mod}_B^{w_1,c}(x,a)}(f, u))x\right)$$

and so

$$\mathrm{MOD}(f, x) = \begin{cases} \mathrm{mod}(f, x) & \text{if } f \in u, \\ w_1 & \text{if } f \notin u \text{ and } x \in a, \\ c & \text{if } f \notin u \text{ and } x \notin a. \end{cases}$$

Hence for an arbitrary set f we have $\mathrm{MOD} f \in u$.

Now we set $g := (\lambda f.\mathrm{MOD} f)$ and $h := \mathsf{fix}(g)$. Therefore $h(x) \simeq gh(x)$, and because $gh \in u$ we have also $h \in u$. It follows

$$h(v) = gh(v) = \mathrm{MOD}(h, v) = \mathrm{mod}(h, v) = \begin{cases} w_1 & \text{if } hv = w_2, \\ w_2 & \text{if } hv \neq w_2. \end{cases}$$

Because $w_1 \neq w_2$ this is a contradiction. Therefore the assumption is false and u is not a set. □

We can also restrict the class $\{f : (f : a \to b)\}$ such that the members of the restricted class does only take values in a. Such a class is also not a set in OST (if a and b contains at least one element, two elements respectively). The proof of this fact is very similar to the proof of the previous theorem as you can see in the following.

Theorem 4.5. *Given a set a containing at least one element and a set b containing at least two elements. Then OST refutes that the class*

$$\{f : (f : a \to b) \land \forall x(x \notin a \to fx\uparrow)\}$$

is a set.

Proof. Let a, b, v, w_1 and w_2 be sets where v is an element of a and w_1 and w_2 are two different elements of b.

Let's assume that

$$\{f : (f : a \to b) \land \forall x(x \notin a \to fx\uparrow)\}$$

is a set and denote it by u.

Let c be an operation which is nowhere defined (by theorem 3.2 such an operation exists). We denote by $A[x, y]$ the Δ_0 formula $x = y$ and by $B[x, y]$ the Δ_0 formula $x \in y$ and define the term mod as

$$\mathrm{mod} := \left(\lambda f x.\mathrm{mod}_B^{\mathrm{mod}_A^{(\lambda x.w_1),(\lambda x.w_2)}(fx,w_2),c}(x, a)\right).$$

Then we have

$$(\mathrm{mod}(f, x))x \simeq \begin{cases} w_1 & \text{if } fx = w_2 \text{ and } x \in a, \\ w_2 & \text{if } fx \neq w_2 \text{ and } x \in a, \\ cx & \text{if } x \notin a. \end{cases}$$

Because $cx\uparrow$ it follows that if $f \in u$ then also $(\lambda x.(\mathrm{mod}(f, x))x) \in u$.

The term MOD is defined as

$$\mathrm{MOD} := \left(\lambda f x.(\mathrm{mod}_B^{\lambda x.\mathrm{mod}(f,x),\lambda x.\mathrm{mod}_B^{(\lambda x.w_1),c}(x,a)}(f, u))xx\right)$$

and so

$$\mathrm{MOD}(f,x) \simeq \begin{cases} (\mathrm{mod}(f,x))x & \text{if } f \in u, \\ w_1 & \text{if } f \notin u \text{ and } x \in a, \\ cx & \text{if } f \notin u \text{ and } x \notin a. \end{cases}$$

Hence for an arbitrary set f we have $\mathrm{MOD}f \in u$.

Now we set $g := (\lambda f.\mathrm{MOD}f)$ and $h := \textsf{\textbf{fix}}(g)$. Therefore $h(x) \simeq gh(x)$, and because $gh \in u$ we have also $h \in u$. It follows

$$h(v) = gh(v) = \mathrm{MOD}(h,v) = (\mathrm{mod}(h,v))v = \begin{cases} w_1 & \text{if } hv = w_2, \\ w_2 & \text{if } hv \neq w_2. \end{cases}$$

Because $w_1 \neq w_2$ this is a contradiction. Therefore the assumption is false and u is not a set. □

Definition 4.6 (Operational extensionality)**.** Extensionality for operations is the assertion

$$(\mathbf{EXT}) \qquad \forall x(fx \simeq gx) \to f = g.$$

In [5] and [4] Cantini and Crosilla showed that the theories COST and EST refutes extensionality for operations. They argued that every total operation f in an extensional partial combinatory algebra possess a fixed point (i. e. there is an x such that $fx = x$) and proved that this is not the case in models of COST and EST. Here we deduce from theorem 4.4 that extensionality for operations fails also in the theory OST.

Corollary 4.7. OST *refutes* **EXT***.*

Proof. Given the set $a := \{\emptyset\}$, the set $b := \{\emptyset, a\}$, an arbitrary set c, and the class $D := \{f : (f : a \to b) \wedge \forall x(x \notin a \to fx = c)\}$. Let f_1 and f_2 be operations, such that

$$f_1 x = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ c & \text{if } x \neq \emptyset, \end{cases}$$

and

$$f_2 x = \begin{cases} a & \text{if } x = \emptyset, \\ c & \text{if } x \neq \emptyset. \end{cases}$$

By Lemma 3.4 such operations exist (take $f_1 = (\lambda x.\mathrm{mod}_A^{\emptyset,c}(x,\emptyset))$ and $f_2 = (\lambda x.\mathrm{mod}_A^{a,c}(x,\emptyset))$ where $A[x,y]$ is the formula $x = y$) and we have that $f_1, f_2 \in D$. With **EXT** we can even prove that $D = \{f_1, f_2\}$ and so D ist a set by the axiom for unordered pairs. This is a contradiction to theorem 4.4, and so OST plus **EXT** is not consistent. □

5 Functions as operations

In this section expressions like $\langle x, y\rangle = z$ do not represent formulas like $\mathbf{p}(x, y) = z$, but they denote Δ_0 formulas wich express statements like „z is identical with the ordered pair $\langle x, y\rangle$“ (how such Δ_0 formulas can be defined, is written for example in [1]).

We introduce two versions of an axiom which is also mentioned in [3]. The weaker version of the axiom states that if f is a set-theoretic function, then the values of f as function agree with the values of f as operation on the set-theoretic domain of f. The stronger version of the axiom says in addition that f as operation does not take any values outside of the set-theoretic domain of f.

Definition 5.1 (Functions are operations)**.** The weaker version of the axiom „functions are operations“ is given by

$$(\mathbf{FO}1) \qquad Fun(f) \to \forall x \forall y(\langle x, y\rangle \in f \to fx = y),$$

and the stronger one is given by

$$(\mathbf{FO}2) \qquad Fun(f) \to \forall x \forall y(\langle x, y\rangle \in f \leftrightarrow fx = y).$$

In this section we are interested in the question whether the theory OST plus **FO**1 (plus **FO**2 respectively) is consistent. Since the axioms of OST does not specify the behaviour of any object as operation and the properties of the same object as set at the same time, it would be a surprise, if OST and **FO**1 (**FO**2 respectively) got not along with each other. The way to answer the question of this section is similar to the way to identify the consistency strength of OST in [8].

5.1 The theory $\mathsf{KP}\omega$

For answer the main question of this section we introduce the theory $\mathsf{KP}\omega$, the Kripke-Platek set theory without urelements (see [1]) plus infinity. $\mathsf{KP}\omega$ is formulated in our basic language $\mathcal{L}$. The logic of $\mathsf{KP}\omega$ is the classical first order logic with equality.

Definition 5.2 (Axioms of $\mathsf{KP}\omega$)**.** The non-logical axioms of $\mathsf{KP}\omega$ are: extensionality, pair, union, infinity (ω is the first infinite ordinal), $\in$-induciton for arbitrary formulas $A[x]$ of $\mathcal{L}$,

$$(\mathcal{L}\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \to A[x]) \to \forall x A[x],$$

as well as Δ_0 separation and Δ_0 collection, i.e.

$$(\Delta_0\text{-}\mathbf{Sep}) \qquad \exists x(x = \{y \in a : B[y]\}),$$

(Δ_0-**Col**) $\qquad (\forall x \in a)(\exists y C[x, y]) \to \exists z (\forall x \in a)(\exists y \in z) C[x, y]$

for arbitrary Δ_0 formulas $B[u]$ and $C[u, v]$ of $\mathcal{L}$.
Note that the first four axioms are formulated as in 2.3.2.

