Master Thesis

Proof-theoretic aspects of weak König's lemma

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1 Introduction

Second order arithmetic Z_2 is a theory formulated in a two sorted language which consists of some basic axioms, full induction scheme and full comprehension scheme. It is a simple approach to formalize natural numbers and their subsets. Although this construction seems to be very primitive and limited, it is indeed powerful enough to formalize a huge branch of mathematics. For example it is possible to define real numbers in Z_2. Even though there is no way in defining the set of real numbers, since we can deal only with countable sets, single real numbers can be coded as some distinguished Cauchy sequences. In addition within second order arithmetic, common concepts from analysis such as continuous functions or Riemann integrability can be formulated. Further, one can also code concepts from algebra, linear algebra or even mathematical logic. Because such codings do not need the full induction scheme nor the full comprehension scheme, it makes sense to consider several subsystems of Z_2. The basic subtheory of second order arithmetic is RCA_0. The acronym RCA stands for "recursive comprehension axiom" and the subscript 0 indicates restricted induction. It may be interpreted as a constructive respectively recursive theory which formalizes the naturals.

Weak König's lemma is a non-constructive axiom, ensuring the existence of infinite paths in binary trees. One can prove that weak König's lemma is not a valid theorem of RCA_0, hence the theory WKL_0, which consists of RCA_0 and weak König's lemma, is a proper supertheory of RCA_0. Surprisingly, WKL_0 is equivalent over RCA_0 to several existential theorems from ordinary mathematics such as the Heine-Borel covering lemma, the extreme value theorem, Brower's fixed point theorem, the separable Hahn-Banach theorem or Gödel's completeness theorem (see [6] for more details). In this thesis we will study conservation results, especially Harrington's conservation theorem. This theorem says that formulas which have no set quantifiers are provable in WKL_0 if and only if they are already provable in RCA_0. This means that there is a huge set of theorems of WKL_0 which do not depend on the non-constructive weak König's lemma. Harrington's proof is model theoretic, making use of a forcing argument. In [4], a pure proof-theoretic proof of Harrington's theorem is presented, using a cut-elimination argument. The aim of this thesis is to elaborate this proof. We start with a preliminary section, where generally speaking we introduce a sequent calculus of WKL_0 and present the usual background. In the second section we prove Harrington's theorem. This is the mainpart of this thesis. In the last section we combine Harrington's theorem with a further conservation result, namely Parsons' theorem. This will show that WKL_0 is Π^0_2-conservative over primitive recursive arithmetic. The appendix contains proofs of partial cut elimination and of Parsons' theorem.

To understand this thesis, basic knowledge of mathematical logic, recursion theory and proof theory is required.

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2 Preliminaries

2.1 The language of second order arithmetic

Definition 2.1. *Let* L_2 *denote the language of second order arithmetic which contains the following symbols:*

(i) countably many free number variables $a_0, a_1, \ldots,$

(ii) countably many bound number variables $x_0, x_1, \ldots,$

(iii) countably many free set variables $A_0, A_1, \ldots,$

(iv) countably many bound set variables $X_0, X_1, \ldots,$

(v) a constant symbols 0,

(vi) a unary function symbol S *and two binary function symbols* $+$ *and* $\cdot$*,*

(vii) three binary relation symbols $=$*,* $<$ *and* $\in$*,*

(viii) the logical symbols $\forall$*,* $\exists$*,* $\wedge$*,* $\vee$ *and* $\neg$*.*

As usual for the notion of terms and formulas we add auxiliary symbols and use infix notation to improve readability. *Numerical terms* are the constant 0, the free number variables and $\mathrm{S}(t_1)$ and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms. *Atomic formulas* are $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in A$ where t_1, t_2 are numerical terms and A is a free set variable. In the following we will simply say *term* instead of numerical term.

Definition 2.2. *The set of* L_2*-formulas is defined inductively by:*

(i) every atomic formula is a formula,

(ii) if F*,* G *are formulas, so are* $F \wedge G$*,* $F \vee G$ *and* $\neg F$*,*

(iii) if F *is a formula,* a *is a free number variable and* x *is a bound number variable not included in* F *and* t *is a term, then* $\forall x\, F[x/a]$*,* $\exists x\, F[x/a]$*,* $\forall x \leq t\, F[x/a]$ *and* $\exists x \leq t\, F[x/a]$ *are formulas,*

(iv) if F *is a formula,* A *is a free set variable and* X *is a bound set variable not included in* F*, then* $\forall X\, F[X/A]$ *and* $\exists X\, F[X/A]$ *are formulas.*

Here $F[x/a]$ *is the result of substituting every occurrence of* a *in* F *by* x*, similarly for* $F[X/A]$*.*

A formula without free number or set variables is called *sentence*. If F, G are formulas and t_1, t_2 are terms, then we can define $F \rightarrow G$, $F \leftrightarrow G$, $t_1 \neq t_2$, $t_1 \leq t_2$ etc. in the obvious way. Substituting terms for free number variables is defined as usual; if G is the result of substituting t for a in F we will simply write $F(a)$ instead of F and $F(t)$ instead of G to denote this circumstance. Furthermore the quantifiers $\forall x \leq t$ and $\exists x \leq t$ are denoted as *bounded number quantifiers* or simply *bounded quantifiers*.

2.2 Models and theories of L_2

The semantics of the language L_2 are given by the following definition.

Definition 2.3. *A model of* L_2*, also called a structure for* L_2 *or a* L_2*-structure, is an ordered 7-tuple*

$$M = \{|M|, \mathscr{S}_M, \mathrm{S}_M, +_M, \cdot_M, 0_M, <_M\}$$

where $|M|$ *is a set which serves as the range of the number variables,* $\mathscr{S}_M$ *is a non-empty set of subsets of* $|M|$ *serving as the range of the set variables,* S *is a unary and* $+_M$ *and* $\cdot_M$ *are binary operations on* M*,* 0_M *is a distinguished element of* $|M|$*, and* $<_M$ *is a binary relation on* $|M|$*. We always assume that the sets* $|M|$ *and* $\mathscr{S}_M$ *are disjoint.*

If M a L_2-structure then a M*-assignment* is a function α assigning an element of $|M|$ to every free number variable and an element of $\mathscr{S}_M$ to every free set variable. We say that a L_2-formula F is *true* in M with respect to α, written $M \vDash F[\alpha]$, if F is true in M where the free variables of F are interpreted according to α and where the other symbols of F are interpreted in the obvious way. We say that M *is a model of* F,written $M \vDash F$, if F is true in M with respect to any M-assignment. A set T of formulas is *true in* M respectively M *is a model of* T, written $M \vDash T$, if every formula contained in T is true in M. Further we set $T \vDash F$ if every model of T is a model of F.
Now we discuss the syntactic counterpart. As in the first order case one can define a Hilbert calculus on L_2 in the following way: our axioms are *propositional tautologies*, the *equality axioms*

$$\begin{gathered}
a = a\,,\\
a = b \to \mathrm{S}(a) = \mathrm{S}(b)\,,\\
a_0 = b_0 \wedge a_1 = b_1 \to a_0 + a_1 = b_0 + b_1\,,\\
a_0 = b_0 \wedge a_1 = b_1 \to a_0 \cdot a_1 = b_0 \cdot b_1\,,\\
a_0 = b_0 \wedge a_1 = b_1 \wedge a_0 < a_1 \to b_0 < b_1\,,\\
a = b \wedge a \in X \to b \in X\,,
\end{gathered}$$

and all formulas of the form

$$\begin{gathered}
\forall x\, F(x) \to F(t) \quad , \quad F(t) \to \exists x\, F(x) \quad , \quad \forall x \leq t\, F(x) \wedge (s \leq t) \to F(s)\\
F(s) \wedge (s \leq t) \to \exists x \leq t\, F(x) \quad , \quad F(A) \to \exists X\, F(X) \quad \text{and} \quad \forall X\, F(X) \to F(A),
\end{gathered}$$

where s,t are arbitrary terms. The rules of the Hilbert systems are *modus ponens* and the six inference rules

$$\begin{array}{cc}
\dfrac{F \to G(a)}{F \to \forall x\, G(x)} & \dfrac{F(a) \to G}{\exists x\, F(x) \to G}\\
\dfrac{F \wedge (a \leq t) \to G(a)}{F \to \forall x \leq t\, G(x)} & \dfrac{F(a) \wedge (a \leq t) \to G}{\exists x \leq t\, F(x) \to G}\\
\dfrac{F \to G(A)}{F \to \forall X\, G(X)} & \dfrac{F(A) \to G}{\exists X\, F(X) \to G},
\end{array}$$

where in the first four rules resp. in the last two rules, a resp. A does not appear in the lower sequent, and where t is a term. If F can be derived in this Hilbert calculus with the formulas of T as further axioms, we simply write $T \vdash F$. As in first order logic we have Gödel's completeness theorem, which says that $\vdash$ and $\models$ are equivalent.

Theorem 2.4. *For any set of* L_2*-formulas* T *and any* L_2*-formula* F *we have*

$$T \models F \quad \text{iff} \quad T \vdash F\,.$$

Later we will formulate this calculus as a sequent calculus. The basic axioms we will always assume to be true in this thesis are the axioms of Robinson arithmetic. They regulate the function symbols and the relation $<$.

Definition 2.5. *The Robinson arithmetic* Q *is given by the following eight* L_2*-formulas:*

(i) $\mathrm{S}(a) \neq 0$

(ii) $\mathrm{S}(a) = \mathrm{S}(b) \rightarrow a = b$

(iii) $a + 0 = a$

(iv) $a + \mathrm{S}(b) = \mathrm{S}(a+b)$

(v) $a \cdot 0 = 0$

(vi) $a \cdot \mathrm{S}(b) = (a \cdot b) + a$

(vii) $\neg a < 0$

(viii) $a < \mathrm{S}(b) \leftrightarrow (a < b \vee a = b)\,.$

Although these axioms imply several number theoretic facts, they don't suffice to prove elementary theorems such as commutativity of addition. The axioms of Q together with the induction axiom

$$\forall X\ (0 \in X \wedge \forall x\ (x \in X \rightarrow \mathrm{S}(x) \in X) \rightarrow \forall x\ (x \in X))$$

and the comprehension scheme

$$\exists X\ \forall x\ (x \in X \leftrightarrow F(x))\,,$$

where $F(a)$ is any formula of L_2, build the axioms of second order arithmetic Z_2. In the following we will discuss special subsystems of Z_2 with restricted induction and restricted comprehension. Our basic system is the theory RCA_0. First an auxiliary definition.

Definition 2.6. *A formula* F *is called bounded if all quantifiers that occur in* F *are bounded number quantifier. The set of all bounded* L_2*-formulas is denoted as* Δ^0_0*. A formula* F *is called a* Σ^0_n*-formula (resp. a* Π^0_n*-formula) if there exists a bounded formula* G *such that* F *is* $\exists x_0 \forall x_1 \ldots x_{n-1} G$ *(resp.* $\forall x_0 \exists x_1 \ldots x_{n-1} G$*). Note that* $\Delta^0_0 = \Sigma^0_0 = \Pi^0_0$*.*

Definition 2.7. *The Σ^0_1-induction scheme, Σ^0_1-IND, consists of all formulas of the form*

$$F(0) \wedge \forall x\,(F(x) \rightarrow F(\mathrm{S}(x))) \rightarrow \forall x\, F(x)\,,$$

where $F(a)$ is a Σ^0_1-formula.
The Δ^0_1-comprehension scheme consists of all formulas of the form

$$\forall x\,(F(x) \leftrightarrow G(x)) \rightarrow \exists X\, \forall x\,(x \in X \leftrightarrow F(x))\,,$$

where $F(a)$ is any Σ^0_1-formula and $G(a)$ is any Π^0_1-formula.
The theory RCA_0 consists of the axioms of Q, the Σ^0_1-induction scheme and the Δ^0_1-comprehension scheme. If we replace the Δ^0_1-comprehension scheme by the bounded comprehension scheme, i.e. every formula $\exists X\, \forall x\,(x \in X \leftrightarrow F(x))$, where $F(a)$ is a bounded formula, we get the theory RCA_0^-.

