Σ^1_1 choice in a theory of sets and classes

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Dedicated to Wolfram Pohlers on his retirement

Abstract

Several decades ago Friedman showed that the subsystem Σ^1_1-AC of second order arithmetic is proof-theoretically equivalent – and thus equiconsistent – to $(\Pi^1_0\text{-CA})_{<\varepsilon_0}$. In this article we prove the analogous result for Σ^1_1 choice in the context of the von Neumann-Bernays-Gödel theory NBG of sets and classes.

Keywords: Proof theory, theories of sets and classes.

1 Introduction

Several decades ago Friedman showed that the subsystem Σ^1_1-AC of second order arithmetic is proof-theoretically equivalent – and thus equiconsistent – to $(\Pi^1_0\text{-CA})_{<\varepsilon_0}$ (cf. Friedman [7]). Later Feferman [2, 3], Tait [16], Feferman and Sieg [6] and Cantini [1] reproved and extended this result, always making use of different proof-theoretic techniques.

In this article we start off from the von Neumann-Bernays-Gödel theory NBG of sets and classes, extend it by the schema $(\mathcal{L}_2\text{-I}_\in)$ of $\in$-induction for arbitrary formulas of the language $\mathcal{L}_2$ of NBG and study the effect of adding Σ^1_1 choice and Σ^1_1 collection,

$$(\Sigma^1_1\text{-AC}) \qquad \forall x \exists Y A[x,Y] \;\rightarrow\; \exists Z \forall x A[x,(Z)_x],$$

$$(\Sigma^1_1\text{-Col}) \qquad \forall x \exists Y A[x,Y] \;\rightarrow\; \exists Z \forall x \exists y A[x,(Z)_y],$$

where A is an elementary formula of $\mathcal{L}_2$, i.e. an $\mathcal{L}_2$ formula which does not contain bound class variables. We will show that the resulting theories are

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equiconsistent to the system $\mathsf{NBG}_{<E_0}$ which is obtained from $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in)$ by adding iterations of elementary comprehension along all initial segments of the notation system $(E_0, \lhd)$. E_0 is an elementarily definable class and $\lhd$ an elementary binary class relation on E_0 which, provably in NBG, well-orders all initial segments of E_0. The notation system $(E_0, \lhd)$ may be seen as the analogue of $(\varepsilon_0, <)$ with the ordinal ω replaced by the collection of all ordinals. In this sense, our result is the perfect analogue of Friedman's result mentioned above with natural numbers and sets of natural numbers replaced by sets and classes, respectively.

Our characterization of $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ is also interesting in connection with Feferman's operational set theory OST, introduced in Feferman [4, 5]. As shown in Jäger [11], the extension $\mathsf{OST}(\mathbf{E}, \mathbb{P})$ of OST with unbounded existential quantification and power set is equiconsistent to $\mathsf{NBG}_{<E_0}$ and therefore, in view of the results of this paper, also to the more familiar system $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$. The results of this paper are discussed from a broader perspective in Jäger [12].

The embedding of $\mathsf{NBG}_{<E_0}$ into $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ is straightforward. The difficult part of this paper is the reduction of $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ to $\mathsf{NBG}_{<E_0}$, and here an asymmetric interpretation plays a major rôle. Similar forms of asymmetric interpretations have been used, for example, in Cantini [1] to deal with subsystems of second order arithmetic and in Jäger [9, 10, 11] and Jäger and Strahm [13] in the context of theories of admissible sets, explicit mathematics and operational set theory.

First we observe that $(\Sigma^1_1\text{-}\mathsf{AC})$ can be replaced by $(\Sigma^1_1\text{-}\mathsf{Col})$. Then, in order to get rid of $(\mathcal{L}_2\text{-}\mathsf{I}_\in)$, we develop (within $\mathsf{NBG}_{<E_0}$) an infinitary sequent style reformulation G^∞ of $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{Col})$ in which constants for all sets are available. By making use of an infinitary rule for universal quantification over sets, we show

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col}) \vdash A \quad\Longrightarrow\quad \mathsf{NBG}_{<E_0} \vdash \text{“}\mathsf{G}^\infty \text{ proves } A\text{”}.$$

A next step is to strengthen this assertion by a partial cut elimination argument for G^∞ to

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col}) \vdash A \quad\Longrightarrow$$
$$\mathsf{NBG}_{<E_0} \vdash \text{“}\mathsf{G}^\infty \text{ proves } A \text{ with simple cuts”}.$$

Now the technical part begins: we have to go back from provability in G^∞ to provability in $\mathsf{NBG}_{<E_0}$. This is achieved in two further steps:

(i) Introduction of a sort of constructible hierarchy of classes and a truth definition based on this hierarchy which reflects all closed elementary formulas A,

$$\mathsf{NBG}_{<E_0} \vdash Tr[A] \leftrightarrow A.$$

(ii) An asymmetric interpretation of a suitable fragment of G^∞ with respect to this hierarchy such that, for all closed elementary formulas A of G^∞,

$$\mathsf{NBG}_{<E_0} \vdash (\text{“}\mathsf{G}^\infty \text{ proves } A \text{ with simple cuts”} \rightarrow Tr[A]).$$

Altogether, we thus have

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col}) \vdash A \quad \Longrightarrow \quad \mathsf{NBG}_{<E_0} \vdash A$$

for all closed elementary formulas, and this is the required reduction. The definitions of our analogue of the constructible hierarchy and the associated notion of truth – although conceptually clear – require some care since everything has to be carried through within the restricted framework of $\mathsf{NBG}_{<E_0}$.

2 Von Neuman-Bernays-Gödel set theory

The von Neumann-Bernays-Gödel set theory NBG is a theory of sets and classes conservative over the system ZFC of Zermelo-Fraenkel set theory with the axiom of choice. NBG is known to be finitely axiomatizable although the version we are going to present below permits axiom schemas and as such is an infinite axiomatization.

Let $\mathcal{L}_1$ be a typical first order language of set theory with countably many set variables $a, b, c, f, g, u, v, w, x, y, z, \ldots$ and a single symbol for the element relation, but without function or individual constants.

$\mathcal{L}_2$, the language of NBG, augments $\mathcal{L}_1$ by a second sort of countably many variables $U, V, W, X, Y, Z, \ldots$ for classes; its *formulas* $(A, B, C, \ldots)$ are inductively generated as follows:

1. If a, b are set variables and if U is a class variable, then all expressions of the form $(a \in b)$ and $(a \in U)$ are (atomic) formulas of $\mathcal{L}_2$.

2. If A and B are formulas of $\mathcal{L}_2$, then so are are $\neg A$, $(A \vee B)$ and $(A \wedge B)$.

3. If A is a formula of $\mathcal{L}_2$, then $\exists x A$, $\forall x A$, $\exists X A$ and $\forall X A$ are formulas of $\mathcal{L}_2$.

The denotations for set variables, class variables and $\mathcal{L}_2$ formulas may be used with and without subscripts. Since we will be working within classical logic, the remaining logical connectives can be defined as usual.

We will often omit parentheses and brackets whenever there is no danger of confusion. Moreover, we frequently make use of the vector notation $\vec{a}$ as shorthand for a finite string $a_1, \ldots, a_n$ of set variables whose length is not important or evident from the context.

Equalities between sets/sets, sets/classes, classes/sets and classes/classes are not atomic formulas of $\mathcal{L}_2$ but defined as

$$(\mathit{Var}_1 = \mathit{Var}_2) \ := \ \forall x(x \in \mathit{Var}_1 \ \leftrightarrow \ x \in \mathit{Var}_2)$$

where Var_1 and Var_2 denote set or class variables. A formula of $\mathcal{L}_2$ is called *elementary* if it does not contain bound class variables; free class variables, however, are permitted. The Σ^1_1 formulas of $\mathcal{L}_2$ are those of the form $\exists X A$ with elementary A. Finally, an $\mathcal{L}_2$ formula A is called Σ^1 if all positive occurrences of class quantifiers are existential and all negative occurrences of class quantifiers are universal; it is called Π^1 if all positive occurrences of class quantifiers are universal and all negative occurrences of class quantifiers are existential. By a closed formula we mean one which does not contain free set or class variables.

The logic of NBG is classical two-sorted logic with equality for the first sort. The non-logical axioms of NBG are given in six groups. To increase readability, we freely use standard set-theoretic terminology.

I. Elementary comprehension For any elementary formula $A[u]$ of $\mathcal{L}_2$ and any class variable X not free in $A[u]$:

(ECA) $$\exists X \forall y(y \in X \ \leftrightarrow \ A[y]).$$

Hence every elementary NBG formula $A[u]$ defines a class, which is typically written as $\{x : A[x]\}$. It may be (extensionally equal to) a set, but this is not necessarily the case.

II. Basic set existence

(Pair) $$\forall x \forall y \exists z(z = \{x, y\}),$$

(Union) $$\forall x \exists y(y = \cup x),$$

(Power set) $$\forall x \exists y \forall z(z \in y \leftrightarrow z \subset x),$$

(Infinity) $$\exists x(\emptyset \in x \ \wedge \ (\forall y \in x)(y \cup \{y\} \in x)).$$

In the following we write $\langle a, b\rangle$ for the ordered pair of the sets a and b à la Kuratowski. Class relations are classes which consist of ordered pairs only, and class functions are class relations which are right unique; i.e. for all U we set:

$$\begin{aligned} Rel[U] &:= (\forall x \in U)\exists y \exists z(x = \langle y, z\rangle), \\ Dom[U] &:= \{x : \exists y(\langle x, y\rangle \in U)\}, \\ Fun[U] &:= Rel[U] \ \wedge\ \forall x \forall y \forall z(\langle x, y\rangle \in U \ \wedge\ \langle x, z\rangle \in U \ \rightarrow\ y = z). \end{aligned}$$

If U is a function and x an element of $Dom[U]$, we write $U(x)$ for the unique y such that $\langle x, y\rangle \in U$. Replacement states that the range of a set under a function is a set.

III. Replacement For any class variable U:

(REP) $$Fun[U] \ \rightarrow\ \forall x \exists y(y = \{U(z) : z \in Dom[U] \cap x\}).$$

Global choice is a very uniform principle of choice which claims the existence of a class function which picks an element of any non-empty set.

IV. Global choice

(GC) $$\exists X(Fun[X] \ \wedge\ Dom[X] = \{y : y \neq \emptyset\} \ \wedge\ \forall y(y \neq \emptyset \ \rightarrow\ X(y) \in y)).$$

To complete the list of axioms of NBG, we add foundation. In NBG it is claimed that the element relation is well-founded with respect to classes.