Definition 5.3 (Σ formulas and Π formulas)**.** The Σ formulas of $\mathcal{L}$ are inductively defined as follows:

(i) If A is a Δ_0 formula of $\mathcal{L}$, then A is a Σ formula of $\mathcal{L}$.

(ii) If A and B are Σ formulas of $\mathcal{L}$, then so are $(A \lor B)$ and $(A \land B)$.

(iii) If A is a Σ formula of $\mathcal{L}$, x is a variable and t a constant or a variable of $\mathcal{L}$, then $(\exists x \in t)A$ and $\exists x A$ are Σ formulas of $\mathcal{L}$.

(iv) If A is a Σ formula of $\mathcal{L}$, x is a variable and t a constant or a variable of $\mathcal{L}$, then $(\forall x \in t)A$ is a Σ formula of $\mathcal{L}$.

On the other hand, the Π formulas of $\mathcal{L}$ are inductively defined as follows:

(i) If A is a Δ_0 formula of $\mathcal{L}$, then A is a Π formula of $\mathcal{L}$.

(ii) If A and B are Π formulas of $\mathcal{L}$, then so are $(A \lor B)$ and $(A \land B)$.

(iii) If A is a Π formula of $\mathcal{L}$, x is a variable and t a constant or a variable of $\mathcal{L}$, then $(\exists x \in t)A$ is a Π formula of $\mathcal{L}$.

(iv) If A is a Π formula of $\mathcal{L}$, x is a variable and t a constant or a variable of $\mathcal{L}$, then $(\forall x \in t)A$ and $\forall x A$ are Π formulas of $\mathcal{L}$.

Remark 1. The negation of any Σ formula of $\mathcal{L}$ is logically equivalent to a Π formula of $\mathcal{L}$ and vice versa.

The proof of the next theorem can be looked up in [1].

Theorem 5.4 (The Σ reflection principle)**.** *If A is a Σ formula, then* $\mathsf{KP}\omega$ *proves*

$$A \leftrightarrow \exists u A^u.$$

Definition 5.5 (Δ formulas)**.** An $\mathcal{L}$ formula A is Δ over $\mathsf{KP}\omega$ if there is some Σ formula B of $\mathcal{L}$ and some Π formula C of $\mathcal{L}$, such that both contain exactly the same free variables as A and $\mathsf{KP}\omega$ proves $A \leftrightarrow B$ and $A \leftrightarrow C$.

Example 2. Of course all Δ_0 formulas are Δ.

We use lower case Greek letters α, β, γ, ... (possibly with subscripts) for ordinals (a set x is an ordinal, if $\mathrm{Ord}(x)$ holds, see table 1) and write $(\alpha < \beta)$ for $(\alpha \in \beta)$. Further $(a \in L_\alpha)$ states that a is an element of the αth level L_α of the constructible hierarchy and $(a <_L b)$ means that a is smaller than b according to the well-ordering $<_L$ on the constructible universe L. It is well-known that the assertions $a \in L_\alpha$ and $a <_L b$ are Δ over $\mathsf{KP}\omega$ (see for example [1] or [12]).

Definition 5.6 $(V = L)$**.** The axiom $(V = L)$ says:

$(V = L)$ All sets are constructible.

Important here is: the axiom $(V = L)$ guarantees that the universe is well-ordered by $<_L$ and for every set a there is an α such that $a \in L_\alpha$.

5.2 The embedding of OST plus FO1/2 into $\mathsf{KP}\omega$ plus $(V = L)$

There are Δ_0 formulas which express that the set a is an ordered pair and that the first or second component of a is x (see [1]). Generally for any natural number n greater than 0 we select a Δ_0 formula $Tup_n(a)$ formalising that a is an ordered n-tuple and Δ_0 formulas $(a)_i = b$, $(a)_i \in b$ and $(a)_i \notin b$ formalising that its ith component is b, is an element of b and is no element of b respectively. So we have

$$Tup_n(a) \wedge (a)_1 = b_1 \wedge ... \wedge (a)_n = b_n \rightarrow a = \langle b_1, ..., b_n \rangle.$$

In addition we fix a Δ_0 formula $\overline{Tup_3}(a)$ which formalises that a is a special sort of ordered triple: a is a set containing two usual ordered triples, namely $a = \{\langle x, y, z\rangle, \langle x, y, \{z\}\rangle\}$ for some sets x, y and z. Then we write $a = [x, y, z]$. Notice that usual ordered triples have the form $\langle\langle x, y\rangle, z\rangle$ (for example in [11]). If you wonder why the form of the ordered triples matters in the following, you have to be patient until remark 3 (in [8] such unusual triples are not used). We fix also a Δ_0 formula $[a]_i = b$, formalising that the ith component of the special triple a is b. So we have for example if $a = [x, y, z]$, then $[a]_2 = y$. We can define $\overline{Tup_3}(a)$ and $[a]_i = b$ like in the following table.

Table 2: Special abbreviations

Abbreviation	*Formula*
$\overline{Tup_3}(a)$	$(\exists x, y \in a)((\forall z \in a)((z = x \vee z = y) \wedge Tup_3(x) \wedge Tup_3(y)$ $\wedge\ (x)_1 = (y)_1 \wedge (x)_2 = (y)_2 \wedge (y)_3 = \{(x)_3\}))$
$[a]_i = b$ for $i \in \{1, 2\}$	$(\forall x \in a)((a)_i = b)$
$[a]_3 = b$	$(\exists x, y \in a)((x)_3 = b \wedge (y)_3 = \{b\})$

Further we fix pairwise different sets $\widehat{\mathsf{k}}$, $\widehat{\mathsf{s}}$, $\widehat{\top}$, $\widehat{\bot}$, $\widehat{\mathsf{el}}$, $\widehat{\mathsf{non}}$, $\widehat{\mathsf{dis}}$, $\widehat{\mathsf{e}}$, $\widehat{\mathbb{S}}$, $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}$ which do all contain infinitely many elements and which are all not set theoretic functions, i.e. $\mathsf{KP}\omega$ proves $\neg Fun(a)$ if a is a set of the list (in [8] these sets only must not be ordered pairs and triples, the stronger conditions here will be used in the proofs of the lemmas 5.9 and 5.10). We will use them as codes of the corresponding constants of $\mathcal{L}^\circ$. $\mathcal{L}^\circ$ terms like $\mathbf{k}x$, $\mathbf{s}x$ or $\mathbf{s}xy$ will be coded by the ordered tuples $\langle\widehat{\mathbf{k}}, x\rangle$, $\langle\widehat{\mathbf{s}}, x\rangle$ and $[\widehat{\mathbf{s}}, x, y]$.

Definition 5.7 (The language $\mathcal{L}(R)$)**.** Let R be a fresh 4-place relation symbol. The language $\mathcal{L}(R)$ is the extension of the language $\mathcal{L}$ which we get if we permit expressions $R(\alpha, a, b, c)$ as additional atomic formulas. We abbreviate

$$R^{<\alpha}(a, b, c) := (\exists\beta < \alpha)R(\beta, a, b, c).$$

The $\mathcal{L}(R)$ formula introduced in the next definition will be used for the interpretation of applying operations of OST to others within the theory $\mathsf{KP}\omega$ plus $(V = L)$.