2.3 Mathematics within RCA_0^-

Within RCA_0^-, by bounded comprehension, there exists a set $\dot{\omega}$ such that $\forall x\, x \in \dot{\omega}$. The set $\dot{\omega}$ is unique, meaning that if X satisfies $\forall x\, x \in X$ then $X = \dot{\omega}$, where $X = Y :\equiv \forall x\,(x \in X \leftrightarrow x \in Y)$. The axioms of RCA_0^- suffice to prove elementary properties of the natural numbers. Let's write $\dot{0} := 0$, $\dot{1} := \mathrm{S}(\dot{0})$, etc.

Proposition 2.8. RCA_0^- *proves that $\dot{\omega}, +, \cdot, 0, \dot{1}, <$ is a commutative, ordered semiring with cancellation.*

Proof. See [6] Lemma II.2.1, p. 65. □

As a standard result we can define in RCA_0^- primitive recursive functions. As we will use them over and over again, it is easier to interpret them syntactically. This can be done with the following definition:

Definition 2.9. *Let $\mathrm{L}_2[PR]$ denote the language obtained from L_2 by replacing in 2.1 the definition of the function symbols with the following inductive definition:*

(i) S *is a unary function symbol,*

(ii) for all natural numbers n,m and k, where $0 \leq k \leq n$, Cs^n_m and Pr^n_k are n-ary function symbols,

(iii) if f is an m-ary function symbol and $g_1, \ldots, g_m$ are n-ary function symbols, then the composition $\mathrm{Comp}^n(f, g_1, \ldots, g_m)$ is an n-ary function symbol,

(iv) if f is an n-ary function symbol and g an $(n+2)$-ary function symbol, then $\mathrm{Rec}^{n+1}(f,g)$ is an $(n+1)$-ary function symbol.

The notions of terms, formulas, semantics, Q, Σ^0_1-formulas, Σ^0_1-IND etc. can easily be transformed from L_2 to $\mathrm{L}_2[PR]$. The next definition gives the usual meaning to the function symbols of $\mathrm{L}_2[PR]$. Let's denote P the L_2-theory Q enriched with the Σ^0_1-IND scheme.

Definition 2.10. *The $\mathrm{L}_2[PR]$-theory $\mathsf{P}[PR]$ consists of the axioms of Q, the Σ^0_1-IND scheme (in $\mathrm{L}_2[PR]$) plus* *(i)* $\mathrm{Cs}^n_m(\vec{a}) = \dot{m}$,

(ii) $\mathrm{Pr}^n_k(\vec{a}) = a_k$,

(iii) $\mathrm{Comp}^n(f, g_1, \ldots, g_m)(\vec{a}) = f(g_1(\vec{a}), \ldots, g_m(\vec{a}))$,

(iv) $\mathrm{Rec}^{n+1}(f,g)(\vec{a}, 0) = f(\vec{a})$,

(v) $\mathrm{Rec}^{n+1}(f,g)(\vec{a}, \mathrm{S}(b)) = g(\vec{a}, b, \mathrm{Rec}^{n+1}(f,g)(\vec{a}, b))$.

It is a well known fact, that for any $\mathrm{L}_2[PR]$ formula F there exists a translation F^- in the language L_2 such that F is provable in $\mathsf{P}[PR]$ iff F^- is provable in P. Furthermore the translation of a Σ^0_1-formula is a Σ^0_1-formula (up to logical equivalence). All our L_2-theories T we will work with contain at least P. Instead of working with T in L_2 we will consider from from now on it's version $\mathsf{T}[PR]$ in $\mathrm{L}_2[PR]$. Note that $\mathsf{T}[PR]$ may be stronger than T. For example $\mathsf{RCA}_0^-[PR]$ has a stronger comprehension scheme than RCA_0^-. On the other side $\mathsf{RCA}_0[PR]$ is equivalent to RCA_0 and thus we can identify these two theories.

Convention 2.11. *Rather than writing* $\mathsf{T}[PR]$ *and* $\mathrm{L}_2[PR]$ *we again write* T *and* L_2.

With this convention we can define a set Seq, sometimes denoted as $\dot{\omega}^{<\dot{\omega}}$, in RCA_0^- which encodes the set of *finite sequences*. For example this can be done in the following way: we let lg be the function symbol associated to the primitive function $\mathrm{lg} : \omega \to \omega$ that maps a code for a sequent to it's length (we use the same symbol for a function symbol and the represented primitive function). Then we set $\mathrm{Seq} := \{x \,:\, x = f(x)\}$ where f is the function symbol associated to the primitive recursive function $n \mapsto \langle (n)_0, \ldots, (n)_{\mathrm{lg}(n) \dot{-} 1} \rangle$. This is a set in the theory RCA_0^- (more exactly in $\mathsf{RCA}_0^-[PR]$) by bounded comprehension. We will frequently use the notation

$$s = \left\langle s_0, \ldots, s_{\mathrm{lg}(s) \dot{-} 1} \right\rangle \quad \text{or} \quad s = \langle s_a \,:\, a < \mathrm{lg}(s) \rangle \,,$$

where s_a is a shortcut for $(s)_a$ i.e. denotes the a-th entry of s whenever $a < \mathrm{lg}(s)$. We write $s \subseteq t$ to mean that s is an initial segment of t, i.e. $\mathrm{lg}(s) \leq \mathrm{lg}(t) \wedge \forall x < \mathrm{lg}(s)\; s_x = t_x$. It is important to note that the expressions $\mathrm{lg}(s) = a$, $s_a = b$, $s \subseteq t$, $s \in \mathrm{Seq}$, etc. are bounded $\mathrm{L}_2[PR]$-formulas with free number variables a, b, s, t. By bounded comprehension, RCA_0^- proves the existence of the set of binary finite sequences $\dot{2}^{<\dot{\omega}} := \{s \,:\, s \in \mathrm{Seq} \wedge \forall x < \mathrm{lg}(s)\; (s_x < \dot{2})\}$ (also denoted $\{0, \dot{1}\}^{<\dot{\omega}}$).From now on we write $X \subseteq Y :\equiv \forall x\, (x \in X \to x \in Y)$ and $a \in \{x \,:\, F(x)\} :\equiv F(a)$.

Definition 2.12. *A set of sequences T is a tree if it is closed under subsequences. If T consists of finite sequences in $\{0, \dot{1}\}$ we call T a binary tree, formally*

$$\mathrm{Tree}(T) :\equiv T \subseteq \dot{2}^{<\dot{\omega}} \wedge \forall x, y\, (y \in T \wedge x \subseteq y \to x \in T)\,.$$

Further we say that a binary tree T is infinite if it is not finite. Formally

$$\mathrm{Tree}_\infty(T) :\equiv \mathrm{Tree}(T) \wedge \neg(\exists x\, \forall y\, (y \in T \to y < x))\,.$$

The next step is to encode functions within RCA_0^-. For this we abbreviate $(s,t) := (s+t)^2 + s$ for arbitrary terms s and t. Then $X \times Y$ is the set of all (x,y) where $x \in X$ and $y \in Y$. This set exists by

Δ^0_0-comprehension. Then we write $f : X \to Y$ for any second order variable f, when f encodes a function from X to Y, i.e.

$$f \subseteq X \times Y \wedge \forall x,y,z\, \big(((x,y) \in f \wedge (x,z) \in f \to y = z) \wedge \forall x\, \exists y\, ((x,y) \in f)\big)\,.$$

Note that $f : X \to \{0, \dot{1}\}$ (also denoted as $f \in \dot{2}^X$) is a formula with free second order variables f and X, equivalent to a Π^0_1-formula with the same free variables, since the function value of f is bounded.

Definition 2.13. *A function $f \in \dot{2}^{\dot{\omega}}$ is an infinite path through a binary tree T if*

$$\mathrm{Path}_\infty(f,T) :\equiv \forall x\, \langle f(0),\ldots,f(x)\rangle \in T\,,$$

where $f(i)$ denotes the value of f at i, i.e. $f(i) = 0$ if $(i,0) \in f$ and $f(i) = \dot{1}$ otherwise. More precisely $\mathrm{Path}_\infty(f,T)$ *is the formula*

$$\forall x\, \forall s\, \big(s \in \mathrm{Seq} \wedge \mathrm{lg}(s) = x+1 \wedge \forall i < \mathrm{lg}(s)\, ((s_i = 0 \leftrightarrow (i,0) \in f) \wedge (s_i = \dot{1} \leftrightarrow (i,0) \notin f)) \to s \in T\big).$$

It can be seen, that $\mathrm{Path}_\infty(f,T)$ *is equivalent to a Π^0_1-formula with free second order variables f and T.*

Now we are ready to write down the definition of weak König's lemma.

Definition 2.14. *Weak König's lemma states that for every infinite binary tree there exists an infinite path through it. Formally*

$$\forall T\ (\mathrm{Tree}_\infty(T) \to \exists f \in \dot{2}^{\dot{\omega}}\ \mathrm{Path}_\infty(f,T))\,.$$

The theory WKL_0 *is the theory* RCA_0 *plus weak König's lemma.*

An important equivalent reformulation of weak König's lemma is the FAN_0 rule

$$\forall X\ \exists x\, F(X,x) \to \exists w\, \forall X\ \exists x \leq w\, F(X,x)$$

where F is a bounded formula. The FAN_0 rule is sometimes denoted as strict Π^1_1-*reflection*. To see the equivalence we need first a lemma.

Lemma 2.15. *Let $F(A,\vec{b})$ be a bounded formula with a distinguished second-order parameter A and with the first-order parameters $\vec{b}$ as shown. One can effectively associate a term $t_F(\vec{b})$, with its free variables as shown, such that the theory* RCA_0^- *proves:*

$$\forall s \in \dot{2}^{t_F(\vec{b})} \big(\forall x < t_F(\vec{b})(x \in A \leftrightarrow s_x = 0) \to (F(A,\vec{b}) \leftrightarrow F^*(s,\vec{b}))\big)\,,$$

where F^* is obtained from F by replacing its atomic subformulas of the form $q \in A$ by the expression $s_q = 0$, and where $\dot{2}^{t_F(\vec{b})}$ is the set of binary sequences of length $t_F(\vec{b})$.

Proof. We proceed by induction on the length of F and argue in RCA_0^-. If $F(A,\vec{b})$ is of the form $q(\vec{b}) = r(\vec{b})$, $q(\vec{b}) \leq r(\vec{b})$ or $q(\vec{b}) \in B$, where $B \neq A$, then we set $t_F(\vec{b}) := 0$. Otherwise, if $F(A,\vec{b})$ is $q(\vec{b}) \in A$, we let $t_F(\vec{b}) := q(\vec{b}) + 1$. Now assume that we have already constructed $t_G(\vec{b})$ and $t_H(\vec{c})$ for some bounded formulas $G(A,\vec{b})$ and $H(A,\vec{c})$. Then easily the claim holds for $t_{\neg G}(\vec{b}) := t_G(\vec{b})$ and $t_{G\wedge H}(\vec{b},\vec{c}) := t_{G\vee H}(\vec{b},\vec{c}) := t_G(\vec{b}) + t_H(\vec{c})$. In the last case assume that $F(A,\vec{b})$ equals $\forall x \le q(\vec{b})\, G(A,x,\vec{b})$ or $\exists x \le q(\vec{b})\, G(A,x,\vec{b})$. The term $t_G(a,\vec{b})$ is built up from the constant 0 and the variables $a,\vec{b}$ by primitive recursive function symbols and can therefore be interpreted as $f(a,\vec{b})$ where f is the obvious primitive recursive function symbol, i.e. $t_G(a,\vec{b}) = f(a,\vec{b})$. Let g be the obvious function symbol associated to the primitive recursive function $(n,\vec{m}) \mapsto \sum_{k=0}^{n} f(k,\vec{m})$. Then we set $t_F(\vec{b}) := g(q(\vec{b}),\vec{b})$. One can show by a straightforward Σ^0_1-induction that $\forall y \le q(\vec{b})\ (t_G(y,\vec{b}) \le t_F(\vec{b}))$ and from this the claim follows. $\square$

Proposition 2.16. *The theory generated by* RCA_0^- *and weak König's lemma proves the* FAN_0 *rule.*