V. Class foundation For any class variable U:

($\mathsf{C\text{-}I}_\in$) $$U \neq \emptyset \ \rightarrow\ (\exists x \in U)(\forall y \in x)(y \notin U).$$

A set a is called an *ordinal* if a itself and all its elements are transitive, On stands for the class of all ordinals; i.e.

$$On \ := \ \{x : Tran(x) \wedge (\forall y \in x)\, Tran(y)\}.$$

The axioms (Infinity) and ($\mathsf{C\text{-}I}_\in$) imply that there exists a least infinite ordinal, which we denote by ω, as usual. The elements of ω are identified with the natural numbers in the sense that $0 := \emptyset,\ 1 := \{0\},\ 2 := 1 \cup \{1\}$ and so on. In the following small Greek letters are supposed to range over On.

One important property of NBG is the subset property: the intersection of a set a with a class is a subset of a. Its proof is standard.

There exist various alternative presentations of NBG. So it is an appealing feature of NBG that the schema of elementary comprehension can be replaced by finitely many axioms and thus a finite axiomatization of NBG is possible. Furthermore, according to a well-known result, see, e.g., Levy [14], NBG is a conservative extension of ZFC.

Theorem 1 *A sentence of the language $\mathcal{L}_1$ is provable in* NBG *if and only if it is provable in* ZFC.

In the following we will be mainly concerned with extensions of NBG. The first of those consists in adding to NBG the schema of $\in$-induction for arbitrary $\mathcal{L}_2$ formulas $A[u]$,

$$(\mathcal{L}_2\text{-}\mathsf{I}_\in) \qquad \forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x].$$

Further interesting principles are the schemas of Σ^1_1 choice and Σ^1_1 collection which consists of all formulas

$$(\Sigma^1_1\text{-}\mathsf{AC}) \qquad \forall x \exists Y A[x, Y] \rightarrow \exists Z \forall x A[x, (Z)_x],$$

$$(\Sigma^1_1\text{-}\mathsf{Col}) \qquad \forall x \exists Y A[x, Y] \rightarrow \exists Z \forall x \exists y A[x, (Z)_y]$$

where $A[u, V]$ is an elementary $\mathcal{L}_2$ formula and $(Z)_a$ is the class given by

$$(Z)_a := \{x : \langle a, x\rangle \in Z\}.$$

Clearly, every instance of $(\Sigma^1_1\text{-}\mathsf{Col})$ follows from $(\Sigma^1_1\text{-}\mathsf{AC})$. However, in NBG also the converse is the case.

Theorem 2 *If $A[u, V]$ is an elementary $\mathcal{L}_2$ formula, then we have*

$$\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{Col}) \vdash \forall x \exists Y A[x, Y] \rightarrow \exists Z \forall x A[x, (Z)_x].$$

Proof. We work within $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{Col})$. Following the pattern of the usual proof of the well-ordering theorem in ZFC and exploiting the fact that we have global choice, it is easy to show that there exist a bijective class function W from On to the collection of all sets. We write W^{-1} for the inverse of W.

Now suppose $\forall x \exists Y A[x, Y]$, where $A[u, V]$ is an elementary $\mathcal{L}_2$ formula. Then by $(\Sigma^1_1\text{-}\mathsf{Col})$ there exists a class Z such that

$$(\star) \qquad \forall x \exists y A[x, (Z)_y].$$

Now the function W^{-1} comes into play in order to associate to any x a unique y for which $A[x, (Z)_y]$. Namely, by elementary comprehension and $(\star)$

$$Sel := \{\langle x, y\rangle : A[x, (Z)_y] \wedge \forall z(A[x, (Z)_z] \rightarrow W^{-1}(y) \leq W^{-1}(z))\}$$

is a class function whose domain is the collection of all sets. Finally, if we write S for the class $\{\langle x, y\rangle : y \in (Z)_{Sel(x)}\}$, which exists by elementary comprehension, we have $(S)_x = (Z)_{Sel(x)}$ for all sets x. Hence $\forall x A[x, (S)_x]$. In other words, S is the required witness for $(\Sigma^1_1\text{-}\mathsf{AC})$. □

Corollary 3 *The theories* $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{AC})$ *and* $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{Col})$ *prove the same formulas.*

In this paper we are interested in the consistency strength of the theories $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ and $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col})$. The much simpler analysis of $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{AC})$ and $\mathsf{NBG} + (\Sigma^1_1\text{-}\mathsf{Col})$ will be presented elsewhere.

3 The notation system $(E_0, \lhd)$

In this section we work within $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in)$ and set up the notation system $(E_0, \lhd)$. The underlying idea is very simple: $(E_0, \lhd)$ is designed to be the analogue of $(\varepsilon_0, <)$ with the set of the natural numbers, i.e. the ordinal ω, replaced by the class of all ordinals. All we have to do is to follow one of the standard introductions of the ordinal notation system up to ε_0 as, for example, in Schütte [15], taking care of the few additional complications arising by the fact that we now have all elements of On as basic entities.

Definition 4 *By* finite sequences *we mean those functions whose domain is a finite ordinal; FS is defined to be the class of all finite sequences,*

$$FS \;:=\; \{f : Fun[f] \wedge (\exists n < \omega)(Dom[f] = n)\}.$$

If we are given n sets $a_0, \dots, a_{n-1}$ for some natural number n, we often write $(a_0, \dots, a_{n-1})$ for that element f of FS which satisfies $Dom[f] = n$ and $(\forall i < n)(f(i) = a_i)$.

By elementary comprehension it can be easily shown in NBG that there exists a binary class relation $\lhd$ on FS satisfying the property (I) below. To simplify the formulation of this property, we abbreviate:

$$a \lhd b \;:=\; \langle a, b\rangle \in \lhd \qquad \text{and} \qquad a \unlhd b \;:=\; a \lhd b \vee a = b.$$

In addition, let $\lhd_{lex}$ be the lexicographic extension of $\lhd$; i.e. if a and b are finite sequences of sets, then $a \lhd_{lex} b$ is written for

$$(Dom[a] < Dom[b] \;\wedge\; (\forall i < Dom[a])(a(i) = b(i)) \;\vee$$
$$(\exists i < Dom[a])(i < Dom[b] \;\wedge\; a(i) \lhd b(i) \;\wedge\; (\forall j < i)(a(j) = b(j)).$$

(I) The binary relation $\lhd$ on FS. For all elements a and b of FS we have $a \lhd b$ if and only if $Dom[a]$ and $Dom[b]$ are at least 2 and one of the following cases holds:

(1) $a(0) = b(0) = 0 \ \wedge \ a(1) < b(1)$,

(2) $a(0) = 0 \ \wedge \ 0 < b(0)$,

(3) $a(0) = 1 \ \wedge \ 2 \leq b(0)$,

(4) $a(0) = b(0) = 2 \ \wedge \ a(1) \lhd b(1)$,

(5) $a(0) = 2 \ \wedge \ b(0) = 3 \ \wedge \ a \unlhd b(1)$,

(6) $a(0) = 3 \ \wedge \ b(0) = 2 \ \wedge \ a(1) \lhd b$,

(7) $a(0) = b(0) = 3 \ \wedge \ a \lhd_{lex} b$.

For the time being, this is a rather weird binary relation on finite sequences. Its real meaning will become transparent when restricted to the subclass E_0 of FS which is introduced in (III) and whose definition is based on $\lhd$.

For every ordinal α we let $\overline{\alpha}$ be the finite sequence $(0, \alpha)$. In addition, Ω is defined to be the finite sequence $(1, 0)$.

(II) The ω-exponentiation of elements of FS. There exists a class function $\widetilde{\omega}$ which is described by $Dom[\widetilde{\omega}] = FS$ and, for all elements a of FS,

$$\widetilde{\omega}(a) \ = \ \begin{cases} \overline{\omega^\alpha} & \text{if } \ a = \overline{\alpha} \ \text{ for some ordinal } \alpha, \\ \Omega & \text{if } \ a = \Omega, \\ (2, a) & \text{otherwise.} \end{cases}$$

In the following, the function $\widetilde{\omega}$ will be interesting four us only when restricted to those finite sequences which act as notations. They are collected in the class E_0 which can be defined be elementary comprehension and is characterized as follows.

(III) The class E_0 of notations. E_0 is defined to be the smallest subclass of FS which satisfies the following closure properties:

(1) For all ordinals α we have $\overline{\alpha} \in E_0$.

(2) $\Omega \in E_0$.

(3) If $a \in E_0$, then $\widetilde{\omega}(a) \in E_0$.

(4) If $a_0, \ldots, a_{n+1} \in E_0$ and $\Omega \unlhd a_{n+1} \unlhd \ldots \unlhd a_1 \unlhd a_0$, then

$$(3, \widetilde{\omega}(a_0), \widetilde{\omega}(a_1), \ldots, \widetilde{\omega}(a_{n+1})) \in E_0.$$

(5) If $a_0, \ldots, a_n \in E_0$ and $\Omega \trianglelefteq a_n \trianglelefteq \ldots \trianglelefteq a_1 \trianglelefteq a_0$ and $\alpha \neq 0$, then

$$(3, \widetilde{\omega}(a_0), \widetilde{\omega}(a_1), \ldots, \widetilde{\omega}(a_n), \overline{\alpha}) \in E_0.$$

The elements of E_0 of the form $(0, a)$ code the ordinals, the element $(1, 0) = \Omega$ is the least element greater than the codes of all ordinals, $(2, a)$ codes the ω-exponentiation of a and $(3, a_0, \ldots, a_{n-1})$ is for the sum of ω-powers and possibly the code of an ordinal, given in decreasing order. The proof of the following lemma is without any problems.

Lemma 5 *The relation $\triangleleft$ is a strict linear ordering on the class E_0.*

In the following we use the small Gothic type letters $\mathfrak{a}, \mathfrak{b}, \ldots$ (possibly with subscripts) for elements of E_0. Expressions like $\exists\mathfrak{a}(\ldots)$ and $\forall\mathfrak{a}(\ldots)$ are then to be read as $(\exists a \in E_0)(\ldots)$ and $(\forall a \in E_0)(\ldots)$, respectively. For simplicity of notation, we also write $\omega^{\mathfrak{a}}$ instead of $\widetilde{\omega}(\mathfrak{a})$.

Definition 6 *For all positive natural numbers n and all $\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1} \in E_0$ we set*

$$[\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}] \;:=\; \begin{cases} \mathfrak{a}_0 & \textit{if } \; n = 1 \wedge (\mathfrak{a}_0 \trianglelefteq \Omega \vee \exists\mathfrak{b}(\mathfrak{a}_0 = \omega^{\mathfrak{b}})), \\ (3, \mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}) & \textit{if } \; (3, \mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}) \in E_0. \end{cases}$$

In all other cases $[\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}]$ may be taken to be undefined or to have the value $\emptyset$.

So every element $\mathfrak{a}$ of E_0 can be uniquely written as $[\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}]$. This representation is useful for a compact description of the addition of ordinal terms. Once more, it can be introduced as a binary class function by elementary comprehension and is characterized by the following properties.