Definition 5.8. The $\mathcal{L}(R)$ formula $\mathfrak{A}[R, \alpha, a, b, c]$ is defined as

$$\mathfrak{A}[R, \alpha, a, b, c] \quad := \quad c \in L_\alpha \wedge \mathfrak{B}[R, \alpha, a, b, c],$$

where $\mathfrak{B}[R, \alpha, a, b, c]$ is the $\mathcal{L}(R)$ formula given as the disjunction of the following clauses:

(1) $a = \widehat{\mathbf{k}} \wedge c = \langle\widehat{\mathbf{k}}, b\rangle$,

(2) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{k}} \wedge (a)_2 = c$,

(3) $a = \widehat{\mathbf{s}} \wedge c = \langle\widehat{\mathbf{s}}, b\rangle$,

(4) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{s}} \wedge c = [\widehat{\mathbf{s}}, (a)_2, b]$,

(5) $\overline{Tup_3}(a) \wedge [a]_1 = \widehat{\mathbf{s}}$
$\wedge (\exists x, y \in L_\alpha)(R^{<\alpha}([a]_2, b, x) \wedge R^{<\alpha}([a]_3, b, y) \wedge R^{<\alpha}(x, y, c))$,

(6) $a = \widehat{\mathbf{el}} \wedge c = \langle\widehat{\mathbf{el}}, b\rangle$,

(7) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{el}} \wedge (a)_2 \in b \wedge c = \widehat{\top}$,

(8) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbf{el}} \wedge (a)_2 \notin b \wedge c = \widehat{\bot}$,

(9) $a = \widehat{\mathbf{non}} \wedge b = \widehat{\top} \wedge c = \widehat{\bot}$,

(10) $a = \widehat{\mathbf{non}} \wedge b = \widehat{\bot} \wedge c = \widehat{\top}$,

(11) $a = \widehat{\mathbf{dis}} \wedge c = \langle\widehat{\mathbf{dis}}, b\rangle$,

(12) $Tup_2(a) \wedge (a)_1 = \widehat{\textbf{dis}} \wedge (a)_2 = \widehat{\top} \wedge c = \widehat{\top}$,

(13) $Tup_2(a) \wedge (a)_1 = \widehat{\textbf{dis}} \wedge (a)_2 = \widehat{\bot} \wedge b = \widehat{\top} \wedge c = \widehat{\top}$,

(14) $Tup_2(a) \wedge (a)_1 = \widehat{\textbf{dis}} \wedge (a)_2 = \widehat{\bot} \wedge b = \widehat{\bot} \wedge c = \widehat{\bot}$,

(15) $a = \widehat{\textbf{e}} \wedge c = \langle \widehat{\textbf{e}}, b \rangle$,

(16) $Tup_2(a) \wedge (a)_1 = \widehat{\textbf{e}} \wedge (\exists x \in b)(R^{<\alpha}((a)_2, x, \widehat{\top})) \wedge c = \widehat{\top}$,

(17) $Tup_2(a) \wedge (a)_1 = \widehat{\textbf{e}} \wedge (\forall x \in b)(R^{<\alpha}((a)_2, x, \widehat{\bot})) \wedge c = \widehat{\bot}$,

(18) $a = \widehat{\mathbb{S}} \wedge c = \langle \widehat{\mathbb{S}}, b \rangle$,

(19) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbb{S}} \wedge (\forall x \in b)(R^{<\alpha}((a)_2, x, \widehat{\top}) \vee R^{<\alpha}((a)_2, x, \widehat{\bot}))$
$\wedge (\forall x \in c)(x \in b \wedge R^{<\alpha}((a)_2, x, \widehat{\top}))$
$\wedge (\forall x \in b)(R^{<\alpha}((a)_2, x, \widehat{\top}) \to x \in c)$,

(20) $a = \widehat{\mathbb{R}} \wedge c = \langle \widehat{\mathbb{R}}, b \rangle$,

(21) $Tup_2(a) \wedge (a)_1 = \widehat{\mathbb{R}} \wedge (\forall x \in b)(\exists y \in c)(R^{<\alpha}((a)_2, x, y))$
$\wedge (\forall y \in c)(\exists x \in b)(R^{<\alpha}((a)_2, x, y))$,

(22) $a = \widehat{\mathbb{C}} \wedge R^{<\alpha}(b, c, \widehat{\top}) \wedge (\forall x \in L_\alpha)(x <_L c \to \neg R^{<\alpha}(b, x, \widehat{\top}))$
$\wedge (\forall \beta < \alpha)(\forall x \in L_\beta)(\neg R^{<\beta}(b, x, \widehat{\top}))$,

(23) $Fun(a) \wedge \langle b, c \rangle \in a$.

Remark 2. The formula $\mathfrak{A}[R, \alpha, a, b, c]$ is Δ over $\mathsf{KP}\omega$ with respect to the language $\mathcal{L}(R)$.

Lemma 5.9. *If we have $Fun(a)$ for a set a, then no one of the clauses (1)-(22) of the previous definition can be satisfied for this a and any α, b and c.*

The rightness of this lemma is less trivial as it may seem, because there are ordered pairs which are also set theoretic functions, as the following example shows.

Example 3. Because

$$\langle x, x \rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\}$$

we have

$$\langle \langle x, x \rangle, \langle x, x \rangle \rangle = \langle \{\{x\}\}, \{\{x\}\} \rangle = \{\{\{\{x\}\}\}\}.$$

But we have also

$$\{\langle \{x\}, \{x\} \rangle\} = \{\{\{\{x\}\}\}\}.$$

Thus $Tup_2(\{\{\{\{x\}\}\}\})$ as well as $Fun(\{\{\{\{x\}\}\}\})$.

Proof of lemma 5.9. In all the clauses of the previous definition, in which a is an ordered pair, we have that $(a)_1 = \widehat{c}$, where $\widehat{c}$ is one of the sets for coding the constants of $\mathcal{L}^\circ$ other than ω. So at first we show that an ordered pair $\langle \widehat{c}, x\rangle$ is never a set theoretic function. We distinguish between the case $\widehat{c} = x$ and the case $\widehat{c} \neq x$.
First assume $\widehat{c} = x$. Then we have $\langle \widehat{c}, x\rangle = \{\{\widehat{c}\}\}$, and this is a set containing exactly one element. Thus if this set were a function f, we would have

$$f = \{\langle y_1, y_2\rangle\} = \{\{\{y_1\}, \{y_1, y_2\}\}\} = \{\{\widehat{c}\}\}$$

for some sets y_1, y_2. By extensionality it would follow $\{\{y_1\}, \{y_1, y_2\}\} = \{\widehat{c}\}$ and thus $y_1 = y_2$ (because $\{\widehat{c}\}$ contains only one set) and so $\{y_1\} = \widehat{c}$. This is a contradiction since $\{y_1\}$ contains only one set and $\widehat{c}$ is an infinite set.
On the other hand if $\widehat{c} \neq x$, then $\langle \widehat{c}, x\rangle$ contains two sets. That is, if it were equal to a function f, we would have

$$f = \{\langle y_1, y_2\rangle, \langle z_1, z_2\rangle\} = \{\{\{y_1\}, \{y_1, y_2\}\}, \{\{z_1\}, \{z_1, z_2\}\}\} = \{\{\widehat{c}\}, \{\widehat{c}, x\}\}$$

for some sets y_1, y_2, z_1, z_2 with $y_1 \neq z_1$. It would follow

$$\{\widehat{c}\} = \{\{y_1\}, \{y_1, y_2\}\} \quad \text{or} \quad \{\widehat{c}\} = \{\{z_1\}, \{z_1, z_2\}\}.$$

Without loss of generality we can assume $\{\widehat{c}\} = \{\{y_1\}, \{y_1, y_2\}\}$ and get the same contradiction as above.
Thus, and since the sets, which act as codes, are all not set theoretic functions, no one of the clauses (1)-(4) and (6)-(22) of the previous definition can be satisfied if $Fun(a)$.

It remains to show that if $a = [\widehat{s}, x, y]$ for arbitrary x and y, then a is not a set theoretic function. But this is a direct consequence of the definition of $[\widehat{s}, x, y]$ (a is a relation but not a function; if it were a function, we would have $a'(\langle \widehat{s}, x\rangle) = y$ and $a'(\langle \widehat{s}, x\rangle) = \{y\}$ at the same time). □

Lemma 5.10. *If $\mathfrak{A}[R, \alpha, a, b, c]$ holds in $\mathsf{KP}\omega + (V = L)$, then exactly one of the clauses (1)-(23) of the previous definition is satisfied for these α, a, b and c in $\mathsf{KP}\omega + (V = L)$.*

Proof. Since the sets, which act as codes, are all infinite sets, they are all not ordered pairs. Thus by means of lemma 5.9 it is easy to verify, that at most one of the clauses (1)-(4) and (6)-(23) of the previous definition can be satisfied for fixed α, a, b and c.