Proof. Arguing in the theory generated by RCA_0^- and weak König's lemma, assume that the formula $\forall w\, \exists X\, \forall x \le w\, F(X,x)$ holds where F is a bounded formula. We must prove that the formula $\exists X\, \forall x\, F(X,x)$ holds too. Define the class term

$$T := \{s \in \dot{2}^{<\dot{\omega}} \,:\, \exists s'\, s \subseteq s' \wedge \forall x \le \mathrm{lg}(s)\, F^*(s',x)\}\,,$$

where F^* is obtained as in the last Lemma. Then for $s \in T$, if we write $G(A,s) = \forall x \le \mathrm{lg}(s)\, F(A,x)$, we get from Lemma 2.15 that $G^*(s',s) \leftrightarrow G^*(s'',s)$ if $s' \restriction t_G(s) = s'' \restriction t_G(s)$, i.e. $\exists s' \in \dot{2}^{t_G(s)}\ s \subseteq s' \wedge G^*(s',s)$ if $\mathrm{lg}(s) \le t_G(s)$ or $G^*(s,s)$ if $t_G(s) < \mathrm{lg}(s)$. But this means that T may be defined by a bounded formula and thus is a set by Δ^0_0-comprehension. Obviously T is a tree. To see that T is infinite take w arbitrary and choose X such that $H(X,w) :\equiv \forall x \le w\, F(X,x)$. Let s' be a piecewise code for X of length $\mathrm{lg}(s') \ge \max(w, t_H(w))$. Then $\forall x \le w\, F^*(s',x)$ and so $s' \restriction w$ has length w and is contained in T. Now by WKL_0 there exists a path $f \in \dot{2}^{\dot{\omega}}$ through T. We set $X := \{n \,:\, f(n) = 0\}$ and let x be arbitrary. Then $s := \langle f(0), \dots, f(y)\rangle$, where $y = \max(x, t_F(x))$, is in T, that is $\exists s'\, s \subseteq s' \wedge \forall z \le \mathrm{lg}(s)\, F^*(s',z)$. This means in particular $F^*(s,x)$ since $\mathrm{lg}(s) \ge t_F(x)$, so again with Proposition 2.15 we get $F(X,x)$. $\square$

Proposition 2.17. *The theory* $\mathsf{RCA}_0^- + \mathsf{FAN}_0$ *proves weak König's lemma.*

Proof. We assume that T is an infinite binary tree, i.e. $\mathrm{Tree}_\infty(T)$. As T is infinite we find for every w a sequence $s \in T$ of length w. As the sequence s is a code for a set X we see that $\forall w\, \exists X\, \forall x \le w\, \big(\exists s \in \dot{2}^x\ (\forall i < \mathrm{lg}(s)\ (s_i = 0 \leftrightarrow i \in X)) \wedge s \in T\big)$. As the formula in brackets is bounded, we conclude by an application of FAN_0 that $\exists X\, \forall x\, \big(\exists s \in \dot{2}^x\ (\forall i < \mathrm{lg}(s)\ (s_i = 0 \leftrightarrow i \in X)) \wedge s \in T\big)$. This means that the characteristic function of X is an infinite path through T. $\square$

Corollary 2.18. *The theories* $\mathsf{RCA}_0^- + \mathsf{FAN}_0$ *and* WKL_0 *are the same.*

Proof. As FAN_0 and weak König's lemma are equivalent over RCA_0^- we only need to check that $\mathsf{RCA}_0^- + \mathsf{FAN}_0$ implies Δ^0_1-comprehension. Thus suppose that $\forall u\, (\exists y\, F(u,y) \leftrightarrow \forall z\, G(u,z))$ where F and G are bounded formulas. Given w the set $X := \{u \,:\, \exists y \le w\, F(u,y)\}$ exists by bounded comprehension. This X victims that

$$\forall w\, \exists X\, \forall x \le w\, \forall u,y,z \le x\, \big((F(u,y) \to u \in X) \wedge (u \in X \to G(u,z))\big)$$

is true. Applying the FAN_0 rule gives

$$\exists X\, \forall x\, \forall u,y,z \le x\, \big((F(u,y) \to u \in X) \wedge (u \in X \to G(u,z))\big)$$

and this entails the desired result. $\square$

2.4 Sequent calculus reformulation of WKL_0

To analyse subsystems of second order arithmetic from a proof-theoretic approach we have to reformulate the Hilbert style calculus as a Gentzen style calculus. The calculus described first is denoted as LK_Q. If we denote the closure under term substitution of the equality axioms and the axioms of Q as $\mathscr{Q}$ then the initial sequents of LK_Q are given by $\mathscr{Q}$ (Note that a L_2-formula F is interpreted in LK_Q as $\longrightarrow F$) and all sequents of the form $F \longrightarrow F$ where F is any formula. The rules of LK_Q consist of the weak structural rules, the cut rule, the propositional rules and the quantifier rules. They are stated below.

Weak Structural Rules
The weak structural rules consists of exchange, contraction and weakening.

$$(\text{Eleft})\quad \frac{\Gamma, F, G, \Pi \longrightarrow \Delta}{\Gamma, G, F, \Pi \longrightarrow \Delta} \qquad (\text{Eright})\quad \frac{\Gamma \longrightarrow \Delta, F, G, \Lambda}{\Gamma \longrightarrow \Delta, G, F, \Lambda}$$

$$(\text{Cleft})\quad \frac{F, F, \Gamma \longrightarrow \Delta}{F, \Gamma \longrightarrow \Delta} \qquad (\text{Cright})\quad \frac{\Gamma \longrightarrow \Delta, F, F}{\Gamma \longrightarrow \Delta, F}$$

$$(\text{Wleft})\quad \frac{\Gamma \longrightarrow \Delta}{F, \Gamma \longrightarrow \Delta} \qquad (\text{Wright})\quad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, F}$$

The cut rule

$$(\text{Cut})\quad \frac{\Gamma \longrightarrow \Delta, F \quad \Gamma, F \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

The Propositional Rules

$$(\neg\text{left})\quad \frac{\Gamma \longrightarrow \Delta, F}{\neg F, \Gamma \longrightarrow \Delta} \qquad (\neg\text{right})\quad \frac{F, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg F}$$

$$(\wedge\text{left})\quad \frac{F, G, \Gamma \longrightarrow \Delta}{F \wedge G, \Gamma \longrightarrow \Delta} \qquad (\wedge\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F \quad \Gamma \longrightarrow \Delta, G}{\Gamma \longrightarrow \Delta, F \wedge G}$$

$$(\vee\text{left})\quad \frac{F, \Gamma \longrightarrow \Delta \quad G, \Gamma \longrightarrow \Delta}{F \vee G, \Gamma \longrightarrow \Delta} \qquad (\vee\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F, G}{\Gamma \longrightarrow \Delta, F \vee G}$$

The First Order Quantifier Rules
For all terms t and all eigenvariables a (i.e. a occurs only in the upper sequent) we have

$$(\forall\text{left})\quad \frac{F(t), \Gamma \longrightarrow \Delta}{\forall x\, F(x), \Gamma \longrightarrow \Delta} \qquad (\forall\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F(a)}{\Gamma \longrightarrow \Delta, \forall x\, F(x)}$$

$$(\exists\text{left})\quad \frac{F(a), \Gamma \longrightarrow \Delta}{\exists x\, F(x), \Gamma \longrightarrow \Delta} \qquad (\exists\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F(t)}{\Gamma \longrightarrow \Delta, \exists x\, F(x)}\,.$$

The Bounded Quantifier Rules
For all terms s, t and all eigenvariables a (i.e. a occurs only in the upper sequent) we have

$$(\forall^{\leq}\text{left})\quad \frac{F(t), \Gamma \longrightarrow \Delta}{t \leq s, \forall x \leq s\, F(x), \Gamma \longrightarrow \Delta} \qquad (\forall^{\leq}\text{right})\quad \frac{a \leq s, \Gamma \longrightarrow \Delta, F(a)}{\Gamma \longrightarrow \Delta, \forall x \leq s\, F(x)}$$

$$(\exists^{\le}\text{left})\quad \frac{a \le s, F(a), \Gamma \longrightarrow \Delta}{\exists x \le s\, F(x), \Gamma \longrightarrow \Delta} \qquad (\exists^{\le}\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F(t)}{t \le s, \Gamma \longrightarrow \Delta, \exists x \le s\, F(x)}\,.$$

The Second Order Quantifier Rules
For all free second order variables B and all eigenvariables A (i.e. A occurs only in the upper sequent) we have

$$(\forall^2\text{left})\quad \frac{F(B), \Gamma \longrightarrow \Delta}{\forall X\, F(X), \Gamma \longrightarrow \Delta} \qquad (\forall^2\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F(A)}{\Gamma \longrightarrow \Delta, \forall X\, F(X)}$$

$$(\exists^2\text{left})\quad \frac{F(A), \Gamma \longrightarrow \Delta}{\exists X\, F(X), \Gamma \longrightarrow \Delta} \qquad (\exists^2\text{right})\quad \frac{\Gamma \longrightarrow \Delta, F(B)}{\Gamma \longrightarrow \Delta, \exists X\, F(X)}\,.$$

One easily checks that $\mathsf{LK_Q}$ is equivalent to Q.

Theorem 2.19. *A L_2-formula F is provable in Q if and only if $\longrightarrow F$ is derivable in $\mathsf{LK_Q}$.*

As described in the appendix the calculus $\mathsf{LK_Q}$ admits *partial cut elimination*, i.e. for any sequent derivable in $\mathsf{LK_Q}$ there exists a proof of this sequence in which the cut formula applies only to Σ^0_1-formulas. Now we enrich $\mathsf{LK_Q}$ to obtain sequent calculi for RCA_0^- and WKL_0. First we have to include the induction rule. It can be stated as follows.

The Σ^0_1-Induction Rule
For all Σ^0_1-formulas F and any term t we add the rule

$$(\text{Ind})\quad \frac{F(a), \Gamma \longrightarrow \Delta, F(a+1)}{F(0), \Gamma \longrightarrow \Delta, F(t)}\,.$$

The next goal is to implement Δ^0_1-comprehension. We could for example take it as an axiom. But as we don't want any quantifier in our axioms we do a little trick. For this we have to introduce the notion of an *abstract*.

Definition 2.20. *Let $G(a)$ be a formula with a distinguished free number variable a. Then the meta-expression $\{x : G(x)\}$ is called an abstract for G. If $F(A)$ is a formula and V an abstract for a formula $G(a)$ then $F(V)$ is the formula obtained from F by simultaneously replacing every subformula of the form $q \in A$ by $G(q)$.*

Now we exchange the second order quantifier rules by the following rules.

The Second Order Quantifier Rules*
For all Abstracts V for a bounded formula and all eigenvariables A (i.e. A occurs only in the upper sequent) we have

$$(\forall^2\text{left})^*\quad \frac{F(V), \Gamma \longrightarrow \Delta}{\forall X\, F(X), \Gamma \longrightarrow \Delta} \qquad (\forall^2\text{right})^*\quad \frac{\Gamma \longrightarrow \Delta, F(A)}{\Gamma \longrightarrow \Delta, \forall X\, F(X)}$$

$$(\exists^2\text{left})^*\quad \frac{F(A), \Gamma \longrightarrow \Delta}{\exists X\, F(X), \Gamma \longrightarrow \Delta} \qquad (\exists^2\text{right})^*\quad \frac{\Gamma \longrightarrow \Delta, F(V)}{\Gamma \longrightarrow \Delta, \exists X\, F(X)}$$

These new second order quantifier rules are equivalent to the former second order quantifier rules together with bounded comprehension. Indeed the formula $\exists X\ \forall x\ (x \in X \leftrightarrow F(x))$, where F is bounded, is a direct consequence of $(\exists^2\text{right})^*$ with upper sequent $\longrightarrow \forall x\ (F(x) \leftrightarrow F(x))$ and abstract $V := \{x\ :\ F(x)\}$.

Definition 2.21. *The sequent calculus obtained in this way is denoted as* $\mathsf{LK}_{\mathsf{RCA}_0^-}$.

Now the calculus $\mathsf{LK}_{\mathsf{RCA}_0^-}$ is equivalent to the theory RCA_0^-. If we add the FAN_0 rule we get the sequent calculus reformulation $\mathsf{LK}_{\mathsf{WKL}_0}$ of WKL_0 we are looking for.

The FAN Rule
For all bounded formulas F and all eigenvariables A

$$(\mathrm{Fan}_0)\quad \frac{\Gamma \longrightarrow \Delta, \exists x\, F(A, x, \vec{b})}{\Gamma \longrightarrow \Delta, \exists v\ \forall s \in 2^{t(v,\vec{b})}\ \exists x \le v\ F^*(s, x, \vec{b})},$$

where $t(v, \vec{b})$ is the term associated to $\exists x \le v\ F(A, x, \vec{b})$ according to Lemma 2.15.