(IV) Addition of elements of E_0. For all $\mathfrak{a}$ and $\mathfrak{b}$ we have:

(1) If $\mathfrak{a} = \overline{0}$, then $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$, if $\mathfrak{b} = \overline{0}$, then $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$.

(2) If $\mathfrak{a} = [\mathfrak{a}_0, \ldots, \mathfrak{a}_{m-1}, \overline{\alpha}]$ and $\mathfrak{b} = \overline{\beta}$ for some ordinals α and β greater than 0, then
$$\mathfrak{a} + \mathfrak{b} \;=\; [\mathfrak{a}_0, \ldots, \mathfrak{a}_{m-1}, \overline{\alpha + \beta}].$$

(3) If $\mathfrak{a} = [\mathfrak{a}_0, \ldots, \mathfrak{a}_{m-1}]$ such that $\Omega \trianglelefteq \mathfrak{a}_{m-1}$ and $\mathfrak{b} = \overline{\beta}$ for some ordinal β greater than 0, then

$$\mathfrak{a} + \mathfrak{b} \;=\; [\mathfrak{a}_0, \ldots, \mathfrak{a}_{m-1}, \mathfrak{b}].$$

(4) If $\mathfrak{a} = [\mathfrak{a}_0, \dots, \mathfrak{a}_{m-1}]$ and $\mathfrak{b} = [\mathfrak{b}_0, \dots, \mathfrak{b}_{n-1}]$ such that $\Omega \trianglelefteq \mathfrak{b}_0$, then, if k is the largest natural number i for which $\mathfrak{b}_0 \trianglelefteq \mathfrak{a}_i$,

$$\mathfrak{a} + \mathfrak{b} \ = \ [\mathfrak{a}_0, \dots, \mathfrak{a}_k, \mathfrak{b}_0, \dots, \mathfrak{b}_{n-1}].$$

Before turning to the well-ordering of initial parts of E_0, a further class function, describing the finite addition of ω-powers of elements of E_0, has to be introduced.

(V) The function $\widehat{\omega}$ on elements of E_0 and finite numbers. There exists a class function $\widehat{\omega}$ which is described by $Dom[\widetilde{\omega}] = E_0 \times \omega$ and, for all $\mathfrak{a}$ and all $n < \omega$,

$$\widehat{\omega}(\mathfrak{a}, n) \ = \ \begin{cases} 0 & \text{if } \ n = 0, \\ \widehat{\omega}(\mathfrak{a}, n-1) + \omega^{\mathfrak{a}} & \text{if } \ 0 < n < \omega. \end{cases}$$

We omit the proof of the following lemma since it is in complete analogy to the case of the notation system for $(\varepsilon_0, <)$.

Lemma 7 *The following assertions can be proved in* NBG*:*

1. $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} \ = \ \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$.
2. $\mathfrak{a} \lhd \mathfrak{b} + \omega^{\mathfrak{c}} \wedge \overline{0} \lhd \mathfrak{c} \ \rightarrow \ (\exists \mathfrak{d} \lhd \mathfrak{c})(\exists n < \omega)(\mathfrak{a} \lhd \mathfrak{b} + \widehat{\omega}(\mathfrak{d}, n))$.

Starting with $\Omega + 1$ a sequence of terms which is cofinal in E_0 is obtained by simply iterating ω-exponentiation.

Definition 8 *For all natural numbers n, the ordinal terms Ω_n are inductively defined by*

$$\Omega_0 \ := \ \Omega + 1 \qquad \textit{and} \qquad \Omega_{n+1} \ := \ \omega^{\Omega_n}.$$

The purpose of the next paragraphs is to show that $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_{\in})$ proves the well-ordering of the relation $\lhd$ on E_0 up to each term Ω_k for k being any standard natural number. To do so, we need the following notations.

Definition 9 *Let $A[u]$ be an arbitrary formula of the language $\mathcal{L}_2$ of* NBG*. Then we set:*

$$\begin{aligned} Prog_{\lhd}[A] \ &:\Longleftrightarrow \ \forall \mathfrak{u}((\forall \mathfrak{v} \lhd \mathfrak{u})A[\mathfrak{v}] \ \rightarrow \ A[\mathfrak{u}]), \\ TI_{\lhd}[\mathfrak{u}, A] \ &:\Longleftrightarrow \ Prog_{\lhd}[A] \rightarrow (\forall \mathfrak{v} \lhd \mathfrak{u})A[\mathfrak{v}]. \\ A^*[\mathfrak{u}] \ &:\Longleftrightarrow \ \forall \mathfrak{v}((\forall \mathfrak{w} \lhd \mathfrak{v})A[\mathfrak{w}] \ \rightarrow \ (\forall \mathfrak{w} \lhd \mathfrak{v} + \omega^{\mathfrak{u}})A[\mathfrak{w}]). \end{aligned}$$

The first two of these formulas express, as usual, the progressiveness of A with respect to $\lhd$ and transfinite induction for A along $\lhd$ up to $\mathfrak{u}$, respectively; A^* is the *jump* of A. The core of the well-ordering proofs up to Ω_k, for any standard natural number k, is provided by the following two properties of the jump-operation.

Lemma 10 *For any formula $A[u]$ of the language $\mathcal{L}_2$, we can prove in* NBG*:*

1. $Prog_{\lhd}[A] \rightarrow Prog_{\lhd}[A^*]$.
2. $TI_{\lhd}[\mathfrak{u}, A^*] \rightarrow TI_{\lhd}[\omega^{\mathfrak{u}}, A]$.

All our notations are chosen such that the proof of this lemma can be taken literally from the proof of the corresponding lemma for notations less than ε_0 in Schütte [15].

Theorem 11 *For any standard natural number k and for any formula $A[u]$ of the language $\mathcal{L}_2$ we have*

$$\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_{\in}) \vdash TI_{\lhd}[\Omega_k, A].$$

Proof. We work informally in $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_{\in})$ and prove this theorem by metainduction on k. Assume that $k = 0$. Then $\Omega_k = \Omega + 1$ and $\in$-induction on the ordinals yields, for arbitrary $\mathcal{L}_2$ formulas $A[u]$,

$$Prog_{\lhd}[A] \rightarrow (\forall \mathfrak{u} \lhd \Omega) A[\mathfrak{u}].$$

By the definition of progressiveness, this implies

$$Prog_{\lhd}[A] \rightarrow (\forall \mathfrak{u} \lhd \Omega + 1) A[\mathfrak{u}],$$

i.e. $TI_{\lhd}[\Omega_0, A]$. For $k > 0$ we have in view of the induction hypothesis for any $\mathcal{L}_2$ formulas $A[u]$ that $TI_{\lhd}[\Omega_{k-1}, A^*]$. Now we simply have to apply Lemma 10 in order to obtain $TI_{\lhd}[\Omega_k, A]$. $\square$

In connection with the notation system $(E_0, \lhd)$ it only remains to introduce a few further notations which will be taken up again towards the end of Section 5.

Definition 12 *The classes of limit notations and strong limit notations are defined by*

$$Lim := \{x \in E_0 : x \neq \overline{0} \wedge (\forall y \in E_0)(x \neq y + \overline{1}\},$$

$$SLim := \{x \in Lim : (\forall y \in E_0)(x \neq y + \overline{\omega}\}.$$

In addition, we define $Lim_0 := \{\overline{0}\} \cup Lim$ and $SLim_0 := \{\overline{0}\} \cup SLim$ and, for any $U \subset E_0$ and $\mathfrak{a}, \mathfrak{b} \in E_0$,

$$\mathfrak{a} \in U \cap \mathfrak{b} := \mathfrak{a} \in U \wedge \mathfrak{a} \lhd \mathfrak{b}.$$

This means that the elements of Lim are the analogues of limit ordinals and the elements of $SLim$ correspond to those limit ordinals which cannot be obtained by adding ω. Clearly, any Ω_n belongs to $SLim$.

4 Elementary hierarchies

This section begins with introducing the theory $\mathsf{NBG}_{<E_0}$ which permits the iteration of elementary comprehension up to any Ω_k with k a standard natural number. It is easily verified afterwards that $\mathsf{NBG}_{<E_0}$ is contained in the system $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$.

Definition 13 *Let $A[U, V, u, v]$ be an elementary $\mathcal{L}_2$ formula with at most the variables U, V, u, v free. Then we write $Hier_A[\mathfrak{a}, U, V]$ for the elementary $\mathcal{L}_2$ formula*

$$(\forall \mathfrak{b} \lhd \mathfrak{a})((V)_\mathfrak{b} = \{x : A[U, \Sigma(V, \mathfrak{b}), x, \mathfrak{b}]\}).$$

Here $\Sigma(V, \mathfrak{b})$ stands for the class $\{\langle x, \mathfrak{c}\rangle \in V : \mathfrak{c} \lhd \mathfrak{b}\}$ representing the disjoint union of the projections of V up to $\mathfrak{b}$.

$\mathsf{NBG}_{<E_0}$ is the theory of sets and classes which extends $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in)$ by claiming the existence of such hierarchies along each initial segment of E_0. Hence the axioms of $\mathsf{NBG}_{<E_0}$ comprise the axioms of NBG, the schema $(\mathcal{L}_2\text{-}\mathsf{I}_\in)$ plus

$$(\mathsf{It\text{-}ECA}) \qquad \forall X \exists Y\, Hier_A[\Omega_n, X, Y]$$

for arbitrary elementary $\mathcal{L}_2$ formulas $A[U, V, u, v]$ with at most the variables U, V, u, v free and all standard natural numbers n.

Employing $(\Sigma^1_1\text{-}\mathsf{AC})$, the following lemma is proved by transfinite induction along $\lhd$ up to Ω_n, which is available in $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in)$ according to Theorem 11. The argument is very similar to that of second order arithmetic, establishing that $\Pi^0_1\text{-}\mathsf{CA}_{<\varepsilon_0}$ is a subsystem of $\Sigma^1_1\text{-}\mathsf{AC}$, and left to the reader.

Lemma 14 *Let $A[u, v, U, V]$ be an elementary $\mathcal{L}_2$ formula with at most the variables u, v, U, V free. For all standard natural numbers n and all class variables X, the theory $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ then proves*

$$(\forall \mathfrak{a} \lhd \Omega_n) \exists Y\, Hier_A[\mathfrak{a}, X, Y].$$

From this lemma we conclude that all axioms $(\mathsf{It\text{-}ECA})$ are provable in the system $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$. Therefore, the embedding of $\mathsf{NBG}_{<E_0}$ into $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ is an immediate consequence.