In order to show that the fifth clause of the previous definition can not be satisfied at the same time as one of the clauses (1)-(4) and (6)-(22) for fixed α, a, b and c, it is enough to show that a special ordered triple $[\widehat{\mathsf{s}}, x, y]$ is never an ordered pair $\langle \widehat{c}, v\rangle$, where $\widehat{c}$ is one of the sets for coding the constants of $\mathcal{L}^\circ$ other than ω and x, y and v are arbitrary sets. Assume that $[\widehat{\mathsf{s}}, x, y]$ is the ordered pair $\{\{\widehat{c}\}, \{\widehat{c}, v\}\}$. The set $[\widehat{\mathsf{s}}, x, y]$ contains exactly two different sets, so $\widehat{c} \neq v$. We have

$$\begin{aligned} [\widehat{\mathsf{s}}, x, y] &= \{\langle \widehat{\mathsf{s}}, x, y\rangle, \langle \widehat{\mathsf{s}}, x, \{y\}\rangle\} \\ &= \{\langle\langle \widehat{\mathsf{s}}, x\rangle, y\rangle, \langle\langle \widehat{\mathsf{s}}, x\rangle, \{y\}\rangle\} \\ &= \{\{\{\langle \widehat{\mathsf{s}}, x\rangle\}, \{\langle \widehat{\mathsf{s}}, x\rangle, y\}\}, \{\{\langle \widehat{\mathsf{s}}, x\rangle\}, \{\langle \widehat{\mathsf{s}}, x\rangle, \{y\}\}\}\}, \end{aligned}$$

and thus $\{\widehat{c}\} = \{\{\langle \widehat{\mathsf{s}}, x\rangle\}, \{\langle \widehat{\mathsf{s}}, x\rangle, y\}\}$ or $\{\widehat{c}\} = \{\{\langle \widehat{\mathsf{s}}, x\rangle\}, \{\langle \widehat{\mathsf{s}}, x\rangle, \{y\}\}\}$ which is impossible since $\widehat{c}$ contains infinitely many elements. Hence $[\widehat{\mathsf{s}}, x, y]$ is not an ordered pair $\langle \widehat{c}, v\rangle$. $\square$

Remark 3. If we used usual ordered triples in clause (5) of the definition above, we would have the following problem: Of course the set $f = \{\langle\{\widehat{\mathsf{s}}\}, \{\widehat{\mathsf{s}}\}\rangle\}$ is a set theoretic function. In a model of OST plus **FO**1 we have therefore $f(\{\widehat{\mathsf{s}}\}) = \{\widehat{\mathsf{s}}\}$. On the other hand we have

$$f = \{\{\{\{\widehat{\mathsf{s}}\}\}\}\} = \langle\langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}\rangle, \langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}\rangle\rangle.$$

If we used usual ordered triples, the set $\langle\langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}\rangle, \langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}\rangle\rangle = \langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}, \langle \widehat{\mathsf{s}}, \widehat{\mathsf{s}}\rangle\rangle$ would be a code for the $\mathcal{L}^\circ$ term $\mathsf{ss}(\mathsf{ss})$. But we don't want to enforce $(\mathsf{ss}(\mathsf{ss}))(\{\mathsf{s}\}) = \{\mathsf{s}\}$ in our model.

Definition 5.11. For any $\mathcal{L}$ formula $B[\alpha, a, b, c]$ with at most the indicated free variables we write $\mathfrak{A}[B, \alpha, a, b, c]$ for the $\mathcal{L}$ formula resulting by replacing each occurrence of an atomic formula of the form $R(\alpha, r, s, t)$ in $\mathfrak{A}[R, \alpha, a, b, c]$ by $B[\alpha, r, s, t]$.

The following theorem is a special case of „definition by Σ recursion“ (or more precisely a corollary of it) developed in [1] (notice that the transitive closure of an ordinal α is α itself).

Theorem 5.12. *There exists a Σ formula $B[\alpha, a, b, c]$ of $\mathcal{L}$ with at most the variables α, a, b and c free such that $\mathsf{KP}\omega$ proves*

$$(\Sigma\text{-}\mathsf{Rec}/\mathfrak{A}) \qquad B[\alpha, a, b, c] \leftrightarrow \mathfrak{A}[B, \alpha, a, b, c].$$

Definition 5.13. Let $B_{\mathfrak{A}}$ be a Σ formula of $\mathcal{L}$ associated to the operator from $\mathfrak{A}[R, \alpha, a, b, c]$ according to $(\Sigma\text{-}\mathsf{Rec}/\mathfrak{A})$ of the previous theorem. We define

$$B_{\mathfrak{A}}^{<\alpha}[a, b, c] := (\exists\beta < \alpha)B_{\mathfrak{A}}[\beta, a, b, c] \text{ and}$$

$$Ap_{\mathfrak{A}}[a, b, c] := \exists\alpha B_{\mathfrak{A}}[\alpha, a, b, c].$$

The next step is to show that $Ap_{\mathfrak{A}}[a,b,c]$ is functional in its third argument, i.e. $Ap_{\mathfrak{A}}[a,b,x]$ and $Ap_{\mathfrak{A}}[a,b,y]$ implies $x=y$.

Lemma 5.14. *The theory* $\mathsf{KP}\omega$ *proves*

$$B_{\mathfrak{A}}[\alpha,\widehat{\mathbb{C}},f,a] \wedge B_{\mathfrak{A}}[\beta,\widehat{\mathbb{C}},f,b] \to \alpha=\beta \wedge a=b.$$

Proof. From the left hand side of the claimed assertion and by lemma 5.10 we have:

(a) $a \in L_\alpha \wedge b \in L_\beta$,

(b) $B_{\mathfrak{A}}^{<\alpha}[f,a,\top] \wedge B_{\mathfrak{A}}^{<\beta}[f,b,\top]$,

(c) $(\forall x \in L_\alpha)(x <_L a \to \neg B_{\mathfrak{A}}^{<\alpha}(f,x,\widehat{\top}))$,

(d) $(\forall x \in L_\beta)(x <_L b \to \neg B_{\mathfrak{A}}^{<\beta}(f,x,\widehat{\top}))$,

(e) $(\forall \gamma < \alpha)(\forall x \in L_\gamma)(\neg B_{\mathfrak{A}}^{<\gamma}(f,x,\widehat{\top}))$,

(f) $(\forall \gamma < \beta)(\forall x \in L_\gamma)(\neg B_{\mathfrak{A}}^{<\gamma}(f,x,\widehat{\top}))$.

From (a), (b), (e) and (f) it follows $\alpha=\beta$ (because $<$ is a linear ordering). But then (a)-(d) imply $a=b$ (because $<_L$ is a linear ordering too). $\square$

Lemma 5.15. *The theory* $\mathsf{KP}\omega$ *proves*

(i) $B_{\mathfrak{A}}^{<\alpha}[a,b,u] \wedge B_{\mathfrak{A}}^{<\alpha}[a,b,v] \to u=v$,

(ii) $Ap_{\mathfrak{A}}[a,b,u] \wedge Ap_{\mathfrak{A}}[a,b,v] \to u=v$.

Proof. Equipped with the lemmas 5.9, 5.10 and 5.14 the first assertion is easily proved by induction on α. The second assertion is a direct consequence of the first. $\square$

The next thing to do is associating to each term t of $\mathcal{L}^\circ$ a formula $[\![t]\!]_{\mathfrak{A}}(u)$ of $\mathcal{L}$ expressing that u is the value of t under the interpretation of the OST-application via the Σ formula $Ap_{\mathfrak{A}}$.