Theorem 2.22. *The calculi* $\mathsf{LK}_{\mathsf{RCA}_0^-}$ *and* $\mathsf{LK}_{\mathsf{WKL}_0}$ *admit partial cut elimination.*

A proof of this theorem can be found in the appendix.

3 Harrington's conservation theorem

If a formula F contains no set quantifier it is called an *arithmetical formula*. The second order counterpart of the Σ_n^0-formulas are the Σ_n^1-formulas given in the next definition.

Definition 3.1. *A formula F is called called a Σ_n^1-formula (resp. a Π_n^1-formula) if there exists an arithmetical formula G such that F is $\exists X_0 \forall X_1 \dots X_{n-1} G$ (resp. $\forall X_0 \exists X_1 \dots X_{n-1} G$). Note that $\Sigma_0^1 = \Pi_0^1$ is the set of arithmetical formulas.*

Now we can state the main theorem of this thesis, namely Harringston's conservation theorem.

Theorem 3.2 (Harrington)**.** *The theory* WKL_0 *is* Π_1^1*-conservative over* RCA_0^-*, i.e. if* WKL_0 *proves the sentence* $\forall X\ F(X)$*, where* $F(A)$ *is an arithmetical formula, then already* RCA_0^- *proves* $\forall X\ F(X)$.

We proof this with a cut elimination approach. If we suppose that WKL_0 proves the sentence $\forall X\ F(X)$ then there is a proof of $\longrightarrow F(A)$ in the sequent calculus $\mathsf{LK}_{\mathsf{WKL}_0}$ described in the last section. By the partial cut elimination theorem 2.22 there is a proof of $\longrightarrow F(A)$ in $\mathsf{LK}_{\mathsf{WKL}_0}$ in which the cut rule applies only to bounded or Σ_1^0-formulas. As a consequence, this proof has no occurrences of second-order quantifiers. Modulo some exchange rules, every sequence in the proof has the form

$$\Gamma, \exists w_1\ H_1(w_1, \vec{A}), \dots, \exists w_n\ H_n(w_n, \vec{A}) \longrightarrow \Delta, \exists y_1\ G_1(y_1, \vec{A}), \dots, \exists y_m\ G_m(y_m, \vec{A}) \tag{1}$$

where:

(i) the Hs and the Gs are bounded formulas and where we allow the absence of the existential quantifiers $\exists w_i$ or $\exists y_j$ in order to accommodate plain bounded formulas in the above sequent;

(ii) there are no bounded or Σ^0_1-formulas in Γ or Δ;

(iii) the tuple $\vec{A}$ displays exactly the second-order parameters which occur in the Hs or in the Gs without occurring neither in Γ nor Δ. These are called the *special parameters* of the sequent;

(iv) we are not displaying other (first or second order) parameters. In particular, we are not displaying second-order parameters that occur in Γ or in Δ (and which may concurrently occur in the Hs or in the Gs).

If the (Fan_0) rule is not applied in the proof, then of course, $\forall X\, F(X)$ is a theorem of RCA_0^-. Otherwise, it occurs for a first time in some branch of the proof tree. At this point we need a lemma in order to eliminate this occurrence.

Lemma 3.3. *Let be given a proof of a sequent of the form (1) in the sequent calculus $\mathsf{LK}_{\mathsf{RCA}_0^-}$. Suppose further that this proof is normal in the following sense: every cut formula is bounded or a Σ^0_1-formula; and, no formula of the proof has second-order quantifiers. Under these conditions, the theory RCA_0^- proves*

$$\begin{cases} \Gamma \wedge \neg\Delta \to \big(\forall w_1, \ldots, w_n\, \exists v\, \forall \vec{A}\, (H_1(w_1, \vec{A}) \wedge \ldots \wedge H_n(w_n, \vec{A}) \to \\ \exists y_1 \leq v\, G_1(y_1, \vec{A}) \vee \ldots \vee \exists y_m \leq v\, G_m(y_m, \vec{A}))\big)\,, \end{cases}$$

where v is a new bounded number variable not occurring in (1).

We will first show how this lemma implies the theorem.

Proof of theorem 3.2. When we arrive at the top sequent of a first application of the (Fan_0) rule in our normal proof of $F(A)$, RCA_0^- proves

$$\Gamma(\vec{b}) \wedge \neg\Delta(\vec{b}) \to \big(\forall w\, \exists v\, \forall A (H(w, \vec{b}) \to \exists x \leq v\, F(A, x, \vec{b}) \vee \exists y \leq v\, G(y, \vec{b})\big)\,,$$

where we are not showing any special parameters besides A and where for simplicity $n = 1$ and $m = 2$. Note that A is an eigenvariable and only shows up in the auxiliary formula of the (Fan_0) rule. Note, also, that the universal quantifications over the other special parameters can safely cross over the quantifier $\exists v$. We are displaying the first order parameters $\vec{b}$ that appear in the auxiliary formula. Hence, RCA_0^- proves

$$\Gamma(\vec{b}) \wedge \neg\Delta(\vec{b}) \wedge \exists w\, H(w, \vec{b}) \to \exists v\, \forall A\, \big(\exists x \leq v\, F(A, x, \vec{b}) \vee \exists y \leq v\, G(y, \vec{b})\big)\,.$$

As a consequence, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma(\vec{b}) \wedge \neg\Delta(\vec{b}) \wedge \exists w\, H(w, \vec{b})$ ans whose consequent is $\big(\exists v\, \forall s \in 2^{t(v,\vec{b})} \exists x \leq v\, F^*(s, x, \vec{b})\big) \vee \exists y\, G(y, \vec{b})$.
We have arrived at the conclusion of a first application of the (Fan_0) rule in a normal proof in $\mathsf{LK}_{\mathsf{WKL}_0}$ via a proof in $\mathsf{LK}_{\mathsf{RCA}_0^-}$. With a partial cut elimination argument this proof may be taken as a normal proof, in the sense of Lemma 3.3. If we repeat this procedure enough times, we arrive at a (normal) proof of $\longrightarrow F(A)$ in $\mathsf{LK}_{\mathsf{RCA}_0^-}$. Hence, the theory RCA_0^- already proves the sentence $\forall X\, F(X)$. $\square$

It remains to prove the lemma. At various points, the proof of the lemma makes appeal to the so-called *bounded collection* scheme $\mathsf{B}\Sigma^0_1$.

Proposition 3.4. *The theory* RCA_0^- *proves*

$$\forall x \le a\, \exists y\, F(x,y) \to \exists z\, \forall x \le a\, \exists y \le z\, F(x,y)\,,$$

for all bounded formulas F.

Proof. The formula $G(a') :\equiv a' \le a \to \exists z\, \forall x \le a'\, \exists y \le z\, F(x,y)$ is obviously equivalent to a Σ_1^0-formula. If we assume $H(a) :\equiv \forall x \le a\, \exists y\, F(x,y)$ we get $G(0)$. By assuming additionally $G(x')$ we find $y_{x'+1}$ and $z_{x'}$ such that $F(x'+1, y_{x'+1})$ and $\forall x \le x'\, \exists y \le z_{x'}\, F(x,y)$ whenever $x' < a$. If we set $z_{x'+1} := y_{x'+1} + z_{x'}$ we immediately get $\forall x \le x'+1\, \exists y \le z_{x'+1}\, F(x,y)$. Hence we have proven $H(a) \to G(0) \wedge \forall x'\, (G(x') \to G(x'+1))$. So by Σ_1^0-induction we conclude $H(a) \to \forall x\, G(x)$ and by a further substitution we get the desired result $H(a) \to G(a)$. $\square$

Definition 3.5. *The depth* $\mathrm{depth}(P)$ *of a given proof tree is defined as the maximal length from the root to a leaf.*

Proof of Lemma 3.3. The proof is by induction on the depth of the given normal proof. There is nothing to prove regarding initial sequents, since they are quantifier-free. One must check that the induction hypothesis is carried over by every rule of $\mathsf{LK}_{\mathsf{RCA}_0^-}$ except for the second order quantifier rules which do not occur in the proof. This is trivial for the weak structural rules. The induction step for the propositional rules is proved by distinguishing whether the auxiliary formula is bounded, Σ_1^0 or none of them, and then a straightforward propositional argumentation.
$(\neg\text{left})$. If the auxiliary formula is neither bounded nor Σ_1^0 then the claim is trivial. Assume that F is bounded and that the inference looks like

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x\, F(x,\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \neg\exists x\, F(x,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,,$$

where, for simplicity, we consider only one bounded formula H and one bounded formula G, and where we distinguish between the special parameters which occur in the auxiliary formula (the parameters $\vec{B}$) and those that do not occur there (the parameters $\vec{A}$). Note that the former are no longer special parameters of the lower sequent. The induction hypothesis tells us that RCA_0^- proves

$$\Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \exists x \le v\, F(x,\vec{B}))\big)\,.$$

But this gives immediately

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \neg\exists x\, F(x,\vec{B}) \to \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)$$

for fixed $\vec{B}$. If we omit the quantifiers $\forall x$, $\exists x$, i.e. if we consider a bounded auxiliary formula, we get instead

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \wedge \neg F(\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,,$$

and this is the desired result.
$(\neg\text{right})$. Again w.l.o.g. we can assume that the auxiliary formula is bounded or Σ_1^0. In the Σ_1^0 case we have an inference of the form

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists x\, F(x,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \neg\exists x\, F(x,\vec{B})}\,,$$

where F is bounded. The induction hypothesis is now that RCA_0^- proves

$$\Gamma \wedge \neg\Delta \to \big(\forall w\,\forall x\,\exists v\,\forall \vec{A}\,\forall \vec{B}\,(H(w,\vec{A},\vec{B}) \wedge F(x,\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

From this we conclude

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \neg\neg\exists x\, F(x,\vec{B}) \to \big(\forall w\,\exists v\,\forall \vec{A}\,(H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

If we omit the quantifiers $\forall x$, $\exists x$, i.e. if we consider a bounded auxiliary formula, we get instead

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \to \big(\forall w\,\exists v\,\forall \vec{A}\,\forall \vec{B}\,(H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \neg F(\vec{B}))\big)\,.$$

($\wedge$left). First we assume that the two auxiliary formulas are either bounded or Σ_1^0. Then the rule is given by

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists x\, F(x,\vec{B}), \exists z\, E(z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists x\, F(x,\vec{B}) \wedge \exists z\, E(z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,,$$

where we allow the absence of the existential quantifiers in front of the bounded formulas E and F. The induction hypothesis is

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \to \big(\forall w\,\forall x\,\forall z\,\exists v\,\forall \vec{A}\,\forall \vec{B}\,(H(w,\vec{A},\vec{B}) \wedge F(x,\vec{B}) \wedge E(z,\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

If both auxiliary formulas are bounded then we are done. Otherwise the conjunction $\exists x\, F \wedge \exists z\, E$ is neither bounded nor Σ_1^0 and we have

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge (\exists x\, F(x,\vec{B}) \wedge \exists z\, E(z,\vec{B})) \wedge \neg\Delta \to \big(\forall w\,\exists v\,\forall \vec{A}\,(H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

The claim is trivial if both auxiliary formulas are neither bounded nor Σ_1^0. In the last case we have an unbounded non Σ_1^0-formula F and a bounded or Σ_1^0-formula $\exists z\, E(z,\vec{B})$, where we allow the absence of $\exists z$, with inference rule

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), F, \exists z\, E(z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), F \wedge \exists z\, E(z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,,$$

and induction hypothesis

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge F \wedge \neg\Delta \to \big(\forall w\,\forall z\,\exists v\,\forall \vec{A}\,\forall \vec{B}\,(H(w,\vec{A},\vec{B}) \wedge E(z,\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

From this we conclude

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge (F \wedge \exists z\, E(z,\vec{B})) \wedge \neg\Delta \to \big(\forall w\,\exists v\,\forall \vec{A}\,(H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

($\wedge$right). Assume first that the auxiliary formulas are given by bounded or Σ_1^0-formulas. The inference ($\wedge$right) says that from the two sequents

$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x\, F(x,\vec{B})$$

$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists z\, E(z,\vec{B})$$

one can infer

$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \to \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x\, F(x,\vec{B}) \wedge \exists z\, E(z,\vec{B})\,,$$

where we again allow the absence of the quantifier associated with E and F. We have the induction hypothesises that RCA_0^- proves both

$$\Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \exists x \le v\, F(x,\vec{B}))\big) \quad \text{and}$$
$$\Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \exists z \le v\, E(z,\vec{B}))\big)\,.$$