Theorem 15 *The theory* $\mathsf{NBG}_{<E_0}$ *is contained in* $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$*; i.e. for all* $\mathcal{L}_2$ *formulas* A *we have*

$$\mathsf{NBG}_{<E_0} \vdash A \quad \Longrightarrow \quad \mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC}) \vdash A.$$

For the reduction of $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col})$ to $\mathsf{NBG}_{<E_0}$ it is convenient to have a global well-ordering of the set-theoretic universe at our disposal. Therefore, let $\mathcal{L}_\mathcal{W}$ be the extension of $\mathcal{L}_2$ by a fresh binary relation symbol $\mathcal{W}$ and include formulas $\mathcal{W}(u,v)$ into the list of atomic formulas. Then the global well-ordering axiom states

$$\text{(GWO)} \qquad \forall x \exists! \alpha \mathcal{W}(x,\alpha) \ \land \ \forall x \forall y \forall \alpha (\mathcal{W}(x,\alpha) \land \mathcal{W}(y,\alpha) \ \rightarrow \ x = y).$$

We write NBGW for the theory NBG – now all schemas formulated for $\mathcal{L}_\mathcal{W}$ formulas – in which the axiom of global choice (GC) has been replaced by the axiom global well-ordering (GWO). Accordingly, $\mathsf{NBGW}_{<E_0}$ is the theory $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in)$ extended by the iteration axiom ($\mathsf{It\text{-}ECA}$), now formulated for all elementary $\mathcal{L}_\mathcal{W}$ formulas.

It goes without saying that NBG and $\mathsf{NBG}_{<E_0}$ are contained in NBGW and $\mathsf{NBGW}_{<E_0}$, respectively. Moreover, with little effort and by making use of standard techniques it can even be shown that we have the following theorem.

Theorem 16 NBGW *is a conservative extension of* NBG *and* $\mathsf{NBGW}_{<E_0}$ *is a conservative extension of* $\mathsf{NBG}_{<E_0}$*, in both cases with respect to all* $\mathcal{L}_2$ *formulas.*

5 Reducing $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ to $\mathsf{NBG}_{<E_0}$

The eventual aim of this section is to show that $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{AC})$ can be reduced to $\mathsf{NBG}_{<E_0}$. In order to achieve this it is sufficient – in view of what we have achieved so far – to reduce the theory $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col})$ to $\mathsf{NBGW}_{<E_0}$, where in this context $(\Sigma^1_1\text{-}\mathsf{Col})$ is for $\mathcal{L}_\mathcal{W}$ formulas.

In the following we develop, within $\mathsf{NBGW}_{<E_0}$, an infinitary sequent calculus G^∞ for $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma^1_1\text{-}\mathsf{Col})$. For this purpose we code the set variables as pairs $\langle 0, n \rangle$ and the class variables as pairs $\langle 1, n \rangle$, n always a natural number. Moreover, to any set a we assign the set constant $\langle 2, a \rangle$. For natural numbers n and sets a we set

$$h_n := \langle 0, n \rangle, \qquad H_n := \langle 1, n \rangle, \qquad p_a := \langle 2, a \rangle.$$

We also fix several elementary class functions defined, for arbitrary sets a, b, c, by (some are written in infix or another mnemonically suitable notation):

$$(a \mathrel{\dot{\in}} b) \ := \ \langle 3, a, b \rangle, \qquad\qquad \dot{\mathcal{W}}(a,b) \ := \ \langle 4, a, b \rangle,$$

$$\dot{\neg}\, a := \langle 5, a\rangle, \qquad (a \mathbin{\dot{\vee}} b) := \langle 6, a, b\rangle,$$

$$(a \mathbin{\dot{\wedge}} b) := \langle 7, a, b\rangle, \qquad \dot{\exists}\, a\, b := \langle 8, a, b\rangle,$$

$$\dot{\forall}\, a\, b := \langle 9, a, b\rangle.$$

We proceed with our development of G^∞ within $\mathsf{NBGW}_{<E_0}$ and present all formulas of G^∞ as sets, mimicking the build up of the formulas of $\mathcal{L}_\mathcal{W}$.

Definition 17 *The class For^∞ is defined to be the smallest class which satisfies the following closure properties:*

(1) For all natural numbers m, n and all sets a, b the class For^∞ contains

$$(h_m \mathbin{\dot{\in}} h_n), \quad (h_m \mathbin{\dot{\in}} p_a), \quad (p_a \mathbin{\dot{\in}} h_m), \quad (p_a \mathbin{\dot{\in}} p_b).$$

(2) For all natural numbers m, n and all sets a, the class For^∞ contains

$$(h_m \mathbin{\dot{\in}} H_n), \quad (p_a \mathbin{\dot{\in}} H_n).$$

(3) For all natural numbers m, n, all sets a, b, the class For^∞ contains

$$\dot{\mathcal{W}}(h_m, h_n), \quad \dot{\mathcal{W}}(h_m, p_a), \quad \dot{\mathcal{W}}(p_a, h_m), \quad \dot{\mathcal{W}}(p_a, p_b).$$

(4) For all $x, y \in For^\infty$, the class For^∞ also contains

$$\dot{\neg}\, x, \quad (x \mathbin{\dot{\vee}} y), \quad (x \mathbin{\dot{\wedge}} y).$$

(5) For all $x \in For^\infty$ and all natural numbers n, the class For^∞ also contains

$$\dot{\exists}\, h_n\, x, \quad \dot{\forall}\, h_n\, x, \quad \dot{\exists}\, H_n\, x, \quad \dot{\forall}\, H_n\, x.$$

This definition could be reformulated as an explicit elementary formula, for the prize of being less perspicuous. We are not going to work out the details, only formulate the corresponding assertion.

Lemma 18 *For^∞ is an elementarily definable class of $\mathsf{NBGW}_{<E_0}$.*

Clearly, for any sets a and b, $(a \mathbin{\dot{\rightarrow}} b)$ stands for $(\dot{\neg}\, a \mathbin{\dot{\vee}} b)$ and $(a \mathbin{\dot{\leftrightarrow}} b)$ for $((a \mathbin{\dot{\rightarrow}} b) \mathbin{\dot{\wedge}} (b \mathbin{\dot{\rightarrow}} a))$; other abbreviations of this sort are used as expected.

It is also elementarily decidable whether a set or class variable occurs freely (in the usual sense) within an element of For^∞. Moreover, there is an elementary class function *Sub* taking care of all sorts of simultaneous substitutions of free occurrences of set and class variables within an element of For^{∞} by constants and variables of the appropriate sort. For instance, given a $\varphi \in For^{\infty}$, a set a and $i_1, i_2, j, m, n < \omega$,

$$Sub(\langle p_a, h_m, H_n \rangle, \langle h_{i_1}, h_{i_2}, H_j \rangle, \varphi)$$

is the element of For^{∞} obtained from φ by simultaneously replacing all free occurrences of h_{i_1}, h_{i_2} and H_j by p_a, h_m and H_n, respectively. Also, if φ is given in the form $\psi[h_{i_1}, h_{i_2}, H_j]$, we often simply write $\psi[p_a, h_m, H_n]$ instead of $Sub(\langle p_a, h_m, H_n \rangle, \langle h_{i_1}, h_{i_2}, H_j \rangle, \varphi)$.

The previous definition is so that Gödel numbers, all belonging to For^{∞}, can be canonically assigned to the formulas of $\mathcal{L}_{\mathcal{W}}$. For this purpose we begin with fixing an mapping $\natural$ which assigns natural numbers to all set and class variables, making sure that different variables are mapped onto different natural numbers.

If u, v are set variables and if U is a class variable of $\mathcal{L}_{\mathcal{W}}$, we define

$$\ulcorner (u \in v) \urcorner := (h_{\natural(u)} \,\dot{\in}\, h_{\natural(v)}), \qquad \ulcorner (u \in U) \urcorner := (h_{\natural(u)} \,\dot{\in}\, H_{\natural(U)}),$$

$$\ulcorner \mathcal{W}(u, v) \urcorner := \dot{\mathcal{W}}(h_{\natural(u)}, h_{\natural(v)}).$$

The Gödel numbers of the non-atomic formulas of $\mathcal{L}_{\mathcal{W}}$ are inductively calculated in compliance with the equations

$$\begin{aligned}
\ulcorner \neg A \urcorner &:= \dot{\neg} \ulcorner A \urcorner, \\
\ulcorner (A \vee B) \urcorner &:= (\ulcorner A \urcorner \,\dot{\vee}\, \ulcorner B \urcorner), \\
\ulcorner (A \wedge B) \urcorner &:= (\ulcorner A \urcorner \,\dot{\wedge}\, \ulcorner B \urcorner), \\
\ulcorner \exists x A \urcorner &:= \dot{\exists}\, h_{\natural(x)} \ulcorner A \urcorner, \\
\ulcorner \forall x A \urcorner &:= \dot{\forall}\, h_{\natural(x)} \ulcorner A \urcorner, \\
\ulcorner \exists X A \urcorner &:= \dot{\exists}\, H_{\natural(X)} \ulcorner A \urcorner, \\
\ulcorner \forall X A \urcorner &:= \dot{\forall}\, H_{\natural(X)} \ulcorner A \urcorner.
\end{aligned}$$

The elements of For^{∞} are called $\mathcal{L}_{\mathcal{W}}^{\infty}$ formulas and will be denoted by the small Greek letters θ, φ, χ and ψ (possibly with subscripts). To increase the readability we often omit the dots when it is clear from the context that we speak about elements of For^{∞}.

The *set-closed* formulas are those $\mathcal{L}_{\mathcal{W}}^{\infty}$ formulas which do not contain free set variables (but they may contain free class variables and set constants); the closed formulas of $\mathcal{L}_{\mathcal{W}}^{\infty}$ are those $\mathcal{L}_{\mathcal{W}}^{\infty}$ formulas which contain neither free set variables nor free class variables. We collect the set-closed formulas in the class SC^{∞} and the closed formulas of $\mathcal{L}_{\mathcal{W}}^{\infty}$ in the class $CFor^{\infty}$; both classes are elementarily definable.

The capital Greek letters $\Theta, \Phi, \Psi, \ldots$ (possibly with subscripts) denote finite sequences of set-closed formulas. If Φ is the sequence of set-closed formulas $\varphi_1, \ldots, \varphi_m$ and Ψ the sequence of set-closed formulas $\psi_1, \ldots, \psi_n$, then

$$\langle 12, m, n, \varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \rangle$$

is the sequent with antecedent Φ and succedent Ψ; typically, it will be written as $(\Phi \supset \Psi)$ or simply as $\Phi \supset \Psi$.

The elementary, Σ_1^1, Σ^1 and Π^1 formulas of $\mathcal{L}_{\mathcal{W}}^{\infty}$ are defined analogously to the corresponding classes of $\mathcal{L}_{\mathcal{W}}$ formulas; set constants are now, of course, permitted as parameters.