Definition 5.16 ($[\![t]\!]_{\mathfrak{A}}(u)$ formula)**.** Let t be an $\mathcal{L}^\circ$ term with u not occurring in t. We define the $\mathcal{L}$ formula $[\![t]\!]_{\mathfrak{A}}(u)$ inductively as follows:

(i) If t is a variable or the constant ω, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(t=u)$.

(ii) If t is another constant, then $[\![t]\!]_{\mathfrak{A}}(u)$ is the formula $(\widehat{t}=u)$.

(iii) If t is the term (rs), then we set

$$[\![t]\!]_{\mathfrak{A}}(u) \ := \ \exists x \exists y([\![r]\!]_{\mathfrak{A}}(x) \wedge [\![s]\!]_{\mathfrak{A}}(y) \wedge Ap_{\mathfrak{A}}[x, y, u]).$$

Remark 4. For every term t of $\mathcal{L}^\circ$ its translation $[\![t]\!]_{\mathfrak{A}}(u)$ is a Σ formula.

Now we are able to translate arbitrary formulas of $\mathcal{L}^\circ$ into formulas of $\mathcal{L}$.

Definition 5.17 (Translation of $\mathcal{L}^\circ$ formulas)**.** Let A be a formula of $\mathcal{L}^\circ$. The $\mathcal{L}$ formula A^* is inductively defined as follows:

(i) For the atomic formulas of $\mathcal{L}^\circ$ we set

$$\begin{aligned} (t{\downarrow})^* \ &:= \ \exists x [\![t]\!]_{\mathfrak{A}}(x), \\ (s \in t)^* \ &:= \ \exists x \exists y([\![s]\!]_{\mathfrak{A}}(x) \wedge [\![t]\!]_{\mathfrak{A}}(y) \wedge x \in y), \\ (s = t)^* \ &:= \ \exists x \exists y([\![s]\!]_{\mathfrak{A}}(x) \wedge [\![t]\!]_{\mathfrak{A}}(y) \wedge x = y). \end{aligned}$$

(ii) If A is the formula $\neg B$, then A^* is $\neg B^*$.

(iii) If A is the formula $(B \wedge C)$, then A^* is $(B^* \wedge C^*)$.

(iv) If A is the formula $\forall x B[x]$, then A^* is $\forall x B^*[x]$.

The translations of the axioms of the logic of partial terms are provable in $KP\omega$ plus $(V = L)$, the proof is an easy exercise.

Lemma 5.18. *If A is the logical operations axiom about bounded existential quantification of OST, then we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

Proof. The translation of the premise $(f : a \to \mathbb{B})$ of the bounded existential quantification axiom is equivalent to

$$(\forall x \in a)(Ap_{\mathfrak{A}}[f, x, \widehat{\top}] \vee Ap_{\mathfrak{A}}[f, x, \widehat{\bot}]), \tag{2}$$

and thus by the Σ reflection principle (theorem 5.4) there must be an ordinal α such that

$$(\forall x \in a)(B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\top}] \vee B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\bot}]), \tag{3}$$

and by lemma 5.15 we have

$$(\forall x \in a)(Ap_{\mathfrak{A}}[f, x, \widehat{\top}] \leftrightarrow B^{<\alpha}_{\mathfrak{A}}[f, x, \widehat{\top}]). \tag{4}$$

By the clauses (16) and (17) of definition 5.8 the assertion (3) also implies

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathsf{e}}, f \rangle, a, \widehat{\top}] \vee \mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathsf{e}}, f \rangle, a, \widehat{\bot}] \quad \text{and} \tag{5}$$

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \leftrightarrow (\exists x \in a) B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]. \tag{6}$$

From the assertions (4)-(6) we can conclude together with theorem 5.12 and lemma 5.15 that

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \vee Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\bot}] \quad \text{and} \tag{7}$$

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbf{e}}, f \rangle, a, \widehat{\top}] \leftrightarrow (\exists x \in a) Ap_{\mathfrak{A}}[f, x, \widehat{\top}], \tag{8}$$

which is equivalent to

$$[\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\top}) \vee [\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\bot}) \quad \text{and} \tag{9}$$

$$[\![\mathbf{e}(f, a)]\!]_{\mathfrak{A}}(\widehat{\top}) \leftrightarrow (\exists x \in a) Ap_{\mathfrak{A}}[f, x, \widehat{\top}]. \tag{10}$$

So we have showed that (2) implies (9) and (10). This implication is equivalent to the translation of the axiom about bounded existential quantification, which is thus proved in $\mathsf{KP}\omega + (V = L)$. □

Lemma 5.19. *If A is the operational set-theoretic axiom about separation for definite operations of OST, then we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

Proof. The axiom about operational separation for definite operations has again the premise $(f : a \to \mathbb{B})$ which is translated into a formula equivalent to (2). From this we can deduce by the Σ reflection principle that there is a set b such that

$$(\forall x \in a)((Ap_{\mathfrak{A}}[f, x, \widehat{\top}])^b \vee (Ap_{\mathfrak{A}}[f, x, \widehat{\bot}])^b). \tag{11}$$

By Δ_0-**Sep** we can introduce a set c satisfying

$$\forall x(x \in c \leftrightarrow (x \in a \wedge (Ap_{\mathfrak{A}}[f, x, \widehat{\top}])^b)). \tag{12}$$

Now we select an ordinal α such that a, b and c belong to L_α. Then from (11), (12), lemma 5.15 and Σ persistence (see in [1] corollary 8.6) it follows

$$c = \{x \in a : B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}]\} = \{x \in a : Ap_{\mathfrak{A}}[f, x, \widehat{\top}]\} \quad \text{and} \tag{13}$$

$$(\forall x \in a)(B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\top}] \vee B_{\mathfrak{A}}^{<\alpha}[f, x, \widehat{\bot}]). \tag{14}$$

If we remember clause (19) of definition 5.8, we see that (13) and (14) imply

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle \widehat{\mathbb{S}}, f \rangle, a, c],$$

and hence by theorem 5.12

$$Ap_{\mathfrak{A}}[\langle \widehat{\mathbb{S}}, f \rangle, a, c]. \tag{15}$$

Then we get by (13), (15) and lemma 5.15 that

$$\exists y [\![\mathbb{S}(f,a)]\!]_{\mathfrak{A}}(y) \quad \text{and} \tag{16}$$

$$\exists y([\![\mathbb{S}(f,a)]\!]_{\mathfrak{A}}(y) \wedge \forall x(x \in y \leftrightarrow (x \in a \wedge Ap_{\mathfrak{A}}[f,x,\widehat{\top}]))). \tag{17}$$

So we have showed again an implication ((2) implies (16) and (17)) which is equivalent to the translation of the axiom, this time the axiom about operational separation for definite operations. That is, $\mathsf{KP}\omega + (V = L)$ proves this axiom too. □

Lemma 5.20. *If A is the operational set-theoretic axiom about replacement of* OST*, then we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

Proof. The premise ($f : a \to \mathbb{V}$) of the operational replacement axiom is translated into an $\mathcal{L}$ formula equivalent to

$$(\forall x \in a)(\exists y Ap_{\mathfrak{A}}[f,x,y]). \tag{18}$$

Hence, by the Σ reflection principle, there exists a set b satisfying

$$(\forall x \in a)(\exists y \in b)(Ap_{\mathfrak{A}}[f,x,y])^b, \tag{19}$$

and by Δ_0-**Sep** there is a set c such that

$$\forall y(y \in c \leftrightarrow (y \in b \wedge (\exists x \in a)(Ap_{\mathfrak{A}}[f,x,y])^b)). \tag{20}$$

Let α be an ordinal such that a, b and c are all in L_α. Then because of lemma 5.15 and Σ persistence (see in [1] corollary 8.6) we can deduce from (19) and (20) that

$$c = \{y \in b : (\exists x \in a)B^{<\alpha}_{\mathfrak{A}}[f,x,y]\} = \{y : (\exists x \in a)Ap_{\mathfrak{A}}[f,x,y]\} \quad \text{and} \tag{21}$$

$$(\forall x \in a)(\exists y \in c)B^{<\alpha}_{\mathfrak{A}}[f,x,y] \wedge (\forall y \in c)(\exists x \in a)B^{<\alpha}_{\mathfrak{A}}[f,x,y]. \tag{22}$$