If both existential quantifier are absent, i.e. if the two auxiliary formulas are bounded, the conclusion of the induction step is immediate. Otherwise $\exists x\, F \wedge \exists z\, E$ is neither bounded nor Σ^0_1 and we get

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \neg(\exists x\, F(x,\vec{B}) \wedge \exists z\, E(z,\vec{B})) \to \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

If both auxiliary formulas are not included in $\Delta^0_0 \cup \Sigma^0_1$ the claim is trivial. In the last case one auxiliary formula is included in $\Delta^0_0 \cup \Sigma^0_1$ and the other is not. Then the rule says that from

$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), F$$
$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists z\, E(z,\vec{B})\,,$$

where $F \notin \Delta^0_0 \cup \Sigma^0_1$, $E \in \Delta^0_0$ and where we allow the absence of $\exists z$, one can infer

$$\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \to \Delta, \exists y\, G(y,\vec{A},\vec{B}), F \wedge \exists z\, E(z,\vec{B})\,.$$

From the hypothesises that RCA_0^- proves both

$$\Gamma \wedge \neg\Delta \wedge \neg F \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big) \quad \text{and}$$
$$\Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \exists z \le v\, E(z,\vec{B}))\big)\,,$$

we conclude

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \neg(F \wedge \exists z\, E(z,\vec{B})) \to \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

The remaining propositional rules can be proved very similar. Alternatively we could formulate our calculus only with the propositional operator $\neg, \wedge$ and since nothing else changes we are done.
($\forall$left). The study of this rule is only interesting when the auxiliary formula is Σ^0_1 or bounded (otherwise the claim follows from $\forall x\, F(x) \to F(t)$). In the former case we have an inference of the form

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists x\, F(x,t,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \forall z\, \exists x\, F(x,z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,,$$

where we are distinguishing between the special variables. By induction hypothesis, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma \wedge \neg\Delta$ and whose consequent is

$$\forall w\, \forall x\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \wedge F(x,t,\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\,.$$

We reason inside RCA_0^-. Fix $\vec{B}'$ and assume the conjunction of Γ with $\neg\Delta$ and $\forall z\, \exists x\, F(x,z,\vec{B}')$. Take x' such that $F(x',t,\vec{B}')$. Fix w. It is now clear that there is v such that $\forall \vec{A}\, (H(w,\vec{A},\vec{B}') \to \exists y \leq v\, G(y,\vec{A},\vec{B}'))$. The case where the auxiliary formula is bounded follows by simply by omitting the quantifiers in the proof of the Σ^0_1 case.
($\forall$right). This rule is only interesting when the auxiliary formula is bounded or Σ^0_1. The former case follows easily from the proof of the second case by admitting a quantifier. So let F be bounded and let the rule be given by

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x\, F(x,a,\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \forall z\, \exists x\, F(x,z,\vec{B})},$$

where $\vec{B}$ are the special variable occurring in the auxiliary formula. Then the induction hypothesis and the fact that a is an eigenvariable say that

$$\Gamma \wedge \neg\Delta \to \big(\forall a\, \forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \to \exists y \leq v\, G(y,\vec{A},\vec{B}) \vee \exists x \leq v\, F(x,a,\vec{B}))\big)$$

is provable in RCA_0^-. We argue inside RCA_0^-. Fix $\vec{B}'$ and assume $\Gamma \wedge \neg\Delta \wedge \neg\forall z\, \exists x\, F(x,z,\vec{B}')$. We fix w and choose z' such that $\forall x\, \neg F(x,z',\vec{B}')$. Hence we get the desired result $\exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}') \to \exists y \leq v\, G(y,\vec{A},\vec{B}'))$.
($\exists$left). Similar to the ($\forall$right) rule the interesting cases are those where the auxiliary formula is bounded or Σ^0_1; otherwise it follows from the fact that a is an eigenvariable. Let's first assume, that the auxiliary formula F is bounded an thus the inference reads

$$\frac{\Gamma, \exists w\, H(w,\vec{A}), F(a,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A})}{\Gamma, \exists w\, H(w,\vec{A}), \exists z\, F(z,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A})}.$$

From the induction hypothesis we get

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A}) \wedge F(a,\vec{A}) \to \exists y \leq v\, G(y,\vec{A}))\big).$$

The conclusion of the induction step is now immediate, as a is an eigenvariable and thus we can quantify over a on the expression on the right side of the implication arrow. Now we assume that the auxiliary formula is Σ^0_1. This time the inference looks like

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists x\, F(x,a,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), \exists z\, \exists x\, F(x,z,\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})},$$

where we again distinguish between the special variables occurring in F and those that don't. By induction hypothesis, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma \wedge \neg\Delta$ and whose consequent is

$$\forall w\, \forall x\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \wedge F(x,a,\vec{B}) \to \exists y \leq v\, G(y,\vec{A},\vec{B})).$$

Let's reason inside RCA_0^-. We fix $\vec{B}'$ and assume $\Gamma \wedge \neg\Delta \wedge \exists z\, \exists x\, F(x,z,\vec{B}')$. Now we fix w. According to our assumption we find x', z' such that $F(x',z',\vec{B}')$. Since a is an eigenvariable we find with the induction hypothesis v such that $\forall \vec{A}\, (H(w,\vec{A},\vec{B}') \to \exists y \leq v\, G(y,\vec{A},\vec{B}'))$ and this was to prove.

($\exists$right). Again w.l.o.g. we assume that the auxiliary formula is bounded or Σ^0_1. In the former case the inference is

$$\frac{\Gamma, \exists w\, H(w,\vec{A}), \longrightarrow \Delta, \exists y\, G(y,\vec{A}), F(t,\vec{A})}{\Gamma, \exists w\, H(w,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A}), \exists z\, F(z,\vec{A})}\,.$$

From the induction hypothesis

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A}) \rightarrow \exists y \leq v\, G(y,\vec{A}) \vee F(t,\vec{A}))\big)\,,$$

we immediately get the conclusion of the induction step, as $F(t,\vec{A}) \rightarrow \exists z \leq t\ F(z,\vec{A})$. Now we assume that the auxiliary formula is Σ^0_1. Then we have the rule

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x\, F(x,t,\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists z\, \exists x\, F(x,z,\vec{B})}\,,$$

and the induction hypothesis says that within RCA_0^- the conditional $\Gamma \wedge \neg\Delta$ implies

$$\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}) \vee \exists x \leq v\, F(x,t,\vec{B}))\,.$$

Let's reason inside RCA_0^-. We fix $\vec{B}'$ and assume $\Gamma \wedge \neg\Delta \wedge \neg\exists z\, \exists x\, F(x,z,\vec{B}')$. Fix w. Then we find v with $\forall \vec{A}\, (H(w,\vec{A},\vec{B}') \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}'))$.
($\forall^{\leq}$left). This rule has the form

$$\frac{F(t,\vec{B}), \Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{t \leq s, \forall x \leq s\, F(x,\vec{B}), \Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,.$$

In the case $F \notin \Delta^0_0 \cup \Sigma^0_1$ we get the induction hypothesis

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge F(t,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

From this we conclude by distinguishing within RCA_0^- the cases $t \leq s$ and $s < t$ that also

$$\Gamma \wedge \neg\Delta \wedge \forall x \leq s\, F(x,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (t \leq s \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)$$

is provable in RCA_0^-. The second case is $F \in \Sigma^0_1$, i.e. $F(t,\vec{B}) \equiv \exists z\, E(t,z,\vec{B})$ for $E \in \Delta^0_0$. The induction hypothesis is now that RCA_0^- proves

$$\Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \forall z\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \wedge E(t,z,\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

As before we conclude

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \forall x \leq s\, F(x,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (t \leq s \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

In the third case, i.e. if $F \in \Delta^0_0$, our induction hypothesis is given by

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \wedge F(t,\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,,$$

and this implies

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (t \leq s \wedge H(w,\vec{A},\vec{B}) \wedge \forall x \leq s\, F(x,\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

($\forall^{\leq}$right). The inference is given by

$$\frac{\Gamma, a \leq t, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), F(a,\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \forall x \leq t\, F(x,\vec{B})}\,.$$

First assume $F \notin \Delta^0_0 \cup \Sigma^0_1$. With the induction hypothesis we have that RCA_0^- proves

$$\Gamma \wedge \neg\Delta \wedge \neg F(a,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (a \leq t \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

We argue in RCA_0^-. If we fix $\vec{B}'$ and assume $\Gamma \wedge \neg\Delta \wedge \neg\forall x \leq t\, F(x,\vec{B}')$ we can choose $x' \leq t$ such that $\neg F(x',\vec{B}')$. For fixed w we find v such that $\forall \vec{A}\, (H(w,\vec{A},\vec{B}') \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}'))$. Now let $F \in \Sigma^0_1$ i.e. $F = \exists z\, E(a,z,\vec{B})$ for some bounded E. Within RCA_0^- we have in this case

$$\Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (a \leq t \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}) \vee \exists z \leq v\, E(a,z,\vec{B}))\big)\,.$$

We reason in RCA_0^-. Fix $\vec{B}'$ and assume $\Gamma \wedge \neg\Delta \wedge \neg\forall x \leq t\, F(x,\vec{B}')$. Again by assigning an adequate value to a and the fact that this is an eigenvariable we get the conclusion $\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}') \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}'))$. Lastly let $F \in \Delta^0_0$. This time RCA_0^- proves the formula

$$\Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (a \leq t \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}) \vee F(a,\vec{A}))\big)\,.$$

Since a is an eigenvariable we can quantify over the expression on the right side of the implication. We argue as usual in RCA_0^-, assume $\Gamma \wedge \neg\Delta$ and fix w. Then we have

$$\forall a \leq t\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}') \vee F(a,\vec{B}))\,.$$

The subformula which begins with $\exists v$ is logically equivalent to a Σ^0_1-formula according to Lemma 2.15. Therefore we can apply $\mathsf{B}\Sigma^0_1$ collection and thus get

$$\exists v\, \forall a \leq t\, \forall \vec{A}\, \forall \vec{B}\, (H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}') \vee F(a,\vec{B}))\,.$$

Now the claim follows immediately.
($\exists^{\leq}$left). This rule has the form

$$\frac{\Gamma, a \leq t, F(a,\vec{B}), \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}{\Gamma, \exists x \leq t\, F(x,\vec{B}), \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B})}\,,$$

with the usual conditions. We distinguish three cases. First let F be neither bounded nor Σ^0_1. Then according to the induction hypothesis RCA_0^- proves

$$\Gamma \wedge \neg\Delta \wedge F(a,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (a \leq t \wedge H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

From this we conclude that RCA_0^- proves also

$$\Gamma \wedge \neg\Delta \wedge \exists x \leq t\, F(x,\vec{B}) \rightarrow \big(\forall w\, \exists v\, \forall \vec{A}\, (H(w,\vec{A},\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,,$$

since a is an eigenvariable. The second case is $F \in \Sigma^0_1$ i.e. F equals $\exists z\, E(a,z,\vec{B})$ for some bounded formula E. This time RCA_0^- proves

$$\Gamma \wedge \neg\Delta \rightarrow \big(\forall w\, \forall z\, \exists v\, \forall \vec{A}\, \forall \vec{B}\, \forall (a \leq t \wedge H(w,\vec{A},\vec{B}) \wedge E(a,z,\vec{B}) \rightarrow \exists y \leq v\, G(y,\vec{A},\vec{B}))\big)\,.$$

Again since a is an eigenvariable RCA_0^- proves also

$$\Gamma \wedge \neg\Delta \wedge \exists x \le t\, F(x,\vec{B}) \to \big(\forall w\, \exists v\, \forall \vec{A}\; (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

In the last case F is bounded. Then RCA_0^- proves the

$$\Gamma \wedge \neg\Delta \to \big(\forall a\, \forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\; (a \le t \wedge F(a,\vec{B}) \wedge H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,,$$

where we have used that a is an eigenvariable. We work within RCA_0^- and assume $\Gamma \wedge \neg\Delta$ and fix w. We have with the induction hypothesis $\forall a \le t\, \exists v\, \forall \vec{A}\, \forall \vec{B}\; (F(a,\vec{B}) \wedge H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))$. We can use Lemma 2.15 and $\mathsf{B}\Sigma^0_1$ collection to see that

$$\exists v\, \forall a \le t\, \forall \vec{A}\, \forall \vec{B}\; (F(a,\vec{B}) \wedge H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\,.$$