Looking at the basic set existence and replacement axioms and at the global well-ordering axiom (GWO) of NBGW, we can convince ourselves that the corresponding axioms, formulated within the language $\mathcal{L}_{\mathcal{W}}^{\infty}$, are elementary $\mathcal{L}_{\mathcal{W}}^{\infty}$ formulas. We collect the resulting set-closed formulas in the class AX^{∞}.

Definition 19 *The* degree $dg(\varphi)$ *of a set-closed formula* φ *is inductively defined as follows:*

1. *If* φ *is a set-closed elementary or* Σ_1^1 *formula of* $\mathcal{L}_{\mathcal{W}}^{\infty}$*, then* $dg(\varphi) := 0$.

2. *For all set-closed formulas which are neither elementary nor* Σ_1^1 *we set*

$$\begin{aligned}
dg(\neg\psi) &:= dg(\psi) + 1, \\
dg(\psi_1 \vee \psi_2) &:= \max(dg(\psi_1), dg(\psi_2)) + 1, \\
dg(\psi_1 \wedge \psi_2) &:= \max(dg(\psi_1), dg(\psi_2)) + 1, \\
dg(\exists h_n \psi[h_n]) &:= dg(\psi[p_\emptyset]) + 1, \\
dg(\forall h_n \psi[h_n]) &:= dg(\psi[p_\emptyset]) + 1, \\
dg(\exists H_n \psi[H_n]) &:= dg(\psi[H_n]) + 1, \\
dg(\forall H_n \psi[H_n]) &:= dg(\psi[H_n]) + 1.
\end{aligned}$$

G^{∞} is an extension of the classical Gentzen sequent calculus LK (cf., e.g., Girard [8] or Takeuti [17]) by additional axioms and rules of inference which take care of the non-logical axioms of NBGW. Universal set quantification in the succedent and the corresponding existential set quantification in the antecedent are infinitary rules branching over the collection of all sets. The axioms and rules of G^∞ can be grouped as follows.

I. Axioms. For all set-closed elementary formulas φ, all elements ψ of AX^∞, all sets a, b, all set-closed elementary formulas $\theta[p_\emptyset]$ and all H_m, h_n so that no variable conflicts arise:

(A1) $\varphi \supset \varphi$,

(A2) $\supset \psi$,

(A3) $\supset (p_a \in p_b)$ if $a \in b$,

(A4) $\supset (p_a \notin p_b)$ if $a \notin b$,

(A5) $\supset \exists H_m \forall h_n (h_n \in H_m \leftrightarrow \theta[h_n])$.

II. Structural rules. The structural rules of G^∞ consist of the usual weakening, exchange and contraction rules.

III. Propositional rules. The propositional rules of G^∞ consist of the usual rules for introducing the propositional connectives on the left and right hand sides of sequents.

IV. Quantifier rules for sets. Formulated only for succedents; there are also corresponding rules for the antecedents. For all set variables h_n, all set constants p_a and all set-closed formulas $\varphi[p_\emptyset]$:

$$\frac{\Phi \ \supset \ \Psi,\, \varphi[p_a]}{\Phi \ \supset \ \Psi,\, \exists h_n \varphi[h_n]}\ , \quad \frac{\Phi \ \supset \ \Psi,\, \varphi[p_b] \quad \text{for all sets b}}{\Phi \ \supset \ \Psi,\, \forall h_n \varphi[h_n]}\ .$$

V. Quantifier rules for classes. Formulated only for succedents; there are also corresponding rules for the antecedents. By $(\star)$ we mark those rules where the designated free class variables are not to occur in the conclusion. For all set-closed formulas $\varphi[H_0]$and all class variables H_m, H_n so that no variable conflicts arise:

$$\frac{\Phi \ \supset \ \Psi,\, \varphi[H_m]}{\Phi \ \supset \ \Psi,\, \exists H_n \varphi[H_n]}\ , \quad \frac{\Phi \ \supset \ \Psi,\, \varphi[H_m]}{\Phi \ \supset \ \Psi,\, \forall H_n \varphi[H_n]}\ (\star).$$

VI. Σ^1_1 collection rules. For all set-closed elementary formulas $\varphi[p_\emptyset, H_0]$ and all variables h_m, H_n, H_k so that no variable conflicts arise:

$$\frac{\Phi \;\supset\; \Psi,\, \forall h_m \exists H_n \varphi[h_m, H_n]}{\Phi \;\supset\; \Psi,\, \exists H_i \forall h_m \exists h_n \varphi[h_m, (H_i)_{h_n}]}\,.$$

VII. Cuts. For all set-closed formulas φ:

$$\frac{\Phi \;\supset\; \Psi, \varphi \qquad \Phi, \varphi \;\supset\; \Psi}{\Phi \;\supset\; \Psi}\,.$$

The formula φ is called the cut formula of this cut; the degree of a cut is the degree of its cut formula.

Since G^∞ has inference rules which branch over all sets, namely the rules for introducing universal quantification over sets in the succedents and existential quantification over sets in the antecedents, infinite proof trees may occur. We confine ourselves to those whose depths are bounded by initial segments of E_0.

Definition 20 *Let k be an arbitrary standard natural number. For any notation $\mathfrak{a} \lhd \Omega_k$, any $n < \omega$ and any sequent $\Phi \supset \Psi$, we define $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_n \Phi \supset \Psi$ by induction on $\mathfrak{a}$.*

1. *If $\Phi \supset \Psi$ is an axiom of G^∞, then we have $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_n \Phi \supset \Psi$ for all $n < \omega$.*

2. *If $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}_x}_n \Phi_x \supset \Psi_x$ and $\mathfrak{a}_x \lhd \mathfrak{a}$ for every premise of a rule which is not a cut, then we have $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_n \Phi \supset \Psi$ for the conclusion $\Phi \supset \Psi$ of this rule.*

3. *If $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}_i}_n \Phi_i \supset \Psi_i$ and $\mathfrak{a}_i \lhd \mathfrak{a}$ for the two premises $\Phi_i \supset \Psi_i$ of a cut $(i = 1, 2)$ whose degree is less than n, then we have $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_n \Phi \supset \Psi$ for the conclusion $\Phi \supset \Psi$ of this cut.*

To be precise, given a standard natural number k, we employ axiom (It-ECA) to introduce a class U such that, for any $\mathfrak{a} \lhd \Omega_k$, the projection $(U)_{\mathfrak{a}}$ consists of all pairs $(\Phi \supset \Psi, n)$ for which we have $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_n \Phi \supset \Psi$.

$\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_0 \Phi \supset \Psi$ says that there exists a cut-free proof in G^∞ whose depth is bounded by the notation $\mathfrak{a}$ and $\mathfrak{a} \lhd \Omega_k$. If we have $\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_1 \Phi \supset \Psi$, then only set-closed formulas which are elementary or Σ^1_1 are permitted as cut formulas.

Since the main formulas of all axioms and the main formulas of the conclusions of all Σ^1_1 collection rules are elementary or Σ^1_1 formulas of $\mathcal{L}^\infty_{\mathcal{W}}$, partial cut elimination – eliminating all cuts whose cut formulas are neither elementary nor Σ^1_1 formulas – can be proved following standard patterns; see, for example, Schütte [15].

Theorem 21 (Partial cut elimination) *Let k be a standard natural number. Then $\mathsf{NBGW}_{<E_0}$ proves for all $n < \omega$, all $\mathfrak{a} \in E_0$ such that $\omega^{\mathfrak{a}} \lhd \Omega_k$ and all sequents $\Phi \supset \Psi$ that*

$$\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_{n+2} \Phi \supset \Psi \quad \rightarrow \quad \mathsf{G}^\infty_k \vdash^{\omega^{\mathfrak{a}}}_{n+1} \Phi \supset \Psi.$$

The axioms and rules of G^∞ are so that apart from $\in$-induction, all axioms of $\mathsf{NBGW} + (\Sigma^1_1\text{-}\mathsf{Col})$ are directly verified within G^∞. For proving the instances of $(\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_\in)$ infinite derivations are required in general.

Lemma 22 *Let k be a standard natural number. Then $\mathsf{NBGW}_{<E_0}$ proves for all set-closed formulas $\varphi[p_\emptyset]$:*

1. *For all ordinals α, all sets a of set-theoretic rank α and all ordinals β such that $\beta = \omega^\alpha + \omega + 2$,*

$$\mathsf{G}^\infty_k \vdash^{\overline{\beta}}_0 \forall h_m((\forall h_n \in h_m)\varphi[h_n] \rightarrow \varphi[h_m]) \supset \varphi[p_a].$$

2. $\mathsf{G}^\infty_k \vdash^{\Omega}_0 \forall h_m((\forall h_n \in h_m)\varphi[h_n] \rightarrow \varphi[h_m]) \supset \forall h_m \varphi[h_m]$.

Proof. We let ψ be the formula $\forall h_m((\forall h_n \in h_m)\varphi[h_n] \rightarrow \varphi[h_m])$ and show the first assertion by induction on α. Given a set a of rank α, the induction hypothesis implies for all $b \in a$

$$\mathsf{G}^\infty_k \vdash^{\overline{\gamma}}_0 \psi \supset \varphi[p_b] \tag{1}$$

where $\gamma := \omega^\alpha$. If $b \notin a$, then according to (A4) and weakening

$$\mathsf{G}^\infty_k \vdash^{\overline{1}}_0 \psi \supset p_b \notin p_a. \tag{2}$$

From (1) and (2) we conclude, for any set b,

$$\mathsf{G}^\infty_k \vdash^{\overline{\gamma+1}}_0 \psi \supset p_b \notin p_a \vee \varphi[p_b].$$

By universal set quantification we thus have

$$\mathsf{G}^\infty_k \vdash^{\overline{\gamma+2}}_0 \psi \supset (\forall h_n \in p_a)\varphi[h_n],$$

and from this, simple manipulations within G^∞ also lead to

$$\mathsf{G}^\infty_k \vdash^{\overline{\gamma+\omega}}_0 \psi, (\forall h_n \in p_a)\varphi[h_n] \rightarrow \varphi[p_a] \supset \varphi[p_a].$$

Universal set quantification and contraction within the antecedent therefore finish the proof of our first assertion. The second assertion follows from the first by a universal set quantification in the succedent. $\square$

It is now routine to verify by induction on the lengths of the proofs in the system $\mathsf{NBGW} + (\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_{\in}) + (\Sigma^1_1\text{-}\mathsf{Col})$ that every theorem of $\mathsf{NBGW} + (\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_{\in}) + (\Sigma^1_1\text{-}\mathsf{Col})$ is derivable in G^∞.