Thus by clause (21) of definition 5.8 we obtain

$$\mathfrak{A}[B_{\mathfrak{A}}, \alpha, \langle\widehat{\mathbb{R}}, f\rangle, a, c]$$

from (22). Then by theorem 5.12 we have

$$Ap_{\mathfrak{A}}[\langle\widehat{\mathbb{R}}, f\rangle, a, c]. \tag{23}$$

In view of lemma 5.15, the assertions (21) and (23) we get

$$\exists z [\![\mathbb{R}(f,a)]\!]_{\mathfrak{A}}(z) \quad \text{and} \tag{24}$$

$$\exists z([\![\mathbb{R}(f,a)]\!]_{\mathfrak{A}}(z) \wedge \forall y(y \in z \leftrightarrow (\exists x \in a)Ap_{\mathfrak{A}}[f,x,y])). \tag{25}$$

Thus $\mathsf{KP}\omega + (V = L)$ proves the implication from (18) to (24) and (25). So $\mathsf{KP}\omega + (V = L)$ proves the axiom about operational replacement, since its translation is equivalent to this implication. □

Lemma 5.21. *If A is the operational set-theoretic axiom about choice of* OST*, then we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

Proof. If we translate the premise $\exists x(fx = \top)$ of the axiom about operational choice, we get a formula equivalent to

$$\exists x Ap_{\mathfrak{A}}[f,x,\widehat{\top}]. \tag{26}$$

This satement and $\in$-induction implies that there is a least ordinal α such that

$$(\exists x \in L_\alpha)B_{\mathfrak{A}}^{<\alpha}[f,x,\widehat{\top}]. \tag{27}$$

Because $<_L$ well-orders the universe, statement (27) implies that we can pick the least set a with respect to $<_L$ satisfying

$$a \in L_\alpha \wedge B_{\mathfrak{A}}^{<\alpha}[f,a,\widehat{\top}]. \tag{28}$$

According to clause (22) of definition 5.8 we therefore have

$$\mathfrak{A}[B_{\mathfrak{A}},\alpha,\widehat{\mathbb{C}},f,a],$$

and by theorem 5.12

$$Ap_{\mathfrak{A}}[\widehat{\mathbb{C}},f,a]. \tag{29}$$

Of course statement (28) also implies

$$Ap_{\mathfrak{A}}[f,a,\widehat{\top}], \tag{30}$$

and we get by (29) and (30)

$$\exists x([\![\mathbb{C}f]\!]_{\mathfrak{A}}(x) \wedge Ap_{\mathfrak{A}}[f,x,\widehat{\top}]). \tag{31}$$

So (26) implies (31). This implication is equivalent to the translation of the axiom about operational choice, which is thus also proved in $\mathsf{KP}\omega + (V = L)$. □

Lemma 5.22. *For every axiom A of* OST *we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

PROOF. The basic set-theoretic axioms of OST are not affected by this translation and are availabele in $\mathsf{KP}\omega + (V = L)$ too. We have defined the formula $\mathfrak{A}[R, \alpha, a, b, c]$ such that this lemma goes through. This is more or less trivial for all applicative axioms and for the first four logical operations axioms. The lemmas 5.18-5.21 say that the assertion is also true, if A is one of the four remaining axioms of OST. $\square$

Lemma 5.23. *If A is the axiom $\mathbf{FO}1$ or $\mathbf{FO}2$, then we have*

$$\mathsf{KP}\omega + (V = L) \vdash A^*.$$

PROOF. Surely we can prove in $\mathsf{KP}\omega + (V = L)$ that the translation of the premise $Fun(f)$ is equivalent to $Fun(f)$ itself, since it is an $\mathcal{L}$ formula.
There is an ordinal α such that $f \in L_\alpha$. Of course also $(\forall x \in f)(x \in L_\alpha)$ and $c \in L_\alpha$ if $\langle b, c\rangle \in f$. Thus for arbitrary b, c with $(\langle b, c\rangle \in f)^*$ (which is equivalent to $\langle b, c\rangle \in f$) the translated premise implies $\mathfrak{A}[B_\mathfrak{A}, \alpha, f, b, c]$ and thus $Ap_\mathfrak{A}[f, b, c]$ which is equivalent in $\mathsf{KP}\omega + (V = L)$ to $(fb = c)^*$. Thus $\mathsf{KP}\omega + (V = L)$ proves $(\mathbf{FO}1)^*$.
On the other hand if we have $(fb = c)^*$ and so $Ap_\mathfrak{A}[f, b, c]$, i.e. there is an ordinal α such that $\mathfrak{A}[B_\mathfrak{A}, \alpha, f, b, c]$, then by lemma 5.9 the translated premise implies $\langle b, c\rangle \in f$ and thus $(\langle b, c\rangle \in f)^*$. Hence $\mathsf{KP}\omega + (V = L)$ also proves $(\mathbf{FO}2)^*$. $\square$

Since the theory $\mathsf{KP}\omega + (V = L)$ is closed under all rules of inference available in OST, the lemmas 5.22 and 5.23 directly implies the next lemma.

Lemma 5.24. *The theories $\mathsf{OST} + \mathbf{FO}1$ and $\mathsf{OST} + \mathbf{FO}2$ can be embedded into $\mathsf{KP}\omega + (V = L)$; i.e. for all $\mathcal{L}^\circ$ formulas A we have*

$$\mathsf{OST} + \mathbf{FO}1 \vdash A \implies \mathsf{KP}\omega + (V = L) \vdash A^* \text{ and}$$

$$\mathsf{OST} + \mathbf{FO}2 \vdash A \implies \mathsf{KP}\omega + (V = L) \vdash A^*.$$

Formulas, which are Δ over $\mathsf{KP}\omega$, are also called absolute formulas. The next theorem is well-known.

Theorem 5.25. *The theory $\mathsf{KP}\omega + (V = L)$ is a conservative extension of $\mathsf{KP}\omega$ for absolute formulas.*

Now it is easy to give an answer to the main question of this section.

Theorem 5.26. *Supposed that the theory $\mathsf{KP}\omega$ is consistent, the theories $\mathsf{OST} + \mathbf{FO}1$ and $\mathsf{OST} + \mathbf{FO}2$ are consistent.*

PROOF. The theorem is a direct consequence of lemma 5.24 and theorem 5.25. $\square$

6 Beeson's theory **ZFR**

Beeson's theory ZFR is presented in [3]. The letters stand for "Zermolo-Fraenkel set theory with rules". The theory ZFR guarantees not only the existence of sets, but also the existence of natural numbers as urelements, i.e. natural numbers which are not sets. Like the theory OST, ZFR is a set theory which allows applying objects to other ones, i.e. all objects are not only sets or numbers, but they are also operations. In [3], operations are called rules, but in this thesis we use the name operation also in respect of ZFR.

In section 4 we've seen that in a model of OST there is no set of all operations from a to b (if a is a set containing at least one element, b a set containing at least two elements). In the proofs of the theorems 4.4 and 4.5 we required amongst others the second logical operations axiom of OST about the operation **el** (the axiom is required for proving the lemmas 3.4 and 2.23). In Beeson's theory ZFR there is no such axiom about an operation like **el** (we will see later that we can define such an operation anyway). The main question of this section is: Can we prove although a theorem like theorem 4.5 within the theory ZFR? In [3] this question is declared as an open question. In the next subsection we introduce the theory ZFR.

6.1 The theory **ZFR**

As noted obove, in a model of the theory ZFR there are objects, which are not sets, but they are natural numbers. We will use the predicates $S(x)$ and $N(x)$ for expressing "x is a set" and "x is natural a number" respectively.