The conclusion of the induction step is an immediate consequence of the above.
($\exists^{\le}$right). We have the rule

$$\frac{\Gamma, \exists w\, H(w,\vec{A},\vec{B}) \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), F(t,\vec{B})}{\Gamma, \exists w\, H(w,\vec{A},\vec{B}), t \le s \longrightarrow \Delta, \exists y\, G(y,\vec{A},\vec{B}), \exists x \le s\, F(x,\vec{B})}$$

and we distinguish again three cases. The first case is $F \notin \Delta^0_0 \cup \Sigma^0_1$. The induction hypothesis equals

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \wedge \neg F(t,\vec{B}) \to \big(\forall w\, \exists v\, \forall \vec{A}\; (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

A straightforward argumentation within RCA_0^- shows

$$\Gamma \wedge \neg\Delta \wedge \neg\exists x \le s\, F(x,\vec{B}) \to \big(\forall w\, \exists v\, \forall \vec{A}\; (t \le s \wedge H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}))\big)\,.$$

This formula follows within RCA_0^- also in the case where F is a Σ^0_1-formula, i.e. $F(t,\vec{B}) \equiv \exists z\, E(t,z,\vec{B})$, from the hypothesis that

$$\Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\; (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee \exists z \le v\, E(t,z,\vec{B}))\big)$$

is valid in RCA_0^-. In the last case, i.e. F is bounded, we have the hypothesis

$$\mathsf{RCA}_0^- \vdash \Gamma \wedge \neg\Delta \to \big(\forall w\, \exists v\, \forall \vec{A}\, \forall \vec{B}\; (H(w,\vec{A},\vec{B}) \to \exists y \le v\, G(y,\vec{A},\vec{B}) \vee F(t,\vec{B}))\big)\,.$$

The conclusion of the induction step is an easy consequence of this.
(Ind). The induction rule equals (modulo renaming of bound variables)

$$\frac{\Gamma, \exists w\, H(w,\vec{A}), \exists x\, F(x,a,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A}), \exists x'\, F(x',a+1,\vec{A})}{\Gamma, \exists w\, H(w,\vec{A}), \exists x\, F(x,0,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A}), \exists x'\, F(x',t,\vec{A})}$$

under the usual conditions, and where a is an eigenvariable and t is an arbitrary term. By induction hypothesis and the fact that a is an eigenvariable, the theory RCA_0^- proves the conditional whose antecedent is $\Gamma \wedge \neg\Delta$ and whose consequent is

$$\forall a\, \forall w\, \forall x\, \exists v\, \forall \vec{A}\; (H(w,\vec{A}) \wedge F(x,a,\vec{A}) \to \exists x' \le v\, F(x',a+1,\vec{A}) \vee \exists y \le v\, G(y,\vec{A}))\,. \qquad (2)$$

Let us reason in RCA_0^-. Assume $\Gamma \wedge \neg\Delta$ and fix elements w and x. We claim that, for all elements a,

$$\exists v\, \forall \vec{A}\ (H(w,\vec{A}) \wedge F(x,0,\vec{A}) \to \exists x' \leq v\, F(x',a,\vec{A}) \vee \exists y \leq v\, G(y,\vec{A}))\,.$$

This solves or problem (substitute t for a). The claim is proved by induction on a. Note that this induction is permissible since by Lemma 2.15, the above formula is equivalent to a Σ_1^0-formula. The base case $a = 0$ is clear. To argue for the induction step, assume that there is v such that

$$\forall \vec{A}\ (H(w,\vec{A}) \wedge F(x,0,\vec{A}) \to \exists x'' \leq v\, F(x'',a,\vec{A}) \vee \exists y \leq v\, G(y,\vec{A}))\,.$$

By (2) we have:

$$\forall x'' \leq v\, \exists v'\, \forall \vec{A}\ (H(w,\vec{A}) \wedge F(x'',a,\vec{A}) \to \exists x' \leq v'\, F(x',a+1,\vec{A}) \vee \exists y \leq v'\, G(y,\vec{A}))$$

By bounded collection $\mathsf{B}\Sigma_1^0$ and Lemma 2.15, there is v' such that,

$$\forall x'' \leq v\, \forall \vec{A}\ (H(w,\vec{A}) \wedge F(x'',a,\vec{A}) \to \exists x' \leq v'\, F(x',a+1,\vec{A}) \vee \exists y \leq v'\, G(y,\vec{A}))\,.$$

It clearly follows that

$$\forall \vec{A}\ (H(w,\vec{A}) \wedge F(x,0,\vec{A}) \to \exists x' \leq v'''\, F(x',a+1,\vec{A}) \vee \exists y \leq v'''\, G(y,\vec{A}))\,,$$

where $v''' := \max(v,v')$.
(Cut). The cut rule says that from the two sequents

$$\Gamma, \exists w\, H(w,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A}), \exists x\, F(x,\vec{A},\vec{B}) \quad \text{and}$$
$$\exists x\, F(x,\vec{A},\vec{B}), \Gamma, \exists w\, H(w,\vec{A}) \longrightarrow \Delta, \exists y\, G(y,\vec{A})$$

one can infer the sequent $\Gamma, \exists w\, H(w,\vec{A}) \longrightarrow \Gamma, \exists y\, G(y,\vec{A})$. We are distinguishing the special parameters which only occur in the cut-formula (the parameters $\vec{B}$). By induction hypothesis, the theory RCA_0^- proves that $\Gamma \wedge \neg\Delta$ implies both

$$\forall w\, \exists v_1\, \forall \vec{A}\ (H(w,\vec{A}) \to \exists y \leq v_1\, G(y,\vec{A}) \vee \forall \vec{B}\, \exists x \leq v_1\, F(x,\vec{A},\vec{B})) \quad \text{and}$$
$$\forall w\, \forall x\, \exists v_2\, \forall \vec{A}\, \forall \vec{B}\ (F(x,\vec{A},\vec{B}) \wedge H(w,\vec{A}) \to \exists y \leq v_2\, G(y,\vec{A}))\,.$$

Let us fix w. Take v_1 according to the first assertion above. An application of Lemma 2.15 and bounded collection $\mathsf{B}\Sigma_1^0$ to the second assertion above yields v_2 such that

$$\forall \vec{B}\, \forall x \leq v_1\, \forall \vec{A}\ (F(x,\vec{A},\vec{B}) \wedge H(w,\vec{A}) \to \exists y \leq v_2\, G(y,\vec{A}))\,.$$

It is now clear that $\forall \vec{A}\ (H(w,\vec{A}) \to \exists y \leq \max(v_1,v_2)\, G(y,\vec{A}))$ follows as wanted. □

4 Further Results

There is no improvement in Harrington's conservation theorem; we show below that WKL_0 is already not Π_2^1-conservative over RCA_0. This is an immediate consequence of the fact that weak König's lemma is not a theorem of RCA_0.

Proposition 4.1. *Weak König's lemma is not provable in* RCA_0.

Proof. We show that König's lemma is false in the standard model $R = \{\omega, \mathscr{R}, \mathrm{S}, +, \cdot, 0, <\}$ of RCA_0, where $\mathscr{R}$ is the set of all recursive subsets of ω. Let A and B be a disjoint pair of recursively inseparable, recursively enumerable subsets of ω. Further let f and g be recursive functions such that $\mathrm{im}(f) = A$ and $\mathrm{im}(g) = B$. Since A and B are recursively inseparable, if follows that for any recursive function $h \in 2^\omega$ we have either $h(n) = 0$ for some $n \in A$ or $h(n) = 1$ for some $n \in B$. Let

$$T := \{s \in 2^{<\omega} \,:\, \forall m, n < \mathrm{lg}(s)\ ((f(m) = n \to s_n = 1) \wedge (g(m) = n) \to s_n = 0)\}\,.$$

Obviously $T \in \mathscr{R}$ and T is an infinite binary tree. Moreover $h \in 2^\omega$ is a path through T if and only if $h(n) = 1$ for all $n \in A$ and $h(n) = 0$ for all $n \in B$. Thus T has no recursive path. $\square$

In the following we will present further conservation results where we consider also first order theories. Therefore we introduce a first order language.

Definition 4.2. *Let* L_1 *denote the language of first order arithmetic which contains the following symbols:*

(i) countably many free number variables $a_0, a_1, \ldots,$

(ii) countably many bound number variables $x_0, x_1, \ldots,$

(iii) the function symbols are defined inductively by :

   (a) 0 is a 0-ary function symbol (i.e. a constant) and S *is a unary function symbol,*

   (b) for all natural numbers n,m and k, where $0 \le k \le n$, Cs^n_m *and* Pr^n_k *are n-ary function symbols,*

   (c) if f *is an m-ary function symbol and* $g_1, \ldots, g_m$ *are n-ary function symbols, then the composition* $\mathrm{Comp}^n(f, g_1, \ldots, g_m)$ *is an n-ary function symbol,*

   (d) if f is an n-ary function symbol and g an $(n+2)$*-ary function symbol, then* $\mathrm{Rec}^{n+1}(f, g)$ *is an* $(n+1)$*-ary function symbol.*

(iv) two binary relation symbols $=$ *and* $<$,

(v) the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$ *and* $\neg$.

Terms, Formulas, Models, Semantics, Hilbert calculi, Sequent calculi, Arithmetical Hierarchies etc. for L_1 can be obtained as for L_2 by obvious modifications. Next we formulate two important L_1-theories.

Definition 4.3. *The* L_1*-theory of primitive recursive arithmetic* PRA *is* Q *enriched with the five axioms from Definition 2.10 and with the induction scheme*

$$F(0) \wedge \forall x\, (F(x) \to F(x+1)) \to \forall x\, F(x)$$

for all quantifier free L_1*-formulas* F. *If we take the induction scheme where* F *runs over all* Σ^0_1*-formulas of* L_1 *we get the theory* $\mathsf{I\Sigma}_1$.

The first order part of RCA_0 is exactly $\mathsf{I}\Sigma_1$. This circumstance is formulated in the next proposition.

Proposition 4.4. *The theory* RCA_0 *is conservative over* $\mathsf{I}\Sigma_1$*, i.e. any* L_1*-formula F which is provable in* RCA_0 *is already provable in* $\mathsf{I}\Sigma_1$.

Proof. Let F be a formula which is not valid in $\mathsf{I}\Sigma_1$. According to the completeness theorem we find a first order model $M = \{|M|, \mathrm{S}_M, +_M, \cdot_M, 0_M, <_M\}$ of $\mathsf{I}\Sigma_1$ in which F is false. We aim to extend M to a second order model M^* of RCA_0 by declaring a set $\mathscr{S}_{M^*}$. We let $\mathscr{S}_{M^*}$ be the set of all $S \subseteq |M|$ such that there exists a Σ^0_1-formula $G(a,\vec{b}) \in \mathrm{L}_1$ and a Π^0_1-formula $H(a,\vec{c}) \in \mathrm{L}_1$ with free variables as shown, such that

$$S = \{x \,:\, M \vDash G(x,\vec{m})\} = \{x \,:\, M \vDash H(x,\vec{n})\}$$

for some parameters $\vec{m},\vec{n} \in |M|$. Then we set $M^* = \{|M|, \mathscr{S}_{M^*}, \mathrm{S}_M, +_M, \cdot_M, 0_M, <_M\}$. Obviously Q is true in M^*. We next show that M^* satisfies Δ^0_1-comprehension and Σ^0_1-induction. Let K be an arithmetical L_2-formula with parameters from $|M| \cup \mathscr{S}_{M^*}$. We construct a L_1-translation K_1 of K. First K is logically equivalent to a formula K' with the same parameters such that the negation $\neg$ occurs exclusively in front of atomic formulas. The translation K_1 is obtained by substituting every occurrence of $x \in S$ in K' by $G(x,\vec{m})$ and every occurrence of $x \notin S$ in K' by $\neg H(x,\vec{n})$, where G resp. H is a Σ^0_1 resp. Π^0_1 formula of L_1 which defines S. Clearly K and K_1 are equivalent in M^* and the translation of a Σ^0_1 resp. Π^0_1-formula is again a Σ^0_1 resp. a Π^0_1-formula (modulo logical equivalence). This implies that M^* proves Δ^0_1-comprehension and Σ^0_1-induction. Hence $M^* \vDash \mathsf{RCA}_0$.
As F is a first order formula with $M \vDash \neg F$ we have also $M^* \vDash \neg F$. Thus F is not provable in RCA_0. ☐

The next theorem is due to Parsons. It's proof can be found in the appendix.