Theorem 23 *Let k be a standard natural number and A a formula of $\mathcal{L}_{\mathcal{W}}$ without free set variables. If A is derivable in $\mathsf{NBGW} + (\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_{\in}) + (\Sigma^1_1\text{-}\mathsf{Col})$, then there exist standard natural numbers m and n such that $\mathsf{NBGW}_{<E_0}$ proves*

$$\mathsf{G}^\infty_k \vdash^{\Omega+\overline{m}}_{n} \; \supset \; \ulcorner A \urcorner.$$

Applying Theorem 21 finitely often we can strengthen this theorem to an interpretation of $\mathsf{NBGW} + (\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_{\in}) + (\Sigma^1_1\text{-}\mathsf{Col})$ in G^∞ with proofs whose cut formulas are either elementary or Σ^1_1 formulas and whose depths are bounded by Ω_k for suitable standard natural numbers k.

Corollary 24 *Let A be a formula of $\mathcal{L}_{\mathcal{W}}$ without free set variables. If A is derivable in $\mathsf{NBGW} + (\mathcal{L}_{\mathcal{W}}\text{-}\mathsf{I}_{\in}) + (\Sigma^1_1\text{-}\mathsf{Col})$, then there exists a standard natural number k such that $\mathsf{NBGW}_{<E_0}$ proves that there is a notation $\mathfrak{a} \lhd \Omega_k$ such that*

$$\mathsf{G}^\infty_k \vdash^{\mathfrak{a}}_{1} \supset \ulcorner A \urcorner.$$

The next step is to introduce a truth definition for the set-closed formulas. This truth definition will always depend on a class U such that the class parameters are interpreted as projections $(U)_a$ (a any set) of U and the class quantifiers range over all projections of U; the set quantifiers range over the universe of all sets.

In the following we let Lh be the elementary class function which assigns to any element φ of For^∞ the number $Lh(\varphi) < \omega$ of occurrences of logical connectives in φ. Also, F_ω is defined to be the class of all functions with domain ω; i.e. we set

$$F_\omega \; := \; \{f : Fun[f] \, \wedge \, Dom[f] = \omega\}.$$

For an $f \in F_\omega$, a set a and an $n < \omega$, we write $f_{(a|n)}$ for the element of F_ω which maps n to a and otherwise agrees with f.

Definition 25

1. *$Sat[U, V, u, v]$ is defined to be the elementary $\mathcal{L}_{\mathcal{W}}$ formula*

$$(\exists \varphi \in SC^\infty)(\exists f \in F_\omega)(u = \langle \varphi, f \rangle \; \wedge \; Lh(\varphi) = v \; \wedge \; A[U, V, f, \varphi]),$$

where $A[U, V, f, \varphi]$ is the auxiliary formula taken to be the disjunction of the following clauses:

(1) $\exists x \exists y(\varphi = (p_x \,\dot{\in}\, p_y) \wedge x \in y)$,

(2) $\exists x(\exists n < \omega)(\varphi = (p_x \,\dot{\in}\, H_n) \wedge x \in (U)_{f(n)})$,

(3) $\exists x \exists y(\varphi = \dot{\mathcal{W}}(p_x, p_y) \wedge \mathcal{W}(x, y))$,

(4) $\exists x(\varphi = \dot{\neg}\, x \wedge \langle x, f\rangle \notin V)$,

(5) $\exists x \exists y(\varphi = (x \,\dot{\vee}\, y) \wedge (\langle x, f\rangle \in V \vee \langle y, f\rangle \in V))$,

(6) $\exists x \exists y(\varphi = (x \,\dot{\wedge}\, y) \wedge \langle x, f\rangle \in V \wedge \langle y, f\rangle \in V)$,

(7) $\exists x(\exists n < \omega)(\varphi = \dot{\exists}\, h_n\, x \wedge \exists y(\langle Sub(p_y, h_n, x), f\rangle \in V))$,

(8) $\exists x(\exists n < \omega)(\varphi = \dot{\forall}\, h_n\, x \wedge \forall y(\langle Sub(p_y, h_n, x), f\rangle \in V))$,

(9) $\exists x(\exists n < \omega)(\varphi = \dot{\exists}\, H_n\, x \wedge \exists y(\langle x, f_{(y|n)}\rangle \in V))$,

(10) $\exists x(\exists n < \omega)(\varphi = \dot{\forall}\, H_n\, x \wedge \forall y(\langle x, f_{(y|n)}\rangle \in V))$.

2. *A class V is called a* satisfaction hierarchy *with respect to U if it satisfies iterating this formula Sat along the natural numbers; i.e.*

$$SH[U, V] \;:=\; (\forall n < \omega)((V)_n = \{x : Sat[U, \bigcup\{(V)_i : i < n\}, x, n]\}).$$

In this definition, the parameter U codes a universe of classes; the class V collects those pairs $\langle \varphi, f\rangle \in SC^\infty \times F_\omega$ such that φ is satisfied with respect to U if its class parameters are interpreted according to f. This leads directly to the definition of the truth of set-closed formulas with respect to a class U and an $f \in F_\omega$.

Definition 26 *For all classes U and sets f, φ we set*

$$Tr[U, f, \varphi] \;:=\; \varphi \in SC^\infty \wedge f \in F_\omega \wedge \exists X(SH[U, X] \wedge \langle \varphi, f\rangle \in (X)_{Lh(\varphi)}).$$

Note that the principle ($\mathsf{It\text{-}ECA}$) makes sure that, provable in $\mathsf{NBGW}_{<E_0}$, for every class U there exists a satisfaction hierarchy with respect to U which is essentially unique: if $SH[U, V_1]$ and $SH[U, V_2]$, then $(V_1)_n = (V_2)_n$ for all $n < \omega$. It is now an easy exercise to verify that this definition of truth has the expected closure properties

Lemma 27 *The theory $\mathsf{NBGW}_{<E_0}$ proves, for all classes U, all $f \in F_\omega$, all set-closed formulas φ, ψ, all sets x, y and all $n < \omega$, that*

$$Tr[U, f, (p_x \,\dot{\in}\, p_y)] \;\leftrightarrow\; x \in y,$$

$$Tr[U, f, (p_x \,\dot{\in}\, H_n)] \;\leftrightarrow\; x \in (U)_{f(n)},$$

$$
\begin{aligned}
Tr[U,f,\dot{\mathcal{W}}(p_x,p_y)] &\leftrightarrow \mathcal{W}(x,y),\\
Tr[U,f,\dot{\neg}\,\varphi] &\leftrightarrow \neg\, Tr[U,f,\varphi],\\
Tr[U,f,(\varphi \mathbin{\dot{\vee}} \psi)] &\leftrightarrow (Tr[U,f,\varphi] \vee Tr[U,f,\psi]),\\
Tr[U,f,(\varphi \mathbin{\dot{\wedge}} \psi)] &\leftrightarrow (Tr[U,f,\varphi] \wedge Tr[U,f,\psi]),\\
Tr[U,f,\dot{\exists}\, h_n\varphi] &\leftrightarrow \exists x\, Tr[U,f,Sub(p_x,h_n,\varphi)],\\
Tr[U,f,\dot{\forall}\, h_n\varphi] &\leftrightarrow \forall x\, Tr[U,f,Sub(p_x,h_n,\varphi)],\\
Tr[U,f,\dot{\exists}\, H_n\varphi] &\leftrightarrow \exists x\, Tr[U,f_{(x|n)},\varphi],\\
Tr[U,f,\dot{\forall}\, H_n\varphi] &\leftrightarrow \forall x\, Tr[U,f_{(x|n)},\varphi].
\end{aligned}
$$

A further expected property of this truth definition is that the truth of an set-closed elementary formula only depends on the interpretation of its class parameters. The following is obvious from, for example, the previous lemma.

Lemma 28 *In* $\mathsf{NBGW}_{<E_0}$ *we have, for all classes* U, V*, all* $f, g \in F_\omega$ *and all set-closed elementary formulas* φ*, that*

$$(\forall n < \omega)((U)_{f(n)} = (V)_{g(n)}) \rightarrow (Tr[U,f,\varphi] \leftrightarrow Tr[V,g,\varphi]).$$

This definition of truth reflects $\mathcal{L}_\mathcal{W}$ formulas without bound class variables in the appropriate way. To simplify the formulation of the following lemma, we state it only for formulas without class parameters.

Lemma 29 (Truth reflection) *Let* A *be a closed elementary formula of* $\mathcal{L}_\mathcal{W}$ *and* B *a closed* Π^1 *formula of* $\mathcal{L}_\mathcal{W}$*. Then the theory* $\mathsf{NBGW}_{<E_0}$ *proves, for any* U *and* $f \in F_\omega$*:*

1. $A \leftrightarrow Tr[U,f,\ulcorner A \urcorner]$.
2. $B \rightarrow Tr[U,f,\ulcorner B \urcorner]$.

In the following *Elm* stands for the class of all elementary $\mathcal{L}_\mathcal{W}^\infty$ formulas which contain h_0 as the only free set variable; additional free occurrences of class variables are permitted. Then we write

$$Def[U,V,u] := Def_1[U,u] \vee Def_2[U,V,u],$$

where

$$Def_1[U,u] := \exists v(u = \langle 0, v\rangle \wedge v \in U),$$

$$Def_2[U,V,u] \ := \ \begin{cases} \exists z(\exists \varphi \in Elm)(\exists f \in F_\omega)(u = \langle\langle \varphi, f\rangle, z\rangle \\ \qquad\qquad \wedge\ Sat[U,V,\langle Sub(p_z,h_0,\varphi),f\rangle, Lh(\varphi)]). \end{cases}$$

For carrying through an asymmetric interpretation of the (quasi cut-free) derivations of the systems G_k^∞ in Theorem 37 below, we need hierarchies of classes with sufficiently strong closure properties. One possible approach to provide such hierarchies is to turn to an analogue of the constructible hierarchy.

Definition 30 *Let k be a standard natural number. Then a class W is said to be a* k-constructible hierarchy *if, for all $\mathfrak{a} \in SLim_0 \cap \Omega_k$, $\mathfrak{b} \in Lim_0 \cap \Omega_k$ and $n < \omega$, we have:*

$$\begin{aligned} (W)_{\mathfrak{a}} &= \{\langle\langle x,y\rangle,z\rangle : x \in Lim_0 \cap \mathfrak{a} \wedge \langle y,z\rangle \in (W)_x\}, \\ (W)_{\mathfrak{b}+\overline{(n+1)}} &= \{x : Sat[(W)_{\mathfrak{b}}, \bigcup\{(W)_{\mathfrak{b}+\overline{y}} : 0 < y < (n+1)\}, x, n]\}, \\ (W)_{\mathfrak{b}+\overline{\omega}} &= \{x : Def[(W)_{\mathfrak{b}}, \bigcup\{(W)_y : \mathfrak{b} \lhd y \lhd \mathfrak{b}+\overline{\omega}\}, x]\}. \end{aligned}$$

The following lemma follows more or less directly, by coding two formulas into one, from the hierarchy axiom of $\mathsf{NBGW}_{<E_0}$; its proof can therefore be omitted.