6.1.1 Language and logic

The language of ZFR, let's call it $\mathcal{L}^\circ_{\mathsf{ZFR}}$, is an extension of our basic language $\mathcal{L}$, but without the constant ω. $\mathcal{L}^\circ_{\mathsf{ZFR}}$ contains the unary relation symbols $\downarrow$ (like in $\mathcal{L}^\circ$), S for sets and N for numbers. The binary relation symbols of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ are those of $\mathcal{L}$. Like $\mathcal{L}^\circ$, $\mathcal{L}^\circ_{\mathsf{ZFR}}$ posses the binary function symbol $\circ$ for partial term application. The constants of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ are k, s, $\mathsf{s}_{\mathbb{N}}$, 0, $\emptyset$, $\mathbb{P}$, d, **uopa**, **un**, $\mathbb{N}$, **im** and countable many c_ϕ (one c_ϕ for every primitive formula, primitive formulas are defined later).
The terms and formulas of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ are inductively defined, the same way as the terms and formulas of $\mathcal{L}^\circ$, but with the additional atomic formulas $S(t)$ and $N(t)$ for $\mathcal{L}^\circ_{\mathsf{ZFR}}$ terms t.
We will use the same abbreviations for $\mathcal{L}^\circ_{\mathsf{ZFR}}$ terms and formulas as we have defined for $\mathcal{L}^\circ$ terms and formulas.

Definition 6.1 (Primitive formulas)**.** A formula of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ is called primitive if it does not contain the function symbol $\circ$ or any constant c_ϕ.

ZFR has the same logic as OST, the classical *logic of partial terms.*

6.1.2 Non-logical axioms

The first axioms of ZFR are the applicative axioms like in OST (see section 2.3.1). Thus λ-abstraction and the recursion theorem (lemmas 2.10 and 2.13) are available in ZFR.

Convention 6.2. To abbreviate some formulas we will use in this section a, b, c, and u as (meta)variables for sets and n and m for natural numbers (all possibly with subscripts). That is to say $\exists a\phi(a)$ means $\exists a(S(a) \wedge \phi(a))$ and so on. The variables f, g, v, w, x, y and z (of course possibly with subscripts) will be used for arbitrary objects. So $\exists x\phi(x)$ means „there is an object x which is a set or a number such that $\phi(x)$" and so on.

The remaining axioms are:

(A1)	*Extensionality:*	$\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b.$
(A2)	*Pairing:*	$S(\mathbf{uopa}yz) \wedge \forall x(x \in \mathbf{uopa}yz \leftrightarrow x = y \vee x = z).$
(A3)	*Union:*	$S(\mathbf{un}a) \wedge \forall x(x \in \mathbf{un}a \leftrightarrow \exists u(u \in a \wedge x \in u)).$
(A4)	*Empty set:*	$S(\emptyset) \wedge \forall x(x \notin \emptyset).$
(A5)	*Infinity:*	$S(\mathbb{N}) \wedge \forall x(x \in \mathbb{N} \leftrightarrow N(x)).$
(A6)	*Separation:*	For every primitive formula ϕ:
		$S(\mathbf{c}_\phi(a, y_1, ..., y_n))$
		$\wedge \forall x(x \in \mathbf{c}_\phi(a, y_1, ..., y_n) \leftrightarrow (x \in a \wedge \phi(x, y_1, ..., y_n))).$
(A7)	*Images:*	$(\forall x \in a)(fx\downarrow) \rightarrow S(\mathbf{im}(a, f))$
		$\wedge \forall z(z \in \mathbf{im}(a, f) \leftrightarrow (\exists x \in a)(fx = z)).$
(A8)	*Powerset:*	$S(\mathbb{P}a) \wedge \forall x(x \in \mathbb{P}a \leftrightarrow (S(x) \wedge (\forall z \in x)(z \in a))).$
(A9)	$\in$*-induction:*	For every formula ϕ:
		$\forall u((\forall x \in u)(\phi(x)) \rightarrow \phi(u)) \rightarrow \forall u(\phi(u)).$
(B1)		$y \in x \rightarrow S(x).$
(B2)	*Cases:*	$\mathbf{d}nnxy = x \wedge (n \neq m \rightarrow \mathbf{d}nmxy = y).$
(B3)	*Successor:*	$N(0) \wedge N(\mathbf{s}_\mathbb{N}n) \wedge (\mathbf{s}_\mathbb{N}n = \mathbf{s}_\mathbb{N}m \rightarrow n = m) \wedge \mathbf{s}_\mathbb{N}n \neq 0.$
(B4)	*Induction:*	For every formula ϕ:
		$(\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(\mathbf{s}_\mathbb{N}n))) \rightarrow \forall n\phi(n).$

We will need 1 as an abbreviation for $\mathbf{s}_\mathbb{N}0$.

6.2 Sets of operations

For proving a similar theorem like theorem 4.5, we need a lemma like lemma 3.4 and a theorem like theorem 3.2 which are given in the following.

Lemma 6.3. *Let $\vec{y}$ be a sequence of variables $y_1, ..., y_n$, $A[\vec{y}]$ a primitive formula of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ with at most the variables $\vec{y}$ free. Further let t and s be two terms of $\mathcal{L}^\circ_{\mathsf{ZFR}}$. Then there exists a term $\mathrm{ite}^{t,s}_A$ of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ such that ZFR proves*

$$(t{\downarrow} \wedge s{\downarrow}) \;\rightarrow\; \mathrm{ite}^{t,s}_A{\downarrow} \;\wedge\; (\mathrm{ite}^{t,s}_A : \mathbb{V}^n \rightarrow \{\{t\},\{s\}\}) \;\; \textit{and}$$

$$(t{\downarrow} \wedge s{\downarrow}) \;\rightarrow\; ((A[\vec{y}] \rightarrow \mathrm{ite}^{t,s}_A(\vec{y}) = \{t\}) \;\wedge\; (\neg A[\vec{y}] \rightarrow \mathrm{ite}^{t,s}_A(\vec{y}) = \{s\})).$$

Further there exists a term $\mathrm{mod}^{t,s}_A$ of $\mathcal{L}^\circ_{\mathsf{ZFR}}$ such that ZFR proves

$$(S(t) \wedge S(s)) \;\rightarrow\; \mathrm{mod}^{t,s}_A{\downarrow} \;\wedge\; (\mathrm{mod}^{t,s}_A : \mathbb{V}^n \rightarrow \{t,s\}) \;\; \textit{and}$$

$$(S(t) \wedge S(s)) \;\rightarrow\; ((A[\vec{y}] \rightarrow \mathrm{mod}^{t,s}_A(\vec{y}) = t) \;\wedge\; (\neg A[\vec{y}] \rightarrow \mathrm{mod}^{t,s}_A(\vec{y}) = s)).$$

If t and s are closed terms, then so are $\mathrm{ite}^{t,s}_A$ and $\mathrm{mod}^{t,s}_A$.

Proof. Assume that $t{\downarrow}$ and $s{\downarrow}$ and let a be the set $\{t, s\}$ and $\phi(x, y_1, ..., y_n)$ be the formula

$$(x = t \wedge A[\vec{y}]) \vee (x = s \wedge \neg A[\vec{y}]).$$

We have

$$x \in \mathbf{c}_\phi(a, \vec{y}) \leftrightarrow x \in a \wedge \phi(x, \vec{y}).$$

We set $\mathrm{ite}^{t,s}_A := \lambda y_1 ... y_n.\mathbf{c}_\phi(a, y_1, ..., y_n)$ and we have

$$\mathrm{ite}^{t,s}_A(\vec{y}) = \begin{cases} \{t\} & \text{if } A[\vec{y}], \\ \{s\} & \text{if } \neg A[\vec{y}]. \end{cases}$$

Further we set $\mathrm{mod}^{t,s}_A := \lambda y_1 ... y_n.\mathbf{un}(\mathrm{ite}^{t,s}_A(\vec{y}))$. If t and s are sets, then of course $\mathbf{un}(\{t\}) = t$ and $\mathbf{un}(\{s\}) = s$ and so

$$\mathrm{mod}^{t,s}_A(\vec{y}) = \begin{cases} t & \text{if } A[\vec{y}], \\ s & \text{if } \neg A[\vec{y}]. \end{cases}$$

□

Remark 5. It would be easy to prove the assertions about the term $\mathrm{mod}^{t,s}_A$ of the previous lemma also for the case that t and s are not sets, if there were an operation f such that $f(\{x\}) = x$ for all singletons $\{x\}$. But for the following reason ZFR doesn't prove the existence of such an operation: Beeson presents in [3] a model M of ZFR, in which all operations from sets to numbers are constant operations. I.e. if a and b are two sets in M and g is an operation in M with $g(a) \in \mathbb{N}$ and $g(b){\downarrow}$, then M satisfies $g(a) = g(b)$. Thus if we have in this model $f(\{n\}) = n$ for a number n, then $f(\{m\}) = n$ for all numbers m with $f(\{m\}){\downarrow}$.