Theorem 4.5 (Parsons)**.** *The theory* $\mathsf{I}\Sigma_1$ *is* Π^0_2*-conservative over* PRA.

If we combine Harrington's theorem, Parsons' theorem and Proposition 4.4 we finally conclude the following corollary.

Corollary 4.6. *The theory* WKL_0 *is* Π^0_2*-conservative over* PRA.

A Partial cut elimination

The aim of this paragraph is to give a proof of partial cut elimination. In order to simplify the notation we make some changes in our sequent calculi. The main difference is that we interpret from now on a *sequence* $\Gamma \longrightarrow \Delta$ no longer as an ordered pair of two finite sequences of formulas Γ and Δ, but as an ordered pair of two finite sets of formulas Γ and Δ. This gives us the exchange rules and the contraction rules for free. The next change is the notion of an *initial sequent*.

Definition A.1. *An initial sequent is either $\Gamma \longrightarrow \Delta, F$, where F is an axiom of $\mathscr{Q}$ or a sequent of the form $\Gamma \longrightarrow \Delta$, where $\Gamma \cap \Delta$ is nonempty.*

Now we can formulate the definition of the new sequent calculi.

Definition A.2. *The sequent calculi $\mathsf{LK}^*_{\mathsf{Q}}$, $\mathsf{LK}^*_{\mathsf{RCA}_0^-}$ and $\mathsf{LK}^*_{\mathsf{WKL}_0}$ are obtained from the calculi LK_{Q}, $\mathsf{LK}_{\mathsf{RCA}_0^-}$ and $\mathsf{LK}_{\mathsf{WKL}_0}$ by interpreting sequences and initial sequences as described above and by omitting the weak structural rules. Note that an auxiliary formula on the right (resp. left) side of an upper sequent in an inference rule may occur additionally as a side formula on the right (resp. left) side of this sequent. We write $\mathscr{T}$ to mean one of the calculi $\mathsf{LK}^*_{\mathsf{Q}}$, $\mathsf{LK}^*_{\mathsf{RCA}_0^-}$ or $\mathsf{LK}^*_{\mathsf{WKL}_0}$.*

These modifications are certainly equivalent to the versions without *. This is justified as weakening is admissible (see Lemma A.4). The next step is to assign to each L_2-formula a rank.

Definition A.3. *The rank $|F|$ of a formula F is defined as 0 if F is either bounded or Σ^0_1 and otherwise defined inductively by*

(i) $|\neg F| = |F| + 1$,

(ii) $|F \wedge G| = |F \vee G| = \max(|F|, |G|) + 1$,

(iii) $|\forall x\, F| = |\exists x\, F| = |\forall x \leq t\, F| = |\exists x \leq t\, F| = |\forall X\, F| = |\exists X\, F| = |F| + 1$.

We write $\mathscr{T} \vdash^n \Gamma \longrightarrow \Delta$ if there exists a $\mathscr{T}$-proof of $\Gamma \longrightarrow \Delta$ with depth less or equal n. Furthermore we write $\mathscr{T} \vdash_r \Gamma \longrightarrow \Delta$ if there exists a $\mathscr{T}$-proof of $\Gamma \longrightarrow \Delta$ where every cut formula has rank less than r. Sometimes we will combine these two notations. Clearly we have the following lemma.

Lemma A.4 (Weakening)**.** *If $\mathscr{T} \vdash^n_r \Gamma \longrightarrow \Delta$ then also $\mathscr{T} \vdash^n_r \Gamma, \Gamma' \longrightarrow \Delta, \Delta'$.*

The idea is now that given a proof of a sequent we try to modify this into a proof of the same sequent where every cut formula has rank 0. We aim to prove the following theorem.

Theorem A.5 (Partial Cut Elimination)**.** *If $\mathscr{T} \vdash \Gamma \longrightarrow \Delta$ then also $\mathscr{T} \vdash_1 \Gamma \longrightarrow \Delta$.*

This theorem is a direct consequence of the following lemma.

Lemma A.6. *Assume that $\mathscr{T} \vdash^m_r \Gamma, F \longrightarrow \Delta$ and $\mathscr{T} \vdash^n_r \Lambda \longrightarrow \Pi, F$ where F is a formula with $0 < |F| \leq r$. Then we have $\mathscr{T} \vdash^{2(m+n)}_r \Gamma, \Lambda \longrightarrow \Delta, \Pi$.*

Proof of Theorem A.5. Assume that $\mathscr{T} \vdash^n_{r+2} \Gamma \longrightarrow \Delta$. A straightforward induction on n shows that $\mathscr{T} \vdash^{4^n}_{r+1} \Gamma \longrightarrow \Delta$. If we apply this procedure $r+1$ times we finally get $\mathscr{T} \vdash_1 \Gamma \longrightarrow \Delta$. □

Proof of Lemma A.6. The proof is by induction on $m+n$. We use the abbreviations $\mathscr{R} :\equiv \Gamma, F \longrightarrow \Delta$ and $\mathscr{S} :\equiv \Lambda \longrightarrow \Pi, F$. If $m = 0$ then $\mathscr{R}$ is an initial sequent, i.e. $\Gamma \cap \Delta \neq \emptyset$, $\mathscr{Q} \cap \Delta \neq \emptyset$ or $F \in \Delta$. In the first two cases $\Gamma, \Lambda \longrightarrow \Delta, \Pi$ is again an initial sequent. Otherwise $\Gamma, \Lambda \longrightarrow \Delta, \Pi$ equals $\Gamma, \Lambda \longrightarrow \Delta, \Pi, F$ and so the conclusion of the induction step follows directly from Lemma A.4. If $n = 0$ then the argumentation is similar (note that all elements of $\mathscr{Q}$ have rank 0, so $F \notin \mathscr{Q}$). So let $m, n > 0$. Now assume that $\mathscr{R}$ is the consequence of an inference $\mathscr{I}$ with upper sequent(s) $\Gamma', F \longrightarrow \Delta'$ (and $\Gamma'', F \longrightarrow \Delta''$) such that F is not a principal formula of this inference. According to the induction hypothesis we have $\mathscr{T} \vdash_r^{2((m-1)+n)} \Gamma', \Lambda \longrightarrow \Delta', \Pi$ (and $\mathscr{T} \vdash_r^{2((m-1)+n)} \Gamma'', \Lambda \longrightarrow \Delta'', \Pi$). After applying $\mathscr{I}$ we get the desired result $\mathscr{T} \vdash_r^{2(m+n)-1} \Gamma, \Lambda \longrightarrow \Delta, \Pi$. If $\mathscr{S}$ is the consequence of an inference rule such that F is not a principal formula of this inference, the argumentation is similar. Hence we can assume that both $\mathscr{R}$ and $\mathscr{S}$ are consequences of inference rules with principal formula F. To prove the theorem we distinguish between the possible inference rules which can be applied to get $\mathscr{R}$. Without loss of generality we can assume $F \notin \Gamma \cup \Pi$.
($\neg$left). The formula F equals $\neg G$. This means that $\mathscr{S}$ is the consequence of a ($\neg$right) rule. Thus we have

$$\frac{\Gamma' \longrightarrow \Delta, G}{\Gamma', \neg G \longrightarrow \Delta} \quad \text{and} \quad \frac{\Lambda, G \longrightarrow \Pi'}{\Lambda \longrightarrow \Pi', \neg G},$$

such that $\Gamma, \neg G = \Gamma', \neg G$ and $\Pi, \neg G = \Pi', \neg G$. By Lemma A.4 we can assume that F is included in both Γ' and Π', that is $\Gamma' = \Gamma, \neg G$ and $\Pi' = \Pi, \neg G$. Applying cross-cuts, the induction hypothesis and weakening imply $\mathscr{T} \vdash_r^{2(m-1)+n)} \Gamma, \Lambda \longrightarrow \Delta, \Pi, G$ and $\mathscr{T} \vdash_r^{2(m+(n-1))} \Gamma, \Lambda, G \longrightarrow \Delta, \Pi$. As G has rank less than r we can apply a cut inference and we get $\mathscr{T} \vdash_r^{2(m+n)-1} \Gamma, \Lambda, \longrightarrow \Delta, \Pi$.
($\wedge$left). We find formulas G, H such that F equals $G \wedge H$. The sequence $\mathscr{S}$ has to be the consequence of of a ($\wedge$right) rule. This means that we have

$$\frac{\Gamma', G, H \longrightarrow \Delta}{\Gamma', G \wedge H \longrightarrow \Delta} \quad \text{and} \quad \frac{\Lambda \longrightarrow \Pi', G \quad \Lambda \longrightarrow \Pi', H}{\Lambda \longrightarrow \Pi', G \wedge H},$$

where $\Gamma, G \wedge H = \Gamma', G \wedge H$ and $\Pi, G \wedge H = \Pi', G \wedge H$. As before we can assume $\Gamma' = \Gamma, G \wedge H$ and $\Pi' = \Pi, G \wedge H$. Again we apply cross-cuts and with the hypothesis and Lemma A.4 we get

$$\mathscr{T} \vdash_r^{2((m-1)+n)} \Gamma, \Lambda, G, H \longrightarrow \Delta, \Pi \quad , \quad \mathscr{T} \vdash_r^{2(m+(n-1))} \Gamma, \Lambda, H \longrightarrow \Delta, \Pi, G$$
$$\text{and} \quad \mathscr{T} \vdash_r^{2(m+(n-1))} \Gamma, \Lambda \longrightarrow \Delta, \Pi, H\,.$$

If we first apply a cut with cut formula G to the first two sequents, and then apply again the cut rule to this result and the third sequent above we finally get $\mathscr{T} \vdash_r^{2(m+n)} \Gamma, \Lambda \longrightarrow \Delta, \Pi$.
($\vee$left). This is proved very similar.
($\forall$left). In this case F equals $\forall x\, G(x)$ and so $\mathscr{S}$ is the result of a ($\forall$right) inference. That is we have

$$\frac{\Gamma', G(t) \longrightarrow \Delta}{\Gamma', \forall x\, G(x) \longrightarrow \Delta} \quad \text{and} \quad \frac{\Lambda \longrightarrow \Pi', G(a)}{\Lambda \longrightarrow \Pi', \forall x\, G(x)},$$

where $\Gamma, \forall x\, G(x) = \Gamma', \forall x\, G(x)$ and $\Pi, \forall x\, G(x) = \Pi', \forall x\, G(x)$. Again we can assume $\Gamma' = \Gamma, \forall x\, G(x)$ and $\Pi' = \Pi, \forall x\, G(x)$. We have according to our assumption that $\mathscr{T} \vdash_r^{n-1} \Lambda \longrightarrow \Pi', G(a)$. If we modify the proof of this sequence by replacing, if necessary, some free variables that occur as eigenvariables and by substituting t for a we get $\mathscr{T} \vdash_r^{n-1} \Lambda \longrightarrow \Pi', G(t)$. Now we can applying weakening and cross-cuts to get

$$\mathscr{T} \vdash_r^{2((m-1)+n)} \Gamma, \Lambda, G(t) \longrightarrow \Delta, \Pi \quad \text{and} \quad \mathscr{T} \vdash_r^{2(m+(n-1))} \Gamma, \Lambda \longrightarrow \Delta, \Pi, G(t)\,.$$

Using one application of (Cut) we finally get $\mathscr{T} \vdash_r^{2(m+n)-1} \Gamma, \Lambda \longrightarrow \Delta, \Pi$.
($\exists$left). This time $F \equiv \exists x\, G$ and $\mathscr{S}$ is the result of a ($\exists$right) rule. This case is proved similar to the last case. Note that since F has a positive rank even if $\mathscr{T}$ equals $\mathsf{LK}^*_{\mathsf{RCA}_0^-}$ or $\mathsf{LK}^*_{\mathsf{WKL}_0}$ it's not possible that $\mathscr{S}$ is the result of a (Ind) or a (Fan_0) rule.
($\forall^{\leq}$left). We have that F is $\forall x \leq s\, G(x)$. The case $F \equiv t \leq s$ is impossible because F has a rank different from 0. This means that $\mathscr{S}$ is the result of a ($\forall^{\leq}$right) rule, i.e.

$$\frac{\Gamma', G(t) \longrightarrow \Delta}{\Gamma', t \leq s, \forall x \leq s\, G(x) \longrightarrow \Delta} \quad \text{and} \quad \frac{\Lambda, a \leq s \longrightarrow \Pi', G(a)}{\Lambda \longrightarrow \Pi', \forall x \leq s\, G(x)}\,,$$

where $\Gamma, \forall x \leq s\, G(x) = \Gamma', t \leq s, \forall x \leq s\, G(x)$ and $\Pi, \forall x \leq s\, G(x) = \Pi', \forall x \leq s\, G(x)$. As before we can assume $\Gamma' = \Gamma, t \leq s, \forall x \leq s\, G(x)$ and $\Pi' = \Pi, \forall x \leq s\, G(x)$. Similar to the unbounded quantifier case we can substitute t for a. We apply cross-cuts and get

$$\mathscr{T} \vdash_r^{2((m-1)+n)} \Gamma, \Lambda, t \leq s, G(t) \longrightarrow \Delta, \Pi \quad \text{and} \quad \mathscr{T} \vdash_r^{2(m+(n-1))} \Gamma, \Lambda, t \leq s, \longrightarrow \Delta, \Pi, G(t)\,.$$

Considering $t \leq s \in \Gamma$ and using one more cut-rule with cut-formula $G(t)$ we get the desired result

$$\mathscr{T} \vdash_r^{2(m+n)-1} \Gamma, \Lambda \longrightarrow \Delta, \Pi\,.$$

($\exists^{\leq}$left). This is proved very similar.
The second order quantifier rules* are proved similar to their first order counterparts. The essential point is here that if we replace in a formula an abstract corresponding to a bounded formula for a free variable, the rank of the formula does not change. □

B Parsons' Theorem

The aim of this paragraph is to show the proof of Parsons' theorem which can be found in [5]. We work exclusively with first order logic. The first step is Herbrand's theorem stated below. Remember that a universal theory is a theory consisting of universal formulas.