Lemma 31 *Let k be a standard natural number. Then $\mathsf{NBGW}_{<E_0}$ proves the existence of a k-constructible hierarchy.*

Now assume that W is a k-constructible hierarchy. For any $\mathfrak{a} \in Lim_0$, the class $(W)_{\mathfrak{a}}$ may be considered as a code of the collection of all classes $((W)_{\mathfrak{a}})_u$, where u is an arbitrary set. The idea of this hierarchy then is as follows:

(i) $(W)_0$ codes the empty collection of classes.

(ii) For any $\mathfrak{b} \in Lim_0$, the successor stages $\mathfrak{b} + \overline{(n+1)}$ are used to collect all set-closed formulas of length n together with $f \in F_\omega$ which are true if their class parameters are interpreted by projections of $(W)_{\mathfrak{b}}$ via f and their class quantifiers range over the projections of $(W)_{\mathfrak{b}}$.

(iii) At limit stages of the form $\mathfrak{b} + \overline{\omega}$ the class $(W)_{\mathfrak{b}+\omega}$ collects $(W)_{\mathfrak{b}}$ and all classes which are definable by elementary formulas and interpretations of class parameters as projections of $(W)_{\mathfrak{b}}$.

(iv) At strong limits simply all projections of the previous limit stages are coded together.

Lemma 32 *Let k be a standard natural number. Then $\mathsf{NBGW}_{<E_0}$ proves for all k-constructible hierarchies W, all $f \in F_\omega$, all $\mathfrak{a} \in Lim_0 \cap \Omega_k$ and all set-closed formulas φ with $Lh(\varphi) = n$ and all $\psi \in Elm$:*

1. $Tr[(W)_\mathfrak{a}, f, \varphi] \ \leftrightarrow\ \langle \varphi, f\rangle \in (W)_{\mathfrak{a}+\overline{(n+1)}}$.
2. $((W)_{\mathfrak{a}+\overline{\omega}})_{\langle \psi, f\rangle} \ =\ \{x : Tr[(W)_\mathfrak{a}, f, Sub(p_x, h_0, \psi)]\}$.

The proof of this lemma is by carefully carrying out the informal considerations above; its details can be left out. Some further useful properties of hierarchies of this sort are listed in the following lemma. For its formulation and for later use we introduce the abbreviations

$$\begin{aligned} U \mathrel{\dot{\in}} V &:= \exists x(U = (V)_x), \\ U \mathrel{\dot{\subset}} V &:= \forall x((U)_x \mathrel{\dot{\in}} V), \\ U \mathrel{\dot{\subset}_\omega} V &:= (\forall n < \omega)((U)_n \mathrel{\dot{\in}} V). \end{aligned}$$

Lemma 33 *Let k be a standard natural number. Then $\mathsf{NBGW}_{<E_0}$ proves for all k-constructible hierarchies W, all $\mathfrak{a} \in Lim_0 \cap \Omega_k$ and all $\mathfrak{b} \in Lim_0 \cap \mathfrak{a}$:*

1. $(W)_\mathfrak{a} \mathrel{\dot{\in}} (W)_{\mathfrak{a}+\overline{\omega}}$ *and* $(W)_\mathfrak{a} \mathrel{\dot{\subset}} (W)_{\mathfrak{a}+\overline{\omega}}$.
2. $(W)_\mathfrak{b} \mathrel{\dot{\subset}} (W)_\mathfrak{a}$ *and* $(W)_\mathfrak{b} \mathrel{\dot{\in}} (W)_\mathfrak{a}$.

Proof. Assume that W, $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the assumptions of this lemma. Then $(W)_\mathfrak{a} \mathrel{\dot{\in}} (W)_{\mathfrak{a}+\overline{\omega}}$ follows from $(W)_\mathfrak{a} = ((W)_{\mathfrak{a}+\omega})_0$. In order to show $(W)_\mathfrak{a} \mathrel{\dot{\subset}} (W)_{\mathfrak{a}+\overline{\omega}}$, pick any set x and an $f \in F_\omega$ such that $f(0) = x$. If φ is the elementary $\mathcal{L}^\infty_\mathcal{W}$ formula $(h_0 \in H_0)$, then $((W)_\mathfrak{a})_x = ((W_{\mathfrak{a}+\overline{\omega}})_{\langle\varphi, f\rangle}$. This establishes the first assertion.

If $\mathfrak{a}$ is an element of $SLim_0$ and $\mathfrak{b} \in Lim_0 \cap \mathfrak{a}$, then $(W)_\mathfrak{b} \mathrel{\dot{\subset}} (W)_\mathfrak{a}$ directly follows from the definition of $(W)_\mathfrak{a}$. From $\mathfrak{a} \in SLim_0$ and $\mathfrak{b} \in Lim_0 \cap \mathfrak{a}$ it also follows that $\mathfrak{b} + \overline{\omega} \in Lim_0 \cap \mathfrak{a}$, hence $(W)_{\mathfrak{b}+\overline{\omega}} \mathrel{\dot{\subset}} (W)_\mathfrak{a}$. In view of the first assertion this implies $(W)_\mathfrak{b} \mathrel{\dot{\in}} (W)_\mathfrak{a}$. A simple transfinite induction on $\mathfrak{a}$, combined with the first assertion, finishes the proof of the second. □

The formula $Tr[U, f, \varphi]$ interprets the class parameters of φ by projections of U which are provided by the element f of F_ω. Sometimes it is more practical to have them coded into a class V.

Definition 34 *For classes U, V and set-closed formulas φ we set*

$$TR[U, V, \varphi] \ :=\ (\exists f \in F_\omega)((\forall n < \omega)((V)_n = (U)_{f(n)}) \ \wedge\ Tr[U, f, \varphi]).$$

For classes V, X, Y and an $n < \omega$ we write $Y = V(X|n)$ to express that $(Y)_n = X$ and $(Y)_m = (V)_m$ for any $m < \omega$ which is different from n. Then

$$TR[U, V, \varphi(X/H_n)] \;:=\; X \mathrel{\dot\in} U \;\wedge\; \exists Y(Y = V(X|n) \;\wedge\; TR[U, Y, \varphi]).$$

Hence in $TR[U, V, \varphi(X/H_n)]$ all free occurrences of the class variable H_n within φ are interpreted by X and all others according to V. Naturally, the predicate $TR[U, V, \varphi]$ inherits the closure properties stated in Lemma 27 from $Tr[U, f, \varphi]$. We collect them for later reference.

Lemma 35 *The theory* $\mathsf{NBGW}_{<E_0}$ *proves, for all classes* U, V*, all set-closed formulas* φ, ψ*, all sets* x, y *and all* $n < \omega$*, that*

$$\begin{aligned}
TR[U, V, (p_x \mathrel{\dot\in} p_y)] &\;\leftrightarrow\; x \in y, \\
TR[U, V, (p_x \mathrel{\dot\in} H_n)] &\;\leftrightarrow\; x \in (V)_n, \\
TR[U, V, \dot{\mathcal{W}}(p_x, p_y)] &\;\leftrightarrow\; \mathcal{W}(x, y), \\
TR[U, V, \dot\neg\, \varphi] &\;\leftrightarrow\; \neg\, TR[U, V, \varphi], \\
TR[U, V, (\varphi \mathrel{\dot\vee} \psi)] &\;\leftrightarrow\; (TR[U, V, \varphi] \vee TR[U, V, \psi]), \\
TR[U, V, (\varphi \mathrel{\dot\wedge} \psi)] &\;\leftrightarrow\; (TR[U, V, \varphi] \wedge TR[U, V, \psi]), \\
TR[U, V, \dot\exists\, h_n \varphi] &\;\leftrightarrow\; \exists x\, TR[U, V, Sub(p_x, h_n, \varphi)], \\
TR[U, V, \dot\forall\, h_n \varphi] &\;\leftrightarrow\; \forall x\, TR[U, V, Sub(p_x, h_n, \varphi)], \\
TR[U, V, \dot\exists\, H_n \varphi] &\;\leftrightarrow\; (\exists X \mathrel{\dot\in} U)\, TR[U, V, \varphi(X/H_n)], \\
TR[U, V, \dot\forall\, H_n \varphi] &\;\leftrightarrow\; (\forall X \mathrel{\dot\in} U)\, TR[U, V, \varphi(X/H_n)].
\end{aligned}$$

Utilizing these properties, it is routine to show (by simultaneous induction on the length of φ and ψ) that set-closed Σ^1 formulas are upward persistent and set-closed Π^1 formulas downward persistent.

Lemma 36 *Let* k *be a standard natural number. Then* $\mathsf{NBGW}_{<E_0}$ *proves for all* k*-constructible hierarchies* W*, all classes* U*, all set-closed* Σ^1 *formulas* φ*, all set-closed* Π^1 *formulas* ψ *and all* $\mathfrak{a}, \mathfrak{b} \in Lim_0 \cap \Omega_k$*:*

1. $\mathfrak{a} \lhd \mathfrak{b} \;\wedge\; TR[(W)_{\mathfrak{a}}, U, \varphi] \;\rightarrow\; TR[(W)_{\mathfrak{b}}, U, \varphi]$.
2. $\mathfrak{a} \lhd \mathfrak{b} \;\wedge\; U \mathrel{\dot\subset_\omega} (W)_{\mathfrak{a}} \;\wedge\; TR[(W)_{\mathfrak{b}}, U, \psi] \;\rightarrow\; TR[(W)_{\mathfrak{a}}, U, \psi]$.

If Φ and Ψ are finite sequences of set-closed formulas, $(\Phi \supset \Psi)^\bullet$ denotes (the Gödel number of) the disjunction whose disjuncts are the negated formulas of Φ and the formulas of Ψ.

Theorem 37 *Let k be a standard natural number. In $\mathsf{NBGW}_{<E_0}$ we can prove that, for all k-constructible hierarchies W, all classes U, all finite sequences Φ of set-closed Π^1 formulas, all finite sequences Ψ of set-closed Σ^1 formulas, all $\mathfrak{a} \lhd \Omega_k$ and all $\mathfrak{b}, \mathfrak{c} \in Lim_0 \cap \Omega_k$, we have the implication*

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}} \Phi \supset \Psi \;\wedge\; \mathfrak{b} + \omega^{\mathfrak{a}+\bar{1}} \unlhd \mathfrak{c} \;\wedge\; U \mathrel{\dot{\subset}_\omega} (W)_{\mathfrak{b}} \quad\rightarrow\quad TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi)^\bullet].$$

Proof. We show this theorem by induction on $\mathfrak{a}$, which is justified by Theorem 11, and distinguish the following cases:

1. $\Phi \supset \Psi$ is an axiom (A1)–(A4) or a conclusion of a structural rule, a propositional rule, a quantifier rule for set or a quantifier rule for classes. Then the assertion is trivially satisfied, is a consequence of Lemma 29 and Lemma 35 or follows from the induction hypothesis.