Theorem 6.4. *There is an operation which is nowhere defined. More precisely, there exists a closed $\mathcal{L}^{\circ}_{\mathsf{ZFR}}$ term t such that* ZFR *proves $t\downarrow$ and $\forall x(tx\uparrow)$.*

PROOF. The proof is like the proof of theorem 3.2. □

Now we are able to formulate and prove the desired theorem.

Theorem 6.5. *Given a set a containing at least one element and a set b containing at least two sets. Then* ZFR *refutes that the class*

$$\{f : (f : a \to b) \wedge (\forall x)(x \notin a \to fx\uparrow)\}$$

is a set.

PROOF. Let v be an element of a and w_1 and w_2 be two different elements of b wich are sets. The rest of the proof is like the proof of theorem 4.5. □

Remark 6. Note that the set b in the previous theorem must contain at least two sets. The proof doesn't work if b contains two or more numbers but less than two sets.

6.3 Logical operations

As noted above, ZFR does not contain axioms like the logical operations axioms of OST. Although we can define closed terms of $\mathcal{L}^{\circ}_{\mathsf{ZFR}}$ which act as logigal operatons.

Theorem 6.6. *There are closed terms $\widetilde{\top}$, $\widetilde{\bot}$, $\widetilde{\mathbf{el}}$, $\widetilde{\mathbf{non}}$, $\widetilde{\mathbf{dis}}$ and $\widetilde{\mathbf{e}}$ of $\mathcal{L}^{\circ}_{\mathsf{ZFR}}$ such that* ZFR *proves for $\widetilde{\mathbb{B}} := \{\widetilde{\top}, \widetilde{\bot}\}$*

(i) $\widetilde{\top} \neq \widetilde{\bot}$,

(ii) $(\widetilde{\mathbf{el}} : \mathbb{V}^2 \to \widetilde{\mathbb{B}}) \wedge \forall x \forall y(\widetilde{\mathbf{el}}(x, y) = \widetilde{\top} \leftrightarrow x \in y)$,

(iii) $(\widetilde{\mathbf{non}} : \widetilde{\mathbb{B}} \to \widetilde{\mathbb{B}}) \wedge (\forall x \in \widetilde{\mathbb{B}})(\widetilde{\mathbf{non}}(x) = \widetilde{\top} \leftrightarrow x = \widetilde{\bot})$,

(iv) $(\widetilde{\mathbf{dis}} : \widetilde{\mathbb{B}}^2 \to \widetilde{\mathbb{B}}) \wedge (\forall x, y \in \widetilde{\mathbb{B}})(\widetilde{\mathbf{dis}}(x, y) = \widetilde{\top} \leftrightarrow (x = \widetilde{\top} \vee y = \widetilde{\top}))$,

(v) $(f : a \to \widetilde{\mathbb{B}}) \to (\widetilde{\mathbf{e}}(f, a) \in \widetilde{\mathbb{B}} \wedge (\widetilde{\mathbf{e}}(f, a) = \widetilde{\top} \leftrightarrow (\exists x \in a)(fx = \widetilde{\top})))$.

PROOF. For the first assertion we set for example $\widetilde{\top} := \{1\}$ and $\widetilde{\bot} := \{0\}$. For later use we set b as the set $\{0, 1\}$.

For the assertions two to four we define in each case the requested terms as $\lambda y_1 ... y_n . \mathbf{c}_{\phi_j}(b, y_1, ..., y_n)$ where n is 1 or 2, j is the number of the assertion and ϕ_j is one of the formulas given below.

$$\phi_2(x, y_1, y_2) \text{ is } (x = 1 \wedge y_1 \in y_2) \vee (x = 0 \wedge y_1 \notin y_2).$$

$$\phi_3(x,y) \text{ is } (x = 1 \wedge y = \widetilde{\bot}) \vee (x = 0 \wedge y = \widetilde{\top}).$$

$$\phi_4(x,y_1,y_2) \text{ is } (x = 1 \wedge (y_1 = \widetilde{\top} \vee y_2 = \widetilde{\top})) \vee (x = 0 \wedge \neg(y_1 = \widetilde{\top} \vee y_2 = \widetilde{\top})).$$

For the fifth assertion let $\phi_5(x,y)$ be the formula

$$(x = 1 \wedge (\exists z \in y)(z = \widetilde{\top})) \vee (x = 0 \wedge \neg(\exists z \in y)(z = \widetilde{\top})).$$

The term $\widetilde{\mathsf{e}}$ is given by $\lambda f a.c_{\phi_5}(b, \mathsf{im}(a,f))$. □

Remark 7. In a model M which is presented in [3] (the same model which is also mentioned in remark 5), all operations from $\mathbb{N}$ to $\mathbb{N}$ are recursive. This may be confusing, because equipped with the operation $\widetilde{\mathsf{el}}$ we can construct operations which are not computable. For example we can construct a characteristic operation f of an arbitrary set a, i.e. an operation f such that $f(x) = \widetilde{\top}$ if $x \in a$ and $f(x) = \widetilde{\bot}$ if $x \notin a$ for all x (take $f := \lambda x.\widetilde{\mathsf{el}}(x,a)$). We would say that f is non-recursive if a is non-recursive. Although this is not contradictory to the assertion that every operation from $\mathbb{N}$ to $\mathbb{N}$ is recursive, since f is not from $\mathbb{N}$ to $\mathbb{N}$.

References

[1] Jon Barwise. *Admissible sets and structures.* Springer-Verlag, Berlin, 1975. An approach to definability theory, Perspectives in Mathematical Logic.

[2] Michael J. Beeson. *Foundations of constructive mathematics*, volume 6 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1985. Metamathematical studies.

[3] Michael J. Beeson. Towards a computation system based on set theory. *Theoret. Comput. Sci.*, 60(3):297–340, 1988.

[4] Andrea Cantini and Laura Crosilla. Elementary constructive operational set theory. To appear in: Festschrift for Wolfram Pohlers, Ontos Verlag.

[5] Andrea Cantini and Laura Crosilla. Constructive set theory with operations. In *Logic Colloquium 2004*, volume 29 of *Lect. Notes Log.*, pages 47–83. Assoc. Symbol. Logic, Chicago, IL, 2008.

[6] Solomon Feferman. Notes on operational set theory, i. generalization of "small" large cardinals in classical and admissible set theory. Draft, http://math.stanford.edu/ feferman/papers/OperationalST-I.pdf, 2001.

[7] Solomon Feferman. Operational set theory and small large cardinals. Draft, http://math.stanford.edu/ feferman/papers/ostcards.pdf, December 2006.

[8] Gerhard Jäger. On Feferman's operational set theory OST. *Ann. Pure Appl. Logic*, 150(1-3):19–39, 2007.

[9] Gerhard Jäger. Full operational set theory with unbounded existential quantification and power set. To appear in: Annals of Pure and Applied Logic, 2009.

[10] Gerhard Jäger and Marc Wirz. Applikative Theorien und explizite Mathematik. Skript zur Vorlesung im Sommersemester 1996 an der Universität Bern.

[11] Thomas Jech. *Set theory.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. Pure and Applied Mathematics.

[12] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics.* North-Holland Publishing Co., Amsterdam, 1980. An introduction to independence proofs.