Theorem B.1 (Herbrand). *Let $\mathscr{U}$ be a universal theory in the (countable) first-order language* L.

(i) Suppose $\exists \vec{x}\, F(\vec{x}, \vec{a})$ is a consequence of $\mathscr{U}$, where F is a quantifier-free formula with it's variables as shown. Then there are terms $\vec{t}_1(\vec{a}), \ldots, \vec{t}_k(\vec{a})$ with at most the variables $\vec{a}$ such that

$$\mathscr{U} \vDash F(\vec{t}_1(\vec{a}), \vec{a}) \vee \cdots \vee F(\vec{t}_k(\vec{a}), \vec{a})\,.$$

(ii) Suppose $\exists \vec{x}\, \forall \vec{y}\, F(\vec{x}, \vec{y}, \vec{a})$ is a consequence of $\mathscr{U}$, where F is an existential formula, with its free variables as shown. Then there are terms $\vec{t}_1(\vec{a}), \vec{t}_2(\vec{a}, \vec{y}_1), \ldots, \vec{t}_k(\vec{a}, \vec{y}_1, \ldots, \vec{y}_{k-1})$ with it's variables among the ones shown such that

$$\mathscr{U} \vDash F(\vec{t}_1(\vec{a}), \vec{y}_1, \vec{a}) \vee F(\vec{t}_2(\vec{a}, \vec{y}_1), \vec{y}_2, \vec{a}) \vee \cdots \vee F(\vec{t}_k(\vec{a}, \vec{y}_1, \ldots, \vec{y}_{k-1}), \vec{y}_k, \vec{a})\,.$$

Proof. In order to make the proof more readable we consider only single variables x, y and a rather than vectors $\vec{x}, \vec{y}$ and $\vec{a}$.
(i). This is a particular case of (ii): just insert two dummy quantifiers and substitute the y_i's by the variable a in the term.
(ii). Assume that no conjunction as in (ii) is a consequence of the theory $\mathscr{U}$. Let a_0, a_1, a_2 be a list of the (free) variables and fix an enumeration $t_1, t_2, t_3, \ldots$ of all terms of the language such that the variables of $t_i(a_0, \ldots a_{i-1})$ are among $a_0, \ldots, a_{i-1}$. Consider the set

$$\mathscr{U} \cup \{\neg F(t_1(c), d_1, c), \neg F(t_2(c, d_1), d_2, c), \ldots, \neg F(t_i(c, d_1, \ldots, d_{i-1}), d_i, c), \ldots\}$$

of $\mathrm{L}(c, d_1, d_2, \ldots)$-formulas, where $c, d_1, d_2, \ldots$ are new constants. It follows from our assumption that this set is finitely satisfiable. By compactness, it has a $\mathrm{L}(a, d_1, d_2, \ldots)$-model $\mathscr{M}$. Let us consider the following subset of the domain of $\mathscr{M}$:

$$M^* := \{t_1^{\mathscr{M}}(c), t_2^{\mathscr{M}}(c, d_1), \ldots, t_i^{\mathscr{M}}(c, d_1, \ldots, d_{i-1}), \ldots\}.$$

Note that all elements $c^{\mathscr{M}}, d_1^{\mathscr{M}}, d_2^{\mathscr{M}}, \ldots$ are members of M^* because the variables a_i appear in the enumeration of terms. It is also clear that M^* defines a substructure $\mathscr{M}^*$ of $\mathscr{M}$. Using the fact that $\mathscr{U}$ is a universal theory, $\mathscr{M}^*$ is a model of $\mathscr{U}$. But

$$\mathscr{M}^* \vDash \forall x\, \exists y\, \neg F(x, y, c).$$

In fact, for $x = t_i(c, d_1, \ldots, d_{i-1})$ take $y = d_i$ and use the fact, that $\neg F$ is a universal formula, and therefore, downward absolute between $\mathscr{M}$ and $\mathscr{M}^*$. If we interpret $\mathscr{M}^*$ as a L-model this gives a contradiction. $\square$

We will apply this theorem to the L_1-theory PRA. This is allowed, since PRA may be interpreted as a universal theory. As PRA admits definition by cases, in part 1 of the above theorem we may simply take $k = 1$. We are now ready to prove Parsons' theorem.

Theorem B.2 (Parsons)**.** *The theory* $\mathsf{I}\Sigma_1$ *is* Π_2^0*-conservative over* PRA.

Proof. Suppose that the Π_2^0-sentence $\forall u\, \exists v\, F(u, v)$ is a consequence of $\mathsf{I}\Sigma_1$, where F is a bounded formula. Without loss of generality we can assume that F is quantifier free since by using primitive recursive functions, bounded quantifiers can be eliminated (for example if G has no quantifiers then $\exists x \le t\, G(x)$ is in PRA equivalent to $G(f(t))$, where $f(x) = \mu y \le x\, G(y)$). By compactness, the given Π_2^0-sentence is a consequence of PRA and finitely many instances of the Σ_1^0-induction scheme. If G is a L_1-formula we write Ind_G for the universal closure of an induction instance with induction formula G. Clearly $\mathrm{Ind}_J \to \mathrm{Ind}_G \wedge \mathrm{Ind}_H$ is derivable, where $J(i, a) :\equiv (i = 0 \to G(a)) \wedge (i \neq 0 \to H(a))$. This means, that the finitely many instances of Σ_1^0-induction can be subsumed by a single instance, i.e. together with the deduction theorem we have

$$\mathsf{PRA} \vDash \mathrm{Ind}_G \to \forall u\, \exists v\, F(u, v),$$

where Ind_G can be identified with

$$\forall c\, \forall z\, (G(c, 0) \wedge \forall x\, (G(c, x) \to G(c, x+1)) \to G(c, z)),$$

for a certain Σ^0_1-formula $G(c,x) :\equiv \exists y\, H(c,x,y)$, H quantifier-free (it is all right to consider only a single parameter c because PRA has a pairing function). We now put the sentence $\mathrm{Ind}_G \to \forall u\, \exists v\, F(u,v)$ in prenex normal form and obtain

$$\mathsf{PRA} \vDash \exists v, c, z, y_0\, \forall x, y, w\, \exists y'\, (F(u,v) \vee J(c,z,y_0,x,y,w,y'))\,, \tag{3}$$

where $J(c,z,y_0,x,y,w,y')$ is the quantifier-free formula

$$H(c,0,y_0) \wedge (H(c,x,y) \to H(c,x+1,y')) \wedge \neg H(c,z,w)\,.$$

Lemma B.3. *Let $t(\vec{p})$, $s(\vec{p})$, $r(\vec{p})$ and $q(\vec{p},x,y,w)$ be terms of* L_1*, with variables as shown. Then*

$$\mathsf{PRA} \vDash \forall \vec{p}\, \exists x, y, w\, \neg J(t(\vec{p}), s(\vec{p}), r(\vec{p}), x, y, w, q(\vec{p},x,y,w))\,.$$

Proof. We reason inside PRA. In order to get a contradiction, suppose that there is $\vec{p}$ such that $\forall x, y, w\, J(t(\vec{p}), s(\vec{p}), r(\vec{p}), x, y, w, q(\vec{p},x,y,w))$. We get

(i) $H(t(\vec{p}), 0, r(\vec{p}))$,

(ii) $\forall x, y, w\, (H(t(\vec{p}), x, y) \to H(t(\vec{p}), x+1, q(\vec{p},x,y,w)))$, and

(iii) $\forall w \neg H(t(\vec{p}), s(\vec{p}), w)$.

Define h by primitive recursion according to the following clauses:

$$\begin{aligned} h(0,\vec{p}) &= r(\vec{p})\,, \\ h(x+1,\vec{p}) &= q(\vec{p}, x, h(x,\vec{p}), 0)\,. \end{aligned}$$

By (i), (ii) and bounded-induction, it follows that $\forall x\, H(t(\vec{p}), x, h(x,\vec{p}))$. In particular we have $\exists w\, H(t(\vec{p}), s(\vec{p}), w)$. This is a contradiction to (iii). $\square$

Now we apply the second part of Herbrand's theorem to (3). That is we find terms $r_1(u), \vec{t}_1(u)$, $r_2(u,\vec{z}_1)$, $\vec{t}_2(u,\vec{z}_1), \ldots, r_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1})$, $\vec{t}_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1})$ such that the disjunction of the following formulas is a consequence of PRA:

$$\begin{gathered} F(u, r_1(u)) \vee \exists y'\, J(\vec{t}_1(u), \vec{z}_1, y') \\ F(u, r_2(u,\vec{z}_1)) \vee \exists y'\, J(\vec{t}_2(u,\vec{z}_1), \vec{z}_2, y') \\ \vdots \\ F(u, r_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1})) \vee \exists y'\, J(\vec{t}_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1}), \vec{z}_k, y')\,, \end{gathered}$$

where each $\vec{z}_i$ abbreviates a triple of variables and each $\vec{t}_i$ abbreviates a triple of terms. Hence , the disjunction of $\exists v\, F(u,v)$ with the formula

$$\exists y'\, J(\vec{t}_1(u), \vec{z}_1, y') \vee \exists y'\, J(\vec{t}_2(u,\vec{z}_1), \vec{z}_2, y') \vee \cdots \vee \exists y'\, J(\vec{t}_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1}), \vec{z}_k, y')\,,$$

is a consequence of PRA. By the first part of Herbrand's theorem applied to this disjunction, there is a term $q(u,\vec{z}_1,\ldots,\vec{z}_k)$ such that the last clause in this disjunction can be substituted by

$$J(\vec{t}_k(u,\vec{z}_1,\ldots,\vec{z}_{k-1}), \vec{z}_k, q(u,\vec{z}_1,\ldots,\vec{z}_k))\,.$$

By the above lemma

$$\exists \vec{z}_k \, \neg J(\vec{t}_k(u, \vec{z}_1, \ldots, \vec{z}_{k-1}), \vec{z}_k, q(u, \vec{z}_1, \ldots, \vec{z}_k)) \, .$$

is valid in PRA. Therefore the disjunction of $\exists v \, F(u, v)$ with the formula

$$\exists y' \, J(\vec{t}_1(u), \vec{z}_1, y') \vee \exists y' \, J(\vec{t}_2(u, \vec{z}_1), \vec{z}_2, y') \vee \cdots \vee \exists y' \, J(\vec{t}_{k-1}(u, \vec{z}_1, \ldots, \vec{z}_{k-2}), \vec{z}_{k-1}, y') \, ,$$

is also a consequence of PRA. If we repeat the previous argument $k-1$ times we eventually conclude that $\mathsf{PRA} \models \exists v \, F(u, v)$. $\square$

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