2. $\Phi \supset \Psi$ is an axiom (A5). Then Φ is empty and Ψ consists of a single formula $\exists H_m \forall h_n (h_n \in H_m \leftrightarrow \varphi[h_n])$, where $\varphi[p_\emptyset]$ is a set-closed elementary formula. In this case, the assertion is a consequence of Lemma 32, Lemma 35, and Lemma 36.

3. $\Phi \supset \Psi$ is a conclusion of a Σ^1_1 collection rule. Then the sequence Ψ is of the form $\Psi_0, \exists H_i \forall h_m \exists h_n \theta[h_m, (H_i)_{h_n}]$ for some set-closed elementary formula $\theta[p_\emptyset, H_0]$, and there exists an $\mathfrak{a}_0 \lhd \mathfrak{a}$ such that

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}_0} \Phi \;\supset\; \Psi_0, \forall h_m \exists H_n \theta[h_m, H_n].$$

For $\mathfrak{c}_0 := \mathfrak{b} + \omega^{\mathfrak{a}_0+\bar{1}}$ the induction hypothesis gives us

$$TR[(W)_{\mathfrak{c}_0}, U, (\Phi \supset \Psi_0, \forall h_m \exists H_n \theta[h_m, H_n])^\bullet].$$

Clearly, $\mathfrak{c}_0 \lhd \mathfrak{c}$, and therefore Lemma 33 implies

$$(W)_{\mathfrak{c}_0} \mathrel{\dot{\subset}} (W)_{\mathfrak{c}} \qquad \text{and} \qquad (W)_{\mathfrak{c}_0} \mathrel{\dot{\in}} (W)_{\mathfrak{c}}. \tag{1}$$

Now we set $\theta_1[h_m] := \theta[h_m, H_n]$ and $\theta_2[h_m, h_n] := \theta[h_m, (H_i)_{h_n}]$. Then by Lemma 35

$$TR[(W)_{\mathfrak{c}_0}, U, (\Phi \supset \Psi_0)^\bullet] \;\vee\; \forall x \exists y\, TR[(W)_{\mathfrak{c}_0}, U, \theta_1[p_x]((W)_{\mathfrak{c}_0})_y / H_n)],$$

and a simple persistency argument, see Lemma 36, together with (1) yields

$$TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi_0)^\bullet] \;\vee\; \forall x \exists y\, TR[(W)_{\mathfrak{c}}, U, \theta_1[p_x]((W)_{\mathfrak{c}_0})_y / H_n)].$$

This can also be written as

$$TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi_0)^\bullet] \ \vee \ \forall x \exists y \, TR[(W)_{\mathfrak{c}}, U, \theta_2[p_x, p_y]((W)_{\mathfrak{c}_0}/H_i)].$$

In view of $(W)_{\mathfrak{c}_0} \mathrel{\dot\in} (W)_{\mathfrak{c}}$, see (1), we continue with

$$TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi_0)^\bullet] \ \vee \ (\exists Z \mathrel{\dot\in} (W)_{\mathfrak{c}}) \forall x \exists y \, TR[(W)_{\mathfrak{c}}, U, \theta_2[p_x, p_y](Z/H_i)].$$

By Lemma 35 this tells us

$$TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi_0, \exists H_i \forall h_m \exists h_n \theta[h_m, (H_i)_{h_n}])^\bullet],$$

completing the treatment of this case.

4. $\Phi \supset \Psi$ is a conclusion of a cut. By assumption, its cut formula has to be a set-closed elementary formula or a set-closed formula of the form $\exists H_n \theta$, where θ is set-closed elementary. In the remainder we concentrate on the second and more complicated case. Then there exists $\mathfrak{a}_1, \mathfrak{a}_2 \lhd \mathfrak{a}$ such that

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}_1} \Phi \ \supset \ \Psi, \ \exists H_n \theta, \tag{2}$$

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}_2} \Phi, \ \exists H_n \theta \ \supset \ \Psi. \tag{3}$$

Set $\mathfrak{c}_1 := \mathfrak{b} + \omega^{\mathfrak{a}_1 + \bar{1}}$ and apply the induction hypothesis to (2). Then we obtain

$$TR[(W)_{\mathfrak{c}_1}, U, (\Phi \supset \Psi, \exists H_n \theta)^\bullet]$$

and from that, because of Lemma 35,

$$TR[(W)_{\mathfrak{c}_1}, U, (\Phi \supset \Psi)^\bullet] \ \vee \ (\exists X \mathrel{\dot\in} (W)_{\mathfrak{c}_1}) \, TR[(W)_{\mathfrak{c}_1}, U, \theta(X/H_n)]. \tag{4}$$

Furthermore, by an inversion argument (we did not formulate it explicitly but it can be proved in a straightforward way), assertion (3) gives

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}_2} \Phi, \ Sub(\langle H_m \rangle, \langle H_n \rangle, \theta) \ \supset \ \Psi, \tag{5}$$

where H_m is a fresh class variable which does not occur in $\Phi \supset \Psi$ and $\exists H_n \theta$. For $\mathfrak{c}_2 := \mathfrak{c}_1 + \omega^{\mathfrak{a}_2 + \bar{1}}$ and all $V \mathrel{\dot\subset}_\omega (W)_{\mathfrak{c}_1}$ the induction hypothesis applied to (5) – with $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ replaced by $\mathfrak{a}_2$, $\mathfrak{c}_1$ and $\mathfrak{c}_2$, respectively – yields

$$TR[(W)_{\mathfrak{c}_2}, V, (\Phi, Sub(\langle H_m \rangle, \langle H_n \rangle, \theta) \supset \Psi)^\bullet].$$

In particular, this is the case for any $V \mathrel{\dot\subset}_\omega (W)_{\mathfrak{c}_1}$ satisfying $(V)_m \mathrel{\dot\in} (W)_{\mathfrak{c}_1}$ as well as $(V)_i = (U)_i$ if $i < \omega$ and $i \neq m$. Once more we apply Lemma 35 and deduce

$$\begin{aligned} & TR[(W)_{\mathfrak{c}_2}, U, (\Phi \supset \Psi)^\bullet] \ \vee \\ & \qquad (\forall X \mathrel{\dot\in} (W)_{\mathfrak{c}_1}) \neg TR[(W)_{\mathfrak{c}_2}, U, Sub(\langle H_m \rangle, \langle H_n \rangle, \theta)(X/H_m)]. \end{aligned}$$

In view of the persistency properties formulated in Lemma 36 and an obvious exchange of variables, $TR[(W)_{\mathfrak{c}_2}, U, Sub(\langle H_m \rangle, \langle H_n \rangle, \theta)(X/H_m)]$ is equivalent, for $X \dot{\in} (W)_{\mathfrak{c}_1}$, to $TR[(W)_{\mathfrak{c}_1}, U, \theta(X/H_n)]$, and it follows that

$$TR[(W)_{\mathfrak{c}_2}, U, (\Phi \supset \Psi)^\bullet] \ \vee \ (\forall X \dot{\in} (W)_{\mathfrak{c}_1}) \neg TR[(W)_{\mathfrak{c}_1}, U, \theta(X/H_n)].$$

Together with (4) this implies

$$TR[(W)_{\mathfrak{c}_1}, U, (\Phi \supset \Psi)^\bullet] \ \vee \ TR[(W)_{\mathfrak{c}_2}, U, (\Phi \supset \Psi)^\bullet].$$

Since $\mathfrak{c}_2 = \mathfrak{c}_1 + \omega^{\mathfrak{a}_2 + \bar{1}} = \mathfrak{b} + \omega^{\mathfrak{a}_1 + \bar{1}} + \omega^{\mathfrak{a}_2 + \bar{1}} \lhd \mathfrak{b} + \omega^{\mathfrak{a} + \bar{1}} \unlhd \mathfrak{c}$, Lemma 36 proves $TR[(W)_{\mathfrak{c}}, U, (\Phi \supset \Psi)^\bullet]$, as desired.

Therefore all possible cases for deriving the sequent $\Phi \supset \Psi$ within G_k^∞ have been considered, proving our theorem. □

Corollary 38 *Let k be a standard natural number and A a closed elementary $\mathcal{L}_\mathcal{W}$ formula. Then the theory $\mathsf{NBGW}_{<E_0}$ proves, for all $\mathfrak{a} \lhd \Omega_k$, that*

$$\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}} \supset \ulcorner A \urcorner \quad \to \quad A.$$

Proof. First of all, Lemma 31 implies that there exists a k-constructible hierarchy W. Then, assuming $\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}} \supset \ulcorner A \urcorner$ and setting $\mathfrak{c} := \omega^{\mathfrak{a}+\bar{1}}$, the previous theorem implies $TR[(W)_{\mathfrak{c}}, \emptyset, \ulcorner A \urcorner]$. Because of truth reflection, c.f. Lemma 29, we therefore also have A. □

Theorem 39 (Reduction) *The theory $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$ can be reduced to the theory $\mathsf{NBGW}_{<E_0}$ with respect to all closed elementary $\mathcal{L}_\mathcal{W}$ formulas; i.e. for all closed elementary $\mathcal{L}_\mathcal{W}$ formulas A we have*

$$\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col}) \vdash A \quad \Longrightarrow \quad \mathsf{NBGW}_{<E_0} \vdash A.$$

Proof. Let A be a closed elementary $\mathcal{L}_\mathcal{W}$ formula provable in the theory $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$. According to Corollary 24 we thus have

$$\mathsf{NBGW}_{<E_0} \vdash (\exists \mathfrak{a} \lhd \Omega_k)(\mathsf{G}_k^\infty \vdash_1^{\mathfrak{a}} \supset \ulcorner A \urcorner)$$

for a suitable standard natural number k. Hence the previous corollary yields $\mathsf{NBGW}_{<E_0} \vdash A$. □

Corollary 40 (Final result) *The four theories $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{AC})$, $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$, $\mathsf{NBGW}_{<E_0}$ and $\mathsf{NBG}_{<E_0}$ are equiconsistent.*

To prove this summary, we simply recall what we have shown before: In view of Theorem 15, $\mathsf{NBG}_{<E_0}$ is contained in $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{AC})$, which, according to Corollary 3, is equivalent to $\mathsf{NBG} + (\mathcal{L}_2\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$. However, this system is obviously contained in $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$. The above reduction theorem provides the reduction of $\mathsf{NBGW} + (\mathcal{L}_\mathcal{W}\text{-}\mathsf{I}_\in) + (\Sigma_1^1\text{-}\mathsf{Col})$ to $\mathsf{NBGW}_{<E_0}$, a conservative extension of $\mathsf{NBG}_{<E_0}$ by Theorem 16. Thus the circle is closed.

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